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Citation

CHRONOPOULOS, Michail; DE REYCK, Bert; and SIDDIQUI, Afzal. Optimal investment under operational flexibility, risk aversion, and uncertainty. (2011). *European Journal of Operational Research*. 213, (1), 221-237.

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Optimal investment under operational flexibility, risk aversion, and uncertainty

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A B S T R A C T

Traditional real options analysis addresses the problem of investment under uncertainty assuming a risk-neutral decision maker and complete markets. In reality, however, decision makers are often risk averse and markets are incomplete. We confirm that risk aversion lowers the probability of investment and demonstrate how this effect can be mitigated by incorporating operational flexibility in the form of embedded suspension and resumption options. Although such options facilitate investment, we find that the likelihood of investing is still lower compared to the risk-neutral case. Risk aversion also increases the likelihood that the project will be abandoned, although this effect is less pronounced. Finally, we illustrate the impact of risk aversion on the optimal suspension and resumption thresholds and the interaction among risk aversion, volatility, and optimal decision thresholds under complete operational flexibility.

Keywords:

Decision analysis
Investment under uncertainty
Real options
Operational flexibility
Risk aversion

1. Introduction

Fluctuating global economic conditions require responsive strategies in order to ensure the effectiveness of investment decisions. The withdrawal of Honda in 2008 from Formula One ([Financial Times, 2008](#)), for instance, was made in light of the rapidly deteriorating conditions facing the global auto industry and reflects the impact of the global financial and economic crisis. Indeed, when market uncertainty increases and decision makers are risk averse, the discretion to abandon, modify, or suspend existing projects becomes of greater importance. In this paper, we examine the impact of such operational flexibility, in terms of being able to suspend and resume the project at any time, on optimal investment policies and option values. We analyze the case where the decision maker exhibits risk aversion and has perpetual options to suspend and resume a project at no cost. Under these conditions, we address the question of how investment decisions are affected by risk aversion, operational flexibility, and uncertainty. Hence, the contribution of this paper is threefold. First, we develop a theoretical framework for investment under uncertainty with risk aversion and operational flexibility in order to derive optimal investment and operational thresholds. Second, we show how risk aversion interacts with operational flexibility to affect optimal investment policy. Third, we provide managerial insights for operational decisions based on analytical and numerical results.

We proceed by discussing some related work in Section 2. In Section 3, we formulate the problem using the nested optimal stopping time approach and a constant relative risk aversion (CRRA) utility function to determine the optimal time of investment that maximizes the decision maker's expected utility of future profits. The impact of operational flexibility, in terms of having the ability to suspend and resume operations, is examined in Section 4. We first analyze the case where the investment is irreversible (4.1) and then introduce operational flexibility in the form of a single abandonment option (4.2), a combined suspension-resumption option (4.3), and finally complete flexibility where the decision maker has an infinite number of perpetual options to suspend and resume operations (4.4). Section 5 provides numerical examples for each case and examines the effects of volatility and risk aversion on the optimal investment, suspension, and resumption thresholds. We illustrate the interaction among risk aversion, uncertainty, and operational flexibility and present managerial insights to enable more informed investment and operational decisions. Section 6 concludes and offers directions for future research.

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2. Related work

Dixit and Pindyck (1994) and McDonald and Siegel (1985, 1986) address the problem of optimal entry to and exit from a project assuming a risk-neutral decision maker with a perpetual option to invest. This canonical real options problem can be solved via either the contingent claims approach, assuming that either markets are complete or the project's unique risk can be diversified, or via dynamic programming, using a subjective discount rate. Contingent claims analysis, however, cannot be used in cases where the project's risk is not diversifiable. This occurs, for example, in research and development (R&D) projects with technical risk that is idiosyncratic, or in nascent markets that may not have sufficiently developed financial instruments. Furthermore, the decision maker may be inherently risk averse due to the firm's ownership structure, e.g., in the case of a municipal authority or due to costs of financial distress. Dynamic programming can then be used to maximize the expected discounted utility of the lifetime profits of a risk-averse decision maker.

In the real options literature, different types of operational flexibility have been studied mainly under the assumption of a risk-neutral decision maker. For example, Majd and Pindyck (1987) analyze the flexibility that lies within the time it takes to build an investment project. Their analysis is based on the fact that the rate at which the construction of an investment project proceeds is flexible and can, therefore, be adjusted as new information becomes available. Applying contingent claims analysis, they show how traditional discounted cash flow methods understate the value of the project by ignoring this flexibility. However, their analysis is restricted by the assumptions of market completeness and a risk-neutral decision maker. Dangl (1999) examines the flexibility that arises not only from the firm's ability to wait for more information but also from the ability to adjust the capacity of the investment project. Thus, under conditions of irreversible investment expenditures and uncertainty in future demand, he addresses the problem of a firm that has to determine the optimal investment timing and the optimal capacity choice. The flexibility to choose between two alternative investment projects of different scales under output price uncertainty has been studied by Décamps et al. (2006). Their analysis extends the results of Dixit (1993) where the irreversible choice among mutually exclusive projects under output price uncertainty is considered. Nevertheless, each of these papers assumes a risk-neutral decision maker.

Since the assumptions of risk neutrality and market completeness are not particularly relevant to most real-life situations, it is important to examine the implications that arise when these assumptions are relaxed. A utility-based framework has been adopted, for example, by Henderson and Hobson (2002), who extend the real options approach to pricing and hedging assets by taking the perspective of a risk-averse decision maker facing incomplete markets. Their analysis is based on Merton (1969) who studies a decision maker facing complete markets seeking to maximize the expected utility of terminal wealth over a fixed and continuous time horizon using a CRRA utility function. Henderson and Hobson (2002) extend Merton's analysis by introducing a second risky asset on which no trading is allowed. In that case, the decision maker has a claim on units of the non-traded asset, and the question is how to price and hedge this random payoff. Furthermore, Henderson (2007) investigates the impact of risk aversion and incompleteness on investment timing and option value by a risk-averse decision maker with an exponential utility function who can choose at any time to undertake an irreversible investment project and receive a risky payoff. To offset some of the risk associated with the unknown investment payoff, the decision maker also trades in a risk-free bond and a risky asset that is correlated with the investment payoff. Results indicate that the higher the decision maker's risk aversion or the lower the correlation between the project value and hedging asset, the lower will the investment threshold and option value be. In particular, there is a parameter region within which the assumptions of complete and incomplete markets yield different results. In this region, and under the assumption of complete markets, the option is never exercised (and investment never occurs), whereas the decision maker exercises the option in the incomplete setting.

