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Market for manipulable information

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Abstract

We study how investors, firms, and information sellers interact in a market with manipulable information. To better predict the firm characteristics they care about, investors can buy a score from a monopolistic information seller, which aggregates signals that are subject to firm manipulation. The average degree of signal manipulability has no effect on the equilibrium, while the uncertainty about manipulability becomes a new source of noise. Its contribution depends on firms' incentive to manipulate the signals, which in turn depends on the equilibrium price sensitivity to the score. The optimal design of the score weighs signal precision against the endogenous uncertainty due to manipulation. The introduction of mandate investors, who care about the scores on the characteristics and not the characteristics themselves, generates an incentive for information sellers to inflate the scores. When applied to green investing, our model implies that the effectiveness of impact investing on the cost of capital could actually decline as the fraction of green investors or the strength of the mandate keeps rising, because they generate stronger incentives for manipulation.

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1 Introduction

Financial markets are in an age of information explosion. An area that has witnessed particularly striking growth is the so-called "alternative data" (examples include social media data, satellite or aerial imagery, and geolocation data), which are widely used to forecast company performances and stock returns, as well as for certification, such as credit ratings or ESG ratings. Compared to "standard data" (such as financial and accounting data, or government statistics), which are typically subject to well-established reporting and auditing requirements, alternative data could be more timely, but also more prone to manipulation. The goal of this paper is to study how investors, firms, and information sellers interact in a market with manipulable information, and their impact on asset prices.

Our model builds on the framework of Admati and Pfleiderer (1986) and introduces two new features: information manipulability and investor heterogeneity. In our baseline model, the financial market consists of a risk-free bond and a stock issued by a firm. The firm is characterized by two private independent attributes that investors value: $\tilde{x}$ and $\tilde{g}$, which could represent the firm's cash dividend and carbon footprint, for example. Two types of investors trade in the market: type-N and type-G. Type-N investors only care about the attribute $\tilde{x}$, while type-G investors care about both $\tilde{x}$ and $\tilde{g}$. A monopolistic information intermediary collects signals about $\tilde{g}$, designs a score, and sells it to investors. The firm can privately and costly manipulate the signals that are used to produce the score, with the goal of maximizing its equilibrium stock price in the short run.[1] Investors only know the distribution of signal manipulability (i.e., the cost of manipulation) across firms.

In equilibrium, the decisions of investors, the firm, and the information seller interact in rich ways. Investors must factor in the firm's potential for signal manipulation as well as the information seller's incentive for profit maximization when assessing the informativeness of the score sold by the information seller. How responsive investors (and the stock price) are to the score is a major factor in the firm's incentive to manipulate the signal. Finally, the firm's tendency for signal manipulation and investors' willingness to pay will influence how the information seller designs and prices the score.

We fully characterize the market equilibrium, the firm's signal manipulation strategy, and the information seller's optimal design and pricing of the score. Several interesting results emerge. First, while the average degree of signal manipulability can change the level of the score (e.g., stronger manipulability could lead to more score inflation), it does not alter the

[1]For example, social media reactions towards new products are being used to predict company performances. A company could influence such sentiment in various ways, from paying "influencers" to promote their products to using "bots" to post fake reviews.

informativeness of the score or the equilibrium stock price. This is because investors who care about the true value of the attribute $\tilde{g}$ already anticipate the average degree of manipulation and the bias it induces in the score, and thus can undo its effect in equilibrium. Instead, the key variable of concern about information manipulation is the uncertainty about signal manipulability across firms, which is a new source of noise in the market.

The effect of this uncertainty on the equilibrium endogenously depends on investor composition and the incentives of the information seller. As Admati and Pfleiderer (1986) show, a monopolistic information seller may want to add extra noise to a score to prevent it from becoming too revealing, since a more informative stock price will reduce investor demand for the score. In our model, uncertainty about manipulability can substitute for the need of the information seller to add noise to the score, which leads to two possibilities. First, when the uncertainty about manipulation is low, we could obtain a "manipulation irrelevance" equilibrium, in which an increase in the uncertainty about manipulability is exactly offset by a reduction in noise added by the information seller, leaving all equilibrium outcomes unaffected by the possibility of signal manipulation.

Second, when the uncertainty about manipulability is sufficiently high, the information seller does not add any extra noise and instead adjusts the price of the score to maximize profit. In this case, the informativeness of the score and the stock price will both decline when the uncertainty about manipulability increases. Furthermore, the informativeness of the score and the stock price could also decline as the share of type-G investors in the market increases, because it raises firms' incentive to manipulate information. Since price informativeness about the attribute $\tilde{g}$ could also be interpreted as how much a firm's cost of capital aligns with $\tilde{g}$, this result has an important implication for impact investing, wherein the objective often involves influencing firms' cost of capital in concordance with non-cash-flow attributes. It shows that a continued rise in impact investing (as represented by the share of type-G investors) could actually reduce its effectiveness in the presence of information manipulation.

We then examine the optimal design of the score in a setting with multi-dimensional signals. Consider first the score that is the most accurate predictor of $\tilde{g}$ (in the MSE sense). The most informative score is a weighted average of the individual signals that minimizes the total variance of the score. With the possibility of manipulation, the best predictor needs to balance the intrinsic noise of individual signals (and their covariance structure) against their uncertainty about manipulability (the average level of manipulability is again irrelevant). Importantly, the relative contribution of the uncertainty about manipulability is endogenous. It depends on the firm's incentive to manipulate individual signals, which in turn depends on the sensitivity of the stock price to the score. As a result, not only do the signals' statistical properties matter, but also the uncertainties about their manipulability and the investor composition.

Next, recall that the goal of the information seller is not to design the most informative score, but to maximize profit. This makes the optimal weights deviate from the ones that are the most informative, again to avoid making the stock price too informative. In fact, the optimal weights are generally not unique. This result suggests that different information sellers could produce different scores even with the same set of signal inputs. It offers new perspectives on the empirically documented disagreements in credit ratings (Cantor and Packer, 1997) or ESG ratings (Chatterji et al., 2016; Berg et al., 2022) across agencies.

We also examine whether the information market helps allocate stocks with higher expected values for characteristic $\tilde{g}$ to investors who value it more. In the baseline model where type-G investors care about the true characteristic $\tilde{g}$, despite the presence of manipulation, we show that the stronger these investors' preferences for $\tilde{g}$, the higher the ex-ante expected value for $\tilde{g}$ in their portfolio holdings. In other words, the information market does help tilt type-G investors' portfolios towards stocks with higher $\tilde{g}$.

In the second part of the paper, we replace the type-G investors with investors who care about the score for $\tilde{g}$, not $\tilde{g}$ itself. This assumption is meant to capture investors who face mandates that are tied to the scores for certain characteristics. We refer to this type of investors "mandate investors." For example, insurance companies face regulatory restrictions (from the NAIC) when investing in corporate bonds, whereby buying bonds with lower credit ratings (which might be different from credit quality) would subject them to higher capital requirements. Similarly, an ESG-themed mutual fund could be required to maintain a minimum ESG score (which might be different from the actual ESG attributes) for its portfolio holding. Moreover, we assume that the mandate investors can invest in the stock only if they purchase the score from the information seller.

A key distinction exists between type-G investors and mandate investors; the former assumes a disciplinary role in mitigating firm manipulation, a responsibility not shared by the latter. In the baseline model, intense manipulation by the firm leads to a decrease in the score's informativeness. As a result, the type-G investors pay less attention to the score, making the price less sensitive to the score, which ultimately reduces the firm's incentive to manipulate. This discipline effect does not exist in the mandate model. There, even if the score and the equilibrium price become less informative due to the firm's manipulation, the price may still remain highly responsive to the score, due to the fact that mandate investors care about the score itself, and not how much information it conveys about the true attribute. Therefore, the firm's manipulation incentive can be stronger in the investment mandate model.

When the information seller's score design is fixed, a stronger investment mandate, i.e., when mandate investors put a bigger weight on the score in their preferences, has a non-monotonic effect on both price informativeness and the alignment of mandate investors' portfolio with true characteristics $\tilde{g}$. Both the price informativeness and the ex-ante expected level of $\tilde{g}$ in the portfolios of mandate investors initially rise but ultimately decrease. These results again demonstrate the downside of pushing too hard for mandates on certain stock attribute in the presence of information manipulation: not only could the alignment between the attribute and the cost of capital deteriorate, but the mandate investors may end up holding less of the desired stocks.

The information seller's optimal design of the score also differs significantly in the mandate model. In the baseline model, type-G investors prefer more informative scores, but the information seller tends to avoid selling the most informative score. The average degrees of manipulability for individual signals are irrelevant and do not affect the optimal weights. In the mandate model, however, mandate investors care about the level of the score but not how informative it is. This generates an incentive for score inflation on the part of the information seller, as they put higher weights on signals with higher degrees of manipulability. At the same time, the information seller tries to keep the noise in the score low because the mandate investors' willingness to pay for the score is generally decreasing in its uncertainty. These two considerations lead to distinct trade-offs for the information seller in the mandate model. For example, a signal that is both noisy and has high uncertainty of manipulability would be considered inferior in the baseline model, but it would receive higher weights in the mandate model if it has high degree of manipulability on average.

Finally, we also consider an extension of our baseline model with sufficiently many homogeneous information sellers. We show that the firm's manipulation remains relevant and important in the equilibrium outcome, and does not disappear due to competition. Specifically, in the symmetric equilibrium, all information sellers choose to add no noise to their scores, and the price of their scores are so low, such that all type-G investors choose to buy all scores. In this case, the equilibrium price is a linear function of the average score from all information sellers. Notably, the firm's manipulation incentive does not disappear in this equilibrium, and in fact, the firm's manipulation becomes the sole source of additional noise in the market that will affect the price informativeness.

Related literature

First, our paper is closely related to the literature on information sales. Building upon the seminal work of Admati and Pfleiderer (1986), which examines a monopolist information seller designing and selling payoff-relevant information to traders, we extend this framework by considering the firm's information manipulation behavior. As a result, the information seller's input in our baseline model is endogenous to its pricing and design of information, in contrast to the exogenous input in Admati and Pfleiderer (1986). Following their work, a growing body of recent literature investigates various settings involving monopolistic information intermediaries. For example, Bergemann and Bonatti (2015) explore the pricing problem of a data broker who provides data to facilitate marketing efforts. Segura-Rodriguez (2021) examines a profit-maximizing data broker who sells data to firms seeking to forecast different consumer characteristics. Meanwhile, Yang (2022) investigates a profit-maximizing data broker who sells data to firms with private production costs. The unique aspect of our model, in which the information intermediary's actions (pricing and design) may endogenously influence the input of the information that the seller can collect, does not show up in these prior studies.

With the development of alternative data, there has been growing interest in understanding how a firm's data can influence the learning dynamics of its quality. Begenau et al. (2018) and Farboodi and Veldkamp (2022) emphasize the role of data (information) in a firm's production and lifecycle. They examine how information can affect a firm's lifecycle, highlighting the "data feedback role," where increased data availability can encourage firm growth. Our focus on how data availability changes a firm's incentives is different from their perspective. In our model, the equilibrium price is (partially) determined by the firm's information available to the information seller, which incentivizes the firm to manipulate its data, consequently reducing the informativeness of the data and price.

Another related stream of literature focuses on information manipulation by agents (firms) with strategic concerns. Numerous studies investigate the manipulation of earnings announcements (eg., Dutta and Fan (2014); Crocker and Slemrod (2007); Laux and Laux (2009)) and the incentives of managers. Among these papers, our work is most closely related to Fischer and Verrecchia (2000), which is also based on a Gaussian setting. However, our paper differs from these studies because the information in our context is neither public nor free, making the role of the information seller crucial. Another difference is that we examine how the pricing and design of information by the information seller can alter the firm's incentive to manipulate, while these earnings manipulation studies mostly focus on incentive contracts. Our framework is also related to the signaling model of Frankel and

Kartik (2019), which also discusses the firm's ability to manipulate its signals and thereby distort payoff-relevant information. However, our discussion on the role of the information seller does not have a counterpart in their paper. Both Ball (2019) and Goldman et al. (2022) consider an information intermediary's problem where information can be endogenously manipulated. However, the information intermediary's objectives in their papers significantly differ from ours, distinguishing our study from existing literature.

Our model predictions are also related to the growing literature on ESG rating divergence and greenwashing. Previous studies confirm substantial disagreements in ESG ratings (e.g., Berg et al. (2022); Chatterji et al. (2016)) due to differences in measurement and methodology. Besides, firms can strategically change their behavior to attract fund flows (e.g., Cooper et al. (2005)), while these behaviors have no real impact on firm performance, as evidenced by the greenwashing behavior documented in the ESG literature (eg., Kaustia and Yu (2021); Liang et al. (2022)). Our paper can rationalize the empirical observations of these studies. In particular, our model can explain why ESG rating agencies may choose different methodologies and why they sometimes include features that are easy to manipulate. Our model highlights the importance of the interactions between ESG rating divergence and greenwashing activities. Both Pástor et al. (2021) and Goldstein et al. (2022) consider the financial market with ESG investors. Apart from the key difference of manipulable information in our baseline model, our Section 3 on mandate investment also addresses the possibility that ESG investment operates as a mandate in an investor's problem rather than reflecting investor preferences. Our discussion on this part highlights different predictions of this angle of ESG investment on the ESG rating market.

2 Baseline Model

2.1 The Setup

The model has a single period, from time 0 to time 1. There are two assets traded in the financial market, a riskless bond with unlimited supply and a risky asset (the stocks of a firm) with random supply $\tilde{s} \sim N(\bar{s}, \sigma_s^2)$. We assume that the average supply of the stock is positive, i.e., $\bar{s} > 0$. The riskless bond is the numeraire, with the risk-free rate normalized to zero.

The stock generates two types of "payoffs" $\tilde{x}$ and $\tilde{g}$ at time 1, where $\tilde{x}$ and $\tilde{g}$ are independent and normally distributed, with $\tilde{x} \sim N(\bar{x}, \sigma_x^2)$ and $\tilde{g} \sim N\left(\bar{g}, \sigma_g^2\right)$ respectively. We consider two different interpretations for $\tilde{x}$ and $\tilde{g}$. In the first case, $\tilde{x}$ and $\tilde{g}$ are the two parts of the cash-flow payoff at $t = 1$, with $\tilde{g}$ representing the cash-flow component that is predictable by public signals (and $\tilde{x}$ the orthogonal component). Examples of such signals include accounting and market information, as well as alternative data such as app downloads, social media sentiment, or retail traffic. Under the second interpretation, $\tilde{x}$ represents cash-flow payoff at $t = 1$, while $\tilde{g}$ represents the monetary equivalent of other firm characteristics that investors might care about, such as the firm's credit rating or ESG attributes.[2] The realizations of $\tilde{x}$ and $\tilde{g}$ are unknown to all market participants at time 0.

Information seller. As in Admati and Pfleiderer (1986), we assume that there is a monopolistic information seller who collects data produced by the firm that are potentially informative about $\tilde{g}$. The information seller aggregates the data to construct a score, g_r, and sells it to investors at price Φ. The score is endogenously designed by the information seller. We assume full transparency in the seller's scoring algorithm, which, in general, includes the signals used and weights given to individual signals (linear scores are optimal in our setting), as well as the potential of additional independent noise added.

For simplicity, we first consider the case of a one-dimensional signal. In the absence of any manipulation by the firm, the signal about $\tilde{g}$ takes the form $\tilde{g} + \epsilon$, where $\epsilon \sim N(0, \underline{\sigma}^2)$. Admati and Pfleiderer (1986) show that a profit-maximizing information seller might sometimes add additional noise (that is independent of ϵ) to the score, and thus the optimal score can be viewed as $\tilde{g} + u$, where $u \sim N(0, \sigma_u^2)$ with $\sigma_u^2 \in [\underline{\sigma}^2, \infty)$. The information seller will choose both the noise in the score σ_u^2 and the price of the score Φ (which is equivalent to choosing the fraction of investors who will buy the score) to maximize their profit. In the case of multiple signals, the information seller also decides on the optimal weights given to the different signals. We study that problem in Section 2.3.

Investors. Besides the noise traders responsible for the random supply of the stock $\tilde{s}$, there are two types of active investors in the financial market, which we label as G and N, whose investor bases are θ and $1 - \theta$, respectively, with $\theta \in [0, 1]$. Both types of investors start with an initial wealth W_0 and have CARA utility. If a type-i ($i \in \{G, N\}$) investor holds ϕ units of the stock and l units of the bond, their expected utility at time 0 is represented by

$$U_G = \mathbb{E}\left[-e^{-A(\phi(\tilde{x}+\beta\tilde{g})+l)}|\mathcal{F}_G\right], \tag{1}$$

$$U_N = \mathbb{E}\left[-e^{-A(\phi\tilde{x}+l)}|\mathcal{F}_N\right], \tag{2}$$

[2] Consider for example, an investor who cares about not only the cash-flow payoff but also the firm's carbon footprint. Then we can view $\tilde{g}$ as the investor's willingness to pay to reduce the firm's carbon emission. For simplicity, we assume $\tilde{x}$ and $\tilde{g}$ are independent. The model can be extended to allow for correlation between the two, which could be more natural for certain firm characteristics.

where A is the coefficient of absolute risk aversion, and $\mathcal{F}_i$ is type-i investor's information set at time 0. The key difference between the two types of investors is that type-G investors care about both $\tilde{x}$ and $\tilde{g}$, with the coefficient $\beta \geqslant 0$ capturing the importance of $\tilde{g}$ relative to $\tilde{x}$, whereas type-N investors only care about $\tilde{x}$. Our reason to introduce type-N investors is to allow for heterogeneity in investor preferences towards non-cash-flow characteristics, such as carbon footprint. In that case, type-G investors correspond to "green investors."[3] Among the type-G investors, a fraction λ will endogenously decide to buy the score from the information seller, and the remainders (fraction $1-\lambda$) will not.

Firm. The main difference of our baseline model from Admati and Pfleiderer (1986) is the possibility for firms to manipulate the signals. The firm manager would like to maximize its market value p at $t=0$. Intuitively, in the absence of any signal manipulation, when the score g_r is higher, type-G investors will expect a higher $\tilde{g}$ at $t=1$, and they will bid up the stock price at $t=0$. This effect creates an incentive for the firm to engage in signal manipulation to increase the stock price and reduce the cost of capital. For example, one could run sentiment analysis on product reviews posted on social media to predict revenue growth. This could provide companies with the incentive to fabricate positive reviews.[4]

Specifically, the firm manager can change the level of the signal by δ at a private cost of $\frac{1}{2q}\delta^2$, where q represents signal manipulability and is privately observed by the firm – the higher q is, the easier it is for a firm to alter the signal. Only the firm manager knows q; the public (including all investors and the information seller) belief about q follows $q \sim N\left(\bar{q}, \sigma_q^2\right)$, where $\bar{q} \gg \sigma_q^2$. This assumption should be viewed as an approximation of $q = \max(0, \tilde{q})$ where $\tilde{q} \sim N\left(\bar{q}, \sigma_q^2\right)$, which ensures that q stays positive.The cost of manipulation can include direct costs of manipulation activities as well as indirect costs, such as the expected costs of punishment if getting caught of cheating. After signal manipulation, the score generated by the information seller becomes

$$g_r = \tilde{g} + u + \delta. \tag{3}$$

While active investors and the information seller are aware of the firm's incentives to manipulate the signals, they cannot precisely forecast the amount of manipulation by the firm and remove its effect from the score entirely due to the lack of precise knowledge of q.

The firm manager's objective is to maximize the stock price net of the cost of information

[3] Indeed, we will eliminate type-N investors by setting $\theta = 1$ when interpreting $\tilde{g}$ as the predictable component of firm cash flow.

[4] As another example, see Elon Musk's dispute with Twitter regarding its number of legitimate users before acquisition.

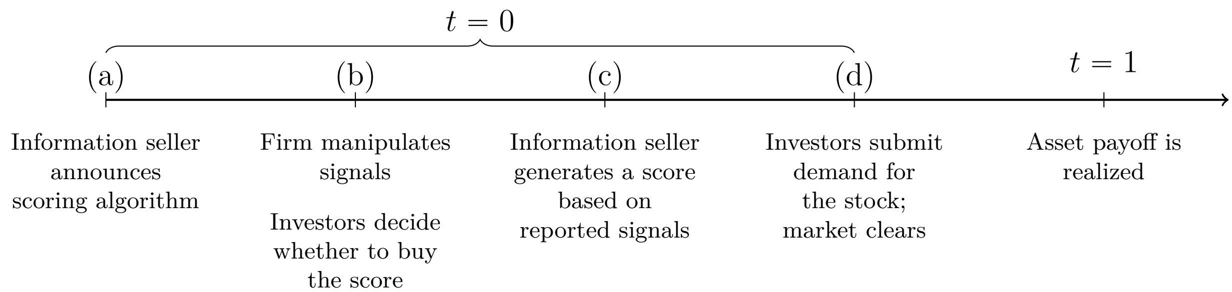

Figure 1: Model timeline

manipulation,

$$\max_{\delta} \mathbb{E}\left(p\right) - \frac{1}{2q}\delta^2. \tag{4}$$

Figure 1 shows the timeline of the model: At time 0, the information seller chooses both the noise σ_u^2 and the price of the score Φ. Then the firm decides on the level of manipulation δ, and investors decide whether to buy the score from the information seller. After that, all investors submit their demand functions and the price p is endogenously determined in equilibrium. At time 1, all random variables are realized, and all players collect their payoffs.

Next, we define the market equilibrium in the presence of information manipulation.

Definition 1 (Noisy Rational Expectations Equilibrium)**.** *We define an equilibrium as* $\{\Phi^*, \sigma_u^*, \lambda^*, p^*, \delta^*\}$*, where* Φ^* *is the price of the score chosen by the information seller,* σ_u *is the standard deviation of the noise in the score chosen by the information seller,* λ^* *is the fraction of type-G investors who opt to purchase the score, and* δ^* *is the manipulation level that the firm selects, such that the following conditions hold:*

- *a fraction* λ^* *of the type-G investors purchase the score, while the remaining fraction* $(1-\lambda^*)$ *choose not to purchase the score, to maximize their utility* (2)*;*
- *the firm selects* δ^* *to maximize the objective* (4)*;*
- *the information seller sets the price of the score* Φ^* *and the standard deviation of the noise in the score* σ_u^* *to maximize its profit;*
- *the equilibrium price* p^* *clears the market, i.e., the total demand for the stock by all active investors equals supply* $\tilde{s}$*.*

2.2 Model Solution

We solve the market equilibrium with manipulation in two steps. First, we solve the partial equilibrium while taking the information seller's strategy (noise level σ_u and fraction of score buyers among type-G investors λ) as given. The price of the score Φ is determined endogenously in this partial equilibrium. Second, we solve the information seller's problem.

Given the fraction of score buyers among type-G investors λ and the noise of the score σ_u, we conjecture that the equilibrium price of the stock has a linear structure, i.e., there exist endogenous coefficients a_0, a_r, and a_s such that

$$p = a_0 + a_r g_r + a_s \tilde{s}. \tag{5}$$

Since only g_r is observable (not $\tilde{g}$), the equilibrium price is linear in the score g_r, and a_r represents the sensitivity of stock price to the score.

From (3) and (5), it is easy to show that the firm's optimization problem (4) also has a linear solution:

$$\delta^* = a_r q, \tag{6}$$

and thus the equilibrium score is

$$g_r = \tilde{g} + u + a_r q. \tag{7}$$

Intuitively, controlling for the cost of manipulation q, the more sensitive the equilibrium price is to changes in the score g_r, the stronger the firm's incentive to manipulate the score. Manipulation in turn has two effects on the score: it inflates the score by $a_r\bar{q}$ on average, and makes it less informative by adding additional noise with variance $a_r^2\sigma_q^2$.

The score in (7) is uninformative about $\tilde{x}$. Since type-N investors only care about $\tilde{x}$ in their utilities, they will never buy the score. With CARA utility, their optimal demand for the stock is

$$\phi_N^* = \frac{\mathbb{E}\left(\tilde{x}|\mathscr{F}_N\right) - p}{A\mathrm{Var}\left(\tilde{x}|\mathscr{F}_N\right)}. \tag{8}$$

Type-G investors face different inference problems about $\tilde{g}$ depending on their decision to purchase the score. Those who buy the score observe both the equilibrium price p and the score g_r, while those who do not buy the score only observe the equilibrium price. Their optimal demand for the stock is

$$\phi_i^* = \frac{\mathbb{E}\left(\tilde{x} + \beta\tilde{g}|\mathscr{F}_i\right) - p}{A\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|\mathscr{F}_i\right)}, \tag{9}$$

where $i \in \{I, U\}$ represents type-G investors who buy and do not buy the score, respectively. Both (8) and (9) show that investors will trade more aggressively in response to news about stock payoff when the uncertainty about the payoff is lower.

The market-clearing condition is

$$(1-\theta)\,\phi_N^* + \theta\left[(1-\lambda)\phi_U^* + \lambda\phi_I^*\right] = \tilde{s}. \tag{10}$$

Before presenting the solution, we define a few variables of interest, which also help simplify the exposition. First, we measure the informativeness of the score g_r about the payoff component $\tilde{g}$ with the correlation $\mathrm{corr}(\tilde{g}, g_r)$, and define

$$r_1 \equiv \mathrm{corr}^2(\tilde{g}, g_r) = \frac{\sigma_g^2}{\sigma_g^2 + \sigma_u^2 + a_r^2\sigma_q^2} \in (0,1) \tag{11}$$

which is also the R^2 of the regression that uses g_r to predict $\tilde{g}$.[5] Notice that the additional noise in the score comes from three sources: the intrinsic noise in the signals and potential noise added by the information seller, which together has total variance σ_u^2, and the uncertainty about signal manipulability, $a_r^2\sigma_q^2$.

We measure the informativeness of the stock price p about $\tilde{g}$ with $\mathrm{corr}(\tilde{g}, p)$, and define

$$r_2 \equiv \mathrm{corr}^2(\tilde{g}, p) = \frac{\sigma_g^2}{\sigma_g^2 + \sigma_u^2 + a_r^2\sigma_q^2 + \left(\frac{a_s}{a_r}\right)^2\sigma_s^2} \in (0,1) \tag{12}$$

which is also the R^2 of the regression that uses p to predict $\tilde{g}$. Compared to the score, the price contains additional noise due to noise trader supply. Finally, we use

$$n \equiv \frac{\mathrm{Var}(\beta\tilde{g})}{\mathrm{Var}(\tilde{x} + \beta\tilde{g})} = \frac{\beta^2\sigma_g^2}{\sigma_x^2 + \beta^2\sigma_g^2} \tag{13}$$

to denote how much the predictable component $\tilde{g}$ contributes to the total variance of stock payoff for type-G investors.

The following proposition summarizes the solution in the partial equilibrium when the

[5] Notice that the correlation $\mathrm{corr}(\tilde{g}, g_r)$ is always non-negative in our setting. More generally, one would use r_1 to measure the informativeness of the score g_r when $\mathrm{corr}(\tilde{g}, g_r)$ could potentially be negative.

information seller's strategy is taken as given.

Proposition 1. *Suppose that the information seller chooses noise level σ_u^2 and sets the price for the score such that a fraction λ of type-G investors choose to buy the score. Then there exists an equilibrium such that:*

1. *the firm's manipulation strategy is $\delta^* = a_r q$,*
2. *the equilibrium score is $g_r = \tilde{g} + u + a_r q$,*
3. *the equilibrium price is $p = a_0 + a_r g_r + a_s \tilde{s}$,*

where a_0, $a_r > 0$ and $a_s < 0$ are uniquely solved by a system of equations:

$$n - n\theta \frac{\lambda}{\beta A \sigma_g^2} \frac{a_s}{a_r} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)} \right)^2 \frac{\sigma_q^2}{\sigma_g^2}, \tag{14a}$$

$$a_r = \frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}, \tag{14b}$$

$$a_0 = \bar{x} + \frac{\lambda \frac{\beta \bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + (1-\lambda)\frac{\beta \bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} - A\bar{s}}{\frac{1-\theta}{\theta\sigma_x^2} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + \frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}} - \left(a_r\left(\bar{g} + a_r\bar{q}\right) + a_s\bar{s}\right). \tag{14c}$$

The key variable that affects the firm's manipulation incentive is a_r. First, an increase in a_r tends to increase the correlation between the price p and the score g_r, assuming that a_s is unchanged. This makes the price more informative. Second, an increase in a_r also intensifies the firm's manipulation, as δ^* is an increasing function of a_r. This tends to reduce the informativeness of the score, consequently reducing the informativeness of the price. The trade-off between these two opposing effects is the key to our main analysis.

Notice that $\bar{q}$, which represents the average manipulability of the signal, has no effect on either the equilibrium price or the informativeness of the score and the price. Higher $\bar{q}$ does lead to more score inflation on average, but this effect on the equilibrium price is exactly offset by a_0. Intuitively, type-G investors care about $\tilde{g}$; when they update beliefs about $\tilde{g}$ through the score, they will take the perceived manipulation activities into account. Since the average degree of signal manipulation is fully anticipated, it does not change these investors' learning or decision making. Instead, what matters for the informativeness of score and price is the uncertainty about manipulability across firms, as measured by σ_q^2. The following corollaries show how signal manipulability affects the informativeness of scores and prices.

Corollary 1. *Suppose that the information seller chooses noise level σ_u^2 and sets the price for the score such that a fraction of λ type-G investors choose to buy the score. When $\bar{q}$ increases, both a_r and $Var(\delta^*) = a_r^2\sigma_q^2$ are unchanged. When σ_q^2 increases, a_r decreases and $Var(\delta^*) = a_r^2\sigma_q^2$ increases.*

Corollary 2. *Suppose that the information seller chooses noise level σ_u^2 and sets the price for the score such that a fraction of λ type-G investors choose to buy the score. Then*

- *$corr(g_r, \tilde{g})$, $corr(p, \tilde{g})$, and $corr(p, g_r)$ are independent of average manipulability $\bar{q}$;*
- *$corr(g_r, \tilde{g})$, $corr(p, \tilde{g})$, and $corr(p, g_r)$ are all decreasing in the uncertainty about manipulability σ_q^2.*

Corollary 1 shows that higher uncertainty about signal manipulability σ_q^2 reduces the sensitivity of the price to the score, while Corollary 2 shows that higher σ_q^2 makes both the score g_r and price p less informative about $\tilde{g}$, and that the correlation between p and g_r also declines. To understand these results, let us consider how a rise in the uncertainty about signal manipulability changes the noise in the score. A direct effect of higher σ_q^2 is to make the score more noisy, due to the fact that the amount of signal manipulation in the score is proportional to q (holding a_r constant; see (6) and (7)). An indirect effect of higher σ_q^2 is that it makes the price less responsive to the score (a_r decreases) and lowers the correlation between the two, because investors' beliefs about $\tilde{g}$ will become less responsive to the score as it becomes more noisy. This in turn reduces the firm's incentive for signal manipulation, i.e., δ^* decreases when holding q constant. However, since the direct effect is stronger, the overall uncertainty in the amount of manipulation, as captured by $\text{Var}(\delta^*) = a_r^2\sigma_q^2$, still increases. This is why the score becomes a noisier predictor for payoff component $\tilde{g}$. Finally, since the information in price about $\tilde{g}$ ultimately stems from the score, it follows that the correlation between price and $\tilde{g}$ also decreases with σ_q^2.

Next, we turn to the information seller's optimization problem, which involves determining the optimal price of the score Φ^* and the noise level σ_u^{*2}.

A key difference between our setting and Admati and Pfleiderer (1986) is the possibility of signal manipulation, which influences the information seller's incentive to sell the score. Selling the score to more investors can increase the informativeness of the price regarding payoff component $\tilde{g}$. However, the firm's manipulation incentive may also increase, which makes the equilibrium score nosier and weaken the information spillover effect. The following lemma shows that the information spillover effect still dominates the manipulation incentive. Therefore, selling the score to more investors always leads to a more informative price regarding the true fundamental information.

Lemma 1. *Suppose in equilibrium, a fraction λ of type-G investors choose to buy the score and the information seller chooses noise level σ_u^2. Then when λ increases, the informativeness of the price regarding the fundamental information $\tilde{g}$, represented by $corr(p, \tilde{g})$, increases.*

Those type-G investors who buy the score must pay a cost Φ in exchange for more information about $\tilde{g}$. In equilibrium, a type-G investor should be indifferent between purchasing the score or not, which leads to the following classic result in the literature regarding the price of the score.

Lemma 2. *In equilibrium, let $\mathcal{F}_U$ and $\mathcal{F}_I$ be the information sets of type-G investors who buy and do not buy the score, respectively. Then the equilibrium price of the score for any given λ is*

$$\Phi(\lambda) = \frac{1}{2A} \ln \left(\frac{Var(\tilde{x} + \beta\tilde{g}|\mathscr{F}_U)}{Var(\tilde{x} + \beta\tilde{g}|\mathscr{F}_I)} \right). \tag{15}$$

Notice that the dependence of the price of the score Φ on λ in (15) is implicit. As the fraction of investors who choose to purchase the score changes, so does the informativeness of the price relative to that of the score, which determines the right-hand-side of (15).

Lemma 3. *Suppose the information seller chooses a noise level σ_u^2 and price $\Phi(\lambda)$ such that a fraction λ of type-G investors buy the score. There exists a threshold $\underline{\sigma}_u^2$, such that:*

- *when $\sigma_u^2 \leqslant \underline{\sigma}_u^2$, $\Phi(\lambda)$ first increases and then decreases in σ_q^2;*
- *when $\sigma_u^2 > \underline{\sigma}_u^2$, $\Phi(\lambda)$ monotonically decreases in σ_q^2.*

The intuition for the non-monotonicity result in Lemma 3 is as follows. Holding σ_u^2 fixed, as σ_q decreases, the score becomes less noisy, and one would expect investors' willingness to pay for the score to increase, pushing the price for the score higher. However, a competing source of information for investors is the stock price. When the score is very precise, investors who buy the score will trade aggressively and impound large amount of information into the stock price. This will tend to reduce the investors' willingness to pay for the score. As Lemma 3 shows, this latter effect dominates when σ_q and σ_u^2 are both low.

