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Valuation of risky projects and other illiquid investments using portfolio selection models

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Chapter 4
Valuation of Risky Projects and Illiquid Investments Using Portfolio Selection Models

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Abstract We develop a portfolio selection framework for the valuation of projects and other illiquid investments for an investor who can invest in a portfolio of private, illiquid investment opportunities as well as in securities in financial markets, but who cannot necessarily replicate project cash flows using financial intruments. We demonstrate how project values can be solved using an inverse optimization procedure and prove several general analytical properties for project values. We also provide an illustrative example on the modeling and pricing of multiperiod projects that are characterized by managerial flexibility.

4.1 Introduction

Project valuation and selection has attracted plenty of attention among researchers and practitioners over the past few decades. Suggested methods for this purpose include (1) discounted cash flow analysis (DCF, see e.g. Brealey and Myers 2000) to account for the time value of money, (2) project portfolio optimization (see Luenberger 1998) to account for limited resources and several competing projects, and (3) options pricing analysis, which has focused on the recognition of the managerial flexibility embedded in projects (Dixit and Pindyck 1994; Trigeorgis 1996). Despite the research efforts that have been made to address challenges in project valuation, traditional methods tend to suffer from shortcomings which limit their practical use and theoretical relevance. For example, DCF analysis does not specify how the discount rate should be derived so as to properly account for (a) the time value of money, (b) risk adjustment implied by (1) the investor's risk aversion (if any) and (2) the risk of the project and its impact on the aggregate risk faced

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by the investor through diversification and correlations; and (c) opportunity costs imposed by alternative investment opportunities such as financial instruments and competing project opportunities. Traditionally, it is proposed that a project's cash flows should be discounted at the rate of return of a publicly traded security that is equivalent in risk to the project (Brealey and Myers 2000). However, the definition of what constitutes an "equivalent" risk is problematic; in effect, unless a publicly traded instrument (or a trading strategy of publicly traded instruments) exactly replicates the project cash flows in all future states of nature, it is questionable whether a true equivalence exists. (Also, it is not clear if such a discount rate would account for anything else than opportunity costs implied by securities.) Likewise, traditional options pricing analysis requires that the project's cash flows are replicated with financial instruments. Still, most private projects and other similar illiquid investments are, by their nature, independent of the fluctuations of the prices of publicly traded securities. Thus, the requirement of the existence of a replicating portfolio for a private project seems unsatisfactory as a theoretical assumption.

In this chapter, we approach the valuation of private investment opportunities through portfolio optimization models. We examine a setting where an investor can invest both in market-traded, infinitely divisible assets as well as lumpy, nonmarket-traded assets so that the opportunity costs of both classes of investments can be accounted for. Examples of such lumpy assets include corporate projects, but in general the analysis extends to any nontraded, all-or-nothing-type investments. Market-traded assets include relevant assets that are available to the investor, such as equities, bonds, and the risk-free asset which provides a risk-free return from one period to the next. In particular, we demonstrate how portfolio optimization models can be used to determine the value of each nontraded lumpy asset within the portfolio. We also show that the resulting values are consistent with options pricing analysis in the special case that a replicating portfolio (or trading strategy) exists for such a private investment opportunity.

Our procedure for the valuation of a project resembles the traditional net present value (NPV) analysis in that we determine what amount of money at present the investor equally prefers to the project, given all the alternative investment opportunities. Because this equivalence can be determined in two similar but different ways (where the choice of the appropriate way depends on the setting being modeled), we present two pricing concepts: breakeven selling price (BSP) and breakeven buying price (BBP), which have been used frequently in decision analytic approaches to investment decision making and even other settings (Luenberger 1998; Raiffa 1968; Smith and Nau 1995). Regarding the investor's preferences, we seek to keep the treatment quite generic by allowing the use of virtually any rational preference model. The preference model merely influences the objective function of the portfolio model and possibly introduces some risk constraints, which do not alter the core of the valuation procedure, on condition that the preference model makes it possible to determine when two investment portfolios are equally preferred. In particular, we show that the required portfolio models can be formulated for both expected utility maximizers and mean-risk optimizers.

To illustrate how the project valuation procedure can be implemented in a multiperiod setting with projects that are characterized by varying degrees of managerial flexibility, we use contingent portfolio programming (CPP, see Gustafsson and Salo 2005) to formulate a multiperiod project-security portfolio selection model. This approach uses project-specific decision trees to capture real options that are embedded in projects. Furthermore, we provide a numerical example that parallels the example in Gustafsson and Salo (2005). This example suggests that while there is wide array of issues to be addressed in multiperiod settings, it is still possible to deal with them all with the help of models that remain practically tractable.

Relevant applications of our valuation methodology include, for instance, pharmaceutical development projects, which are characterized by large nonmarket-related risks and which are often illiquid due to several factors (see Chapter 13 in this book). For example, if the project is very specialized, the expertise to evaluate it may be available only within the company, in which case it may be difficult for outsiders to verify the accuracy and completeness of evaluation information. Also, details concerning pharmaceutical compounds may involve corporate secrets that can be delivered only to trusted third parties, which may lower the number of potential buyers and decrease the liquidity of the project.

This chapter is structured as follows. Section 4.2 introduces the basic structure of integrated project-security portfolio selection models and discusses different formulations of portfolio selection problems. In Section 4.3, we introduce our valuation concepts and examine their properties. Section 4.4 discusses the valuation of investment opportunities such as real options embedded in the project. Section 4.5 gives an example of the framework in a multiperiod setting, which allows us to compare the results with a similar example in Gustafsson and Salo (2005). In Section 4.6, we summarize our findings and discuss implications of our results.

4.2 Integrated Portfolio Selection Models for Projects and Securities

4.2.1 Basic Model Structure

We consider investment opportunities in two categories: (1) *securities*, which can be bought and sold in any quantities and (2) *projects*, lumpy all-or-nothing type investments. From a technical point of view, the main difference between these two types of investments is that the projects' decision variables are binary, while those of the securities are continuous. Another difference is that the cost, or price, of securities is determined by a market equilibrium model, such as the Capital Asset Pricing Model (CAPM, see Lintner 1965; Sharpe 1964), while the investment cost of a project is an endogenous property of the project.

Portfolio selection models can be formulated either in terms of rates of return and portfolio weights, like in Markowitz-type formulations, or by specifying a budget constraint, expressing the *initial wealth level*, subject to which the investor's *terminal wealth level* is maximized. The latter approach is more appropriate to project portfolio selection, because the investor is often limited by a budget constraint and it is natural to characterize projects in terms of cash flows rather than in terms of portfolio weights and returns.

4.2.2 Types of Preference Models

Early portfolio selection formulations (see, e.g., Markowitz 1952) were bi-criteria decision problems minimizing risk while setting a target for expected return. Later, the mean-variance model was formulated in terms of expected utility theory (EUT) using a quadratic utility function. However, there are no similar utility functions for most other risk measures, including the widely used absolute deviation (Konno and Yamazaki 1991). In effect, due to the lexicographic nature of bi-criteria decision problems, most mean-risk models cannot be represented by a real-valued functional, thus being distinct from the usual preference functional models such as the expected utility model. Therefore, we distinguish between two classes of preference models: (1) preference functional models, such as the expected utility model, and (2) bi-criteria optimization models or mean-risk models.

For example, for EUT, the preference functional is $U[X] = E[u(X)]$, where $u(\cdot)$ is the investor's von Neumann–Morgenstern utility function. Many other kinds of preference functional models, such as Choquet-expected utility models, have also been proposed. In addition to preference functional models, mean-risk models have been widely used in the literature. These models are important, because much of the modern portfolio theory, including the CAPM, is based on a mean-risk model, namely the Markowitz mean-variance model (Markowitz 1952).

Table 4.1 describes three possible formulations for mean-risk models: *risk minimization*, where risk is minimized for a given level of expectation (Luenberger 1998), *expected value maximization*, where expectation is maximized for a given level of risk (Eppen et al. 1989), and the *additive formulation*, where the weighted sum of mean and risk is maximized (Yu 1985) and which is effectively a preference functional model. The models employed by Sharpe (1970) and Ogryczak and Ruszczynski (1999) are special cases of this model. The latter one, in particular, is important, because it can represent the investor's certainty equivalent. In Table 4.1, ρ is the investor's risk measure, μ is the minimum level for expectation, and R is the maximum level for risk. The parameters λ are tradeoff coefficients.

