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Abstract

We study how derivatives (with nonlinear payoffs) affect the underlying asset's liquidity. In a rational expectations equilibrium, informed investors expect low conditional volatility and sell derivatives to the others. These derivative trades affect different investors' utility differently, possibly amplifying liquidity risk. As investors delta hedge their derivative positions, price impact in the underlying drops, suggesting improved liquidity, because informed trading is diluted. In contrast, effects on price reversal are ambiguous, depending on investors' relative delta hedging sensitivity, i.e., the gamma of the derivatives. The model cautions of potential disconnections between illiquidity measures and liquidity risk premium due to derivatives trading.

Keywords: derivatives, options, liquidity risk premium, liquidity measure, price impact, price reversal

I Introduction

Investors often receive liquidity shocks, such as hedging needs and/or (time-sensitive) information. They then rush to trade to fulfill liquidity needs, leaving traces in the market: They generate (temporary) price pressure when hedging (Grossman and Miller (1988)) and (permanent) price impacts when speculating on private signals (Kyle (1985)). Anticipating such liquidity shocks, ex ante, investors require a certain premium when pricing assets. The literature has extensively studied the association between various illiquidity measures and such liquidity risk premium. See, e.g., reviews by Amihud, Mendelson, and Pedersen (2005) and Vayanos and Wang (2013) and the recent issue of *Critical Finance Review* (2019).

This paper examines market (il)liquidity and the associated risk premium from a novel angle: derivatives. Derivatives markets are huge. The Bank for International Settlements (BIS) reports that by the second quarter of 2021, the open interest of all exchange-traded options alone exceeded US$50 trillion. The outstanding notional amount of over-the-counter derivatives was close to US$600 trillion by 2020. It is natural to ask whether such significant derivatives activity affects its liquidity and risk premium; and, if so, how.

To this end, we develop a two-period rational expectations equilibrium (REE) model: There is one risky asset in an economy populated by ex ante homogeneous investors. Then a liquidity shock strikes a fraction of randomly selected investors, with two effects. First, a shocked investor will receive a future endowment correlated with the risky asset. Second, she also observes a private signal about the risky asset payoff. To hedge the endowment shock and to (timely) exploit the private information, the shocked investors demand liquidity to trade with the other unshocked investors, who serve as liquidity suppliers. The ex-ante (pre-shock) equilibrium asset price therefore commands two risk premia, one for the asset's fundamental risk and the other for such a liquidity (shock) risk. We then introduce volatility derivatives (e.g., variance swaps) and study both the post-shock market liquidity and the pre-shock liquidity risk premium.

Our first finding is that the liquidity demanders always write (or sell) the derivative to the liquidity suppliers. This is because the derivative is valued according to an investor's conditional expectation of the *nonlinear* payoff component. (The linear part of the payoff, always replicable by the underlying, is redundant.) In such a conditional expectation, the leading term is (approximately) the conditional volatility of the underlying. Since the demanders are more informed, thanks to their private signal, they always face lower conditional volatility—and hence value the derivative to be cheaper—than do the suppliers. As an extreme example, if a demander has a perfect signal, she knows exactly the underlying' payoff and expects zero volatility.

This result seems to defy the conventional wisdom that more informed investors tend to buy volatility derivatives. For example, knowing that a firm will engage in a lawsuit, an informed investor can profit from future price swings—whether the firm wins or loses—by buying a straddle. Smith (2019) captures this conventional intuition by letting some agents be informed of, and only of, the variance of the underlying payoff; that is, whether there is a lawsuit or not (but not the outcome of the lawsuit). Instead, in our framework, the informed demanders receive a noisy signal of the *exact* payoff: no lawsuit, winning, or losing. As such, our informed demanders always have a more precise posterior belief and sell volatility.

Our model shows that *with information asymmetry*, derivatives can serve as a channel for investors to "bet" on the underlying volatility, with the demanders always selling—like writing insurance—to the suppliers. Empirical evidence seems to support our prediction. For example, Gârleanu, Pedersen, and Poteshman (2009) document that non-market-makers (public customers and firm proprietary traders) mostly write equity options, which can be constructed as spreads and straddles for volatility bets, to market makers. To the extent that these market makers are relatively less informed, this evidence is consistent with our model prediction that more informed liquidity demanders sell volatility to suppliers. Additionally, Mixon and Onur (2014) use regulatory data and find that hedge funds—often considered as more informed—overall sell volatility as their net position is short in VIX futures.

Our second result is about how the derivative-induced volatility bets affect the underlying's trading. Specifically, we show analytically that investors adjust their underlying positions precisely to delta hedge their derivative holdings. This is because a derivative's payoff can always be thought of as a bundle of linear and nonlinear functions of the underlying payoff. An investor trades the derivative only to bet on the underlying's volatility (the nonlinear part) but in doing so, she also gains the exposure to the linear part, which appears as an inventory shock of the underlying. She then delta hedges such a shock by trading oppositely in the underlying's market.

Importantly, such a "delta hedging channel" can distort frequently used empirical measures of illiquidity, such as price impact and price reversal.[1] Price impact is defined as the regression coefficient of price returns on (informed investors') trading volume, à la Kyle (1985) and Amihud (2002). Because delta hedging trades are not information driven, the overall informative portion of the trading volume is smaller with than without derivatives. In other words, delta hedging trades reduce the information-to-noise ratio in the underlying's order flow. Therefore, such price impact-based measures would decrease with derivative trading, *suggesting improved market liquidity*.

Price reversal is defined as the negative return autocovariance, à la Roll (1984). The idea is that short-run price returns are negatively autocorrelated because the price mean-reverts to the long-run fundamental (Grossman and Miller (1988)). For example, an initial price concession due to strong selling pressure will eventually see a positive reversal, as the selling pressure dissipates over time. The larger this temporary deviation, the more negative is the autocovariance and the less liquid is the market. With derivatives, the suppliers' and the demanders' delta hedging trades always differ in sign (because the suppliers are always long and the demanders always short on the derivative). Thus, one group's delta hedging always contributes to such price reversal, while the other group dampens it. We show that they neutralize one another only in special cases, suggesting

[1] Empirical studies using "price impact" to measure illiquidity include, e.g., Brennan and Subrahmanyam (1996), Acharya and Pedersen (2005), Sadka (2006), and Collin-Dufresne and Fos (2015). Empirical studies using "price reversal" to measure illiquidity include, e.g., Campbell, Grossman, and Wang (1993), Llorente, Michaely, Saar, and Wange (2002), Pástor and Stambaugh (2003), and Hasbrouck (2009).

that derivatives can either exacerbate or attenuate price reversal.

Our third result is that derivatives can either exacerbate or alleviate the liquidity *risk* of the underlying asset. We show that while overall derivatives improve the post-shock trading gains (by allowing the volatility bets), the investors split the enlarged "pie" differently. A key parameter is the pervasiveness of the liquidity shock—how much of the population demand liquidity and how much supply it. For example, if there are very few suppliers, their per capita trading gain is huge, while the demanders split a negligibly small piece of the pie. From a pre-shock point of view, becoming a demander (receiving a liquidity shock), is a very serious risk because a demander will be *relatively* much worse off than a supplier. Anticipating such a utility wedge in the future, investors require a large ex-ante risk premium for the liquidity shock. That is, derivatives amplify liquidity risk and thus increase liquidity risk premium in this case. If the shock only affects a small population, the reasoning runs in the opposite direction, and the ex-ante liquidity risk premium declines.

With this result, our model helps reconcile the mixed empirical findings. Consider options, arguably the most common derivatives (with nonlinear payoffs), for example. Earlier works, like Branch and Finnerty (1981), Conrad (1989), and Detemple and Jorion (1990), document underlying price increases after option listings, while negative effects are shown in later works using more recent data, like Mayhew and Mihov (2000) and Danielsen and Sorescu (2001). Our model provides a novel insight to this time-series trend from the angle of the changing investor demographics.

In summary, our analysis highlights that derivatives can lead to conflicting interpretations of various illiquidity measures and attenuate their association with the liquidity risk premium. We prove the above findings analytically with a specific "variance swap" derivative and also study options to provide additional robustness to the results. The analyses of options also yield additional testable predictions. For example, we show that a key determinant of the illiquidity measures—price impact and price reversal—is the moneyness of the options: If an out-of-the-money call option is introduced, both liquidity demanders and suppliers know that the call is unlikely to be exercised, but the more informed demanders are *surer* of the low moneyness. As a result, the demanders' buy delta hedging turns less aggressive than the suppliers' sell delta hedging, and the two net to a selling pressure in the underlying. If the call option is in-the-money, the above effects reverse, creating a positive pressure.

Such moneyness induced price pressures further affect market liquidity measures. For example, we find that the price impact in the underlying asset is U-shaped in the option moneyness. This is because, as moneyness increases, the demanders' increased delta hedging buys have two effects. First, as they are not information driven, these buys reduce the information-to-noise ratio and lower price impact. Second, they generate buying pressure that adds to the price impact. The latter effect is negligible when the call is deep out-of-the-money (the demanders only buy little to delta hedge) but becomes dominant when the call is more in-the-money, hence the U-shaped price impact.

Contributions. Our paper primarily contributes to understanding the links between derivatives and the liquidity of the underlying assets. A large volume of the literature has studied the impact of derivatives. For example, the seminal work of Brennan and Cao (1996) studies the contemporaneous impacts of derivatives on the underlying; Cao (1999), Massa (2002), and Huang (2015) examine the effects on agents' information acquisition; and more recently, Smith (2019) studies variance derivatives and variance risk premium. Compared to these works, our model builds and analyzes the connection between derivatives to the underlying's liquidity risk premium. Our angle is similar to Vayanos and Wang (2012), who, however, do not study derivatives.

Chabakauri, Yuan, and Zachariadis (2017) share with our framework the feature of derivatives being "informationally irrelevant;" that is, introducing derivatives does not affect learning or inference about fundamentals. Gao and Wang (2017) study options' implications for volatility, and their framework also assumes this irrelevance. This feature distinguishes our mechanism from, e.g., Dow (1998), who shows that introducing new securities can worsen market liquidity due to exacerbated adverse selection. The migration of informed trading between the underlying and its derivatives, e.g., Biais and Hillion (1994) and Easley, O'Hara, and Srinivas (1998), is also muted in our model. Smith (2019) introduces information asymmetry about the asset payoff's variance. The learning channel, which is affected by the introduction of the variance derivative, plays a significant role in that framework. Compared to these works, our contribution lies in the novel, non-informational delta-hedging channel, which drive various market liquidity measures in possibly divergent directions, disconnecting with the underlying's ex-ante liquidity risk premium.

Our analysis of options contributes to the REE models with non-normal asset payoffs. See, for example, Barlevy and Veronesi (2003), Albagli, Hellwig, and Tsyvinski (2012), Breon-Drish (2015), Malamud (2015), Chabakauri et al. (2017), and Han (2018), among many others. When the REE models deviate from the standard CARA-normal assumption, the characterization of the equilibrium and its uniqueness become challenging. For example, Bernardo and Judd (2000) use numerical approaches to solve models with general distributions and preferences numerically and show that the REE in Grossman and Stiglitz (1980) is not robust to certain parametric assumptions. Breon-Drish (2015) obtains a general characterization of price in an economy where investors have CARA utilities and the payoff of the risky asset belongs to the exponential distribution family. Different from this strand of the literature, in our model, it is the option derivatives that create non-normality, while the underlying payoff is normally distributed. We prove the existence and the uniqueness of the equilibrium in this setup.

Structure of the paper. We first set up the model in Section II, where we also highlight a general information irrelevance result. The equilibrium is then analyzed backwardly. Section III characterizes the post-shock equilibrium and then examines the implications of derivatives on market (il)liquidity measures. Section IV turns to the pre-shock equilibrium and studies the impact of derivatives on the ex-ante liquidity risk premium. In each section, we compare three cases: the no-derivative benchmark, introducing a variance swap, and introducing options. We summarize our findings in Section V before concluding in Section VI.

II Model setup

We first set up the general model in Section II.A and then study agents' information and learning in Section II.B, where we highlight the information irrelevance of derivatives.

II.A Setup

Timeline. There are three dates: $t \in \{0, 1, 2\}$. At $t = 0$, homogeneous investors arrive and trade. Between $t = 0$ and $t = 1$, a liquidity shock strikes a subset of investors. Then all investors trade again at $t = 1$. At $t = 2$, payoffs realize and investors consume.

Assets. There is a risk-free consumption good in perfectly elastic supply serving as the numéraire. The risk-free rate is normalized to be zero. There is also a risky asset in supply of $\bar{X} > 0$ units, each paying off a random amount of D units of the consumption good at $t = 2$. The asset's price at time t is denoted by $\{P_t\}$.

In addition, there is a derivative maturing at $t = 2$, when its long side receives from the short side a payoff of $f(D)$. The payoff structure $f(D)$ is exogenous and the key restriction is that it does *not* depend on the underlying's intermediate price P_1 (but can depend on the initial price P_0 and, of course, the terminal price $P_2 = D$). In other words, we rule out path-dependent derivatives like lookback, barrier, or Asian options.[2] When there is no derivative, our framework reduces to that of Vayanos and Wang (2012), who share with us the same focus on market liquidity and liquidity risk but study the effects of information asymmetry and imperfect competition.

Investors. There is a continuum of investors of measure one, indexed by $i \in [0, 1]$. They derive constant absolute risk aversion (CARA) utility, with the same risk-aversion parameter α (> 0), over their $t = 2$ consumption. At $t = 0$, the investors are homogeneous, endowed with the per capita

[2] Such path-dependent derivatives create uninteresting "mechanical" (delta-)hedging motives. For example, suppose $f(D) = g(D) + P_1$. Then buying a unit of $f(D)$ gives exposures to both $g(D)$ and P_1. The exposure to P_1 is "redundant" as it can always be replicated by trading the underlying and can be offset by selling a unit of the underlying at $t = 1$—a mechanical hedging trade. The internet appendix S1 considers a tractable example of such path-*dependent* derivatives and examines the robustness of our result in view of mechanical (delta-)hedging motives.

supply of the risky asset.

Liquidity shock. A liquidity shock hits a fraction $\pi \in (0, 1)$ of the investors between $t = 0$ and $t = 1$. These shocked investors are referred to as "liquidity demanders" and the rest $(1 - \pi)$ as "liquidity suppliers." Specifically, the shocked want to trade the risky asset for two reasons: First, they each will receive an amount of $(D - \bar{D})z$ units of the consumption good at $t = 2$, where $\bar{D} := \mathbb{E}[D]$, and so they want to hedge this shock at $t = 1$. The realization of the shock z is their private information. The suppliers only know the distribution of z.

Second, the shocked also receive a private signal $s := D + \varepsilon$, where ε is some noise specified below. To make use of this signal timely, they therefore demand liquidity to buy or sell the risky asset at $t = 1$. For simplicity, we let the endowment shock z and the signal s affect the same group of investors (as in, e.g., Biais, Bossaerts, and Spatt (2010) and Vayanos and Wang (2012)).

Trading. Investors submit demand schedules to maximize their expected utility from consumption at $t = 2$. We introduce the following notations:

- Prices: $\{P_t\}$ for the risky asset and $\{Q_t\}$ for the derivative, where $t \in \{0, 1, 2\}$.
- Demand schedules at $t = 0$: $X_0(\cdot)$ for the risky asset and $Y_0(\cdot)$ for the derivative.
- The liquidity demanders at $t = 1$: $X_{1d}(\cdot)$ for the risky asset and $Y_{1d}(\cdot)$ for the derivative.
- The liquidity suppliers at $t = 1$: $X_{1s}(\cdot)$ for the risky asset and $Y_{1s}(\cdot)$ for the derivative.

The demand schedules are functions of the assets' market clearing prices, conditional on all (other) assets' current and past prices and the investors' private information and endowment shocks (if any). Omitting these arguments for brevity, the market clearing conditions are:

$$X_0(\cdot) = \bar{X} \quad \text{and} \quad \pi X_{1d}(\cdot) + (1 - \pi) X_{1s}(\cdot) = \bar{X}; \tag{1}$$

$$Y_0(\cdot) = 0 \quad \text{and} \quad \pi Y_{1d}(\cdot) + (1 - \pi) Y_{1s}(\cdot) = 0. \tag{2}$$

Note that the demanders' endowment shock $(D - \bar{D})z$ does not materialize until $t = 2$. These market clearing conditions then pin down the equilibrium asset prices.

Equilibrium. There are four prices (two assets, two trading rounds) and six demand schedules in total (two assets, homogeneous investors at $t = 0$ and demanders vs. suppliers at $t = 1$). To fully characterize the equilibrium, we would need to solve for all these ten endogenous objects. Some shortcuts can be taken: We will not explicitly derive the functional form of $X_0(\cdot)$ or $Y_0(\cdot)$, as in equilibrium market clearing implies that, at $t = 0$, the *homogeneous* investors must hold the per capita supply, i.e., $X_0(\cdot) = \bar{X}$ and $Y_0(\cdot) = 0$. Further, since there is no demand for the derivative at $t = 0$ ($Y_0 = 0$), we will not study its price Q_0 either. As such, we will only focus on heterogeneous investors' trading at $t = 1$, i.e., the demand schedules $\{X_{1d}(\cdot), Y_{1d}(\cdot), X_{1s}(\cdot), Y_{1s}(\cdot)\}$, as well as the three asset prices $\{P_0, P_1, Q_1\}$. (Clearly, upon liquidation at $t = 2$, $P_2 = D$ and $Q_2 = f(D)$.)

Parameters and distributions. There are three fundamental random variables in this economy, $\{D, z, \varepsilon\}$, which are jointly normal and pairwise independent, with means $\{\bar{D}, 0, 0\}$ and variances $\{G_0^{-1}, \tau_z^{-1}, \tau_\varepsilon^{-1}\}$. To ensure bounded utility (Vayanos and Wang (2012)), we assume that

$$\alpha^2 G_0^{-1} \tau_z^{-1} < 1. \tag{3}$$

Intuitively, without this cap, the endowment shock z might be too severe (in ex-ante expectation).

II.B Information and learning

Before diving into the equilibrium analysis, we first characterize the agents' learning post-shock at $t = 1$. Given the private signal s, the informed demanders have that D is conditionally normal with

$$\operatorname{var}_{1d}[D]^{-1} = \operatorname{var}[D|\, s]^{-1} = G_0 + \tau_\varepsilon := G_{1d} \text{ and } \mathbb{E}_{1d}[D] = \mathbb{E}[D|\, s] = \frac{1}{G_{1d}}\left(G_0 \bar{D} + \tau_\varepsilon s\right). \tag{4}$$

Unsurprisingly, this posterior is unaffected by $f(D)$. Below we study the learning by the suppliers, first without and then with the derivative $f(D)$.

The suppliers' learning, without the derivative. Without the derivative, only the underlying asset is traded and the suppliers obtain a noisy signal of the demanders' private signal s from the asset's equilibrium price P_1^{nd} (the superscript "nd" emphasizes that there is no derivative):

$$\eta := s - \frac{\alpha}{\tau_\varepsilon} z.$$

This is a standard result and we defer the formal proof to that of the more general Lemma 1 below. Therefore, conditional on η, the suppliers obtain

$$\text{(5)} \qquad \mathrm{var}_{1s}^{nd}[D]^{-1} = \mathrm{var}[D|\,\eta\,]^{-1} = G_0 + \frac{\tau_\varepsilon^2 \tau_z}{\tau_\varepsilon \tau_z + \alpha^2} := G_{1s} \text{ and } \mathbb{E}_{1s}^{nd}[D] = \frac{G_0}{G_{1s}} \bar{D} + \left(1 - \frac{G_0}{G_{1s}}\right)\eta.$$

The suppliers' learning, with the derivative. We now reintroduce a derivative with some generic payoff $f(D)$. In addition to the underlying price P_1, the suppliers now also observe the derivative price Q_1. Does the new information set $\{P_1, Q_1\}$ affect the suppliers' learning about D? No.