The total profit for the information seller is $\theta\lambda\Phi(\lambda)$. With the equilibrium price of the score in (15), we can state the information seller's problem as follows:

$$\max_{\lambda\in[0,1],\sigma_u^2\geqslant\underline{\sigma}^2} \lambda\frac{1}{2A}\ln\left(\frac{\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right)}{\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right)}\right). \tag{16}$$

Before presenting the full solution to the equilibrium, we first discuss a quantity of interest, $-\frac{a_s}{a_r}$. Notice that the equilibrium stock price can be rewritten as

$$p = a_0 + a_r\left(g_r + \frac{a_s}{a_r}\tilde{s}\right),$$

where the term $-\frac{a_s}{a_r}$ measures the effect of the noisy supply $\tilde{s}$ on stock price.[6] A larger $-\frac{a_s}{a_r}$ makes the price a more noisy predictor of $\tilde{g}$.

According to Proposition 1, when the information seller chooses λ and σ_u^2, the equilibrium quantity $-\frac{a_s}{a_r}$ is solved by (14a). It is obvious that $-\frac{a_s}{a_r}$ is an increasing function of σ_u^2. For any given λ, there is a minimum level for $-\frac{a_s}{a_r}$, $M(\lambda)$, which is the solution to the following equation,

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}M\left(\lambda\right) = 1 + \frac{\underline{\sigma}^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)M\left(\lambda\right)\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}{1+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{17}$$

This is essentially the same equation as (14a) but with σ_u^2 replaced by its lower bound $\underline{\sigma}^2$. Thus, the noise in the equilibrium price contributed by the noisy supply is minimized when $-\frac{a_s}{a_r} = M\left(\lambda\right)$, and the information seller can effectively achieve any level $-\frac{a_s}{a_r} \geqslant M(\lambda)$ by raising σ_u^2 from $\underline{\sigma}^2$.

We now characterize the information seller's optimal choice of score price Φ^* and noise level σ_u^{*2} in the following theorem.

Theorem 1. *Define*

$$\overline{\lambda} \equiv \frac{\lambda_0 A\sigma_s}{\theta}\sqrt{\sigma_x^2+\beta^2\sigma_g^2}, \tag{18}$$

where λ_0 is a constant number solved in the appendix. Then the information seller's optimal choice Φ^* and σ_u^{*2} are characterized as follows:

[6] Naturally, $a_r > 0$ and $a_s < 0$ and thus $|\frac{a_s}{a_r}| = -\frac{a_s}{a_r}$.

1. *If* $\overline{\lambda} \leqslant 1$ *and* $M\left(\overline{\lambda}\right) \leqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$*, then the information seller chooses* σ_u^{*2} *satisfying*

$$Var(g_r) = \sigma_g^2 + \sigma_u^{*2} + a_r^{*2}\sigma_q^2 = n\left(\sigma_g^2 + \frac{\theta\bar{\lambda}}{A\beta}\frac{\sigma_g}{\sigma_s}\sqrt{n}\right), \tag{19}$$

   where a_r^* *is the equilibrium* a_r *in the price equation and is defined by* (A.93) *in the appendix. The price* Φ^* *is set such that a fraction* $\lambda^* = \overline{\lambda}$ *of type-G investors choose to buy the score.*

2. *If* $\overline{\lambda} > 1$ *and* $M(1) \leqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$*, then the information seller chooses* σ_u^{*2} *satisfying*

$$Var(g_r) = \sigma_g^2 + \sigma_u^{*2} + a_r^{**2}\sigma_q^2 = n\left(\sigma_g^2 + \frac{\theta}{A\beta}\frac{\sigma_g}{\sigma_s}\sqrt{n}\right), \tag{20}$$

   where a_r^{**} *is the equilibrium* a_r *in the price equation, and is defined by* (A.102) *in the appendix. The price* Φ^* *is the highest price under which all type-G investors choose to buy the score* ($\lambda^* = 1$)*.*

3. *Otherwise, in equilibrium, the information seller always chooses* $\sigma_u^2 = \underline{\sigma}^2$*. The price* Φ^* *is the highest price under which a fraction* $\lambda^* = \lambda_c$ *of type-G investors choose to buy the score, where* λ_c *satisfies*

$$\lambda_c = \arg\max_{\lambda\in[0,1]} \lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{M(\lambda)} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}M\left(\lambda\right)\right)}\right). \tag{21}$$

Theorem 1 shows that, depending on the model parameters, there are three possible equilibria (see Figure 2 for an illustration). In the first unconstrained equilibrium, type-G investors who buy and do not buy the score coexist, and the information seller chooses an interior noise level to add to the score. The other two equilibria are constrained. In the second possible equilibrium, the information seller sells the score to all type-G investors, resulting in no ex-post heterogeneity among them. Nevertheless, the information seller still opts to add additional noise to the score. In the third possible equilibrium, the information seller does not add any noise to the score (σ_u^2 is constrained to $\underline{\sigma}^2$), such that the total noise level is determined by the intrinsic noise in the signal, and a fraction $\lambda_c < 1$ of the investors buy the score.

An interesting property of the unconstrained equilibrium (Case 1 of Theorem 1) is that the equilibrium mass of score buyers $\theta\lambda^*$ and the price of the score are both independent of the fraction of type-G investors θ. The first part of this result immediately follows from (18).

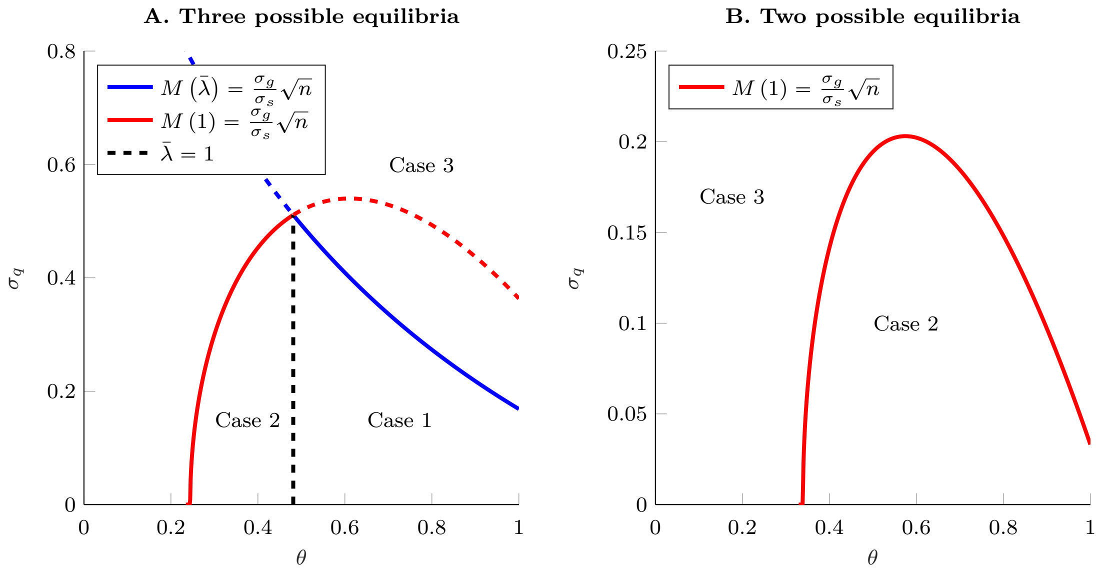

Figure 2: **Possible senarios of Theorem 1.** We consider how (θ, σ_q) affect the equilibrium structure. Parameters values are $\sigma_g = 0.3$, $\sigma_s = 1$, $\sigma_x = 0.2$, $A = 1.5$, and $\underline{\sigma} = 0.1$. We set $\beta = 1.5$ for the left panel and $\beta = 6$ for the right panel. There is no unconstrained equilibrium in the right panel because $\bar{\lambda} > 1$.

The score price, as (15) shows, depends on the ratio of posterior beliefs $\text{Var}(\tilde{x} + \beta\tilde{g}|\mathcal{F}_U)$ and $\text{Var}(\tilde{x} + \beta\tilde{g}|\mathcal{F}_I)$, both of which depend only on the mass of score buyers $\lambda\theta$ and the impact of noisy supply on the price $-\frac{a_s}{a_r}$, which in turn depends on σ_u^2. Thus, in the unconstrained equilibrium, when θ changes, the information seller will adjust σ_u^2 such that the same $\theta\lambda^*$ and Φ^* maximize their profit, $\theta\lambda^*\Phi^*$.

In fact, the intuition that the information seller can adjust σ_u^2 to offset the effect from signal manipulation leads to a more general irrelevance result in the unconstrained equilibrium, which we summarize in Proposition 2 below.

Proposition 2. *(Manipulation Irrevalance) In the unconstrained equilibrium, the optimal price Φ^*, the variance of the score $Var(g_r)$, and the informativeness of both the score g_r and price p are identical to those in the model where the firm cannot manipulate.*

The next proposition shows how the uncertainty about signal manipulability σ_q^2 affects the information seller's strategy.

Proposition 3. *Starting from the unconstrained equilibrium, as the uncertainty about signal manipulability σ_q^2 increases, the information seller first responds by decreasing the noise level σ_u^2 while keeping the score price Φ^* unchanged. Once the noise level reaches the lowest possible level $\underline{\sigma}^2$, the information seller then responds by adjusting the score price.*

Unlike in Lemma 3, where we hold the noise level σ_u^2 and the fraction λ of type-G investors who buy the score fixed, in Proposition 3 we examine the relationship between score price and signal manipulability under optimal choices of σ_u^2 and λ. When the signal is sufficiently precise to begin with and the noise resulting from the firm's signal manipulation is not too high, the information seller will want to add extra noise to the score to avoid making the score and the price too informative about $\tilde{g}$. This result is qualitatively similar to Admati and Pfleiderer (1986), and it has an important implication. It means that, as the uncertainty due to signal manipulation rises, initially it simply reduces the need for the information seller to add more noise. In fact, the information seller is indifferent about the fact that the signal is prone to manipulation under this scenario. Only when σ_q^2 becomes so high that the information seller is no longer adding any noise to the score will they begin to care about the noise added due to signal manipulation, which will reduce their profits. Such preferences become important in the score design problem when there are multiple signals, which we investigate next.

In the following proposition, we examine how investors' risk aversion A affects the equilibrium.

Proposition 4. *In the unconstrained equilibrium characterized by Theorem 1, when the risk aversion A increases locally,*

- *the fraction of type-G investors who buy the score, λ^*, increases;*
- *the value of a_r^*, which represents the sensitivity of the equilibrium price to the score ($\partial p/\partial g_r$), increases. Consequently, the firm's incentive to manipulate the signal increases, as does the noise that signal manipulation adds to the score, $Var(\delta^*)$;*
- *the information seller's choice σ_u^{*2} decreases. The total variance of the score $Var(g_r)$ and the informativeness of the score about $\tilde{g}$, as measured by $corr(g_r, \tilde{g})$, are unchanged;*
- *the variance of the equilibrium price $Var(p)$ increases, and the informativeness of the equilibrium price about $\tilde{g}$, measured by $corr(p, \tilde{g})$, is unchanged.*

Proposition 4 shows that, in the unconstrained case, the informativeness of the score and the price are unchanged when investors become more risk-averse. Intuitively, with higher risk aversion, investors are trading less aggressively, which leads to less information spillover, and thus the information seller is willing to sell the score to more investors (larger λ^*), which leads to more information spillover. These two effects cancel out, and thus the information seller chooses to keep the informativeness of the score unchanged to maximize the profit. The informativeness of the price is also unchanged since the two opposite effects cancel out.

More investors purchasing the score tends to make the price more sensitive to the score. At the same time, the portfolio rules (8)-(9) show that higher A reduces investors' demand for the stock, all else equal, which could make the price less responsive to changes in the score. It turns out that the first effect dominates, resulting in higher price sensitivity to the score a_r^*. And the higher price sensitivity to the score strengthens the firm's incentive to manipulate the signal.

We finish this section by examining how type-G investors' preferences for the payoff component $\tilde{g}$ affects their equilibrium portfolio holdings. In particular, does the market for information help allocate stocks with higher expected value for $\tilde{g}$ towards investors who value such characteristics more? Proposition 5 below shows that the answer is yes, even in the presence of signal manipulation (at least when β is sufficiently high).

Before stating the results, we first define an ex-ante measure for the expected value of $\tilde{g}$ in type-G investors' stock holding,

$$\mathbb{E}[\tilde{g}|G] \equiv \frac{\mathbb{E}\left[\left(\theta\left((1-\lambda)\,\phi_U^* + \lambda\phi_I^*\right)\right)\tilde{g}\right]}{\mathbb{E}\left[\theta\left((1-\lambda)\,\phi_U^* + \lambda\phi_I^*\right)\right]}. \tag{22}$$

This is the dollar-weighted expected payoff $\tilde{g}$ across type-G investors' holdings, which shows how much type-G investors' investment tilts towards stocks with higher expected value for $\tilde{g}$. For reference, in the absence of the market for information about $\tilde{g}$, the holdings of all investors will have $\tilde{g}$ randomly drawn, and thus $\mathbb{E}[\tilde{g}|G] = \bar{g}$.

With the market for information about $\tilde{g}$, type-G investors will buy more of the stock when its expected value for $\tilde{g}$ is higher. This is especially true for those investors who buy the score, which will further reduce the uncertainty about $\tilde{g}$. As type-G investors start to care more about $\tilde{g}$, as captured by a higher β, it should result in more type-G investors buying the score, as well as stronger demand for stocks with higher expected $\tilde{g}$. An offsetting force in our model is that firms' incentive to manipulate the signal also strengthens, which will tend to make the score more noisy. However, as Proposition 5 shows, the overall effect is that higher β still makes the portfolio of type-G investors tilting more towards stocks with higher $\tilde{g}$ in expectation.

Proposition 5. *Suppose $\underline{\sigma} = 0$. In the case when β is sufficiently high, when β increases locally, the expected $\tilde{g}$ of the risky asset held by all type-G investors, represented by $\mathbb{E}[\tilde{g}|G]$, increases. Besides, in this case, the information seller will always choose $\sigma_u = \underline{\sigma} = 0$ and set the price optimally such that $\lambda = 1$.*

Figure 3 plots how the price of the equilibrium score Φ, the number of buyers of the score $\theta\lambda^*$ and the price informativeness $\text{corr}(p, \tilde{g})$ change with the uncertainty of the score

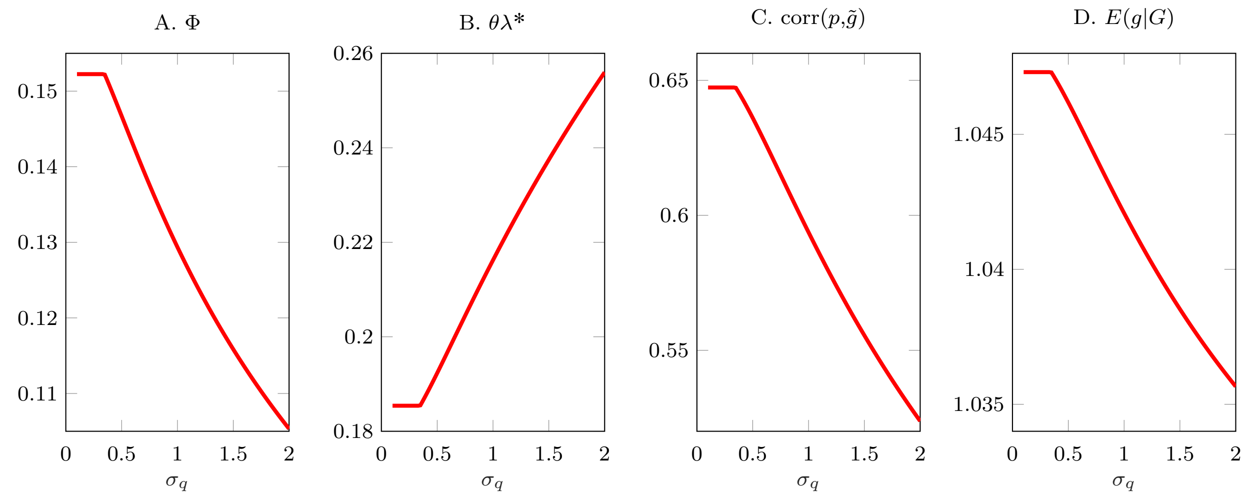

Figure 3: **Equilibrium outcomes under different σ_q.** Parameters values are $\sigma_g = 0.3$, $\sigma_s = 0.3$, $\sigma_x = 0.2$, $A = 1.5$, $\underline{\sigma} = 0.2$, $\beta = 2$, $\theta = 0.5$, $\bar{g} = 1$ and $\bar{s} = 1$.

manipulability σ_q. At low leves of σ_q, the additional noise from score manipulation is limited. Due to the manipulation irrelevance result Proposition 2, equilibrium outcomes such as score price Φ, total mass of score buyers $\theta\lambda^*$ and price informativeness $\text{corr}(p, \tilde{g})$ are independent of σ_q. When σ_q continues to increase, the equilibrium will enter the third case in Theorem 1. An increase in σ_q makes the score more noisy, which makes the score less valuable to the score buyers, and also reduces the information spillover externality. The numerical results suggest that, to maximize profit, the information seller chooses a lower score price and sells the score to more investors. It is also intuitive that the price informativeness is lower due to higher uncertainty in the manipulability. The following proposition formalizes some of the results.

Proposition 6. *There exists $k_1 > k_2$, such that*

1. *when $\sigma_q \leqslant k_2$, all of $corr(g_r, \tilde{g})$, $corr(p, \tilde{g})$, and $\mathbb{E}(\tilde{g}|G)$ are locally independent of σ_q;*
2. *when $\sigma_q \geqslant k_1$, all of $corr(g_r, \tilde{g})$, $corr(p, \tilde{g})$, and $\mathbb{E}(\tilde{g}|G)$ are locally decreasing in σ_q.*

Figure 4 plots how the price of the equilibrium score Φ, the number of buyers of the score $\theta\lambda^*$ and the price informativeness $\text{corr}(p, \tilde{g})$ change with the mass of type-G investors θ. The blue dotted line represents the benchmark case in which manipulation is impossible. These numerical results suggest that, for lower values of θ, due to the limited information spillover externality, the information seller optimally chooses to sell the most informative score to all type-G investors. When θ increases in this region, the price informativeness will increase because more investors are trading on information about $\tilde{g}$. When θ lies within an intermediate range, the equilibrium corresponds to the unconstrained equilibrium described

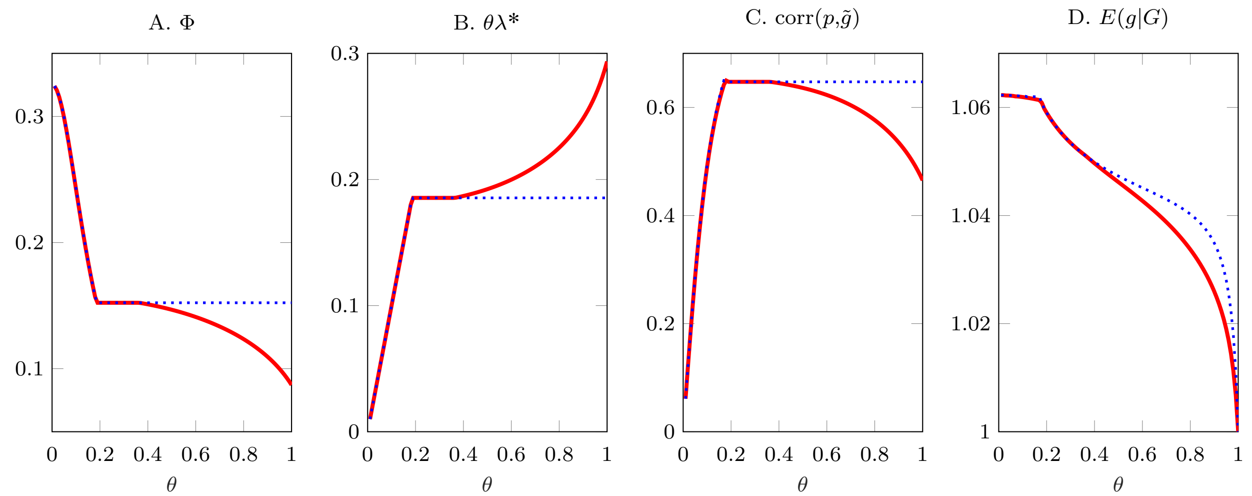

Figure 4: **Equilibrium outcomes under different θ.** Parameters values are $\sigma_g = 0.3$, $\sigma_s = 0.3$, $\sigma_x = 0.2$, $A = 1.5$, $\underline{\sigma} = 0.2$, $\beta = 2$, $\sigma_q = 0.34$, $\bar{g} = 1$ and $\bar{s} = 1$.

in Theorem 1. In this case, the information seller maintains a constant target level for both the price and the number of score buyers. However, an overly high θ makes the price too sensitive to the score, which leads to too much score manipulation by the firm. In this case, the information seller will choose to sell the most informative score again. When θ increases in this region, the score is sold to more investors, but the price informativeness actually decreases due to excessive manipulation by the firm. The following proposition formalizes this result.

Proposition 7. *For any θ satisfying $\theta > \frac{1}{3}\left(1 + \frac{\sigma_x^2}{\beta^2 \sigma_g^2}\right)$, there exists $k_4 > 0$, such that when $\sigma_q > k_4$, $\partial corr(p, \tilde{g}) / \partial\theta < 0$.*

Figure 5 plots how the price of the equilibrium score Φ, the number of buyers of the score $\theta\lambda^*$ and the price informativeness $\text{corr}(p, \tilde{g})$ change with the risk aversion of investors A. Again, the blue dotted line represents the results when manipulation is not feasible. First, under the chosen parameters, we are always in the unconstrained equilibrium in the absence of manipulation but always in the constrained equilibrium (the third case) when manipulation is possible. As a result, the price informativeness is independent of the risk aversion A in the absence of manipulation, while it is decreasing with A when manipulation is possible. This confirms the importance of manipulation in determining the equilibrium structures. Second, when the risk aversion A increases in this case, the information seller will choose to sell the most informative score, and sell the score to more investors. However, the low trading motives make the price less informative. When A is too high, the information seller will sell the score to all type-G investors, and the price informativeness will continue to decrease due to lower trading motives.

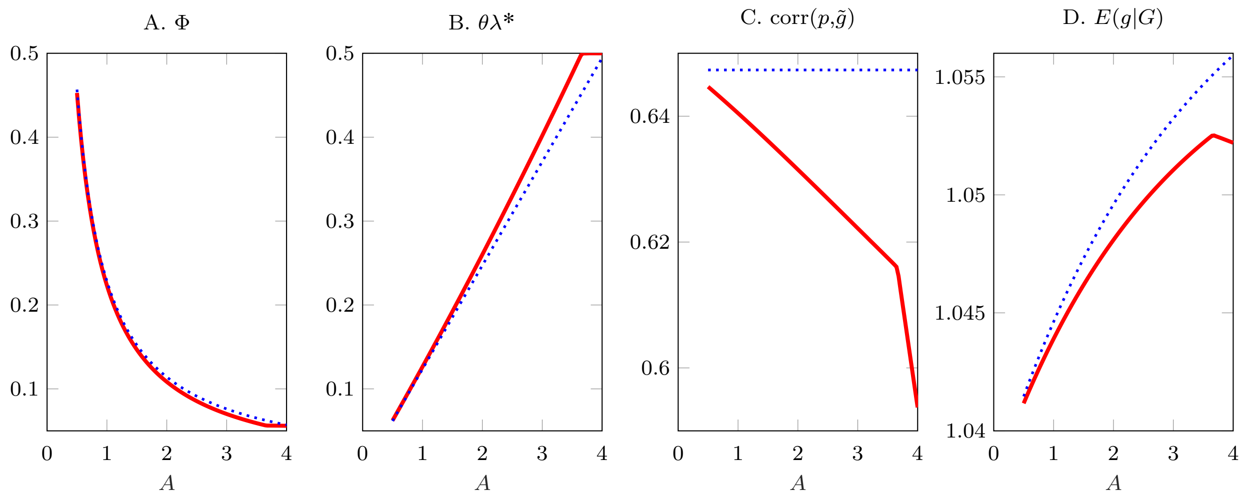

Figure 5: **Equilibrium outcomes under different A.** Parameters values are $\sigma_g = 0.3$, $\sigma_s = 0.3$, $\sigma_x = 0.2$, $\sigma_q = 0.34$, $\underline{\sigma} = 0.2$, $\beta = 2$, $\theta = 0.5$, $\bar{g} = 1$ and $\bar{s} = 1$.

Figure 6 plots how the price of the equilibrium score Φ, the number of buyers of the score $\theta\lambda^*$ and the price informativeness $\text{corr}(p, \tilde{g})$ change with β. When β is low, type-G investors in general have low incentive to trade on information about $\tilde{g}$, so the information seller will choose to sell the most informative score, which leads to the third case in Theorem 1. In this region, when β increases from a low level, type-G investors care more about the signal and thus the information seller can charge a higher price. Since the price is still relatively low when β is low, to avoid the excessive information spillover effect, the information seller will reduce the mass of score buyers to maximize the total profit. When β is relatively high in this third case region, the price of the score is relatively high, so the profit of selling the score to more investors is significant. As the best response, when β continues to increase in this region, the information seller will choose to sell the score to more investors, leading to a U-shaped strategy shown in the plot. In this third case region, the price informativeness monotonically increases, because when β is higher, investors are more willing to trade on the score, which tends to make the price more informative. However, when β is overly high, the equilibrium becomes the first case in Theorem 1. In this case, the information seller will start to add noise to the score to avoid too much information spillover. When β increases, the information seller chooses to add more noise to the score, keeping the score price unchanged, and selling the score to more investors. Due to increasing noise added by the information seller, the price informativeness decreases in this region when β increases.

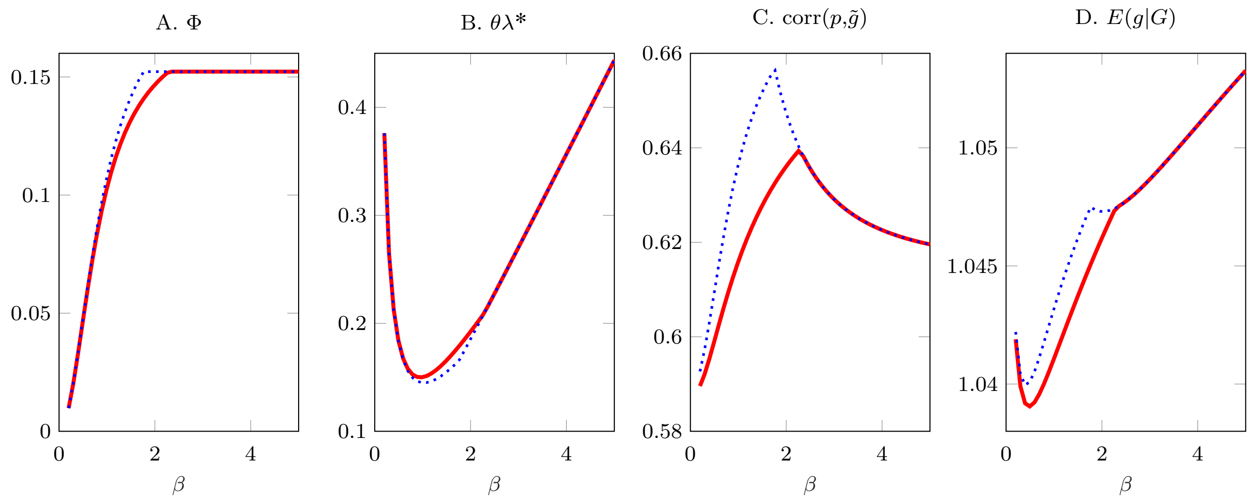

Figure 6: **Equilibrium outcomes under different β.** Parameters values are $\sigma_g = 0.3$, $\sigma_s = 0.3$, $\sigma_x = 0.2$, $\sigma_q = 0.34$, and $\underline{\sigma} = 0.2$, $A = 1.5$, $\theta = 0.5$, $\bar{g} = 1$ and $\bar{s} = 1$.

2.3 Score Design

In this section, we delve into the problem of endogenous score design. In practice, information sellers often design the scores they sell to the market by modifying the methodology used for score calculation. For instance, a firm may have multiple relevant signals, each associated with a specific intrinsic noise level and different degrees of manipulability. Certain signals, such as accounting information, may be more difficult to manipulate (due to regulated reporting standards and auditing requirements) but contain a higher level of intrinsic noise (for example, accounting reports tend to be infrequent and backward-looking). In contrast, signals constructed using alternative data have been shown to be more accurate predictors of firm performance in the absence of manipulation. The challenge is that these signals could also be more vulnerable to manipulation.

We consider an information seller designing a score by assigning weights to a variety of signals. These weights influence both the score's overall intrinsic noise level and manipulability. As in the case of a one-dimensional signal, we can allow the seller to add additional noise to the score by including an extra "signal" that is pure noise.

We assume that a firm has N signals (or attributes), all of which are potentially informative about the true value of $\tilde{g}$. In the absence of manipulation, the N attributes can be represented as $\tilde{g} + u_i$, where $i \in \{1, 2, \ldots, N\}$. We assume that $\mathbf{u} \equiv (u_1, u_2, \ldots, u_N)^T \sim N(\mathbf{0}, \mathbf{\Sigma_u})$. Next, for each attribute i, the firm can increase its level by δ_i at a cost of $\frac{1}{2q_i}\delta_i^2$, where $q_i \sim N\left(\bar{q}_i, \sigma_{q_i}^2\right)$ represents the manipulability of attribute i. We assume that the q_i across attributes are mutually independent and independent of all other random variables in the model.[7]

Given the normality structure, it is optimal for the information seller to use a linear combination of the N attributes to predict $\tilde{g}$. They choose a vector $\mathbf{w} = (w_1, w_2, \ldots, w_N)^T$ such that $\mathbf{w'1} = 1$ and thus the score is

$$g_r = \sum_{i=1}^{N} w_i \left(\tilde{g} + u_i + \delta_i\right) = \tilde{g} + \sum_{i=1}^{N} w_i u_i + \sum_{i=1}^{N} w_i \delta_i. \tag{23}$$

We still conjecture that there is a linear equilibrium price of the risky asset, i.e., there exists a_0, a_r and a_s such that

$$p = a_0 + a_r g_r + a_s \tilde{s}. \tag{24}$$

The firm's problem is

$$\max_{\delta=(\delta_1,\ldots,\delta_N)} \mathbb{E}\left(p\right) - \sum_{i=1}^{N} \frac{1}{2q_i} \delta_i^2, \tag{25}$$

and the optimal manipulation level for the ith attribute is

$$\delta_i^* = w_i a_r q_i, \tag{26}$$

and thus the equilibrium score is

$$g_r = \tilde{g} + \sum_{i=1}^{N} w_i u_i + a_r \sum_{i=1}^{N} w_i^2 q_i. \tag{27}$$

Thus, we can decompose the noise in the score into two parts, the first due to intrinsic noise, the second due to uncertainty about signal manipulability.

$$\sigma_u^2\left(\mathbf{w}\right) = \mathbf{w}^T \mathbf{\Sigma_u} \mathbf{w}, \tag{28}$$

$$\sigma_q^2\left(\mathbf{w}\right) = \sum_{i=1}^{N} w_i^4 \sigma_{q_i}^2. \tag{29}$$

Before presenting the equilibrium solution, let us first consider a related problem where the information seller tries to maximize the informativeness of the score conditional on a

[7] Again, the normality assumption for q_i should be viewed as an approximation of $q_i = \max(0, \tilde{q})_i$ where $\tilde{q}_i \sim N\left(\bar{q}_i, \sigma_{q_i}^2\right)$, which ensures that q stays positive. Instead of independence, we can also allow for general covariance structure between $\mathbf{u}$ and $\mathbf{q}$.

fraction λ of type-G investors buying the score. Let $\mathbf{w}_{\max}(\lambda)$ be the optimal weights to this most-informative score,

$$\mathbf{w}_{\max}(\lambda) = \arg\max_{\mathbf{w}} \operatorname{corr}(g_r, \tilde{g})$$
$$= \arg\min_{\mathbf{w}} \sigma_g^2 + \sigma_u^2(\mathbf{w}) + \left[\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right) M_b(\lambda) \left(\frac{\lambda + \frac{\beta A \sigma_s^2}{n\theta} M_b(\lambda)}{1+\frac{\beta A \sigma_s^2}{n\theta} M_b(\lambda)}\right)}\right]^2 \sigma_q^2(\mathbf{w}), \tag{30}$$

where $M_b(\lambda)$ is the analog of $M(\lambda)$ from the baseline model (which is defined as the solution to (17)),

$$n\sigma_g^2 + n\theta \frac{\lambda}{A\beta} M_b(\lambda) = \min_{\mathbf{w}} \sigma_g^2 + \sigma_u^2(\mathbf{w}) + \left[\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right) M_b(\lambda) \left(\frac{\lambda + \frac{\beta A \sigma_s^2}{n\theta} M_b(\lambda)}{1+\frac{\beta A \sigma_s^2}{n\theta} M_b(\lambda)}\right)}\right]^2 \sigma_q^2(\mathbf{w}). \tag{31}$$

To maximize the informativeness of the score, one needs to minimize the total variance of all noises in the score. The optimal weights are not the ones that minimize the intrinsic noise $\sigma_u^2(\mathbf{w})$, the standard objective of a forecasting problem in the absence of manipulation (minimizing the mean squared error). Nor do they minimize the amount of noise generated by signal manipulation. Instead, it strikes a balance between the two considerations. Thus, a signal that is highly informative in the absence of manipulation but shows significant uncertainty in manipulability across firms may not receive a high weight in the score. Importantly, the degree to which uncertainty about manipulability contributes to the total noise, and thus its impact on $\mathbf{w}_{\max}$, is endogenous in this model. It depends on the price sensitivity to the score, a_r, which in turn depends on the investor composition, as well as the distribution of signal precision and manipulability. For example, the share of type-G investors in the market θ and investors' risk aversion coefficient A, which are neither related to the statistical properties of the signals nor their manipulability, will both have important effects on $\mathbf{w}_{\max}$.