In our setting, the sole requirement for the applicable preference model is that it can uniquely identify equally preferable portfolios. By construction, models based on preference functionals have this property. Also, risk minimization and expected value maximization models can be employed if we define that equal preference

Table 4.1 Formulations of mean-risk preference models

	Objective	Constraints
Risk minimization	$\min \rho[X]$	$E[X] \geq \mu$
Expected value maximization	$\max E[X]$	$\rho[X] \leq R$
General additive	$\max \lambda_1 \times E[X] - \lambda_2 \times \rho[X]$	
Sharpe (1970)	$\max \lambda \times E[X] - \rho[X]$	
Ogryczak and Ruszczynski (1999)	$\max E[X] - \lambda \times \rho[X]$	

prevails whenever the objective function values are equal and the applicable constraints are satisfied (but not necessarily binding).

In general, there no particular reason to favor any one of these models, because the choice of the appropriate model may depend on the setting. For example, in settings involving the CAPM, the mean-variance model may be more appropriate, while decision theorists may prefer to opt for the expected utility model.

4.2.3 Single-Period Example Under Expected Utility

A single-period portfolio model under expected utility can be formulated as follows. Let there be n risky securities, a risk-free asset (labeled as the 0th security), and m projects. Let the price of asset i at time 0 be S_i^0 and let the corresponding (random) price at time 1 be $\tilde{S}_i^1$. The price of the risk-free asset at time 0 is 1 and $1 + r_\mathrm{f}$ at period 1, where r_f is the risk-free interest rate. The amounts of securities in the portfolio are denoted by $x_i,\ i = 0, \ldots, n$. The investment cost of project k in time 0 is C_k^0 and the (random) cash flow at time 1 is $\tilde{C}_k^1$. The binary variable z_k indicates whether project k is started or not. The investor's budget is b. We can then formulate the model using utility function u as follows:

1. maximize utility at time 1:

$$\max_{\mathbf{x},\mathbf{z}} E\left[u\left(\sum_{i=0}^{n} \tilde{S}_i^1 x_i + \sum_{k=1}^{m} \tilde{C}_k^1 z_k\right)\right]$$

subject to

2. budget constraint at time 0: $\sum_{i=0}^{n} S_i^0 x_i + \sum_{k=1}^{m} C_k^0 z_k \geq b$
3. binary variables for projects: $z_k \in \{0, 1\}, \quad k = 1, \ldots, m$
4. continuous variables for securities: x_i free $i = 0, \ldots, n$

In typical settings, the budget constraint could be formulated as an equality, because in the presence of a risk-free asset all of the budget will normally be expended at the optimum. In this model and throughout the chapter, it is assumed that there are no transaction costs or taxes on capital gains, and that the investor is able to borrow and lend at the risk-free interest rate without limit.

4.3 Valuation of Projects and Illiquid Investments

4.3.1 Breakeven Buying and Selling Prices

Because we consider projects as illiquid, nontradable investment opportunities, there is no market price that can be used to value the project. In such a setting, it is reasonable to define the value of the project as the cash amount at present that is equally preferred to the project. In a portfolio context, this can be interpreted so that the investor is indifferent between the following two portfolios: (A1) a portfolio with the project and (B1) a portfolio without the project and additional cash equal to the value of the project. Alternatively, however, we may define the value of a project as the indifference between the following two portfolios: (A2) a portfolio without the project and (B2) a portfolio with the project and a reduction in available cash equal to the value of the project. The project values obtained in these two ways will not, in general, be the same. Analogous to (Luenberger 1998; Raiffa 1968; Smith and Nau 1995), we refer to the first value as the "*breakeven selling price*," as the portfolio comparison can be understood as a selling process, and the second type of value as the "*breakeven buying price*."

A central element in BSP and BBP is the determination of equal preference for two different portfolios, which holds in many portfolio optimization models when the optimal values for the objective function match for the two portfolios. This works straightforwardly under preference functional models where the investor is, by definition, indifferent between two portfolios with equal utility scores, but the situation becomes slightly more complicated for mean-risk models where equal preference might be regarded to hold only when two values – mean and risk – are equal for the two portfolios. However, as we discussed before, if the risks are modeled as constraints, the investor can be said to be indifferent if the expectations of the two portfolios are equal and they both satisfy the risk constraints. Thus, we can establish equal preference by comparing the optimal objective function values also in this case.

Table 4.2 describes the four portfolio selection settings and, in particular, the necessary modifications to the base portfolio selection model for the calculation of breakeven prices. Here, the base portfolio selection model is simply the model that is appropriate for the setting being modeled; for example, it can be the simple one-period project-security model given in Section 4.2.3 or the complex multiperiod model described in Section 4.5. All that is required for the construction of the underlying portfolio selection problem is that it makes it possible to establish equal preference between two portfolios (in this case through the objective function value) and to have a parameter that describes the initial budget. Here, v_j^s and v_j^b are unknown modifications to the budget in Problem 2 such that the optimal objective function value in Problem 2 matches that of Problem 1. The aim of our valuation methodology is to determine the values of these unknown parameters.

Table 4.2 Definitions of the value of project j

	A. Breakeven selling price	B. Breakeven buying price
Definition	v_j^s such that $W_s^+ = W_s^-$	v_j^b such that $W_b^+ = W_b^-$
Problem 1	*Problem A1* Mandatory investment in the project Optimal objective function value: W_s^+ Budget at time 0: b_0	*Problem B1* Project is excluded from the portfolio (investment in the project is prohibited) Optimal objective function value: W_b^- Budget at time 0: b_0
Problem 2	*Problem A2* Project is excluded from the portfolio (investment in the project is prohibited) Optimal objective function value: W_s^- Budget at time 0: $b_0 + v_j^s$	*Problem B2* Mandatory investment in the project Optimal objective function value: W_b^+ Budget at time 0: $b_0 - v_j^b$

4.3.2 Inverse Optimization Procedure

Finding a BSP and BBP is an *inverse optimization problem* (see e.g. Ahuja and Orlin 2001): one has to find for what budget the optimal value of the Problem 2 matches a certain desired value (the optimal value of Problem 1). Indeed, in an inverse optimization problem, the challenge is to find the values for a set of parameters, typically a subset of all model parameters, that yield the desired optimal solution. Inverse optimization problems can broadly be classified into two groups: (a) finding an optimal value for the objective function and (b) finding a solution vector. The problem of finding a BSP or BBP falls within the first class.

In principle, the task of finding a BSP is equivalent to finding a root to the function $f^s\left(v_j^s\right) = W_s^-\left(v_j^s\right) - W_s^+$, where W_s^+ is the optimal value of Problem 1 and $W_s^-\left(v_j^s\right)$ is the corresponding optimal value in Problem 2 as the function of parameter v_j^s. Similarly, the BBP can be obtained by finding the root to the function $f^b\left(v_j^b\right) = W_b^- - W_b^+\left(v_j^b\right)$. In typical portfolio selection problems where the investor exhibits normal kind of risk preferences and a risk-free asset is available, these functions are normally increasing with respect to their parameters. To solve such root-finding problems, we can use any of the usual root-finding algorithms (see, e.g. Belegundu and Chandrupatla 1999) such as the bisection method, the secant method, and the false position method. These methods do not require knowledge of the functions' derivatives which are not typically known. If the first derivatives are known, or when approximated numerically, we can also use the Newton–Raphson method.

Solution of BSP and BBP in more complex settings with discontinuous or nonincreasing functions may require more sophisticated methods. It may be noted that, in some extreme, possibly unrealistic settings, equal preference cannot be established for any budget amount, which would imply that BSP and BBP would not exist.

4.3.3 General Analytical Properties

4.3.3.1 Sequential Consistency

Breakeven selling and buying prices are not, in general, equal to each other. While this discrepancy is accepted as a general property of risk preferences in EUT (Raiffa 1968), it may also seem to contradict the rationality of these valuation concepts. It can be argued that if the investor were willing to sell a project at a lower price than at which he/she would be prepared to buy it, the investor would create an arbitrage opportunity and lose an infinite amount of money when another investor repeatedly bought the project at its selling price and sold it back at the buying price. In a reverse situation where the investor's selling price for a project is greater than the respective buying price, the investor would be irrational in the sense that he/she would not take advantage of an arbitrage opportunity – if such an opportunity existed – where it would be possible to buy the project repeatedly at the investor's buying price and to sell it at a slightly higher price below the investor's BSP.

