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O.R. Applications

Project options valuation with net present value and decision tree analysis

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Abstract

Real options analysis (ROA) has been developed to correctly value projects with inherent flexibility, including the possibility to abandon, defer, expand, contract or switch to a different project. ROA allows computing the correct discount rate using the replicating portfolio technique or risk-neutral probability method. We propose an alternative approach for valuing Real Options based on the certainty-equivalent version of the net present value formula, which eliminates the need to identify market-priced twin securities. In addition, our approach can be extended to the case of multinomial trees, a useful tool for modeling uncertainty in projects. We introduce within decision tree analysis (DTA) a method to derive the different discount rates that prevail at different chance nodes. We illustrate the valuation method with an application presented in "A Scenario Approach to Capacity Planning" [Eppen, G.D., Martin, R.K., Schrage, L.E., 1989. A scenario approach to capacity planning. Operations Research, 37 (4)], in which the authors state that for the capacity configuration investment decision studied at General Motors, "... there is no scientific way to determine the appropriate discount rate based on estimated demand." Our method allows deriving the scientifically correct discount rates. A major result of the analysis is that the discount rates are endogenously derived from the project structure and its behavior in light of prevailing market conditions, instead of being exogenously imposed.
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1. Introduction

Enormous research efforts have already been devoted to the analysis and valuation of investment projects. Traditional financial theory proposes the net present value (NPV) concept, using a cost of capital based on the inherent project risk. The NPV framework has been criticized because it is claimed that it cannot cope with the potential flexibility that comes with investment projects, resulting in changes in the original cash flow pattern.

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Trigeorgis (1996) claims that traditional capital budgeting methods or discounted cash flow approaches cannot cope with the operation flexibility options and other strategic aspects of various projects but that the application of option techniques results in the correct solution. Also, Dixit and Pindyck (1995) state "The net present value rule is easy, but it makes the false assumption that the investment is either reversible or that it cannot be delayed". In fact, even if a project has a positive net present value, this does not necessarily mean that the project should be taken on immediately. Sometimes delaying a positive NPV project can further improve the value of the project. Smith and McCardle (1999) write "... using the cost-of-capital-based discounting rule may ... lead to trouble when applied to projects that are significantly different from the firm as a whole. If you are going to use risk-adjusted discount rates for different projects, you should use different discount rates for different projects, evaluating each on the basis of their own cost of capital. ... Given a flexible project, you might need to go one step further and use different discount rates for different time periods and different scenarios as the risk of a project may change over time, depending on how uncertainties unfold and management reacts. ... While in principle, one could use time- and state-varying discount rates to value flexible projects, it becomes very difficult to determine the appropriate discount rates to be used in this framework." Brealey and Myers (2000) note "Most projects produce cash flows for several years. Firms generally use the same risk-adjusted rate to discount each of these cash flows. When they do this, they are implicitly assuming that cumulative risk increases at a constant rate as you look further into the future. That assumption is usually reasonable. ... But exceptions sometimes prove the rule. Be on the alert for projects where risk clearly does not increase steadily. In these cases, you should break the project into segments within which the same discount rate can be reasonably used."

These critiques to the NPV methodology for valuing projects have led to the emergence of real options analysis (ROA) for valuing managerial flexibility in projects. The contingent claims analysis approach to ROA uses market-priced securities to construct a portfolio that replicates the payoffs of the project and determines the project value using a no-arbitrage argument. The risk-neutral probability approach is an equivalent method that computes adjusted probabilities that allow valuing the project using risk-free discount rates. Both methods use geometric Brownian motion processes or binomial trees to model the project's uncertainty. In this paper, we present an alternative approach for valuing Real Options based on the certainty-equivalent version of the Net Present Value formula. Our method eliminates the need to identify market-priced twin securities to value real options, which, although theoretically sound and easy to do in valuing financial options, is rather difficult for real projects in practice. We also show that project valuation based on the net present value method, if correctly applied, is still valid in the case of project flexibility. In addition, our approach can be extended to the case of multinomial trees, whereas ROA has traditionally been restricted to binomial trees. Although binomial trees are useful for modeling financial assets, real projects are frequently modeled using multinomial trees. We will also outline how to extend decision tree analysis (DTA), a pragmatic tool to model management's beliefs about the future, to value flexibility in projects by deriving the appropriate discount rates that prevail at each chance node. This will enable the valuation of project options and flexibility in general, removing the need to distinguish between an underlying inflexible project and real options in the form of defer, abandon, accelerate, expand, contract, switch or other types of real options. Finally, we illustrate the valuation method using an application presented in "A Scenario Approach to Capacity Planning" (Eppen et al., 1989). Table 1 illustrates the contribution of this paper, and refers to the sections in the paper where this is discussed.

In Section 2, we discuss how the value of real options in binomial trees can be determined using a modified version of the net present value method as an alternative next to the replicating portfolio or

Table 1
Real options valuation approaches

	Replicating portfolio approach	Risk-neutral probability approach	Our approach
Binomial trees	Cox and Ross (1976) Cox et al. (1979) Dixit and Pindyck (1995)	Cox and Ross (1976) Harrison and Kreps (1979) Dixit and Pindyck (1995)	Section 2
Multinomial trees	–	–	Section 3
Decision tree analysis	–	–	Section 4

certainty-equivalent probability methods. Section 3 discusses how this can be extended to multinomial trees, a more versatile tool to represent uncertainty and flexibility in projects. In Section 4, we describe how the proposed method can also be used for valuing projects in general, with or without flexibility, by integrating it with decision tree analysis. A practical application is presented in Section 5. Finally, in Section 6 we give some conclusions and outline ideas for future research.

2. Valuing real options using binomial trees

2.1. The fallacy of net present value analysis and decision tree analysis for project valuation

Trigeorgis (1996) argues that traditional discounted cash flow approaches to the appraisal of capital investment projects, such as the standard NPV rule, cannot properly capture management's flexibility to adapt and revise later decisions in response to unexpected market developments. The resulting asymmetry caused by managerial adaptability calls for an expanded or strategic investment criterion and that an options approach to capital budgeting results in the correct solution.

Traditional NPV, which was initially developed to value bonds or stocks, implicitly assumes that corporations hold a collection of real assets passively. The value of active management, i.e. the value of flexibility, is better captured using decision tree analysis. Here flexibility is modeled through decision nodes allowing future managerial decisions to be made after some uncertainty has been resolved and more information has been obtained, before proceeding to the next stage. The presence of flexibility, however, changes the payoff structure and therefore the risk characteristics of an actively managed asset in a way that invalidates the use of a constant discount rate. Unfortunately, classic DTA is in no better position than discounted cash flow techniques to provide any recommendations concerning the appropriate discount rate.