The score design problem is further complicated by the fact that the information seller's objective is not to maximize the informativeness of the score, but to maximize the profit. We characterize the equilibrium as follows.

Theorem 2. *In the model of score design, the information seller's optimal choice of* Φ_b^* *and*

$\mathbf{w}^*$ *are characterized as follows:*

1. *If* $\overline{\lambda} \leqslant 1$ *and* $M_b\left(\overline{\lambda}\right) \leqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$*, then the information seller chooses* $\mathbf{w}^*$ *satisfying*

$$Var(g_r) = \sigma_g^2 + \sigma_u^2\left(\mathbf{w}^*\right) + a_r^{*2}\sigma_q^2\left(\mathbf{w}^*\right) = n\sigma_g^2 + n\theta\frac{\overline{\lambda}}{A\beta}\frac{\sigma_g}{\sigma_s}\sqrt{n}, \tag{32}$$

where a_r^* *is the equilibrium* a_r *in the price equation and takes the same value as* a_r^* *in Theorem 1 (as defined by* (A.93) *in the appendix), and the price* Φ_b^* *is set such that a fraction of* $\lambda^* = \overline{\lambda}$ *investors choose to buy the score.*

2. *If* $\overline{\lambda} > 1$ *and* $M_b(1) \leqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$*, then the information seller chooses* $\mathbf{w}^*$ *satisfying*

$$Var(g_r) = \sigma_g^2 + \sigma_u^2\left(\mathbf{w}^*\right) + a_r^{**2}\sigma_q^2\left(\mathbf{w}^*\right) = n\sigma_g^2 + n\theta\frac{1}{A\beta}\frac{\sigma_g}{\sigma_s}\sqrt{n}, \tag{33}$$

where a_r^{**} *is the equilibrium* a_r *in the price equation, and is defined by* (A.102) *in the appendix, and the price* Φ_b^* *is the highest price that all investors choose to buy the score (*$\lambda^* = 1$*).*

3. *Otherwise, in equilibrium, the price* Φ_b^* *is the highest price that a fraction of* $\min(\lambda_b, 1)$ *investors choose to buy the signal where* λ_b *satisfies*

$$\lambda_b = \arg\max_{\lambda} \lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{M_b(\lambda)} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}M_b\left(\lambda\right)\right)}\right). \tag{34}$$

The information seller chooses $\mathbf{w}^* = \mathbf{w}_b\left(\lambda_b\right)$*.*

In the first unconstrained equilibrium in Theorem 2, the information seller attains the highest possible payoff. Notice also that in both case 1 and 2, the optimal choice of scoring weights $\mathbf{w}^*$ differ from $\mathbf{w}_b\left(\lambda^*\right)$, the weights that maximize the informativeness of the score. Moreover, the optimal weight is not unique. This result is demonstrated in the following corollary. Intuitively, the non-uniqueness of optimal weights is due to the fact that the information seller is indifferent about the various ways to add additional noise to the score that is the most informative about $\tilde{g}$.

Corollary 3. *In the unconstrained equilibrium characterized by Theorem 2, the solution* $\mathbf{w}^*$ *satisfying (32) is generically not unique.*

The information seller only chooses the weights to maximize the informativeness of the score in the third equilibrium. This occurs when the level of intrinsic noise embedded in the signals is not too low. Again, the average degree of signal manipulability does not matter for the design of the score.

3 A Model of Investment Mandate

In the previous section, we study the market for manipulable information when the information is used by investors to forecast asset payoffs. The score provided by the information seller is valuable only because it helps improve investors' forecasts. In some cases, an investor may care about the score itself, even if the score is noisy or biased (Baghai et al. (2023)). This is often because the investor faces an investment mandate that specifically depends on the score. For example, certain institutional investors are only allowed to invest in corporate bonds that have an investment grade rating, while an ESG fund may be required to hold a portfolio that exceeds a minimum ESG score. We refer to these investors as "mandate investors." In this section, we use a variation of the baseline model to demonstrate how the presence of mandate investors can significantly change the incentives of the investors and the information seller in a market with manipulable information.

We still consider two types of investors in the financial market, a fraction $(1-\theta)$ of type-N investors who have the same preference as those in our baseline model, and a fraction θ of type-M investors, which stand for "mandate investors." To comply with the investment mandate, type-M investors must first acquire the score before constructing their portfolios.

As in the baseline model, both types of investors have initial wealth of W_0 and CARA utility. The preferences of the type-N investors are identical to those in (2); in particular, they only care about the payoff component $\tilde{x}$. Mandate investors, on the other hand, care about both $\tilde{x}$ and the level of the score g_r. For example, fund managers may face pressure from their investors to incorporate ESG considerations into their investment processes. This pressure can come from individual investors, pension funds, endowments, and other institutional investors who are interested in aligning their investments with their values and ethical beliefs. To fulfill the investment mandate, we assume that if mandate investors do not buy the score from the information seller, they can only invest in the risk-free bond. If they buy the score, they can invest in both the stock and the risk-free bond, with time-0 utility

$$U_M = \mathbb{E}\left[-e^{-A(\phi(\tilde{x}+\beta g_r)+l)}|\mathcal{F}_M\right], \tag{35}$$

where ϕ is his holding of the stock, l is the holding of the risk-free bond and $\mathcal{F}_M$ is the information set of mandate investors at time 0. We assume $\beta > 0$ to highlight that mandate investors prefer to hold high-rated assets. It is worth emphasizing that the key distinction between the preferences of the mandate investors in (35) and those of the type-G investors in the baseline model (1) is that the former replaces the actual characteristic $\tilde{g}$ with the score g_r in their utility. This distinction has significant implications for the model.

3.1 Model Solution

In this investment mandate model, we first consider the market equilibrium when σ_u^2 (the variance of the intrinsic noise u) and the price of the score Φ are given, and then we consider the information seller's optimal choice. This partial equilibrium consists of investors' optimal portfolio holdings and the firm's optimal manipulation strategy. Specifically, mandate investors first decide whether to buy the score, and if they do, they choose the optimal portfolio holding that maximizes their expected utility given by (35). Type-N investors will not buy the score in equilibrium, and they choose the optimal portfolio holding to maximize their utility given by (2). Anticipating the market equilibrium price $\widehat{p}$, the firm manager selects its manipulation level $\widehat{\delta}$ to maximize the same objective as (4).

Suppose in this partial equilibrium, a fraction λ of mandate investors choose to buy the score, and the time-0 price of the stock, $\hat{p}$, is a linear combination of the equilibrium score $\widehat{g}_r$ and the noisy supply $\tilde{s}$. Given the equilibrium stock price $\widehat{p}$, the optimal stock demand of type-N investors are the same as (8). The optimal stock demand of mandate investors who buy the score is

$$\phi_M^* = \frac{E\left(\tilde{x} + \beta\widehat{g}_r | \mathscr{F}_M\right) - \widehat{p}}{A\mathrm{Var}\left(\tilde{x}|\mathscr{F}_M\right)}, \tag{36}$$

which differs from the demand of type-G investors in (9) due to the fact that mandate investors care only about the score $\widehat{g}_r$ (which is known at the time of the investment), not the actual payoff component $\tilde{g}$ or the uncertainty associated with it.

The market clear condition implies that

$$(1-\theta)\frac{E\left(\tilde{x}|\mathscr{F}_N\right) - \widehat{p}}{A\mathrm{Var}\left(\tilde{x}|\mathscr{F}_N\right)} + \lambda\theta\frac{E\left(\tilde{x} + \beta\widehat{g}_r|\mathscr{F}_M\right) - \widehat{p}}{A\mathrm{Var}\left(\tilde{x}|\mathscr{F}_M\right)} = \tilde{s}. \tag{37}$$

As in the baseline model, the equilibrium stock price $\widehat{p}$ still have a linear structure, as do the firm manipulation level $\widehat{\delta}$ and the equilibrium score $\widehat{g}_r$. These results are summarized in the proposition below.

Proposition 8. *In the model of investment mandate, when the variance of intrinsic noise σ_u^2 and the price of the score Φ are given, and a fraction λ mandate investors choose to buy* *the score, then there exists an equilibrium, such that the firm's manipulation level is*

$$\widehat{\delta} = \widehat{a}_r q, \tag{38}$$

where

$$\widehat{a}_r = \frac{\lambda\theta}{1-\theta+\lambda\theta}\beta, \tag{39}$$

the equilibrium stock price is

$$\widehat{p} = \bar{x} + \widehat{a}_r \widehat{g}_r - \frac{A\sigma_x^2 \tilde{s}}{1-\theta+\lambda\theta}, \tag{40}$$

and the equilibrium score is

$$\widehat{g}_r = \tilde{g} + u + \widehat{\delta}. \tag{41}$$

The firm's manipulation incentive significantly differs between the baseline model and the investment mandate model, with the key parameters being a_r and $\widehat{a}_r$ in each respective model. In the mandate model, $\widehat{a}_r$ is independent of various parameters describing the information environment, such as σ_x, σ_g, and in particular, σ_q. In the baseline model, an increase in σ_q leads to a decrease in a_r. The rationale here is that a higher σ_q tends to make the score noisier in the baseline model, and thus investors rely less on the score for trading decisions. Then the price becomes less sensitive to the score. In contrast, in the mandate model, even though a higher σ_q would reduce the informativeness of the score, mandate investors only care about the level of the score and not its informativeness. As a result, changes in σ_q do not affect investors' incentive to purchase the score or trade the stocks based on the score, nor do they affect the firm's manipulation incentive.

Next, we show the comparative statics results about the informativeness of the price and the score.

Proposition 9. *In the partial equilibrium characterized by Proposition 8, suppose λ is fixed, when the strength of investment mandate β increases, or when the fraction of mandate investors θ increases,*

1. *the informativeness of the score $\widehat{g}_r$, measured by $corr(\widehat{g}_r, \tilde{g})$, decreases;*
2. *the correlation between equilibrium price and the score $\widehat{g}_r$, $corr(\widehat{p}, \widehat{g}_r)$, increases;*
3. *the correlation between equilibrium price and true $\tilde{g}$, $corr(\widehat{p}, \tilde{g})$, first increases and then*

decreases.

As β increases, mandate investors have a stronger incentive to hold high-rated assets, driving up the price of those assets, and at the same time making the price more correlated with the score $\widehat{g}_r$. However, the higher price for these high-rated assets can also incentivize firms to manipulate their score to boost their prices even further. Then the net impact of an increase in the strength of the investment mandate, measured by β, on the price informativeness can be positive or negative.

Using a similar argument, the overall impact of an increase in the fraction of mandate investors, denoted by θ, on the price informativeness is also uncertain. On the one hand, the direct effect of greater mandate investor participation makes asset prices and the score more correlated. On the other hand, the indirect effect of higher θ is that it encourages firms to manipulate their signal more, which reduces the score's accuracy. As a result, the impact of θ on the informativeness of the price is similar to that of the strength of the investment mandate β.

An interesting result here is that, when a stricter market-wide investment mandate is implemented, i.e., when β is higher, the expected $\tilde{g}$ of the risky asset that mandate investors hold, denoted by $\mathbb{E}[\tilde{g}|M] = \frac{\mathbb{E}\left(\phi_M^* \tilde{g}\right)}{\mathbb{E}\left(\phi_M^*\right)}$, does not necessarily increase.

Proposition 10. *We have*

$$\mathbb{E}[\tilde{g}|M] = \frac{\mathbb{E}\left(\phi_M^* \tilde{g}\right)}{\mathbb{E}\left(\phi_M^*\right)} = \frac{\sigma_g^2}{\frac{A\sigma_x^2 \bar{s}}{\beta(1-\theta)} + \bar{g} + \frac{\lambda\theta\beta}{1-\theta+\lambda\theta}\bar{q}} + \bar{g}. \tag{42}$$

Then $\mathbb{E}[\tilde{g}|M]$ increases in β when $\beta < \sqrt{\frac{A\sigma_x^2 \bar{s}(1-\theta+\lambda\theta)}{\lambda\theta(1-\theta)\bar{q}}}$, and decreases in β when $\beta \geqslant \sqrt{\frac{A\sigma_x^2 \bar{s}(1-\theta+\lambda\theta)}{\lambda\theta(1-\theta)\bar{q}}}$.

3.2 Information Seller's Optimal Choice

Now we consider the information seller's score design and pricing. Similar to the baseline model, in this section, we assume that the manipulability features $(\bar{q}, \sigma_q^2)$ are given, and solve for the information seller's optimal choice, $\widehat{\sigma}_u^2$, and the price of the signal $\widehat{\Phi}$.

In equilibrium, the score does not provide any information about the monetary payoff $\tilde{x}$, so type-N investors will never purchase the score. Then the information seller's problem is

$$\max_{\lambda, \sigma_u^2} \lambda \widehat{\Phi}\left(\lambda; \sigma_u^2\right), \tag{43}$$

where $\widehat{\Phi}\left(\lambda;\sigma_u^2\right)$ is the highest price the information seller can charge to have a fraction of λ mandate investors buy the score.

The score price must satisfy an indifferent condition, which means that mandate investors are indifferent between purchasing the score and not. This implies

$$-e^{-A\left(W_0-\widehat{\Phi}\right)}\mathbb{E}\left[e^{-A\left[\phi_M^*\mathbb{E}(\tilde{x}-\widehat{p}+\beta\widehat{g}_r|\mathscr{F}_M)-\frac{1}{2}A\phi_M^{*2}\mathrm{Var}(\tilde{x}|\mathscr{F}_M)\right]}\right]=-e^{-AW_0}. \tag{44}$$

The following lemma provides a closed-form expression for the price $\widehat{\Phi}$.

Lemma 4. *The price of the score is*

$$\widehat{\Phi}\left(\lambda;\sigma_u^2\right)=\frac{1}{2A}\left[\frac{\mu_z^2}{1+\sigma_z^2}+\log\left(1+\sigma_z^2\right)\right], \tag{45}$$

where

$$\mu_z=\frac{(1-\theta)\,\beta\bar{g}+A\sigma_x^2\bar{s}+(1-\theta)\,\frac{\lambda\theta}{1-\theta+\lambda\theta}\beta^2\bar{q}}{(1-\theta+\lambda\theta)\sigma_x}, \tag{46}$$

and

$$\sigma_z=\frac{\sqrt{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2+\left(\frac{\lambda\theta}{1-\theta+\lambda\theta}\right)^2\beta^2\sigma_q^2\right)+A^2\sigma_x^4\sigma_s^2}}{(1-\theta+\lambda\theta)\sigma_x}. \tag{47}$$

For the following analysis in this section, we make two additional assumptions. First, we assume that the information seller can only choose noise level within some range, $\sigma_u\in[\underline{\sigma},\bar{\sigma}]$, where $\bar{\sigma}<\infty$. This is because in the above maximization problem (43), the information seller can always choose σ_u^2 which is infinitely high, to obtain an infinite payoff. This result is mechanical because we didn't impose any restrictions on mandate investors' short position. In practice, the information seller will not design an infinitely noisy score due to other independent concerns. Second, we focus on the case when $\bar{q}$ is large enough. This is consistent with our assumption $\bar{q}\gg\sigma_q^2$ which guarantees that $\tilde{q}$ is mostly positive. Under these two assumptions, we obtain the following result.

Proposition 11. *If the information seller can choose any $\sigma_u\in[\underline{\sigma},\bar{\sigma}]$, there exists a threshold $\bar{q}_m$, such that if $\bar{q}>\bar{q}_m$, the solution to the information seller's problem (43), $\widehat{\sigma}_u$, is $\widehat{\sigma}_u=\underline{\sigma}$.*

Although the information seller has limited incentive to add noise in this case, the informativeness of the equilibrium can still be affected by the firm's manipulation behavior, as shown in Proposition 9 and Proposition 10.

3.3 Score Design

In this part, we consider the information seller's score design problem in the investment mandate model. Similarly to our formulation in Section 2.3, we consider an information seller who designs a score by adjusting the weights to a variety of signals.

For simplicity in exposition, we borrow the notation used in Section 2.3, and present only the final optimization problem of the information seller here.

In this model, the information seller's problem is

$$\max_{\lambda;\mathbf{w}'\mathbf{1}=1,w_i\geqslant 0} \frac{\lambda}{2A}\left[\frac{\mu_z^2}{1+\sigma_z^2}+\log\left(1+\sigma_z^2\right)\right], \tag{48}$$

such that

$$\mu_z = \frac{(1-\theta)\,\beta\bar{g} + A\sigma_x^2\bar{s} + (1-\theta)\,\frac{\lambda\theta}{1-\theta+\lambda\theta}\beta^2\sum_{i=1}^N w_i\bar{q}_i}{(1-\theta+\lambda\theta)\sigma_x} \tag{49}$$

and

$$\sigma_z = \frac{\sqrt{(1-\theta)^2\,\beta^2\left(\sigma_g^2+\sigma_u^2\left(\mathbf{w}\right)+\left(\frac{\lambda\theta}{1-\theta+\lambda\theta}\right)^2\beta^2\sigma_q^2\left(\mathbf{w}\right)\right)+A^2\sigma_x^4\sigma_s^2}}{(1-\theta+\lambda\theta)\sigma_x}. \tag{50}$$

The definitions of $\sigma_u^2\left(\mathbf{w}\right)$ and $\sigma_q^2\left(\mathbf{w}\right)$ are also the same as in Section 2.3. Let $\left(\widehat{\lambda},\widehat{\mathbf{w}}\right)$ denote the solution to the above problem. To establish a benchmark, let's introduce the weight $\widehat{\mathbf{w}}_{\max}$ that maximizes the informativeness of the score, i.e.,

$$\widehat{\mathbf{w}}_{\max} = \arg\max_{\mathbf{w}} \operatorname{corr}\left(\hat{g}_r,\tilde{g}\right), \tag{51}$$

which is equivalent to minimizing the σ_z in the above problem:

$$\widehat{\mathbf{w}}_{\max} = \arg\min_{\mathbf{w}} \sigma_z = \arg\min_{\mathbf{w}} \sigma_g^2+\sigma_u^2\left(\mathbf{w}\right)+\left(\frac{\lambda\theta}{1-\theta+\lambda\theta}\right)\beta^2\sigma_q^2\left(\mathbf{w}\right). \tag{52}$$

Since $\widehat{\mathbf{w}}$ and $\widehat{\mathbf{w}}_{\max}$ solve two different problems, it's clear that they are generically different. Besides, we also find a score inflation result in the mandate investment model.

Proposition 12. *(Score Inflation) If $\theta < \frac{3}{4}$, there exists $\bar{q}_w$, such that if $\bar{q}_i > \bar{q}_w$ for all i,*

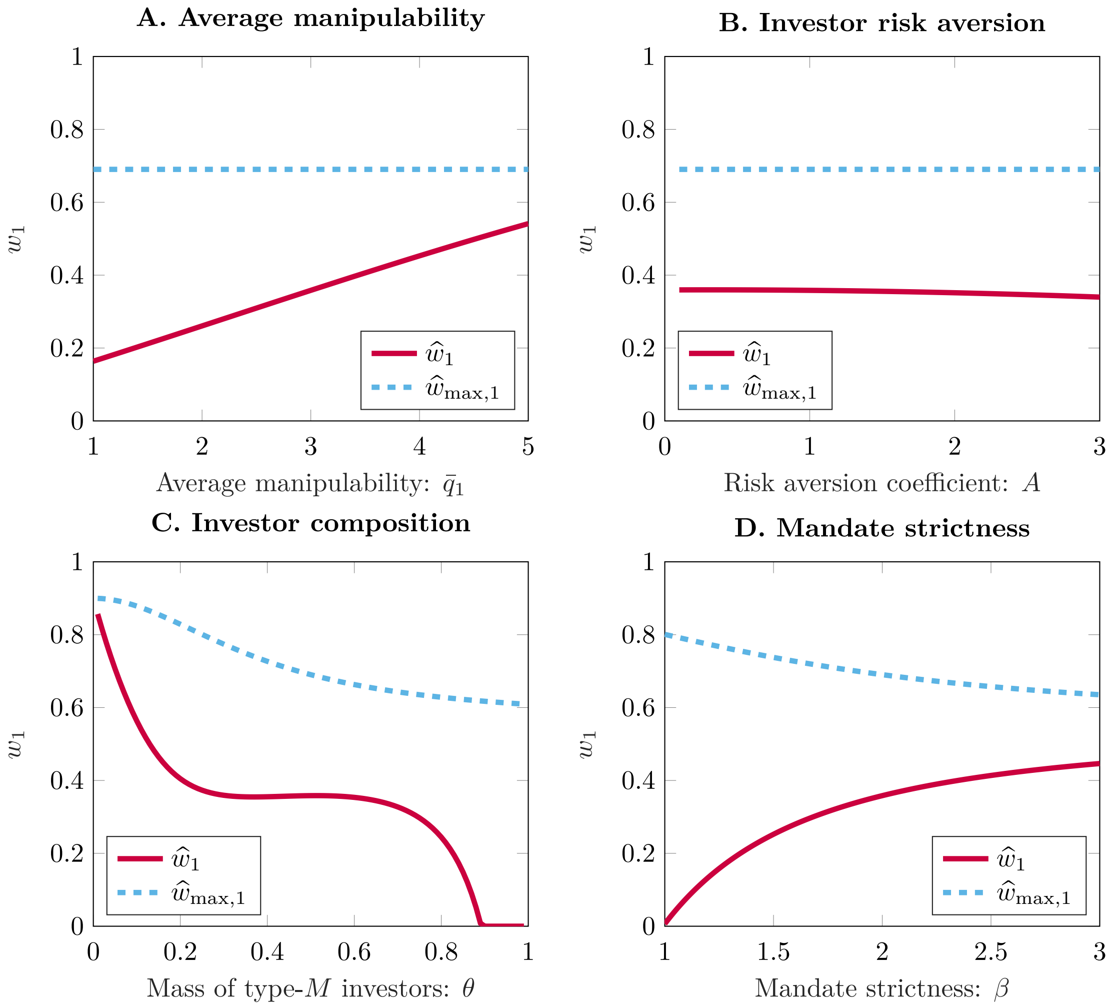

Figure 7: **Optimal score design in the mandate model.** This figure plots the optimal weight $\widehat{w}$ and most-informative weight $\widehat{w}_{max}$ for the first signal in a two-signal example.

the solution to the information seller's problem, $\widehat{\lambda}$ *and* $\widehat{\mathbf{w}}$*, satisfy* $\widehat{\lambda} = 1$ *and*

$$\frac{\partial \widehat{w}_i}{\partial \bar{q}_i} > 0$$

for all i*.*

Proposition 12 implies that, in the mandate model, the information seller indeed has an incentive to overweight signals that are on average easier to manipulate. This is in contrast to the baseline model, where the average manipulability does not play a role in the information seller's optimization problem.

To illustrate the effect of mandate investors on the score design, we consider a numerical example with two signals. In this case, it is sufficient to study the weight for the first signal, w_1. The parameter values for the attributes $\tilde{x}$ and $\tilde{g}$ are $\sigma_x = 0.2$, $\bar{g} = 2$ and $\sigma_g = 0.3$ ($\bar{x}$ is irrelevant for the score design); for the security supply, we have $\bar{s} = 1$ and $\sigma_s = 1$; the fraction of mandate investors is $\theta = 0.5$, all investors have risk aversion coefficient $A = 1$, and mandate investors' preference for the score is $\beta = 2$. For the two signals, we assume $\bar{q}_1 = 3$, $\sigma_{u_1} = 0.1$, $\sigma_{q_1} = 0.2$, and $\bar{q}_2 = 7$, $\sigma_{u_2} = 0.3$, $\sigma_{q_2} = 0.3$. It implies that the second signal is intrinsically more noisy ($\sigma_{u_1} < \sigma_{u_2}$), on average easier to manipulate ($\bar{q}_1 < \bar{q}_2$), but also has higher uncertainty about its manipulability ($\sigma_{q_1} < \sigma_{q_2}$).

In Figure 7, we compare $\widehat{w}_1$, which is profit-maximizing, against $\widehat{w}_{\max,1}$, which is the most informative weight, while changing signal manipulability (Panel A), investor risk aversion (Panels B), investor composition (Panel C), and mandate strictness (Panel D). For the whole parameter space in this figure, we always have $\widehat{\lambda} = 1$. In all cases, the profit-maximizing weight $\widehat{w}_1$ is lower than $\widehat{w}_{\max,1}$, meaning that the weight has been shifted towards the second signal, which is on average more manipulable. It is easy to see from (52) that the most informative weights (blue dotted lines) are unaffected by changes in the average signal manipulability $\bar{q}_1$ or risk aversion coefficient A. Not surprisingly, $\widehat{w}_1$ increases as the manipulability of the first signal improves.

Next, as we increase the share of mandate investors, both $\widehat{w}_{\max,1}$ and $\widehat{w}_1$ decrease. The reason that $\widehat{w}_{\max,1}$ decreases is that an increase in θ raises firm's incentive to manipulate, which makes the noise due to uncertainty about manipulability more important. To get the most informative score, one need to shift more weight towards the second signal, which, although intrinsically more noisy than the first signal, does not have as big a gap in the uncertainty about manipulability. Increasing β, which proxies for the strength of the investment mandate, has a similar effect on $\widehat{w}_{\max,1}$. The fact that $\widehat{w}_1$ declines even faster with θ is due to the stronger incentive for the information seller to inflate the score.

In contrast, with higher β, $\widehat{w}_1$ becomes higher. This is because higher uncertainty about the score will reduce the mandate investors' willingness to pay for it, and this effect is stronger with larger β. To offset this effect, the information seller shifts more weight to the first signal, which is both less noisy and has lower uncertainty about manipulability.

Taken together, the set of comparative statics in the two-signal example highlights the distinct effects that mandate investors have on the optimal score design. It is clear that the presence of such investors can significantly distort the incentives of the information seller and prevent them from producing the most informative scores.

4 Competing Information Sellers

In this section, we consider an extension in which N information sellers design and sell conditionally independent signals to investors. Suppose each seller sells a score

$$g_r^i = \tilde{g} + u_i + \delta, \tag{53}$$

where δ is the manipulation level chosen by the firm, which is the same for all scores, and u_i is the noise added by the info seller i, and the noise is independent across all information sellers. At the beginning, each information seller i chooses price Φ_i and the standard deviation of the noise u_i, $\sigma_i \in [\underline{\sigma}, \infty)$, simultaneously. Then the firm chooses the manipulation level δ, and investors decide which scores to buy. The rest of the timeline is the same as in Figure 1. Let λ_i be the fraction of type-G investors who buy the score from information seller i.

We focus on symmetric equilibria, in which all information sellers choose the same level of noise $\sigma_i = \sigma_u$ and the same price $\Phi_i = \Phi$.

Proposition 13. *When N is sufficiently high, there exists an equilibrium, such that all information sellers choose $\sigma_i = \underline{\sigma}$, and a strictly positive price $\Phi_i = \Phi^* > 0$.*

To gain intuitions about the above results, note that in this equilibrium, since all investors buy all scores, the sufficient statistics of the N scores is

$$G_r = \tilde{g} + \frac{1}{N}\sum_{i=1}^{N} u_i + \delta.$$

Then the equilibrium price equation must have the following functional form:

$$\begin{aligned} p &= a_0 + a_r G_r + a_s \tilde{s} \\ &= a_0 + a_r \left(\tilde{g} + \frac{1}{N}\sum_{i=1}^{N} u_i + \delta + \frac{a_s}{a_r}\tilde{s} \right), \end{aligned}$$

where a_0, a_r and a_s are constant. Then the optimal level of manipulation is

$$\delta^* = a_r q.$$

For any investor, the total cost he would like to pay for m scores, $P(m)$, is given by

$$P(m) = \frac{1}{2A} \ln \left(\frac{\text{Var}(\tilde{x} + \beta \tilde{g} | p)}{\text{Var}\left(\tilde{x} + \beta \tilde{g} | p, \{g_r^i\}_{i=1}^m\right)} \right)$$
$$= \frac{1}{2A} \ln \left(\text{Var}(\tilde{x} + \beta \tilde{g} | p)\right) - \frac{1}{2A} \ln \left(\text{Var}\left(\tilde{x} + \beta \tilde{g} | p, \{g_r^i\}_{i=1}^m\right)\right).$$

So we can consider the following objective function of an type-G who buys m scores

$$-\frac{1}{2A} \ln \left(\text{Var}\left(\tilde{x} + \beta \tilde{g} | p, \{g_r^i\}_{i=1}^m\right)\right) - m\Phi^*.$$

Let

$$V(m) = -\frac{1}{2A} \ln \left(\text{Var}\left(\tilde{x} + \beta \tilde{g} | p, \{g_r^i\}_{i=1}^m\right)\right).$$

Then the marginal price that the investor would like to pay for the m-th score is

$$V(m) - V(m-1).$$

Lemma 5. *There exists $\underline{N}_1$ such that when $N > \underline{N}_1$,*

$$V(N) - V(N-1) = \min_{i \in \{2,3 \ldots N\}} \left(V(i) - V(i-1)\right).$$

The price Φ^* is given by

$$\Phi^* = V(N) - V(N-1)$$

in equilibrium. This is the marginal price that any investor would like to pay for the N-th score. Φ^* is a positive number, and converges to zero when N converges to infinity.

There are two effects of competition among information sellers. On the one hand, when there are more information sellers, the aggregate score G_r becomes more precious if $\text{Var}(\delta)$ is unchanged, which tends to improve the informativeness of the price. On the other hand, the presence of more information sellers also incentivizes the firm to manipulate the score more intensively, as shown by the following lemma.

Lemma 6. *When N is sufficiently high, if N increases, the firm's manipulation intensity, measured by the variance $Var(\delta) = a_r^2 \sigma_q^2$, increases.*

However, the following proposition shows that the additional noise in the price from more manipulation is dominated by the reduced noise in G_r, so competition of information sellers will improve the informativeness of the price.

Proposition 14. *When N is sufficiently high, if N increases, the informativeness of the price, measured by the correlation $corr(p, \tilde{g})$, increases.*

5 Conclusion

We study how investors, firms, and information sellers interact in a market with manipulable information. Our model builds on the framework of Admati and Pfleiderer (1986) and introduces two new features: information manipulability and investor heterogeneity. In the baseline model where investors care about actual characteristics, the average degree of signal manipulability has no effect on the equilibrium, whereas the uncertainty about signal manipulability plays a key role. Its contribution depends on firms' incentive to manipulate the signals that are used to generate the score, which in turn depends on the equilibrium price sensitivity to the score. The optimal design of the score in this setting weights the precision of different signals against the endogenous uncertainty from manipulation. The introduction of mandate investors, who care about the scores on the characteristics and not the characteristics themselves, generates a new incentive for information sellers to inflate the scores. Pushing too strongly on the mandate could lead to reduction in the informativeness of the score and the equilibrium price, and could even result in mandate investors holding less of the desired stocks.

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Appendix

A A microfoundation of mandate investors' preference

In this section, we present a simple delegation framework that rationalizes the utility function of mandate investors. Consider a delegated investment model with one principal and one agent. There are one risk free asset and N risky assets with two attributes $\tilde{x}_i$ and $\tilde{g}_i$, where $i \in \{1, 2...N\}$. The principal and the agent have different preferences over $\tilde{x}_i$ and $\tilde{g}_i$, both are unobservable to all players, and all of x_i and g_i follow normal distributions and are mutually independent.

Specifically, if the agent purchases ϕ_i units of the risky asset i, then the agent's total utility is

$$u_A = \sum_{i=1}^{N} \phi_i \left[E\left(\tilde{x}_i\right) - p_i\right] - \sum_{i=1}^{N} \frac{A}{2} \phi_i^2 \mathrm{Var}\left(\tilde{x}_i\right), \tag{A.1}$$

and the principal's utility is

$$u_P = \sum_{i=1}^{N} \phi_i \left[E\left(\tilde{x}_i + k_i \tilde{g}_i\right) - p_i\right] - \sum_{i=1}^{N} \frac{A}{2} \phi_i^2 \mathrm{Var}\left(\tilde{x}_i + k_i \tilde{g}_i\right), \tag{A.2}$$

where p_i is the equilibrium price of risky asset i and k_i are constant numbers.[8] In the ESG investing framework, the principal is the fund investor who cares about both the monetary return $\tilde{x}_i$ and the greenness of the holding $\tilde{g}_i$, while the agent, who is the fund manager, only cares about the monetary return $\tilde{x}_i$. There are N independent public scores $g_{r,i} = \tilde{g}_i + \epsilon_i$, which are informative about the true greenness $\tilde{g}_i$. To make the agent's holding aligned with the principal's preference, the principal imposes a "greenness requirement" on the portfolio, which requires the average greenness of the portfolio to satisfy the following constraint,

$$\sum_{i=1}^{N} \phi_i g_{r,i} \leqslant \bar{g}. \tag{A.3}$$

Let β be the Lagrange multiplier of the greenness constraint. Then the agent's optimization problem is

$$\begin{aligned} &\max_{\phi_i} \sum_{i=1}^{N} \phi_i \left[E\left(\tilde{x}_i\right) - p_i\right] - \sum_{i=1}^{N} \frac{A}{2} \phi_i^2 \mathrm{Var}\left(\tilde{x}_i\right) + \beta \left(\sum_{i=1}^{N} \phi_i g_{r,i} - \bar{g}\right) \\ &= \sum_{i=1}^{N} \phi_i \left[E\left(\tilde{x}_i + \beta g_{r,i}\right) - p_i\right] - \sum_{i=1}^{N} \frac{A}{2} \phi_i^2 \mathrm{Var}\left(\tilde{x}_i\right) - \beta \bar{g}. \end{aligned} \tag{A.4}$$

The first-order condition on any ϕ_i is

$$E\left(\tilde{x}_i + \beta g_{r,i}\right) - p_i - A \phi_i \mathrm{Var}\left(\tilde{x}_i\right) = 0, \tag{A.5}$$

and the Lagrange multiplier β is a function of the greenness threshold $\bar{g}$.