Lemma 1 (Information irrelevance of the derivative). *Fix the realizations of the fundamental random variables $\{D, \varepsilon, z\}$. Suppose that there is an equilibrium with the derivative of $f(D)$. In this equilibrium, the suppliers' posterior distribution of $\{D, \varepsilon, z\}$ conditional on $\{P_1, Q_1\}$ is the same as the posterior in the no-derivative equilibrium conditional only on $\{P_1^{nd}\}$. That is, $D|_{\{P_1,Q_1\}}$ is normally distributed with $\mathbb{E}_{1s}[D] = \mathbb{E}_{1s}^{nd}[D]$ and $\mathrm{var}_{1s}[D] = \mathrm{var}_{1s}^{nd}[D]$ as given in (5).*

Learning from the price $\{P_1, Q_1\}$ is equivalent to learning from the quantities $\{X_{1d}, Y_{1d}\}$, via the market clearing conditions (1) and (2). While there seem to be two equations and two unknowns (s and z), as the proof of Lemma 1 shows, the way s and z enter X_{1d} is always exactly the same as they enter Y_{1d}—in the form of $\eta = s - \alpha z/\tau_\varepsilon$. As such, the suppliers always find it equivalent to learn from either X_{1d} or Y_{1d} and there is no additional information from the derivative's trading.

This result is not unique to our model. Similar features are also seen in, among many other contributions, Brennan and Cao (1996), Cao (1999), Huang (2015), Chabakauri et al. (2017), and Gao and Wang (2017). In particular, Chabakauri et al. (2017) coin the term "informational irrelevance" and study the conditions for such irrelevance to hold. The key assumption that ensures such irrelevance is that there are only one informed type in the economy. The uninformed, therefore, can always and only infer a jammed signal from that informed type's demand schedule.

More generally, if there are more informed types, each will reveal a differently jammed signal to the uninformed and in such cases introducing derivatives might help the uninformed to learn more.

We do note that such informational irrelevance ignores certain aspects of real-world trading. For example, it is well known that options can be used to bet on information (Biais and Hillion (1994), Easley et al. (1998)). In addition, when the new asset's payoff itself is informative of the existing asset's payoff, as in Dow (1998), investors' learning will be affected. Smith (2019) finds private information about a stock's risk generates the variance risk premium. these effects, Lemma 1 highlights the novelty of the channels in our model—they are *not* about investors' learning, thus differentiating our contribution from the above literature.

III The after-shock equilibrium at $t = 1$

We first derive the equilibrium in Section III.A and then study the implications for market illiquidity measures—price impact and price reversal—in Section III.B. Three cases are compared:

- no derivative as a benchmark (Section III.A.1 and III.B.1);
- a variance swap to articulate the novel channel of delta hedging (Section III.A.2 and III.B.2); and
- a call option to demonstrate robustness and generality (Section III.A.3 and III.B.3).

III.A Equilibrium characterization

III.A.1 The no-derivative benchmark

As a benchmark, we first switch off the derivative by setting $f(D) = 0$. The model essentially degenerates to Section 3 of Vayanos and Wang (2012).

Note that given $f(D) = 0$, the derivative price is trivially $Q_1 = 0$ and its demand $\{Y_{1d}, Y_{1s}\}$ are undefined. We thus only focus on the demand and price for the underlying: $\{X_{1d}^{nd}, X_{1s}^{nd}, P_1^{nd}\}$. At $t = 1$, the liquidity demanders have the same information set of $\{P_0^{nd}, P_1^{nd}, s, z\}$, while all suppliers only observe $\{P_0^{nd}, P_1^{nd}\}$. Standard analysis gives the CARA investors' demand schedules

$$X_{1d}^{nd} = \frac{\mathbb{E}_{1d}^{nd}[D] - P_1^{nd}}{\alpha \text{var}_{1d}^{nd}[D]} - z \text{ and } X_{1s}^{nd} = \frac{\mathbb{E}_{1s}^{nd}[D] - P_1^{nd}}{\alpha \text{var}_{1s}^{nd}[D]},$$

where $\mathbb{E}_{1j}^{nd}[\cdot]$ and $\text{var}_{1j}^{nd}[\cdot]$ have been given in Equations (4) and (5). Through the market clearing condition (1), we solve for the equilibrium and summarize it in the following proposition:

Proposition 1 (Benchmark equilibrium at $t = 1$**).** *At* $t = 1$*, there is a unique equilibrium. The liquidity demanders' demand schedule is*

$$X_{1d}^{nd}(p; s, z) = \frac{G_{1d}}{\alpha}\left[\left(\frac{G_0}{G_{1d}}\bar{D} + \frac{G_{1d} - G_0}{G_{1d}}s\right) - p\right] - z; \tag{6}$$

the liquidity suppliers' demand schedule is

$$X_{1s}^{nd}(p) = \frac{G_{1s}}{\alpha}\left[\left(\frac{G_0}{G_{1s}}\bar{D} + \frac{G_{1s} - G_0}{G_{1s}} \cdot \frac{G_1 p - G_0\bar{D} + \alpha\bar{X}}{G_1 - G_0}\right) - p\right];$$

and the market clears at

$$P_1^{nd} = \left(\frac{G_0}{G_1}\bar{D} + \frac{G_1 - G_0}{G_1}\eta\right) - \frac{\alpha}{G_1}\bar{X}, \tag{7}$$

where G_{1d} *and* G_{1s} *are a demanders' and a suppliers' respective posterior precision, as given in Equations* (4) *and* (5)*;* $G_1 := \pi G_{1d} + (1-\pi)G_{1s}$ *is an average investor's precision; and* $\eta := s - \frac{\alpha}{\tau_\varepsilon}z$ *is the signal about* D *learned by the market. Note that* $G_0 < G_{1s} < G_1 < G_{1d}$ *because the suppliers only imperfectly infer* s *from* P_1^{nd}*.*

We briefly explain the interpretation of the equilibrium. A liquidity demander trades on the difference between her private valuation (a weighted average between the unconditional mean $\bar{D}$ and the private signal s) and the market price P_1^{nd}. Her trading aggressiveness, G_{1d}/α, on this difference is increasing in her precision G_{1d} and decreasing in her risk aversion α. She also offloads the endowment shock z (hedging). The same interpretation holds for a liquidity supplier, except that 1) she does not see the private signal s but infers it from P_1^{nd}, and 2) she does not have hedging needs. The market clearing price P_1^{nd} is a weighted average between the expected payoff $\bar{D}$ and the signal η, and is further adjusted for a risk premium of $\alpha\bar{X}/G_1$.

III.A.2 Variance swap

We now introduce a "variance swap" derivative, which was written at $t = 0$ and will pay the realized variance of the underlying, i.e., $f(D) = (D - P_0)^2$, at $t = 2$. The purpose is to present our novel economic forces in the most tractable and transparent way. The results and intuition generalize to other nonlinear $f(D)$ as later shown with options.[3]

Consider an investor of type j at $t = 1$, where $j = d$ for a liquidity demander and $j = s$ for a supplier. Her terminal wealth is

$$W_{2j} = W_0 + (P_1 - P_0)X_0 + (D - P_1)X_{1j} + (f(D) - Q_1)Y_{1j} + (D - \bar{D})z_j, \tag{8}$$

where z_j is her type-specific endowment shock, with $z_d = z$, $z_s = 0$, and $f(D) = (D - P_0)^2$. She chooses her demand X_{1j} and Y_{1j} to maximize her conditional expected utility, $\mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right]$, taking the prices $\{P_1 = p, Q_1 = q\}$ as given. For the demanders, their X_{1d} and Y_{1d} can also depend on the private signal s and endowment shock z.

Lemma 1 provides the conditional distribution of D for both the demanders and suppliers. The optimization problems can then be evaluated in closed form, yielding the following proposition.

Proposition 2 (Equilibrium at $t = 1$ with variance swap). *There exists a unique equilibrium at $t = 1$. The demand schedules for the underlying are*

$$X_{1d}(p, q; s, z) = X^{nd}_{1d}(p; s, z) - 2(p - P_0)Y_{1d}(p, q; s, z); \text{ and } X_{1s}(p, q) = X^{nd}_{1s}(p) - 2(p - P_0)Y_{1s}(p, q).$$

The demand schedules for the variance swap are

$$Y_{1d}(p, q; s, z) = \frac{1}{2\alpha}\left(\left(q - (p - P_0)^2\right)^{-1} - G_{1d}\right); \text{ and } Y_{1s}(p, q) = \frac{1}{2\alpha}\left(\left(q - (p - P_0)^2\right)^{-1} - G_{1s}\right).$$

The underlying's market clears at $P_1 = P^{nd}_1$, the same as in the benchmark (Equation 7). The

[3] In general, any derivative payoff $f(D)$ can be decomposed into a linear and a nonlinear component. Cao (1999) shows that the linear component is redundant in such frameworks—it is simply a combination of the numéraire and the underlying. We therefore focus on such a variance swap, which is arguably the simplest nonlinear derivative. In the proof for Proposition 2 , we consider a more general quadratic derivative and demonstrate that all results presented here with $f(D) = (D - P_0)^2$ are robust.

derivative's market clears at $Q_1 = (P_1 - P_0)^2 + G_1^{-1}$. The conditional precision $\{G_{1d}, G_{1s}, G_1\}$ are the same as those defined in Proposition 1.

The formal proof is deferred to the appendix. We discuss below the equilibrium insights.

(1) Betting on "volatility." We begin by investigating investors' derivative demand $Y_{1j}(p,q)$—why they trade the derivative. With some rearrangement, it can be seen that

$$Y_{1j} \propto \left(G_{1j}^{-1} + (P_1 - P_0)^2 - Q_1\right) = \left(G_{1j}^{-1} - G_1^{-1}\right),$$

where the equality follows the equilibrium derivative price $Q_1 = (P_1 - P_0)^2 + G_1^{-1}$. Recall from Proposition 1 that G_1^{-1} is the market's average conditional variance of the underlying return $D - P_0$. Therefore, we see that each investor is trading on the difference between her and the market's valuations of the variance of $D - P_0$, as if she engages in a "volatility bet" against the market average. Such a bet is similar to the underlying trading in the benchmark: after adjusting for the endowment shock z_j, one's demand for the underlying is $X_{1j}^{nd} + z_j \propto \left(\mathbb{E}_{1j}[D] - P_1\right)$; that is, everyone bets her valuation of D, $\mathbb{E}_{1j}[D]$, against the market average P_1.

(2) The (more informed) demanders write the derivative to the suppliers. In equilibrium,

$$Y_{1d} = -\frac{G_{1d} - G_1}{2\alpha} < 0 < Y_{1s} = \frac{G_1 - G_{1s}}{2\alpha},$$

because $G_{1s} < G_1 < G_{1d}$. That is, a liquidity demander always takes a short position in the derivative, betting that the volatility is low, while a supplier always takes a long position, betting on high volatility. Intuitively, this is because in our framework, the demanders' signal always points to a more precise posterior, hence lower posterior volatility.

To compare, in Smith (2019), an agent informed of the underlying's volatility might either buy or sell such volatility derivatives. This is because of the different payoff and information structure assumed: In Smith (2019), the underlying's payoff D is decomposed into a mean component $\tilde{\mu}$ and a variance component $\tilde{V}$, for example, as $D = \tilde{\mu} + \tilde{V}v$, where v follows a standard normal distribution, so that $D|\tilde{V}$ is normally distributed. An investor informed of $\tilde{V}$, therefore, has a better understanding only about the *scale* of the payoff D, unlike in the current paper where the informed have a "pointy" posterior of $D|s$. With a high (low) posterior of the scale, the investors in Smith (2019) then buy (sell) volatility derivatives.

Our model reveals that, investors more informed *of the exact payoff* tend to sell "volatility," as if selling protection or insurance, to the less informed investors. For example, Gârleanu et al. (2009) document that non-market-markers, including public customers and firm proprietary trader, write equity options to the arguably less informed market makers. Since options can be used to replicate volatility (e.g., through strangles or straddles, etc.), this finding is consistent with our prediction here. In addition, Mixon and Onur (2014) document from regulatory data that hedge funds—often considered as informed—have an overall net short position in VIX futures. That is, they tend to sell volatility to relatively less informed investors.

(3) Trading in the underlying is affected: delta hedging. Proposition 2 shows that investors' trading in the underlying is affected by the derivative. In particular, compared to the benchmark, there is a new term of $-2(p - P_0)Y_{1j}$. That is, for every unit of the derivative, the investor trades *against* it by $2(p - P_0)$ units of the underlying, where $p = P_1$ is the price of the underlying.

This new term turns out to be the investor's delta hedging and it is the main economic force driving the subsequent results about illiquidity measures. Write a type-j investor's terminal utility as $u(W_{2j})$, where W_{2j} is a function of the underlying payoff D (see Equation (8)). Therefore, her expected exposure to a small fluctuation in D is

$$\begin{aligned}\mathbb{E}_{1j}\left[\frac{\partial u(W_{2j})}{\partial D}\right] = \mathbb{E}_{1j}\left[u^{'}(W_{2j})\frac{\partial W_{2j}}{\partial D}\right] &= \mathbb{E}_{1j}\left[u^{'}(W_{2j})\Big(X_{1j} + z_j + Y_{1j}f^{'}(D)\Big)\right] \\ &= \mathbb{E}_{1j}\left[u^{'}(W_{2j})\right]\Big(X_{1j} + z_j + Y_{1j}\hat{\mathbb{E}}_{1j}\left[f^{'}(D)\right]\Big),\end{aligned} \tag{9}$$

where we write[4]

$$\hat{\mathbb{E}}_{1j}\left[f'(D)\right] := \mathbb{E}_{1j}\left[\frac{u'(W_{2j})}{\mathbb{E}_{1j}\left[u'(W_{2j})\right]}f'(D)\right]. \tag{10}$$

That is, measured in multiples of the expected marginal utility $\mathbb{E}_{1j}\left[u'(W_{2j})\right]$, the investor expects a total exposure to fluctuations in D of

$$X_{1j} + z_j + Y_{1j}\hat{\mathbb{E}}_{1j}\left[f'(D)\right].$$

In words, she holds X_{1j} units of the underlying and will receive an endowment of z_j, and her Y_{1j} units of the derivative have a per-unit exposure of $\hat{\mathbb{E}}_{1j}\left[f'(D)\right]$, which is precisely the definition of the "delta" hedging ratio of the derivative.

Applied to our current example of variance swaps, the hedging ratio is $\Delta_{1j} := \hat{\mathbb{E}}_{1j}[f'(D)] = \hat{\mathbb{E}}_{1j}[2(D - P_0)] = 2(P_1 - P_0)$, and an investor's delta hedging trade is

$$-\Delta_{1j}Y_{1j}(P_1, Q_1) = -2(P_1 - P_0)Y_{1j}(P_1, Q_1) = X_{1j}(P_1, Q_1) - X^{nd}_{1j}(P_1). \tag{11}$$

Intuitively, as the investors bet on the volatility through the derivative, they also receive extra exposure to the underlying through the derivative position Y_{1j}. This exposes them to "too much" D, and they therefore trade against Y_{1j} to neutralize the exposure. Note that the above analysis is generic for any distribution of D, any utility function (subject to standard regularity), and any (piecewise differentiable) derivative payoff $f(\cdot)$.

The expression shown in Equation (11) connects an investor's demand with and without the derivative to her delta hedging trade. This is, in fact, a robust result for any type of derivatives $f(D)$:

[4] Note that the ratio $\Lambda_{1j} := \frac{u'(W_{2j})}{\mathbb{E}_{1j}\left[u'(W_{2j})\right]}$ is a function of D, is strictly positive, and satisfies $\mathbb{E}_{1j}[\Lambda_{1j}] = 1$. It therefore serves as a Radon-Nikodym derivative and changes the expectation $\mathbb{E}_{1j}[\cdot]$ to a risk-neutral pricing measure $\hat{\mathbb{E}}_{1j}[\cdot]$. For example, the first-order condition regarding the underlying holding X_{1j} is

$$\mathbb{E}_{1j}\left[u'(W_{2j})\left(\frac{\partial W_{2j}}{\partial X_{1j}}\right)\right] = \mathbb{E}_{1j}\left[u'(W_{2j})(D - P_1)\right] = 0 \implies \mathbb{E}_{1j}\left[u'(W_{2j})D\right] = \mathbb{E}_{1j}\left[u'(W_{2j})\right]P_1.$$

Dividing both sides by the expected marginal utility $\mathbb{E}_{1j}\left[u'(W_{2j})\right]$ yields the risk-neutral pricing formula $P_1 = \hat{\mathbb{E}}_{1j}[D]$.

Proposition 3 (Net exposure to the underlying). *Given the asset prices p_1 and q_1, a type-j investor's net exposure to the underlying asset is the same, with or without the derivative:*

$$X_{1j}(p_1, q_1) + \Delta_{1j}(p_1, q_1) Y_{1j}(p_1, q_1) = X_{1j}^{nd}(p_1). \tag{12}$$

Below in Section III.A.3 we will show that this also holds true for options.

(4) A knife-edge result: the underlying price is unaffected, i.e., $P_1 = P_1^{nd}$. While all investors delta hedge their derivative positions, the net delta hedging trade (across all investors) is zero:

$$\pi \cdot \left(X_{1d} - X_{1d}^{nd}\right) + (1-\pi)\left(X_{1s} - X_{1s}^{nd}\right) = -2(P_1 - P_0) \cdot \underbrace{(\pi Y_{1d} + (1-\pi) Y_{1s})}_{\text{=0, by the derivative's market clearing}} = 0. \tag{13}$$

As such, the net demand for the underlying asset remains exactly the same as in the benchmark. In other words, there is no price pressure pushing P_1 away from the benchmark P_1^{nd}.

However, we would like to emphasize that this is a *knife-edge result* due to the specific derivative payoff $f(D) = (D - P_0)^2$. In particular, both the demanders and the suppliers have the same delta hedging ratio of $\Delta_{1d} = \Delta_{1s} = 2(P_1 - P_0)$. More generally, however, different investors' hedging ratios are not always the same, and the net delta hedging will be nonzero, pushing the equilibrium P_1 away from P_1^{nd}. Such additional price pressure has asset pricing implications and, in particular, affects empirical measures of illiquidity. Section III.A.3 below shows such examples.

III.A.3 Options

We now turn to options, the most common (nonlinear) derivatives. Specifically, there are n ($< \infty$) call options written at $t = 0$, each with strike price K_i, $i \in \{1, 2, ..., n\}$. (The results from this section generalize to arbitrary combinations of calls or puts, thanks to the put-call parity.) That is, the i-th call pays $\max\{0, D - K_i\}$ at $t = 2$. While the nonlinear payoffs from the options add to the complication of the analysis, we show that there exists a unique equilibrium characterized by a set of nonlinear first-order conditions:

Proposition 4 (Equilibrium with options). *At $t = 1$, a type-j investor's optimal demand for the asset and the options exists and is uniquely characterized by the unique solution to Equation* (A.6). *The asset and the option prices are the unique solutions of market clearing conditions of Equation* (A.7).

The equilibrium characterization contributes to the literature of REE models with non-normal payoffs. In particular, in our framework, it is the option derivatives that necessarily introduce the non-normality in our model (even though the underlying is still normally distributed).