2.2. Binomial trees and real options

The binomial tree in Fig. 1 represents a one-period project resulting in cash flows $\tilde{c} = (c_1, c_2)$, with probabilities p and $1 - p$. An investment I is required at the start of the project. r_f denotes the risk-free rate, and the project is discounted using the project's specific cost of capital k, obtained through the market valuation of a security or project with exactly the same payoff pattern.

The present value of the project is

$$P(\tilde{c}) = \frac{E(\tilde{c})}{1+k} = \frac{pc_1 + (1-p)c_2}{1+k},$$

where $E(\tilde{c})$ denotes the expected value of the project's cash flows. Subtracting the investment costs gives the project's net present value:

$$\mathrm{NPV}(\tilde{c}) = -I + P(\tilde{c}).$$

In the literature, various types of real options are defined: abandon, expand, contract, defer, switch, as well as compound options and rainbow options relying on multiple underlying projects. These real options result in different project cash flows, denoted by $\tilde{o} = (o_1, o_2)$. For instance, a deferral option, where the firm has a

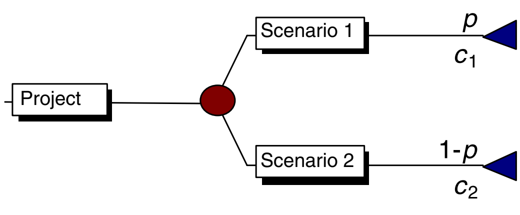

Fig. 1. Binomial tree with the cash flows of a project.

license granting it the exclusive right to defer undertaking the project for one period, will change the project's cash flows to:

$$o_s = \max(c_s - I', 0), \quad s = 1, 2,$$

with I' the investment cost next year. An expand option will change the project's cash flows to:

$$o_s = \max(fc_s - e, 0), \quad s = 1, 2,$$

with f the expand factor and e the cost of the expansion or the option's exercise price. Therefore, the risk has changed, invalidating the use of the discount rate k to compute the NPV of the option's cash flows, which would lead to the following erroneous result:

$$P(\tilde{o}) = \frac{E(\tilde{o})}{1+k} = \frac{po_1 + (1-p)o_2}{1+k}.$$

2.3. Valuing real options using contingent-claims analysis

To value real options, we can construct an equivalent replicating portfolio consisting of buying a particular number, x, of shares of the underlying asset, further denoted as the *underlying* or *inflexible project*, and borrowing against them an appropriate amount, b, at the risk free rate, that would exactly replicate the future payoffs of the project with the option, denoted as the *flexible project*, in any state of nature:

$$c_s x - (1 + r_f)b = o_s, \quad s = 1, 2.$$

Solving for x and b gives:

$$x = \frac{o_1 - o_2}{c_1 - c_2} \text{ and } b = \frac{c_1 x - o_1}{1 + r_f}.$$

The present value of the cash flows of the flexible project is then as follows:

$$P(\tilde{o}) = P(\tilde{c}) * x - b.$$

This method relies on the Market Asset Disclaimer (MAD) assumption (Copeland and Antikarov, 2001), that states that the present value of the cash flows of the project without flexibility is the best unbiased estimate of the market value of the project. This allows using the underlying project as the twin security in valuing real options.

Alternatively, we could use the risk-neutral probability approach by constructing a hedge portfolio composed of the underlying project's cash flows and a short position of m shares of the project with the option. The hedge ratio m in this example would be $(c_1 - c_2)/(o_1 - o_2)$. Since the hedge portfolio yields the same payoffs in both states, it can be discounted at r_f. Therefore:

$$P(\tilde{c}) - mP(\tilde{o}) = \frac{c_1 - mo_1}{1 + r_f} = \frac{c_2 - mo_2}{1 + r_f},$$

from which can be derived:

$$P(\tilde{o}) = \frac{P(\tilde{c})(1 + r_f) - c_1 + mo_1}{m(1 + r_f)}, \quad \text{or:}$$

$$P(\tilde{o}) = \frac{p_1 o_1 + p_2 o_2}{1 + r_f}$$

with $p_1 = \frac{P(\tilde{c})(1+r_f) - c_2}{c_1 - c_2}$ and $p_2 = \frac{c_1 - P(\tilde{c})(1+r_f)}{c_1 - c_2}$ the so-called risk neutral probabilities.

2.4. An alternative real option valuation method

Using the NPV method in a naïve manner for valuing real options is incorrect because as the option has changed the cash flow pattern of the project, so has it changed its risk profile and therefore adjustment for risk should be done appropriately. Clearly, the standard deviation of the cash flows of the underlying project and the project with the option are not necessarily identical. However, it does not follow that the NPV methodology is inappropriate for option valuation. In Proposition 1, we show that the NPV methodology, if correctly applied, can be used for real option valuation.

Proposition 1. *Real options in binomial trees can be valued using formula (2.1) below*:

$$P(\tilde{o}) = \frac{E(\tilde{o}) - \rho(\tilde{o},\tilde{c})(k - r_{\mathrm{f}})P(\tilde{c})\frac{\sigma(\tilde{o})}{\sigma(\tilde{c})}}{1 + r_{\mathrm{f}}}, \ \rho(\tilde{o},\tilde{c}) \in \{-1, 1\}. \tag{2.1}$$

Proof. The capital asset pricing model (Sharpe, 1965; Lintner, 1965) allows to value the cash flows of a project with an option using certainty equivalents and the following formula (Brealey and Myers, 2000, p. 248):

$$P(\tilde{o}) = \frac{E(\tilde{o}) - \lambda \operatorname{cov}(\tilde{o}, r\tilde{m})}{1 + r_{\mathrm{f}}} \quad \text{with } \lambda = \frac{E(r\tilde{m}) - r_{\mathrm{f}}}{\sigma^2(r\tilde{m})}, \tag{2.2}$$

where $r\tilde{m}$ denotes the market return and λ is the so-called market price of risk. This formula can be derived from the present value of the project with the option:

$$P(\tilde{o}) = \frac{E(\tilde{o})}{1 + E(r\tilde{o})}, \tag{2.3}$$

with $r\tilde{o} = (ro_1, ro_2)$ the returns of the project with the option, and $E(r\tilde{o})$ the expected return:

$$E(r\tilde{o}) = r_{\mathrm{f}} + \beta(\tilde{o})(E(r\tilde{m}) - r_{\mathrm{f}}) = r_{\mathrm{f}} + \frac{\operatorname{cov}(r\tilde{o}, r\tilde{m})}{\sigma^2(r\tilde{m})}(E(r\tilde{m}) - r_{\mathrm{f}}), \tag{2.4}$$

where $\beta(\tilde{o})$ denotes the beta of the cash flows associated with the flexible project. Substituting (2.4) in (2.3) and using

$$ro_s = \frac{o_s}{P(\tilde{o})} - 1, \quad s = 1, 2,$$

such that

$$\operatorname{cov}(r\tilde{o}, r\tilde{m}) = \frac{\operatorname{cov}(\tilde{o}, r\tilde{m})}{P(\tilde{o})}, \text{ gives:}$$

$$P(\tilde{o}) = \frac{E(\tilde{o})}{1 + \left(r_{\mathrm{f}} + \frac{\operatorname{cov}(\tilde{o}, r\tilde{m})}{P(\tilde{o})\sigma^2(r\tilde{m})}(E(r\tilde{m}) - r_{\mathrm{f}})\right)},$$

which results into the CAPM valuation formula (2.2).