More pertinent to our analysis is the working paper by Hugonnier and Morellec (2007), who extend the work of Dixit and Pindyck (1994) and McDonald and Siegel (1986) by illustrating how risk aversion affects investment under uncertainty when the decision maker faces incomplete markets. Instead of using contingent claims, they use an optimal stopping time approach to allow for the decision maker's risk aversion to be incorporated via a CRRA utility function. Their framework is based on a closed-form expression for the expected discounted utility of stochastic cash flows derived by Karatzas and Shreve (1999). The results indicate that risk aversion lowers the likelihood of investment and erodes the value of investment projects. In this paper, we extend Hugonnier and Morellec (2007) by incorporating operational flexibility in the form of suspension and resumption options that can be exercised at any time at no cost. We will show how this flexibility can mitigate the effect of risk aversion and offer insights on how to exercise optimally such suspension and resumption options.

3. Problem formulation and assumptions

We assume that a risk-averse decision maker holds the perpetual option to invest in a project that yields stochastic revenues over an infinite lifetime. Prior to investment, the decision maker's initial wealth is invested in a risk-free asset with rate of return $r > 0$. Let K be the amount of wealth the decision maker gives up in order to cover the fixed and irrecoverable cost of investment and c be the deterministic variable operating cost of the project. As the operating cost c is incurred in perpetuity, the present value of these costs at the time of investment equals $K + \frac{c}{r}$, which we assume is the decision maker's initial wealth. Also, time is continuous and denoted by $t \geq 0$, and the value of the project's exogenous output price, P_t , follows a geometric Brownian motion (GBM):

$$dP_t = \mu P_t dt + \sigma P_t dZ_t, \quad P_0 > 0 \quad (1)$$

Here, μ is the growth rate, σ is the proportional variance, Z_t is the standard Brownian motion, and P_0 is the initial value of the project's output price. All values and rates are expressed in real terms. The decision maker's preferences are described by an increasing and concave utility function, $U(\cdot)$. Hence, our analysis can accommodate hyperbolic absolute risk aversion (HARA), constant absolute risk aversion (CARA), and constant relative risk aversion (CRRA) utility functions. To enable comparisons with Hugonnier and Morellec (2007), we apply the same utility function, i.e., a CRRA utility function:

$$U(P_t) = \begin{cases} \frac{P_t^{1-\gamma}}{1-\gamma} & \text{if } \gamma \geq 0 \text{ \& } \gamma \neq 1 \\ \ln(P_t) & \text{if } \gamma = 1 \end{cases} \quad (2)$$

The relative risk aversion parameter, γ , is restricted to $[0, 1)$ for the purposes of our analysis and reflects greater risk aversion as it increases.

We follow the framework of Hugonnier and Morellec (2007) for decomposing cash flows into disjoint time intervals. We denote by $P_{\tau_j}^i$ the output price at time $\tau_j, j = 1, 2, 3, \dots$, at which we exercise an investment ($j = 1$), suspension ($j = 2, 4, 6, \dots$), or resumption option ($j = 3, 5, 7, \dots$) when $i = 0, 1, 2, 3, \dots$ subsequent embedded options exist. For example, $P_{\tau_1}^0$ is the price at which we exercise an investment option without operational flexibility, $P_{\tau_2}^0$ is the price at which we exercise an abandonment option, and $P_{\tau_2}^1$ is the price at which we exercise a suspension option with a resumption option still available, etc. Suppose now that we have a perpetually operating project that we start at random time τ_1 . Thus, up to time τ_1 , we earn an instantaneous cash flow of $c + rK$ per time unit with utility $U(c + rK)$ discounted at our subjective rate of time preference, $\rho > \mu$. Once we invest in the project, we swap this certain cash flow for a risky one, P_t per time unit, with utility $U(P_t)$, as shown in Fig. 1.

Using the law of iterated expectations and the strong Markov property of the GBM, which states that price values after time τ_1 are independent of the values before τ_1 and depend only on the value of the process at τ_1 , the time-zero discounted expected utility of the cash flows is:

$$\int_0^{\tau_1} e^{-\rho t} U(c + rK) dt + \mathbb{E}_{P_0} \left[\int_{\tau_1}^{\infty} e^{-\rho t} U(P_t) dt \right] = \int_0^{\infty} e^{-\rho t} U(c + rK) dt + \mathbb{E}_{P_0} [e^{-\rho \tau_1}] V_1(P_{\tau_1}^0) \quad (3)$$

where

$$V_1(P_{\tau_1}^0) = \mathbb{E}_{P_{\tau_1}^0} \left[\int_0^{\infty} e^{-\rho t} [U(P_t) - U(c + rK)] dt \right] \quad (4)$$

is the expected utility of the project's cash flows, discounted to τ_1 . Here, \mathbb{E}_{P_0} denotes the expectation operator, which is conditional on the initial value, P_0 , of the price process.

Now, we extend this framework by allowing for an abandonment option at random time $\tau_2 > \tau_1$. The value of the output price at which the option to abandon the project is exercised is denoted by $P_{\tau_2}^0$, as shown in Fig. 2.

In this case, the expected discounted utility of all future cash flows equals:

$$\int_0^{\infty} e^{-\rho t} U(c + rK) dt + \mathbb{E}_{P_0} [e^{-\rho \tau_1}] \left[V_1(P_{\tau_1}^1) + \mathbb{E}_{P_{\tau_1}^1} [e^{-\rho(\tau_2 - \tau_1)}] V_2(P_{\tau_2}^0) \right] \quad (5)$$

where

$$V_2(P_{\tau_2}^0) = \mathbb{E}_{P_{\tau_2}^0} \left[\int_0^{\infty} e^{-\rho t} [U(c) - U(P_t)] dt \right] \quad (6)$$

is the expected utility of the project's cash flows, discounted to τ_2 .

Finally, we allow for a subsequent resumption option at random time $\tau_3 > \tau_2$. The output price at which the resumption option is exercised is denoted by $P_{\tau_3}^0$ as shown in Fig. 3.