Let the solution of the above problem be $\phi_i\left(\beta\right)$. Then the principal effectively chooses the

[8] The mean-variance utility can be rationalized in the CARA framework.

"mandate level" β to solve the following problem,

$$\max_{\beta} \sum_{i=1}^{N} \phi_i(\beta) \left[E(\tilde{x}_i + k_i \tilde{g}_i) - p_i \right] - \sum_{i=1}^{N} \frac{A}{2} \phi_i^2(\beta) \operatorname{Var}(\tilde{x}_i + k_i \tilde{g}_i). \tag{A.6}$$

As long as the greenness constraint (A.3) is binding, the principal's choice β will be nonzero. In this case, the agent's utility is equivalent to

$$\hat{u}_A = \sum_{i=1}^{N} \phi_i \left[E(\tilde{x}_i + \beta g_{r,i}) - p_i \right] - \sum_{i=1}^{N} \frac{A}{2} \phi_i^2 \operatorname{Var}(\tilde{x}_i), \tag{A.7}$$

which is also equivalent to the CARA preference in our mandate investment model with only one risky asset.

B Proofs

B.1 Proof of Proposition 1

A fraction of $(1-\theta)$ investors are type-N investors who only care about $\tilde{x}$, and a fraction of θ investors are type-G investors who care about $\tilde{x} + \beta\tilde{g}$. Among the type-$G$ investors, a fraction of λ investors choose to buy the score, and the rest $(1-\lambda)$ investors do not buy the score. The information seller chooses an intrinsic noise level $\sigma_u^2 \in [\underline{\sigma}^2, \infty)$. Let's assume that in equilibrium, the stock price has the following linear structure

$$p = a_0 + a_r g_r + a_s \tilde{s} \tag{A.8}$$

where a_0,a_r and a_s are constant numbers. We know that in equilibrium, we have

$$g_r = \tilde{g} + u + \delta. \tag{A.9}$$

Firm's optimization problem is

$$\max_{\delta} E(p) - \frac{1}{2q}\delta^2 = a_0 + a_r(\bar{g} + \delta) + a_s \bar{s} - \frac{1}{2q}\delta^2. \tag{A.10}$$

Then the optimal manipulation level is

$$\delta^* = a_r q. \tag{A.11}$$

So the equilibrium score is

$$g_r = \tilde{g} + u + a_r q, \tag{A.12}$$

and the equilibrium price is

$$p = a_0 + a_r(\tilde{g} + u + a_r q) + a_s \tilde{s}. \tag{A.13}$$

It's easy to show that

$$
\begin{aligned}
E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right) &= \bar{x}+\beta E\left(\tilde{g}|g_r\right) \\
&= \bar{x}+\beta\bar{g}+\frac{\beta\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2}\left(g_r-\bar{g}_r\right), && \text{(A.14)} \\
\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right) &= \sigma_x^2+\beta^2\text{Var}\left(\tilde{g}|g_r\right) \\
&= \sigma_x^2+\beta^2\sigma_g^2\left(1-\frac{\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2}\right), && \text{(A.15)} \\
E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right) &= \bar{x}+\beta E\left(\tilde{g}|p\right) \\
&= \bar{x}+\beta\bar{g}+\frac{\beta\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}\left(\frac{p-\bar{p}}{a_r}\right), && \text{(A.16)} \\
\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right) &= \sigma_x^2+\beta^2\text{Var}\left(\tilde{g}|p\right) \\
&= \sigma_x^2+\beta^2\sigma_g^2\left(1-\frac{\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}\right). && \text{(A.17)}
\end{aligned}
$$

The market clear condition is

$$
(1-\theta)\frac{\bar{x}-p}{A\sigma_x^2}+\theta\left[\lambda\frac{\bar{x}+\beta\bar{g}+\frac{\beta\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2}\left(g_r-\bar{g}_r\right)-p}{A\left(\sigma_x^2+\beta^2\sigma_g^2\left(1-\frac{\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2}\right)\right)}+(1-\lambda)\frac{\bar{x}+\beta\bar{g}+\frac{\beta\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}\left(\frac{p-\bar{p}}{a_r}\right)-p}{A\left(\sigma_x^2+\beta^2\sigma_g^2\left(1-\frac{\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}\right)\right)}\right]=\tilde{s}. \tag{A.18}
$$

Together with

$$
p=a_0+a_rg_r+a_s\tilde{s}, \tag{A.19}
$$

we can obtain the equilibrium a_0, a_r and a_s. To see how this works, note that

$$
r_1=\frac{\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2}\in(0,1) \tag{A.20}
$$

and

$$
r_2=\frac{\sigma_g^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}\in(0,1) \tag{A.21}
$$

represent the explanatory power of g_r and p, respectively. I't clear that we must have $r_1>r_2$. Then the market clear condition becomes

$$
(1-\theta)\frac{\bar{x}-p}{\sigma_x^2}+\theta\left[\lambda\frac{\bar{x}+\beta\bar{g}+\beta r_1\left(g_r-\bar{g}_r\right)-p}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}+(1-\lambda)\frac{\bar{x}+\beta\bar{g}+\beta r_2\left(\frac{p-\bar{p}}{a_r}\right)-p}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}\right]=A\tilde{s}. \tag{A.22}
$$

So

$$(1-\theta)\frac{-dp}{\sigma_x^2}+\theta\left[\frac{\lambda\left(\beta r_1 dg_r-dp\right)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}+(1-\lambda)\frac{\left(\frac{\beta r_2}{a_r}-1\right)dp}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}\right]=Ad\tilde{s}$$

$$\Longleftrightarrow\left(\frac{1-\theta}{\theta}\right)\frac{-dp}{\sigma_x^2}+\left[\frac{\lambda\left(\beta r_1 dg_r-dp\right)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}+(1-\lambda)\frac{\left(\frac{\beta r_2}{a_r}-1\right)dp}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}\right]=\frac{A}{\theta}d\tilde{s}$$

$$\Longleftrightarrow\left(\frac{1-\theta}{\theta\sigma_x^2}+\frac{(1-\lambda)\left(1-\frac{\beta r_2}{a_r}\right)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}\right)dp=\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}dg_r-\frac{A}{\theta}d\tilde{s}$$

The pricing equation implies

$$dp=a_r dg_r+a_s d\tilde{s}, \tag{A.23}$$

so

$$\frac{1-\theta}{\theta\sigma_x^2}+\frac{(1-\lambda)\left(1-\frac{\beta r_2}{a_r}\right)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}=\frac{1}{a_r}\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}$$

$$\Longleftrightarrow\frac{1-\theta}{\theta\sigma_x^2}a_r+\frac{(1-\lambda)\left(a_r-\beta r_2\right)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}+\frac{\lambda a_r}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}=\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}$$

$$\Longleftrightarrow\left(\frac{1-\theta}{\theta\sigma_x^2}+\frac{(1-\lambda)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}\right)a_r=\frac{(1-\lambda)\beta r_2}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}+\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}.$$

and

$$\frac{a_s}{a_r}=-\frac{A}{\theta}\frac{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}{\lambda\beta r_1} \tag{A.24}$$

$$=-\frac{A}{\theta}\sigma_g^2\frac{\left(\frac{\sigma_x^2}{\sigma_g^2}+\beta^2\right)\frac{1}{r_1}-\beta^2}{\lambda\beta}<0. \tag{A.25}$$

Note that

$$n=\frac{\beta^2\sigma_g^2}{\sigma_x^2+\beta^2\sigma_g^2}, \tag{A.26}$$

then (**??**) becomes

$$\left(\frac{1-\theta}{\theta\sigma_x^2}+\frac{(1-\lambda)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}\right)a_r=\frac{(1-\lambda)\beta r_2}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}+\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}, \tag{A.27}$$

$$\left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2+\frac{(1-\lambda)}{\frac{\beta^2}{n}-\beta^2 r_2}+\frac{\lambda}{\frac{\beta^2}{n}-\beta^2 r_1}\right)a_r=\beta\left[\frac{(1-\lambda)r_2}{\frac{\beta^2}{n}-\beta^2 r_2}+\frac{\lambda r_1}{\frac{\beta^2}{n}-\beta^2 r_1}\right] \tag{A.28}$$

and

$$\frac{a_s}{a_r} = -\frac{A}{\theta}\sigma_g^2 \frac{\frac{\beta^2}{n} - \beta^2 r_1}{\lambda \beta r_1} = -\frac{A}{\theta}\beta\sigma_g^2 \frac{\frac{1}{n} - r_1}{\lambda r_1}. \tag{A.29}$$

We can write r_1 as a function of $\frac{a_s}{a_r}$:

$$r_1 = \frac{\frac{\beta^2}{n}}{\beta^2 - \theta \frac{\lambda\beta}{A\sigma_g^2}\frac{a_s}{a_r}} = \frac{\frac{\beta}{n}}{\beta - \theta\frac{\lambda}{A\sigma_g^2}\frac{a_s}{a_r}}. \tag{A.30}$$

Then

$$\frac{1}{r_2} = \frac{1}{r_1} + \left(\frac{a_s}{a_r}\right)^2 \frac{\sigma_s^2}{\sigma_g^2} = n - n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r} + \left(\frac{a_s}{a_r}\right)^2\frac{\sigma_s^2}{\sigma_g^2}. \tag{A.31}$$

The market clear condition implies that

$$\begin{aligned}
&\left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{(1-\lambda)}{\frac{\beta^2}{n} - \beta^2 r_2} + \frac{\lambda}{\frac{\beta^2}{n} - \beta^2 r_1}\right) a_r = \beta\left[\frac{(1-\lambda)\, r_2}{\frac{\beta^2}{n} - \beta^2 r_2} + \frac{\lambda r_1}{\frac{\beta^2}{n} - \beta^2 r_1}\right] \\
\Longleftrightarrow &\left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{(1-\lambda)\frac{n}{\beta^2}(1 - nr_2 + nr_2)}{1 - nr_2} + \frac{\lambda\frac{n}{\beta^2}(1 - nr_1 + nr_1)}{1 - nr_1}\right) a_r = \beta\left[\frac{(1-\lambda)\, r_2}{\frac{\beta^2}{n} - \beta^2 r_2} + \frac{\lambda r_1}{\frac{\beta^2}{n} - \beta^2 r_1}\right] \\
\Longleftrightarrow &\left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{n}{\beta^2} + \frac{(1-\lambda)\frac{n^2}{\beta^2} r_2}{1 - nr_2} + \frac{\lambda\frac{n^2}{\beta^2} r_1}{1 - nr_1}\right) a_r = \beta\left[\frac{(1-\lambda)\, r_2}{\frac{\beta^2}{n} - \beta^2 r_2} + \frac{\lambda r_1}{\frac{\beta^2}{n} - \beta^2 r_1}\right] \\
\Longleftrightarrow &\left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{n}{\beta^2} + \frac{(1-\lambda)\frac{n^2}{\beta^2}}{\frac{1}{r_2} - n} + \frac{\lambda\frac{n^2}{\beta^2}}{\frac{1}{r_1} - n}\right) a_r = \beta\frac{n}{\beta^2}\left[\frac{(1-\lambda)}{\frac{1}{r_2} - n} + \frac{\lambda}{\frac{1}{r_1} - n}\right] \\
\Longleftrightarrow &\left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{n}{\beta^2} + \frac{n^2}{\beta^2}\left(\frac{(1-\lambda)}{\frac{1}{r_2} - n} + \frac{\lambda}{\frac{1}{r_1} - n}\right)\right) a_r = \frac{n}{\beta}\left[\frac{(1-\lambda)}{\frac{1}{r_2} - n} + \frac{\lambda}{\frac{1}{r_1} - n}\right]. \\
\Longleftrightarrow & a_r = \frac{\frac{n}{\beta}\left[\frac{(1-\lambda)}{\frac{1}{r_2}-n} + \frac{\lambda}{\frac{1}{r_1}-n}\right]}{\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{n}{\beta^2} + \frac{n^2}{\beta^2}\left(\frac{(1-\lambda)}{\frac{1}{r_2}-n} + \frac{\lambda}{\frac{1}{r_1}-n}\right)} \\
\Longleftrightarrow & a_r = \frac{\frac{n}{\beta}}{\left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{n}{\beta^2}\right)\frac{1}{\frac{(1-\lambda)}{\frac{1}{r_2}-n} + \frac{\lambda}{\frac{1}{r_1}-n}} + \frac{n^2}{\beta^2}} \\
\Longleftrightarrow & a_r = \frac{\frac{n}{\beta}}{\frac{n^2}{\beta^2} - \left(\frac{1-\theta}{\theta\sigma_x^2}\sigma_g^2 + \frac{n}{\beta^2}\right)\frac{n\theta\left(\frac{a_s}{a_r}\right)}{\beta A\sigma_g^2}\left(\frac{\lambda - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1 - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)} \\
\Longleftrightarrow & a_r = \frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2 + \beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1 - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}.
\end{aligned}$$

Here we used the following result

$$\begin{aligned}
\frac{(1-\lambda)}{\frac{1}{r_2}-n}+\frac{\lambda}{\frac{1}{r_1}-n} &= \frac{(1-\lambda)}{-n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r}+\left(\frac{a_s}{a_r}\right)^2\frac{\sigma_s^2}{\sigma_g^2}}+\frac{\lambda}{-n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r}} \\
&= \frac{1}{-\frac{n}{\beta^2}\theta\left(\frac{a_s}{a_r}\right)}\left(\frac{(1-\lambda)}{\frac{\lambda\beta}{A\sigma_g^2}-\frac{\beta^2\sigma_s^2}{n\theta\sigma_g^2}\left(\frac{a_s}{a_r}\right)}+\frac{\lambda}{\frac{\lambda\beta}{A\sigma_g^2}}\right) \\
&= \frac{1}{-\frac{n}{\beta^2}\theta\left(\frac{a_s}{a_r}\right)}\left(\frac{(1-\lambda)\frac{\lambda\beta}{A\sigma_g^2}+\lambda\left(\frac{\lambda\beta}{A\sigma_g^2}-\frac{\beta^2\sigma_s^2}{n\theta\sigma_g^2}\left(\frac{a_s}{a_r}\right)\right)}{\left[\frac{\lambda\beta}{A\sigma_g^2}-\frac{\beta^2\sigma_s^2}{n\theta\sigma_g^2}\left(\frac{a_s}{a_r}\right)\right]\frac{\lambda\beta}{A\sigma_g^2}}\right) \\
&= \frac{1}{-\frac{n}{\beta^2}\theta\left(\frac{a_s}{a_r}\right)}\left(\frac{\frac{\lambda\beta}{A\sigma_g^2}-\lambda\frac{\beta^2\sigma_s^2}{n\theta\sigma_g^2}\left(\frac{a_s}{a_r}\right)}{\left[\frac{\lambda\beta}{A\sigma_g^2}-\frac{\beta^2\sigma_s^2}{n\theta\sigma_g^2}\left(\frac{a_s}{a_r}\right)\right]\frac{\lambda\beta}{A\sigma_g^2}}\right) \\
&= \frac{\beta A\sigma_g^2}{-n\theta\left(\frac{a_s}{a_r}\right)}\left(\frac{1-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{\lambda-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right).
\end{aligned}$$

Then we have

$$\frac{1}{r_1}=\frac{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2}{\sigma_g^2}=1+\frac{\sigma_u^2}{\sigma_g^2}+a_r^2\frac{\sigma_q^2}{\sigma_g^2} \tag{A.32}$$

$$\Longleftrightarrow n-n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r}=1+\frac{\sigma_u^2}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}-\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2} \tag{A.33}$$

The LHS is increasing in $-\frac{a_s}{a_r}$ while the RHS is decreasing in $-\frac{a_s}{a_r}$, so $\frac{a_s}{a_r}$ is uniquely pinned down by the above condition. We can also solve a_0 by:

$$(1-\theta)\frac{\bar{x}-p}{\sigma_x^2}+\theta\left[\lambda\frac{\bar{x}+\beta\bar{g}+\beta r_1(g_r-\bar{g}_r)-p}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+(1-\lambda)\frac{\bar{x}+\beta\bar{g}+\beta r_2\left(\frac{p-\bar{p}}{a_r}\right)-p}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}\right]=A\tilde{s}. \tag{A.34}$$

Note that

$$\begin{aligned}
&(1-\theta)\frac{\bar{x}-p}{\sigma_x^2}+\theta\left[\lambda\frac{\bar{x}+\beta\bar{g}+\beta r_1(g_r-\bar{g}_r)-p}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+(1-\lambda)\frac{\bar{x}+\beta\bar{g}+\beta r_2\left(\frac{p-\bar{p}}{a_r}\right)-p}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}\right]=A\tilde{s} \\
\Longleftrightarrow\ &(1-\theta)\frac{\bar{x}-\bar{p}}{\sigma_x^2}+\theta\left[\lambda\frac{\bar{x}+\beta\bar{g}-\bar{p}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+(1-\lambda)\frac{\bar{x}+\beta\bar{g}-\bar{p}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}\right]=A\bar{s} \\
\Longleftrightarrow\ &\left(\frac{1-\theta}{\theta\sigma_x^2}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}\right)\bar{p}=\begin{array}{c}\left(\frac{1-\theta}{\theta}\right)\frac{\bar{x}}{\sigma_x^2}+\lambda\frac{\bar{x}+\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+ \\ (1-\lambda)\frac{\bar{x}+\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}-A\bar{s}\end{array}
\end{aligned}$$

or

$$\bar{p}=\bar{x}+\frac{\lambda\frac{\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+(1-\lambda)\frac{\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}-A\bar{s}}{\frac{1-\theta}{\theta\sigma_x^2}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}}. \tag{A.35}$$

Since $\bar{p}=a_0+a_r\bar{g}_r+a_s\bar{s}=a_0+a_r\left(\bar{g}+a_r\bar{q}\right)+a_s\bar{s}$, we have

$$a_0=\bar{x}+\frac{\lambda\frac{\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+(1-\lambda)\frac{\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}-A\bar{s}}{\frac{1-\theta}{\theta\sigma_x^2}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}}-\left(a_r\left(\bar{g}+a_r\bar{q}\right)+a_s\bar{s}\right). \tag{A.36}$$

B.2 Proof of Corollary 1

First, from Proposition 1, it's clear that $\bar{q}$ has no impact on the informativeness of the equilibrium, so when $\bar{q}$ increases, both a_r and $\mathrm{Var}\left(\delta^*\right)$ will be unchanged. To consider the effect of σ_q^2, note that the equilibrium $\left(-\frac{a_s}{a_r}\right)$ is solved by

$$n-n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r}=1+\frac{\sigma_u^2}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}-\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.37}$$

Then when σ_q increases, it's clear that $\left(-\frac{a_s}{a_r}\right)$ increases. And a_r is solved by

$$a_r=\frac{1}{\frac{n}{\beta}-\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}. \tag{A.38}$$

So when $\left(-\frac{a_s}{a_r}\right)$ increases, a_r decreases. Besides, the condition A.37 can be rewritten as

$$n-n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r}=1+\frac{\sigma_u^2}{\sigma_g^2}+\frac{a_r^2\sigma_q^2}{\sigma_g^2}. \tag{A.39}$$

Then when $\left(-\frac{a_s}{a_r}\right)$ increases, it's clear that $a_r^2\sigma_q^2$ increases.

B.3 Proof of Corollary 2

First, it's clear that $\bar{q}$ has no impact on the correlations, as it has no impact on the informativeness of the price and the score. Next let's consider the effect of σ_q. The equilibrium condition is

$$n - n\theta \frac{\lambda}{\beta A \sigma_g^2} \frac{a_s}{a_r} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)} \right)^2 \frac{\sigma_q^2}{\sigma_g^2}. \tag{A.40}$$

When σ_q increases, since $\left(\frac{\lambda - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)$ is an increasing function of $\left(-\frac{a_s}{a_r}\right)$, it's clear that $\left(-\frac{a_s}{a_r}\right)$ will increase. a_r is solved by

$$a_r = \frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}, \tag{A.41}$$

then a_r will decrease. The equilibrium score is

$$g_r = \tilde{g} + u + a_r q, \tag{A.42}$$

the correlation

$$\operatorname{corr}(g_r, \tilde{g}) = \sqrt{\frac{\operatorname{Cov}^2(g_r, \tilde{g})}{\operatorname{Var}(g_r)\operatorname{Var}(\tilde{g})}} \tag{A.43}$$

$$= \sqrt{\frac{\sigma_g^4}{\left(\sigma_g^2 + \sigma_u^2 + a_r^2\sigma_q^2\right)\sigma_g^2}}. \tag{A.44}$$

We have already shown in Corollary 1 that when σ_q increases, $\operatorname{Var}(\delta^*) = a_r^2\sigma_q^2$ increases, which implies that $\operatorname{corr}(g_r, \tilde{g})$ will decrease. The equilibrium price can be written as

$$p = a_0 + a_r\left(\tilde{g} + u + a_r q + \frac{a_s}{a_r}\tilde{s}\right). \tag{A.45}$$

When σ_q increases, since both $\operatorname{Var}(\delta^*) = a_r^2\sigma_q^2$ and $\left(-\frac{a_s}{a_r}\right) > 0$ increase, we can conclude that $\operatorname{corr}(p, \tilde{g})$ decreases. The equilibrium price can also be written as

$$p = a_0 + a_r\left(g_r + \frac{a_s}{a_r}\tilde{s}\right). \tag{A.46}$$

The correlation

$$\operatorname{corr}(g_r, p) = \sqrt{\frac{\operatorname{Cov}^2\left(g_r, g_r + \frac{a_s}{a_r}\tilde{s}\right)}{\operatorname{Var}(g_r)\operatorname{Var}\left(g_r + \frac{a_s}{a_r}\tilde{s}\right)}} = \sqrt{\frac{\operatorname{Var}(g_r)}{\operatorname{Var}\left(g_r + \frac{a_s}{a_r}\tilde{s}\right)}} = \sqrt{\frac{\operatorname{Var}(g_r)}{\operatorname{Var}(g_r) + \left(\frac{a_s}{a_r}\right)^2\sigma_s^2}}. \tag{A.47}$$

Note that $\text{Var}(g_r) = \sigma_g^2 + \sigma_u^2 + a_r^2\sigma_q^2$, and the equilibrium condition (A.40) implies that

$$\sigma_g^2 + \sigma_u^2 + a_r^2\sigma_q^2 = n\sigma_g^2 - n\theta\frac{\lambda}{\beta A}\frac{a_s}{a_r}, \tag{A.48}$$

then

$$\begin{aligned}\text{corr}(g_r, p) &= \sqrt{\frac{\text{Var}(g_r)}{\text{Var}(g_r) + \left(\frac{a_s}{a_r}\right)^2 \sigma_s^2}} = \sqrt{\frac{1}{1 + \frac{\left(\frac{a_s}{a_r}\right)^2}{\sigma_g^2+\sigma_u^2+a_r^2\sigma_q^2}\sigma_s^2}} \\ &= \sqrt{\frac{1}{1 + \frac{\left(\frac{a_s}{a_r}\right)^2}{n\sigma_g^2 - n\theta\frac{\lambda}{\beta A}\frac{a_s}{a_r}}\sigma_s^2}} = \sqrt{\frac{1}{1 + \frac{\left(-\frac{a_s}{a_r}\right)}{\frac{n\sigma_g^2}{\left(-\frac{a_s}{a_r}\right)} + n\theta\frac{\lambda}{\beta A}}\sigma_s^2}}. \end{aligned} \tag{A.49}$$

When σ_q increases, we've already shown that $\left(-\frac{a_s}{a_r}\right)$ increases, then $\frac{\left(-\frac{a_s}{a_r}\right)}{\frac{n\sigma_g^2}{\left(-\frac{a_s}{a_r}\right)} + n\theta\frac{\lambda}{\beta A}}\sigma_s^2$ increases and thus $\text{corr}(g_r, p)$ decreases.

B.4 Proof of Lemma 1

Since $p = a_0 + a_r\left(\tilde{g} + u + a_r q + \frac{a_s}{a_r}\tilde{s}\right)$, the correlation $\text{corr}(p, \tilde{g})$ is

$$\begin{aligned}\text{corr}(p, \tilde{g}) &= \sqrt{\frac{\text{Cov}^2\left(\tilde{g} + u + a_r q + \frac{a_s}{a_r}\tilde{s}, \tilde{g}\right)}{\text{Var}(\tilde{g})\,\text{Var}\left(\tilde{g} + u + a_r q + \frac{a_s}{a_r}\tilde{s}\right)}} \\ &= \sqrt{\frac{\text{Var}(\tilde{g})}{\text{Var}\left(\tilde{g} + u + a_r q + \frac{a_s}{a_r}\tilde{s}\right)}} \\ &= \sqrt{\frac{\text{Var}(\tilde{g})}{\text{Var}\left(\tilde{g} + u + a_r q + \frac{a_s}{a_r}\tilde{s}\right)}} \\ &= \sqrt{r_2}. \end{aligned} \tag{A.50}$$

Let

$$z = \frac{1}{r_2} = n - n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r} + \left(\frac{a_s}{a_r}\right)^2\frac{\sigma_s^2}{\sigma_g^2}. \tag{A.51}$$

Note the equilibrium condition is

$$n - n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1 - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.52}$$

When λ increases, it's clear that $\left(-\frac{a_s}{a_r}\right)$ will decrease.

The above condition also implies that

$$z = \frac{\sigma_s^2}{\sigma_g^2}\left(\frac{a_s}{a_r}\right)^2 + 1 + \frac{\sigma_u^2}{\sigma_g^2} + \frac{\beta^2}{n^2}\left(\frac{n - \frac{\beta A \sigma_s^2}{\theta}\left(\frac{a_s}{a_r}\right)}{\left(\frac{1-\theta}{\sigma_x^2}\frac{\beta^2\sigma_g^2}{n\theta} + 1\right)(z-n) + n - \frac{\beta A \sigma_s^2}{\theta}\left(\frac{a_s}{a_r}\right)}\right)^2 \frac{\sigma_q^2}{\sigma_g^2}. \tag{A.53}$$

The definition of z (A.51) implies that

$$z - n > 0. \tag{A.54}$$

Then for the equation (A.53),the LHS is increasing in z, the RHS is decreasing in z and increasing in $\left(-\frac{a_s}{a_r}\right)$. Then we conclude that when λ increases, z decreases. So when λ increases, $\text{corr}\,(p, \tilde{g})$ increases.

B.5 Proof of Lemma 2

This proof replicates Admati and Pfleiderer (1987). Suppose investor i's information set is $\mathscr{F}_i$, then investor i's expected utility at time 0 is

$$\begin{aligned} U_i &= \mathbb{E}\left[-e^{-A[\phi_i(\tilde{x}+\beta\tilde{g})+l_i]}|\mathscr{F}_i\right] \\ &= -e^{-A\mathbb{E}[\phi_i(\tilde{x}+\beta\tilde{g})+l_i|\mathscr{F}_i]+\frac{1}{2}A^2\phi_i^2\text{Var}(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i)}. \end{aligned} \tag{A.55}$$

The budget balance condition is

$$W_0 = \phi_i p + l_i, \tag{A.56}$$

where p is the equilibrium price of the stock. So

$$U_i = -e^{-AW_0}e^{-A\left[\phi_i E(\tilde{x}+\beta\tilde{g}-p|\mathscr{F}_i) - \frac{1}{2}A\phi_i^2\text{Var}(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i)\right]}. \tag{A.57}$$

It's clear that the demand curve is characterized by

$$\phi_i^* = \frac{E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right) - p}{A\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}. \tag{A.58}$$

Then investor i's time 0 utility is

$$U_i = -e^{-AW_0}e^{-\frac{\left(E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p\right)^2}{2\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}}. \tag{A.59}$$

By our normality assumption, $\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)$ is a constant, so

$$\mathbb{E}\left[\text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)\right] = \text{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right). \tag{A.60}$$

To calculate the value of the score, let's consider the investor i's ex ante utility. Investor i's ex ante utility is

$$-e^{-AW_0}\mathbb{E}\left[e^{-\frac{\left(E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p\right)^2}{2\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}}\right]. \tag{A.61}$$

By our normality assumption, $E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p$ follows normal distribution. Let

$$\mu_z=\mathbb{E}\left[E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p\right] \tag{A.62}$$

and

$$\sigma_z^2=\mathrm{Var}\left[E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p\right]. \tag{A.63}$$

By the moment-generating function of noncentral chi-squared distribution, we have

$$\begin{aligned}
&-e^{-AW_0}\mathbb{E}\left[e^{-\frac{\left(E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p\right)^2}{2\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}}\right]\\
=&-e^{-AW_0}\mathbb{E}\left[e^{-\frac{\sigma_z^2}{2\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}\left(\frac{E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p}{\sigma_z}\right)^2}\right]\\
=&-e^{-AW_0}e^{\frac{\mu_z^2\cdot\left(-\frac{1}{2\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}\right)}{1+\frac{\sigma_z^2}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}}}\left(1+\frac{\sigma_z^2}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}\right)^{-1/2}\\
=&-e^{-AW_0}e^{-\frac{1}{2}\frac{\mu_z^2\cdot}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)+\sigma_z^2}}\left(\frac{\sigma_z^2+\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}\right)^{-1/2}\\
=&-e^{-AW_0}e^{-\frac{1}{2}\frac{\mu_z^2\cdot}{\mathbb{E}\left[\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)\right]+\sigma_z^2}}\left(\frac{\sigma_z^2+\mathbb{E}\left[\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)\right]}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}\right)^{-1/2}
\end{aligned}$$

By the law of total variance,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)\right]+\sigma_z^2&=\mathbb{E}\left[\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)\right]+\mathrm{Var}\left[E\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)-p\right] &\text{(A.64)}\\
&=\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}-p\right). &\text{(A.65)}
\end{aligned}$$

Denote

$$\sigma_0^2=\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}-p\right). \tag{A.66}$$

Then

$$
\begin{aligned}
&- e^{-AW_0}\mathbb{E}\left[e^{-\frac{(E(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i)-p)^2}{2\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i)}}\right] \\
&= - e^{-AW_0} e^{-\frac{1}{2}\frac{\mu_{z\cdot}^2}{\sigma_0^2}} \left(\frac{\sigma_0^2}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)}\right)^{-1/2} \\
&= - e^{-AW_0}\sqrt{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_i\right)} \cdot \left(e^{-\frac{1}{2}\frac{\mu_{z\cdot}^2}{\sigma_0^2}}\frac{1}{\sigma_0}\right).
\end{aligned}
$$

At time 0, the information set of score buyers is

$$
\mathscr{F}_I = \{g_r, p\} = \{g_r\} \tag{A.67}
$$

and the information set of other type-G investors is

$$
\mathscr{F}_U = \{p\}. \tag{A.68}
$$

Let the price of the score be Φ, then the equilibrium condition is

$$
e^{-AW_0}\sqrt{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right)} = e^{-A(W_0-\Phi)}\sqrt{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right)}, \tag{A.69}
$$

which is

$$
\Phi = \frac{1}{2A}\ln\left(\frac{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right)}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right)}\right). \tag{A.70}
$$

B.6 Proof of Lemma 3

From Lemma 2, the score price is

$$
\begin{aligned}
\ln\left(\frac{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_{un}\right)}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_{in}\right)}\right) &= \ln\left(\frac{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_2\right)}{\sigma_x^2+\beta^2\sigma_g^2\left(1-r_1\right)}\right) \\
&= \ln\left(\frac{1-nr_2}{1-nr_1}\right) = \ln\left(1+n\frac{r_1-r_2}{1-nr_1}\right) \\
&= \ln\left(1+n\frac{r_1r_2}{1-nr_1}\left(\frac{a_s}{a_r}\right)^2\frac{\sigma_s^2}{\sigma_g^2}\right) = \ln\left(1+n\frac{r_2}{\frac{1}{r_1}-n}\left(\frac{a_s}{a_r}\right)^2\frac{\sigma_s^2}{\sigma_g^2}\right) \\
&= \ln\left(1+n\frac{\left(\frac{a_s}{a_r}\right)^2\frac{\sigma_s^2}{\sigma_g^2}}{\left(n-\frac{n}{\beta^2}\theta\frac{\lambda\beta}{A\sigma_g^2}\frac{a_s}{a_r}+\frac{\sigma_s^2}{\sigma_g^2}\left(\frac{a_s}{a_r}\right)^2\right)\left(-\frac{n}{\beta^2}\theta\frac{\lambda\beta}{A\sigma_g^2}\frac{a_s}{a_r}\right)}\right) \\
&= \ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right),
\end{aligned}
$$

where $-\frac{a_s}{a_r}$ is solved in equilibrium by

$$n - n\theta \frac{\lambda}{\beta A \sigma_g^2} \frac{a_s}{a_r} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1 - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)} \right)^2 \frac{\sigma_q^2}{\sigma_g^2}. \tag{A.71}$$

In the above intermediate steps, we use the following two results from the proof of Proposition 1:

$$\frac{1}{r_1} = n - n\theta \frac{\lambda}{\beta A \sigma_g^2} \frac{a_s}{a_r} \tag{A.72}$$

and

$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{r_1 - r_2}{r_1 r_2} = n - n\theta \frac{\lambda}{\beta A \sigma_g^2} \frac{a_s}{a_r} + \left(\frac{a_s}{a_r}\right)^2 \frac{\sigma_s^2}{\sigma_g^2}. \tag{A.73}$$

It's clear that when σ_q increases, $\left(-\frac{a_s}{a_r}\right)$ will increase. When $\sigma_q^2 = 0$, $-\frac{a_s}{a_r}$ reaches to its minimum value $-\frac{a_s}{a_r} = \frac{\sigma_g^2+\sigma_u^2-n\sigma_g^2}{\frac{n\lambda\theta}{\beta A}}$. From the expression **??**, the score price is decreasing in $-\frac{a_s}{a_r}$ when $-\frac{a_s}{a_r} \leqslant \sqrt{\frac{n\sigma_g^2}{\sigma_s^2}}$ and is increasing in $-\frac{a_s}{a_r}$ when $-\frac{a_s}{a_r} > \sqrt{\frac{n\sigma_g^2}{\sigma_s^2}}$. Then if

$$\frac{\sigma_g^2 + \sigma_u^2 - n\sigma_g^2}{\frac{n\lambda\theta}{\beta A}} \leqslant \sqrt{\frac{n\sigma_g^2}{\sigma_s^2}} \Longleftrightarrow \sigma_u^2 \leqslant \underline{\sigma}_u^2 = \frac{n\lambda\theta}{\beta A}\sqrt{\frac{n\sigma_g^2}{\sigma_s^2}} + (n-1)\,\sigma_g^2 \tag{A.74}$$

$$\Longleftrightarrow \sigma_u \leqslant \underline{\sigma}_u = \sqrt{\frac{n\lambda\theta}{\beta A}\sqrt{\frac{n\sigma_g^2}{\sigma_s^2}} + (n-1)\,\sigma_g^2}, \tag{A.75}$$

when σ_q^2 increases from zero to ∞, $-\frac{a_s}{a_r}$ will increase from a value lower than $\sqrt{\frac{n\sigma_g^2}{\sigma_s^2}}$ to ∞, as a result, the score price will first increase and then decrease . If $\sigma_u^2 > \underline{\sigma}_u^2$, when σ_q^2 increases from zero to ∞, $-\frac{a_s}{a_r}$ will increase from a value above $\sqrt{\frac{n\sigma_g^2}{\sigma_s^2}}$ to ∞, as a result, the score price will monotonically decrease.