However, these arguments ignore the fact that the breakeven prices are affected by the budget and that therefore these prices may change after obtaining the project's selling price and after paying its buying price. Indeed, it can be shown that in a sequential setting where the investor first sells the project, adds the selling price to the budget, and then buys the project back, the investor's selling price and the respective (sequential) buying price are always equal to each other. This observation is formalized as the following proposition. The proof is given in the Appendix. It is assumed in this proof and throughout Section 4.3.3 that the objective function is continuous and strictly increasing with respect to the investor's budget (money available at present). Thus, an increase in the budget will always result in an increase in the objective function value.

Proposition 1. *A project's breakeven selling (buying) price and its sequential breakeven buying (selling) price are equal to each other.*

4.3.3.2 Consistency with Contingent Claims Analysis

Option pricing analysis or *contingent claims analysis* (CCA; see, e.g., Brealey and Myers 2000; Luenberger 1998), can be applied to value projects whenever the cash flows of a project can be replicated using financial instruments. According to CCA, the value of project j is given by the market price of the replicating portfolio (a portfolio required to initiate a replicating trading strategy) less than the investment cost of the project:

$$v_j^{\mathrm{CCA}} = I_j^* - I_j^0$$

where I_j^0 is the time-0 investment cost of the project and I_j^* is the cash needed to initiate the replicating trading strategy. A replicating trading strategy is a trading strategy using financial instruments that exactly replicates the cash flows of the project in each future state of nature.

It is straightforward to show that, when CCA is applicable, i.e., if there exists a replicating trading strategy, then the breakeven buying and selling prices are equal to each other and yield the same v_j^{CCA} result as CCA (cf. Smith and Nau 1995). This follows from the fact that a portfolio consisting of (A) the project and (B) a shorted replicating portfolio is equal to getting a cash flow for sure now and zero cash flows at all future states of nature. Therefore, whenever a replicating portfolio exists, the project can effectively be reduced to obtaining a sure amount of money at present.

To illustrate this, suppose first that v_j^{CCA} is positive. Then, *any* rational investor will invest in the project (and hence its value must be positive for any such investors), since it is possible to make money for sure, amounting to v_j^{CCA}, by investing in the project and shorting the replicating portfolio. Furthermore, any rational investor will start the project even when he/she is forced to pay a sum v_j^b less than v_j^{CCA} to gain a license to invest in the project, because it is now possible to gain $v_j^{\mathrm{CCA}} - v_j^b$ for sure. On the other hand, if v_j^b is greater than v_j^{CCA}, the investor will not start the project, because the replicating portfolio makes it possible to obtain the project cash flows at a lower cost. A similar reasoning applies to BSPs. These observations are formalized in Proposition 2. The proof is straightforward and is given in the Appendix. Due to the consistency with CCA, the breakeven prices can be regarded as a *generalization* of CCA to incomplete markets.

Proposition 2. *If there is a replicating trading strategy for a project, the breakeven selling price and breakeven buying price are equal to each other and yield the same result as CCA.*

4.3.3.3 Sequential Additivity

The BBP and BSP for a project depend on what other assets are in the portfolio. The value obtained from breakeven prices is, in general, an *added value*, which is determined relative to the situation without the project. When there are no other projects in the portfolio, or when we remove them from the model before determining the value of the project, we speak of the *isolated value* of a project. We define the respective values for a set of projects as the *joint added value* and *joint value*. Figure 4.1 illustrates the relationship between these concepts.

Isolated project values are, in general, *non-additive*; they do not sum up to the value of the project portfolio composed of the same projects. However, in a sequential setting where the investor buys the projects one after the other at the prevailing buying price at each time, the obtained project values do add up to the joint value of the project portfolio. These prices are the projects' *added* values in a sequential buying process, where the budget is reduced by the buying price after each step. We refer to these values as *sequential added values*. This *sequential additivity* property holds regardless of the order in which the projects are bought. Individual projects can, however, acquire different added values depending on the sequence in which they are bought. These observations are formalized in the following proposition. The proof is in the Appendix.

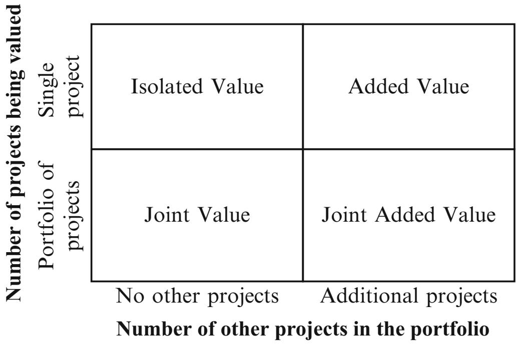

Fig. 4.1 Different types of valuations for projects

Proposition 3. *The breakeven buying (selling) prices of sequentially bought (sold) projects add up to the breakeven buying (selling) price of the portfolio of the projects regardless of the order in which the projects are bought (sold).*

4.4 Valuation of Opportunities

4.4.1 Investment Opportunities

When valuing a project, we can either value an already started project or an opportunity to start a project. The difference is that, although the value of a started project can be negative, that of an opportunity to start a project is always nonnegative, because a rational investor does not start a project with a negative value. While BSP and BBP are appropriate for valuing started projects, new valuation concepts are needed for valuing opportunities.

Since an opportunity entails the right but not the obligation to take an action, we need selling and buying prices that rely on the comparison of settings where the investor *can* and *cannot* invest in the project, instead of *does* and *does not*. The lowest price at which the investor would be willing to sell an opportunity to start a project can be obtained from the definition of the BSP by removing the requirement to invest in the project in Problem A1. We define this price as the *opportunity selling price* (OSP) of the project. Likewise, the *opportunity buying price* (OBP) of a project can be obtained by removing the investment requirement in Problem B2. It is the highest price that the investor is willing to pay for a license to start the project. Opportunity selling and buying prices have a lower bound of zero; it is also straightforward to show that the opportunity prices can be computed by taking a maximum of 0 and the respective breakeven price. Table 4.3 gives a summary of opportunity selling and buying prices.

Table 4.3 Definitions of a value of an opportunity

	A. Opportunity selling price	B. Opportunity buying price
Definition	v_*^s such that $W_s^+ = W_s^-$	v_*^b such that $W_b^+ = W_b^-$
Problem 1	*Problem A1* No alteration to base portfolio model Optimal objective function value: W_s^+ Budget at time 0: b_0	*Problem B1* Opportunity is excluded from the portfolio Optimal objective function value: W_b^- Budget at time 0: b_0
Problem 2	*Problem A2* Opportunity is excluded from the portfolio Optimal objective function value: W_s^- Budget at time 0: $b_0 + v_*^s$	*Problem B2* No alterations to base portfolio model Optimal objective function value: W_b^+ Budget at time 0: $b_0 - v_*^b$

4.4.2 Real Options and Managerial Flexibility

Opportunity buying and selling prices can also be used to value *real options* (Brosch 2008; Trigeorgis 1996) contained in the project portfolio. These options result from the managerial flexibility to adapt later decisions to unexpected future developments. Typical examples include possibilities to expand production when markets are up, to abandon a project under bad market conditions, and to switch operations to alternative production facilities.

Real options can be valued much in the same way as opportunities to start projects. However, instead of comparing portfolio selection problems with and without the possibility to start a project, we will compare portfolio selection problems with and without the real option. This can typically be implemented by preventing the investor from taking a particular action (e.g., expanding production) when the real option is not present. Since breakeven prices are consistent with CCA, also opportunity prices have this property, and can thus be regarded as a generalization of the standard CCA real option valuation procedure to incomplete markets.

4.4.3 Opportunity to Sell the Project to Third Party Investor

From the perspective of finance theory, an important application of the real options concept is the management's ability to sell the project to a third party investor (or to the market). Indeed, a valuation where the option to sell the project to a third party investor is accounted for, in addition to the opportunity costs implied by other investment opportunities, can be regarded as a *holistic valuation* that fully accounts for both private and market factors that influence the value of the project. Projects where options to sell are important include pharmaceutical development projects, where the rights to develop compounds further can be sold to bigger companies after a certain stage in clinical trials is reached.

The possibility to sell the project to the market is effectively an American put option embedded in the project. When selling the project is a relevant management option, the related selling decisions typically need to be implemented as a part of the project's decision tree, which necessitates the use of an approach where projects are modeled through decision trees, such as CPP (Gustafsson and Salo 2005). That is, at each state, in addition to any other options available to the firm, the firm can opt to sell the project at the highest price than being offered by any third party investor. The offer price may depend on several factors and who the investor is, but if a market-implied pricing is used, then it may be possible to compute the offer price by using standard market pricing techniques, such as the market-implied risk-neutral probability distribution. Like any other real option, the opportunity to sell the project can increase but cannot decrease the value of the project. Also, such an American put option also sets a lower bound for the value of the project.