For the case of binomial trees, the cash flows of a project with a real option are always perfectly positively or negatively correlated with the underlying project's cash flows[1]:

$$\rho(\tilde{o},\tilde{c}) \in \{-1, 1\}.$$

Consequently:

$$\rho(\tilde{o}, r\tilde{m}) = \rho(\tilde{o},\tilde{c})\rho(\tilde{c}, r\tilde{m}), \text{ or:}$$

$$\operatorname{cov}(\tilde{o}, r\tilde{m}) = \rho(\tilde{o},\tilde{c})\operatorname{cov}(\tilde{c}, r\tilde{m})\frac{\sigma(\tilde{o})}{\sigma(\tilde{c})}. \tag{2.5}$$

[1] Note that this is also the case for multi-period binomial trees, as the tree can be collapsed as the valuation proceeds from right to left in a backward pass fashion.

Moreover:

$$k = r_{\mathrm{f}} + \beta(\tilde{c})(E(r\tilde{m}) - r_{\mathrm{f}}), \text{ or:}$$
$$\beta(\tilde{c}) = \frac{k - r_{\mathrm{f}}}{E(r\tilde{m}) - r_{\mathrm{f}}} \tag{2.6}$$

and also:

$$\beta(\tilde{c}) = \frac{\mathrm{cov}(r\tilde{c}, r\tilde{m})}{\sigma^2(r\tilde{m})} = \frac{\mathrm{cov}(\tilde{c}, r\tilde{m})}{P(\tilde{c})\sigma^2(r\tilde{m})}. \tag{2.7}$$

where $r\tilde{c} = (rc_1, rc_2)$ denote the project returns in the different scenarios. Effectively, $k = E(r\tilde{c})$. Using (2.6) and (2.7), we obtain:

$$\mathrm{cov}(\tilde{c}, r\tilde{m}) = \frac{k - r_{\mathrm{f}}}{E(r\tilde{m}) - r_{\mathrm{f}}} P(\tilde{c})\sigma^2(r\tilde{m}). \tag{2.8}$$

Substituting (2.8) into (2.5) and the result into (2.2), gives, after some algebra, the result (2.1). □

This valuation formula (2.1) provides a general way of valuing real options based on the underlying project, and substantiates the claim that the NPV methodology, if correctly applied, is appropriate for valuing real options. Our valuation formula uses the present value of the project without options, $P(\tilde{c})$, to arrive at the value of the project with options, $P(\tilde{o})$. This is exactly the MAD assumption adopted by Copeland and Antikarov (2001), which is also used in Contingent Claims Analysis, because by designating the project itself as the underlying marketed security, the market becomes complete for this project. However, observe that in the option valuation formula (2.1), the term b, representing the amount borrowed at the risk-free rate in the replicating portfolio, does not appear. Consequently, (2.1) allows valuing a real option directly from the project's cash flows without the need to search for market-priced twin securities. Also, (2.1) shows that the risk premium is proportional to the ratio of the volatilities of the flexible project and the inflexible project. In other words, the risk premium changes proportionally with the percentage change in volatility introduced by the real option.

Note also that although the valuation formula (2.2) is derived from the CAPM, and therefore also depends on the applicability of the CAPM assumptions (Lintner, 1965), formula (2.1) directly values the project with real options based on the underlying inflexible project, and therefore does not require any additional assumptions than those of the replicating portfolio approach. In fact, for binomial trees, the valuation formula (2.1) yields the same result as the replicating portfolio approach. Table 2 summarizes the results.

3. Valuing real options using multinomial trees

3.1. Real options in multinomial trees

Consider a multinomial tree representing a one-period project with s states and cash flows $\tilde{c} = (c_1, \ldots, c_s)$ with probabilities $p_1, p_2, \ldots, p_s$. For valuing a real option, the replicating portfolio technique requires identifying a number of shares, $y_i, i = 1, \ldots, s-2$, of $s-2$ additional priced securities with payoffs $c\tilde{y}_i = (cy_{i1}, \ldots, cy_{is})$, so that the following system of equations can be solved:

$$\forall s : c_s x + \sum_{i=1}^{s-2} cy_{is} y_i - (1 + r_{\mathrm{f}})b = o_s,$$

Table 2
Valuing real options using binomial trees

	Replicating portfolio approach	Risk-neutral probability approach	Our approach
Binomial trees	$P(\tilde{o}) = P(\tilde{c}) * x - b$ $x = \frac{o_1 - o_2}{c_1 - c_2}$ $b = \frac{c_1 x - o_1}{1 + r_{\mathrm{f}}}$	$P(\tilde{o}) = \frac{p_1 o_1 + p_2 o_2}{1 + r_{\mathrm{f}}}$ $p_1 = \frac{P(\tilde{c})(1 + r_{\mathrm{f}}) - c_2}{c_1 - c_2}$ $p_2 = \frac{c_1 - P(\tilde{c})(1 + r_{\mathrm{f}})}{c_1 - c_2}$	$P(\tilde{o}) = \frac{E(\tilde{o}) - \rho(\tilde{o},\tilde{c})(k - r_{\mathrm{f}})P(\tilde{c})\frac{\sigma(\tilde{o})}{\sigma(\tilde{c})}}{1 + r_{\mathrm{f}}}$ $\rho(\tilde{o}, \tilde{c}) \in \{-1, 1\}$

where x denotes the number of shares of the underlying project. These priced securities, however, are generally not readily available.

So when markets are incomplete, neither the replicating portfolio approach nor the risk-neutral approach is able to determine uniquely the value of a real option. Also the valuation formula (2.1) cannot be used directly, because in general the cash flows of the project with the abandon option are not perfectly positively or negatively correlated with the inflexible project's cash flows, i.e. $\rho(\tilde{o},\tilde{c}) \notin \{1,-1\}$. This assumption, which is automatically satisfied in binomial trees, is required to determine the correlation of the cash flows of the project with real options with the market, $\rho(\tilde{o},r\tilde{m})$. Also note that breaking down multinomial trees into multi-stage binomial trees is not a solution to this problem because the appropriate correlation coefficients for each of the stages cannot be determined.