B.7 Proof of Theorem 1

The information seller's problem is

$$\max_{(\Phi,\sigma_u)} \theta\lambda \cdot \frac{1}{2A} \ln\left(\frac{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|\mathscr{F}_U\right)}{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|\mathscr{F}_I\right)}\right). \tag{A.76}$$

From the proof of Lemma 3, the objective function is equivalent to

$$\theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta} \frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right). \tag{A.77}$$

where $\frac{a_s}{a_r}$ satisfies:

$$n - n\theta \frac{\lambda}{\beta A \sigma_g^2} \frac{a_s}{a_r} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2 + \beta^2 \sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1 - \frac{\beta A \sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)} \right)^2 \frac{\sigma_q^2}{\sigma_g^2}. \tag{A.78}$$

The unconstrained solution of

$$\min \frac{1}{-\frac{a_s}{a_r}}\beta + \theta \frac{\lambda}{A\sigma_g^2} + \left(-\frac{a_s}{a_r}\right)\frac{\beta \sigma_s^2}{n\sigma_g^2} \tag{A.79}$$

satisfies

$$\frac{1}{-\frac{a_s}{a_r}}\beta = \left(-\frac{a_s}{a_r}\right)\frac{\beta \sigma_s^2}{n\sigma_g^2} \Longleftrightarrow \left(\frac{a_s}{a_r}\right)^2 = n\frac{\sigma_g^2}{\sigma_s^2}, \tag{A.80}$$

which is

$$\left(-\frac{a_s}{a_r}\right) = \frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.81}$$

Then

$$\frac{1}{-\frac{a_s}{a_r}}\beta + \theta \frac{\lambda}{A\sigma_g^2} + \left(-\frac{a_s}{a_r}\right)\frac{\beta \sigma_s^2}{n\sigma_g^2} = \theta \frac{\lambda}{A\sigma_g^2} + 2\sqrt{n}\frac{\sigma_g}{\sigma_s}\frac{\beta}{n}\frac{\sigma_s^2}{\sigma_g^2} = \theta \frac{\lambda}{A\sigma_g^2} + 2\beta \frac{\sigma_s}{\sigma_g \sqrt{n}}. \tag{A.82}$$

Then

$$\begin{aligned}
&\theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right) \\
=&\theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\theta\frac{\lambda}{A\sigma_g^2} + 2\beta\frac{\sigma_s}{\sigma_g\sqrt{n}}}\right) \\
=&\theta\lambda \ln\left(1 + \frac{\frac{A\beta^2\sigma_s^2}{n\lambda\theta}}{\theta\frac{\lambda}{A\sigma_g^2} + 2\beta\frac{\sigma_s}{\sigma_g\sqrt{n}}}\right) \\
=&\theta\lambda \ln\left(1 + \frac{1}{\theta\frac{\lambda}{A\sigma_g^2}\frac{n\lambda\theta}{A\beta^2\sigma_s^2} + \frac{n\lambda\theta}{A\beta^2\sigma_s^2}2\frac{\sigma_s}{\sigma_g\sqrt{n}}}\right) \\
=&\theta\lambda \ln\left(1 + \frac{1}{\frac{\lambda^2}{A^2\sigma_g^2}\frac{n\theta^2}{\beta^2\sigma_s^2} + 2\frac{\sqrt{n}\lambda\theta}{\beta A\sigma_s\sigma_g}}\right) \\
=&\theta\lambda \ln\left(1 + \frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2 + 2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right).
\end{aligned}$$

Admati and Pfleiderer (1986) show that there exists a unique solution to the following problem

$$\max_k k \ln\left(1+\frac{1}{k(k+2)}\right), \tag{A.83}$$

denote the solution as λ_0.[9] Then the information seller's optimal choice $\bar{\lambda}$ is

$$\bar{\lambda} = \frac{\lambda_0 A\sigma_s}{\theta}\sqrt{\sigma_x^2+\beta^2\sigma_g^2}.$$

Let $M(\lambda)$ be the solution of

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}M(\lambda) = 1 + \frac{\sigma^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)M(\lambda)\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}{1+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.84}$$

Lemma 7. *$M(\lambda)$ is a decreasing function of λ , and $\lim_{\lambda\to 0} M(\lambda) = \infty$.*

Proof. First, it's clear that the RHS of (A.84) must be greater than 1. When $\lambda \to 0$, we must have

$$\lim_{\lambda\to 0}\frac{n\lambda}{\beta A\sigma_g^2}M(\lambda) \geqslant 1+\frac{\sigma^2}{\sigma_g^2} - n = 1 + \frac{\sigma^2}{\sigma_g^2} - \frac{\beta^2\sigma_g^2}{\sigma_x^2+\beta^2\sigma_g^2} > 0. \tag{A.85}$$

which implies that

$$\lim_{\lambda\to 0} M(\lambda) = \infty. \tag{A.86}$$

The LHS is an increasing function of both M and λ, the RHS is a decreasing function of both M and λ. We have

$$LHS(M,\lambda) = RHS(M,\lambda). \tag{A.87}$$

Then

$$\frac{\partial LHS}{\partial M}dM + \frac{\partial LHS}{\partial \lambda}d\lambda = \frac{\partial RHS}{\partial M}dM + \frac{\partial RHS}{\partial \lambda}d\lambda \tag{A.88}$$

which leads to

$$\frac{dM}{d\lambda} = \frac{\frac{\partial RHS}{\partial\lambda} - \frac{\partial LHS}{\partial\lambda}}{\frac{\partial LHS}{\partial M} - \frac{\partial RHS}{\partial M}}. \tag{A.89}$$

Since $\frac{\partial RHS}{\partial\lambda} < 0$, $\frac{\partial RHS}{\partial M} < 0$, $\frac{\partial LHS}{\partial\lambda} > 0$ and $\frac{\partial LHS}{\partial M} > 0$, we must have $\frac{dM}{d\lambda} < 0$. □

[9] $\lambda_0 \approx 0.651461$.

As we discussed earlier, the information seller's problem is

$$\max_{(\Phi,\sigma_u)} \theta\lambda \ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right), \tag{A.90}$$

where $\frac{a_s}{a_r}$ satisfies

$$n-n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r}=1+\frac{\sigma_u^2}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}-\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.91}$$

When $\bar{\lambda}\leqslant 1$ and $M\left(\bar{\lambda}\right)\leqslant\frac{\sigma_g}{\sigma_s}\sqrt{n}$, there must exist $\sigma_u\geqslant\underline{\sigma}$ such that

$$n+n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n}=1+\frac{\sigma_u^2}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.92}$$

Then in this case, $\bar{\lambda}$, together with σ_u^2 satisfying the above condition, must be the solution of the information seller's optimization problem, as this is the highest possible utility for the information seller. It's easy to see that

$$\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\bar{\lambda}+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)} \tag{A.93}$$

is the equilibrium a_r in this equilibrium. Denote it as a_r^*, then the above equilibrium condition becomes

$$n+n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n}=1+\frac{\sigma_u^2}{\sigma_g^2}+a_r^{*2}\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.94}$$

When $\bar{\lambda}>1$ and $M(1)\leqslant\frac{\sigma_g}{\sigma_s}\sqrt{n}$, since $M(\cdot)$ is a decreasing function with $M(0)=\infty$, there must exist

$$\lambda_1\in[0,1] \tag{A.95}$$

such that

$$M(\lambda_1)=\frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.96}$$

First, for any $\lambda < \lambda_1$, we must have

$$\theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right) \leqslant \theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{\frac{\sigma_g}{\sigma_s}\sqrt{n}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(\frac{\sigma_g}{\sigma_s}\sqrt{n}\right)\right)}\right). \tag{A.97}$$

For any $\lambda \in [\lambda_1, 1]$, there must exist $\sigma_u \geqslant \underline{\sigma}$ such that

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda + \frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1 + \frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.98}$$

We know in equilibrium if we can obtain $\left(-\frac{a_s}{a_r}\right) = \frac{\sigma_g}{\sigma_s}\sqrt{n}$, the objective function becomes

$$\theta\lambda \ln\left(1 + \frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2 + 2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right), \tag{A.99}$$

which is a unimodal function of λ.[10] When $\bar{\lambda} > 1$, we must have that

$$\theta\lambda \ln\left(1 + \frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2 + 2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right) \leqslant \theta \ln\left(1 + \frac{1}{\left(\frac{\sqrt{n}\theta}{A\beta\sigma_s\sigma_g}\right)^2 + 2\frac{\sqrt{n}\theta}{A\beta\sigma_s\sigma_g}}\right) \tag{A.100}$$

for any $\lambda \in [0, 1]$. Then we conclude that in this case, the information seller chooses $\lambda = 1$ and chooses $\sigma_u \geqslant \underline{\sigma}$ such that

$$n + n\theta\frac{1}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.101}$$

It's clear that

$$\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}} \tag{A.102}$$

is the equilibrium a_r in this equilibrium. Denote this as a_r^{**}. Then the above condition becomes

$$n\sigma_g^2 + n\theta\frac{1}{A\beta}\frac{\sigma_g}{\sigma_s}\sqrt{n} = \sigma_g^2 + \sigma_u^2 + a_r^{**2}\sigma_q^2. \tag{A.103}$$

If $\bar{\lambda} \leqslant 1$ and $M\left(\bar{\lambda}\right) > \frac{\sigma_g}{\sigma_s}\sqrt{n}$, we want to show that the information seller must choose $\sigma_u = \underline{\sigma}$. Suppose the information seller chooses λ_2. Let λ_1 be the solution of $M\left(\lambda_1\right) = \frac{\sigma_g}{\sigma_s}\sqrt{n}$. If $\lambda_2 > \lambda_1$,

[10]We can show that function $k \ln\left(1 + \frac{1}{k(k+2)}\right)$ is a unimodal function.

we know that

$$M(\lambda_2) < M(\lambda_1) = \frac{\sigma_g}{\sigma_s}\sqrt{n}, \tag{A.104}$$

and the information seller is able to find σ_u^2 such that in equilibrium $\left(-\frac{a_s}{a_r}\right) = \frac{\sigma_g}{\sigma_s}\sqrt{n}$. However, since in this case $\bar{\lambda} \leqslant 1$ and $M\left(\bar{\lambda}\right) > \frac{\sigma_g}{\sigma_s}\sqrt{n}$, and since the objective function

$$\theta\lambda \ln\left(1 + \frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2 + 2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right) \tag{A.105}$$

is a unimodal function of λ, we know that λ_2 is suboptimal, and is dominated by λ_1. So the information seller must choose a $\lambda_2 \leqslant \lambda_1$. When $\lambda_2 \leqslant \lambda_1$, we know that

$$M(\lambda_2) \geqslant M(\lambda_1) = \frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.106}$$

Since in equilibrium $-\frac{a_s}{a_r} \geqslant M(\lambda_2) \geqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$, and we know that

$$\theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right) \tag{A.107}$$

is decreasing in $\left(-\frac{a_s}{a_r}\right) \geqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$, the information seller must choose $\sigma_u = \underline{\sigma}$ which leads to $-\frac{a_s}{a_r} = M(\lambda_2)$.

If $\bar{\lambda} > 1$ and $M(1) > \frac{\sigma_g}{\sigma_s}\sqrt{n}$, it's clear that no matter what λ the information seller chooses, we must have

$$M(\lambda) > \frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.108}$$

Since

$$\theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right) \tag{A.109}$$

is decreasing in $\left(-\frac{a_s}{a_r}\right) \geqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$, we conclude that the information seller must choose $\sigma_u = \underline{\sigma}$, so $-\frac{a_s}{a_r} = M(\lambda)$.

B.8 Proof of Proposition 2

To show the results in this proposition, we just need to show that in the unconstrained equilibrium, the optimal price Φ^*, the variance of the score $\text{Var}(g_r)$, and the informativeness of both the score g_r and price p are independent of $\bar{q}$ and σ_q. First, the equation (19) implies that $\text{Var}(g_r)$ is independent of $\bar{q}$ and σ_q in the unconstrained equilibrium. Note that if $\text{Var}(g_r)$ is independent of $\bar{q}$ and σ_q, the informativeness of g_r, which is represented by

$$r_2 = \frac{\sigma_g^2}{\text{Var}\left(g_r\right)}, \tag{A.110}$$

is also independent of $\bar{q}$ and σ_q. Since in the unconstrained equilibrium,

$$\frac{a_s}{a_r} = \frac{\sigma_g}{\sigma_s}\sqrt{n}, \tag{A.111}$$

then the informativeness of p, which is represented by

$$r_1 = \frac{\sigma_g^2}{\text{Var}\left(g_r\right) + \left(\frac{a_s}{a_r}\right)^2 \sigma_s^2} = \frac{\sigma_g^2}{\text{Var}\left(g_r\right) + n\sigma_g^2}, \tag{A.112}$$

is also independent of $\bar{q}$ and σ_q. Finally, the optimal score price Φ^* satisfies

$$\Phi^* = \frac{1}{2A}\ln\left(\frac{\text{Var}\left(\tilde{x} + \beta\tilde{g}|p\right)}{\text{Var}\left(\tilde{x} + \beta\tilde{g}|g_r\right)}\right) = \frac{1}{2A}\ln\left(\frac{\sigma_x^2 + \beta^2\sigma_g^2\left(1 - r_2\right)}{\sigma_x^2 + \beta^2\sigma_g^2\left(1 - r_1\right)}\right), \tag{A.113}$$

and it's clear that Φ^* is also independent of $\bar{q}$ and σ_q in the unconstrained equilibrium.

B.9 Proof of Proposition 3

From the proof of Theorem 1, in the unconstrained equilibrium, the information seller chooses σ_u such that

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda + \frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1 + \frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.114}$$

If $\sigma_u > \underline{\sigma}$, when σ_q increases, it's clear that the information seller can still make the above condition hold by choosing a lower σ_u. And it's clear that the price of the score is independent of σ_u in the unconstrained equilibrium. When σ_u reaches the lowest level $\underline{\sigma}$, the information seller can not make the above condition hold anymore if σ_q continues to increase. Based on our results in Theorem 1, this is the case when $\bar{\lambda} \leqslant 1$ and $M\left(\bar{\lambda}\right) > \frac{\sigma_g}{\sigma_s}\sqrt{n}$, and this is the third case in Theorem 1. In this case, the information seller will always choose $\sigma_u = \underline{\sigma}$, and will adjust the price as a response to the increase of σ_q.

B.10 Proof of Proposition 4

In the unconstrained equilibrium, $\lambda^* = \frac{\lambda_0 A\sigma_s}{\theta}\sqrt{\sigma_x^2 + \beta^2\sigma_g^2}$. It's obvious that when A increases, λ^* increases. a_r^* in this equilibrium is

$$\begin{aligned} a_r^* &= \frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda + \frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)} \\ &= \frac{1}{\frac{n}{\beta} + \left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\frac{\lambda}{A} + \frac{\beta\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)} \end{aligned} \tag{A.115}$$

So when A increases, a_r^* increases, and thus$\text{Var}(\delta^*) = a_r^2\sigma_q^2$ also increases. The equilibrium condition in this case is

$$n + n\theta\frac{\lambda^*}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n} = n + \lambda_0 n = 1 + \frac{\sigma_u^{*2}}{\sigma_g^2} + a_r^{*2}\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.116}$$

Since a_r^* is increasing in A, we can conclude that when A increases, σ_u^* must decrease. The total variance of g_r is

$$\text{Var}(g_r) = \sigma_g^2 + \sigma_u^{*2} + a_r^{*2}\sigma_q^2 = n\sigma_g^2(1+\lambda_0), \tag{A.117}$$

which is independent of A. The correlation

$$\text{corr}(g_r, \tilde{g}) = \sqrt{\frac{\text{Cov}^2(g_r, \tilde{g})}{\text{Var}(g_r)\,\text{Var}(\tilde{g})}} = \sqrt{\frac{\text{Var}(\tilde{g})}{\text{Var}(g_r)}} \tag{A.118}$$

must also be independent of A. Since $\text{Var}(g_r)$ is independent of A, $\text{corr}(g_r, \tilde{g})$ must also be independent of A. The variance of the price is

$$\text{Var}(p) = a_r^{*2}\left(\sigma_g^2 + \sigma_u^{*2} + a_r^{*2}\sigma_q^2 + \frac{\sigma_g^2}{\sigma_s^2}n\sigma_s^2\right) = a_r^{*2}\left(n\sigma_g^2(1+\lambda_0) + n\sigma_g^2\right). \tag{A.119}$$

Since a_r^* is increasing in A, $\text{Var}(p)$ must also be increasing in A. The correlation

$$\text{corr}(p, \tilde{g}) = \sqrt{\frac{\text{Cov}^2\left(g_r + \sqrt{\frac{\sigma_g^2}{\sigma_s^2}}n\tilde{s}, \tilde{g}\right)}{\text{Var}\left(g_r + \sqrt{\frac{\sigma_g^2}{\sigma_s^2}}n\tilde{s}\right)\text{Var}(\tilde{g})}} = \sqrt{\frac{\text{Var}(\tilde{g})}{\text{Var}\left(g_r + \sqrt{\frac{\sigma_g^2}{\sigma_s^2}}n\tilde{s}\right)}} = \sqrt{\frac{\sigma_g^2}{n\sigma_g^2(1+\lambda_0) + n\sigma_g^2}}, \tag{A.120}$$

which is independent of A.

B.11 Proof of Proposition 5

We consider the limiting case when $\beta \to \infty$, or equivalently, when $\frac{1}{\beta} \to 0$. Based on Theorem 1, $\bar{\lambda} = \frac{\lambda_0 A \sigma_s}{\theta}\sqrt{\sigma_x^2 + \beta^2\sigma_g^2} \to \infty$, so $\lim_{\beta\to\infty} \bar{\lambda} > 1$. Now we consider the value of $\lim_{\beta\to\infty} M(\lambda)$. Note that

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}M(\lambda) = 1 + \frac{\underline{\sigma}^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)M(\lambda)\left(\frac{\lambda + \frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}{1+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \quad \text{(A.121)}$$

Since $\underline{\sigma} = 0$, we have

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}M(\lambda) = 1 + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)M(\lambda)\left(\frac{\lambda + \frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}{1+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \quad \text{(A.122)}$$

Let $d = \frac{1}{\beta}$, then when $\beta \to \infty$, $d \to 0$ and

$$n = \frac{\beta^2\sigma_g^2}{\sigma_x^2 + \beta^2\sigma_g^2} = 1 - \frac{\sigma_x^2}{\sigma_x^2 + \beta^2\sigma_g^2} = 1 - \frac{\sigma_x^2}{\sigma_g^2}d^2 + o\left(d^2\right). \quad \text{(A.123)}$$

Then (A.122) becomes

$$\begin{aligned}
&1 - \frac{\sigma_x^2}{\sigma_g^2}d^2 + o\left(d^2\right) + \left(1 - \frac{\sigma_x^2}{\sigma_g^2}d^2 + o\left(d^2\right)\right)\theta\frac{\lambda}{A\sigma_g^2}M(\lambda)\,d = \\
&1 + \left(\frac{1}{d - \frac{\sigma_x^2}{\sigma_g^2}d^3 + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_g^2}d^2 + o(d^2)\right)M(\lambda)\left(\frac{\lambda d + \frac{A\sigma_s^2}{n\theta}M(\lambda)}{d+\frac{A\sigma_s^2}{n\theta}M(\lambda)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2} \\
\iff &-\frac{\sigma_x^2}{\sigma_g^2}d^2 + o\left(d^2\right) + \left(1 - \frac{\sigma_x^2}{\sigma_g^2}d^2 + o\left(d^2\right)\right)\theta\frac{\lambda}{A\sigma_g^2}M(\lambda)\,d = \\
&\left(\frac{1}{d - \frac{\sigma_x^2}{\sigma_g^2}d^3 + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_g^2}d^2 + o(d^2)\right)M(\lambda)\left(\frac{\lambda d + \frac{A\sigma_s^2}{n\theta}M(\lambda)}{d+\frac{A\sigma_s^2}{n\theta}M(\lambda)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}.
\end{aligned}$$

It's clear that $\lim_{d\to 0} M(\lambda) \to \infty$ for any $\lambda > 0$.Otherwise, when $d \to 0$, the LHS converges to 0, while the RHS converges to a positive number or ∞. Besides, we must also have $\lim_{d\to 0} dM(\lambda) \to 0$ for any $\lambda > 0$. Otherwise, when $d \to 0$, the LHS converges to a positive number, while the RHS converges to 0. Then the above condition can be simplified as

$$(1+o(d))\,\theta\frac{\lambda}{A\sigma_g^2}M(\lambda)\,d+o(d)=\left(\frac{1}{\frac{1}{A}\frac{1-\theta}{\sigma_x^2}M(\lambda)(1+o(d))+o(d)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}, \tag{A.124}$$

which is

$$[(1+o(d))\,M(\lambda)]^3=\frac{1}{\theta\frac{\lambda}{A\sigma_g^2}d}\left(\frac{1}{\frac{1}{A}\frac{1-\theta}{\sigma_x^2}}\right)^2\frac{\sigma_q^2}{\sigma_g^2}+o(d)\,. \tag{A.125}$$

Then

$$\lim_{d\to 0}M(\lambda)=\lim_{d\to 0}A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda d}\right)^{1/3}\to\infty \tag{A.126}$$

for any $\lambda>0$. Since $\lim_{d\to 0}M(1)\to\infty$, based on [Theorem 1], we know that in equilibrium, the information seller must choose

$$\sigma_u=\underline{\sigma}=0. \tag{A.127}$$

And

$$\lambda_c=\max\lambda\ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{M(\lambda)}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}M(\lambda)\right)}\right). \tag{A.128}$$

When $d = \frac{1}{\beta} \to 0$, the above objective function becomes

$$\lambda \ln \left(1 + \frac{A\sigma_s^2}{(1 + o(d))\lambda\theta} \frac{1}{\left(\frac{d}{M(\lambda)} + \theta \frac{\lambda}{A\sigma_g^2} d^2 + \frac{d\sigma_s^2}{(1+o(d))\sigma_g^2} M(\lambda) \right)} \right)$$

$$= \lambda \ln \left(1 + \frac{A\sigma_s^2}{(1 + o(d))\lambda\theta} \frac{1}{\left(\begin{array}{c} \frac{d}{A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda d}\right)^{1/3} + o(1)} + \theta \frac{\lambda}{A\sigma_g^2} d^2 + \\ \frac{d\sigma_s^2}{(1+o(d))\sigma_g^2} \left[A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda d}\right)^{1/3} + o(1) \right] \end{array} \right)} \right)$$

$$= \lambda \ln \left(1 + \frac{A\sigma_s^2}{(1 + o(d))\lambda\theta} \frac{1}{\left(\begin{array}{c} \frac{d^{4/3}}{A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3} + o\left(d^{1/3}\right)} + \theta \frac{\lambda}{A\sigma_g^2} d^2 + \\ \frac{d^{2/3}\sigma_s^2}{(1+o(d))\sigma_g^2} \left[A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3} + o\left(d^{1/3}\right) \right] \end{array} \right)} \right)$$

$$= \lambda \ln \left(1 + \frac{A\sigma_s^2}{(1 + o(d))\lambda\theta} \frac{1}{\left(\frac{d^{2/3}\sigma_s^2}{\sigma_g^2} \left[A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3} \right] + o(d^{2/3}) \right)} \right)$$

$$= \lambda \ln \left(1 + \frac{1}{\lambda^{2/3}} \frac{A\sigma_s^2}{\theta \left(\frac{d^{2/3}\sigma_s^2}{\sigma_g^2} \left[A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3} \right] + o(d^{2/3}) \right)} \right).$$

Let

$$K = \frac{A\sigma_s^2}{\theta \left(\frac{d^{2/3}\sigma_s^2}{\sigma_g^2} \left[A\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3} \right] + o(d^{2/3}) \right)}. \tag{A.129}$$

When $\beta \to \infty$, $d \to 0$ and thus $K \to \infty$. And it's easy to show that the function

$$\lambda \ln \left(1 + \frac{K}{\lambda^{2/3}} \right) \tag{A.130}$$

is an increasing function of λ on $\lambda \in [0, 1]$ if K is sufficiently large. Then we conclude that when β is sufficiently large (or d is sufficiently small), $\lambda_c = 1$.

Now let's turn to the analysis on $\frac{\mathbb{E}\left[\left(\theta\left((1-\lambda)\phi_U^* + \lambda\phi_I^*\right)\right)\tilde{g}\right]}{\mathbb{E}\left[\theta\left((1-\lambda)\phi_U^* + \lambda\phi_I^*\right)\right]}$.

$$
\begin{aligned}
&\frac{\mathbb{E}\left[\left(\theta\left(\left(1-\lambda\right)\phi_U^*+\lambda\phi_I^*\right)\right)\tilde{g}\right]}{\mathbb{E}\left[\theta\left(\left(1-\lambda\right)\phi_U^*+\lambda\phi_I^*\right)\right]}\\
=&\frac{\mathrm{Cov}\left(\left(1-\lambda\right)\phi_U^*+\lambda\phi_I^*,\tilde{g}\right)+\mathbb{E}\left[\left(1-\lambda\right)\phi_U^*+\lambda\phi_I^*\right]\mathbb{E}\left(\tilde{g}\right)}{\mathbb{E}\left[\left(1-\lambda\right)\phi_U^*+\lambda\phi_I^*\right]}\\
=&\frac{\mathrm{Cov}\left(\left(1-\lambda\right)\frac{\mathbb{E}[\tilde{x}+\beta\tilde{g}|\mathcal{F}_U]-p}{A\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_U)}+\lambda\frac{\mathbb{E}[\tilde{x}+\beta\tilde{g}|\mathcal{F}_I]-p}{A\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_I)},\tilde{g}\right)}{\mathbb{E}\left[\left(1-\lambda\right)\frac{\mathbb{E}[\tilde{x}+\beta\tilde{g}|\mathcal{F}_U]-p}{A\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_U)}+\lambda\frac{\mathbb{E}[\tilde{x}+\beta\tilde{g}|\mathcal{F}_I]-p}{A\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_I)}\right]}+\bar{g}\\
=&\frac{\frac{(1-\lambda)\mathrm{Cov}\left(\bar{x}+\beta\bar{g}+\beta r_2\left(\frac{p-\bar{p}}{a_r}\right)-a_0-a_rg_r-a_ss,\tilde{g}\right)}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_U)}+\frac{\lambda\mathrm{Cov}(\bar{x}+\beta\bar{g}+\beta r_1(g_r-\bar{g}_r)-a_0-a_rg_r-a_ss,\tilde{g})}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_I)}}{\left[\mathbb{E}\left[\tilde{x}+\beta\tilde{g}\right]-p\right]\left[\frac{1-\lambda}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_U)}+\frac{\lambda}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_I)}\right]}+\bar{g}\\
=&\frac{\frac{1-\lambda}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_U)}\left(\beta r_2-a_r\right)\sigma_g^2+\frac{\lambda}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_I)}\left(\beta r_1-a_r\right)\sigma_g^2}{\left(\bar{x}+\beta\bar{g}-a_0-a_r\left(\bar{g}+a_r\bar{q}\right)-a_s\bar{s}\right)\left[\frac{1-\lambda}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_U)}+\frac{\lambda}{\mathrm{Var}(\tilde{x}+\beta\tilde{g}|\mathcal{F}_I)}\right]}+\bar{g}\\
=&\frac{\sigma_g^2}{\bar{x}+\beta\bar{g}-a_0-a_r\left(\bar{g}+a_r\bar{q}\right)-a_s\bar{s}}\left[\beta\frac{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}r_2+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}r_1}{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}-a_r\right]+\bar{g}\\
=&\sigma_g^2\frac{\frac{1-\theta}{\theta\sigma_x^2}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}}{\beta\bar{g}\frac{1-\theta}{\theta\sigma_x^2}+A\bar{s}}\left[\beta\frac{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}r_2+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}r_1}{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}-a_r\right]+\bar{g}\\
=&\sigma_g^2\frac{\frac{1-\theta}{\theta\sigma_x^2}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}}{\beta\bar{g}\frac{1-\theta}{\theta\sigma_x^2}+A\bar{s}}\frac{\frac{(1-\lambda)\beta r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}-\sigma_g^2\frac{\frac{(1-\lambda)\beta r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}{\beta\bar{g}\frac{1-\theta}{\theta\sigma_x^2}+A\bar{s}}+\bar{g}\\
=&\sigma_g^2\frac{\frac{(1-\lambda)\beta r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}{\beta\bar{g}\frac{1-\theta}{\theta\sigma_x^2}+A\bar{s}}\left[\frac{\frac{1-\theta}{\theta\sigma_x^2}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}+\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}}{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}-1\right]+\bar{g}\\
=&\sigma_g^2\beta\frac{\frac{(1-\lambda)r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}{\beta\bar{g}\frac{1-\theta}{\theta\sigma_x^2}+A\bar{s}}\left[\frac{\frac{1-\theta}{\theta\sigma_x^2}}{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}\right]+\bar{g}\\
=&\frac{\sigma_g^2}{\bar{g}}\frac{\beta\frac{1-\theta}{\theta\sigma_x^2}}{\beta\frac{1-\theta}{\theta\sigma_x^2}+A\frac{\bar{s}}{\bar{g}}}\left[\frac{\frac{(1-\lambda)r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}\right]+\bar{g}.
\end{aligned}
$$

In the above derivation, we use the following two results:

1. $\left(\frac{1-\theta}{\theta\sigma_x^2}+\frac{(1-\lambda)}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}\right)a_r=\frac{(1-\lambda)\beta r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}+\frac{\lambda\beta r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}.$

2. $a_0 = \bar{x} + \frac{\lambda \frac{\beta \bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + (1-\lambda)\frac{\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} - A\bar{s}}{\frac{1-\theta}{\theta\sigma_x^2} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + \frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}} - \left(a_r\left(\bar{g}+a_r\bar{q}\right)+a_s\bar{s}\right)$, then

$$
\begin{aligned}
&\bar{x} + \beta\bar{g} - a_0 - a_r\left(\bar{g}+a_r\bar{q}\right) - a_s\bar{s} \\
=&\beta\bar{g} - \frac{\lambda \frac{\beta \bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + (1-\lambda)\frac{\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} - A\bar{s}}{\frac{1-\theta}{\theta\sigma_x^2} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + \frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}} \\
=&\frac{\beta\bar{g}\left(\frac{1-\theta}{\theta\sigma_x^2} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + \frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}\right) - \lambda\frac{\beta \bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} - (1-\lambda)\frac{\beta\bar{g}}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} + A\bar{s}}{\frac{1-\theta}{\theta\sigma_x^2} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + \frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}} \\
=&\frac{\beta\bar{g}\frac{1-\theta}{\theta\sigma_x^2} + A\bar{s}}{\frac{1-\theta}{\theta\sigma_x^2} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)} + \frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)}}.
\end{aligned}
$$

When β is sufficiently large, the information seller chooses $\lambda^* = 1$, and

$$
\frac{\sigma_g^2}{\bar{g}} \frac{\beta\frac{1-\theta}{\Lambda\sigma_x^2}}{\beta\frac{1-\theta}{\Lambda\sigma_x^2} + A\frac{\bar{s}}{\bar{g}}} \left[\frac{\frac{(1-\lambda)r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} + \frac{\lambda r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}{\frac{1-\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}\right] + \bar{g} = \frac{\sigma_g^2}{\bar{g}} \frac{\frac{1-\theta}{\theta\sigma_x^2}}{\frac{1-\theta}{\theta\sigma_x^2} + \frac{1}{\beta}A\frac{\bar{s}}{\bar{g}}} r_1 + \bar{g}. \tag{A.131}
$$

In equilibrium, we have

$$
\frac{1}{r_1} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + a_r^2\frac{\sigma_q^2}{\sigma_g^2} = n - n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r} = n + n\theta\frac{\lambda}{\beta A\sigma_g^2}M\left(\lambda\right). \tag{A.132}
$$

Note that we introduced $d = \frac{1}{\beta}$, and when $d \to 0$, we have

$$
\begin{aligned}
\frac{1}{r_1} &= n + n\theta\frac{\lambda}{\beta A\sigma_g^2}M\left(\lambda\right) \\
&= \left(1 - \frac{\sigma_x^2}{\sigma_g^2}d^2 + o\left(d^2\right)\right)\left[1 + \frac{\lambda\theta}{\sigma_g^2}\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3}d^{2/3} + o\left(d\right)\right] \\
&= 1 + \frac{\lambda\theta}{\sigma_g^2}\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3}d^{2/3} + o\left(d\right).
\end{aligned}
$$

Then equation (A.131) becomes

$$
\begin{aligned}
&\frac{\sigma_g^2}{\bar{g}}\frac{\frac{1-\theta}{\theta\sigma_x^2}}{\frac{1-\theta}{\theta\sigma_x^2}+dA\frac{\bar{s}}{\bar{g}}}r_1+\bar{g}.\\
=&\frac{\sigma_g^2}{\bar{g}}\frac{1}{1+\frac{A\theta\sigma_x^2\bar{s}}{(1-\theta)\bar{g}}d}\left(1-\frac{\lambda\theta}{\sigma_g^2}\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3}d^{2/3}+o\left(d\right)\right)+\bar{g}\\
=&\frac{\sigma_g^2}{\bar{g}}\left(1-\frac{A\theta\sigma_x^2\bar{s}}{(1-\theta)\,\bar{g}}d+o\left(d\right)\right)\left(1-\frac{\lambda\theta}{\sigma_g^2}\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3}d^{2/3}+o\left(d\right)\right)+\bar{g}\\
=&\frac{\sigma_g^2}{\bar{g}}\left(1-\frac{\lambda\theta}{\sigma_g^2}\left(\frac{\sigma_q^2}{\theta}\frac{\sigma_x^4}{(1-\theta)^2}\right)^{1/3}\left(\frac{1}{\lambda}\right)^{1/3}d^{2/3}+o\left(d^{2/3}\right)\right)+\bar{g}.
\end{aligned}
$$

When β increases, d decreases, and thus $\frac{\mathbb{E}\left[\left(\theta\left((1-\lambda)\phi_U^*+\lambda\phi_I^*\right)\right)\tilde{g}\right]}{\mathbb{E}\left[\theta\left((1-\lambda)\phi_U^*+\lambda\phi_I^*\right)\right]}$ increases.