Here, the expected discounted utility of all future cash flows is:

$$\int_0^{\infty} e^{-\rho t} U(c + rK) dt + \mathbb{E}_{P_0} [e^{-\rho \tau_1}] \left[V_1(P_{\tau_1}^2) + \mathbb{E}_{P_{\tau_1}^2} [e^{-\rho(\tau_2 - \tau_1)}] \left[V_2(P_{\tau_2}^1) + \mathbb{E}_{P_{\tau_2}^1} [e^{-\rho(\tau_3 - \tau_2)}] V_3(P_{\tau_3}^0) \right] \right] \quad (7)$$

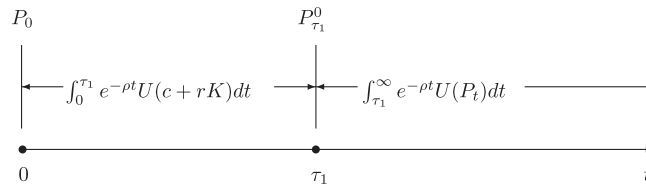


Fig. 1. Irreversible investment under risk aversion.

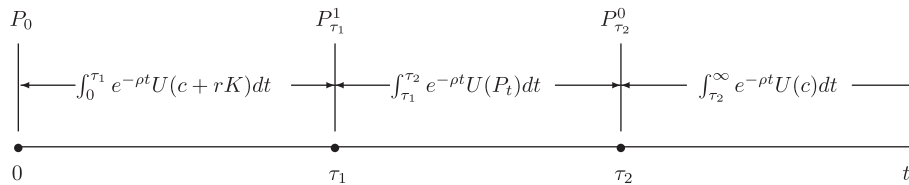


Fig. 2. Investment under risk aversion with a single abandonment option.

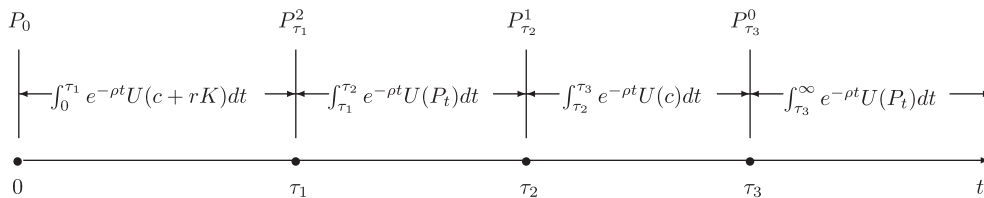


Fig. 3. Investment under risk aversion with one suspension and one resumption option.

where

$$V_3(P_{\tau_3}^0) = \mathbb{E}_{P_{\tau_3}^0} \left[\int_0^{\infty} e^{-\rho t} [U(P_t) - U(c)] dt \right] \quad (8)$$

is the expected utility of the project's cash flows, discounted to τ_3 . Following the same reasoning, we can extend the model to include complete operational flexibility, i.e. infinitely many suspension and resumption options.

4. Analytical results

4.1. Investment without operational flexibility

Since this problem has already been examined by [Hugonnier and Morellec \(2007\)](#), we summarize the results for ease of reference, to allow for comparisons, and to provide further insights. Let $F_{\tau_j}^i(\cdot)$ denote the value of an option that is exercised at time τ_j , $j = 1, 2, 3, \dots$, with $i = 0, 1, 2, 3, \dots$ subsequent embedded options remaining. $F_{\tau_1}^0(\cdot)$ refers to an investment option without operational flexibility, $F_{\tau_2}^0(\cdot)$ refers to an abandonment option, while $F_{\tau_2}^1(\cdot)$ refers to a suspension with one resumption option, and so on. We define the value of the incremental investment opportunity, $F_{\tau_1}^0(P_0)$, as follows:

$$F_{\tau_1}^0(P_0) = \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0} [e^{-\rho \tau_1}] V_1(P_{\tau_1}^0) \quad (9)$$

By \mathcal{S} , we denote the set of stopping times of the filtration generated by the price process.

Using Theorem 9.18 of [Karatzas and Shreve \(1999\)](#) for the CRRA utility function in (2), we find that:

$$\mathbb{E}_{P_0} \int_0^{\infty} e^{-\rho t} U(P_t) dt = \mathcal{A}U(P_0) \quad (10)$$

where $\mathcal{A} = \frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} > 0$ and $\beta_1 > 1$, $\beta_2 < 0$ are the solutions to the following quadratic equation:

$$\frac{1}{2} \sigma^2 x(x-1) + \mu x - \rho = 0 \quad (11)$$

Since the expected discount factor $\mathbb{E}_{P_0} [e^{-\rho \tau_1}] = \left(\frac{P_0}{P_{\tau_1}^0} \right)^{\beta_1}$ ([Karatzas and Shreve, 1999](#)), (9) can be written as follows:

$$F_{\tau_1}^0(P_0) = \max_{P_{\tau_1}^0 \geq P_0} \left(\frac{P_0}{P_{\tau_1}^0} \right)^{\beta_1} \left[\mathcal{A}U(P_{\tau_1}^0) - \frac{U(c+rK)}{\rho} \right] \quad (12)$$

The first-order necessary condition (FONC) for this unconstrained maximization problem may be expressed as follows:

$$\frac{\beta_2}{1-\beta_2-\gamma} P_{\tau_1}^{0*1-\gamma} + (c+rK)^{1-\gamma} = 0 \quad (13)$$

Therefore, under a CRRA utility function, the optimal investment threshold is:

$$P_{\tau_1}^{0*} = (c+rK) \left[\frac{\beta_2 + \gamma - 1}{\beta_2} \right]^{\frac{1}{1-\gamma}} \quad (14)$$

The second-order sufficiency condition (SOSC) requires the objective function to be concave at $P_{\tau_1}^{0*}$, which we show in [Proposition 4.1](#). All proofs can be found in the appendix.

Proposition 4.1. *The objective function (9) is strictly concave at $P_{\tau_1}^{0*}$ iff $\gamma < 1$.*

Clearly, as $\left[\frac{\beta_2 + \gamma - 1}{\beta_2} \right]^{\frac{1}{1-\gamma}} > 1$, this implies that $P_{\tau_1}^{0*} > c + rK$. Thus, (14) implies that the option to invest should be exercised only when the critical value, $P_{\tau_1}^{0*}$, exceeds the amortized investment cost, $c + rK$, by a positive quantity. This, in turn, implies that uncertainty and risk aversion drive a wedge between the optimal investment threshold and the amortized investment cost. The size of this wedge, as we will show later, depends on the levels of uncertainty, risk aversion, and operational flexibility.