3.2. Bounding the value of real options in multinomial trees

Although unique risk-neutral probabilities cannot be determined in multinomial trees, calculating all equivalent probability measures under which the discounted (with r_{f}) underlying project is a martingale allows to derive bounds on a project with real options as follows:

$$\inf_{Q} E^{Q}\left[\frac{\tilde{o}}{1+r_{\mathrm{f}}}\right] \leqslant P(\tilde{o}) \leqslant \sup_{Q} E^{Q}\left[\frac{\tilde{o}}{1+r_{\mathrm{f}}}\right],$$

where Q runs among all equivalent martingale measures and E^{Q} denotes the expectation under measure Q. In effect, this boils down to determining which set of risk-neutral probabilities consistent with the project value $P(\tilde{c})$ minimize and maximize $P(\tilde{o})$.

When $\rho(\tilde{o},\tilde{c}) \notin \{1,-1\}$, our proposed valuation method can be used as long as the correlation of the cash flows of the flexible project with the market returns can be computed. Instead of using formula (2.1), which assumes perfect correlation between the cash flows of the flexible and the underlying inflexible project, formula (2.2) should be used. Valuation formula (2.2) requires determining the covariance $cov(\tilde{o},r\tilde{m})$, or alternatively the correlation $\rho(\tilde{o},r\tilde{m})$, of the cash flows of the flexible project with market returns. This is not possible in general. However, bounds for $\rho(\tilde{o},r\tilde{m})$ can be derived, which result in bounds for the real option value.

Proposition 2. *In multinomial trees, the value of a project with real options is bounded as follows*:

$$\begin{aligned} P(\tilde{o}) &\geqslant \frac{E(\tilde{o}) - \left(\rho(\tilde{o},\tilde{c})\rho(\tilde{c},r\tilde{m}) + \sqrt{(\rho(\tilde{o},\tilde{c})^2-1)(\rho(\tilde{c},r\tilde{m})^2-1)}\right)\frac{\sigma(\tilde{o})}{\sigma(r\tilde{m})}(E(r\tilde{m})-r_{\mathrm{f}})}{1+r_{\mathrm{f}}},\\ P(\tilde{o}) &\leqslant \frac{E(\tilde{o}) - \left(\rho(\tilde{o},\tilde{c})\rho(\tilde{c},r\tilde{m}) - \sqrt{(\rho(\tilde{o},\tilde{c})^2-1)(\rho(\tilde{c},r\tilde{m})^2-1)}\right)\frac{\sigma(\tilde{o})}{\sigma(r\tilde{m})}(E(r\tilde{m})-r_{\mathrm{f}})}{1+r_{\mathrm{f}}}. \end{aligned} \tag{3.1}$$

Proof. Consider correlation matrix

$$C = \begin{bmatrix} \rho(\tilde{o},\tilde{o}) & \rho(\tilde{o},\tilde{c}) & \rho(\tilde{o},r\tilde{m}) \\ \rho(\tilde{c},\tilde{o}) & \rho(\tilde{c},\tilde{c}) & \rho(\tilde{c},r\tilde{m}) \\ \rho(r\tilde{m},\tilde{o}) & \rho(r\tilde{m},\tilde{c}) & \rho(r\tilde{m},r\tilde{m}) \end{bmatrix},$$

$\rho(\tilde{c},r\tilde{m})$ can be computed from the present value or the cost of capital of the underlying project. $\rho(\tilde{o},\tilde{c})$ can be computed directly from both projects' cash flows. $\rho(\tilde{o},r\tilde{m})$ is unknown, and is required to value the project with the option. The matrix C needs to be consistent, or semi positive-definite, which requires $\forall e_i{:}e_i \geqslant 0$ and $\exists e_i{:}e_i > 0$, where e_i, $i = 1,2,3$ denote the eigenvalues of C, which can be determined by solving the following equation, where $|X|$ denotes the determinant of X:

$$\begin{vmatrix} \rho(\tilde{o},\tilde{o}) - e & \rho(\tilde{o},\tilde{c}) & \rho(\tilde{o},r\tilde{m}) \\ \rho(\tilde{c},\tilde{o}) & \rho(\tilde{c},\tilde{c}) - e & \rho(\tilde{c},r\tilde{m}) \\ \rho(r\tilde{m},\tilde{o}) & \rho(r\tilde{m},\tilde{c}) & \rho(r\tilde{m},r\tilde{m}) - e \end{vmatrix} = 0.$$

It follows that:

$$e^3 - 3e^2 - (\rho(\tilde{o}, r\tilde{m})^2 + \rho(\tilde{o}, \tilde{c})^2 + \rho(\tilde{c}, r\tilde{m})^2 - 3)e + (\rho(\tilde{o}, r\tilde{m})^2 + \rho(\tilde{o}, \tilde{c})^2 + \rho(\tilde{c}, r\tilde{m})^2 - 2\rho(\tilde{o}, r\tilde{m})\rho(\tilde{o}, \tilde{c})\rho(\tilde{c}, r\tilde{m}) + 1) = 0.$$

For any third-order polynomial:

$$\prod_{i=1}^{3} e_i = -c,$$

where e_i are the roots of the polynomial, and c denotes the constant term. Because c is positive semi-definite, it follows that:

$$\prod_{i=1}^{3} e_i \geqslant 0, \text{ or } c \leqslant 0.$$

Therefore:

$$\rho(\tilde{o}, r\tilde{m})^2 + \rho(\tilde{o}, \tilde{c})^2 + \rho(\tilde{c}, r\tilde{m})^2 - 2\rho(\tilde{o}, r\tilde{m})\rho(\tilde{o}, \tilde{c})\rho(\tilde{c}, r\tilde{m}) + 1 \leqslant 0, \text{ or :}$$

$$\rho(\tilde{o}, r\tilde{m}) \geqslant \rho(\tilde{o}, \tilde{c})\rho(\tilde{c}, r\tilde{m}) - \sqrt{(\rho(\tilde{o}, \tilde{c})^2 - 1)(\rho(\tilde{c}, r\tilde{m})^2 - 1)} \text{ and}$$

$$\rho(\tilde{o}, r\tilde{m}) \leqslant \rho(\tilde{o}, \tilde{c})\rho(\tilde{c}, r\tilde{m}) + \sqrt{(\rho(\tilde{o}, \tilde{c})^2 - 1)(\rho(\tilde{c}, r\tilde{m})^2 - 1)}.$$