B.12 Proof of Proposition 6

For the first part, from Theorem 1, we know that in both the first and second case, all of $\text{corr}\left(g_r,\tilde{g}\right)$, $\text{corr}\left(p,\tilde{g}\right)$ and $\mathbb{E}\left(\tilde{g}|G\right)$ are independent of σ_q. Then we just need to confirm that there exists a constant k_2, such that when $\sigma_q \leqslant k_2$, the equilibrium must be in the first or second case in Theorem 1. It's clear that for any fixed λ, $M\left(\lambda;\sigma_q\right)$ is increasing in σ_q. When $\sigma_q=0$, $M\left(\lambda;0\right)=\frac{\sigma_g^2+\sigma^2-n\sigma_g^2}{\frac{n\theta\lambda}{A\beta}}>0$. Let $c=\min\left\{\bar{\lambda},1\right\}$. If $M\left(c;0\right)\leqslant\frac{\sigma_g}{\sigma_s}\sqrt{n}$, then there exists a constant $k_2>0$, such that when $\sigma_q\leqslant k_2$, we must have $M\left(c;\sigma_q\right)\leqslant\frac{\sigma_g}{\sigma_s}\sqrt{n}$, and thus the equilibrium must be the first or the second case in Theorem 1. If $M\left(c;0\right)>\frac{\sigma_g}{\sigma_s}\sqrt{n}$, then the equilibrium must be the third case in Theorem 1 for any $\sigma_q\geqslant 0$, in this case, let's define $k_2=-1$.

For the second part, let's first show that when σ_q is sufficiently high, the equilibrium must be in the third case, and the information seller chooses $\lambda_c=1$. Note that for any $\lambda\in(0,1]$, $M\left(\lambda;\sigma_q\right)$ satisfies

$$n+n\theta\frac{\lambda}{A\beta\sigma_g^2}M=1+\frac{\sigma^2}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)M\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}M}{1+\frac{\beta A\sigma_s^2}{n\theta}M}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}.$$

Rewriting, we have

$$\left(n+n\theta\frac{\lambda}{A\beta\sigma_g^2}M-1-\frac{\sigma^2}{\sigma_g^2}\right)\left(\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)M\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}M}{1+\frac{\beta A\sigma_s^2}{n\theta}M}\right)\right)^2=\frac{\sigma_q^2}{\sigma_g^2}.$$

It's clear that we must have $n+n\theta\frac{\lambda}{A\beta\sigma_g^2}M-1-\frac{\sigma^2}{\sigma_g^2}>0$. We also must have $M\left(\lambda;\sigma_q\right)\geqslant M\left(1;\sigma_q\right)$, because $M\left(\lambda;\sigma_q\right)$ is a decreasing function of λ. Besides, it's clear that $M\left(1;\sigma_q\right)$ is an increasing function of σ_q. Then there must exist a constant k_{11}, such that when $\sigma_q>k_{11}$, we have $M\left(1;\sigma_q\right)>\frac{\sigma_g}{\sigma_s}\sqrt{n}$. In this case, the equilibrium must be in the third case.

Lemma 8. *There exists $k_1 \geqslant k_{11}$, such that when $\sigma_q \geqslant k_1$, the equilibrium must be the third case and the information seller always chooses $\lambda_c = 1$ where*

$$\lambda_c = \arg\max_{\lambda\in(0,1]} \lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\frac{\beta}{M(\lambda;\sigma_q)} + \frac{\lambda\theta}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}M(\lambda;\sigma_q)}\right).$$

Proof. Note that $M(\lambda;\sigma_q)$ satisfies

$$\left(n + n\theta\frac{\lambda}{A\beta\sigma_g^2}M - 1 - \frac{\sigma^2}{\sigma_g^2}\right)\left(\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2 + \beta^2\sigma_g^2}\right)M\left(\frac{\lambda + \frac{\beta A\sigma_s^2}{n\theta}M}{1 + \frac{\beta A\sigma_s^2}{n\theta}M}\right)\right)^2 = \frac{\sigma_q^2}{\sigma_g^2},$$

which is equivalent to

$$\left(M\lambda - \frac{1 + \frac{\sigma^2}{\sigma_g^2} - n}{n\theta\frac{1}{A\beta\sigma_g^2}}\right)\left(M + \frac{n}{\beta\left(\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\right)} - \frac{1-\lambda}{\frac{1}{M} + \frac{\beta A\sigma_s^2}{n\theta}}\right)^2 = \frac{1}{n\theta\frac{1}{A\beta\sigma_g^2}}\frac{1}{\left(\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\right)^2}\frac{\sigma_q^2}{\sigma_g^2}.$$

Let $f = \frac{1+\frac{\sigma^2}{\sigma_g^2}-n}{n\theta\frac{1}{A\beta\sigma_g^2}}, r = \frac{1}{n\theta\frac{1}{A\beta\sigma_g^2}}\frac{1}{\left(\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\right)^2}$, and $v(\lambda, M) = \frac{n}{\beta\left(\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\right)} - \frac{1-\lambda}{\frac{1}{M}+\frac{\beta A\sigma_s^2}{n\theta}}$. Both f and r are positive constant. Then $M(\lambda;\sigma_q)$ satisfies

$$(\lambda M - f)(M + v)^2 = r\sigma_q^2,$$

which is equivalent to

$$(\lambda M - f)(\lambda M)^2\left(1 + \frac{v}{M}\right)^2 = r\lambda^2\sigma_q^2.$$

Note that for any σ_q, when λ takes values in $(0,1]$, λM can be arbitrarily close to f.

Lemma 9. *For any $K > 0$, there exists $k_{12}(K) > 0$, such that if $\sigma_q > k_{12}$, $M(\lambda;\sigma_q) > K$ for all $\lambda \in (0,1]$.*

Proof. First, we know $(\lambda M - f)(M + v)^2 = r\sigma_q^2$, when $\lambda = 1$, $v(1) = \frac{n}{\beta\left(\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\right)}$ and $(M(1;\sigma_q) - f)(M(1;\sigma_q) + v(1))^2 = r\sigma_q^2$. It's clear that $M(1;\sigma_q)$ is an increasing function of σ_q, and for any $K > 0$, there exists $k_{12} > 0$, such that if $\sigma_q > k_{12}$, $M(1;\sigma_q) > K$. Since $M(\lambda;\sigma_q)$ is a decreasing function of $\lambda \in (0,1]$, then we conclude that for any $K > 0$, there exists $k_{12}(K) > 0$, such that if $\sigma_q > k_{12}$, $M(\lambda;\sigma_q) > K$ for all $\lambda \in (0,1]$. □

Let

$$\begin{aligned} F(\lambda, M) &= \lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\theta}\frac{1}{\frac{\lambda\beta}{M} + \frac{\lambda^2\theta}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\lambda M}\right) \\ &= \lambda \ln\left(1 + \frac{1}{\frac{n\theta}{A\beta^2\sigma_s^2}\left(\frac{\lambda\beta}{M} + \frac{\lambda^2\theta}{A\sigma_g^2}\right) + \frac{\theta}{A\beta\sigma_g^2}\lambda M}\right). \end{aligned}$$

Let $a(\lambda, M(\lambda)) = \frac{n\theta}{A\beta^2\sigma_s^2}\left(\frac{\lambda\beta}{M} + \frac{\lambda^2\theta}{A\sigma_g^2}\right)$ and $x(\lambda, M(\lambda)) = \frac{\theta}{A\beta\sigma_g^2}\lambda M$. Then $F(\lambda; \sigma_q) = \lambda \ln\left(1 + \frac{1}{a(\lambda,M)+x(\lambda,M)}\right)$.

$$\begin{aligned}
\frac{dF}{d\lambda} &= \ln\left(1+\frac{1}{a+x}\right) + \lambda\frac{1}{1+\frac{1}{a+x}}\frac{-1}{(a+x)^2}\left(\frac{da}{d\lambda}+\frac{dx}{d\lambda}\right) \\
&= \ln\left(1+\frac{1}{a+x}\right) - \frac{\lambda}{(a+x)(1+a+x)}\left(\frac{da}{d\lambda}+\frac{\theta}{A\beta\sigma_g^2}\left(M+\lambda\frac{dM}{d\lambda}\right)\right) \\
&= \ln\left(1+\frac{1}{a+x}\right) - \frac{1}{1+a+x} + \frac{1}{1+a+x} - \frac{\lambda\frac{\theta}{A\beta\sigma_g^2}M}{(a+x)(1+a+x)} - \frac{\lambda\left(\frac{da}{d\lambda}+\frac{\theta\lambda}{A\beta\sigma_g^2}\frac{dM}{d\lambda}\right)}{(a+x)(1+a+x)} \\
&= \left[\ln\left(1+\frac{1}{a+x}\right) - \frac{1}{1+a+x}\right] + \frac{1}{1+a+x}\left[1-\frac{x}{(a+x)}\right] - \frac{\lambda\left(\frac{da}{d\lambda}+\frac{\theta\lambda}{A\beta\sigma_g^2}\frac{dM}{d\lambda}\right)}{(a+x)(1+a+x)} \\
&= \left[\ln\left(1+\frac{1}{a+x}\right) - \frac{1}{1+a+x}\right] + \frac{1}{1+a+x}\frac{a}{a+x} - \frac{\frac{\theta\lambda}{A\beta\sigma_g^2}\lambda\left(\frac{A\beta\sigma_g^2}{\theta\lambda}\frac{da}{d\lambda}+\frac{dM}{d\lambda}\right)}{(a+x)(1+a+x)}.
\end{aligned}$$

By $(\lambda M - f)(M+v)^2 = r\sigma_q^2$, we get

$$(M+\lambda M')(M+v)^2 + (M\lambda - f)\,2(M+v)\left(M' + \frac{M}{1+\frac{\beta A\sigma_s^2}{n\theta}M} - \frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}M'\right) = 0.$$

$$M'\left[\lambda(M+v)^2 + (M\lambda - f)\,2(M+v)\left(1-\frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}\right)\right] + M(M+v)^2 + \frac{(M\lambda-f)\,2(M+v)}{1+\frac{\beta A\sigma_s^2}{n\theta}M}M = 0.$$

Since $M+v>0$ and $M\lambda - f > 0$ for all $\lambda \in (0,1]$, we have

$$M'\left[\frac{\lambda(M+v)}{2(M\lambda-f)} + \left(1-\frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}\right)\right] = -\frac{M(M+v)}{2(M\lambda-f)} - \frac{M}{1+\frac{\beta A\sigma_s^2}{n\theta}M}.$$

So

$$M' = -\frac{\frac{M(M+v)}{2(M\lambda-f)} + \frac{M}{1+\frac{\beta A\sigma_s^2}{n\theta}M}}{\frac{\lambda(M+v)}{2(M\lambda-f)} + \left(1-\frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}\right)}.$$

Then

$$\begin{aligned}\frac{A\beta\sigma_g^2}{\theta\lambda}\frac{da}{d\lambda}+\frac{dM}{d\lambda} &= \frac{n\sigma_g^2}{\lambda\beta\sigma_s^2}\left(\frac{\beta}{M}-\lambda\beta\frac{M'}{M^2}+\frac{2\lambda\theta}{A\sigma_g^2}\right)+M' \\ &= \frac{n\sigma_g^2}{\lambda\beta\sigma_s^2}\left(\frac{\beta}{M}+\frac{2\lambda\theta}{A\sigma_g^2}\right)+\left(1-\frac{n\sigma_g^2}{M^2\sigma_s^2}\right)M' \\ &= \frac{n\sigma_g^2}{\lambda\beta\sigma_s^2}\left(\frac{\beta}{M}+\frac{2\lambda\theta}{A\sigma_g^2}\right)-\left(1-\frac{n\sigma_g^2}{M^2\sigma_s^2}\right)\frac{\frac{(M+v)}{2(M\lambda-f)}+\frac{1}{1+\frac{\beta A\sigma_s^2}{n\theta}M}}{\frac{\lambda(M+v)}{2(M\lambda-f)}+\left(1-\frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}\right)}M \\ &= \frac{n\sigma_g^2}{\lambda\beta\sigma_s^2}\left(\frac{\beta}{M}+\frac{2\lambda\theta}{A\sigma_g^2}\right)-\left(1-\frac{n\sigma_g^2}{M^2\sigma_s^2}\right)\frac{M+v+\frac{2(M\lambda-f)}{1+\frac{\beta A\sigma_s^2}{n\theta}M}}{\lambda(M+v)+2(M\lambda-f)\left(1-\frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}\right)}M.\end{aligned}$$

Based on Lemma 9, there exists $k_{14}>k_{11}$, such that when $\sigma_q>k_{14}$,

$$\lambda(M+v)<2M$$

and

$$2(M\lambda-f)\left(1-\frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}\right)<2M.$$

Then

$$\frac{M+v+\frac{2(M\lambda-f)}{1+\frac{\beta A\sigma_s^2}{n\theta}M}}{\lambda(M+v)+2(M\lambda-f)\left(1-\frac{1-\lambda}{\left(1+\frac{\beta A\sigma_s^2}{n\theta}M\right)^2}\right)}>\frac{M}{2M+2M}=\frac{1}{4}.$$

Then

$$\frac{A\beta\sigma_g^2}{\theta\lambda}\frac{da}{d\lambda}+\frac{dM}{d\lambda}<\frac{n\sigma_g^2}{\lambda\beta\sigma_s^2}\left(\frac{\beta}{M}+\frac{2\lambda\theta}{A\sigma_g^2}\right)-\left(1-\frac{n\sigma_g^2}{M^2\sigma_s^2}\right)\frac{1}{4}M.$$

Again, based on Lemma 9, there exists $k_1\geqslant k_{14}$, such that when $\sigma_q\geqslant k_1$, $\frac{A\beta\sigma_g^2}{\theta\lambda}\frac{da}{d\lambda}+\frac{dM}{d\lambda}<0$. Note that for any positive $(a+x)$, we must have $\left[\ln\left(1+\frac{1}{a+x}\right)-\frac{1}{1+a+x}\right]>0$, then we conclude that when $\sigma_q>k_1$, $\frac{dF}{d\lambda}>0$, and thus $F(1;\sigma_q)>F(\lambda;\sigma_q)$ for any $\lambda>0$. So

$$1=\arg\max_{\lambda\in(0,1]}F(\lambda;\sigma_q)$$

for any $\sigma_q\geqslant k_1$. □

Now let's focus on the region when $\sigma_q\geqslant k_1$. In equilibrium,

$$\sigma_g^2+\underline{\sigma}^2+a_r^2\sigma_q^2=n\sigma_g^2+n\theta\frac{1}{A\beta}\left(-\frac{a_s}{a_r}\right),$$

then in this case, the correlations are

$$\operatorname{corr}(g_r, \tilde{g}) = \sqrt{\frac{\sigma_g^2}{\sigma_g^2 + \underline{\sigma}^2 + a_r^2 \sigma_q^2}} = \sqrt{\frac{\sigma_g^2}{n\sigma_g^2 + n\theta \frac{1}{A\beta}\left(-\frac{a_s}{a_r}\right)}}$$

and

$$\operatorname{corr}(p, \tilde{g}) = \sqrt{\frac{\sigma_g^2}{n\sigma_g^2 + n\theta \frac{1}{A\beta}\left(-\frac{a_s}{a_r}\right) + \left(\frac{a_s}{a_r}\right)^2 \sigma_s^2}},$$

where $\left(-\frac{a_s}{a_r}\right)$ is solved by

$$n + n\theta \frac{1}{A\beta\sigma_g^2}\left(-\frac{a_s}{a_r}\right) = 1 + \frac{\underline{\sigma}^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2 + \beta^2\sigma_g^2}\right)\left(-\frac{a_s}{a_r}\right)}\right)^2 \frac{\sigma_q^2}{\sigma_g^2}.$$

When σ_q increases, it's clear that $\left(-\frac{a_s}{a_r}\right)$ increases, then both $\operatorname{corr}(g_r, \tilde{g})$ and $\operatorname{corr}(p, \tilde{g})$ decrease. Based on our results in Proposition 5, we know

$$\mathbb{E}(\tilde{g}|G) = \frac{\mathbb{E}\left[\theta\left((1-\lambda)\phi_U^* + \lambda\phi_I^*\right)\tilde{g}\right]}{\mathbb{E}\left[\theta\left((1-\lambda)\phi_U^* + \lambda\phi_I^*\right)\right]} = \frac{\sigma_g^2}{\bar{g}} \frac{\beta\frac{1-\theta}{\theta\sigma_x^2}}{\beta\frac{1-\theta}{\theta\sigma_x^2} + A\frac{\bar{s}}{\bar{g}}} \left[\frac{\frac{(1-\lambda)r_2}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} + \frac{\lambda r_1}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}{\frac{(1-\lambda)}{\sigma_x^2+\beta^2\sigma_g^2(1-r_2)} + \frac{\lambda}{\sigma_x^2+\beta^2\sigma_g^2(1-r_1)}}\right] + \bar{g}.$$

Taking $\lambda_c = 1$, we have

$$\mathbb{E}(\tilde{g}|G) = \frac{\sigma_g^2}{\bar{g}} \frac{\beta\frac{1-\theta}{\theta\sigma_x^2}}{\beta\frac{1-\theta}{\theta\sigma_x^2} + A\frac{\bar{s}}{\bar{g}}} r_1 + \bar{g}.$$

Note that $r_1 = \operatorname{corr}^2(g_r, \tilde{g})$. Since we've already shown that $\operatorname{corr}(g_r, \tilde{g}) > 0$ decreases in σ_q, we can conclude that r_1 decreases in σ_q, and thus $\frac{\mathbb{E}\left[\theta\left((1-\lambda)\phi_U^*+\lambda\phi_I^*\right)\tilde{g}\right]}{\mathbb{E}\left[\theta\left((1-\lambda)\phi_U^*+\lambda\phi_I^*\right)\right]}$ decreases in σ_q.

B.13 Proof of Proposition 7

Let's focus on the case when $\sigma_q \geqslant k_1$, where k_1 is the parameter in Proposition 6, and thus in equilibrium, the information seller must choose $\lambda_c = 1$. Note that in this case,

$$\operatorname{corr}(p, \tilde{g}) = \sqrt{\frac{\sigma_g^2}{n\sigma_g^2 + n\theta \frac{1}{A\beta}\left(-\frac{a_s}{a_r}\right) + \left(\frac{a_s}{a_r}\right)^2 \sigma_s^2}},$$

where $\left(-\frac{a_s}{a_r}\right) = M$ is solved by

$$n + n\theta \frac{1}{A\beta\sigma_g^2} M = 1 + \frac{\underline{\sigma}^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2 + \beta^2\sigma_g^2}\right) M}\right)^2 \frac{\sigma_q^2}{\sigma_g^2}. \tag{A.133}$$

To show that $\frac{d\text{corr}(p,\tilde{g})}{d\theta} < 0$, it's sufficient to show that $\frac{dM}{d\theta} > 0$. . Rewriting (A.133), we have

$$\left[n + n\theta \frac{1}{A\beta\sigma_g^2} M - 1 - \frac{\underline{\sigma}^2}{\sigma_g^2}\right] \left(\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2 + \beta^2\sigma_g^2}\right) M\right)^2 = \frac{\sigma_q^2}{\sigma_g^2}$$

$$\Longleftrightarrow \begin{array}{c} \left[\theta M + \frac{A\beta\sigma_g^2}{n}\left(n - 1 - \frac{\underline{\sigma}^2}{\sigma_g^2}\right)\right] \left(\left(\frac{\sigma_x^2}{\beta^2\sigma_g^2} + (1-\theta)\right) M + \frac{An\sigma_x^2}{\beta^3\sigma_g^2}\left(\sigma_x^2 + \beta^2\sigma_g^2\right)\right)^2 \\ = \left[\sigma_x^2\left(\sigma_x^2 + \beta^2\sigma_g^2\right)\right]^2 \frac{A^3\beta\sigma_q^2}{n\left(\beta^2\sigma_g^2\right)^2} \end{array}.$$

Taking derivatives for both sides,

$$\begin{array}{c} \left(M + \frac{dM}{d\theta}\right)\left(\left(\frac{\sigma_x^2}{\beta^2\sigma_g^2} + (1-\theta)\right) M + \frac{An\sigma_x^2}{\beta^3\sigma_g^2}\left(\sigma_x^2 + \beta^2\sigma_g^2\right)\right) + \\ 2\left[\theta M + \frac{A\beta\sigma_g^2}{n}\left(n - 1 - \frac{\underline{\sigma}^2}{\sigma_g^2}\right)\right]\left(-M + \left(\frac{\sigma_x^2}{\beta^2\sigma_g^2} + (1-\theta)\right)\frac{dM}{d\theta}\right) \end{array} = 0$$

$$\Longleftrightarrow LHS \cdot \frac{dM}{d\theta} = M \cdot RHS,$$

where

$$LHS = \left(\frac{\sigma_x^2}{\beta^2\sigma_g^2} + (1-\theta)\right) M + \frac{An\sigma_x^2}{\beta^3\sigma_g^2}\left(\sigma_x^2 + \beta^2\sigma_g^2\right) + 2\left(\frac{\sigma_x^2}{\beta^2\sigma_g^2} + (1-\theta)\right)\left[\theta y + \frac{A\beta\sigma_g^2}{n}\left(n - 1 - \frac{\underline{\sigma}^2}{\sigma_g^2}\right)\right] > 0,$$

and

$$\begin{aligned} RHS &= 2\left[\theta M + \frac{A\beta\sigma_g^2}{n}\left(n - 1 - \frac{\underline{\sigma}^2}{\sigma_g^2}\right)\right] - \left(\frac{\sigma_x^2}{\beta^2\sigma_g^2} + (1-\theta)\right) M - \frac{An\sigma_x^2}{\beta^3\sigma_g^2}\left(\sigma_x^2 + \beta^2\sigma_g^2\right) \\ &= \left(3\theta - 1 - \frac{\sigma_x^2}{\beta^2\sigma_g^2}\right) M + 2\frac{A\beta\sigma_g^2}{n}\left(n - 1 - \frac{\underline{\sigma}^2}{\sigma_g^2}\right) - \frac{An\sigma_x^2}{\beta^3\sigma_g^2}\left(\sigma_x^2 + \beta^2\sigma_g^2\right). \end{aligned}$$

For any θ satisfying $3\theta - 1 - \frac{\sigma_x^2}{\beta^2\sigma_g^2} > 0$, based on Lemma 9, there exists k_{21}, such that when $\sigma_q \geqslant k_{21}$, we must have M large enough satisfying

$$\left(3\theta - 1 - \frac{\sigma_x^2}{\beta^2\sigma_g^2}\right) M + 2\frac{A\beta\sigma_g^2}{n}\left(n - 1 - \frac{\underline{\sigma}^2}{\sigma_g^2}\right) + \frac{An\sigma_x^2}{\beta^3\sigma_g^2}\left(\sigma_x^2 + \beta^2\sigma_g^2\right) > 0.$$

Then we must have

$$\frac{dM}{d\theta} > 0.$$

Together with Lemma 8, we conclude that, for any θ satisfying $3\theta - 1 - \frac{\sigma_x^2}{\beta^2\sigma_g^2} > 0$, there exists $k_4 > 0$, such that when $\sigma_q > k_4$, $\frac{dM}{d\theta} > 0$, which means that

$$\frac{d\text{corr}(p, \tilde{g})}{d\theta} < 0.$$

B.14 Proof of Theorem 2

We almost replicate the proof of Theorem 1. The information seller's objective function is still

$$\lambda \ln\left(\frac{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right)}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right)}\right). \tag{A.134}$$

And the optimization problem is

$$\max_{(\Phi_b,w)} \lambda\theta\cdot\frac{1}{2A}\ln\left(\frac{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right)}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right)}\right). \tag{A.135}$$

It can be shown (similar to the proof in Theorem 1), the objective function is equivalent to

$$\theta\lambda \ln\left(\frac{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_U\right)}{\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\mathscr{F}_I\right)}\right) \tag{A.136}$$

$$=\theta\lambda\ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right), \tag{A.137}$$

where $\frac{a_s}{a_r}$ satisfies:

$$n-n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r}=1+\frac{\sigma_u^2}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}-\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1-\frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \tag{A.138}$$

The unconstrained solution of

$$\min \frac{1}{-\frac{a_s}{a_r}}\beta+\theta\frac{\lambda}{A\sigma_g^2}+\left(-\frac{a_s}{a_r}\right)\frac{\beta\sigma_s^2}{n\sigma_g^2} \tag{A.139}$$

satisfies

$$\frac{1}{-\frac{a_s}{a_r}}\beta=\left(-\frac{a_s}{a_r}\right)\frac{\beta\sigma_s^2}{n\sigma_g^2}\Longleftrightarrow\left(\frac{a_s}{a_r}\right)^2=n\frac{\sigma_g^2}{\sigma_s^2}, \tag{A.140}$$

which is

$$\left(-\frac{a_s}{a_r}\right)=\frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.141}$$

Then

$$\frac{1}{-\frac{a_s}{a_r}}\beta+\theta\frac{\lambda}{A\sigma_g^2}+\left(-\frac{a_s}{a_r}\right)\frac{\beta\sigma_s^2}{n\sigma_g^2}=\theta\frac{\lambda}{A\sigma_g^2}+2\sqrt{n}\frac{\sigma_g}{\sigma_s}\frac{\beta}{n}\frac{\sigma_s^2}{\sigma_g^2}=\theta\frac{\lambda}{A\sigma_g^2}+2\beta\frac{\sigma_s}{\sigma_g\sqrt{n}}. \tag{A.142}$$

Then the objective becomes

$$\theta\lambda\ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right)$$

$$=\theta\lambda\ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\theta\frac{\lambda}{A\sigma_g^2}+2\beta\frac{\sigma_s}{\sigma_g\sqrt{n}}}\right)$$

$$=\theta\lambda\ln\left(1+\frac{\frac{A\beta^2\sigma_s^2}{n\lambda\theta}}{\theta\frac{\lambda}{A\sigma_g^2}+2\beta\frac{\sigma_s}{\sigma_g\sqrt{n}}}\right)$$

$$=\theta\lambda\ln\left(1+\frac{1}{\theta\frac{\lambda}{A\sigma_g^2}\frac{n\lambda\theta}{A\beta^2\sigma_s^2}+\frac{n\lambda\theta}{A\beta^2\sigma_s^2}2\frac{\sigma_s}{\sigma_g\sqrt{n}}}\right)$$

$$=\theta\lambda\ln\left(1+\frac{1}{\frac{\lambda^2}{A^2\sigma_g^2}\frac{n\theta^2}{\beta^2\sigma_s^2}+2\frac{\sqrt{n}\lambda\theta}{\beta A\sigma_s\sigma_g}}\right)$$

$$=\theta\lambda\ln\left(1+\frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2+2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right).$$

Admati and Pfleiderer (1986) shows that there exists a unique solution to the following problem

$$\max_k k\ln\left(1+\frac{1}{k\,(k+2)}\right), \tag{A.143}$$

denote the solution as λ_0[11]. Then the information seller's optimal choice $\bar{\lambda}$ is

$$\bar{\lambda}=\frac{\lambda_0 A\sigma_s}{\theta}\sqrt{\sigma_x^2+\beta^2\sigma_g^2}. \tag{A.144}$$

Let $M_b(\lambda)$ be the solution of

$$n+n\theta\frac{\lambda}{A\beta\sigma_g^2}M_b(\lambda)=\min_{\mathbf{w}}1+\frac{\sigma_u^2(\mathbf{w})}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)M_b(\lambda)\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}{1+\frac{\beta A\sigma_s^2}{n\theta}M(\lambda)}\right)}\right)^2\frac{\sigma_q^2(\mathbf{w})}{\sigma_g^2}. \tag{A.145}$$

By choosing different $\mathbf{w}$, the rating agency can achieve equilibrium with any $\left(-\frac{a_s}{a_r}\right)\geqslant M_b(\lambda)$.

Lemma 10. *$M_b(\lambda)$ is a decreasing function of λ , and $\lim_{\lambda\to 0}M_b(\lambda)=\infty$.*

Proof. First, it's clear that the RHS of A.145 must be greater than 1. When $\lambda\to 0$, we must have

$$\lim_{\lambda\to 0}\frac{n\lambda}{\beta A\sigma_g^2}M_b(\lambda)\geqslant 1+\frac{\underline{\sigma}^2}{\sigma_g^2}-n=1+\frac{\underline{\sigma}^2}{\sigma_g^2}-\frac{\beta^2\sigma_g^2}{\sigma_x^2+\beta^2\sigma_g^2}>0. \tag{A.146}$$

[11] $\lambda_0\approx 0.651461$

which implies that

$$\lim_{\lambda \to 0} M_b(\lambda) = \infty. \tag{A.147}$$

The LHS is an increasing function of both M_b and λ, the RHS is a decreasing function of both M_b and λ. We have

$$LHS(M_b, \lambda) = RHS(M_b, \lambda). \tag{A.148}$$

Then

$$\frac{\partial LHS}{\partial M_b} dM_b + \frac{\partial LHS}{\partial \lambda} d\lambda = \frac{\partial RHS}{\partial M_b} dM_b + \frac{\partial RHS}{\partial \lambda} d\lambda, \tag{A.149}$$

which leads to

$$\frac{dM_b}{d\lambda} = \frac{\frac{\partial RHS}{\partial \lambda} - \frac{\partial LHS}{\partial \lambda}}{\frac{\partial LHS}{\partial M_b} - \frac{\partial RHS}{\partial M_b}}. \tag{A.150}$$

Since $\frac{\partial RHS}{\partial \lambda} < 0$, $\frac{\partial RHS}{\partial M_b} < 0$, $\frac{\partial LHS}{\partial \lambda} > 0$ and $\frac{\partial LHS}{\partial M_b} > 0$, we must have $\frac{dM_b}{d\lambda} < 0$. □

As we discussed earlier, the information seller's problem is

$$\max_{(\Phi_b, \mathbf{w})} \theta\lambda \ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta} \frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right), \tag{A.151}$$

where $\frac{a_s}{a_r}$ satisfies:

$$n - n\theta\frac{\lambda}{\beta A\sigma_g^2}\frac{a_s}{a_r} = 1 + \frac{\sigma_u^2}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} - \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)\left(\frac{\lambda - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}{1 - \frac{\beta A\sigma_s^2}{n\theta}\left(\frac{a_s}{a_r}\right)}\right)}\right)^2 \frac{\sigma_q^2}{\sigma_g^2}. \tag{A.152}$$

When $\bar{\lambda} \leqslant 1$ and $M_b\left(\bar{\lambda}\right) \leqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$, there must exist $\mathbf{w}$ such that

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n} = 1 + \frac{\sigma_u^2(\mathbf{w})}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda + \frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1 + \frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)}\right)^2 \frac{\sigma_q^2(\mathbf{w})}{\sigma_g^2}. \tag{A.153}$$

Then in this case, $\bar{\lambda}$, together with $\mathbf{w}$ satisfying the above condition, must be the solution of the information seller's optimization problem, as this is the highest possible utility for the information seller. It's easy to see that

$$\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)} \tag{A.154}$$

is the equilibrium a_r in this equilibrium, and is the same as the a_r^* in Theorem 1.