Another way of expressing (13) is to relate the marginal benefit (MB) of waiting to invest with its marginal cost (MC):

$$\left(\frac{P_0}{P_{\tau_1}^{0*}} \right)^{\beta_1} \left[\mathcal{A}P_{\tau_1}^{0*-\gamma} + \frac{\beta_1}{P_{\tau_1}^{0*}} \frac{U(c+rK)}{\rho} \right] = \left(\frac{P_0}{P_{\tau_1}^{0*}} \right)^{\beta_1} \frac{\beta_1 \mathcal{A}}{P_{\tau_1}^{0*}} U(P_{\tau_1}^{0*}) \quad (15)$$

The first term on the left-hand side of (15) is positive and represents the incremental project value created by waiting until the price is higher. Multiplied by the discount factor, it is a positive, decreasing function of the output price, as waiting longer enables the project to start at a higher initial price; however, the rate at which this benefit accrues diminishes due to the effect of discounting. The second term is positive and represents the reduction in the MC of waiting to invest due to saved investment and operating cost. Together, these two terms constitute the MB of delaying investment. The MC of waiting to invest on the right-hand side of (15) is positive and reflects the opportunity cost of forgone cash flows discounted appropriately. For low price values, it is worthwhile to postpone investment since the MB is greater than the MC according to [Corollary 4.1](#).

Corollary 4.1. *The MB curve is steeper than the MC curve at $P_{\tau_1}^{0*}$.*

As risk aversion increases, the MC of waiting to invest decreases relatively more than the MB. This happens because the MC consists entirely of risk cash flows and, therefore, is affected more by risk aversion. As a result, the marginal utility of the investment's payoff increases, thereby increasing the incentive to postpone investment. This leads to [Proposition 4.2](#).

Proposition 4.2. *The optimal investment threshold is increasing with risk aversion.*

Finally, for a fixed level of risk aversion, the optimal investment threshold increases as the economic environment becomes more uncertain. This happens because greater uncertainty causes the value of waiting to increase, which in turn increases the opportunity cost of investing. [Proposition 4.3](#) verifies this intuition.

Proposition 4.3. *The optimal investment threshold is increasing with volatility.*

4.2. Investment with a single abandonment option

Here, the value of the investment opportunity is:

$$\begin{aligned} F_{\tau_1}^1(P_0) &\equiv \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0} \left[\int_{\tau_1}^{\infty} e^{-\rho t} [U(P_t) - U(c + rK)] dt + \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}} \left[\int_{\tau_2}^{\infty} e^{-\rho t} [U(c) - U(P_t)] dt \right] \right] \\ &= \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0} \left[e^{-\rho \tau_1} \left[V_1(P_{\tau_1}^1) + \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}^1} \left[e^{-\rho(\tau_2 - \tau_1)} V_2(P_{\tau_2}^0) \right] \right] \right] = \max_{P_{\tau_1}^1 \geq P_0} \left(\frac{P_0}{P_{\tau_1}^1} \right)^{\beta_1} \left[V_1(P_{\tau_1}^1) + F_{\tau_2}^0(P_{\tau_1}^1) \right] \end{aligned} \quad (16)$$

The value of the output price at which we exercise the abandonment option is $P_{\tau_2}^0$, and the maximized value of the option to abandon a just-activated project is denoted by $F_{\tau_2}^0(P_{\tau_1}^1)$, i.e.:

$$F_{\tau_2}^0(P_{\tau_1}^1) = \max_{P_{\tau_2}^0 \leq P_{\tau_1}^1} \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^0} \right)^{\beta_2} V_2(P_{\tau_2}^0) \Rightarrow F_{\tau_2}^0(P_{\tau_1}^1) = \max_{P_{\tau_2}^0 \leq P_{\tau_1}^1} \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^0} \right)^{\beta_2} \left[\frac{U(c)}{\rho} - \mathcal{A}U(P_{\tau_2}^0) \right] \quad (17)$$

We solve this compound real options problem backward by first determining the optimal abandonment threshold price, $P_{\tau_2}^{0*}$. The FONC for this unconstrained maximization problem is expressed as:

$$\frac{\beta_1}{1 - \beta_1 - \gamma} P_{\tau_2}^{0*1-\gamma} + c^{1-\gamma} = 0 \quad (18)$$

Solving (18) with respect to $P_{\tau_2}^{0*}$, we obtain the following expression for the optimal abandonment threshold:

$$P_{\tau_2}^{0*} = c \left[\frac{\beta_1 + \gamma - 1}{\beta_1} \right]^{\frac{1}{1-\gamma}} \quad (19)$$

To ensure the existence of a local maximum at $P_{\tau_2}^{0*}$, the SOSC has to be verified.

Proposition 4.4. *The objective function (17) is strictly concave at $P_{\tau_2}^{0*}$ iff $\gamma < 1$.*

Since $\left[\frac{\beta_1 + \gamma - 1}{\beta_1} \right]^{\frac{1}{1-\gamma}} < 1$, (19) implies that $P_{\tau_2}^{0*} < c$, i.e. the option to abandon operations permanently should be exercised when the operating cost, c , exceeds the critical value, $P_{\tau_2}^{0*}$, by a positive quantity. Hence, uncertainty and risk aversion again drive a wedge between the critical value, $P_{\tau_2}^{0*}$, and the operating cost, c . The size of this wedge is affected by volatility, risk aversion, and operational flexibility.

In contrast to the previous section, we now express (18) by relating the MB from accelerating abandonment of the project with the MC. Note that unlike in the investment stage, an incremental increase in the threshold value implies that abandonment is accelerated:

$$-\left(\frac{P_{\tau_1}^1}{P_{\tau_2}^{0*}} \right)^{\beta_2} \frac{\beta_2}{P_{\tau_2}^{0*}} \frac{U(c)}{\rho} = \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^{0*}} \right)^{\beta_2} \left[\mathcal{A}P_{\tau_2}^{0*-\gamma} - \frac{\beta_2 \mathcal{A}}{P_{\tau_2}^{0*}} U(P_{\tau_2}^{0*}) \right] \quad (20)$$

The left-hand side of (20) is the MB of accelerating abandonment and represents the recovery of the operating cost from shutting down the project. This term is positive, indicating that abandoning operations at a higher price level (i.e. more quickly) increases the expected utility of the salvageable operating cost. The right-hand side of (20) is the MC of accelerating abandonment. The first term corresponds to killing the revenues of the project at a higher price level, while the second term is also positive and corresponds to the increase in the MC from speeding up abandonment. This term represents the increase in the opportunity cost from waiting less, thereby forgoing information. As risk aversion increases, the decision maker appears more willing to terminate operations and, thus, avoid potential losses as [Proposition 4.5](#) states.

Proposition 4.5. *The optimal abandonment threshold is increasing in risk aversion.*

The behavior of the optimal abandonment threshold when the level of uncertainty changes can be determined using the FONC with respect to σ^2 . This leads to the following proposition.