Below we proceed to discuss the properties of the equilibrium, highlighting what inherits from the variance swap in (1)–(3) as well as what differs in (4). For clarity, the discussion focuses on the case of a single call option (i.e., $n = 1$) and we shall drop the subscript i.[5]

(1) Betting on volatility. The nonlinearity in the call payoff $f(D) = \max\{0, D - K\}$ provides investors with a vehicle to trade on the underlying volatility. To see this, suppose that the call is (weakly) out-of-the-money at $t = 1$, i.e., $K \geq P_1$. We can then decompose its payoff as (see Lemma S1 in the internet appendix)

$$\max\{0, D - K\} = \frac{1}{2}|D - P_1| + \frac{1}{2}\big(1 - 2\mathbb{1}_{\{P_1 \leq D \leq K\}}\big)(D - P_1) + \mathbb{1}_{\{D > K\}}(P_1 - K). \tag{14}$$

Note that in expectation, the leading term, $\frac{1}{2}|D - P_1|$, is exactly (half of) the underlying asset's volatility. The same intuition as for the variance swap applies: Because the more informed demanders value volatility less than do the suppliers, the option is traded to "bet on volatility."

(2) The (more informed) demanders write options to the suppliers. Figure 1(a) plots an investor's (risk-neutral) valuation of the asset's volatility against the option's moneyness, defined as $P_1 - K$: The call is in-the-money (ITM) when $P_1 - K > 0$, at-the-money (ATM) when $P_1 - K = 0$, and out-of-the-money (OTM) when $P_1 - K < 0$. Indeed, a demander always believes that the conditional volatility is lower than does a supplier. As a result, the demanders always sell the option to the

[5] As will be shown below, our novel insights are largely based on the moneyness of the single option. When there are multiple options, our results then speak to their average moneyness. In reality, the average moneyness varies across assets. The model implications, therefore, can be empirically examined in the cross-section of different assets.

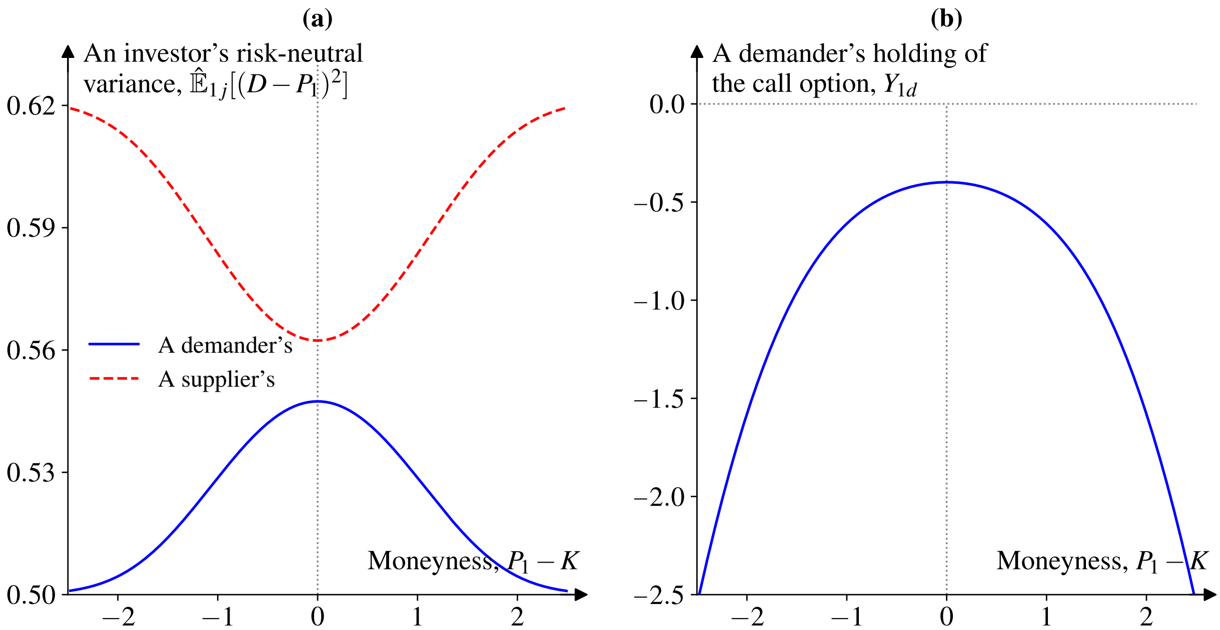

Figure 1: Call option trading at $t = 1$. This figure describes the equilibrium trading of the call option at $t = 1$. Panel (a) plots investors' (risk-neutral) valuation of the underlying volatility in equilibrium. Panel (b) plots a demander's option position. In both panels, the horizontal axes show a range of moneyness of the call option, defined as the difference between the underlying price and the strike price. The call's strike is fixed at $K = -1.0$ and the two state variables—the private signal and the endowment shock—vary to affect the equilibrium price P_1 and the moneyness. The other primitive parameters are set at $\bar{D} = 0.0$, $\bar{X} = 0.8$, $G_0 = 1.0$, $\tau_\varepsilon = 1.0$, $\tau_z = 1.0$, $\pi = 0.5$, and $\alpha = 0.8$.

suppliers, i.e., $Y_{1d} < 0 < Y_{1s}$ as seen in Panel (b). Note that only a demander's equilibrium option holding is plotted in (b) because by market clearing, a suppliers' holding immediately follows to be positive: $Y_{1s} = -\pi Y_{1d}/(1-\pi) > 0$. Comparing both panels, we see that the demanders write more calls to the suppliers (more negative Y_{1d}) when their volatility valuations differ more, i.e., when the call is further away from ATM.

(3) Delta hedging. We next turn to the investors' delta hedging. A type-$j \in \{s, d\}$ investor in equilibrium holds Y_{1j} units of the call, which also gives her additional underlying exposure. Following Equation (10), the extra exposure, per unit of the option position, is

$$\Delta_{1j} := \hat{\mathbb{E}}_{1j}\left[f'(D)\right] = \hat{\mathbb{E}}_{1j}\left[\frac{\partial}{\partial D}\max\{0, D-K\}\right] = \hat{\mathbb{E}}_{1j}\left[\mathbb{1}_{\{D>K\}}\right] = \hat{\mathbb{P}}_{1j}[D > K],$$

which has the usual interpretation of an option's delta: the probability of ending up in-the-money. Therefore, the investor has an incentive to delta hedge against such extra exposure by trading $-\Delta_{1j}Y_{1j}$ units of the underlying, just as in the case of a variance swap. In fact, as shown in Proposition 3, one's delta hedging trade $-\Delta_{1j}Y_{1j}$ is also the only deviation from her no-derivative benchmark: $X_{1j} = X_{1j}^{nd} - \Delta_{1j}Y_{1j}$.

(4) Non-zero net delta hedging trade. Different from the case of a variance swap, aggregating across all investors, the net delta hedging trade

$$-\pi\Delta_{1d}Y_{1d} - (1-\pi)\Delta_{1s}Y_{1s} \tag{15}$$

is in general *not* zero. This is because the delta hedging ratios are no longer the same, $\Delta_{1d} \neq \Delta_{1s}$. To see why, note that $\Delta_{1j} = \hat{\mathbb{P}}_{1j}[D > K]$ depends on the respective information set of the type-j investors. Instead, in the case of the variance swap $f(D) = (D - P_0)^2$, $\Delta_{1j} = \hat{\mathbb{E}}_{1j}[2(D - P_0)] = 2(P_1 - P_0)$ is common for both $j \in \{d, s\}$—a knife-edge special case. Below we provide more discussions to better understand why—and when—$\Delta_{1d} \lessgtr \Delta_{1s}$.

(4.1) $\Delta_{1d} > (<)\ \Delta_{1s}$ **when the call is ITM (OTM).** Figure 2(a) shows the equilibrium call option price Q_1. It has the well-known property of monotonically increasing in the moneyness, with an asymptotic slope equal to zero (one) when the call is extremely OTM (ITM). Figure 2(b) plots the two types of investors' delta hedging ratios. While at first glance the two Δs seem to overlap, they differ by a small amount. We zoom in on their difference $\Delta_{1d} - \Delta_{1s}$ in Figure 2(c), plotting it in the solid line (left axis). It can be seen that $\Delta_{1d} > \Delta_{1s}$ if and only if the call is ITM.

The intuition is as follows. As explained before, a demander always sells the call (betting on low volatility). When the call is ITM, she knows that she will likely receive a negative shock in the underlying by $t = 2$ because the call will likely be exercised. To hedge this expected negative inventory shock, the demander takes a long delta hedge position $-\Delta_{1d}Y_{1d} > 0$, and likewise, a supplier takes $-\Delta_{1s}Y_{1s} < 0$. The difference is that the more informed demanders are "surer" of the moneyness of the call than are the suppliers. Therefore, the demanders' delta hedging ratio

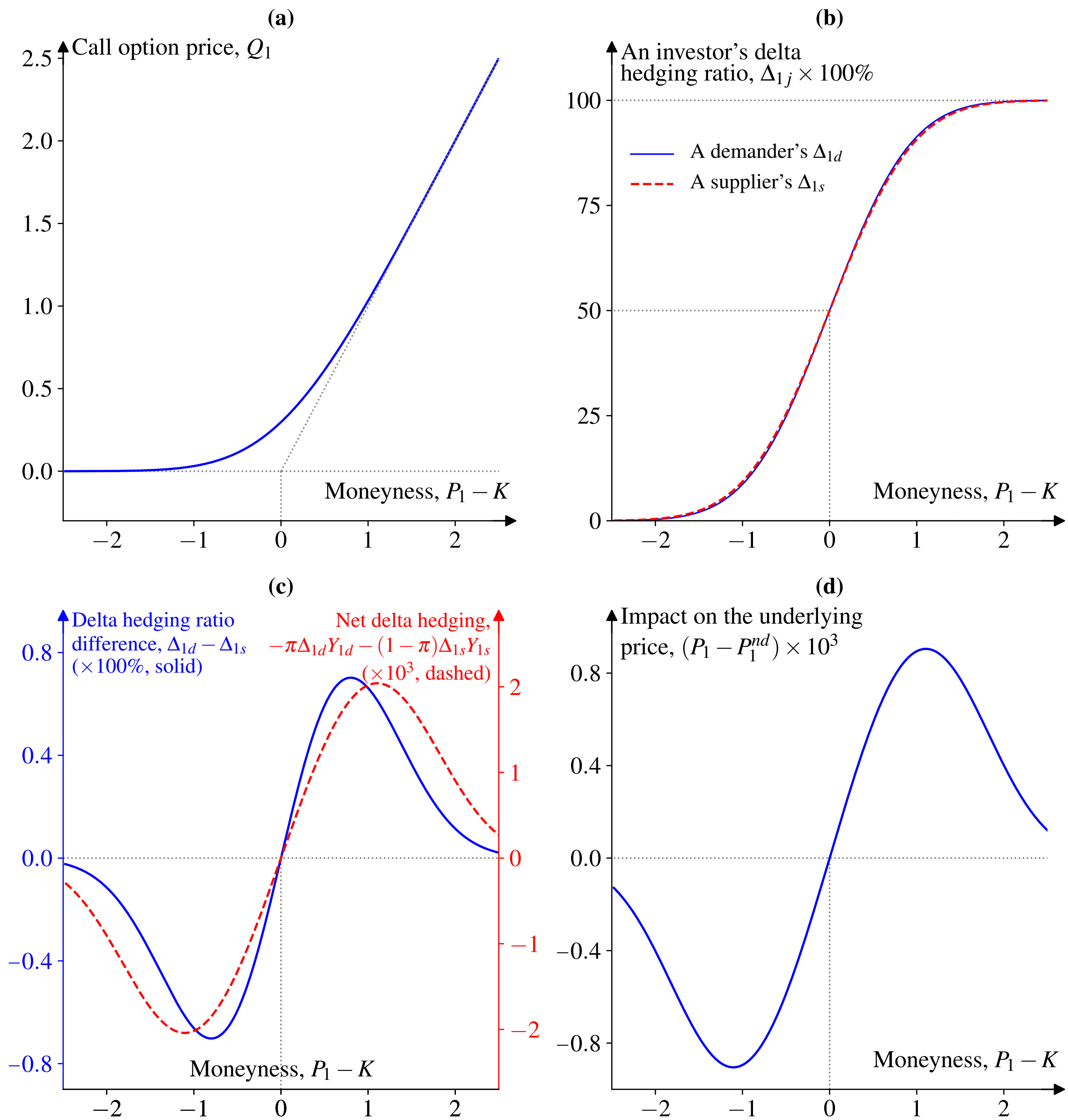

Figure 2: Equilibrium asset prices at $t = 1$, with a call option. This figure describes the $t = 1$ equilibrium with a call option. In all panels, the horizontal axes show a range of moneyness of the call option, defined as the difference between the underlying price and the strike price. The call's strike is fixed at $K = -1.0$ and the two state variables—the private signal and the endowment shock—vary to affect the equilibrium price P_1 and the moneyness. The other primitive parameters are set at $\bar{D} = 0.0$, $\bar{X} = 0.8$, $G_0 = 1.0$, $\tau_\varepsilon = 1.0$, $\tau_z = 1.0$, $\pi = 0.5$, and $\alpha = 0.8$.

is larger, $\Delta_{1d} > \Delta_{1s}$, if the call is ITM. Instead, if the call is OTM (unlikely exercised), the surer demanders will delta hedge less than the not-so-sure suppliers, resulting in $\Delta_{1d} < \Delta_{1s}$. When the call is ATM, we have the well-known result of $\Delta_{1d} = \Delta_{1s} = 1/2$.

The result can also be understood by recalling that an OTM (ITM) call option's Vanna—the second-order derivative of the option price over the underlying price and volatility—is positive (negative). That is, compared to an uninformed supplier (seeing higher volatility), a more informed demander (seeing lower volatility) always thinks of an OTM (ITM) call as less OTM (more ITM). She, therefore, always under-(over-)hedges relative to the supplier if the call is OTM (ITM).

Note that when the moneyness becomes extreme, we see that $\Delta_{1d} - \Delta_{1s} \downarrow 0$ again. This is because the delta of a call is bounded between zero and one: $0 \leq \Delta_{1j} = \hat{\mathbb{P}}_{1j}[D > K] \leq 1$. In particular, for a deep ITM (OTM) call, its delta converges to one (zero), regardless of the information set of the investor. This explains the flattening tails in the solid line in Figure 2(c).

(4.2) The net delta hedging and option moneyness. The dashed line (right axis) in Figure 2(c) plots the net delta hedging trade (15) against the call's moneyness. It tracks the pattern from the delta hedging ratio difference (the solid line, left axis). This is unsurprising because Equation (15) is simply the market clearing condition, $\pi Y_{1d} + (1 - \pi) Y_{1s} = 0$, rescaled by the respective investors' delta hedging ratios. When the call is ITM, the demanders delta hedge (long) more than the suppliers (short), and the net delta hedging is positive; vice versa. Summarizing, the following novel prediction can be empirically tested:

> **Prediction 1 (Net delta hedging and option moneyness).** *Investors' net delta hedging against their volatility bets has the same sign as the moneyness $P_1 - K$. That is, such net delta hedging is positive (negative) when the option is an ITM call or OTM put (OTM call or ITM put).*

Going beyond the call option, Table 1 extends the above prediction to various combinations of option types and moneyness. In any case, the sign of the net delta hedging (15) is the same as the moneyness $\text{sign}[P_1 - K]$, also the same as the sign of the option's Vanna (the last row of the table). This is because what matters for the net delta hedging trade is whether the (informed) demanders

	$P_1 - K < 0$		$P_1 - K > 0$	
	OTM-call	ITM-put	ITM-call	OTM-put
(a) Option holdings				
a demander's position Y_{1d}	< 0	< 0	< 0	< 0
a supplier's position Y_{1s}	> 0	> 0	> 0	> 0
(b) Delta hedging trades				
a demander's $-\Delta_{1d} Y_{1d}$	> 0	$< 0^*$	$> 0^*$	< 0
a supplier's $-\Delta_{1s} Y_{1s}$	$< 0^*$	> 0	< 0	$> 0^*$
(c) Net delta hedging trades				
$-\pi \Delta_{1d} Y_{1d} - (1 - \pi) \Delta_{1s} Y_{1s}$	< 0	< 0	> 0	> 0
(d) Sign of the option's Vanna	< 0		> 0	

Table 1: Investors' option holding and delta hedging trades for calls and puts. This table summarizes investors' option holding and delta hedging trades for four scenarios: out-of-the-money call, in-the-money call, in-the-money put, and out-of-the-money put. Panel (a) shows the equilibrium option positions by the demanders and the suppliers. Panel (b) shows their delta hedging directions. The dominant effect in each scenario (column) is superscripted with $*$, following which Panel (c) shows the signs of the net delta hedging. Panel (d) shows the sign of the option's Vanna, the second-order derivative of the option price over the underlying price and volatility.

expect a positive or negative cash flow from the option: If the option is ITM, as the writer (seller), the surer demanders expect negative $t = 2$ cash flows from the option and engage in considerable delta hedging, which dominates in the aggregate. If the option is OTM, the demanders expect to enjoy the positive sales from writing the options and engage in little delta hedging, thus making the suppliers' delta hedging the dominant force.

(4.3) The effect on the underlying price P_1. The *nonzero* net delta hedging is the key difference from the case of a variance swap, where the delta hedging trades always sum to zero; see the discussion on page 17. Figure 2(d) shows how the nonzero price pressure affects the equilibrium underlying price. Relative to the no-derivative benchmark P_1^{nd}, P_1 increases when the net delta hedging trade shown in Equation (15) is positive. That is, the underlying price is pushed in the direction of the net delta hedging, i.e., the sign of option moneyness. Such "price pressure" from net delta hedging has impacts on the illiquidity measures, which we study later in Section III.B.3.

We would like to emphasize that the price pressure due to delta hedging is more general than the specific model studied here. This is because asymmetrically informed investors tend to assign different values to the necessity (urgency) of delta hedging. The more informed agents (the demanders in the model) will delta hedge more aggressively than their less informed counterparties if and only if their derivative positions expect negative cash flows. This intuition is revealed through two modeling ingredients: the trading of the underlying and the option is *linked* through investors who are *asymmetrically informed*.

III.B Market illiquidity measures

We consider two widely used empirical measures of illiquidity, following Vayanos and Wang (2012). The first is the price impact of liquidity demanders' trading, in the spirit of Kyle (1985). It is defined as the sensitivity of price return with respect to liquidity demanders' signed volume (order flow) at $t = 1$. From an empiricist's point of view, this coefficient is obtained by regressing the price return $P_1 - P_0$ on the order flow $\pi \cdot (X_{1d} - X_0)$ at $t = 1$; that is,

$$\lambda := \frac{\mathrm{cov}[P_1 - P_0, \pi \cdot (X_{1d} - X_0)]}{\mathrm{var}[\pi \cdot (X_{1d} - X_0)]}. \tag{16}$$

The larger is λ, the more sensitive is price to order flows, implying a less liquid market. Many empirical works have used proxies in this spirit, including Glosten and Harris (1988), Brennan and Subrahmanyam (1996), Sadka (2006), and Amihud (2002), just to name a few.

The second measure is price reversal, as in Roll (1984) and Grossman and Miller (1988). The idea is that the risky asset's price will deviate from its "fundamental" D at $t = 1$, due to the price pressure of demanders' hedging. Such price pressure can be equivalently seen as the compensation required by the suppliers to absorb the shocked investors' liquidity demand. Eventually (in the long run, $t = 2$), the price will revert to $P_2 = D$. An empiricist can then compute

$$\gamma = -\mathrm{cov}[D - P_1, P_1 - P_0] \tag{17}$$

to measure market illiquidity. A larger γ implies a stronger price pressure at $t = 1$, hence low liquidity. Empirical applications of this measure include Roll (1984), Campbell et al. (1993), Hasbrouck (2009), and Pástor and Stambaugh (2003).