When $\bar{\lambda} > 1$ and $M_b(1) \leqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$, since $M_b(\cdot)$ is a decreasing function with $M_b(0) = \infty$, there must exist

$$\lambda_2 \in [0,1] \tag{A.155}$$

such that

$$M_b(\lambda_2) = \frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.156}$$

First, for any $\lambda < \lambda_2$, we must have

$$\theta\lambda\ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right)\leqslant\theta\lambda\ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{\frac{\sigma_g}{\sigma_s}\sqrt{n}}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}\left(\frac{\sigma_g}{\sigma_s}\sqrt{n}\right)\right)}\right). \tag{A.157}$$

For any $\lambda \in [\lambda_2, 1]$, there must exist $\mathbf{w}$ such that

$$n+n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n}=1+\frac{\sigma_u^2(\mathbf{w})}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)}\right)^2\frac{\sigma_q^2(\mathbf{w})}{\sigma_g^2}. \tag{A.158}$$

We know in equilibrium if we can obtain $\left(-\frac{a_s}{a_r}\right) = \frac{\sigma_g}{\sigma_s}\sqrt{n}$, the objective function becomes

$$\theta\lambda\ln\left(1+\frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2+2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right), \tag{A.159}$$

which is a unimodal function of λ. When $\bar{\lambda} > 1$, we must have that

$$\theta\lambda\ln\left(1+\frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2+2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right)\leqslant\theta\ln\left(1+\frac{1}{\left(\frac{\sqrt{n}\theta}{A\beta\sigma_s\sigma_g}\right)^2+2\frac{\sqrt{n}\theta}{A\beta\sigma_s\sigma_g}}\right) \tag{A.160}$$

for any $\lambda \in [0,1]$. Then we conclude that in this case, the information seller chooses $\lambda = 1$ and $\mathbf{w}$

such that

$$n+n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n}=1+\frac{\sigma_u^2\left(\mathbf{w}\right)}{\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)}\right)^2\frac{\sigma_q^2\left(\mathbf{w}\right)}{\sigma_g^2}. \tag{A.161}$$

It's clear that

$$\frac{1}{\frac{n}{\beta}+\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}} \tag{A.162}$$

is the equilibrium a_r in this equilibrium. This is the same as the a_r^{**} in Theorem 1. Then the above condition becomes

$$n\sigma_g^2+n\theta\frac{1}{A\beta}\frac{\sigma_g}{\sigma_s}\sqrt{n}=\sigma_g^2+\sigma_u^2\left(\mathbf{w}\right)+a_r^{**2}\sigma_q^2\left(\mathbf{w}\right). \tag{A.163}$$

If $\bar{\lambda}\leqslant 1$ and $M_b\left(\bar{\lambda}\right)>\frac{\sigma_g}{\sigma_s}\sqrt{n}$, we want to show that the information seller must choose $\mathbf{w}_{\max}\left(\lambda\right)$. Suppose that the information seller chooses λ_3. Let λ_2 be the solution of $M_b\left(\lambda_2\right)=\frac{\sigma_g}{\sigma_s}\sqrt{n}$. Since $M_b\left(\cdot\right)$ is a decreasing function, we know $\bar{\lambda}<\lambda_2$. If $\lambda_3>\lambda_2$, we know that

$$M_b\left(\lambda_3\right)<M_b\left(\lambda_2\right)=\frac{\sigma_g}{\sigma_s}\sqrt{n}, \tag{A.164}$$

and the information seller is able to find $\mathbf{w}$ such that $\left(-\frac{a_s}{a_r}\right)=\frac{\sigma_g}{\sigma_s}\sqrt{n}$. However, since in this case $\bar{\lambda}\leqslant 1$ and $M_b\left(\bar{\lambda}\right)>\frac{\sigma_g}{\sigma_s}\sqrt{n}$, and since the objective function

$$\theta\lambda\ln\left(1+\frac{1}{\left(\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}\right)^2+2\frac{\sqrt{n}\lambda\theta}{A\beta\sigma_s\sigma_g}}\right) \tag{A.165}$$

is a unimodal function of λ, we know that λ_3 is suboptimal, and is dominated by λ_2. Since

$$M_b\left(\lambda_2\right)=\frac{\sigma_g}{\sigma_s}\sqrt{n}, \tag{A.166}$$

the information seller will choose $\mathbf{w}_{\max}\left(\lambda\right)$ in this case. If $\lambda_3\leqslant\lambda_2$, we know that

$$M_b\left(\lambda_3\right)\geqslant M_b\left(\lambda_2\right)=\frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.167}$$

Since $-\frac{a_s}{a_r}\geqslant M_b\left(\lambda_3\right)\geqslant\frac{\sigma_g}{\sigma_s}\sqrt{n}$, and we know that

$$\theta\lambda\ln\left(1+\frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}}+\theta\frac{\lambda}{A\sigma_g^2}+\frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right) \tag{A.168}$$

is decreasing in $\left(-\frac{a_s}{a_r}\right) \geqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$, the information seller must choose $\mathbf{w}$ which leads to $-\frac{a_s}{a_r} = M_b\left(\lambda_2\right)$.

If $\bar{\lambda} > 1$ and $M_b\left(1\right) > \frac{\sigma_g}{\sigma_s}\sqrt{n}$, it's clear that no matter what λ the information seller chooses, we must have

$$M_b\left(\lambda\right) > \frac{\sigma_g}{\sigma_s}\sqrt{n}. \tag{A.169}$$

Since

$$\theta\lambda\ln\left(1 + \frac{A\beta^2\sigma_s^2}{n\lambda\theta}\frac{1}{\left(\frac{\beta}{-\frac{a_s}{a_r}} + \theta\frac{\lambda}{A\sigma_g^2} + \frac{\beta\sigma_s^2}{n\sigma_g^2}\left(-\frac{a_s}{a_r}\right)\right)}\right) \tag{A.170}$$

is decreasing in $\left(-\frac{a_s}{a_r}\right) \geqslant \frac{\sigma_g}{\sigma_s}\sqrt{n}$, we conclude that the information seller must choose $\mathbf{w}_{\max}\left(\lambda\right)$, so $-\frac{a_s}{s_r} = M_b\left(\lambda\right)$.

B.15 Proof of Corollary 3

In the unconstrained equilibrium, the optimal is $\mathbf{w}^*$ satisfies

$$n + n\theta\frac{\lambda}{A\beta\sigma_g^2}\frac{\sigma_g}{\sigma_s}\sqrt{n} = 1 + \frac{\sigma_u^2\left(\mathbf{w}^*\right)}{\sigma_g^2} + \left(\frac{1}{\frac{n}{\beta} + \frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2} + \frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\frac{\sigma_g}{\sigma_s}\sqrt{n}\left(\frac{\lambda+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}{1+\frac{\beta A\sigma_s^2}{n\theta}\frac{\sigma_g}{\sigma_s}\sqrt{n}}\right)}\right)^2\frac{\sigma_q^2\left(\mathbf{w}^*\right)}{\sigma_g^2}. \tag{A.171}$$

Since the RHS of the above equation is a quartic equation of $\mathbf{w}^*$, the solution is generically not unique.

B.16 Proof of Proposition 8

Suppose mandate investors purchase the score, as we discussed in the model, the market clear condition is

$$\left(1-\theta\right)\frac{E\left(\tilde{x}|\mathscr{F}_N\right) - \hat{p}}{A\text{Var}\left(\tilde{x}|\mathscr{F}_N\right)} + \lambda\theta\frac{E\left(\tilde{x} + \beta\hat{g}_r|\mathscr{F}_M\right) - \hat{p}}{A\text{Var}\left(\tilde{x}|\mathscr{F}_M\right)} = \tilde{s}. \tag{A.172}$$

Let's conjecture that the price is a linear combination of the score $\hat{g}_r$ and the noisy demand $\tilde{s}$, i.e., there exists c_0, c_1 and c_2, such that

$$p = c_0 + c_1\hat{g}_r + c_2\tilde{s}. \tag{A.173}$$

If an investor doesn't purchase the score, then his information set is

$$\mathcal{F} = \left\{\hat{p}\right\}. \tag{A.174}$$

If an investor purchasese the score, then his information set is

$$\mathcal{F} = \{\hat{p}, \hat{g}_r\} = \{\hat{g}_r, \tilde{s}\}. \tag{A.175}$$

Note that we have

$$\{\hat{p}\} \perp \tilde{x} \tag{A.176}$$

and

$$\{\hat{g}_r, \tilde{s}\} \perp \tilde{x}, \tag{A.177}$$

then we must have

$$E\left(\tilde{x}|\mathscr{F}_N\right) = E\left(\tilde{x}|\mathscr{F}_M\right) = \bar{x} \tag{A.178}$$

and

$$\text{Var}\left(\tilde{x}|\mathscr{F}_N\right) = \text{Var}\left(\tilde{x}|\mathscr{F}_E\right) = \sigma_x^2. \tag{A.179}$$

The market clear condition then becomes the following.

$$(1-\theta)\frac{\bar{x}-\hat{p}}{A\sigma_x^2} + \lambda\theta\frac{\bar{x}+\beta\hat{g}_r-\hat{p}}{A\sigma_x^2} = \tilde{s}, \tag{A.180}$$

which is equivalent to

$$\hat{p} = \bar{x} + \frac{\lambda\theta\beta\hat{g}_r - A\sigma_x^2\tilde{s}}{1-\theta+\lambda\theta}. \tag{A.181}$$

Given the equilibrium price and the index, the firm's problem is the following.

$$\max_{\hat{\delta}} \mathbb{E}\left(p\right) - \frac{1}{2q}\hat{\delta}^2 = \bar{x} + \frac{\lambda\theta}{1-\theta+\lambda\theta}\beta\mathbb{E}\left(\hat{g}_r\right) - \frac{A\sigma_x^2\bar{s}}{1-\theta+\lambda\theta} - \frac{1}{2q}\hat{\delta}^2 \tag{A.182}$$

$$= \bar{x} + \frac{\lambda\theta}{1-\theta+\lambda\theta}\beta\left(\bar{g}+\delta\right) - \frac{A\sigma_x^2\bar{s}}{1-\theta+\lambda\theta} - \frac{1}{2q}\hat{\delta}^2. \tag{A.183}$$

It's clear that the optimal manipulation is

$$\hat{\delta}^* = \frac{\lambda\theta}{1-\theta+\lambda\theta}\beta q. \tag{A.184}$$

Then the equilibrium score is

$$\hat{g}_r = \tilde{g} + \hat{u} + \frac{\lambda\theta}{1-\theta+\lambda\theta}\beta q. \tag{A.185}$$

B.17 Proof of Proposition 9

First, from Proposition 8, the equilibrium score is

$$\hat{g}_r = \tilde{g} + \hat{u} + \frac{\lambda\theta}{1-\theta+\lambda\theta}\beta q. \tag{A.186}$$

Since $\tilde{g} \sim N\left(0, \sigma_g^2\right)$, $\hat{u} \sim N\left(0, \hat{\sigma}_u^2\right)$, and $q \sim N\left(\bar{q}, \sigma_q^2\right)$, the informativeness of the score is measured by $\text{corr}(\tilde{g}, \hat{g}_r)$, or equivalently,

$$\frac{1}{\text{Var}\left(\tilde{g}|\hat{g}_r\right)} = \left(1 + \frac{\sigma_g^2}{\left(\frac{\lambda\theta}{1-\theta+\lambda\theta}\right)^2 \beta^2\sigma_q^2 + \hat{\sigma}_u^2}\right)\frac{1}{\sigma_g^2}. \tag{A.187}$$

Then it's clear that when θ or β increases, $\frac{1}{\text{Var}(\tilde{g}|\hat{g}_r)}$ decreases.

$$\begin{aligned}
\text{corr}\left(\hat{g}_r, \hat{p}\right) &= \frac{\text{Cov}\left(\hat{g}_r \hat{p}\right)}{\sqrt{\text{Var}\left(\hat{g}_r\right)\text{Var}\left(\hat{p}\right)}} \\
&= \frac{\text{Cov}\left(\hat{g}_r \hat{g}_r - \frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\tilde{s}\right)}{\sqrt{\text{Var}\left(\hat{g}_r\right)\text{Var}\left(\hat{g}_r - \frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\tilde{s}\right)}} \\
&= \frac{\text{Var}\left(\hat{g}_r\right)}{\sqrt{\text{Var}\left(\hat{g}_r\right)\left(\text{Var}\left(\hat{g}_r\right) + \left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\right)^2\sigma_s^2\right)}} \\
&= \frac{1}{\sqrt{1 + \frac{1}{\text{Var}(\hat{g}_r)}\left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\right)^2\sigma_s^2}}.
\end{aligned}$$

Note that

$$\frac{\left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\right)^2\sigma_s^2}{\text{Var}\left(\hat{g}_r\right)} = \frac{\left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\right)^2\sigma_s^2}{\sigma_g^2 + \hat{\sigma}_u^2 + \left(\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\right)^2\sigma_q^2}.$$

When β or θ increases, $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}$ increases, and thus $\frac{\left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\right)^2\sigma_s^2}{\text{Var}(\hat{g}_r)}$ decreases. Then $\text{corr}\left(\hat{g}_r, \hat{p}\right)$ increases.

Finally,

$$\begin{aligned}
\operatorname{corr}(\hat{p},\tilde{g}) &= \frac{\operatorname{Cov}(\hat{p},\tilde{g})}{\sqrt{\operatorname{Var}(\hat{p})\operatorname{Var}(\tilde{g})}} \\
&= \frac{\operatorname{Cov}\left(\hat{g}_r - \frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\tilde{s},\tilde{g}\right)}{\sqrt{\operatorname{Var}\left(\hat{g}_r - \frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\tilde{s}\right)\operatorname{Var}(\tilde{g})}} \\
&= \frac{\operatorname{Cov}\left(\tilde{g}+\hat{u}+\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}q - \frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\tilde{s},\tilde{g}\right)}{\sqrt{\operatorname{Var}\left(\tilde{g}+\hat{u}+\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}q - \frac{1-\theta+\lambda\theta}{\theta\lambda\beta}A\sigma_x^2\tilde{s}\right)\operatorname{Var}(\tilde{g})}} \\
&= \frac{\sigma_g^2}{\sqrt{\left(\sigma_g^2+\hat{\sigma}_u^2+\left(\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\right)^2\sigma_q^2+\left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}\right)^2A^2\sigma_x^4\sigma_s^2\right)\sigma_g^2}} \\
&= \frac{1}{\sqrt{1+\frac{\hat{\sigma}_u^2+\left(\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\right)^2\sigma_q^2+\left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}\right)^2A^2\sigma_x^4\sigma_s^2}{\sigma_g^2}}}.
\end{aligned}$$

Note that

$$\frac{\hat{\sigma}_u^2+\left(\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\right)^2\sigma_q^2+\left(\frac{1-\theta+\lambda\theta}{\theta\lambda\beta}\right)^2A^2\sigma_x^4\sigma_s^2}{\sigma_g^2}$$

is decreasing in $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}$ when $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\leqslant\sqrt{\frac{A\sigma_x^2\sigma_s}{\sigma_q}}$ and increasing in $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}$ when $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\geqslant\sqrt{\frac{A\sigma_x^2\sigma_s}{\sigma_q}}$. Then $\operatorname{corr}(\hat{p},\tilde{g})$ is increasing in $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}$ when $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\leqslant\sqrt{\frac{A\sigma_x^2\sigma_s}{\sigma_q}}$ and decreasing in $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}$ when $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\geqslant\sqrt{\frac{A\sigma_x^2\sigma_s}{\sigma_q}}$. Since $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}$ is increasing in β and θ, then when β or θ increase, $\operatorname{corr}(\hat{p},\tilde{g})$ first increases and then decreases.

B.18 Proof of Proposition 10

We know that

$$\begin{aligned}
\phi_M^* &= \frac{\mathbb{E}(\tilde{x}+\beta\hat{g}_r|\mathcal{F}_M)-\hat{p}}{A\operatorname{Var}(\tilde{x})} \\
&= \frac{\bar{x}+\beta\hat{g}_r-\left(\bar{x}+\frac{\theta\lambda\beta\hat{g}_r-A\sigma_x^2\tilde{s}}{1-\theta+\lambda\theta}\right)}{A\sigma_x^2} \\
&= \frac{\beta\hat{g}_r-\frac{\theta\lambda\beta\hat{g}_r-A\sigma_x^2\tilde{s}}{1-\theta+\lambda\theta}}{A\sigma_x^2} \\
&= \frac{\beta(1-\theta)\hat{g}_r}{(1-\theta+\lambda\theta)A\sigma_x^2}+\frac{\tilde{s}}{1-\theta+\lambda\theta}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\mathbb{E}\left(\phi_M^* \tilde{g}\right)}{\mathbb{E}\left(\phi_M^*\right)} &= \frac{\operatorname{Cov}\left(\phi_M^*, \tilde{g}\right) + \mathbb{E}\left(\phi_M^*\right)\mathbb{E}\left(\tilde{g}\right)}{\mathbb{E}\left(\phi_M^*\right)} \\
&= \frac{\operatorname{Cov}\left(\phi_M^*, \tilde{g}\right)}{\mathbb{E}\left(\phi_M^*\right)} + \bar{g} \\
&= \frac{\operatorname{Cov}\left(\frac{\beta(1-\theta)\hat{g}_r}{(1-\theta+\lambda\theta)A\sigma_x^2} + \frac{\tilde{s}}{1-\theta+\lambda\theta}, \tilde{g}\right)}{\mathbb{E}\left(\frac{\beta(1-\theta)\hat{g}_r}{(1-\theta+\lambda\theta)A\sigma_x^2} + \frac{\tilde{s}}{1-\theta+\lambda\theta}\right)} + \bar{g} \\
&= \frac{\operatorname{Cov}\left(\hat{g}_r + \frac{A\sigma_x^2}{\beta(1-\theta)}\tilde{s}, \tilde{g}\right)}{\mathbb{E}\left(\hat{g}_r + \frac{A\sigma_x^2}{\beta(1-\theta)}\tilde{s}\right)} + \bar{g} \\
&= \frac{\sigma_g^2}{\mathbb{E}\left(\tilde{g} + \hat{u} + \frac{\theta\lambda\beta}{1-\theta+\lambda\theta}q + \frac{A\sigma_x^2}{\beta(1-\theta)}\tilde{s}\right)} + \bar{g} \\
&= \frac{\sigma_g^2}{\bar{g} + \frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\bar{q} + \frac{A\sigma_x^2}{\beta(1-\theta)}\bar{s}} + \bar{g}.
\end{aligned}$$

Since $\bar{s} > 0$,

$$\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\bar{q} = \frac{A\sigma_x^2}{\beta(1-\theta)}\bar{s} \Longleftrightarrow \beta = \sqrt{\frac{A\sigma_x^2\bar{s}\left(1-\theta+\lambda\theta\right)}{\theta\lambda\left(1-\theta\right)\bar{q}}}.$$

It's clear that $\frac{\theta\lambda\beta}{1-\theta+\lambda\theta}\bar{q} + \frac{A\sigma_x^2}{\beta(1-\theta)}\bar{s}$ decreases in β if and only if $\beta < \sqrt{\frac{A\sigma_x^2\bar{s}(1-\theta+\lambda\theta)}{\theta\lambda(1-\theta)\bar{q}}}$. Then $\frac{\mathbb{E}\left(\phi_M^*\tilde{g}\right)}{\mathbb{E}\left(\phi_M^*\right)}$ increases in β if and only if $\beta < \sqrt{\frac{A\sigma_x^2\bar{s}(1-\theta+\lambda\theta)}{\theta\lambda(1-\theta)\bar{q}}}$.

B.19 Proof of Lemma 4

If mandate investors buy the score, under optimal holdings, their utility is

$$-e^{-A\left(W_0-\hat{\Phi}\right)}\mathbb{E}\left[e^{-A\left[\phi_M^*\mathbb{E}(\tilde{x}-\hat{p}+\beta\hat{g}_r|\mathscr{F}_M)-\frac{1}{2}A\phi_M^{*2}\operatorname{Var}(\tilde{x}|\mathscr{F}_M)\right]}\right]. \tag{A.188}$$

If they do not buy the score, under optimal holdings, they can only buy the risk-free asset, and thus their utility is

$$-e^{-AW_0}. \tag{A.189}$$

Then the maximum price $\bar{\hat{\Phi}}$ that the information seller can charge satisfies

$$-e^{-A\left(W_0-\bar{\hat{\Phi}}\right)}\mathbb{E}\left[e^{-A\left[\phi_M^*\mathbb{E}(\tilde{x}-\hat{p}+\beta\hat{g}_r|\mathscr{F}_M)-\frac{1}{2}A\phi_M^{*2}\operatorname{Var}(\tilde{x}|\mathscr{F}_M)\right]}\right] = -e^{-AW_0}, \tag{A.190}$$

which is

$$e^{A\bar{\hat{\Phi}}}\mathbb{E}\left[e^{-A\left[\phi_M^*\mathbb{E}(\tilde{x}-\hat{p}+\beta\hat{g}_r|\mathscr{F}_M)-\frac{1}{2}A\phi_M^{*2}\operatorname{Var}(\tilde{x}|\mathscr{F}_M)\right]}\right] = 1. \tag{A.191}$$

Since

$$\phi_M^* = \frac{\mathbb{E}\left(\tilde{x} - \hat{p} + \beta \hat{g}_r | \mathscr{F}_M\right)}{A\text{Var}\left(\tilde{x} | \mathscr{F}_M\right)}, \tag{A.192}$$

$$\mathbb{E}\left(\tilde{x} | \mathscr{F}_E\right) = \bar{x}, \tag{A.193}$$

and

$$\text{Var}\left(\tilde{x} | \mathscr{F}_M\right) = \sigma_x^2, \tag{A.194}$$

we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-A\left[\phi_E^* \mathbb{E}(\tilde{x} - \hat{p} + \beta \hat{g}_r | \mathscr{F}_M) - \frac{1}{2} A \phi_M^{*2} \text{Var}(\tilde{x} | \mathscr{F}_M)\right]}\right] \\
=&\mathbb{E}\left[e^{-\frac{\left[\mathbb{E}\left(\tilde{x} + \beta \hat{g}_r | \mathscr{F}_M\right) - \hat{p}\right]^2}{2\text{Var}\left(\tilde{x} | \mathscr{F}_M\right)}}\right] \\
=&\mathbb{E}\left[e^{-\frac{\left[\tilde{x} + \beta \hat{g}_r - \bar{x} - \frac{\theta \lambda \beta \hat{g}_r - A\sigma_x^2 \tilde{s}}{1 - \theta + \lambda \theta}\right]^2}{2\sigma_x^2}}\right] \\
=&\mathbb{E}\left[e^{-\frac{\left[\beta \hat{g}_r - \frac{\theta \lambda \beta \hat{g}_r - A\sigma_x^2 \tilde{s}}{1 - \theta + \lambda \theta}\right]^2}{2\sigma_x^2}}\right] \\
=&\mathbb{E}\left[e^{-\frac{\left[(1-\theta)\beta \hat{g}_r + A\sigma_x^2 \tilde{s}\right]^2}{(1-\theta+\lambda\theta)^2 2\sigma_x^2}}\right]
\end{aligned}$$

Then

$$\hat{\Phi} = \frac{-1}{A} \ln \mathbb{E}\left[e^{-\frac{\left[(1-\theta)\beta \hat{g}_r + A\sigma_x^2 \tilde{s}\right]^2}{(1-\theta+\lambda\theta)^2 2\sigma_x^2}}\right]$$

Let

$$z = \frac{(1-\theta)\beta \hat{g}_r + A\sigma_x^2 \tilde{s}}{(1-\theta+\lambda\theta)\sigma_x},$$

then z is a normally distributed random variable with mean

$$\mu_z = \frac{(1-\theta)\beta\left(\bar{g} + \frac{\lambda\theta}{1-\theta+\lambda\theta}\beta\bar{q}\right) + A\sigma_x^2 \bar{s}}{(1-\theta+\lambda\theta)\sigma_x},$$

and standard deviation

$$\sigma_z = \frac{\sqrt{(1-\theta)^2 \beta^2 \left(\sigma_g^2 + \widehat{\sigma}_u^2 + \left(\frac{\lambda\theta}{1-\theta+\lambda\theta}\right)^2 \beta^2 \sigma_q^2\right) + A^2 \sigma_x^4 \sigma_s^2}}{(1-\theta+\lambda\theta)\sigma_x}.$$

The expectation

$$\mathbb{E}\left(e^{-\frac{z^2}{2}}\right) \tag{A.195}$$

can be obtained using the moment generating function of noncentral chi-squared distributions. Note that

$$\mathbb{E}\left(e^{-\frac{z^2}{2}}\right) = \mathbb{E}\left(e^{-\frac{1}{2}\sigma_z^2\left(\frac{z}{\sigma_z}\right)^2}\right) \tag{A.196}$$

and here

$$\frac{z}{\sigma_z} \tag{A.197}$$

has unit variance and mean $\frac{\mu_z}{\sigma_z}$. We have

$$\ln\mathbb{E}\left(e^{-\frac{z^2}{2}}\right) = \ln\mathbb{E}\left(e^{-\frac{1}{2}\sigma_z^2\left(\frac{z}{\sigma_z}\right)^2}\right) = -\frac{1}{2}\left[\frac{\mu_z^2}{1+\sigma_z^2} + \ln\left(1+\sigma_z^2\right)\right]. \tag{A.198}$$

Then the price $\hat{\Phi}$ is

$$\hat{\Phi} = \frac{-1}{A}\ln\mathbb{E}\left[e^{-\frac{\left[(1-\theta)\beta\hat{g}_r + A\sigma_x^2\bar{s}\right]^2}{2\sigma_x^2}}\right] = \frac{1}{2A}\left[\frac{\mu_z^2}{1+\sigma_z^2} + \ln\left(1+\sigma_z^2\right)\right]. \tag{A.199}$$

B.20 Proof of Proposition 11

The information seller's problem is equivalent to

$$\max_{\lambda,\sigma_u} H\left(\lambda,\sigma_u\right) = \lambda\left[\frac{\mu_z^2}{1+\sigma_z^2} + \ln\left(1+\sigma_z^2\right)\right]$$

where

$$\mu_z = \frac{(1-\theta)\,\beta\bar{g} + (1-\theta)\,\frac{\lambda\theta}{1-\theta+\lambda\theta}\beta^2\bar{q} + A\sigma_x^2\bar{s}}{(1-\theta+\lambda\theta)\,\sigma_x}$$

and

$$\sigma_z = \frac{\sqrt{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2+\left(\frac{\lambda\theta}{1-\theta+\lambda\theta}\right)^2\beta^2\sigma_q^2\right) + A^2\sigma_x^4\sigma_s^2}}{(1-\theta+\lambda\theta)\,\sigma_x}.$$

For any λ and $\sigma_u \in [\underline{\sigma}, \bar{\sigma}]$,

$$\mu_z \leqslant \bar{\mu}_z = \frac{(1-\theta)\,\beta\bar{g} + \lambda\theta\beta^2\bar{q} + A\sigma_x^2\bar{s}}{(1-\theta)\,\sigma_x},$$

$$\sigma_z \leqslant \bar{\sigma}_z = \frac{\sqrt{(1-\theta)^2\beta^2\left(\sigma_g^2+\bar{\sigma}^2+\theta^2\beta^2\sigma_q^2\right) + A^2\sigma_x^4\sigma_s^2}}{(1-\theta)\,\sigma_x},$$

and

$$\sigma_z \geqslant \underline{\sigma}_z = \frac{\sqrt{(1-\theta)^2 \beta^2 \sigma_g^2 + A^2 \sigma_x^4 \sigma_s^2}}{\sigma_x}$$

Then for any λ and $\sigma_u \in [\underline{\sigma}, \bar{\sigma}]$,

$$H(\lambda, \sigma_u) \leqslant \lambda \left[\frac{\left[\frac{(1-\theta)\beta\bar{g} + \lambda\theta\beta^2\bar{q} + A\sigma_x^2\bar{s}}{(1-\theta)\sigma_x} \right]^2}{1 + \underline{\sigma}_z^2} + \ln\left(1 + \bar{\sigma}_z^2\right) \right].$$

When $\lambda = 1$,

$$\mu_z = \frac{(1-\theta)\beta\bar{g} + (1-\theta)\theta\beta^2\bar{q} + A\sigma_x^2\bar{s}}{\sigma_x}.$$

Then the optimal $\hat{\lambda}$ must satisfy

$$\hat{\lambda} \left[\frac{\left[\frac{(1-\theta)\beta\bar{g} + \hat{\lambda}\theta\beta^2\bar{q} + A\sigma_x^2\bar{s}}{(1-\theta)\sigma_x} \right]^2}{1 + \underline{\sigma}_z^2} + \ln\left(1 + \bar{\sigma}_z^2\right) \right] \geqslant \left[\frac{\left[\frac{(1-\theta)\beta\bar{g} + (1-\theta)\theta\beta^2\bar{q} + A\sigma_x^2\bar{s}}{\sigma_x} \right]^2}{1 + \bar{\sigma}_z^2} + \ln\left(1 + \underline{\sigma}_z^2\right) \right].$$

When $\bar{q} \to \infty$, to make the above condition hold, we must have

$$\hat{\lambda} \frac{\left[\hat{\lambda}\theta\beta^2\bar{q}\right]^2}{(1-\theta)^2 (1 + \underline{\sigma}_z^2)} \geqslant \frac{\left[(1-\theta)\theta\beta^2\bar{q}\right]^2}{1 + \bar{\sigma}_z^2},$$

which is $\hat{\lambda} \geqslant \left(\frac{1+\underline{\sigma}_z^2}{1+\bar{\sigma}_z^2} (1-\theta)^4 \right)^{1/3}$. Then there must exist $\bar{q}_{m1} > 0$, such that when $\bar{q} > \bar{q}_{m1}$, we have

$$\hat{\lambda} > \underline{\lambda} = \frac{1}{2} \left(\frac{1 + \underline{\sigma}_z^2}{1 + \bar{\sigma}_z^2} (1-\theta)^4 \right)^{1/3}.$$

Here $\underline{\lambda}$ is independent of $\bar{q}$.

Note that μ_z is independent of σ_u and σ_z^2 is increasing in σ_u , we first examine how the objective function changes with σ_z^2. It's easy to show that

$$\begin{aligned} \frac{\partial \left[\frac{\mu_z^2}{1+\sigma_z^2} + \ln\left(1 + \sigma_z^2\right) \right]}{\partial (\sigma_z^2)} &= -\frac{\mu_z^2}{(1 + \sigma_z^2)^2} + \frac{1}{1 + \sigma_z^2} \\ &= \frac{1}{1 + \sigma_z^2} \left(1 - \frac{\mu_z^2}{1 + \sigma_z^2} \right). \end{aligned}$$

Then the objective function is increasing in σ_z^2 if and only if $1 + \sigma_z^2 > \mu_z^2$. We view σ_z^2 as a function of σ_u. Then for any $\lambda \geqslant \hat{\lambda}$ and σ_u,

$$\sigma_z^2 \geqslant \sigma_{z,min}^2 = \frac{(1-\theta)^2 \beta^2 \left(\sigma_g^2 + \underline{\sigma} + \left(\frac{\hat{\lambda}\theta}{1-\theta+\hat{\lambda}\theta} \right)^2 \beta^2 \sigma_q^2 \right) + A^2 \sigma_x^4 \sigma_s^2}{(1-\theta)\sigma_x^2},$$

and

$$\sigma_z^2 \leqslant \sigma_{z,max}^2 = \frac{(1-\lambda)^2 \beta^2 \left(\sigma_g^2 + \bar{\sigma}^2 + \theta^2 \beta^2 \sigma_q^2\right) + A^2 \sigma_x^4 \sigma_s^2}{\sigma_x^2}.$$

For any μ_z, let

$$F_L = \frac{\mu_z^2}{1+\sigma_{z,min}^2} + \ln\left(1+\sigma_{z,min}^2\right)$$

and

$$F_R = \frac{\mu_z^2}{1+\sigma_{z,max}^2} + \ln\left(1+\sigma_{z,max}^2\right).$$

It's clear that for any given μ_z, we have

$$\max\{F_L, F_R\} = \max_{\sigma_u \in [\underline{\sigma}, \bar{\sigma}]} \frac{\mu_z^2}{1+\sigma_z^2} + \ln\left(1+\sigma_z^2\right).$$

$F_L \geqslant F_R$ is equivalent to

$$\begin{aligned}
&\frac{\mu_z^2}{1+\sigma_{z,min}^2} + \ln\left(1+\sigma_{z,min}^2\right) \geqslant \frac{\mu_z^2}{1+\sigma_{z,max}^2} + \ln\left(1+\sigma_{z,max}^2\right) \\
\Longleftrightarrow &\mu_z^2 \left(\frac{1}{1+\sigma_{z,min}^2} - \frac{1}{1+\sigma_{z,max}^2}\right) \geqslant \ln \frac{1+\sigma_{z,max}^2}{1+\sigma_{z,min}^2} \\
\Longleftrightarrow &\mu_z^2 \geqslant \frac{\ln \frac{1+\sigma_{z,max}^2}{1+\sigma_{z,min}^2}}{\frac{1}{1+\sigma_{z,min}^2} - \frac{1}{1+\sigma_{z,max}^2}}.
\end{aligned}$$

Since for any $\lambda \geqslant \underline{\lambda}$ and σ_u,

$$\mu_z \geqslant \frac{(1-\theta)\beta\bar{g} + (1-\theta)\frac{\underline{\lambda}\theta}{1-\theta+\underline{\lambda}\theta}\beta^2\bar{q} + A\sigma_x^2\bar{s}}{(1-\theta)\sigma_x},$$

when

$$\left(\frac{(1-\theta)\beta\bar{g} + (1-\theta)\frac{\underline{\lambda}\theta}{1-\theta+\underline{\lambda}\theta}\beta^2\bar{q} + A\sigma_x^2\bar{s}}{(1-\theta)\sigma_x}\right)^2 \geqslant \frac{\ln \frac{1+\sigma_{z,max}^2}{1+\sigma_{z,min}^2}}{\frac{1}{1+\sigma_{z,min}^2} - \frac{1}{1+\sigma_{z,max}^2}},$$

the optimal solution is $\hat{\sigma}_u = \underline{\sigma}$. Let $\bar{q}_m$ be maximum of the the positive solution of the above condition and $\bar{q}_{m1}$, it's clear that when $\bar{q} > \bar{q}_m$, the optimal solution must be $\hat{\sigma}_u = \underline{\sigma}$.