Proposition 4.6. *The optimal abandonment threshold is decreasing in volatility.*

[Proposition 4.6](#) implies that the greater the uncertainty, the more reluctant the decision maker is to abandon an active project. Intuitively, this happens because she would not want to abandon the project due to a temporary downturn, which is more likely when volatility is higher.

By moving back to the investment stage, we now solve the decision maker's investment timing problem given the solution to the optimal exercise of the abandonment option:

$$F_{\tau_1}^1(P_0) = \max_{P_{\tau_1}^1 \geq P_0} \left(\frac{P_0}{P_{\tau_1}^1} \right)^{\beta_1} \left[\mathcal{A}U(P_{\tau_1}^1) - \frac{U(c+rK)}{\rho} + \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^{0*}} \right)^{\beta_2} \left[\frac{U(c)}{\rho} - \mathcal{A}U(P_{\tau_2}^{0*}) \right] \right] \quad (21)$$

Substituting in $P_{\tau_2}^{0*}$ and applying the FONC to (21) leads to the following non-linear equation that gives the optimal investment threshold:

$$\frac{\beta_2}{1-\beta_2-\gamma} P_{\tau_1}^{1*1-\gamma} + (c+rK)^{1-\gamma} - \frac{\rho(\beta_1-\beta_2)}{\beta_1} (1-\gamma) F_{\tau_2}^0(P_{\tau_1}^{1*}) = 0 \quad (22)$$

By comparing (22) and (13), we can show that the optimal investment threshold decreases due to the embedded abandonment option as follows:

Proposition 4.7. *The optimal investment threshold when an abandonment option is available is lower compared to an irreversible investment opportunity, ceteris paribus.*

In order to illustrate Proposition 4.7, we express (22) by relating the MB of waiting to invest to the MC as shown in (23).

$$\left(\frac{P_0}{P_{\tau_1}^{1*}} \right)^{\beta_1} \left[\mathcal{A}P_{\tau_1}^{1*1-\gamma} + \frac{\beta_1}{P_{\tau_1}^{1*}} \frac{U(c+rK)}{\rho} - (\beta_2-\beta_1) \left(\frac{P_{\tau_1}^{1*}}{P_{\tau_2}^{0*}} \right)^{\beta_2} \frac{\mathcal{A}}{P_{\tau_1}^{1*}} U(P_{\tau_2}^{0*}) \right] = \left(\frac{P_0}{P_{\tau_1}^{1*}} \right)^{\beta_1} \frac{1}{P_{\tau_1}^{1*}} \left[\beta_1 \mathcal{A}U(P_{\tau_1}^{1*}) - (\beta_2-\beta_1) \left(\frac{P_{\tau_1}^{1*}}{P_{\tau_2}^{0*}} \right)^{\beta_2} \frac{U(c)}{\rho} \right] \quad (23)$$

Compared to the case of investment without operational flexibility (15), the MB and MC of delaying investment have now increased due to the additional terms on each side of (23). These terms are positive and correspond to the MB and MC from the embedded abandonment option. In fact, the MC increases by more than the MB since, at abandonment, the expected utility of the salvageable operating cost is greater than the expected utility of the forgone cash flows. Thus, the marginal utility of the payoff from delaying investment decreases, thereby increasing the incentive to invest. Intuitively, the abandonment option reduces the decision maker's insecurity since she can now terminate her investment in case the output price drops significantly.

4.3. Investment with a single suspension and resumption option

With a single suspension and resumption option the value of the investment opportunity is:

$$\begin{aligned} F_{\tau_1}^2(P_0) &\equiv \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0} \left[\int_{\tau_1}^{\infty} e^{-\rho t} [U(P_t) - U(c+rK)] dt + \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}^2} \left[\int_{\tau_2}^{\infty} e^{-\rho t} [U(c) - U(P_t)] dt + \sup_{\tau_3 \geq \tau_2} \mathbb{E}_{P_{\tau_2}^1} \left[\int_{\tau_3}^{\infty} e^{-\rho t} [U(P_t) - U(c)] dt \right] \right] \right] \\ &= \sup_{\tau_1 \in \mathcal{S}} \mathbb{E}_{P_0} \left[e^{-\rho \tau_1} \left[V_1(P_{\tau_1}^2) + \sup_{\tau_2 \geq \tau_1} \mathbb{E}_{P_{\tau_1}^2} \left[e^{-\rho(\tau_2-\tau_1)} \left[V_2(P_{\tau_2}^1) + \sup_{\tau_3 \geq \tau_2} \mathbb{E}_{P_{\tau_2}^1} \left[e^{-\rho(\tau_3-\tau_2)} V_3(P_{\tau_3}^0) \right] \right] \right] \right] \right] \\ &= \max_{P_{\tau_1}^2 \geq P_0} \left(\frac{P_0}{P_{\tau_1}^2} \right)^{\beta_1} \left[V_1(P_{\tau_1}^2) + F_{\tau_2}^1(P_{\tau_1}^2) \right] \end{aligned} \quad (24)$$

Here, $P_{\tau_2}^1$ is the threshold price at which we suspend the investment project. The last term, $F_{\tau_2}^1(P_{\tau_1}^2)$, is the maximized value of the option to suspend a just-activated project with a subsequent resumption option and is defined as:

$$\begin{aligned} F_{\tau_2}^1(P_{\tau_1}^2) &\equiv \max_{P_{\tau_2}^1 \leq P_{\tau_1}^2} \left(\frac{P_{\tau_1}^2}{P_{\tau_2}^1} \right)^{\beta_2} \mathbb{E}_{P_{\tau_1}^2} \left[\int_0^{\infty} e^{-\rho t} [U(c) - U(P_t)] dt + \max_{P_{\tau_3}^0 \geq P_{\tau_2}^1} \left(\frac{P_{\tau_2}^1}{P_{\tau_3}^0} \right)^{\beta_1} \mathbb{E}_{P_{\tau_3}^0} \left[\int_0^{\infty} e^{-\rho t} [U(P_t) - U(c)] dt \right] \right] \\ &= \max_{P_{\tau_2}^1 \leq P_{\tau_1}^2} \left(\frac{P_{\tau_1}^2}{P_{\tau_2}^1} \right)^{\beta_2} \left[\frac{U(c)}{\rho} - \mathcal{A}U(P_{\tau_2}^1) + F_{\tau_3}^0(P_{\tau_2}^1) \right] \end{aligned} \quad (25)$$