III.B.1 The no-derivative benchmark

Thanks to Proposition 1, without derivatives, the two liquidity measures can be found as:

$$\lambda^{nd} = \frac{\alpha}{1-\pi}\frac{G_1 - G_0}{G_1 - G_{1s}}\frac{1}{G_0}, \quad \text{and} \quad \gamma^{nd} = \left(1 - \frac{G_0}{G_1}\right)\left(1 - \frac{G_{1s}}{G_1}\right)\frac{1}{G_{1s} - G_0}. \tag{18}$$

Vayanos and Wang (2012) focus on how these two measures are affected by degrees of information asymmetry and imperfect competition. Our focus below turns to the effect of derivatives.

III.B.2 With variance swap

With the variance swap $f(D) = (D - P_0)^2$, the two liquidity measures can be found, following Proposition 2, as

$$\lambda = \frac{\alpha}{1-\pi}\frac{G_1 - G_0}{G_1 - G_{1s}}\frac{1}{G_1} = \frac{G_0}{G_1}\lambda^{nd} \quad \text{and} \quad \gamma = \left(1 - \frac{G_0}{G_1}\right)\left(1 - \frac{G_{1s}}{G_1}\right)\frac{1}{G_{1s} - G_0} = \gamma^{nd}.$$

Comparing with the no-derivative benchmark, we have the following corollary:

> **Corollary 1 (Illiquidity measures with a variance swap).** *After the derivative of $f(D) = (D - P_0)^2$ is introduced, the underlying asset has a lower price impact and the same price reversal; that is, $\lambda < \lambda^{nd}$ and $\gamma = \gamma^{nd}$.*

Price impact λ. Price impact measures the price elasticity with respect to the liquidity demanders' trading. Recall from Proposition 2 that compared to the benchmark X_{1d}^{nd}, a new term of $-2(P_1 - P_0)Y_{1d}$ rises in X_{1d}, reflecting the novel delta hedging by liquidity demanders. Such delta hedging does not reveal additional information (Lemma 1). Indeed, the informed trading component in X_{1d} remains unchanged as in the benchmark X_{1d}^{nd}. Relatively speaking, therefore, the additional delta hedging *reduces* the proportion of informed trading relative to the demanders' overall trading.

Consequently, if an empiricist regresses price changes on informed investors' trading, a lower price impact will obtain after derivatives are introduced.

Price reversal γ. As shown in Proposition 2, the $t = 1$ equilibrium price P_1 remains the same as the benchmark P_1^{nd}. Therefore, the negative return autocovariance, γ, is also unchanged. We again emphasize that this is a knife-edge result, due specifically to the derivative payoff of $f(D) = (D - P_0)^2$, under which all investors' delta hedging trades aggregate to zero (see the discussion on page 17). When options are introduced, for example, this will no longer be the case: the underlying price P_1 will deviate from the benchmark P_1^{nd}, tilting *in the direction of investors' net delta hedging.*

III.B.3 Options

The two illiquidity measures no longer have closed-form expressions when options are introduced. They can still be numerically examined, following their definitions in Equations (16) and (17). Figure 3 illustrates the patterns against various moneyness values of a call option.

Price impact. Figure 3(a) shows a qualitatively similar result to that seen in Section III.B.2: The price impact is lower with a derivative, i.e., $\lambda < \lambda^{nd}$. The key driver of the lower price impact is that the liquidity demanders now delta hedge their volatility bets. Such additional hedging trades are not informed speculation, making the demanders' trades less "toxic" overall, thereby reducing price impact. The effect of such non-informational delta hedging has been examined and recognized through empirical works, like Ni, Pearson, and Poteshman (2005) and Ni, Pearson, Poteshman, and White (2021), whose focus lies largely on the impact on underlying asset prices. Further, after introducing options, the bid-ask spread in the underlying typically also decreases (Damodaran and Lim (1991), Fedenia and Grammatikos (1992), and Kumar, Sarin, and Shastri (1995)). In particular, Kumar, Sarin, and Shastri (1998) find that it is the adverse-selection component of the bid-ask spread that decreases, consistent with our intuition of informed trading being diluted by delta hedging. Recent evidence by Hu (2017) is also consistent with our channel whereby uninformed trading becomes dominant (relative to informed trading) after option listing.

The existing empirical literature typically argues that the price impact or bid-ask spread decreases because informed investors migrate from the underlying to the options (see, e.g., Easley et al. (1998)). While the prediction is the same, our channel of delta hedging is different because in our model, derivatives are purposefully made informationally irrelevant.

To examine our channel, we propose to investigate *how* price impact is affected. Figure 3(a) depicts the patterns regarding how the underlying's price impact changes with respect to the moneyness of the call. Starting from the left side of the graph, where the call is deep OTM (K is very large), we see that $\lambda \approx \lambda^{nd}$. This is intuitive because the deep OTM call is valueless. Its impact on the equilibrium, compared to the no-derivative benchmark, is close to nothing.

As the moneyness increases, we see that λ first decreases and then increases. This is because there are two components of the price return $P_1 - P_0$, and they are affected differently by the demanders' trading $X_{1d} - X_0$. To see them, rewrite the price return as

$$P_1 - P_0 = \underbrace{(\mathbb{E}_1[D] - \mathbb{E}_0[D])}_{\text{(i) change in the fundamental}} + \underbrace{((P_1 - \mathbb{E}_1[D]) - (P_0 - \mathbb{E}_0[D]))}_{\text{(ii) change in the price pressure}},$$

where we essentially nonparametrically decompose a price P_t into (i) the time-t fundamental $\mathbb{E}_t[D]$ and the remainder $P_t - \mathbb{E}_t[D]$, which we refer to as (ii) price pressure. Therefore, the price impact λ is more precisely the sum of the impact on (i) and that on (ii).[6]

As the call becomes more ITM, the demanders know that the call they write is more likely to be exercised eventually, and they delta hedge more. Such *increases* in delta hedging trades are uninformative (Lemma 1) and only increase the volume of the demanders' underlying trading. Therefore, the price impact in (i) is lower—there is more trading but the fundamental price is unaffected. However, the price pressure in (ii) grows and can dominate if the call option becomes deep ITM, thus driving λ higher, resulting in the U-shape seen in Figure 3(a). The following

[6] In the empirical market microstructure literature, trades' impact on (i) and (ii) are referred to as the *permanent* and the *transitory* price impact, respectively, because changes in (i) are driven by information revealed in trades and therefore persist, while changes in (ii) will mean-revert to zero in the long run. See, e.g., Glosten and Harris (1988).

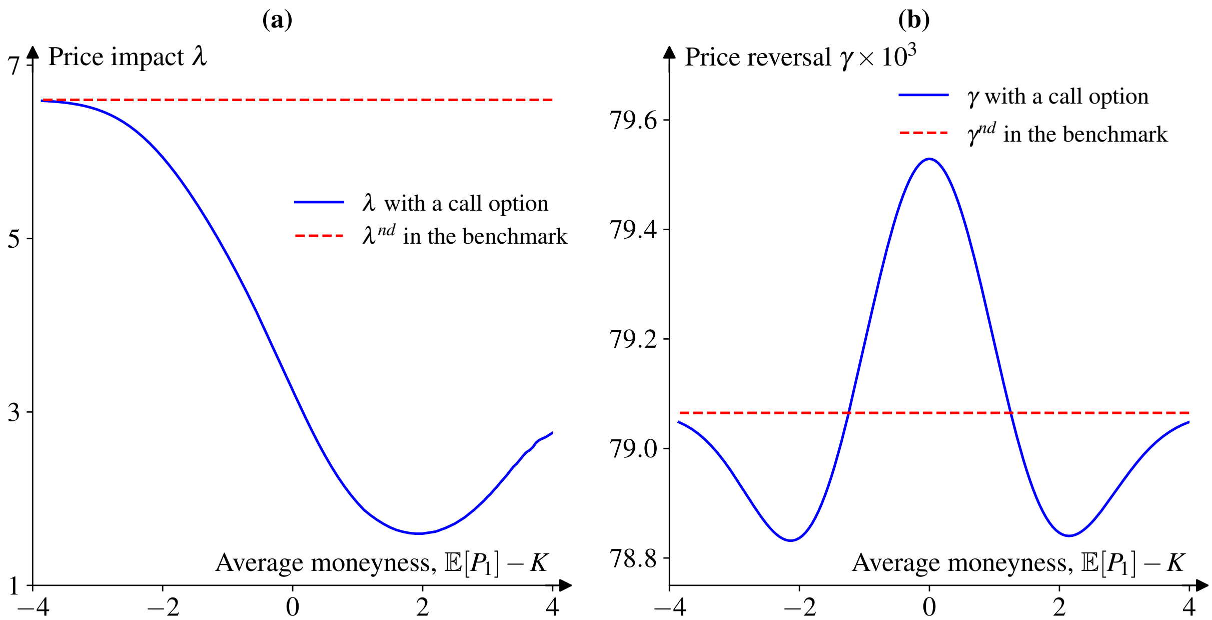

Figure 3: Illiquidity measures, with vs. without a call option. This figure shows how illiquidity measures are affected by a call option. Panel (a) shows the price impact λ (with the call, the solid line) and λ^{nd} (no derivative, dashed), while Panel (b) shows the price reversal γ (with the call, solid) and γ^{nd} (no derivative, dashed). In both panels, the horizontal axis is the average moneyness of the introduced call option, calculated by varying the strike price K, solving the underlying price P_1 at $t = 1$, and then taking expectation of the difference, i.e., $\mathbb{E}[P_1] - K$. The other parameters are set at $\pi = 0.50$, $\bar{D} = 0.0$, $\bar{X} = 0.8$, $G_0 = 1.0$, $\tau_\varepsilon = 1.0$, $\tau_z = 1.0$, and $\alpha = 0.8$.

empirical prediction summarizes the discussion thus far.

Prediction 2 (Price impact and option moneyness). *The price impact λ of the underlying is U-shaped in the option moneyness.*

While the discussion has focused only on a call option, the prediction above does not distinguish between calls and puts, thanks to put-call parity. (While the moneyness flips between a call and a put of the same strike, the predicted U-shape still remains U-shaped after the flip.)

Price reversal. In Section III.B.2, the price reversal is unaffected by the variance swap of $f(D) = (D - P_0)^2$. Such a knife-edge result no longer holds here. Figure 3(b) shows that when the moneyness of the call option is moderate (roughly between -1.7 and +1.7), the price reversal γ is exacerbated ($\gamma > \gamma^{nd}$); however, when either deep ITM or OTM, γ is alleviated ($\gamma < \gamma^{nd}$).

Two features of the model can explain this pattern. First, the demanders' (the suppliers') delta hedging trade, $-\Delta_{1d}Y_{1d}$ ($-\Delta_{1s}Y_{1s}$), always amplifies (dampens) price pressure. To see this, consider a small positive increase in the equilibrium price P_1, which makes the call option more ITM. As such, both types of investors would like to delta hedge more than before, as their Δ_{1j} increases (Figure 2b). In particular, the demanders delta hedge more by *buying* more of the underlying, thus amplifying the small initial price increase, while the suppliers *sell* more, dampening the initial price increase. Whether the net effect is amplifying or dampening depends on whether Δ_{1d} is more or less sensitive than Δ_{1s} to the initial small increase in P_1. That is, the sensitivity of the delta hedging ratios—the option's convexity or Gamma Γ—matters.

This leads to the second feature: The demanders' Δ_{1d} is more sensitive than the suppliers' Δ_{1s} only when the call option's moneyness is moderate. This can be seen by zooming in on Figure 2(b): Δ_{1d} is steeper than Δ_{1s} for moderate moneyness but flatter when the call is deep ITM or OTM. Examining this difference more closely, Figure 4 plots the two types of investors' average delta hedging sensitivity $\mathbb{E}[\Gamma_{1d}]$ and $\mathbb{E}[\Gamma_{1s}]$ in Panel (a) and their difference in Panel (b).

By combining these two features, we see that the difference in the price reversal $\gamma - \gamma^{nd}$ is driven by the delta hedging sensitivity, or gamma, of the call option. Indeed, comparing Figures 3(b) and 4(b), $\gamma - \gamma^{nd} > 0$ precisely when the demanders' delta hedging is more sensitive ($\Gamma_{1d} > \Gamma_{1s}$), i.e., for moderate moneyness of the call. In this region, the demanders' amplifying delta hedging trades respond to price fluctuations more than the suppliers' dampening delta hedging trades. We summarize the discussion in the following empirical prediction.

> **Prediction 3 (Price reversal and option moneyness).** *Compared to the no-derivative benchmark, the underlying's price reversal with an option is higher only when the option's moneyness is moderate (not too OTM or too ITM).*

Barbon and Buraschi (2019) provide consistent evidence. They approximate dealers' (liquidity

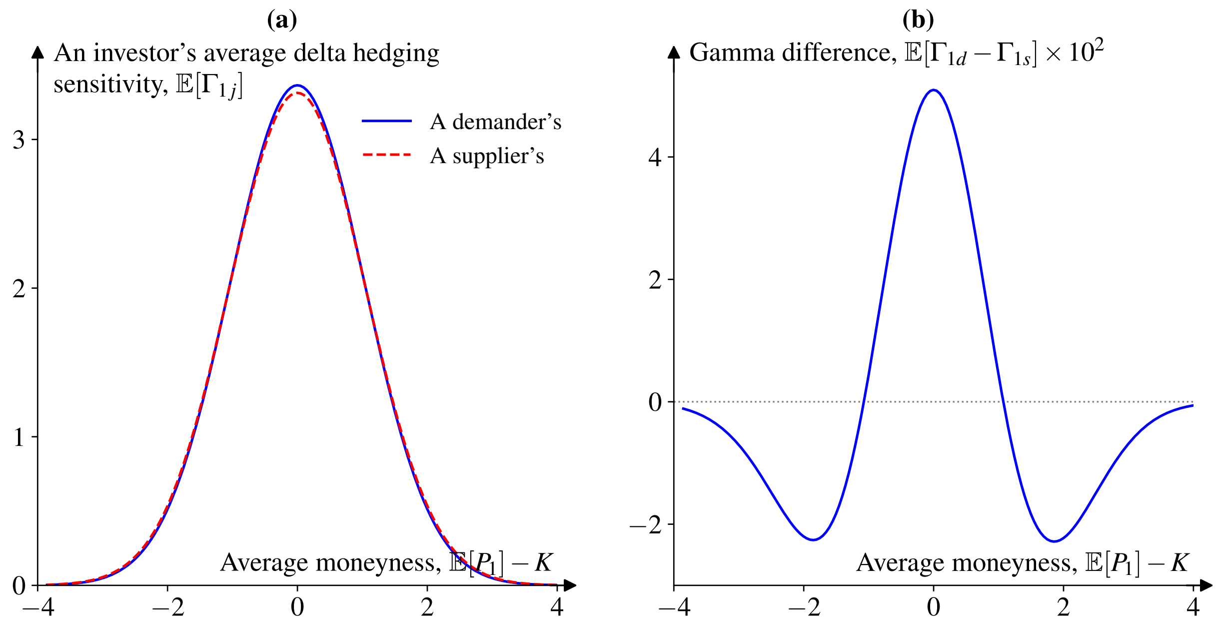

Figure 4: Investors' delta hedging sensitivity, gamma. This figure plots investors' average delta hedging sensitivity to fluctuations in the underlying price P_1, i.e., their gamma. Panel (a) jointly plots both the demanders' and the suppliers' gammas. Panel (b) plots their difference, namely the demanders less the suppliers' gamma. The other parameters are set at $\pi = 0.50$, $\bar{D} = 0.0$, $\bar{X} = 0.8$, $G_0 = 1.0$, $\tau_\varepsilon = 1.0$, $\tau_z = 1.0$, and $\alpha = 0.8$.

suppliers') aggregate Gamma imbalance and find that when it is negative, there is larger intraday momentum, i.e., larger price reversal.

IV The pre-shock equilibrium at $t = 0$

As before, we first consider the no-derivative case as the benchmark. In particular we are interested in the ex-ante ($t = 0$) liquidity risk premium and how derivatives affect it. To this end, we then consider the variance swap of $f(D) = (D - P_0)^2$, with which we can characterize the equilibrium in closed-form. Finally, we illustrate the robustness of the findings from the variance swap by considering options.

The no-derivative benchmark. At $t = 0$, the investors are initially homogeneous: each has the same probability π (resp. $1 - \pi$) of becoming a liquidity demander (resp. supplier) by $t = 1$. They choose a symmetric demand schedule $X_0^{nd}(p)$ to maximize expected utility. The market clearing condition (1) then yields the equilibrium price P_0^{nd}, given by the following proposition.

Proposition 5 (Benchmark asset price at $t = 0$**).** *At $t = 0$, each investor holds the per capita supply $\bar{X}$ units of the risky asset. The equilibrium asset price is*

$$P_0^{nd} = \bar{D} - \left(\alpha G_0^{-1}\right)\bar{X} - \frac{\pi M^{nd}}{1 - \pi + \pi M^{nd}}(\alpha\Sigma)\bar{X},$$

where the parameters M^{nd} and Σ are given in the proof.

The equilibrium P_0^{nd} has two discounts from the unconditional expected payoff $\bar{D}$. First, investors require a risk premium expressed as the product of an investor's risk aversion α and the unconditional payoff variance $\text{var}^{nd}[D] = G_0^{-1}$, scaled by the per capita holding $X_0 = \bar{X}$. Second, there is a "liquidity risk premium." The fraction $\frac{\pi M^{nd}}{(1-\pi)+\pi M^{nd}}$ is the risk-neutral probability for an investor to receive a liquidity shock, where M^{nd} is the ratio of demanders' expected marginal utility over that of the suppliers. For simplicity, M^{nd} will be referred to as the marginal utility ratio, or MUR, henceforth. The term Σ is a variance expression, akin to G_0^{-1} in the first discount.

Introducing a variance swap. We now introduce a variance swap $f(D) = (D - P_0)^2$. In this case, the underlying price P_0 can still be solved in closed-form:

Proposition 6 (Underlying price at $t = 0$**, with variance swap).** *In equilibrium, the underlying asset price at $t = 0$ is*

$$P_0 = \bar{D} - \left(\alpha G_0^{-1}\right)\bar{X} - \frac{\pi M}{1 - \pi + \pi M}(\alpha\Sigma)\bar{X},$$

where M is given in the proof and Σ is the same as in the benchmark (Proposition 5). Taking P_0 as given, both types of investors' pre-trading utility are higher with the variance swap, i.e., $U_{0j} > U_{0j}^{nd}$ for $j \in \{d, s\}$, and so is the unconditional expected utility, $U_0 = \pi U_{0d} + (1 - \pi)U_{0s} > U_0^{nd}$.

Compared with the benchmark P_0^{nd} (Proposition 5), the first two terms—the expected payoff $\bar{D}$

and the risk-premium $\alpha G_0^{-1}\bar{X}$—are the same. Only the third term—the liquidity risk premium—is affected and only through the marginal utility ratio (MUR) M in the risk-neutral probability. We show in the proof of the proposition that M and M^{nd} differ by a factor of

$$\frac{M}{M^{nd}} = \sqrt{\frac{G_{1d}}{G_{1s}}} e^{-\frac{G_{1d}-G_{1s}}{2G_1}} \gtrless 1. \tag{19}$$

Therefore, it is the MUR that determines the liquidity risk premium. Note that P_0 is monotone decreasing in M (and P_0^{nd} in M^{nd}): the underlying's ex-ante price P_0 features a higher liquidity risk premium if and only if $M > M^{nd}$; i.e., $P_0 - P_0^{nd} < 0$, if and only if $M - M^{nd} > 0$.