4.5 Implementation of Multiperiod Project Valuation Model

4.5.1 Framework

We develop an illustrative multiperiod model using the CPP framework (Gustafsson and Salo 2005). In CPP, uncertainties are modeled using a state tree, representing the structure of future states of nature, as depicted in the leftmost chart in Fig. 4.2. The state tree need not be binomial or symmetric; it may also take the form of a multinomial tree with different probability distributions in its branches. In each nonterminal state, securities can be bought and sold in any, possibly fractional quantities. The framework includes budget balance constraints that allow the transfer of cash from one time period to the next, adding interest while doing so, so that the accumulated cash and the impact of earlier cash flows can be measured in the terminal states.

Projects are modeled using decision trees that span over the state tree. The two right-most charts in Fig. 4.2 describe how project decisions, when combined with the state tree, lead to project-specific decision trees. The specific feature of these decision trees is that the chance nodes are shared by all projects, since they are generated using the common state tree. Security trading is implemented through state-specific trading variables, which are similar to the ones used in financial models of stochastic programming (e.g. Mulvey et al. 2000) and in Smith and Nau's method (Smith and Nau 1995). Similar to the single-period portfolio selection model in Section 4.2.3, the investor seeks either to maximize the utility of the terminal wealth level, or the expectation of the terminal wealth level subject to a risk constraint.

4.5.2 Model Components

The two main components of the model are (a) states and (b) the investor's investment decisions, which imply the cash flow structure of the model.

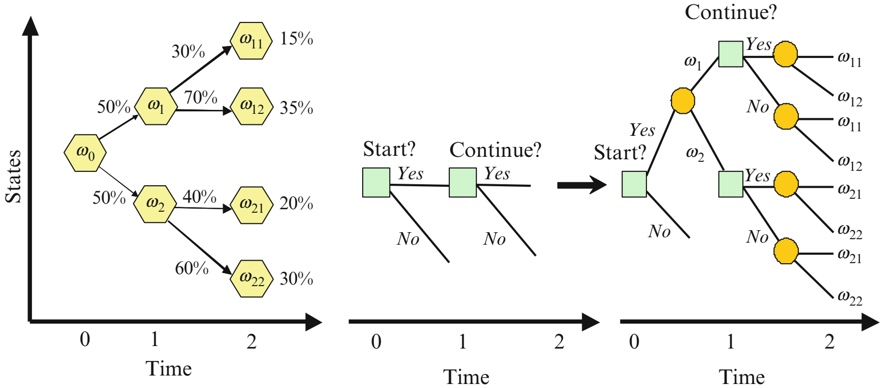

Fig. 4.2 A state tree, decision sequence, and a decision tree for a project

4.5.2.1 States

Let the planning horizon be $\{0, \ldots, T\}$. The set of states in period t is denoted by Ω_t, and the set of all states is $\Omega = \bigcup_{t=0}^{T} \Omega_t$. The state tree starts with base state ω_0 in period 0. Each nonterminal state is followed by at least one state. This relationship is modeled by the function $B : \Omega \to \Omega$ which returns the immediate predecessor of each state, except for the base state, for which the function gives $B(\omega_0) = \omega_0$. The probability of state ω, when $B(\omega)$ has occurred, is given by $p_{B(\omega)}(\omega)$. Unconditional probabilities for each state, except for the base state, can be computed recursively from the equation $p(\omega) = p_{B(\omega)}(\omega) \times p(B(\omega))$. The probability of the base state is $p(\omega_0) = 1$.

4.5.2.2 Investment Decisions

Let there be n securities available in financial markets. The amount of security i bought in state ω is indicated by trading variable $x_{i,\omega}$, $i = 1, \ldots, n$, $\omega \in \Omega$, and the price of security i in state ω is denoted by $S_i(\omega)$. Under the assumption that all securities are sold in the next period, the cash flow implied by security i in nonterminal state $\omega \neq \omega_0$ is $S_i(\omega) \times (x_{i,B(\omega)} - x_{i,\omega})$. In base state ω_0, the cash flow is $-S_i(\omega_0) \times x_{i,\omega_0}$, and in a terminal state ω_T it is $S_i(\omega_\mathrm{T}) \times x_{i,B(\omega_\mathrm{T})}$.

The investor can invest privately in m projects. The decision opportunities for each project, $k = 1, \ldots, m$, are structured as a decision tree, where there are decision points D_k and function $ap(d)$ that gives the action leading to decision point $d \in D_k \backslash \{d_k^0\}$, where d_k^0 is the first decision point of project k. Let A_d be the set of actions that can be taken in decision point $d \in D_k$. For each action a in A_d, a binary action variable z_a indicates whether the action is selected or not. Action variables at each decision point d are bound by the restriction that only one $z_a,\ a \in A_d$, can be equal to one. The state in which the action at decision point d is chosen is denoted by $\omega(d)$.

For a project k, the vector of all action variables z_a relating to the project, denoted by $\mathbf{z}_k$, is called the *project management strategy* of k. The vector of all action variables of all projects, denoted by $\mathbf{z}$, is the *project portfolio management strategy*. We call the pair $(\mathbf{x}, \mathbf{z})$, composed of all trading and action variables, the *aggregate portfolio management strategy*.

4.5.2.3 Cash Flows and Cash Surpluses

Let $\mathrm{CF}_k^p(\mathbf{z}_k, \omega)$ be the cash flow of project k in state ω with project management strategy $\mathbf{z}_k$. When $C_a(\omega)$ is the cash flow in state ω implied by action a, this cash flow is given by

$$\mathrm{CF}_k^p(\mathbf{z}_k, \omega) = \sum_{\substack{d \in D_k: \\ \omega(d) \in \Omega^B(\omega)}} \sum_{a \in A_d} C_a(\omega) \times z_a$$

where the restriction in the summation of the decision points guarantees that actions yield cash flows only in the prevailing state and in the future states that can be reached from the prevailing state. The set $\Omega^B(\omega)$ is defined as $\Omega^B(\omega) = \left\{\omega' \in \Omega \,|\, \exists\, k \geq 0 \text{ such that } B^k(\omega) = \omega'\right\}$, where $B^n(\omega) = B(B^{n-1}(\omega))$ is the nth predecessor of ω $(B^0(\omega) = \omega)$.

The cash flows from security i in state $\omega \in \Omega$ are given by

$$\mathrm{CF}_i^s(\mathbf{x_i}, \omega) = \begin{cases} -S_i(\omega) \times x_{i,\omega} & \text{if } \omega = \omega_0 \\ S_i(\omega) \times x_{i,B(\omega)} & \text{if } \omega \in \Omega_T \\ S_i(\omega) \times (x_{i,B(\omega)} - x_{i,\omega}) & \text{if } \omega \neq \omega_0 \wedge \omega \notin \Omega_T \end{cases}$$

Thus, the aggregate cash flow $\mathrm{CF}(\mathbf{x}, \mathbf{z}, \omega)$ in state $\omega \in \Omega$, obtained by summing up the cash flows for all projects and securities, is

$$\begin{aligned} \mathrm{CF}(\mathbf{x}, \mathbf{z}, \omega) &= \sum_{i=1}^{n} \mathrm{CF}_i^s(\mathbf{x_i}, \omega) + \sum_{k=1}^{m} \mathrm{CF}_k^p(\mathbf{z_k}, \omega) \\ &= \begin{cases} \displaystyle\sum_{i=1}^{n} -S_i(\omega) \times x_{i,\omega} + \sum_{\substack{d \in D_k: \\ \omega(d) \in \Omega^B(\omega)}} \sum_{a \in A_d} C_a(\omega) \times z_a, & \text{if } \omega = \omega_0 \\ \displaystyle\sum_{i=1}^{n} S_i(\omega) \times x_{i,B(\omega)} + \sum_{\substack{d \in D_k: \\ \omega(d) \in \Omega^B(\omega)}} \sum_{a \in A_d} C_a(\omega) \times z_a, & \text{if } \omega \in \Omega_T \\ \displaystyle\sum_{i=1}^{n} S_i(\omega) \times (x_{i,B(\omega)} - x_{i,\omega}) + \sum_{\substack{d \in D_k: \\ \omega(d) \in \Omega^B(\omega)}} \sum_{a \in A_d} C_a(\omega) \times z_a, & \text{if } \omega \neq \omega_0 \wedge \omega \notin \Omega_T \end{cases} \end{aligned}$$

Together with the initial budget in each state, cash flows define *cash surpluses* that would result in state $\omega \in \Omega$ if the investor chose portfolio management strategy $(\mathbf{x}, \mathbf{z})$. Assuming that excess cash is invested in the risk-free asset, the cash surplus in state $\omega \in \Omega$ is given by

$$\mathrm{CS}_{\omega} = \begin{cases} b(\omega) + \mathrm{CF}(\mathbf{x}, \mathbf{z}, \omega) & \text{if } \omega = \omega_0 \\ b(\omega) + \mathrm{CF}(\mathbf{x}, \mathbf{z}, \omega) + (1 + r_{B(\omega)\to\omega}) \times \mathrm{CS}_{B(\omega)} & \text{if } \omega \neq \omega_0 \end{cases},$$

where $b(\omega)$ is the initial budget in state $\omega \in \Omega$ and $r_{B(\omega)\to\omega}$ is the short rate at which cash accrues interest from state $B(\omega)$ to ω. The cash surplus in a terminal state is the investor's terminal wealth level in that state.