Analogously, we define $F_{\tau_3}^0(P_{\tau_2}^1)$ to be the maximized value of the option to resume forever a just-suspended project, and $P_{\tau_3}^0$ the threshold price at which we exercise the option to resume the investment project:

$$F_{\tau_3}^0(P_{\tau_2}^1) = \max_{P_{\tau_3}^0 \geq P_{\tau_2}^1} \left(\frac{P_{\tau_2}^1}{P_{\tau_3}^0} \right)^{\beta_1} V_3(P_{\tau_3}^0) = \max_{P_{\tau_3}^0 \geq P_{\tau_2}^1} \left(\frac{P_{\tau_2}^1}{P_{\tau_3}^0} \right)^{\beta_1} \mathbb{E}_{P_{\tau_3}^0} \left[\int_0^{\infty} e^{-\rho t} (U(P_t) - U(c)) dt \right] = \max_{P_{\tau_3}^0 \geq P_{\tau_2}^1} \left(\frac{P_{\tau_2}^1}{P_{\tau_3}^0} \right)^{\beta_1} \left[\mathcal{A}U(P_{\tau_3}^0) - \frac{U(c)}{\rho} \right] \quad (26)$$

We solve this compound real options problem backward by first determining the optimal resumption threshold price. The FONC yields:

$$P_{\tau_3}^{0*} = c \left[\frac{\beta_2 + \gamma - 1}{\beta_2} \right]^{\frac{1}{1-\gamma}} \quad (27)$$

Differentiating (27) with respect to γ , we obtain the following proposition:

Proposition 4.8. *The optimal resumption threshold is increasing with risk aversion.*

Next, we step back to when the investment project is active in order to decide when to suspend operations, i.e. sub-problem (25). Applying the FONC, we obtain the following non-linear equation that gives the optimal suspension threshold:

$$\frac{\beta_1}{1-\beta_1-\gamma} P_{\tau_2}^{1*1-\gamma} + c^{1-\gamma} - \frac{\rho(\beta_1-\beta_2)}{\beta_2} (1-\gamma) F_{\tau_3}^0(P_{\tau_2}^{1*}) = 0 \quad (28)$$

Proposition 4.9. *The optimal suspension threshold is higher than the optimal abandonment one.*

To illustrate [Proposition 4.9](#), we will examine the relationship between the MB from accelerating suspension and its MC, which is described in the following equation:

$$-\left(\frac{P_{\tau_1}^{2*}}{P_{\tau_2}^{1*}}\right)^{\beta_2} \frac{1}{P_{\tau_2}^{1*}} \left[\frac{\beta_2 U(c)}{\rho} - (\beta_1 - \beta_2) \left(\frac{P_{\tau_2}^{1*}}{P_{\tau_3}^{0*}}\right)^{\beta_1} \mathcal{A}U(P_{\tau_3}^{0*}) \right] = \left(\frac{P_{\tau_1}^{2*}}{P_{\tau_2}^{1*}}\right)^{\beta_2} \left[\mathcal{A}P_{\tau_2}^{1*-\gamma} - \frac{\beta_2}{P_{\tau_2}^{1*}} \mathcal{A}U(P_{\tau_2}^{1*}) + (\beta_1 - \beta_2) \left(\frac{P_{\tau_2}^{1*}}{P_{\tau_3}^{0*}}\right)^{\beta_1} \frac{1}{P_{\tau_2}^{1*}} \frac{U(c)}{\rho} \right] \quad (29)$$

The left-hand side of (29) is the MB of accelerating suspension. The first term is the MB of accelerating abandonment, while the second term represents the MB from the embedded resumption option. Since the latter term is positive, the MB of suspension has increased compared to the case of abandonment in (20). The right-hand side of (29) is the MC of accelerating suspension. The first two terms correspond to the MC of accelerating abandonment, while the third term represents the MC from the embedded option to resume operations. Since this term is always positive, it causes the MC of abandonment to increase. Although both the MB and MC increase due to the embedded resumption option, the former increases relatively more since at resumption, the expected utility of the risky cash flows is greater than the expected utility of the operating cost. Thus, the marginal utility of the payoff from suspending operations increases, which in turn increases the incentive to suspend operations. As a result, MB and MC curves intersect at a higher level of the output price, thereby indicating that the embedded resumption option facilitates suspension. Intuitively, the decision maker is more willing to suspend operations since now, unlike in the case of permanent abandonment, she can recover the lost cash flows by exercising her resumption option.

Finally, we move to the investment stage to solve the complete problem taking $P_{\tau_3}^{0*}$ and $P_{\tau_2}^{1*}$ as fixed:

$$F_{\tau_1}^2(P_0) = \max_{P_{\tau_1} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^2}\right)^{\beta_1} \left[\mathcal{A}U(P_{\tau_1}^2) - \frac{U(c+rK)}{\rho} + \left(\frac{P_{\tau_1}^2}{P_{\tau_2}^{1*}}\right)^{\beta_2} \left[\frac{U(c)}{\rho} - \mathcal{A}U(P_{\tau_2}^{1*}) + F_{\tau_3}^0(P_{\tau_2}^{1*}) \right] \right] \quad (30)$$

The optimal investment threshold is obtained numerically by solving the following non-linear equation resulting from the FONC:

$$\frac{\beta_2}{1 - \beta_2 - \gamma} P_{\tau_1}^{2*1-\gamma} + (c+rK)^{1-\gamma} - \frac{\rho(\beta_1 - \beta_2)}{\beta_1} (1-\gamma) F_{\tau_2}^1(P_{\tau_1}^{2*}) = 0 \quad (31)$$

Proposition 4.10. *The optimal investment threshold when a single suspension and a single resumption option are available is lower than for an investment opportunity with a single abandonment option.*

Intuitively, the suspension and resumption options facilitate investment because they provide the decision maker the subsequent option to halt the project in case of a downturn and then to resume it. [Propositions 4.9 and 4.10](#) lead to the insight that additional flexibility facilitates investment and operational decisions, thereby resulting in an increase of the optimal suspension threshold and a decrease of the optimal investment threshold.