Whether $M > M^{nd}$ or $M < M^{nd}$ is driven by model parameters. We focus on π, the pervasiveness of the liquidity shock, as it characterizes a key attribute of investor composition.

Corollary 2 (The liquidity risk premium and investor composition). *Introducing the variance swap alleviates the liquidity risk if and only if the liquidity shock is not very pervasive. That is, there exists a unique* $\pi^* \in (0,1)$ *such that* $M \leq M^{nd}$ *and* $P_0 \geq P_0^{nd}$ *if and only if* $\pi \leq \pi^*$.

To understand the corollary, recall that Proposition 6 also finds that the investors' ex-ante expected utility is higher with the variance swap ($U_0 > U_0^{nd}$). This higher utility arises from the new trading gains available at $t = 1$: the variance swap allows the demanders to sell volatility, or insurance, to the suppliers. However, such trading gain is split *unevenly* between the demanders and the suppliers, depending on their relative market share. For example, if π is close to 0, this small group of demanders will extract most of the trading gain, because they are the only derivative writers. In this case, at $t = 0$, investors expect a substantial boost in their terminal wealth should they receive a liquidity shock, hence also a lower *marginal* expected utility. Compared to the benchmark, therefore, for sufficiently small π, the MUR decreases, i.e., $M < M^{nd}$. (When π is large, close to 1, the derivative trades in a buyers' market at $t = 1$, and the opposite follows.)

In fact, the difference in the MUR, $M - M^{nd}$, is monotonically increasing in shock pervasiveness π. Figure 5(a) shows the pattern. As the liquidity shock becomes more likely, the MUR difference increases and crosses zero once at some threshold $\pi^* \in (0,1)$. That is, introducing the

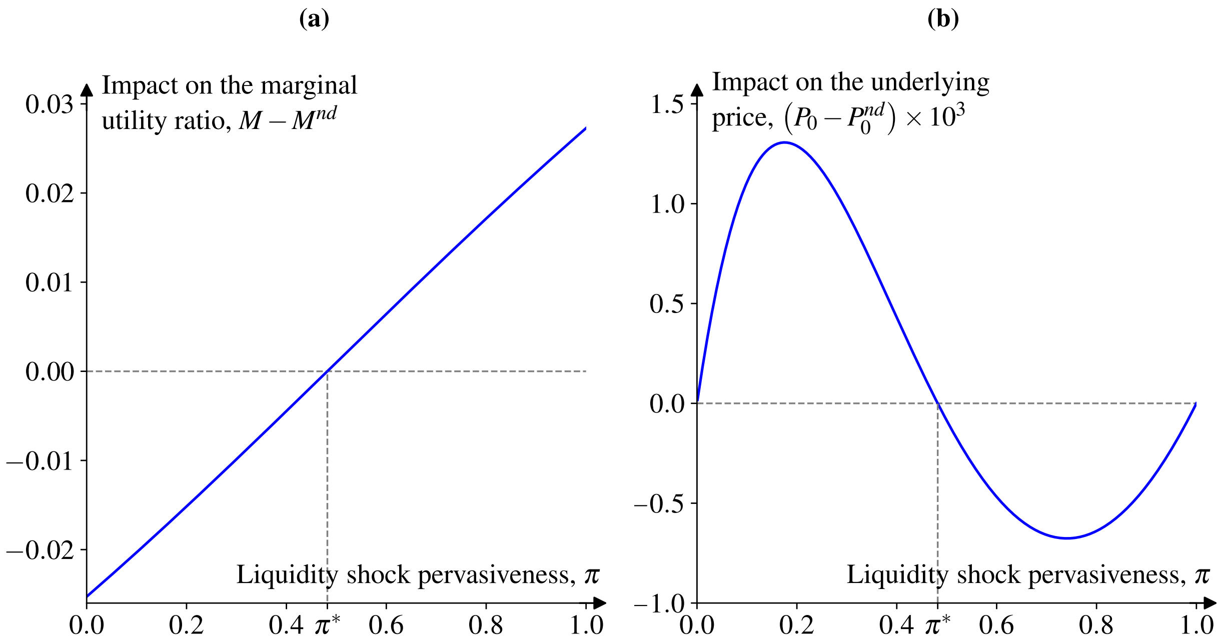

Figure 5: Liquidity risk and liquidity shock. This figure illustrates how the introduction of the derivative of $f(D) = D^2$ affects the equilibrium at $t = 0$, for various levels of liquidity shock pervasiveness $\pi \in (0, 1)$. Panel (a) plots difference in the marginal utility ratio (MUR), with the derivative less without the derivative, against π. The threshold π^* is where the two MUR are equal. Translating this into the ex-ante asset price, Panel (b) plots the difference in the underlying asset price P_0 and P_0^{nd} against π. The other primitive parameters are set at $\bar{D} = 0.0$, $\bar{X} = 0.8$, $G_0 = 1.0$, $\tau_\varepsilon = 1.0$, $\tau_z = 1.0$, and $\alpha = 0.8$.

derivative alleviates the ex-ante liquidity risk if and only if $\pi < \pi^*$ (the MUR between liquidity demanders and suppliers is reduced). When $\pi > \pi^*$, the liquidity risk is exacerbated instead.

Panel (b) illustrates how the $t = 0$ underlying price is affected. When $\pi < \pi^*$, $P_0 > P_0^{nd}$ because investors require a smaller risk premium, and vice versa. Note that at the two extremes of $\pi \downarrow 0$ and $\pi \uparrow 1$, $P_0 = P_0^{nd}$. This is because in addition to the effect on MUR, π is also the physical probability of receiving a liquidity shock. Recall from Proposition 6 that the risk-neutral probability of a liquidity shock is $\pi M/(1 - \pi + \pi M)$. Therefore, in these two extremes, all investors will be of the same type at $t = 1$, and there will be no open interest for the derivative ($Y_{1d} = Y_{1s} = 0$). The prices with and without the derivative therefore converge.

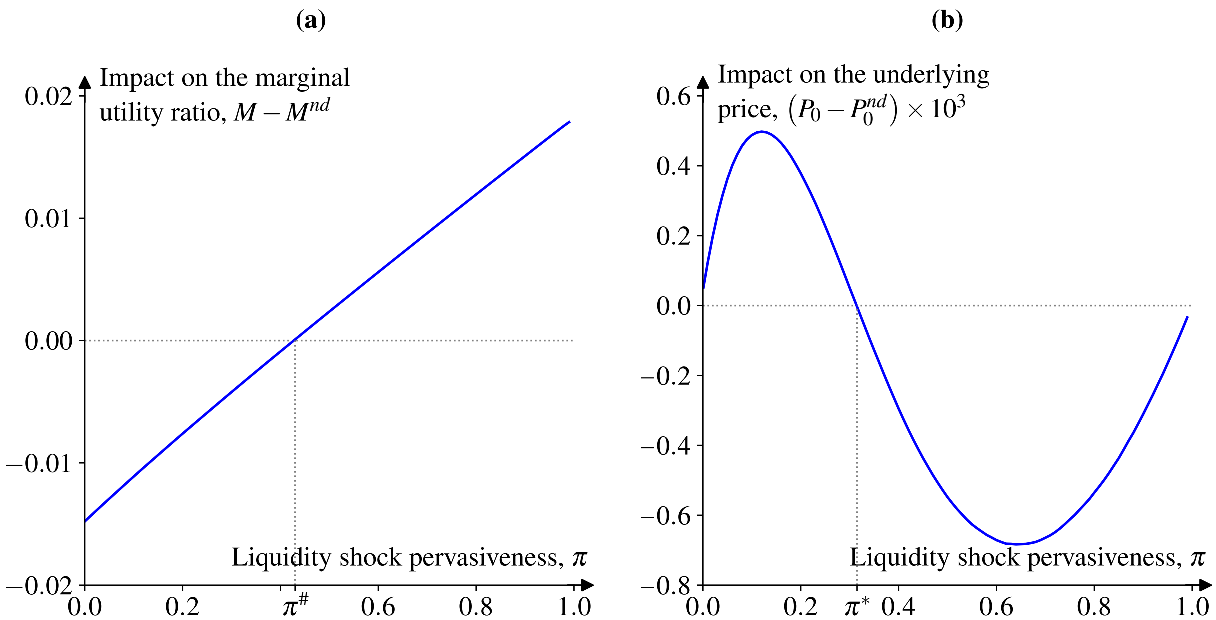

Figure 6: Liquidity risk and liquidity shock, with one call option. This figure qualitatively replicates the patterns shown in Figure 5. The difference is that here we only introduce a single call option at strike price $K = -1.0$. The other primitive parameters are set at $\bar{D} = 0.0$, $\bar{X} = 0.8$, $G_0 = 1.0$, $\tau_\varepsilon = 1.0$, $\tau_z = 1.0$, and $\alpha = 0.8$.

Introducing options. Finally, we introduce options to the benchmark. Corollary 2 shows that the ex-ante liquidity risk premium can be either alleviated or worsened after a variance swap of $(D - P_0)^2$ is introduced, largely depending on the pervasiveness π of the liquidity shock. This result remains robust and the intuition is the same: With options, investors are now able to trade on the volatility (through the options' nonlinear payoffs) of the underlying asset. When the liquidity shock pervasiveness π changes, the ex-ante marginal utility ratio (MUR) might either increase or decrease, depending on whether the liquidity demanders or the suppliers benefit more.

As an example, Figure 6 qualitatively replicates Figure 5. As π increases, the liquidity shock affects more of the population, the MUR M monotonically increases, and there is a threshold $\pi^{\#}$ only above which $M > M^{nd}$. Similarly, there is a threshold π^* only above which the liquidity risk premium increases (the underlying price P_0 decreases) with the introduction of the call option.[7]

Empirical evidence. The empirical evidence is mixed regarding how derivatives affect their underlying prices. Earlier works, like Branch and Finnerty (1981), Conrad (1989), and Detemple and Jorion (1990), find prices of the underlying increase after option listings, while evidences from more recent data, like Mayhew and Mihov (2000), document the opposite. More specifically, Sorescu (2000) find the effect to be positive before 1981 but negative after. Danielsen and Sorescu (2001) attribute the cutoff to the effective mitigation of short sale constraints due to the introduction of options. Our model instead provides a novel explanation to the switch of the signs: The switch coincides in time with the well-known boom of mutual funds in the 1980s. The mutual funds can be interpreted as the liquidity demanders in our model, as they have private information (at least the active funds do) and are subject to liquidity shocks driven by fund flows. Thus, the rise of mutual funds—an increase of parameter π—can push the price effect of option listings from positive to negative, as shown in Figure 5(b).

V Disconnection between (il)liquidity measures and liquidity risk premium

We have seen two sets of implications of derivatives on the underlying: the ex-post ($t = 1$) *illiquidity measures* like the price impact λ and the price reversal γ in Section III; and the ex-ante ($t = 0$) *liquidity risk premium* in Section IV. In summary, we find that derivatives might disconnect the empirical illiquidity measures from the liquidity risk premium in asset prices:

(1) illiquidity measures would potentially disagree, with $\lambda < \lambda^{nd}$ but $\gamma \lesseqgtr \gamma^{nd}$.

(2) ex-ante MUR might either increase or decrease, i.e., $M \lesseqgtr M^{nd}$;

[7] Note that unlike Figure 5, the two thresholds $\pi^{\#}$ for MUR and π^{*} for the underlying price are not the same here. This is because in the case of a variance swap, only the risk-neutral probability of receiving a liquidity shock, $\pi M/(1 - \pi + \pi M)$, is affected by π. In general, the parameter π also affects other equilibrium components in the ex-ante underlying price P_0, the closed-form solution of which we unfortunately do not obtain.

As such, it is important that the trading in the derivatives be properly accounted for when empirically analyzing liquidity risk and market illiquidity.

The disconnection. The disconnection between the illiquidity measures and the liquidity risk premium calls for caution in interpreting empirical findings. For example, the Amihud (2002) measure, like price impact, is found to associate with stock returns less strongly in recent years (Drienko, Smith, and von Reibnitz (2019), Harris and Amato (2019), and Amihud (2019)). Our theory suggests that the rise of derivatives trading in the meantime could contribute to this trend—the increased delta hedging volume in the underlying dilutes illiquidity measures like price impact.

On the other hand, the market-wide liquidity measure of Pástor and Stambaugh (2003), related to price reversal, is found to associate with a higher liquidity risk premium in more recent data (Li, Novy-Marx, and Velikov (2019), Pontiff and Singla (2019), and Pástor and Stambaugh (2019)). This evidence is also consistent with our prediction from Section III.B.3: the investors' net delta hedging trades can exacerbate price pressure and amplify price reversal. Ben-Rephael, Kadan, and Whol (2015) separately study the "characteristic" liquidity premium and the "systematic" liquidity premium and find that in recent years, both exist only in (small) NASDAQ stocks but not in NYSE or AMEX stocks. We argue that the lack of derivatives trading on relatively small NASDAQ stocks (Mayhew and Mihov (2004)) could explain why their liquidity measures and liquidity premium differ from those of relatively large stocks.

The novel underlying channel: delta hedging. Our model explains such disconnection through a novel channel of investors' delta hedging trades, $-\Delta_{1j}Y_{1j}$. (To compare, Vayanos and Wang (2012) discuss in great detail the impact of other market conditions, like adverse selection and competition, without derivatives.) Such delta hedging is the joint result of 1) investors' intrinsic demand Y_{1j} for the derivative *and* 2) the nonzero delta hedging ratio $\Delta_{1j} \neq 0$. The demand for the derivative arises from how heterogeneously informed investors bet against each other on the asset's variance. While such bets are due to the information asymmetry across investors, their learning is unaffected by our derivative, thus differentiating this paper from the extant literature, like Biais and Hillion (1994), Easley et al. (1998), and Dow (1998).

It is through the nonzero delta hedging that derivatives affect the illiquidity measures: Investors delta hedge their volatility bets by buying and selling in the underlying, creating uninformed price pressure, thus lowering the price impact and pushing the price reversal in either direction. We argue that such a delta hedging effect is a robust feature of the current financial market, as the derivative contracts almost always have nonzero deltas.[8]

VI Conclusion

This paper studies how derivatives (with nonlinear payoffs) affect their underlying securities' liquidity. Through a rational expectations equilibrium model, we examine such derivatives' implications for the liquidity risk premium and for empirical illiquidity measures. The key element of the model is a liquidity shock that randomly strikes a fraction of the otherwise homogeneous population. The shocked investors then demand liquidity for both information and hedging reasons, while the rest provide liquidity to them. We contrast equilibrium outcomes with and without derivatives.

In terms of the liquidity risk premium, we find that derivatives have ambiguous effects. Introducing derivatives allows investors to trade on some nonlinear structures of the underlying's future payoff, creating additional trading gains. However, the split of such trading gains between liquidity demanders and suppliers depends on model parameters. In general, derivatives affect the wedge between liquidity demanders' and suppliers' ex-ante marginal expected utilities. As such, before the shock, investors adjust—sometimes amplify—the risk-neutral probability of receiving the liquidity shock and might demand higher liquidity risk premia.

[8] Recall that we focus on derivatives written ex-ante of the liquidity shock, i.e., written at $t = 0$, and are not path-dependant. That is, $\mathbb{E}[\partial f(D)/\partial P_1] = 0$ almost surely. Therefore, ex-post of the liquidity shock, the delta hedging ratio $\Delta_{1j} = \hat{\mathbb{E}}_{1j}[f'(D)]$ becomes a function of the random P_1, and is almost surely nonzero. For example, consider the variance swap of $f(D) = (D - P_0)^2$, implying $\Delta_{1j} = 2\hat{\mathbb{E}}_{1j}[D - P_0] = 2(P_1 - P_0)$, which is zero when P_1 realizes to be exactly P_0—a zero probability event.

In terms of empirical illiquidity measures, we find seemingly "contradictory" messages, depending on the specific measure chosen. The key channel is investors' delta hedging of their derivative positions in the underlying. Such new hedging trades dilute the informed trading in the total trading volume, lowering price impacts—hence yielding better liquidity. On the other hand, the equilibrium price is now more subject to the price pressure of delta hedging trades, sometimes amplifying but sometimes dampening the price reversal.

Taken together, the results of our model emphasize the potential disconnect between assets' liquidity risk premium and their empirical illiquidity measures because of the trading of derivatives. Empirical studies associating the two sides should carefully control for the activity in the assets' derivative markets.

Appendix

A Proofs

Lemma 1

Proof. Clearly, the demanders' learning is unaffected by the derivative: $D|_s$ still normally distributed with $\text{var}_{1d}[D] = G_{1d}^{-1}$ and $\mathbb{E}_{1d}[D] = \frac{G_0}{G_{1d}}\bar{D} + \frac{G_{1d}-G_0}{G_{1d}}s$. We need to show that the suppliers do not learn more than they do in the benchmark. That is, fixing the same realizations of $\{D, \varepsilon, z\}$, with or without the derivative, the suppliers hold the same posterior that D is normally distributed with $\text{var}_{1s}[D] = \text{var}_{1s}^{nd}[D] = G_{1s}^{-1}$ and $\mathbb{E}_{1s}[D] = \mathbb{E}_{1s}^{nd}[D] = \frac{G_0}{G_{1s}}\bar{D} + \frac{G_{1s}-G_0}{G_{1s}}\eta$.

Consider an arbitrary investor in this economy with type-$j \in \{d, s\}$. Her terminal wealth is given in Equation (8). She chooses her demand X_{1j} and Y_{1j} to maximize her conditional expected utility at $t = 1$, i.e., $U_{1j} := \mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right]$, taking both prices $\{P_1, Q_1\}$ as given. It is well-known that such an optimization problem is strictly concave.[9] Therefore, if an equilibrium exists, it must

[9] Formally, Hessian H of the optimization problem is a symmetric 2-by-2 matrix, with diagonal terms $\frac{\partial^2 U_{1j}}{\partial {X_{1j}}^2} = \mathbb{E}_{1j}\left[-\alpha^2 \cdot (D - P_1)^2 e^{-\alpha W_{2j}}\right] =: \mathbb{E}_{1j}[-a(D)^2]$ and $\frac{\partial^2 U_{1j}}{\partial {Y_{1j}}^2} = \mathbb{E}_{1j}\left[-\alpha^2 \cdot (f(D) - Q_1)^2 e^{-\alpha W_{2j}}\right] =: \mathbb{E}_{1j}[-b(D)^2]$, both strictly negative. The off-diagonal term is $\frac{\partial^2 U_{1j}}{\partial X_{1j} Y_{1j}} = \mathbb{E}_{1j}\left[-\alpha^2 \cdot (D - P_1)(f(D) - Q_1)e^{-\alpha W_{2j}}\right] = \mathbb{E}_{1j}[-a(D)b(D)]$. It follows that for any nonzero $x = [x_1, x_2]^\top \in \mathbb{R}^2$, $x^\top H x = \mathbb{E}_{1j}[-a(D)^2 x_1^2 - 2a(D)b(D)x_1 x_2 - b(D)^2 x_2^2] = \mathbb{E}_{1j}[-(a(D)x_1 + b(D)x_2)^2] < 0$, i.e., the Hessian is negative definite. Therefore, the optimization problem is convex.

be interior and the first-order conditions must hold:

$$\text{(A.1)} \qquad \frac{\partial U_{1j}}{\partial X_{1j}} = \mathbb{E}_{1j}\left[\alpha \cdot (D - P_1)e^{-\alpha W_{2j}}\right] = 0; \text{ and, } \frac{\partial U_{1j}}{\partial Y_{1j}} = \mathbb{E}_{1j}\left[\alpha \cdot (f(D) - Q_1)e^{-\alpha W_{2j}}\right] = 0.$$

Recalling that from a demander's point of view, D is conditionally normal. Using the conditional density, her first-order conditions can be simplified to a two-equation-two-unknown system:

$$\text{(A.2)} \qquad 0 = \int_{\mathbb{R}} (D - P_1) \cdot e^{-\alpha f(D) Y_{1d}} \cdot e^{-\frac{G_{1d}}{2}\left(D - \bar{D} - \hat{\eta} + \frac{\alpha}{G_{1d}} X_{1d}\right)^2} \mathrm{d}D; \text{ and}$$

$$\text{(A.3)} \qquad 0 = \int_{\mathbb{R}} (f(D) - Q_1) \cdot e^{-\alpha f(D) Y_{1d}} \cdot e^{-\frac{G_{1d}}{2}\left(D - \bar{D} - \hat{\eta} + \frac{\alpha}{G_{1d}} X_{1d}\right)^2} \mathrm{d}D.$$

where $\hat{\eta} := \frac{\tau_\varepsilon}{\tau_\varepsilon + G_0}\left(s - \bar{D} - \frac{\alpha}{\tau_\varepsilon} z\right) = \frac{\tau_\varepsilon}{\tau_\varepsilon + G_0}\left(\eta - \bar{D}\right)$ is informationally equivalent to what suppliers can infer in the benchmark (c.f., Equation A.5).