Using version (2.2) of the valuation formula:

$$P(\tilde{o}) = \frac{E(\tilde{o}) - \rho(\tilde{o}, r\tilde{m})\frac{\sigma(\tilde{o})}{\sigma(r\tilde{m})}(E(r\tilde{m}) - r_{\mathrm{f}})}{1 + r_{\mathrm{f}}},$$

the result (3.1) follows. □

To examine the tightness of the bounds, we relate the magnitude of the range between the upper and lower bound to the standard deviation of the cash flows of the project with the options. Below, we derive the worst-case tightness. The range R between the upper and lower bound equals:

$$R = \frac{2\sqrt{(\rho(\tilde{o}, \tilde{c})^2 - 1)(\rho(\tilde{c}, r\tilde{m})^2 - 1)}\frac{\sigma(\tilde{o})}{\sigma(r\tilde{m})}(E(r\tilde{m}) - r_{\mathrm{f}})}{1 + r_{\mathrm{f}}},$$

which reaches a maximum when $\rho(\tilde{o}, \tilde{c}) = 0$ and $\rho(\tilde{c}, r\tilde{m}) = 0$, so that:

$$R^{\text{worst-case}} = \frac{2\frac{\sigma(\tilde{o})}{\sigma(r\tilde{m})}(E(r\tilde{m}) - r_{\mathrm{f}})}{1 + r_{\mathrm{f}}}.$$

Therefore, the tightness, defined as the worst-case range divided by the standard deviation of the project cash flows, equals:

$$\text{Tightness} = \frac{2(E(r\tilde{m}) - r_{\mathrm{f}})}{\sigma(r\tilde{m})(1 + r_{\mathrm{f}})}.$$

For instance, when the average market return is 12% with a standard deviation of 20% and the risk-free rate is 5%, the tightness equals 67%. This means that in the worst-case, the range of the bound is equal to 67% of the standard deviation of the project cash flows.

Consider for example a project with three possible outcomes, \$1.2, \$1 or \$0.8, with 25%, 50% and 25% probability, respectively. With a risk-free interest rate of 5% and a cost of capital of 10%, we find that $E(\tilde{c}) = \$1.00$ and $P(\tilde{c}) = \$0.909$. An abandon option with a payoff of \$1 results in $\rho(\tilde{o}, \tilde{c}) = 0.816$. Assuming average market returns of 12% with a standard deviation of 20% (Brealey and Myers, 2000), we find $P(\tilde{o}) \in [\$0.972; \$0.985]$. The abandon option, therefore, is worth between \$0.063 and \$0.076. The risk-neutral approach would result in $P(\tilde{o}) \in [\$0.952; \$1.026]$ or an option value between \$0.043 and \$0.117.

4. Valuing project options

4.1. Project options and decision tree analysis

We have shown how the NPV methodology can be modified to value real options. A characteristic of real options is that they are based on an underlying project, so that their value can be derived indirectly from the value of the underlying project. In ROA, it is generally assumed that the value of the underlying project is known, or that a particular discount rate is appropriate. However, the challenge remains to value the underlying project. Therefore, the distinction between a flexible version of a project and inflexible underlying project is rather artificial, since every project will inherently contain decisions to be made that depend on the circumstances prevailing at that particular time, which in essence are no different from real options. Therefore, since every project has options and flexibility embedded, one may argue that this is a vicious cycle and that no project can ever be valued. We believe that it is not necessary, nor advisable, to use an indirect valuation approach and distinguish between an inflexible version of the project and an enhanced version including real options, but that one should value projects with embedded options, which we will refer to as *project options* instead of real options, directly.

When the probability distribution of market returns are known and the conditional probability distribution of the project returns to these market returns can be estimated, then using our proposed valuation method in conjunction with decision tree analysis, an appropriate discount rate at each chance node in the decision tree can be determined.

4.2. Separating unique and systematic project risks

When the covariance or correlation between a project's cash flows and the market returns is known, the present value of the uncertain cash flows can be readily determined using formula (2.2). For instance, uncertain cash flows that depend solely on a project's unique risk, are not correlated with market returns, and can therefore be discounted at the risk-free rate. Cash flows that are completely dependent on market conditions can sometimes be assumed to exhibit a perfect correlation with market returns, which can then be valued using formula (2.2) with $\rho = 1$, resulting in a locally appropriate discount rate that depends on the volatility of the cash flows. All other situations can be tackled by either estimating the market correlation directly, or by splitting up the uncertainty factors in market-unrelated and market-related ones, or in other words, in unique and systematic risks.

In accordance to Smith and Nau (1995), keeping unique and market risks separate whenever possible, will enable a valuation of projects and project options, and also clarifies how the project's success depends on internal and external factors. The project cash flows can be tied to macro-economic growth or other market-related factors, by determining what happens to the project's cash flows depending on specific external circumstances. By creating different future market states, and evaluating the project in each of those states, we will be able to determine the correlation between the project's cash flows and the market returns, and hence the project's value.

The characteristics of long-term market returns, including their distribution, are quite well known. We can approximate this continuous distribution by a discrete distribution using the moment-matching methods of Smith (1993) and Miller and Rice (1983), or the equal-area approach described in McNamee and Celona (1987). Then, for each market state rm_i $(i = 1, \ldots, t)$, the possible outcomes for the project and their probability should be determined. This is depicted in Fig. 2. This will in effect split up the unique project risk from the market risk involved, and will enable the valuation of the project using a risk-free discount rate for the unique risks, and an appropriate discount rate for the market-related risks, depending on the volatility of the cash flows. The unique project risks can be valued in each market state by computing the expected values $\tilde{e} = (e_1, \ldots, e_n)$ with

$$e_i = E(\tilde{c}_i) = \sum_{s=1}^{t} p_{is} c_{is},$$

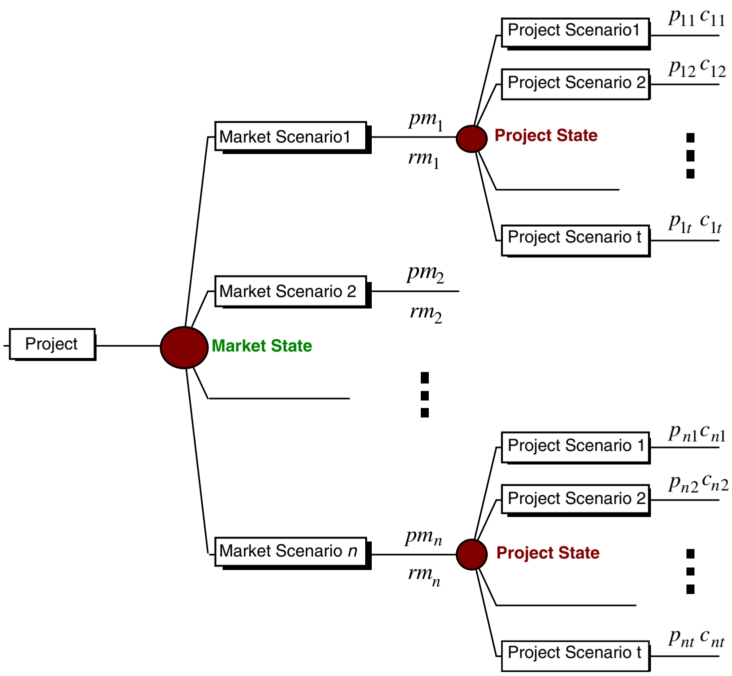

Fig. 2. Separating systematic and unique risks.