4.4. Investment with complete operational flexibility

Following the methodology of [McDonald \(2006\)](#), suppose that we are now operating an investment project with infinitely many perpetual options to suspend and resume operations. The symmetry of the problem suggests that the optimal values of the output prices at which these options are exercised will not be affected by additional flexibility, i.e. each time we suspend or resume operations, we still have infinitely many options left. Therefore, each resumption and suspension threshold will be affected equally by flexibility. We let $P_{\tau_e}^{\infty}$, where e stands for *even* (i.e. 2, 4, 6, ...), denote the common threshold at which all suspension options are exercised, and $P_{\tau_o}^{\infty}$, where o stands for *odd* (i.e. 3, 5, 7, ...), denote the common threshold at which all resumption options are exercised. Hence, the value of an operating project activated at $P_{\tau_o}^{\infty}$ can be written as follows:

$$V_3(P_{\tau_o}^{\infty}) + \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} V_2(P_{\tau_e}^{\infty}) + \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_o}^{\infty}}\right)^{\beta_1} V_3(P_{\tau_o}^{\infty}) + \dots = \sum_{i=0}^{\infty} \left\{ \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_1} \right\}^i \left\{ V_3(P_{\tau_o}^{\infty}) + \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} V_2(P_{\tau_e}^{\infty}) \right\} \quad (32)$$

Since $\sum_{i=0}^{\infty} \left\{ \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_1} \right\}^i$ is a geometric series with $\left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_1} < 1$, we know that:

$$\sum_{i=0}^{\infty} \left\{ \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_1} \right\}^i = \frac{1}{1 - \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_1}} \quad (33)$$

Therefore, the decision maker's problem in an active state is:

$$F_{\tau_e}^{\infty}(P_{\tau_o}^{\infty}) = \max_{P_{\tau_e}^{\infty} \leq P_{\tau_o}^{\infty}} \frac{1}{1 - \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_1}} \left\{ V_3(P_{\tau_o}^{\infty}) + \left(\frac{P_{\tau_o}^{\infty}}{P_{\tau_e}^{\infty}}\right)^{\beta_2} V_2(P_{\tau_e}^{\infty}) \right\} \quad (34)$$

It follows that the option to resume a currently suspended project with infinitely many resumption and suspension options, given that the current value of the output price is $P_{\tau_e}^{\infty}$, is:

$$F_{\tau_o}^{\infty}(P_{\tau_e}^{\infty}) = \max_{P_{\tau_o}^{\infty} \geq P_{\tau_e}^{\infty}} \left(\frac{P_{\tau_e}^{\infty}}{P_{\tau_o}^{\infty}}\right)^{\beta_1} F_{\tau_e}^{\infty}(P_{\tau_o}^{\infty}) \quad (35)$$

In order to solve for $P_{\tau_1}^{\infty*}$, $P_{\tau_0}^{\infty*}$, and $P_{\tau_e}^{\infty*}$, we first substitute (34) into (35) and use as an initial guess for $P_{\tau_e}^{\infty}$ the price at which the option to abandon the investment project is exercised, i.e. $P_{\tau_e}^{\infty} = P_{\tau_2}^{0*}$. Thus, we obtain an equation, which we then maximize with respect to $P_{\tau_0}^{\infty}$. The estimate of $P_{\tau_0}^{\infty}$ we obtain this way is then substituted into (34), which we maximize with respect to $P_{\tau_e}^{\infty}$. This procedure is iterated until each solution converges. As we will demonstrate numerically in Section 5.4, the optimal suspension and resumption thresholds converge toward the operating cost c . Intuitively, each time that additional flexibility becomes available, the optimal suspension threshold increases and the optimal resumption threshold decreases. Assuming that $P_{\tau_{2j}}^i < c$ and $P_{\tau_{2j+1}}^i > c$, $\forall i < \infty$ and $\forall j = 1, 2, 3, \dots$, this implies that $\lim_{i \rightarrow \infty} P_{\tau_{2j}}^i = c$ and $\lim_{i \rightarrow \infty} P_{\tau_{2j+1}}^i = c$, $\forall j$.

Finally, we take $P_{\tau_0}^{\infty*}$, $P_{\tau_e}^{\infty*}$, $F_{\tau_0}^{\infty}(P_{\tau_e}^{\infty*})$, and $F_{\tau_e}^{\infty}(P_{\tau_0}^{\infty*})$ as given and solve the investment problem for investment threshold $P_{\tau_1}^{\infty}$ assuming investment cost K :

$$F_{\tau_1}^{\infty}(P_0) \equiv \max_{P_{\tau_1}^{\infty} \geq P_0} \left(\frac{P_0}{P_{\tau_1}^{\infty}} \right)^{\beta_1} \left[V_1(P_{\tau_1}^{\infty}) + \left(\frac{P_{\tau_1}^{\infty}}{P_{\tau_e}^{\infty*}} \right)^{\beta_2} [V_2(P_{\tau_e}^{\infty*}) + F_{\tau_0}^{\infty}(P_{\tau_e}^{\infty*})] \right] \quad (36)$$

5. Numerical examples

5.1. Investment without operational flexibility

Suppose we have a project with $K = \$100$, $c = \$10$, $\sigma \in [0, 0.2]$, and $P_0 = \$13.6$. We set $r = \rho = 0.05$ and $\mu = 0.01$. Fig. 4 shows that the investment threshold, $P_{\tau_1}^{0*}$, increases in risk aversion γ for a fixed volatility, σ . This happens because the underlying expected utility of the project decreases with γ , thereby raising the required threshold for investment. Hence, increased risk aversion reduces the incentive to invest. Second, $P_{\tau_1}^{0*}$ increases in σ for fixed γ because greater uncertainty increases the value of waiting and, thus, the opportunity cost of investing.

Fig. 5 illustrates the MB and MC of waiting to invest, for $\sigma = 0.2$ and $\gamma = 0.025$. For low prices it is worthwhile to postpone investment as the MB is greater than the MC. As risk aversion increases, the MC, which consists entirely of risky cash flows and, hence, gets affected more by risk aversion, decreases by more than the MB. As a result, the marginal utility of the payoff when investment is delayed increases, which, in turn, decreases the incentive to invest and causes the optimal investment threshold to increase with risk aversion.

Fig. 6 illustrates the impact of volatility, σ , and risk aversion, γ , on the value of the option to invest and the value of the project. In the graph on the left, we plot the value of the project as well as the option value for $\sigma = 0.1, 0.15, 0.2$ holding $\gamma = 0.25$. As uncertainty increases, the project value decreases, but the value of the option to invest, evaluated at the initial level of the output price, increases due to greater waiting value. Consequently, the value of the option to wait also increases, thereby delaying investment. In the graph on the right, we plot the value of the project and the option value for $\gamma = 0, 0.25, 0.5$ holding $\sigma = 0.2$. The graph indicates that as risk aversion increases, the decision maker requires a higher price before exercising the option to invest. This is due to the decreased expected utility of the project, which decreases both the value of the option to invest and the likelihood of investment.