Now turn to the suppliers. In equilibrium, they know that Equations (A.2) and (A.3) must hold. They can condition on the equilibrium prices $\{P_1, Q_1\}$. They also observe the demanders' demand realizations $\{X_{1d}, Y_{1d}\}$. This is because in equilibrium, the suppliers know their own demand $\{X_{1s}, Y_{1s}\}$ and through the market clearing conditions $\pi X_{1d} + (1-\pi) X_{1s} = \bar{X}$ and $\pi Y_{1d} + (1-\pi) Y_{1s} = 0$, the suppliers can thus infer perfectly the realizations of $\{X_{1d}, Y_{1d}\}$. Knowing $\{P_1, Q_1, X_{1d}, Y_{1d}\}$, therefore, from any supplier's perspective, each equation in the system (A.2) and (A.3) has one and only one unknown, $\hat{\eta}$. Therefore, they can infer, *at best*, $\hat{\eta}$.

We further show below that given $\{P_1, X_{1d}, Y_{1d}\}$, the first line of Equation (A.2) has unique solution of $\hat{\eta}$. Hence, the suppliers can fully back out $\hat{\eta}$, learning exactly the same as they do in the benchmark. (Equation A.3 is redundant for learning, in this sense.)

To prove the uniqueness of the solution, we begin by writing

$$a(D; \hat{\eta}) := e^{-\alpha f(D) Y_{1d}} \cdot e^{-\frac{G_{1d}}{2}\left(D - \bar{D} - \hat{\eta} + \frac{\alpha}{G_{1d}} X_{1d}\right)^2} \text{ and } b(D; \hat{\eta}) := D - \bar{D} - \hat{\eta} + \frac{\alpha}{G_{1d}} X_{1d},$$

so that $\frac{\partial a(D;\hat{\eta})}{\partial \hat{\eta}} = G_{1d} a(D; \hat{\eta}) b(D; \hat{\eta})$ and Equation (A.2) can be rearranged as

$$\text{(A.4)} \qquad P_1 = \frac{\int_{\mathbb{R}} D a(D; \hat{\eta}) \mathrm{d}D}{\int_{\mathbb{R}} a(D; \hat{\eta}) \mathrm{d}D}.$$

Note that this is an equation of $\hat{\eta}$ and the suppliers try to solve $\hat{\eta}$ from it. To show the uniqueness of the solution, take derivative with respect to $\hat{\eta}$ on the right-hand side to get

$$\frac{G_{1d}}{\left(\int_{\mathbb{R}} a \mathrm{d}D\right)^2}\left[\int_{\mathbb{R}} Dab \mathrm{d}D \int_{\mathbb{R}} a \mathrm{d}D - \int_{\mathbb{R}} Da \mathrm{d}D \int_{\mathbb{R}} ab \mathrm{d}D\right],$$

where we omit the arguments of $a(\cdot)$ and $b(\cdot)$ for notation simplicity. Note that b can be rewritten as $b = D - c$ where $c := \bar{D} + \hat{\eta} - \frac{\alpha}{G_{1d}} X_{1d}$ is independent of D conditional on $\hat{\eta}$. Plug in $b = D - c$ and simplify to get

$$\frac{G_{1d}}{\left(\int_{\mathbb{R}} a \mathrm{d}D\right)^2} \left[\int_{\mathbb{R}} D^2 a \mathrm{d}D \int_{\mathbb{R}} a \mathrm{d}D - \left(\int_{\mathbb{R}} D a \mathrm{d}D \right)^2 \right].$$

Cauchy-Schwarz inequality has that $\left(\int_{\mathbb{R}} f(x) g(x) \mathrm{d}x\right)^2 \leq \int_{\mathbb{R}} f(x)^2 \mathrm{d}x \int_{\mathbb{R}} g(x)^2 \mathrm{d}x$. Note $D^2 a = D\sqrt{a} \cdot \sqrt{a}$ always holds because $a > 0$. Applying the above Cauchy-Schwarz inequality with $f = D\sqrt{a}$ and $g = \sqrt{a}$ yields

$$\left(\int_{\mathbb{R}} D a \mathrm{d}D \right)^2 \leq \int_{\mathbb{R}} D^2 a \mathrm{d}D \int_{\mathbb{R}} a \mathrm{d}D.$$

Therefore, the right-hand side of Equation (A.4) is monotone increasing in $\hat{\eta}$. That is, so long the equilibrium exists, suppliers can exactly infer the $\hat{\eta}$.[10] Since $\hat{\eta}$ is informationally equivalent to what suppliers can learn without the call options (as in the benchmark), there is no additional information revealed. □

Propositions 1 and 5

Proof. The two propositions (and their proofs) correspond to Propositions 3.1-3.3 in Vayanos and Wang (2012). We highlight some notation differences below. The three constants a, b, and c in Equation (3.1) of Vayanos and Wang (2012) can be expressed in our notation as $a = \bar{D} - \alpha \bar{X}/G_1$, $b = 1 - G_0/G_1$, and $c = \alpha/(G_{1d} - G_0)$. These then help verify our Proposition 1 as a replication of their Propositions 3.1 and 3.2. The stated P_0 in our Proposition 5 has the same form as the one in their Proposition 3.3, where our $\alpha\Sigma$ replaces their Δ_1. That is, our Σ can be expressed in their notation as

$$\Sigma = \frac{\alpha^2 b \sigma^2 (\sigma^2 + \sigma_\epsilon^2) \sigma_z^2}{1 + \Delta_0 (1 - \pi)^2 - \alpha^2 \sigma^2 \sigma_z^2}$$

where $\sigma^2 = G_0^{-1}$, $\sigma_z^2 = \tau_z^{-1}$, $\sigma_\epsilon^2 = \tau_\epsilon^{-1}$, and Δ_0 is given in Their equation (3.7a) and can be equivalently written in our notation as $\Delta_0 = \frac{1}{\pi^2} \frac{G_0}{G_{1s} - G_0} \frac{(G_1 - G_{1s})^2}{G_1^2}$. Finally, the marginal utility ratio

[10] More rigorously, one needs to prove the existence of the solution to Equation (A.4) in solving for $\hat{\eta}$. This is trivial, because Equation (A.4) is a rewriting of the first-order condition (A.1). Since the equilibrium is assumed to exist, the first-order condition necessarily holds.

M^{nd} in Proposition 5 can be expressed as

$$M^{nd} = \exp\left(\frac{\alpha}{2}\Delta_2\bar{\theta}^2\right)\sqrt{\frac{1+\pi^2\Delta_0}{1+\Delta_0(1-\pi)^2-\alpha^2\sigma^2\sigma_z^2}}$$

where $\bar{\theta} = \bar{X}$, $\sigma^2 = G_0^{-1}$, $\sigma_z^2 = \tau_z^{-1}$, and Δ_2 is given in their equation (3.7c). In our notation, Δ_2 can be written as

$$\Delta_2 = \frac{\alpha^3\tau_z^{-1}G_0^{-2}\left[1+\frac{G_0}{G_{1d}-G_0}\frac{(G_1-G_{1d})^2}{G_1^2}\right]}{1+\Delta_0(1-\pi)^2-\alpha^2\tau_z^{-1}G_0^{-1}}.$$

We also highlight how the suppliers learn in this equilibrium. The only source of new information for the suppliers is the risky asset's price P_1^{nd}, whose closed-form solution is spelled out in Equation (7). It can be seen that the only "learnable" component is a linear combination of the private signal s and the endowment shock z:

$$\eta := s - \frac{\alpha}{\tau_\varepsilon}z = D + \varepsilon - \frac{\alpha}{\tau_\varepsilon}z. \qquad \text{(A.5)}$$

That is, the suppliers are only learning from the above noisy signal η, which is a linear combination of all three random variables in this economy. Given this, we have $\text{var}_{1s}^{nd}[D] = \text{var}[D\,|\,P_1^{nd}] = G_{1s}^{-1}$ and $\mathbb{E}_{1s}^{nd}[D] = \mathbb{E}[D\,|\,P_1^{nd}] = \frac{G_0}{G_{1s}}\bar{D} + \frac{G_{1s}-G_0}{G_{1s}}\eta$. □

Proposition 2

Proof. This proof considers a more general quadratic payoff $f(D) = D^2 - aP_0^2 - bDP_0 - cD - eP_0 - f$. In particular, the stated result with a variance swap, $(D-P_0)^2$, is a special case of $a=-1$, $b=2$, and $c=e=f=0$. Consider a type-j investor. Her terminal wealth W_{2j} is given by Equation (8). Lemma 1 ensures that she holds the same posterior distribution for D with or without the derivative. In particular, D remains conditionally normal. Let $z_s = 0$, $z_d = z$, and $W_1 = W_0 + (p - P_0)X_0$. Evaluating the expected utility (e.g., Lemma A.1 of Marín and Rahi (1999)) yields,

$$\begin{aligned}
\mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right] &= -\frac{1}{\sqrt{1+2\alpha\text{var}_{1j}[D]Y_{1j}}} \\
&\cdot \exp\left[\alpha\Big(-W_1 + X_{1j}(p-P_0) + z_j(\bar{D}-P_0) - Y_{1j}\Big((1-a-b)P_0^2 - (c+e)P_0 - f - q\Big)\Big)\right] \\
&\cdot \exp\left[-\alpha\big(X_{1j} + z_j + ((2-b)P_0 - c)Y_{1j}\big)(\mathbb{E}_{1j}[D]-P_0) - \alpha Y_{1j}(\mathbb{E}_{1j}[D]-P_0)^2\right] \\
&\cdot \exp\left[\frac{\alpha^2\text{var}_{1j}[D]\big(X_{1j}+z_j+Y_{1j}(2\mathbb{E}_{1j}[D]-bP_0-c)\big)^2}{2\big(1+2\alpha\text{var}_{1j}[D]Y_{1j}\big)}\right].
\end{aligned}$$

The first-order condition with respect to X_{1j} yields

$$X_{1j} = \frac{\mathbb{E}_{1j}[D] - p}{\alpha \text{var}_{1j}[D]} - z_j - (2p - bP_0 - c)Y_{1j}.$$

Plug this back to $\mathbb{E}_{1j}[-e^{-\alpha W_{2j}}]$ and evaluate the first-order condition with respect to Y_{1j} to get:

$$Y_{1j} = \frac{1}{2\alpha}\left(\frac{1}{q - p^2 + aP_0^2 + bP_0 p + eP_0 + cp + f} - \frac{1}{\text{var}_{1j}[D]}\right).$$

Finally, clearing the market with (1) and (2) yields the equilibrium prices $p = P_1$ and $q = Q_1$ as stated in the proposition. (The utility maximization problem is a strictly concave one. Hence, the above solution implied by the first-order conditions are unique.) □

Proposition 3

Proof. Lemma S2 in the internet appendix shows that a type-j investor's risk-neutral density is $\hat{\phi}_{1j}(D)$, whose derivative with respect to D can be found as

$$\hat{\phi}'_{1j}(D) = \left[-\alpha \cdot (X_{1j} + f'(D)Y_{1j}) - G_{1j}D + (G_0\bar{D} + (G_{1j} - G_0)\eta)\right]\hat{\phi}_{1j}(D).$$

Note also that $\lim_{D\to\pm\infty} \hat{\phi}_{1j} = 0$, for otherwise the first-order conditions (S2) (i.e., the risk-neutral prices) would not be well-defined and the equilibrium would not exist. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{\phi}'_{1j}(D)\mathrm{d}D &= \hat{\mathbb{E}}_{1j}\left[-\alpha \cdot (X_{1j} + f'(D)Y_{1j}) - G_{1j}D + (G_{1j} - G_0)\eta\right] \\ &= -\alpha X_{1j} - \alpha\Delta_{1j}Y_{1j} - G_{1j}p_1 + G_0\bar{D} + (G_{1j} - G_0)\eta = 0,\end{aligned}$$

where the last equality holds because $\int_{-\infty}^{\infty} \hat{\phi}'_{1j}(D)\mathrm{d}D = \lim_{D\to\infty} \hat{\phi}_{1j}(D) - \lim_{D\to-\infty} \hat{\phi}_{1j}(D) = 0$. Hence,

$$X_{1j} + \Delta_{1j}Y_{1j} = \frac{G_{1j}}{\alpha}\left(\frac{G_0}{G_{1j}}\bar{D} + \frac{G_{1j} - G_0}{G_{1j}}\eta - p_1\right).$$

Note from Proposition 1 that the right-hand side above is exactly the demand function $X_{1j}^{nd}(p_1)$. □

Proposition 4

Proof. To begin with, Lemma 1 ensures that these additional call options do not reveal new information.[11] Therefore, both the demanders and the suppliers have the same posterior distribution

[11] Lemma 1 only proves the information redundancy result for a single arbitrary derivative. It can be easily generalized by noting (1) that the demanders' learning is unaffected; (2) that the suppliers can infer from the market

of D as in the benchmark.

Next, we consider investors' optimization. A type-$j \in \{d, s\}$ investor's terminal wealth is

$$W_{2j} = W_1 + (D - P_1)X_{1j} + \sum_{i=1}^{n} \big((D - K_i)^+ - Q_{1i}\big)Y_{1ji} + (D - \bar{D})z_j,$$

where $W_1 = W_0 + (P_1 - P_0)X_0$; Q_{1i} is the price of the call with strike K_i (at $t = 1$); Y_{1ji} is the investor's holding of that call; z_j is her endowment shock ($z_d = z$ and $z_s = 0$). She chooses her demand X_{1j} and $\{Y_{1ji}\}$ to maximizes $U_{1j} := \mathbb{E}_{1j}\big[-e^{-\alpha W_{2j}}\big]$.

We first establish the uniqueness of the optimal demand, taking all prices P_1 and $\{Q_{1i}\}$ as given. This is because the optimization problem is strictly concave:

$$\frac{\partial^2 U_{1j}}{\partial {X_{1j}}^2} = \mathbb{E}_{1j}\big[-\alpha^2 \cdot (D - P_1)^2 e^{-\alpha W_{2j}}\big] < 0$$
$$\frac{\partial^2 U_{1j}}{\partial {Y_{1ji}}^2} = \mathbb{E}_{1j}\Big[-\alpha^2 \cdot \big((D - K_i)^+ - Q_{1i}\big)^2 e^{-\alpha W_{2j}}\Big] < 0, \ \forall i \in \{1, ..., n\}.$$

Therefore, the first-order conditions (if the solution exists) are sufficient for the global optimum:

$$\frac{\partial U_{1j}}{\partial X_{1j}} = \mathbb{E}_{1j}\big[\alpha \cdot (D - P_1) e^{-\alpha W_{2j}}\big] = 0; \text{ and}$$
$$\frac{\partial U_{1j}}{\partial Y_{1ji}} = \mathbb{E}_{1j}\big[\alpha \cdot \big((D - K_i)^+ - Q_{1i}\big) e^{-\alpha W_{2j}}\big] = 0, \ \forall i \in \{1, ..., n\}. \tag{A.6}$$

While not analytically tractable, it is clear that the first-order conditions (A.6) always have a solution (existence): Consider the extreme choices of X_{1j} and its implication for U_{1j} for example. When $|X_{1j}| \to \infty$, $U_{1j} \to -\infty$ because with infinite holding X_{1j}, the uncertainty associated with the risky payoff D becomes infinity. Such infinite payoff risk makes the investor suffer from infinite risk. Given that the optimization is strictly concave and that $\lim_{X_{1j} \to \pm\infty} U_{1j} = -\infty$, there exists a unique X_{1j} that maximizes U_{1j}. The same argument applies to option holdings $\{Y_{1ji}\}$.