which can then be used as cash flows in formula (2.2) to determine the present value of the project. Actually, the expected values e_i represent the possible project states where the unique project risk has been removed, i.e. collapsed in the expected values. The remaining variability in the project's cash flows is due to market-related factors, and determines the project's systematic risk. In other words, we assume that conditional to $r\tilde{m}$, there is no risk premium for the unique project risk. We are now in a position to determine the covariance term in valuation formula (2.2) as follows:

$$\operatorname{cov}(\tilde{e}, r\tilde{m}) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} pm_i pm_j (rm_i - rm_j)(e_i - e_j),$$

and derive the resulting present value.

4.3. An example

Consider the product development project depicted in Fig. 3 with two possible outcomes, $t = 2$, resulting in cash flows $c_1 = \$1$ and $c_2 = \$0.5$, the probabilities of which depend on the market success of the newly developed product. The outcome depends on the market conditions, but also on the market response to the product. If we assume, without loss of generality, that the market returns are normally distributed with an expected return of 12% and a standard deviation equal to 20%, a moment-matching trinomial approximation of the market distribution, $n = 3$, can be derived as follows: $rm_1 = -22.64\%$, $rm_2 = 12.00\%$ and $rm_3 = 46.64\%$ with $pm_1 = 16.67\%$, $pm_2 = 66.67\%$ and $pm_3 = 16.67\%$ (Smith, 1993). Assume that management agrees that a good project outcome will occur under favorable market conditions with 70% probability, under average market conditions with 60% probability and under unfavorable market conditions with 20% probability.

The expected value of the project in each of the three market states is $\tilde{e} = (\$0.60, \$0.80, \$0.85)$, with $E(\tilde{e}) = \$0.775$, $\sigma(\tilde{e}) = \$0.08$ and $\operatorname{cov}(\tilde{e}, r\tilde{m}) = 0.014$. Assuming $r_f = 8\%$ gives $\lambda = 1$ and formula (2.2) yields:

$$P(\tilde{e}) = \frac{\$0.775 - 1(0.014)}{1.08} = \$0.704.$$

The implicit discount rate equals 10.05%.

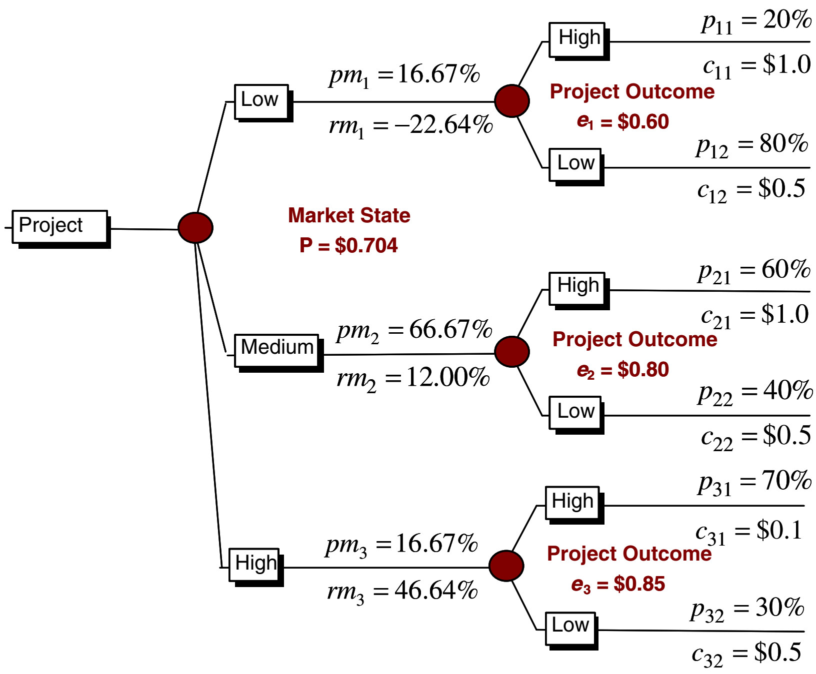

Fig. 3. A project.

5. An application to a capital investment project

5.1. Introduction

In "A Scenario Approach to Capacity Planning" (Eppen et al., 1989), the authors study a multi-product, multi-plant, multi-period capacity configuration decision problem at General Motors with uncertain future demand modeled using scenarios linked to general macro-economic conditions. In the paper, it is stated, "... there is no scientific way to determine the appropriate discount rate based on estimated demand." Consequently, the authors assume a discount rate of 10% to compute the present value of the relevant cash flows. Using the methodology presented in this paper, we can now derive the correct discount rates. A result of this analysis is that the discount rates are endogenously derived from the project structure instead of being exogenously, and rather arbitrarily, imposed.

Table 3
Capacity and production costs (costs measured in $1000)

Sites										
	1		2			3		4		
Configuration	0	1	0	1	2	0	1	0	1	2
Capacity	0	1000	0	1000	2000	0	1000	0	1000	3000
Changeover	10,000	0	10,000	0	16,000	10,000	0	10,000	0	225,000
Fixed costs	0	0	0	0	0	0	0	0	0	0
Marginal production costs										
Product 1	–	10	–	–	10	–	–	–	–	–
Product 2	–	–	–	10	10	–	–	–	–	–
Product 3	–	–	–	–	–	–	10	–	–	6
Product 3	–	–	–	–	–	–	–	–	10	6

Table 4
Demand and sales price data

Product	Quantity demanded			Sales price		
	Scenario 1	Scenario 2	Scenario 3	Scenario 1	Scenario 2	Scenario 3
1	1000	1500	1500	\$12	\$12	\$13
2	700	1400	1450	\$12	\$12	\$12
3	3000	700	400	\$20	\$12	\$12
4	900	700	400	\$12	\$12	\$12

5.2. The problem

In their paper, Eppen et al. provide an example problem. The example includes 4 products and 4 production sites. Management is considering proposals to convert site 2 into a flexible plant with double the original capacity and/or convert site 4 into a flexible plant with triple the original capacity and lower marginal productions costs. Capacity and cost information, as used in the paper, is presented in Table 3.