4.5.3 *Optimization Model*

When using a preference functional U, the objective function for the model can be written as a function of cash surplus variables in the last time period, i.e.

$$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} U(\mathbf{CS}_T),$$

where $\mathbf{CS}_T$ denotes the vector of cash surplus variables in period T. Under the risk-constrained mean-risk model, the objective is to maximize the expectation of the investor's terminal wealth level

$$\max_{\mathbf{x},\mathbf{z},\mathbf{CS}} \sum_{\omega \in \Omega_T} p(\omega) \times \mathrm{CS}_{\omega}.$$

Three types of constraints are imposed on the model: (a) *budget constraints*, (b) *decision consistency constraints*, and (c) *risk constraints* (in the case of risk-constrained models). The formulation of a multiperiod portfolio selection model under both a preference functional and a mean-risk model is given in Table 4.4.

4.5.3.1 Budget Constraints

Budget constraints ensure that there is a nonnegative amount of cash in each state. They can be implemented using continuous *cash surplus variables* CS_{ω}, which measure the amount of cash in state ω. These variables lead to the budget constraints

$$\mathrm{CF}(\mathbf{x}, \mathbf{z}, \omega_0) - \mathrm{CS}_{\omega_0} = -b(\omega_0)$$

$$\mathrm{CF}(\mathbf{x}, \mathbf{z}, \omega) + (1 + r_{B(\omega)\to\omega}) \times \mathrm{CS}_{B(\omega)} - \mathrm{CS}_{\omega} = -b(\omega), \quad \forall \omega \in \Omega \backslash \{\omega_0\}$$

Table 4.4 Multi-period models

	Preference functional model	Mean-risk model
Objective function	$\max\limits_{\mathbf{x,z,CS}} U(\mathbf{CS}_T)$	$\max\limits_{\mathbf{x,y,CS}} \sum_{\omega \in \Omega_T} p(\omega) \times \mathrm{CS}_\omega$
Budget constraints	$\mathrm{CF}(\mathbf{x,z}, \omega_0) - \mathrm{CS}_{\omega_0} = -b(\omega_0)$	
	$\mathrm{CF}(\mathbf{x}, \mathbf{z}, \omega) + (1 + r_{B(\omega)\to\omega}) \times \mathrm{CS}_{B(\omega)} - \mathrm{CS}_\omega = -b(\omega),$ $\forall \omega \in \Omega \backslash \{\omega_0\}$	
Decision consistency constraints	$\sum_{a \in A_{d_k^0}} z_a = 1, \quad k = 1, \ldots, m$	
	$\sum_{a \in A_d} z_a = z_{ap(d)}, \quad \forall d \in D_k \backslash \{d_k^0\}, \quad k = 1, \ldots, m$	
Risk constraints		$\rho\left(\Delta^-, \Delta^+\right) \leq R$ $\mathrm{CS}_\omega - \tau(\mathbf{CS}_T) - \Delta_\omega^+ + \Delta_\omega^- = 0 \forall \omega \in \Omega_T$
Variables	$z_a \in \{0, 1\}, \forall a \in A_d$ $\forall d \in D_k\, k = 1, \ldots, m$ $x_{i,\omega}$free $\forall \omega \in \Omega,\ i = 1, \ldots, n$ CS_ω free $\forall \omega \in \Omega$	$z_a \in \{0, 1\}, \forall a \in A_d$ $\forall d \in D_k\, k = 1, \ldots, m$ $x_{i,\omega}$ free$\forall \omega \in \Omega,$ $i = 1, \ldots, n$ CS_ω free $\forall \omega \in \Omega$ $\Delta_\omega^- \geq 0 \quad \forall \omega \in \Omega_T$ $\Delta_\omega^+ \geq 0 \quad \forall \omega \in \Omega_T$

Note that if CS_ω is negative, the investor borrows money at the risk-free interest rate to cover a funding shortage. Thus, CS_ω can also be regarded as a trading variable for the risk-free asset.

4.5.3.2 Decision Consistency Constraints

Decision consistency constraints ensure the logical consistency of the projects' decision trees. They require that (a) at each decision point reached only one action is selected, and that (b) at each decision point that is not reached, no action is taken. Decision consistency constraints can be written as

$$\sum_{a \in A_{d_k^0}} z_a = 1, \quad k = 1, \ldots, m$$

$$\sum_{a \in A_d} z_a = z_{ap(d)}, \quad \forall d \in D_k \backslash \{d_k^0\}, \quad k = 1, \ldots, m,$$

where the first constraint ensures that one action is selected in the first decision point, and the second implements the above requirements for the other decision points.

4.5.3.3 Risk Constraints

A risk-constrained model includes one or more risk constraints. We focus on the single constraint case. When ρ denotes the risk measure and R the risk tolerance, a risk constraint can be expressed as

$$\rho(\mathbf{CS}_T) \leq R.$$

In addition to variance (V), several other risk measures have been proposed. These include *semivariance* (Markowitz 1959), *absolute deviation* (Konno and Yamazaki 1991), *lower semi-absolute deviation* (Ogryczak and Ruszczynski 1999), and their fixed target value counterparts (Fishburn 1977). Semivariance (SV), absolute deviation (AD) and lower semi-absolute deviation (LSAD) are defined as

$$\text{SV} : \bar{\sigma}_X = \int_{-\infty}^{\mu_X} (x - \mu_X)^2 \, \mathrm{d}F_X(x), \text{AD} : \delta_X = \int_{-\infty}^{\infty} |x - \mu_X| \, \mathrm{d}F_X(x), \text{ and}$$

$$\text{LSAD} : \bar{\delta}_X = \int_{-\infty}^{\mu_X} |x - \mu_X| \, \mathrm{d}F_X(x) = \int_{-\infty}^{\mu_X} (\mu_X - x) \mathrm{d}F_X(x),$$

where μ_X is the mean of random variable X and F_X is the cumulative density function of X. The fixed target value statistics are obtained by replacing μ_X by an appropriate constant target value τ. All these measures can be formulated in an optimization program by introducing *deviation constraints*. In general, deviation constraints are expressed as

$$\text{CS}_\omega - \tau(\mathbf{CS}_T) - \Delta_\omega^+ + \Delta_\omega^- = 0 \quad \forall \omega \in \Omega_T,$$

where $\tau(\mathbf{CS}_T)$ is a function that defines the target value from which the deviations are calculated, and Δ_ω^+ and Δ_ω^- are nonnegative *deviation variables* which measure how much the cash surplus in state $\omega \in \Omega_T$ differs from the target value. For example, when the target value is the mean of the terminal wealth level, the deviation constraints are written as

$$\text{CS}_\omega - \sum_{\omega' \in \Omega_T} p(\omega')\text{CS}_{\omega'} - \Delta_\omega^+ + \Delta_\omega^- = 0, \quad \forall \omega \in \Omega_T.$$

With the help of these deviation variables, some common dispersion statistics can now be written as follows:

$$\text{AD} : \sum_{\omega \in \Omega_T} p(\omega) \times (\Delta_\omega^- + \Delta_\omega^+)$$

$$\text{LSAD} : \sum_{\omega \in \Omega_T} p(\omega) \times \Delta_\omega^-$$

$$V: \sum_{\omega \in \Omega_T} p(\omega) \times (\Delta_\omega^- + \Delta_\omega^+)^2$$

$$\mathrm{SV}: \sum_{\omega \in \Omega_T} p(\omega) \times (\Delta_\omega^-)^2$$

The respective fixed-target value statistics can be obtained with the deviation constraints

$$\mathrm{CS}_\omega - \tau - \Delta_\omega^+ + \Delta_\omega^- = 0, \quad \forall \omega \in \Omega_T,$$

where τ is the fixed target level. Expected downside risk (EDR), for example, can then be obtained from the sum $\sum_{\omega \in \Omega_T} p(\omega) \times \Delta_\omega^-$.