We have shown that the optimal demand is a unique function of the prices P_1 and $\{Q_{1i}\}$. These $n + 1$ prices are pinned down via the $n + 1$ market clearing conditions:

$$\pi X_{1d} + (1 - \pi)X_{1s} - \bar{X} = 0; \text{ and } \pi Y_{1di} + (1 - \pi)Y_{1si} = 0, \ \forall i \in \{1, ..., n\}.$$

clearing conditions the exact quantities of the demanders' demand in all assets; (3) that the suppliers effectively learn from the demanders' first-order conditions; and (4) that the demanders' first-order condition for the underlying always reveals the same information as in the benchmark.

or, defining $F : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$:

$$\text{(A.7)} \qquad F(P_1, Q_{1i}, ..., Q_{1n}) := \begin{bmatrix} \pi X_{1d} + (1-\pi)X_{1s} - \bar{X} \\ \pi Y_{1d1} + (1-\pi)Y_{1s1} \\ \vdots \\ \pi Y_{1dn} + (1-\pi)Y_{1sn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It turns out that the prices pinned down by $F(\cdot) = 0$ are also unique. We prove this by showing that the Jacobian matrix of $F(\cdot)$

$$\text{(A.8)} \qquad \pi \cdot \begin{bmatrix} \frac{\mathrm{d}X_{1d}}{\mathrm{d}P_1} & \frac{\mathrm{d}X_{1d}}{\mathrm{d}Q_{11}} & \cdots & \frac{\mathrm{d}X_{1d}}{\mathrm{d}Q_{1n}} \\ \frac{\mathrm{d}Y_{1d1}}{\mathrm{d}P_1} & \frac{\mathrm{d}Y_{1d1}}{\mathrm{d}Q_{11}} & \cdots & \frac{\mathrm{d}Y_{1d1}}{\mathrm{d}Q_{1n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}Y_{1dn}}{\mathrm{d}P_1} & \frac{\mathrm{d}Y_{1dn}}{\mathrm{d}Q_{11}} & \cdots & \frac{\mathrm{d}Y_{1dn}}{\mathrm{d}Q_{1n}} \end{bmatrix} + (1-\pi) \begin{bmatrix} \frac{\mathrm{d}X_{1s}}{\mathrm{d}P_1} & \frac{\mathrm{d}X_{1s}}{\mathrm{d}Q_{11}} & \cdots & \frac{\mathrm{d}X_{1s}}{\mathrm{d}Q_{1n}} \\ \frac{\mathrm{d}Y_{1s1}}{\mathrm{d}P_1} & \frac{\mathrm{d}Y_{1s1}}{\mathrm{d}Q_{11}} & \cdots & \frac{\mathrm{d}Y_{1s1}}{\mathrm{d}Q_{1n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}Y_{1sn}}{\mathrm{d}P_1} & \frac{\mathrm{d}Y_{1sn}}{\mathrm{d}Q_{11}} & \cdots & \frac{\mathrm{d}Y_{1sn}}{\mathrm{d}Q_{1n}} \end{bmatrix}$$

is negative definite. To do so, consider the underlying asset for example (the argument is generic for any of the $n+1$ securities). The first-order condition (A.6) is an implicit function of the demand X_{1j} and all $n+1$ prices. By implicit function theorem, we have

$$\text{(A.9)} \qquad \frac{\mathrm{d}X_{1j}}{\mathrm{d}P_1} = -\frac{\partial^2 U_{1j}/\left(\partial X_{1j}\partial P_1\right)}{\partial^2 U_{1j}/\partial {X_{1j}}^2} = \frac{\mathbb{E}_{1j}\left[\alpha e^{-\alpha W_{2j}} - \alpha^2 \cdot (D-P_1)X_{1j}e^{-\alpha W_{2j}}\right]}{\mathbb{E}_{1j}\left[-\alpha^2 \cdot (D-P_1)^2 e^{-\alpha W_{2j}}\right]}.$$

Clearly, the denominator is negative (it is the second-order derivative of U_{1j} with respect to X_{1j}). The numerator can be further simplified:

$$\text{(A.10)} \qquad \mathbb{E}_{1j}\left[\alpha e^{-\alpha W_{2j}} - \alpha^2 \cdot (D-P_1)X_{1j}e^{-\alpha W_{2j}}\right] = \mathbb{E}_{1j}\left[\alpha e^{-\alpha W_{2j}}\right] - \alpha X_{1j} \underbrace{\mathbb{E}_{1j}\left[\alpha \cdot (D-P_1)e^{-\alpha W_{2j}}\right]}_{=0\text{, by the first-order condition (A.6)}} > 0.$$

Therefore, we have $\mathrm{d}X_{1j}/\mathrm{d}P_1 < 0$ in equilibrium, which is an intuitive result that the demand for a security strictly decreases with its price, for both $j \in \{d, s\}$. Again by implicit function theorem, $\forall i \in \{1, ..., n\}$, we also have

$$\frac{\mathrm{d}X_{1j}}{\mathrm{d}Q_{1i}} = -\frac{\partial^2 U_{1j}/\left(\partial X_{1j}\partial Q_{1i}\right)}{\partial^2 U_{1j}/\partial {X_{1j}}^2} = \frac{\mathbb{E}_{1j}\left[-\alpha^2 \cdot (D-P_1)Y_{1ji}e^{-\alpha W_{2j}}\right]}{\mathbb{E}_{1j}\left[-\alpha^2 \cdot (D-P_1)^2 e^{-\alpha W_{2j}}\right]}$$

where, like before, the numerator can be further simplified as

$$\mathbb{E}_{1j}\left[-\alpha^2 \cdot (D-P_1)Y_{1ji}e^{-\alpha W_{2j}}\right] = -\alpha Y_{1ji}\mathbb{E}_{1j}\left[\alpha \cdot (D-P_1)e^{-\alpha W_{2j}}\right] = 0$$

following the first-order condition (A.6). Therefore, $\mathrm{d}X_{1j}/\mathrm{d}Q_{1i} = 0, \forall i \in \{1, ..., n\}$.

Following the same steps as above, for each call option i, we have $\mathrm{d}Y_{1ji}/\mathrm{d}P_1 = 0$, $\mathrm{d}Y_{1ji}/\mathrm{d}Q_{1i} < 0$, and $\mathrm{d}Y_{1ji}/\mathrm{d}Q_{1l} = 0$ for $l \neq i$. Therefore, the Jacobian (A.8) has all off-diagonal terms equal to zero and all diagonal terms strictly negative. It is negative definite, implying a unique solution to $F(\cdot) = 0$. This unique solution of prices also implies that investors' demand, pinned down by the first-order conditions (A.6), is unique.

To sum up, we have characterized the $t = 1$ equilibrium under any arbitrary set of (call) options in terms of investors' optimal demand (A.6) and market clearing (A.7). We have also shown that the solution to the equation system (A.6) and (A.7) is unique. □

Proposition 6

Proof. This proof considers a more general quadratic payoff $f(D) = D^2 - aP_0^2 - bDP_0 - cD - eP_0 - f$. In particular, the stated result is a special case of $a = -1$, $b = 2$, and $c = e = f = 0$. The proof of Proposition 2 gives an investor's expected utility at $t = 1$, $\mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right]$, taking $p = P_1$ and $q = Q_1$ as given. Consider a demander ($j = d$) first. Expanding with $W_1 = W_0 + (P_1 - P_0)X_0$ and $\mathbb{E}_{1d}[D]$ with s and z gives

$$\mathbb{E}_{1d}\left[-e^{-\alpha W_{2d}}\right] = -\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1}} \cdot e^{-\alpha\cdot(W_0 + (P_1 - P_0)X_0 + (P_1 - \bar{D})z)} \cdot e^{-\frac{G_{1d}}{2}\left(\bar{D} + \frac{G_{1d} - G_0}{G_{1d}}(s - \bar{D}) - P_1\right)^2}$$

where $P_1 = P_1^{nd}$ can be further written as a linear combination of s and z. Taking the expectation of the above over $\{s, z\}$ yields the "interim" utility U_{0d} of a demander; that is, the expected utility after the type realizes but before the signal and the endowment shock are observed:

$$U_{0d} = -\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1} - \alpha W_{0d}} \left(1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2 \left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right) - \frac{\alpha^2}{G_0 \tau_z}\right)^{-\frac{1}{2}},$$

$$\begin{aligned}\text{where}\quad W_{0d} :=& W_0 + (\bar{D} - P_0)X_0 - \frac{\alpha}{G_0}X_0\bar{X} + \frac{\alpha}{2G_0}\bar{X}^2 - \frac{\alpha}{2}\left[1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2\left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right) - \frac{\alpha^2}{G_0\tau_z}\right]^{-1} \\ &\cdot \left\{\left(\frac{G_1 - G_0}{G_1}\right)^2 \left(\frac{1}{G_0} + \frac{1}{\tau_\varepsilon}\right)\left(1 + \frac{\alpha^2}{\tau_\varepsilon\tau_z}\right)(X_0 - \bar{X})^2 \right. \\ &\left. + \left(\frac{1}{G_0} + \frac{1}{\tau_\varepsilon}\right)^2 \frac{\alpha^2}{\tau_z}\left[2\left(1 - \frac{G_0}{G_1}\right)\left(1 - \frac{G_0}{G_{1d}}\right)X_0\bar{X} + \left[\frac{G_0}{G_{1d}}\left(1 - \frac{G_{1d}}{G_1}\right)^2 - \left(1 - \frac{G_0}{G_1}\right)^2\right]\bar{X}^2\right]\right\}.\end{aligned}$$

Similarly, the interim utility U_{0s} of liquidity suppliers can be derived as

$$U_{0s} = -\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1 - G_{1s}}{2G_1} - \alpha W_{0s}} \left(1 + \frac{G_0}{G_{1s} - G_0}\left(1 - \frac{G_{1s}}{G_1}\right)^2\right)^{-\frac{1}{2}},$$

$$W_{0s} = W_0 + (\bar{D} - P_0)X_0 - \frac{\alpha}{2G_0}X_0^2 + \frac{\alpha G_{1s}}{2G_1^2}\left[1 + \frac{G_0}{G_{1s} - G_0}\left(1 - \frac{G_{1s}}{G_1}\right)^2\right]^{-1} \cdot \left(X_0 - \bar{X}\right)^2.$$

At $t = 0$, investors choose X_0 to maximize $\pi U_{0d} + (1-\pi)U_{0s}$. The first-order condition, together with the market clearing condition $X_0 = \bar{X}$, leads to

$$\pi \cdot \left(\bar{D} - p - \alpha G_0^{-1}\bar{X} - \alpha\Sigma\bar{X}\right)M + (1-\pi)\left(\bar{D} - p - \alpha G_0^{-1}\bar{X}\right) = 0,$$

$$\text{where} \quad M = e^{\frac{G_{1s}-G_{1d}}{2G_1}}\sqrt{\frac{G_{1d}}{G_{1s}}}\exp\left(\frac{\alpha}{2}\Delta_2\bar{X}^2\right)\sqrt{\frac{1+\pi^2\Delta_0}{1+\Delta_0(1-\pi)^2 - \alpha^2/(\tau_z G_0)}},$$

and Δ_0 and Δ_2 are the same coefficients as given in the proof of Propositions 5 and 1. (Note that the second-order conditions are satisfied as well as both U_{0d} and U_{0s} are monotone transformation of quadratic terms in X_0.) It can be seen that the above first-order condition is linear in the market clearing price p, which then uniquely solves the equilibrium P_0 stated in the proposition.

Conditional on the realization of P_0, in the no-derivative benchmark, following Vayanos and Wang (2012), the liquidity demanders' interium utility is

$$U_{0d}^{nd} = -e^{-\alpha W_{0d}^{nd}}\left(1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2\left(1 + \frac{\alpha^2}{\tau_\varepsilon\tau_z}\right) - \frac{\alpha^2}{G_0\tau_z}\right)^{-\frac{1}{2}},$$

where $W_{0d}^{nd} = W_{0d}$. As $0 < \frac{G_1}{G_{1d}} < 1$, $\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1-G_{1d}}{2G_1}}$ is a decreasing function of $\frac{G_{1d}}{G_1}$. Therefore, $\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1-G_{1d}}{2G_1}} < 1$ and $U_{0d} > U_{0d}^{nd}$. Likewise, for the liquidity suppiers, $\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1-G_{1s}}{2G_1}}$ is an increasing function of $\frac{G_{1s}}{G_1}$ because $\frac{G_1}{G_{1s}} > 1$. Then $\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1-G_{1s}}{2G_1}} < 1$ and $U_{0s} > U_{0s}^{nd}$. □

Corollary 1

Proof. This proof considers a more general quadratic payoff $f(D) = D^2 - aP_0^2 - bDP_0 - cD - eP_0 - f$. In particular, the stated result is a special case of $a = -1$, $b = 2$, and $c = e = f = 0$. By market clearing, we have $\pi(X_{1d} - \bar{X}) = -(1-\pi)(X_{1s} - \bar{X})$, where

$$X_{1s} = \frac{\mathbb{E}_{1s}[D] - P_1}{\alpha \mathrm{var}_{1s}[D]} - (2P_1 - bP_0 - c)Y_{1s} \text{ and } Y_{1s} = \frac{1}{2\alpha}(G_1 - G_{1s}).$$

Therefore,

$$\lambda = \frac{\mathrm{cov}[P_1 - P_0, \pi \cdot (X_{1d} - X_0)]}{\mathrm{var}[\pi \cdot (X_{1d} - X_0)]} = \frac{-\mathrm{cov}\left(P_1 - P_0, (1-\pi)\left(X_{1s} - \bar{X}\right)\right)}{\mathrm{var}\left[(1-\pi)(X_{1s} - \bar{X})\right]} = \frac{1}{1-\pi}\frac{(G_1 - G_0)}{(G_1 - G_{1s})}\frac{\alpha}{G_1}.$$

Compared to Equation (18) in the benchmark, we have $\lambda/\lambda^{nd} = G_0/G_1 < 1$.

From Proposition 2, P_1 is unchanged after the introduction of options, $P_1 = P_1^{nd}$. As a result, the price reversal γ is unaffected. □

Corollary 2

Proof. From Equation (19), $\frac{M}{M^{nd}} = \sqrt{\frac{G_{1d}}{G_{1s}}} e^{-\frac{G_{1d}-G_{1s}}{2G_1}} \gtrless 1$. Define $\pi^* \equiv \frac{1}{\log(G_{1d}/G_{1s})} - \frac{G_{1d}}{G_{1d}-G_{1s}}$, we have

(1) $\exp\left(\frac{G_{1s}-G_{1d}}{2G_1}\right)\sqrt{\frac{G_{1d}}{G_{1s}}} > 1$ only if $\frac{G_{1d}-G_{1s}}{G_1} + \log\left(\frac{G_{1s}}{G_{1d}}\right) < 0$, which is equivalent to $\pi > \pi^*$.

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References

"Liquidity: Replications, Extensions, and Critique." In *Critical Finance Review*, Vol. 8, I. Welch, ed. NOW Publisher (2019).

Acharya, V. V., and L. H. Pedersen. "Asset Pricing with Liquidity Risk." *Journal of Financial Economics*, 77 (2005), 375–410.

Albagli, E.; C. Hellwig; and A. Tsyvinski. "A Theory of Asset Prices based on Heterogeneous Information." (2012).

Amihud, Y. "Illiquidity and Stock Returns: Cross-Section and Time-Series Effects." *Journal of Financial Markets*, 5 (2002), 31–56.

Amihud, Y. "Illiquidity and Stock Returns: A Revisit." *Critical Finance Review*, 8 (2019), 203–221.

Amihud, Y.; H. Mendelson; and L. H. Pedersen. "Liquidity and Asset Prices." In *Foundations and Trends in Finance*, Vol. 1, G. M. Constantinides; F. Allen; A. W. Lo; and R. M. Stulz, eds. now Publishers Inc. (2005), 269–364.

Barbon, A., and A. Buraschi. "Gamma Fragility." (2019).

Barlevy, G., and P. Veronesi. "Rational Panics and Stock Market Crashes." *Journal of Economic Theory*, 110 (2003), 234–263.

Ben-Rephael, A.; O. Kadan; and A. Whol. "The Diminishing Liquidity Premium." *Journal of Financial and Quantitative Analysis*, 50 (2015), 197–229.

Bernardo, A. E., and K. L. Judd. "Asset Market Equilibrium with General Tastes, Returns, and Informational Asymmetries." *Journal of Financial Markets*, 3 (2000), 17–43.

Biais, B.; P. Bossaerts; and C. Spatt. “Equilibrium Asset Pricing and Portfolio Choice Under Asymmetri Information.” *The Review of Financial Studies*, 23 (2010), 1503–1543.

Biais, B., and P. Hillion. “Insider and Liquidity Trading in Stock and Options Markets.” *The Review of Financial Studies*, 7 (1994), 743–780.

Branch, B., and J. E. Finnerty. “The impact of option listing on the price and volume of the underlying stock.” *Financial Review*, 16 (1981), 1–15.

Brennan, M. J., and H. H. Cao. “Information, Trade, and Derivative Securities.” *The Review of Financial Studies*, 9 (1996), 163–208.

Brennan, M. J., and A. Subrahmanyam. “Market Microstructure and Asset Pricing: On the Compensation for Illiquidity in Stock Returns.” *Journal of Financial Economics*, 41 (1996), 441–464.

Breon-Drish, B. “On Existence and Uniqueness of Equilibrium in a Class of Noisy Rational Expectations Models.” *The Review of Economic Studies*, 82 (2015), 868–921.

Campbell, J. Y.; S. J. Grossman; and J. Wang. “Trading Volume and Serial Correlation in Stocks Returns.” *The Quarterly Journal of Economics*, 108 (1993), 905–939.

Cao, H. H. “The Effect of Derivative Assets on Information Acquisition and Price Behavior in a Rational Expectations Equilibrium.” *The Review of Financial Studies*, 12 (1999), 131–163.

Chabakauri, G.; K. Yuan; and K. Zachariadis. “Multi-Asset Noisy Rational Expectations Equilibrium with Contingent Claims.” (2017).

Collin-Dufresne, P., and V. Fos. “Do Prices Reveal the Presence of Informed Trading?” *The Journal of Finance*, 70 (2015), 1555–1582.

Conrad, J. “The price effect of option introduction.” *The Journal of Finance*, 44 (1989), 487–498.

Damodaran, A., and J. Lim. “The effects of option listing on the underlying stocks’ return process.” *Journal of Banking & Finance*, 15 (1991), 647–664.

Danielsen, B. R., and S. M. Sorescu. “Why Do Option Introductions Depress Stock Prices? A Study of Diminishing Short Sale Constraints.” *Journal of Financial and Quantitative Analysis*, 36 (2001), 451–484.

Detemple, J., and P. Jorion. “Option listing and stock returns: An empirical analysis.” *Journal of Banking and Finance*, 14 (1990), 781–801.

Dow, J. “Arbitrage, Hedging, and Financial Innovation.” *The Review of Financial Studies*, 11 (1998), 739–755.

Drienko, J.; T. Smith; and A. von Reibnitz. “A Review of the ReturnIlliquidity Relationship.” *Critical Finance Review*, 8 (2019), 127–171.

Easley, D.; M. O’Hara; and P. S. Srinivas. “Option Volume and Stock Prices: Evidence on Where Informed Traders Trade.” *The Journal of Finance*, 53 (1998), 431–465.

Fedenia, M., and T. Grammatikos. “Options Trading and the Bid-Ask Spread of the Underlying Stocks.” *Journal of Business*, 65 (1992), 335–351.

Gao, F., and J. Wang. “The Market Impact of Options.” (2017).

Gârleanu, N.; L. H. Pedersen; and A. M. Poteshman. “Demand-Based Option Pricing.” *The Review of Financial Studies*, 22 (2009), 4259–4299.

Glosten, L. R., and L. E. Harris. “Estimating the Components of the Bid-Ask Spread.” *Journal of Financial Economics*, 21 (1988), 123–142.

Grossman, S. J., and M. H. Miller. “Liquidity and Market Structure.” *The Journal of Finance*, 43 (1988), 617–633.

Grossman, S. J., and J. E. Stiglitz. “On the Impossibility of Informationally Efficient Markets.” *American Economic Review*, 70 (1980), 393–408.

Han, B. Y. “Dynamic Information Acquisition and Asset Prices.” (2018).

Harris, L., and A. Amato. “Illiquidity and Stock Returns: Cross-Section and Time-Series Effects: A Replication.” *Critical Finance Review*, 8 (2019), 173–202.

Hasbrouck, J. “Trading Costs and Returns for U.S. Equities: Estimating Effective Costs from Daily Data.” *The Journal of Finance*, 64 (2009), 1445–1477.

Hu, J. “Option Listing and Information Asymmetry.” *Review of Finance*, 22 (2017), 1153–1194.

Huang, S. “The Effects of Options on Information Acquisition and Asset Pricing.” (2015).

Kumar, R.; A. Sarin; and K. Shastri. “The Impact of Index Options on the Underlying Stocks: The Evidence from the Listing of Nikkei Stock Average Options.” *Pacific-Basin Finance Journal*, 3 (1995), 303–317.

Kumar, R.; A. Sarin; and K. Shastri. “The Impact of Options Tradign on the Market Quality of the Underlying Security: An Empirical Analysis.” *The Journal of Finance*, 53 (1998), 717–732.

Kyle, A. S. “Continuous Auctions and Insider Trading.” *Econometrica*, 53 (1985), 1315–1336.

Li, H.; R. Novy-Marx; and M. Velikov. “Liquidity Risk and Asset Pricing.” *Critical Finance Review*, 8 (2019), 223–255.

Llorente, G.; R. Michaely; G. Saar; and J. Wange. "Dynamic VolumeReturn Relation of Individual Stocks." *The Review of Financial Studies*, 15 (2002), 1005–1047.

Malamud, S. "Noisy Arrow-Debreau Equilibria." (2015).

Marín, J. M., and R. Rahi. "Speculative securities." *Economic Theory*, 14 (1999), 653–668.

Massa, M. "Financial Innovation and Information: The Role of Derivatives When a Market for Information Exists." *The Review of Financial Studies*, 15 (2002), 927–957.

Mayhew, S., and V. T. Mihov. "Another Look at Option Listing Effects." (2000).