In Table 3, a "–" indicates that the given product-configuration pair is not feasible. There is a changeover cost for shutting down a plant as well as for retooling it. Fixed operating costs are incurred if the plant is open, independent of the number of cars produced. The unit production costs vary directly with the number of cars produced. Configuration 1 represents the current situation while configuration 0 corresponds to closing down the plant; configuration 2 is a potential new configuration for the plant, which is flexible in the sense that it can produce a combination of different products. The management of GM decided that the configuration at a site could be changed at most once during a 5-year interval.

In addition to selecting the capacity configuration, production decisions must be made. This leads to a sequential decision problem. It is assumed that the demand is observed before the production decision is made. The production plan is then an optimal decision in view of the known demand and the available capacities. No inventory is carried from period to period. The authors propose to use a scenario approach in which three scenarios or states of nature were specified for each year, with scenario probabilities independent from period to period. A scenario can be thought of as a combination of factors, e.g. the state of the US economy, level of energy prices and level of foreign competition that determine the demand for the GM products. For each scenario, prices and quantity demanded for each product are estimated. Their model maximizes the present value of the expected discounted cash flows subject to a constraint on risk. The example data for the first year are presented in Table 4. Scenarios 1, 2, and 3 occur with probabilities 60%, 30% and 10%, respectively. In the example, the demands, prices and probabilities assumed for year 1 hold in each of the five years.

Observe from Table 4 that the quantity demanded of products 1 and 2 is inversely correlated with the demand for products 3 and 4. The authors clarify the example by indicating that products 3 and 4 could be economy vehicles having high demand in scenario 1, while products 1 and 2 are larger less fuel-efficient vehicles with high demand in scenario 3. We use this key observation to link the scenarios to general macro-economic conditions by assuming that scenario 1 corresponds to a "low" economy or market while scenario 3 would indicate a "high" or booming economy. Scenario 2 represents then a "medium" market state. Therefore at the time the research was done, low macro-economic conditions were forecast with a 60% probability, a medium market with 30% probability while a booming economy was deemed only 10% likely.

We are now in a position to apply the principles outlined in Section 4.1. Assuming long-term market returns are normally distributed with a mean of 12% and a standard deviation of 20% (Brealey and Myers, 2000), we can determine the three discrete market returns that correspond with the three scenarios given in Table 4 using the equal-area approach (McNamee and Celona, 1987) as follows[2]:

[2] We use the equal-area approach instead of the moment-matching method of Smith (1993) or Miller and Rice (1983) because the moment-matching method becomes very complex for three-point asymmetric approximations with pre-specified probabilities as it involves solving systems of nonlinear equations. A very pragmatic approach based on the normal distribution is also available (Keefer, 1994), which may be of interest when applied in practice.

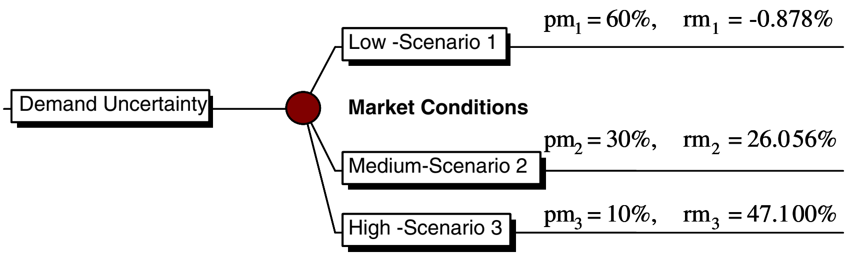

Fig. 4. Scenario representation of demand uncertainty and macro-economic conditions.

$$rm_i = \frac{1}{pm_i}\int_{F^{-1}\left(\sum_{j=1}^{i-1} pm_j\right)}^{F^{-1}\left(\sum_{j=1}^{i} pm_j\right)} \tilde{rm} f(\tilde{rm})\,\mathrm{d}\tilde{rm}, \quad i = 1,2,\ldots,n,$$

where n denotes the degree of the multinomial tree. $f(\tilde{rm})$ and $F^{-1}(\tilde{rm})$ denote the probability density function and inverse cumulative probability distribution of the market returns, and rm_i, $i = 1,\ldots,n$ denote the market returns in the multinomial tree, each with pre-set probability pm_i, $i = 1,\ldots,n$. In this case, we obtain the three market scenarios depicted in Fig. 4.

5.3. Modeling year 5 and the following years

The authors select an infinite planning horizon but assume that the demand process remains stationary after the first four periods, i.e., the characteristics associated with each scenario and the probabilities assigned to each scenario remain constant for periods 5 through infinity. Also, no retooling decisions are possible after period 5. This effectively produces a 5-period model in which period 5 represents periods 5 through infinity, with an annuity factor of 7.51 used discount the cash flows. We will now derive the equivalent of our valuation methodology to implement this idea using a stationary demand process, keeping the characteristics associated with each scenario and the probabilities assigned to each scenario constant from period 5 onwards.

Let $\tilde{c}_t$ represent the vector of scenario cash flows in each time period t, i.e. $\tilde{c}_t = (c_{1t}, c_{2t}, c_{3t})$. In addition, if P_{t-1} indicates the present value of all cash flows from period t to infinity and $\tilde{f}_t$ represents the vector of scenario values in period t to be discounted, i.e. $\tilde{f}_t = (c_{1t} + P_t, c_{2t} + P_t, c_{3t} + P_t)$, the valuation in period 4 yields: $P_4 = \frac{E(\tilde{f}_5) - \lambda\,\mathrm{cov}(\tilde{f}_5, \tilde{rm})}{1 + r_f}$. As period 5 repeats itself through infinity, the scenario cash flows are identical in each period and we can drop the index t resulting into $\tilde{c}_t = (c_{1t}, c_{2t}, c_{3t}) = \tilde{c} = (c_1, c_2, c_3)$ for $t = 5, 6, \ldots, \infty$. In addition, given that P_t is a constant, we obtain that $E(\tilde{f}_t) = E(\tilde{c}) + P_t$, and $\sigma(\tilde{f}_t) = \sigma(\tilde{c})$, and finally, $\mathrm{cov}(\tilde{f}_t, \tilde{rm}) = \mathrm{cov}(\tilde{c}, \tilde{rm})$. Substituting this into the valuation formula and working out the recursion gives: $P_4 = \sum_{t=1}^{\infty} \frac{E(\tilde{c}) - \lambda\,\mathrm{cov}(\tilde{c}, \tilde{rm})}{(1 + r_f)^t} = \frac{E(\tilde{c}) - \lambda\,\mathrm{cov}(\tilde{c}, \tilde{rm})}{r_f}$. This formula is quite intuitive as it represents the perpetuity equivalent of the valuation formula (2.2) where the uncertainty as expressed into the demand scenarios repeats itself infinitely.