4.5.3.4 Other Constraints

Even other constraints can be modeled, including short selling limitations, upper bounds for the number of shares bought, and credit limit constraints (Markowitz 1987). For the sake of simplicity, however, we assume in the following sections that there are no such additional constraints in the model.

4.5.4 Example

We next illustrate project valuation in a multiperiod setting with an example similar to the one in Gustafsson and Salo (2005). In this setting, the investor can invest in projects A and B in two stages as illustrated in Fig. 4.3. At time 0, he/she can start either one or both of the projects. If a project is started, he/she can make a further investment at time 1. If the investment is made, the project generates a positive cash flow at time 2; otherwise, the project is terminated with no further cash flows. In the spirit of the CAPM, it is assumed that the investor can also borrow and lend money at a risk-free interest rate, in this case 8%, and invest in the equity market portfolio. The investor is able to buy and short the market portfolio and the risk-free asset in any quantities. The initial budget is \$9 million. The investor is a mean-LSAD optimizer with a risk (LSAD) tolerance of $R = \$5$ million. Here, we use a risk-constrained model instead of a preference functional model as in Gustafsson and Salo (2005), because otherwise the optimal strategy will be unbounded with the investor investing an infinite amount in the market portfolio and financing this by going short in the risk-free asset, or vice versa, depending on the value of the mean-LSAD model's risk aversion parameter.

Uncertainties are captured through a state tree where uncertainties are divided into market and private uncertainties (see Figs. 4.3 and 4.4). The price of the market portfolio is entirely determined by the prevailing market state, while the projects'

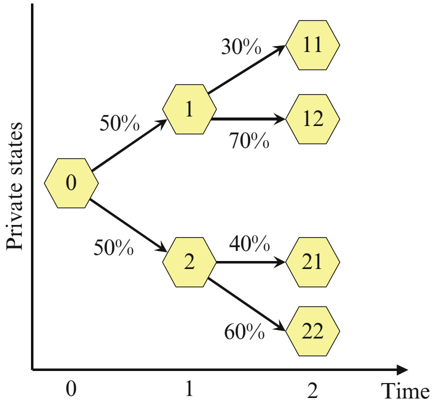

Fig. 4.3 Private states

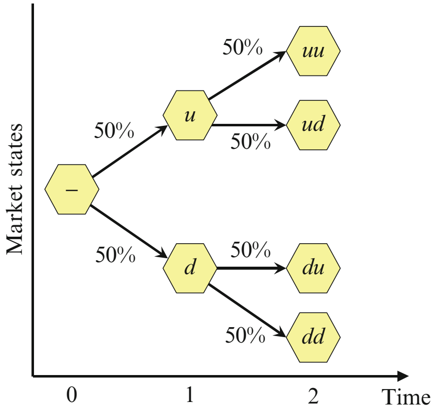

Fig. 4.4 Market states

cash flows depend solely on the private states. At time 1, there are two possible private states, 1 and 2, and two possible market states u and d. These states imply four joint states, $1u$, $1d$, $2u$ and $2d$, for time 1. At time 2, following the time-1 private state 1, private state may be 11 or 12; if the time-1 private state was 2, then the time-2 private state may be either 21 or 22. At time 2, the market state may be either uu or ud, if u obtained at time 1, or du or dd if the time-1 market state was d. These private and market states imply the following 16 joint states for time 2 (the probability of the state is given in parentheses): $1u1u$ (3.75%), $1u1d$ (3.75%), $1d1u$ (3.75%), $1d1d$ (3.75%), $1u2u$ (8.75%), $1u2d$ (8.75%), $1d2u$ (8.75%), $1d2d$ (8.75%), $2u1u$ (5%), $2u1d$ (5%), $2d1u$ (5%), $2d1d$ (5%), $2u2u$ (7.5%), $2u2d$ (7.5%), $2d2u$ (7.5%), $2d2d$ (7.5%).

When project decisions are combined with the state tree implied by Figs. 4.3 and 4.4, we obtain the simplified decision trees in Figs. 4.5 and 4.6, where each action is associated with an indexed binary action variable z and the cash flows it generates. The market states are indicated by "x" meaning that the value can be

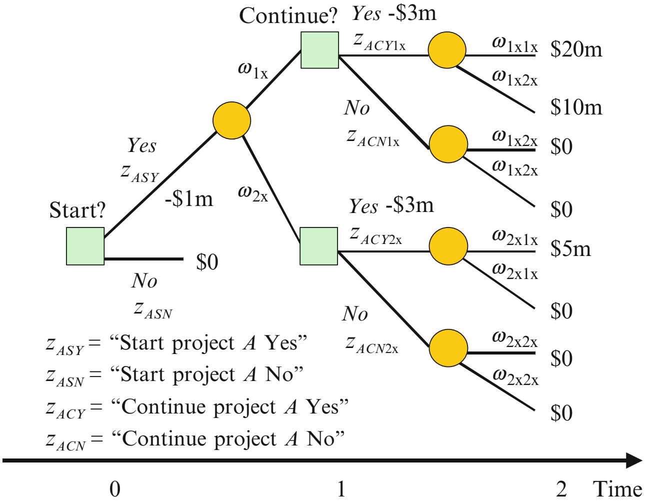

Fig. 4.5 Decision tree of project A

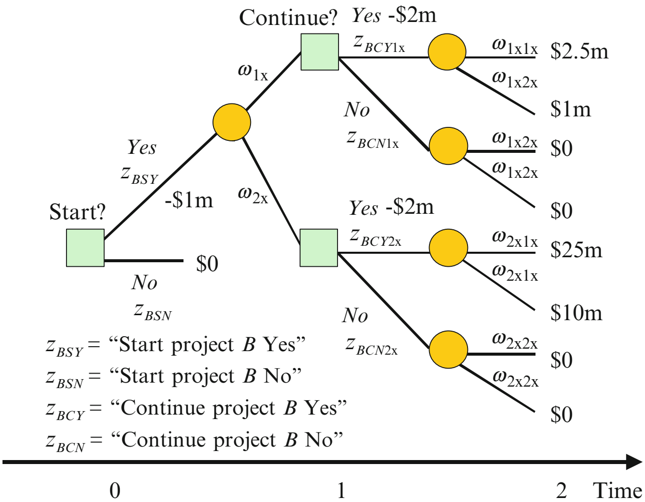

Fig. 4.6 Decision tree of project B

either one of u and d, because the project outcomes are independent of the market state and hence the project will generate the same cash flows regardless of which one of the market states obtains. The market portfolio is assumed to yield a return of 24% if u results and 0% if d results, which are both equally probable events. This implies an expected excess rate of return of 4% for the market portfolio.

Based on Figs. 4.5 and 4.6, budget constraints can now be written as:

$$
\begin{aligned}
&-1z_{\mathrm{ASY}} - 2z_{\mathrm{BSY}} - x_{\omega 0} - \mathrm{CS}_{\omega 0} = -9\\
&-3z_{\mathrm{ACY}1u} - 2z_{\mathrm{BCY}1u} + 1.24x_{\omega 0} - 1.24x_{\omega 1u} + 1.08\mathrm{CS}_{\omega 0} - \mathrm{CS}_{\omega 1u} = 0\\
&-3z_{\mathrm{ACY}2u} - 2z_{\mathrm{BCY}2u} + 1.24x_{\omega 0} - 1.24x_{\omega 2u} + 1.08\mathrm{CS}_{\omega 0} - \mathrm{CS}_{\omega 2u} = 0\\
&-3z_{\mathrm{ACY}1d} - 2z_{\mathrm{BCY}1d} + 1x_{\omega 0} - 1x_{\omega 1d} + 1.08\mathrm{CS}_{\omega 0} - \mathrm{CS}_{\omega 1d} = 0\\
&-3z_{\mathrm{ACY}2d} - 2z_{\mathrm{BCY}2d} + 1x_{\omega 0} - 1x_{\omega 2d} + 1.08\mathrm{CS}_{\omega 0} - \mathrm{CS}_{\omega 2d} = 0\\
&20z_{\mathrm{ACY}1u} + 2.5z_{\mathrm{BCY}1u} + 1.5376x_{\omega 1u} + 1.08\mathrm{CS}_{\omega 1u} - \mathrm{CS}_{\omega 1u1u} = 0\\
&20z_{\mathrm{ACY}1u} + 2.5z_{\mathrm{BCY}1u} + 1.24x_{\omega 1u} + 1.08\mathrm{CS}_{\omega 1u} - \mathrm{CS}_{\omega 1u1d} = 0\\
&20z_{\mathrm{ACY}1d} + 2.5z_{\mathrm{BCY}1d} + 1.24x_{\omega 1d} + 1.08\mathrm{CS}_{\omega 1d} - \mathrm{CS}_{\omega 1d1u} = 0\\
&20z_{\mathrm{ACY}1d} + 2.5z_{\mathrm{BCY}1d} + 1x_{\omega 1d} + 1.08\mathrm{CS}_{\omega 1d} - \mathrm{CS}_{\omega 1d1d} = 0\\
&10z_{\mathrm{ACY}1u} + 1z_{\mathrm{BCY}1u} + 1.5376x_{\omega 1u} + 1.08\cdot CS_{\omega 1u} - CS_{\omega 1u2u} = 0\\
&10z_{\mathrm{ACY}1u} + 1z_{\mathrm{BCY}1u} + 1.24x_{\omega 1u} + 1.08\mathrm{CS}_{\omega 1u} - \mathrm{CS}_{\omega 1u2d} = 0\\
&10z_{\mathrm{ACY}1d} + 1z_{\mathrm{BCY}1d} + 1.24x_{\omega 1d} + 1.08\mathrm{CS}_{\omega 1d} - \mathrm{CS}_{\omega 1d2u} = 0\\
&10z_{\mathrm{ACY}1d} + 1z_{\mathrm{BCY}1d} + 1x_{\omega 1d} + 1.08\mathrm{CS}_{\omega 1d} - \mathrm{CS}_{\omega 1d2d} = 0\\
&5z_{\mathrm{ACY}2u} + 25z_{\mathrm{BCY}2u} + 1.5376x_{\omega 2u} + 1.08\mathrm{CS}_{\omega 2u} - \mathrm{CS}_{\omega 2u1u} = 0\\
&5z_{\mathrm{ACY}2u} + 25z_{\mathrm{BCY}2u} + 1.24x_{\omega 2u} + 1.08\mathrm{CS}_{\omega 2u} - \mathrm{CS}_{\omega 2u1d} = 0\\
&5z_{\mathrm{ACY}2d} + 25z_{\mathrm{BCY}2d} + 1.24x_{\omega 2d} + 1.08\mathrm{CS}_{\omega 2d} - \mathrm{CS}_{\omega 2d1u} = 0\\
&5z_{\mathrm{ACY}2d} + 25z_{\mathrm{BCY}2d} + 1x_{\omega 2d} + 1.08\mathrm{CS}_{\omega 2d} - \mathrm{CS}_{\omega 2d1d} = 0\\
&10z_{\mathrm{BCY}2u} + 1.5376x_{\omega 2u} + 1.08\mathrm{CS}_{\omega 2u} - \mathrm{CS}_{\omega 2u2u} = 0\\
&10z_{\mathrm{BCY}2u} + 1.24x_{\omega 2u} + 1.08\mathrm{CS}_{\omega 2u} - \mathrm{CS}_{\omega 2u2d} = 0\\
&10z_{\mathrm{BCY}2d} + 1.24x_{\omega 2d} + 1.08\mathrm{CS}_{\omega 2d} - \mathrm{CS}_{\omega 2d2u} = 0\\
&10z_{\mathrm{BCY}2d} + 1x_{\omega 2d} + 1.08\mathrm{CS}_{\omega 2d} - \mathrm{CS}_{\omega 2d2d} = 0
\end{aligned}
$$

For each terminal state $\omega_{\mathrm{T}} \in \Omega_T$, there is a deviation constraint $\mathrm{CS}_{\omega_{\mathrm{T}}} - EV - \Delta^{+}_{\omega_{\mathrm{T}}} + \Delta^{-}_{\omega_{\mathrm{T}}} = 0$, where EV is the expected cash balance over all terminal states, viz.

$$\mathrm{EV} = \sum_{\omega_{\mathrm{T}} \in \Omega_T} p_{\omega_{\mathrm{T}}} \mathrm{CS}_{\omega_{\mathrm{T}}}.$$

Table 4.5 Investments in securities ($ million)

State	x_M	CS
0	71.37	−64.37
1u	16.40	−4.35
1d	105.46	−106.61
2u	27.28	−16.85
2d	98.71	−98.86

In addition, the following decision consistency constraints apply:

$$\begin{aligned}
&z_{\mathrm{ASY}} + z_{\mathrm{ASN}} = 1 && z_{\mathrm{BSY}} + z_{\mathrm{BSN}} = 1\\
&z_{\mathrm{ACY1}u} + z_{\mathrm{ACN1}u} = z_{\mathrm{ASY}} && z_{\mathrm{BCY1}u} + z_{\mathrm{BCN1}u} = z_{\mathrm{BSY}}\\
&z_{\mathrm{ACY2}u} + z_{\mathrm{ACN2}u} = z_{\mathrm{ASY}} && z_{\mathrm{BCY2}u} + z_{\mathrm{BCN2}u} = z_{\mathrm{BSY}}\\
&z_{\mathrm{ACY1}d} + z_{\mathrm{ACN1}d} = z_{\mathrm{ASY}} && z_{\mathrm{BCY1}d} + z_{\mathrm{BCN1}d} = z_{\mathrm{BSY}}\\
&z_{\mathrm{ACY2}d} + z_{\mathrm{ACN2}d} = z_{\mathrm{ASY}} && z_{\mathrm{BCY2}d} + z_{\mathrm{BCN2}d} = z_{\mathrm{BSY}}
\end{aligned}$$

The risk constraint is now

$$\sum_{\omega_{\mathrm{T}} \in \Omega_T} p_{\omega_{\mathrm{T}}} \Delta^{-}_{\omega_{\mathrm{T}}} \leq 5,$$

and the objective function is

$$\text{Maximize EV} = \sum_{\omega_{\mathrm{T}} \in \Omega_T} p_{\omega_{\mathrm{T}}} \mathrm{CS}_{\omega_{\mathrm{T}}}.$$

In this portfolio selection problem, the optimal strategy is to start both projects; project A is terminated at time 1 if private state 2 occurs and project B if private state 1 occurs, i.e., variables z_{ASY}, $z_{\mathrm{ACY1}u}$, $z_{\mathrm{ACY1}d}$, z_{BSY}, $z_{\mathrm{BCY2}u}$, $z_{\mathrm{BCY2}d}$ are one and all other action variables are zero. The optimal amounts invested in the market portfolio and the risk-free asset are given in columns 2 and 3 of Table 4.5, respectively. There is an expected cash balance of EV $=$ \$25.63 million and LSAD of \$5.00 million at time 2. The portfolio has its least value, \$0.32 million, in state $1d2d$. It is worth noting that the value of the portfolio can be negative in some terminal states at higher risk levels, because the states' cash surplus variables are not restricted to nonnegative values. Thus, the investor will borrow money at the risk-free rate and invest it in the market portfolio, and may hence default on his/her loan obligations if the market does not go up in either of the time periods and the project portfolio performs poorly.

Breakeven selling and buying prices for projects A and B, as well as for the entire project portfolio, are given in the last row in Table 4.6. These prices are now equal, and we therefore record them in a single cell. (This is a property of the employed preference model.) For the sake of comparison, we also give the terminal wealth levels when the investor does and does not invest in the project/portfolio being valued, denoted by W^+ and W^-, respectively.

The portfolio value differs from the value of \$5.85 million obtained in Gustafsson and Salo (2005), where the investor did not have the possibility to invest in

Table 4.6 Project values ($ million)

	Portfolio	A	B
W^+	$25.63	$25.63	$25.63
W^-	$17.16	$22.17	$20.54
$W^+ - W^-$	$8.47	$3.46	$5.09
$v = v^b = v^s$	$7.26	$2.97	$4.36

the market portfolio. There are two reasons for this difference. First, due to the possibility to invest limitless amounts in the market portfolio, we use a risk-constrained preference model with $R = \$5.00$ million, whereas the example in Gustafsson and Salo (2005) used a preference functional model with $\lambda = 0.5$. With different preference models we also have different risk-adjustment. Second, because here it is possible to invest in the market portfolio, the optimal portfolio mix is likely to be different from the setting where investments in market-traded securities are not possible, and thus the project portfolio is also likely to obtain a value different from the one obtained in Gustafsson and Salo (2005).