Mayhew, S., and V. T. Mihov. "How Do Exchanges Select Stocks for Option Listing?" *The Journal of Finance*, 59 (2004), 447–471.

Mixon, S., and E. Onur. "Volatility Derivatives in Practice: Activity and Impact." (2014).

Ni, S. X.; N. D. Pearson; and A. M. Poteshman. "Stock Price Clustering on Option Expiration Dates." *Journal of Financial Economics*, 78 (2005), 49–87.

Ni, S. X.; N. D. Pearson; A. M. Poteshman; and J. White. "Does Option Trading Have a Pervasive Impact on Underlying Stock Prices?" *The Review of Financial Studies*, 34 (2021), 1952–1986.

Pástor, v., and R. F. Stambaugh. "Liquidity Risk and Expected Returns." *Journal of Political Economy*, 111 (2003), 642–685.

Pástor, v., and R. F. Stambaugh. "Liquidity Risk After 20 Years." *Critical Finance Review*, 8 (2019), 277–299.

Pontiff, J., and R. Singla. "Liquidity Risk?" *Critical Finance Review*, 8 (2019), 257–276.

Roll, R. "A Simple Implicit Measure of the Effective Bid-Ask Spread in an Efficient Market." *The Journal of Finance*, 39 (1984), 1127–39.

Sadka, R. "Momentum and post-earnings-announcement drift anomalies: The role of liquidity risk." *Journal of Financial Economics*, 80 (2006), 309–349.

Smith, K. "Financial Markets with Trade on Risk and Return." *The Review of Financial Studies*, 4041–4078.

Sorescu, S. M. "The Effect of Options on Stock Prices: 1973 to 1995." *The Journal of Finance*, 55 (2000), 487–514.

Vayanos, D., and J. Wang. "Liquidity and Asset Returns Under Asymmetric Information and Imperfect Competition." *The Review of Financial Studies*, 25 (2012), 1339–1365.

Vayanos, D., and J. Wang. "Chapter 19: Market Liquidity—Theory and Empirical Evidence." In

Handbook of the Economics of Finance, Vol. 2, G. M. Constantinides; M. Harris; and R. M. Stulz, eds. Elsevier (2013), 1289–1361.

Derivatives and Market (Il)liquidity

Internet Appendix

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This internet appendix contains the following parts:

- Section S1 studies a path-dependent general quadratic derivative; and
- Section S2 provides additional useful lemmas.

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S1 Introducing a path-dependent derivative

This internet appendix studies a possibly path-dependent quadratic derivative, $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$. In particular, we allow $f(D)$ to depend on the underlying asset price P_1 (even though $f(D)$ is realized at $t = 2$), hence "path-dependent."

Proposition S1. *With $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$, there exists a unique equilibrium at $t = 1$. The demand schedules for the underlying are*

$$X_{1d}(p, q; s, z) = X_{1d}^{nd}(p; s, z) + [(b-2)p + c]Y_{1d}(p, q; s, z); \text{ and}$$
$$X_{1s}(p, q) = X_{1s}^{nd}(p) + [(b-2)p + c]Y_{1s}(p, q).$$

The demand schedules for the general variance swap are

$$Y_{1d}(p, q; s, z) = \frac{1}{2\alpha}\left(\left(q + \left((a+b-1)p^2 + (c+e)p + f\right)\right)^{-1} - G_{1d}\right); \text{ and}$$
$$Y_{1s}(p, q) = \frac{1}{2\alpha}\left(\left(q + \left((a+b-1)p^2 + (c+e)p + f\right)\right)^{-1} - G_{1s}\right).$$

The underlying's market clears at $P_1 = P_1^{nd}$, the same as in the benchmark (Equation (6)). The derivative's market clears at $Q_1 = G_1^{-1} - (a+b-1)P_1^2 - (c+e)P_1 - f$. The conditional precision $\{G_{1d}, G_{1s}, G_1\}$ are the same as those defined in Proposition 1.

Proof. Consider a type-j investor. Her terminal wealth W_{2j} is given by

$$W_{2j} = W_0 + (P_1 - P_0)X_0 + (D - P_1)X_{1j} + (f(D) - Q_1)Y_{1j} + (D - \bar{D})z_j. \quad \text{(S1)}$$

Lemma 1 ensures that she holds the same posterior distribution for D with or without the derivative. In particular, D remains conditionally normal. Let $z_s = 0$, $z_d = z$, and $W_1 = W_0 + (p - P_0)X_0$.

Evaluating the expected utility (e.g., Lemma A.1 of Marín and Rahi (1999)) yields,

$$\begin{aligned}
&\mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right] \\
&= -\frac{1}{\sqrt{1+2\alpha \mathrm{var}_{1j}[D]Y_{1j}}} \exp\left[\alpha\Big(-W_1 + z_j(\bar{D}-p) - Y_{1j}\Big((1-a-b)p^2 - (c+e)p - f - q\Big)\Big)\right] \\
&\quad \cdot \exp\left[-\alpha\big(X_{1j} + z_j + ((2-b)p - c)Y_{1j}\big)(\mathbb{E}_{1j}[D]-p) - \alpha Y_{1j}(\mathbb{E}_{1j}[D]-p)^2\right] \\
&\quad \cdot \exp\left[\frac{\alpha^2 \mathrm{var}_{1j}[D]\big(X_{1j} + z_j + Y_{1j}(2\mathbb{E}_{1j}[D] - bp - c)\big)^2}{2\big(1+2\alpha\mathrm{var}_{1j}[D]Y_{1j}\big)}\right].
\end{aligned}$$

The first-order condition with respect to X_{1j} yields

$$X_{1j} = \frac{\mathbb{E}_{1j}[D]-p}{\alpha \mathrm{var}_{1j}[D]} - z_j - ((2-b)p-c)Y_{1j}.$$

Plug this back to $\mathbb{E}_{1j}[-e^{-\alpha W_{2j}}]$ and evaluate the first-order condition with respect to Y_{1j} to get:

$$Y_{1j} = \frac{1}{2\alpha}\left(\frac{1}{q+(a+b-1)p^2+(c+e)p+f} - \frac{1}{\mathrm{var}_{1j}[D]}\right).$$

Finally, clearing the market yields the equilibrium prices $p = P_1$ and $q = Q_1$ as stated in the proposition. (The utility maximization problem is a strictly concave one. Hence, the above solution implied by the first-order conditions are unique.) □

With the variance swap $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$, the two liquidity measures can be found, following Proposition 2, as

$$\begin{aligned}
\lambda &= \frac{\alpha}{1-\pi}\frac{G_1-G_0}{G_1-G_{1s}}\frac{1}{G_1-0.5b(G_1-G_0)} = \frac{G_0}{G_1-\frac{b}{2}(G_1-G_0)}\lambda^{nd} \quad \text{and} \\
\gamma &= \left(1-\frac{G_0}{G_1}\right)\left(1-\frac{G_{1s}}{G_1}\right)\frac{1}{G_{1s}-G_0} = \gamma^{nd}.
\end{aligned}$$

As can be seen, the possibly path-dependent derivative might change the result of $\lambda < \lambda^{nd}$ from Corollary 1. This happens if and only if the coefficient $b \geq 2$, i.e., the loading on DP_1. This is because with such path-dependent derivatives, the built-in dependence of $f(D)$ on P_1 creates some "mechanical" delta-hedging needs for the investors. In the quadratic example above, the total delta-hedging ratio is $\hat{\mathbb{E}}_{1j}\left[\frac{\partial f}{\partial D}\right] = 2P_1 - bP_1 - c$, and we can see that the term $-bP_1$ contributes to it, simply because of the built-in interaction between the actual terminal payoff D and the intermediate price P_1. In particular, when $b \geq 2$, the sign of the delta-hedging ratio above changes, mechanically affecting the information-to-noise ratio in the underlying and, hence, also the price impact λ.

On the other hand, the price reversal γ is unaffected, because for both types of investors, $j \in \{d, s\}$, the above delta hedging ratio remains the same. Hence, the net delta-hedging trading remains zero, as in the case of a path-independent variance swap in the paper, and there is no additional price pressure, ensuring $P_1 = P^{nd}$. As such, γ remains unaffected.

Proposition S2. *With $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$, the underlying's $t = 0$ equilibrium price remains the same as stated in Proposition 6.*

Proof. The proof of Proposition S1 gives an investor's expected utility at $t = 1$, $\mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right]$, taking $p = P_1$ and $q = Q_1$ as given. Consider a demander ($j = d$) first. Expanding with $W_1 = W_0 + (P_1 - P_0)X_0$ and $\mathbb{E}_{1d}[D]$ with s and z gives

$$\mathbb{E}_{1d}\left[-e^{-\alpha W_{2d}}\right] = -\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1}} \cdot e^{-\alpha\cdot(W_0 + (P_1 - P_0)X_0 + (P_1 - \bar{D})z)} \cdot e^{-\frac{G_{1d}}{2}\left(\bar{D} + \frac{G_{1d} - G_0}{G_{1d}}(s - \bar{D}) - P_1\right)^2}$$

where $P_1 = P_1^{nd}$ can be further written as a linear combination of s and z. Taking the expectation of the above over $\{s, z\}$ yields the "interim" utility U_{0d} of a demander; that is, the expected utility after the type realizes but before the signal and the endowment shock are observed:

$$U_{0d} = -\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1} - \alpha W_{0d}} \left(1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2 \left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right) - \frac{\alpha^2}{G_0 \tau_z}\right)^{-\frac{1}{2}},$$

where

$$\begin{aligned} W_{0d} :=& W_0 + (\bar{D} - P_0)X_0 - \frac{\alpha}{G_0} X_0 \bar{X} + \frac{\alpha}{2G_0}\bar{X}^2 - \frac{\alpha}{2}\left[1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2\left(1 + \frac{\alpha^2}{\tau_\varepsilon\tau_z}\right) - \frac{\alpha^2}{G_0\tau_z}\right]^{-1} \\ &\cdot \left\{\left(\frac{G_1 - G_0}{G_1}\right)^2 \left(\frac{1}{G_0} + \frac{1}{\tau_\varepsilon}\right)\left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right)(X_0 - \bar{X})^2 \right. \\ &\left. + \left(\frac{1}{G_0} + \frac{1}{\tau_\varepsilon}\right)^2 \frac{\alpha^2}{\tau_z}\left[2\left(1 - \frac{G_0}{G_1}\right)\left(1 - \frac{G_0}{G_{1d}}\right)X_0\bar{X} + \left[\frac{G_0}{G_{1d}}\left(1 - \frac{G_{1d}}{G_1}\right)^2 - \left(1 - \frac{G_0}{G_1}\right)^2\right]\bar{X}^2\right]\right\}. \end{aligned}$$

Note that condition $\alpha^2 G_0^{-1}\tau_z^{-1} < 1$ ensures U_{0d} is well-defined; in particular, the term inside the brackets is always positive. Similarly, the interim utility U_{0s} of liquidity suppliers can be derived as

$$U_{0s} = -\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1 - G_{1s}}{2G_1} - \alpha W_{0s}} \left(1 + \frac{G_0}{G_{1s} - G_0}\left(1 - \frac{G_{1s}}{G_1}\right)^2\right)^{-\frac{1}{2}},$$

where

$$W_{0s} = W_0 + (\bar{D} - P_0)X_0 - \frac{\alpha}{2G_0}X_0^2 + \frac{\alpha G_{1s}}{2G_1^2}\left[1 + \frac{G_0}{G_{1s} - G_0}\left(1 - \frac{G_{1s}}{G_1}\right)^2\right]^{-1} \cdot \left(X_0 - \bar{X}\right)^2.$$

At $t = 0$, investors choose X_0 to maximize

$$\pi U_{0d} + (1 - \pi)U_{0s}.$$

The first-order condition, together with the market clearing condition $X_0 = \bar{X}$, leads to

$$\pi \cdot \left(\bar{D} - p - \alpha G_0^{-1}\bar{X} - \alpha \Sigma \bar{X}\right)M + (1 - \pi)\left(\bar{D} - p - \alpha G_0^{-1}\bar{X}\right) = 0,$$

where

$$M = e^{\frac{G_{1s} - G_{1d}}{2G_1}} \sqrt{\frac{G_{1d}}{G_{1s}}} \exp\left(\frac{\alpha}{2}\Delta_2 \bar{X}^2\right)\sqrt{\frac{1 + \pi^2\Delta_0}{1 + \Delta_0(1 - \pi)^2 - \alpha^2/(\tau_z G_0)}},$$

and Δ_0 and Δ_2 are the same coefficients as given in the proof of Propositions 5 and 1. (Note that the second-order conditions are satisfied as well as both U_{0d} and U_{0s} are monotone transformation of quadratic terms in X_0.) It can be seen that the above first-order condition is linear in the market clearing price p, which then uniquely solves the equilibrium P_0 stated in the proposition.

Conditional on the realization of P_0, in the no-derivative benchmark, following Vayanos and Wang (2012), the liquidity demanders' interium utility is

$$U_{0d}^{nd} = -e^{-\alpha W_{0d}^{nd}}\left(1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2\left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right) - \frac{\alpha^2}{G_0 \tau_z}\right)^{-\frac{1}{2}},$$

where $W_{0d}^{nd} = W_{0d}$. As $0 < \frac{G_1}{G_{1d}} < 1$, $\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1}}$ is a decreasing function of $\frac{G_{1d}}{G_1}$. Therefore, $\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1}} < 1$ and $U_{0d} > U_{0d}^{nd}$. Likewise, for the liquidity suppiers, $\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1 - G_{1s}}{2G_1}}$ is an increasing function of $\frac{G_{1s}}{G_1}$ because $\frac{G_1}{G_{1s}} > 1$. Then $\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1 - G_{1s}}{2G_1}} < 1$ and $U_{0s} > U_{0s}^{nd}$. □

As we have seen above, while the path-dependence of the derivative payoff affects an individual investor's delta hedging at $t = 1$, in aggregate, the net delta-hedging trade remains zero. As such, intuitively, the path-dependent derivative does not create additional trading gains and nor does it affect the split of the "pie." One step back to $t = 0$, therefore, the evaluation of the underlying asset is unaffected.

S2 Additional lemmas

Lemma S1

> **Lemma S1 (Decomposition of a call).** *Suppose the underlying price at $t = 1$ is P_1. The $t = 2$ payoff of an out-of-the-money call option with strike $K \geq P_1$ can be decomposed into*
>
> $$\max\{0, D - K\} = \frac{1}{2}|D - P_1| + \frac{1}{2}\left(1 - 2\mathbb{1}_{\{P_1 \leq D \leq K\}}\right)(D - P_1) + \mathbb{1}_{\{D>K\}}(P_1 - K);$$
>
> *and that of an in-the-money call with $K \leq P_1$ can be decomposed into*
>
> $$\max\{0, D - K\} = \frac{1}{2}|D - P_1| + \frac{1}{2}\left(1 + 2\mathbb{1}_{\{K \leq D \leq P_1\}}\right)(D - P_1) + \mathbb{1}_{\{D>K\}}(P_1 - K).$$

Proof. Consider the out-of-the-money call with $K \geq P_1$.

$$\max\{0, D - K\} = \frac{1}{2}|D - K| + \frac{1}{2}(D - K) = \frac{1}{2}|V - (K - P_1)| + \frac{1}{2}(V - (K - P_1))$$

where $V := D - P_1$ as a shorthand notation. Compare $|V - (K - P_1)|$ to $|V|$:

$$|V - (K - P_1)| - |V| = \begin{cases} K - P_1, & \text{if } V < 0 \\ -2V + (K - P_1), & \text{if } 0 \leq V \leq K - P_1 \\ -(K - P_1), & \text{if } V > K - P_1 \end{cases}.$$

Therefore, $|V - (K - P_1)| = |V| + \mathbb{1}_{\{V<0\}}(K - P_1) + \mathbb{1}_{\{0 \leq V \leq K - P_1\}}(-2V + K - P_1) - \mathbb{1}_{\{V>K-P_1\}}(K - P_1)$. Substituting into the call's payoff expression and simplifying gives the expression stated in the lemma. The proof for the decomposition of the in-the-money call repeats the above steps and is omitted. □

Lemma S2

Lemma S2 (Risk-neutral pricing). *The equilibrium underlying price P_1 and the derivative price Q_1 must satisfy*

$$P_1 = \hat{\mathbb{E}}_{1j}[D] = \int_{\mathbb{R}} D\hat{\phi}_{1j}(D)\mathrm{d}D \quad \textit{and} \quad Q_1 = \hat{\mathbb{E}}_{1j}[f(D)] = \int_{\mathbb{R}} f(D)\hat{\phi}_{1j}(D)\mathrm{d}D \tag{S2}$$

where $\hat{\phi}_{1j}(D)$ is a type-j investor's risk-neutral density, defined as

$$\hat{\phi}_{1j}(D) := \frac{h_{1j}(D)}{\int_{\mathbb{R}} h_{1j}(D)\mathrm{d}D}, \quad \textit{with } h_{1j}(D) := e^{-\alpha\cdot\left(DX_{1j}+f(D)Y_{1j}\right)-\frac{G_{1j}}{2}D^2+(G_0\bar{D}+(G_{1j}-G_0)\eta)D}. \tag{S3}$$

Proof. The risk-neutral pricing formulas follow the first-order conditions

$$\frac{\partial U_{1j}}{\partial X_{1j}} = \mathbb{E}_{1j}\left[\alpha\cdot(D-P_1)e^{-\alpha W_{2j}}\right] = 0 \text{ and } \frac{\partial U_{1j}}{\partial Y_{1j}} = \mathbb{E}_{1j}\left[\alpha\cdot(f(D)-Q_1)e^{-\alpha W_{2j}}\right] = 0$$

which imply

$$P_1 = \frac{\mathbb{E}_{1j}[De^{-\alpha W_{2j}}]}{\mathbb{E}[e^{-\alpha W_{2j}}]} = \frac{\int_{\mathbb{R}} De^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}{\int_{\mathbb{R}} e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D} \text{ and } Q_1 = \frac{\mathbb{E}_{1j}[f(D)e^{-\alpha W_{2j}}]}{\mathbb{E}[e^{-\alpha W_{2j}}]} = \frac{\int_{\mathbb{R}} f(D)e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}{\int_{\mathbb{R}} e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}$$

where $\phi_{1j}(D)$ is the type-j investor's posterior density (conditional on the prices) of D. Letting $\hat{\phi}_{1j}(D) := \frac{e^{-\alpha W_{2j}}\phi_{1j}(D)}{\int_{\mathbb{R}} e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}$, one obtains the risk-neutral pricing formula given in the lemma. It remains to simplify the expression of $\hat{\phi}_{1j}(D)$. To do so, recall $W_{2j} = W_1 + (D-P_1)X_{1j} + (f(D) - Q_1)Y_{1j} + (D-\bar{D})z_j$, where $W_1 = W_0 + (P_1 - P_0)X_0$ and z_j is a type-j investor's endowment shock ($z_d = z$ and $z_s = 0$). In addition, by Lemma 1, $\phi_{1d}(D)$ is the normal density with mean $\frac{G_0}{G_{1d}}\bar{D}+\frac{G_{1d}-G_0}{G_{1d}}s$ and variance G_{1d}^{-1}; and $\phi_{1s}(D)$ is the normal density with mean $\frac{G_0}{G_{1s}}\bar{D} + \frac{G_{1s}-G_0}{G_{1s}}\eta$ (with $\eta := s - \frac{\alpha}{\tau_\varepsilon}z$) and variance G_{1s}^{-1}. The simplified expression of $\hat{\phi}_{1j}(D)$ with $h_{1j}(D)$ follows by plugging these expressions into $\hat{\phi}_{1j}(D)$ and offsetting common terms in the numerator and the denominator. □