5.4. The objective function

We use as much as possible the notation introduced by Eppen et al. A major difference is that while all revenues, profits and costs are expressed as present values in Eppen et al., our data are the undiscounted, actual numbers. For instance, in Eppen et al., r_{ikhmt} represents the present value of the variable contribution margin from selling one unit of product i at site k with configuration h under scenario m in period t, while in our model it is the actual, undiscounted value.

Let c_{mt} be the actual, undiscounted scenario contribution for scenario m in period t as follows:

$$c_{mt} = \sum_{i=1}^{N}\sum_{k=1}^{K}\sum_{h=0}^{H_k} r_{ikhmt}x_{ikhmt} - \sum_{k=1}^{K}\sum_{h=0}^{H_k} F_{kht}w_{kht} - \sum_{k=1}^{K}\sum_{h=0}^{H_k} C_{kht}y_{kht}.$$

Table 5
Detailed profits with our approach

Scenario	Probability (%)	$\tilde{rm}$ (%)	$\tilde{f}_1$	$\tilde{f}_2$	$\tilde{f}_3$	$\tilde{f}_4$	$\tilde{c}_5 = \tilde{c}$
1	60	−0.878	286,464	511,464	511,464	511,464	47,200
2	30	26.056	252,264	477,264	477,264	477,264	13,000
3	10	47.100	249,724	474,724	474,724	474,724	10,460
Expected value		12.000	272,530	497,530	497,530	497,530	33,266
Standard deviation		16.792		17,080	17,080	17,080	17,080
Covariance				−2732	−2732	−2732	−2732
Correlation				−0.952	−0.952	−0.952	−0.952
Begin of year value			$P_0 = 272{,}530$	$P_1 = 464{,}264$	$P_2 = 464{,}264$	$P_3 = 464{,}264$	$P_4 = 464{,}264$
Implied discount rate				7.165%	7.165%	7.165%	7.165%

Using the notation defined above, the valuation formula in period t, applicable to the decision horizon of years 1 to 4, can then be written as: $P_{t-1} = \frac{E(\tilde{f}_t) - \lambda \operatorname{cov}(\tilde{f}_t, \tilde{rm})}{1+r_f}$, where $\lambda = \frac{E(\tilde{rm}) - r_f}{\sigma^2(\tilde{rm})}$ denotes the market price of risk. After some algebra, and using indices u and v in addition to m to indicate the market scenarios, we find that: $\operatorname{cov}(\tilde{f}_t, \tilde{rm}) = \sum_{u=1}^{M-1} \sum_{v=u+1}^{M} pm_u pm_v (rm_u - rm_v)(f_{ut} - f_{vt})$, such that: $P_{t-1} = \frac{\sum_{u=1}^{M} pm_u f_{ut} - \lambda \sum_{u=1}^{M-1} \sum_{v=u+1}^{M} pm_u pm_v (rm_u - rm_v)(f_{ut} - f_{vt})}{1+r_f}$. Summarizing, the net present value of the project is then $P = P_0 = \sum_{u=1}^{3} pm_u (c_{u1} + P_1)$. Alternatively, we can avoid the recursion in the valuation formula and use the following objective function:

$$P = P_0 = \sum_{u=1}^{3} pm_u c_{u1} + \frac{E(\tilde{c}_2) - \lambda \operatorname{cov}(\tilde{c}_2, \tilde{rm})}{1 + r_f} + \frac{E(\tilde{c}_3) - \lambda \operatorname{cov}(\tilde{c}_3, \tilde{rm})}{(1 + r_f)^2} + \frac{E(\tilde{c}_4) - \lambda \operatorname{cov}(\tilde{c}_4, \tilde{rm})}{(1 + r_f)^3} + \frac{1}{(1 + r_f)^3} \frac{E(\tilde{c}) - \lambda \operatorname{cov}(\tilde{c}, \tilde{rm})}{r_f}.$$

Optimizing our objective function, we find the same optimal total capacity configuration, i.e. leave sites 1, 2, and 3 in their original configuration and retool site 4 into configuration 2 in the first year. Using a risk-free interest rate $r_f = 8\%$, our expected profit is \$272,530 substantially more than \$141,000 as reported by EMS. The detailed breakdown of our profits over the decision horizon is given in Table 5. The implied discount rate, endogenously determined by the specific project structure and general historical market characteristics, is 7.165%, considerably less than the "assumed" discount rate of 10% in the EMS approach. This confirms the observation made by several researchers that a traditionally determined discount rate typically ignores flexibility and real options in capacity investments, thereby overestimating uncertainty and the required risk premium.

6. Conclusions

We have provided a framework integrating decision tree analysis and a net present value-based analysis of real options, combining the flexibility, elegance and clarity of DTA with a sound valuation of flexibility in projects. The proposed valuation method, in contrast to the replicating portfolio technique, does not rely on the identification of market-priced twin securities, and can also be used when project uncertainty is modeled using multinomial trees, and for valuing project options, rather than just real options, which are rather narrowly defined as being simple extensions of an underlying project for which the value is supposed to be known. This allows the extension of decision tree analysis with a correct valuation of the project based on its inherent uncertainty and its behavior in light of prevailing market conditions.

For an investment analysis of a project with uncertain cash flows and inherent project options, we require:

- a scenario representation of the possible cash flows with associated probabilities;
- an assessment of the correlation of the project's cash flows with the market;
- a computation of the expected value and standard deviation of the project's cash flows;
- the valuation method proposed in this paper.

The scenario representation of uncertain cash flows is a common and pragmatic way for management to represent the future outcomes and deal with risk and uncertainty (see for instance Eppen et al., 1989). The assessment of the correlation of a project's cash flows with the market is essential for any investment appraisal, as it is the key component of the risk-adjusted discount rate. Discounting cash flows at a company's cost of capital implicitly assumes that the project represents average risk to the company and consequently represents an average correlation with the market that can then readily be derived. However, using an average cost of capital-based discount rate for all underlying projects will either under- or overestimate the project's market risk and will result in erroneous valuations. In decision tree analysis, the standard deviation can readily be used in the valuation formula as a measure of the project's risk. Finally, observe that those four requirements are necessary whether or not there are any real options present in the project at hand.

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