4.6 Summary and Conclusions

In this chapter, we have considered the valuation of private projects in a setting where an investor can invest in a portfolio of projects as well as securities traded in financial markets, but where the replication of project cash flows with financial securities may not be possible. Specifically, we have developed a valuation procedure based on the concepts of *breakeven selling and buying prices*. This inverse optimization procedure requires the solution of portfolio selection problems with and without the project that is being valued and determining the lump sum that makes the investor indifferent between the two settings. We have also offered analytical results concerning the properties of breakeven prices. Our results show that the breakeven prices are, in general, consistent valuation measures in that they exhibit sequential additivity and consistency; they are also consistent with CCA.

Quite importantly, the proposed methodology overcomes several deficiencies in earlier approaches to the valuation of projects and other illiquid investments. That is, the methodology accounts systemically for the time value of money, explicates the investor's risk preferences, captures the projects' risk characteristics and their impacts on aggregate risks at the portfolio level. This methodology also accounts for the opportunity costs of alternative investment opportunities which are explicitly included in the portfolio selection model. Overall, the methodology constitutes a new, complete and theoretically well-founded approach to the valuation of nonmarket traded investments.

Furthermore, we have shown that it is possible to include real options and managerial flexibility in the projects through modeling them as decision trees in the CPP framework (Gustafsson and Salo 2005). We have also shown how such real options can be valued using the concepts of opportunity buying and selling prices, and demonstrated that such resulting real option values are consistent with CCA. Indeed, since the present framework does not require the existence of a replicating trading strategy, a key implication is that the proposed methodology makes it possible to generalize CCA to the valuation of private projects in incomplete markets.

This work suggests several avenues for further research. In particular, it is of interest to investigate settings where the investor can opt to sell the project to the market (or to a third party investor), as the resulting valuations would then holistically account for the impact that markets can have on the value of the project (i.e. implicit market pricing and opportunity costs). Analysis of specific preference models such as the mean-variance model also seems appealing, not least because the CAPM (Lintner 1965; Sharpe 1964) is based on such preferences. More work is also needed to facilitate the use of the methodology in practice and to link it to existing theories of pricing.

Appendix

Proof of Proposition 1

Let us prove the proposition first for the BSP and the sequential buying price. Let the BSP for the project be v_j^s. Then, based on Table 4.2, v_j^s will be defined by the portfolio setting in the middle column of Table A.1.

Next, we can observe that Problem B1 in determining the sequential buying price is the same as Problem A2 for the BSP, wherefore also the optimal objective function values will be the same, i.e., $W_b^- = W_s^-$. Since by definition of breakeven prices

Table A.1 Definition of the sequential buying price value of project j

	A. Breakeven selling price	B. Sequential buying price
Definition	v_j^s such that $W_s^+ = W_s^-$	v_j^b such that $W_b^+ = W_b^-$
Problem 1	*Problem A1*	*Problem B1*
	Mandatory investment in the project	Project is excluded from the portfolio (investment in the project is prohibited)
	Optimal objective function value: W_s^+	Optimal objective function value: W_b^-
	Budget at time 0: b_0	Budget at time 0: $b_0 + v_j^s$
Problem 2	*Problem A2*	*Problem B2*
	Project is excluded from the portfolio (investment in the project is prohibited)	Mandatory investment in the project
	Optimal objective function value: W_s^-	Optimal objective function value: W_b^+
	Budget at time 0: $b_0 + v_j^s$	Budget at time 0: $b_0 + v_j^s - v_j^b$

we have $W_{\mathrm{s}}^{+} = W_{\mathrm{s}}^{-}$ and $W_{\mathrm{b}}^{+} = W_{\mathrm{b}}^{-}$, it follows that we also have $W_{\mathrm{s}}^{+} = W_{\mathrm{b}}^{+}$. Since Problem A1 and Problem B2 are otherwise the same, except that the first has the budget of b_0 and the second $b_0 + v_j^s - v_j^b$, and because the optimal objective function value is strictly increasing with respect to the budget, it follows that $b_0 + v_j^s - v_j^b$ must be equal to b_0 to get $W_{\mathrm{s}}^{+} = W_{\mathrm{b}}^{+}$, and therefore $v_j^b = v_j^s$. The proposition for the BBP and the respective sequential selling price is proven similarly. ■

Proof of Proposition 2

A replicating trading strategy for a project is a trading strategy that produces exactly the same cash flows in all future states of nature as the project. Thus, by definition, starting the project and shorting the replicating trading strategy will lead to a situation where cash flows net each other out in each state of except at time 0. At time 0, the cash flow will be $I_k^* - I_k^0$, where I_k^0 is the time-0 investment cost of the project and I_j^* is the cash needed to initiate the replicating trading strategy. Therefore, a setting where the investor starts the project and shorts the replicating trading strategy will be exactly the same as a case where the project is not included in the portfolio and the time-0 budget is increased by $I_k^* - I_k^0$ (and hence the same objective function values). Therefore, the investor's BSP for the project is, by the definition of BSP, $I_k^* - I_k^0$. The proposition for the BBP can be proven similarly. ■

Proof of Proposition 3

Let us begin with the BBP and a setting where the portfolio does not include projects and the budget is b_0. Suppose that there are n projects that the investor buys sequentially. Let us denote the optimal value for this problem by $W_{\mathrm{b},1}^{-}$. Suppose then that the investor buys a project, indexed by 1, at his or her BBP, v_1^b. Let us denote the resulting optimal value for the problem by $W_{\mathrm{b},1}^{+}$. By definition of the BBP, $W_{\mathrm{b},1}^{-} = W_{\mathrm{b},1}^{+}$. Suppose then that the investor buys another project, indexed by 2, at his or her BBP, v_2^b. The initial budget is now $b_0 - v_1^b$, and after the second project is bought, it is $b_0 - v_1^b - v_2^b$. Since the second project's Problem 1 and the first project's Problem 2 are the same, the optimal values for these two problems are the same, i.e., $W_{\mathrm{b},2}^{-}$ is equal to $W_{\mathrm{b},1}^{+}$. Add then the rest of the projects in the same manner, as illustrated in Table A.2.

The resulting budget in the last optimization problem, which includes all the projects, is $b_0 - v_1^b - v_2^b - \cdots - v_n^b$. Because Problem 2 of each project (except for the last) in the sequence is always Problem 1 of the next project and because by the definition of the BBP, for each project, the objective function values in Problems 1 and 2 are equal, we have $W_{\mathrm{b},n}^{+} = W_{\mathrm{b},n}^{-} = W_{\mathrm{b},n-1}^{+} = W_{\mathrm{b},n-1}^{-} = W_{\mathrm{b},n-2}^{+} = \cdots = W_{\mathrm{b},1}^{-}$. Therefore, by the definition of the BBP, the BBP for the portfolio including all the

Table A.2 Buying prices in a sequential buying process

	First project	Second project	Third project	nth project
P1	Optimal value: $W_{b,1}^{-}$	Optimal value: $W_{b,2}^{-}\left(=W_{b,1}^{+}\right)$	Optimal value: $W_{b,3}^{-}\left(=W_{b,2}^{+}\right)$	Optimal value: $W_{b,n}^{-}\left(=W_{b,n-1}^{+}\right)$
	Budget at time 0: b_0	Budget at time 0: $b_0 - v_1^b$	Budget at time 0: $b_0 - v_1^b - v_2^b$	Budget at time 0: $b_0 - v_1^b - v_2^b - \cdots - v_{n-1}^b$
P2	Optimal value: $W_{b,1}^{+}$	Optimal value: $W_{b,2}^{+}$	Optimal value: $W_{b,3}^{+}$	Optimal value: $W_{b,n}^{+}$
	Budget at time 0: $b_0 - v_1^b$	Budget at time 0: $b_0 - v_1^b - v_2^b$	Budget at time 0: $b_0 - v_1^b - v_2^b - v_3^b$	Budget at time 0: $b_0 - v_1^b - v_2^b - \cdots - v_n^b$

project must be $v_{ptf}^b = v_1^b + v_2^b + \cdots + v_n^b$. By re-indexing the projects and using the above procedure, we can change the order in which the projects are added to the portfolio. In doing so, the projects can obtain different values, but they still sum up to the same joint value of the portfolio. Similar logic proves the proposition for BSPs. ■

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