B.21 Proof of Proposition 12

The information seller's problem is

$$\max_{\lambda, \sigma_u} H(\lambda, \mathbf{w}) = \lambda\left[\frac{\mu_z^2}{1+\sigma_z^2} + \ln\left(1+\sigma_z^2\right)\right]$$

where

$$\mu_z = \frac{(1-\theta)\beta\bar{g} + (1-\theta)\frac{\lambda\theta}{1-\theta+\lambda\theta}\beta^2\sum_{i=1}^n w_i\bar{q}_i + A\sigma_x^2\bar{s}}{(1-\theta+\lambda\theta)\sigma_x}$$

and

$$\sigma_z = \frac{\sqrt{(1-\theta)^2 \beta^2 \left(\sigma_g^2 + \sigma_u^2(\mathbf{w}) + \left(\frac{\lambda\theta}{1-\theta+\lambda\theta}\right)^2 \beta^2 \sigma_q^2(\mathbf{w})\right) + A^2 \sigma_x^4 \sigma_s^2}}{(1-\theta+\lambda\theta)\,\sigma_x}.$$

First, let's show that there exists $\bar{q}_w$, such that when $\bar{q}_i > \bar{q}_w$ for all i, we must have $\hat{\lambda} = 1$. We prove this by contradiction. Suppose that there exists $\lambda_1 < 1$, such that for any $M > 0$, there exist $\bar{q}_i > M$ for all i and $\lambda_M \leqslant \lambda_1$, such that

$$\max_{\mathbf{w}} H(\lambda_M, \mathbf{w}) > \max_{\mathbf{w}} H(1, \mathbf{w}). \tag{A.200}$$

For any M, let

$$(\lambda_M, \mathbf{w}_M) = \arg \max_{\lambda \leqslant \lambda_1, \mathbf{w}} H(\lambda, \mathbf{w}).$$

Let's consider the ratio $\frac{H(1,\mathbf{w}_M)}{H(\lambda_M,\mathbf{w}_M)}$. Since $w_i \geqslant 0$ for all i, then both $\sigma_u^2(\mathbf{w})$ and $\sigma_q^2(\mathbf{w})$ are bounded, which means that there exists $C_1 > 0$, such that $\max\left\{\sigma_u^2(\mathbf{w}), \sigma_q^2(\mathbf{w})\right\} < C_1$ for any $\mathbf{w}$.When

$\bar{q}_i > M$ for all i, we must have $\sum_{i=1}^n w_i \bar{q}_i \geqslant M$ for any w. Then

$$
\begin{aligned}
&\lim_{M\to\infty} \frac{H(\lambda_M, \mathbf{w}_M)}{H(1, \mathbf{w}_M)} \\
&= \lim_{M\to\infty} \frac{\lambda_M \left[\dfrac{\left(\frac{(1-\theta)\beta\bar{g}+(1-\theta)\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i+A\sigma_x^2\bar{s}}{(1-\theta+\lambda_M\theta)\sigma_x}\right)^2}{1+\left(\frac{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\left(\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\right)^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}{(1-\theta+\lambda_M\theta)^2\sigma_x^2}\right)} + \ln\left(1+\left(\frac{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\left(\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\right)^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}{(1-\theta+\lambda_M\theta)^2\sigma_x^2}\right)\right)\right]}{\left[\dfrac{\left(\frac{(1-\theta)\beta\bar{g}+(1-\theta)\theta\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i+A\sigma_x^2\bar{s}}{\sigma_x}\right)^2}{1+\left(\frac{\sqrt{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}}{\sigma_x}\right)} + \ln\left(1+\left(\frac{\sqrt{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}}{\sigma_x}\right)\right)\right]} \\
&= \lim_{M\to\infty} \frac{\lambda_M\left[\dfrac{\left(\frac{(1-\theta)\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i}{(1-\theta+\lambda_M\theta)}\right)^2}{\sigma_x^2+\left(\frac{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\left(\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\right)^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}{(1-\theta+\lambda_M\theta)^2}\right)}\right]}{\left[\dfrac{\left((1-\theta)\theta\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i\right)^2}{\sigma_x^2+(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}\right]} \\
&= \lim_{M\to\infty} \frac{\lambda_M\left[\dfrac{\left((1-\theta)\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i\right)^2}{\sigma_x^2(1-\theta+\lambda_M\theta)^2+(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\left(\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\right)^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}\right]}{\left[\dfrac{\left((1-\theta)\theta\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i\right)^2}{\sigma_x^2+(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}\right]} \\
&= \lim_{M\to\infty} \frac{\lambda_M\left[\dfrac{\left((1-\theta)\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i\right)^2}{\sigma_x^2\frac{(1-\theta+\lambda_M\theta)^4}{(\lambda_M\theta)^2}+(1-\theta)^2\beta^4\sigma_q^2(\mathbf{w}_M)+\frac{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}{\left(\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\right)^2}}\right]}{\left[\dfrac{\left((1-\theta)\theta\beta^2\sum_{i=1}^n w_{M,i}\bar{q}_i\right)^2}{\sigma_x^2+(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}\right]} \\
&= \lim_{M\to\infty} \frac{\dfrac{1}{\sigma_x^2\frac{(1-\theta+\lambda_M\theta)^4}{\lambda_M^3}+\frac{(1-\theta)^2\beta^4\sigma_q^2(\mathbf{w}_M)}{\lambda_M}+\frac{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}{\lambda_M\left(\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\right)^2}}}{\left[\dfrac{1}{\sigma_x^2+(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}\right]} \\
&\lim_{M\to\infty} \frac{\sigma_x^2+(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)+\theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}{\sigma_x^2\frac{(1-\theta+\lambda_M\theta)^4}{\lambda_M^3}+\frac{(1-\theta)^2\beta^4\sigma_q^2(\mathbf{w}_M)}{\lambda_M}+\frac{(1-\theta)^2\beta^2\left(\sigma_g^2+\sigma_u^2(\mathbf{w}_M)\right)+A^2\sigma_x^4\sigma_s^2}{\lambda_M\left(\frac{\lambda_M\theta}{1-\theta+\lambda_M\theta}\right)^2}}
\end{aligned}
$$

It's clear that

$$(1-\theta)^2 \beta^2 \left(\sigma_g^2 + \sigma_u^2(\mathbf{w}_M) + \theta^2\beta^2\sigma_q^2(\mathbf{w}_M)\right) + A^2\sigma_x^4\sigma_s^2 > \frac{(1-\theta)^2 \beta^4 \sigma_q^2(\mathbf{w}_M)}{\lambda_M} + \frac{(1-\theta)^2 \beta^2 \left(\sigma_g^2 + \sigma_u^2(\mathbf{w}_M)\right) + A^2\sigma_x^4\sigma_s^2}{\lambda_M \left(\frac{\lambda_M \theta}{1-\theta+\lambda_M\theta}\right)^2}$$

for any $\lambda_M \leqslant \lambda_1 < 1$. Then if function $\frac{(1-\theta+x\theta)^4}{x^3}$ is a decreasing function of x, we can conclude that $\lim_{M\to\infty} \frac{H(\lambda_M, \mathbf{w}_M)}{H(1, \mathbf{w}_M)} < 1$. Note that

$$\left(\ln\left(\frac{(1-\theta+x\theta)^4}{x^3}\right)\right)' = \frac{4\theta}{1-\theta+x\theta} - \frac{3}{x}.$$

And $\frac{4\theta}{1-\theta+x\theta} - \frac{3}{x} < 0 \iff \theta x < 3(1-\theta)$. When $\theta < \frac{3}{4}$, this condition always holds. This is a contradiction to (A.200). Since we always have $\hat{\lambda} = 1$ in this case, it's easy to see that the objective function is a supermodular function of $(w_i, \bar{q}_i)$, and thus we must have

$$\frac{\partial \widehat{w}_i}{\partial \bar{q}_i} > 0.$$

B.22 Proof of Proposition 13

First, we want to verify that when N is large enough, if all information sellers choose $\sigma_u > 0$, and a sufficiently low price $\Phi > 0$ (which is a function of N), there exists an equilibrium such that all type-G investors would like to buy all scores. For any N, if all type-G investors buy all scores, then the sufficient statistic of the N scores is

$$G_r = \tilde{g} + \frac{1}{N}\sum_{i=1}^{N} u_i + \delta. \tag{A.201}$$

Let's assume that the equilibrium price equation is

$$p = a_0 + a_r G_r + a_s \tilde{s} = a_0 + a_r \left(\tilde{g} + \frac{1}{N}\sum_{i=1}^{N} u_i + \delta + \frac{a_s}{a_r}\tilde{s}\right). \tag{A.202}$$

Then the optimal level of manipulation is

$$\delta^* = a_r q. \tag{A.203}$$

For any type-G investor, the total cost he would like to pay for m scores, $P(m)$, is given by

$$\begin{aligned} P(m) &= \frac{1}{2A}\ln\left(\frac{\mathrm{Var}(\tilde{x} + \beta\tilde{g}|p)}{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|p, \{g_r^i\}_{i=1}^m\right)}\right) \\ &= \frac{1}{2A}\ln\left(\mathrm{Var}(\tilde{x} + \beta\tilde{g}|p)\right) - \frac{1}{2A}\ln\left(\mathrm{Var}\Big(\tilde{x} + \beta\tilde{g}|p, \{g_r^i\}_{i=1}^m\Big)\right). \end{aligned} \tag{A.204}$$

So we can consider the following utility function of an type-G who buys m scores

$$-\frac{1}{2A}\ln\left(\mathrm{Var}\Big(\tilde{x} + \beta\tilde{g}|p, \{g_r^i\}_{i=1}^m\Big)\right) - m\Phi^*. \tag{A.205}$$

Let

$$V(m) = -\frac{1}{2A}\ln\left(\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|p,\{g_r^i\}_{i=1}^m\right)\right). \tag{A.206}$$

Then the marginal price that the investor would like to pay for the score m is

$$V(m) - V(m-1). \tag{A.207}$$

Lemma 11. *There exists $\underline{N}_1$ such that when $N > \underline{N}_1$,*

$$V(N) - V(N-1) = \min_{i\in\{2,3\ldots N\}}\left(V(i) - V(i-1)\right). \tag{A.208}$$

Proof. We have

$$\begin{aligned}
-2AV(m) &= \ln\left(\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|p,\{g_r^i\}_{i=1}^m\right)\right)\\
&= \ln\left(\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\tilde{g}+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\delta+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s},\{\tilde{g}+u_i+\delta\}_{i=1}^m\right)\right)\\
&= \ln\left(\mathrm{Var}\left(\tilde{x}+\beta\tilde{g}|\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s},\tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\right)\right)\\
&= \ln\left(\sigma_x^2+\beta^2\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s},\tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s},\tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\right)\\
&= \begin{array}{c}\mathbb{E}\left(\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta\right)|\begin{array}{c}\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s}\\ \tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\end{array}\right)\\ +\mathrm{Var}\left(\mathbb{E}\left(\tilde{g}|\tilde{g}+\delta\right)|\begin{array}{c}\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s}\\ \tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\end{array}\right)\end{array}.
\end{aligned}$$

Since

$$\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta\right) = \frac{1}{\frac{1}{\sigma_g^2}+\frac{1}{a_r^2\sigma_q^2}}, \tag{A.209}$$

which is a constant number in equilibrium, we have

$$\mathbb{E}\left(\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta\right)|\begin{array}{c}\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s}\\ \tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\end{array}\right) = \frac{1}{\frac{1}{\sigma_g^2}+\frac{1}{a_r^2\sigma_q^2}}. \tag{A.210}$$

Second,

$$\mathbb{E}\left(\tilde{g}|\tilde{g}+\delta\right) = \bar{g}+\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\left(\tilde{g}+\delta-\bar{g}-a_r\bar{q}\right). \tag{A.211}$$

Then

$$\begin{aligned}
&\mathrm{Var}\left(\bar{g}+\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\left(\tilde{g}+\delta-\bar{g}-a_r\bar{q}\right) \mid \begin{array}{c}\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s}\\ \tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\end{array}\right)\\
=&\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2\mathrm{Var}\left(\left(\tilde{g}+\delta\right) \mid \begin{array}{c}\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s}\\ \tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\end{array}\right)\\
=&\frac{\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2}{\frac{1}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{1}{\frac{1}{N-m}\sigma_u^2+\left(\frac{N}{N-m}\frac{a_s}{a_r}\right)^2\sigma_s^2}+\frac{m}{\sigma_u^2}}.
\end{aligned}$$

We used the following result in the above derivation.

$$\frac{1}{\mathrm{Var}\left(\left(\tilde{g}+\delta\right) \mid \begin{array}{c}\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s}\\ \tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\end{array}\right)}=\frac{1}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{1}{\frac{1}{N-m}\sigma_u^2+\left(\frac{N}{N-m}\frac{a_s}{a_r}\right)^2\sigma_s^2}+\frac{m}{\sigma_u^2}. \tag{A.212}$$

Then

$$\begin{aligned}
&\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s},\tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\right)\\
=&\frac{1}{\frac{1}{\sigma_g^2}+\frac{1}{a_r^2\sigma_q^2}}+\frac{\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2}{\frac{1}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{1}{\frac{1}{N-m}\sigma_u^2+\left(\frac{N}{N-m}\frac{a_s}{a_r}\right)^2\sigma_s^2}+\frac{m}{\sigma_u^2}}.
\end{aligned}$$

$$\begin{aligned}
&\ln\left(\sigma_x^2+\beta^2\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta+\frac{1}{N-m}\sum_{i=N-m+1}^{N}u_i+\frac{N}{N-m}\frac{a_s}{a_r}\tilde{s},\tilde{g}+\delta+\frac{1}{m}\sum_{i=1}^{m}u_i\right)\right)\\
=&\ln\left(\sigma_x^2+\frac{\beta^2}{\frac{1}{\sigma_g^2}+\frac{1}{a_r^2\sigma_q^2}}+\frac{\beta^2\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2}{\frac{1}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{1}{\frac{1}{N-m}\sigma_u^2+\left(\frac{N}{N-m}\frac{a_s}{a_r}\right)^2\sigma_s^2}+\frac{1}{\frac{\sigma_u^2}{m}}}\right).
\end{aligned}$$

Then

$$\begin{aligned}
V\left(m\right)=&\frac{-1}{2A}\ln\left(\sigma_x^2+\frac{\beta^2}{\frac{1}{\sigma_g^2}+\frac{1}{a_r^2\sigma_q^2}}+\frac{\beta^2\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2}{\frac{1}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{1}{\frac{1}{N-m}\sigma_u^2+\left(\frac{N}{N-m}\frac{a_s}{a_r}\right)^2\sigma_s^2}+\frac{1}{\frac{\sigma_u^2}{m}}}\right)\\
=&\frac{-1}{2A}\ln\left(\sigma_x^2+\frac{\beta^2\sigma_g^2a_r^2\sigma_q^2}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{\beta^2\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2}{\frac{1}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{1}{\frac{1}{N-m}\sigma_u^2+\left(\frac{N}{N-m}\frac{a_s}{a_r}\right)^2\sigma_s^2}+\frac{1}{\frac{\sigma_u^2}{m}}}\right)\\
=&\frac{-1}{2A}\ln\left(\beta^2\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2\right)+\frac{-1}{2A}\ln\left(\frac{\sigma_x^2+\frac{\beta^2\sigma_g^2a_r^2\sigma_q^2}{\sigma_g^2+a_r^2\sigma_q^2}}{\beta^2\left(\frac{\sigma_g^2}{\sigma_g^2+a_r^2\sigma_q^2}\right)^2}+\frac{1}{\frac{1}{\sigma_g^2+a_r^2\sigma_q^2}+\frac{1}{\frac{1}{N-m}\sigma_u^2+\left(\frac{N}{N-m}\frac{a_s}{a_r}\right)^2\sigma_s^2}+\frac{1}{\frac{\sigma_u^2}{m}}}\right)
\end{aligned}$$

Let

$$A = \frac{\sigma_x^2 + \frac{\beta^2 \sigma_g^2 a_r^2 \sigma_q^2}{\sigma_g^2 + a_r^2 \sigma_q^2}}{\beta^2 \left(\frac{\sigma_g^2}{\sigma_g^2 + a_r^2 \sigma_q^2} \right)^2} > 0 \tag{A.213}$$

and

$$\begin{aligned} B(m) &= \frac{1}{\sigma_g^2 + a_r^2 \sigma_q^2} + \frac{1}{\frac{1}{N-m}\sigma_u^2 + \left(\frac{N}{N-m} \frac{a_s}{a_r} \right)^2 \sigma_s^2} + \frac{1}{\frac{\sigma_u^2}{m}} \\ &= \frac{1}{\sigma_g^2 + a_r^2 \sigma_q^2} + \frac{1}{\frac{1}{N-m}\sigma_u^2 + \left(\frac{N}{N-m} \frac{a_s}{a_r} \right)^2 \sigma_s^2} + \frac{m}{\sigma_u^2}. \end{aligned} \tag{A.214}$$

Then

$$V(m) = \frac{-1}{2A} \ln \left(\beta^2 \left(\frac{\sigma_g^2}{\sigma_g^2 + a_r^2 \sigma_q^2} \right)^2 \right) + \frac{-1}{2A} \ln \left(A + \frac{1}{B(m)} \right). \tag{A.215}$$

We consider the following derivative

$$\begin{aligned} \frac{dV(m)}{dm} &= \frac{1}{2A} \frac{1}{A + \frac{1}{B(m)}} \frac{1}{B^2(m)} \frac{dB(m)}{dm} \\ &= \frac{1}{2A} \frac{1}{AB(m) + B(m)} \frac{dB(m)}{dm} \\ &= \frac{1}{2A} \frac{1}{AB^2(m) + B(m)} \left(-\left(\frac{1}{N-m} \right)^2 \frac{\sigma_u^2 + 2\frac{N^2}{N-m} \left(\frac{a_s}{a_r} \right)^2 \sigma_s^2}{\left(\frac{1}{N-m}\sigma_u^2 + \left(\frac{N}{N-m} \frac{a_s}{a_r} \right)^2 \sigma_s^2 \right)^2} + \frac{1}{\sigma_u^2} \right) \\ &= \frac{1}{2A} \frac{1}{AB^2(m) + B(m)} \left(-\frac{\sigma_u^2 + 2\frac{N^2}{N-m} \left(\frac{a_s}{a_r} \right)^2 \sigma_s^2}{\left(\sigma_u^2 + \frac{N^2}{N-m} \left(\frac{a_s}{a_r} \right)^2 \sigma_s^2 \right)^2} + \frac{1}{\sigma_u^2} \right) \\ &= \frac{1}{2A} \frac{1}{AB^2(m) + B(m)} \left(-\frac{(N-m)\sigma_u^2 + 2N^2 \left(\frac{a_s}{a_r} \right)^2 \sigma_s^2}{\left(\sqrt{N-m}\sigma_u^2 + \frac{N^2}{\sqrt{N-m}} \left(\frac{a_s}{a_r} \right)^2 \sigma_s^2 \right)^2} + \frac{1}{\sigma_u^2} \right). \end{aligned}$$

For any N, since $\sigma_u \geqslant \underline{\sigma} > 0$, we must have

$$\lim_{m \to N} \frac{B(m)}{m} = \frac{1}{m} \frac{1}{\sigma_g^2 + a_r^2 \sigma_q^2} + \frac{1}{m} \frac{1}{\frac{1}{N-m}\sigma_u^2 + \left(\frac{N}{N-m} \frac{a_s}{a_r} \right)^2 \sigma_s^2} + \frac{1}{\sigma_u^2} = \frac{1}{\sigma_u^2}, \tag{A.216}$$

and

$$\lim_{m \to N} \left(AB^2(m) + B(m) \right) \frac{dV(m)}{dm} \to \frac{1}{2A} \frac{1}{\sigma_u^2}. \tag{A.217}$$

Note that the above results imply that when both N and m are sufficiently large, $B(m)$ is an increasing function of m, and $B(m) \to \infty$; $\frac{dV(m)}{dm}$ is a decreasing function of m and $\frac{dV(m)}{dm} \to 0$. Then there must exist $\underline{N}_1$, such that when $N > \underline{N}_1$,

$$\left. \frac{dV(m)}{dm} \right|_{m \to N} = \min_{m \in (1,N)} \frac{dV(m)}{dm}, \tag{A.218}$$

which implies

$$V(N) - V(N-1) = \min_{i \in \{2,3...N\}} (V(i) - V(i-1)). \tag{A.219}$$

$\square$

In this case, if all information seller charges

$$\Phi \leqslant \Phi^* = V(N) - V(N-1), \tag{A.220}$$

then all type-G investors would like to buy all scores.

Now let's turn to an arbitrary information seller's problem, WLOG, let's consider information seller N, and show that $\sigma_N = \underline{\sigma}$ and $\Phi_N = \Phi^*$ is indeed the best response if all other information sellers choose this strategy. When information seller N deviates from the above strategy, suppose that it chooses σ_N and Φ_N, and in equilibrium a fraction of λ_N type-G investors buy score N. We focus on symmetric off-equilibrium market after this deviation, such that all scores from information seller 1,2,...$N-1$ are sitll purchased by all type-G investors, as in the proposed equilibrium.[12] Since scores 1 to $N-1$ are purchased by all type-G investors, the sufficient statistics for scores 1 to $N-1$ is

$$G_r = \tilde{g} + \frac{1}{N-1} \sum_{i=1}^{N-1} u_i + \delta. \tag{A.221}$$

The public price has the following functional form

$$\begin{aligned} p &= a_0 + a_G G_r + a_r g_r^N + a_s \tilde{s} \\ &= a_0 + a_G \left(\tilde{g} + \frac{1}{N-1} \sum_{i=1}^{N-1} u_i + \delta \right) + a_r (\tilde{g} + u_i + \delta) + a_s \tilde{s}. \end{aligned} \tag{A.222}$$

Then the manipulation level is

$$\delta = (a_G + a_r) q. \tag{A.223}$$

For type-G investors who do not buy score N, the additional info contained in the price is

[12] So type-G investors are considering a constrained optimization problem: they decide whether to buy score N conditional on purchasing all scores 1 to $N-1$.

equivalent to

$$\frac{p - a_0 - a_G G_r}{a_r} = g_r^N + \frac{a_s}{a_r}\tilde{s}. \tag{A.224}$$

If a type-G investor does not buy the score, his info set is

$$\left\{G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right\}, \tag{A.225}$$

and if a type-G investor buys the score, his info set is

$$\left\{G_r, g_r^N\right\}. \tag{A.226}$$

According to the baseline model, the price of the info seller N's score is

$$\Phi_N = \frac{1}{2A}\ln\left(\frac{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right)}{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N\right)}\right). \tag{A.227}$$

And the info seller N's problem is

$$\max_{\lambda,\sigma_N} \lambda\frac{1}{2A}\ln\left(\frac{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right)}{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N\right)}\right), \tag{A.228}$$

where

$$\begin{aligned}\ln\frac{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right)}{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N\right)} &= \ln\left(1 + \frac{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) - \mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N\right)}{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N\right)}\right) \\ &= \ln\left(1 + \beta^2\frac{\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) - \mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right)}{\sigma_x^2 + \beta^2\mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right)}\right).\end{aligned} \tag{A.229}$$

When N is large, from the above analysis, Φ^* converges to zero, and both $\mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right)$ and $\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right)$ converge to zero. Formally speaking, $\lim_{N\to\infty}\mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right) = o(1)$ and $\lim_{N\to\infty}\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) = o(1)$. Then when $N \to \infty$,

$$\ln\left(1 + \beta^2\frac{\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) - \mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right)}{\mathrm{Var}\left(\tilde{x} + \beta\tilde{g}|G_r, g_r^N\right)}\right) \approx \beta^2\frac{\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) - \mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right)}{\sigma_x^2}. \tag{A.230}$$

Note that

$$\begin{aligned}\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) &= \mathbb{E}\left(\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta\right)|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) + \mathrm{Var}\left(\mathbb{E}\left(\tilde{g}|\tilde{g}+\delta\right)|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) \\ &= \frac{1}{\tau_g+\tau_\delta} + \frac{\tau_{(g+\delta)}^2}{\tau_g^2}\frac{1}{\tau_{(g+\delta)}+\tau_1+\tau_2}, \end{aligned} \tag{A.231}$$

where $\tau_g = \frac{1}{\sigma_g^2}$, $\tau_\delta = \frac{1}{(a_G+a_r)^2\sigma_q^2}$, $\tau_{(g+\delta)} = \frac{1}{\sigma_g^2+(a_G+a_r)^2\sigma_q^2}$, $\tau_1 = \frac{1}{\sigma_N^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}$ and $\tau_2 = \frac{1}{\frac{1}{(N-1)}\underline{\sigma}^2} = \frac{N-1}{\underline{\sigma}^2}$, and

$$\begin{aligned}\mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right) &= \mathbb{E}\left(\mathrm{Var}\left(\tilde{g}|\tilde{g}+\delta\right)|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) + \mathrm{Var}\left(\mathbb{E}\left(\tilde{g}|\tilde{g}+\delta\right)|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) \\ &= \frac{1}{\tau_g+\tau_\delta} + \frac{\tau_{(g+\delta)}^2}{\tau_g^2}\frac{1}{\tau_{(g+\delta)}+\tau_0+\tau_2} \end{aligned} \tag{A.232}$$

Where $\tau_0 = \frac{1}{\sigma_N^2}$. Then

$$\begin{aligned}\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) - \mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right) &= \frac{\tau_{(g+\delta)}^2}{\tau_g^2}\left(\frac{1}{\tau_{(g+\delta)}+\tau_1+\tau_2} - \frac{1}{\tau_{(g+\delta)}+\tau_0+\tau_2}\right) \\ &= \frac{\tau_{(g+\delta)}^2}{\tau_g^2}\frac{\tau_0-\tau_1}{\left(\tau_{(g+\delta)}+\tau_1+\tau_2\right)\left(\tau_{(g+\delta)}+\tau_0+\tau_2\right)} \end{aligned} \tag{A.233}$$

When N is sufficiently large, $\tau_2 >> \max\left\{\tau_1, \tau_0, \tau_{(g+\delta)}\right\}$, so

$$\begin{aligned}\mathrm{Var}\left(\tilde{g}|G_r, g_r^N + \frac{a_s}{a_r}\tilde{s}\right) - \mathrm{Var}\left(\tilde{g}|G_r, g_r^N\right) &= \frac{\tau_{(g+\delta)}^2}{\tau_g^2}\frac{\tau_0-\tau_1}{\left(\tau_{(g+\delta)}+\tau_1+\tau_2\right)\left(\tau_{(g+\delta)}+\tau_0+\tau_2\right)} \\ &\approx \frac{\tau_{(g+\delta)}^2}{\tau_g^2\tau_2^2}\left(\tau_0-\tau_1\right) \end{aligned} \tag{A.234}$$

Then the information seller N's optimization problem becomes

$$\max \lambda \frac{\beta^2}{\sigma_x^2}\frac{\tau_{(g+\delta)}^2}{\tau_g^2\tau_2^2}\left(\tau_0-\tau_1\right). \tag{A.235}$$

Since

$$\tau_0 - \tau_1 = \frac{1}{\sigma_N^2} - \frac{1}{\sigma_N^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2} = \frac{\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}{\left(\sigma_N^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2\right)\sigma_N^2} \tag{A.236}$$

Then

$$\lambda\frac{\beta^2}{\sigma_x^2}\frac{\tau_{(g+\delta)}^2}{\tau_g^2\tau_2^2}(\tau_0-\tau_1)=\lambda\frac{\beta^2}{\sigma_x^2}\frac{\tau_{(g+\delta)}^2}{\tau_g^2\tau_2^2}\frac{\left(\frac{a_s}{a_r}\right)^2\sigma_s^2}{\left(\sigma_N^2+\left(\frac{a_s}{a_r}\right)^2\sigma_s^2\right)\sigma_N^2}$$

$$=\lambda\frac{\beta^2}{\sigma_x^2}\frac{\tau_g^4\underline{\sigma}^4}{\left(\sigma_g^2+(a_G+a_r)^2\sigma_q^2\right)^2(N-1)^2}\frac{1}{\left(\left(\frac{a_r}{a_s}\right)^2\frac{\sigma_N^2}{\sigma_s^2}+1\right)\sigma_N^2}. \quad \text{(A.237)}$$

In this equilibrium, since the number of scores is very large, then any individual score will have little impact on the equilibrium price. But the impact of the aggregate supply $\tilde{s}$ on price should not be negligible, which implies $\lim_{N\to\infty}\frac{a_r}{a_s}=0$ and $\lim_{N\to\infty}\frac{d(a_G+a_r)}{d\sigma_N}=0$. Then it's obvious that to maximize

$$\lambda\frac{\beta^2}{\sigma_x^2}\frac{\tau_g^4\underline{\sigma}^4}{\left(\sigma_g^2+(a_G+a_r)^2\sigma_q^2\right)^2(N-1)^2}\frac{1}{\left(\left(\frac{a_r}{a_s}\right)^2\frac{\sigma_N^2}{\sigma_s^2}+1\right)\sigma_N^2}, \quad \text{(A.238)}$$

the information seller N should choose $\sigma_N=\underline{\sigma}$. Besides, since we also have $\lim_{N\to\infty}\frac{a_r}{a_s}=0$ and $\lim_{N\to\infty}\frac{d(a_G+a_r)}{d\lambda_N}=0$, the information seller N should also choose $\lambda_N=1$. Based on our previous analysis, Φ^* is the highest price that can guarantee $\lambda_N=1$ with $\sigma_N=\underline{\sigma}$, so the information seller N will choose $\Phi_N=\Phi^*$.

B.23 Proof of Lemma 5

The proof is included in the proof of Proposition 13.

B.24 Proof of Lemma 6

As shown in the proof of Proposition 13, when N is sufficiently high, all type-G borrowers buy all the scores, and all information sellers choose $\sigma_i=\underline{\sigma}$. In this case, the market equilibrium can be characterized by Proposition 1. The market equilibrium condition becomes

$$n-n\theta\frac{1}{\beta A\sigma_g^2}\frac{a_s}{a_r}=1+\frac{\underline{\sigma}^2}{N\sigma_g^2}+\left(\frac{1}{\frac{n}{\beta}-\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)}\right)^2\frac{\sigma_q^2}{\sigma_g^2}. \quad \text{(A.239)}$$

When N increases, it's clear that the equilibrium $\frac{-a_s}{a_r}$ will decrease. The equilibrium a_r is solved by

$$a_r=\frac{1}{\frac{n}{\beta}-\frac{1}{A}\left(\frac{1-\theta}{\sigma_x^2}+\frac{\theta}{\sigma_x^2+\beta^2\sigma_g^2}\right)\left(\frac{a_s}{a_r}\right)}. \quad \text{(A.240)}$$

It's clear that when $\frac{-a_s}{a_r}$ decreases, a_r will increase, and thus the variance $\text{Var}(\delta)=a_r^2\sigma_q^2$ will increase.

B.25 Proof of Proposition 14

Note that the equilibrium price is

$$\begin{aligned} p &= a_0 + a_r G_r + a_s \tilde{s} \\ &= a_0 + a_r \left(\tilde{g} + \frac{1}{N} \sum_{i=1}^{N} u_i + \delta + \frac{a_s}{a_r} \tilde{s} \right). \end{aligned}$$

Then the correlation $\mathrm{corr}(\tilde{g}, p)$ is determined by the variance $\sigma_g^2 + \frac{\underline{\sigma}^2}{N} + a_r^2 \sigma_q^2 + \left(\frac{a_s}{a_r}\right)^2 \sigma_s^2$. With the equilibrium conditions (A.239) and (A.240) in the proof of Lemma 6, we have

$$1 + \frac{\underline{\sigma}^2}{N\sigma_g^2} + a_r^2 \frac{\sigma_q^2}{\sigma_g^2} + \left(\frac{a_s}{a_r}\right)^2 \frac{\sigma_s^2}{\sigma_g^2} = n - n\theta \frac{1}{\beta A \sigma_g^2} \frac{a_s}{a_r} + \left(\frac{a_s}{a_r}\right)^2 \frac{\sigma_s^2}{\sigma_g^2}. \tag{A.241}$$

When N increases, we have shown in the proof of Lemma 6 that $\frac{-a_s}{a_r} > 0$ will decrease, so $n - n\theta \frac{1}{\beta A \sigma_g^2} \frac{a_s}{a_r} + \left(\frac{a_s}{a_r}\right)^2 \frac{\sigma_s^2}{\sigma_g^2}$ will also decrease, and thus the correlation $\mathrm{corr}(\tilde{g}, p)$ will increase.