5.2. Investment with a single abandonment option

Increasing flexibility by adding an abandonment option decreases the optimal investment threshold. The proportional increase in option value due to the subsequent abandonment option is larger for higher levels of uncertainty and risk aversion. Both of these results are illustrated in Fig. 7. In the graph on the left, we compare the case of investment without operational flexibility to that of investment with a single abandonment option. We plot the value of the project and the value of the investment opportunity for $\gamma = 0.25$ and $\sigma = 0.2$. The graph on the right illustrates how the proportional increase in option value due to the subsequent abandonment option fluctuates with risk aversion for three levels of volatility.

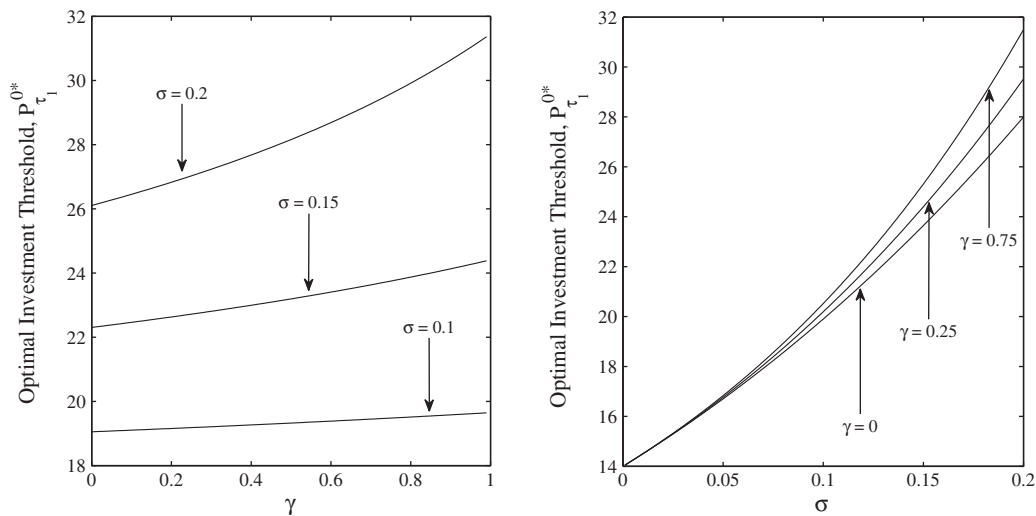


Fig. 4. Optimal investment threshold versus γ for $\sigma = 0.1, 0.15, 0.2$ (left), and optimal investment threshold versus σ for $\gamma = 0, 0.25, 0.5$ (right).

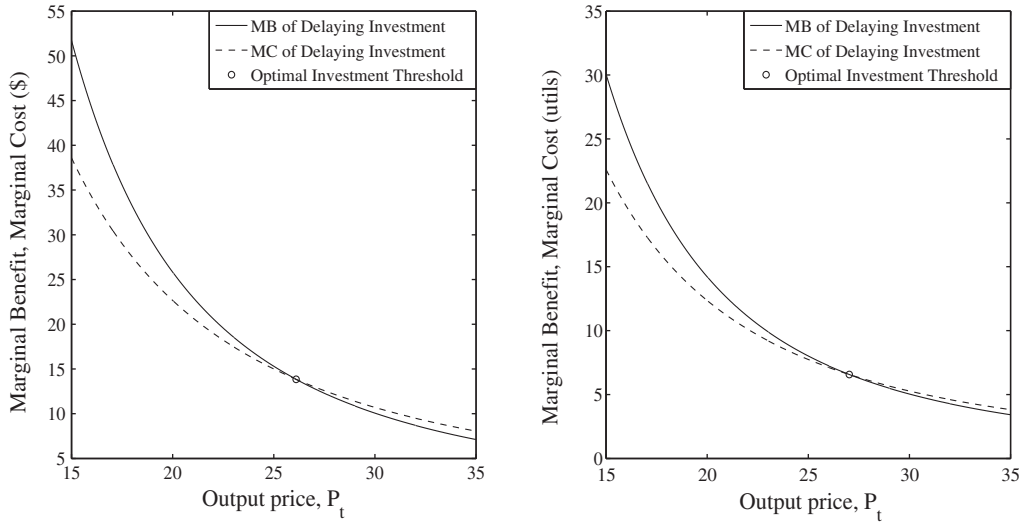


Fig. 5. Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$, (right) for an irreversible investment opportunity.

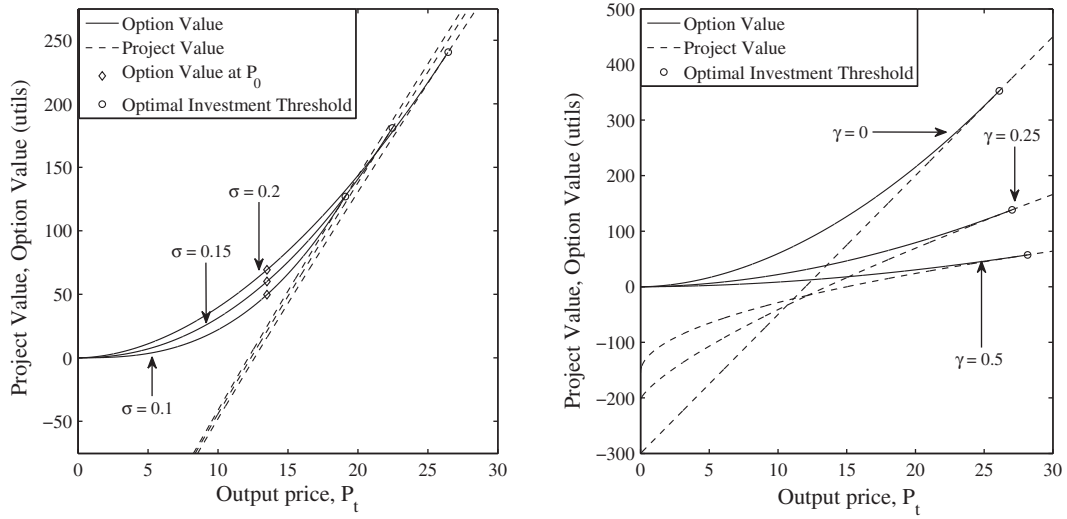


Fig. 6. Option value and project value versus P_t for $\gamma = 0.25$ and $\sigma = 0.1, 0.15, 0.2$ (left), and option value and project value versus P_t for $\sigma = 0.2$ and $\gamma = 0, 0.25, 0.5$ (right).

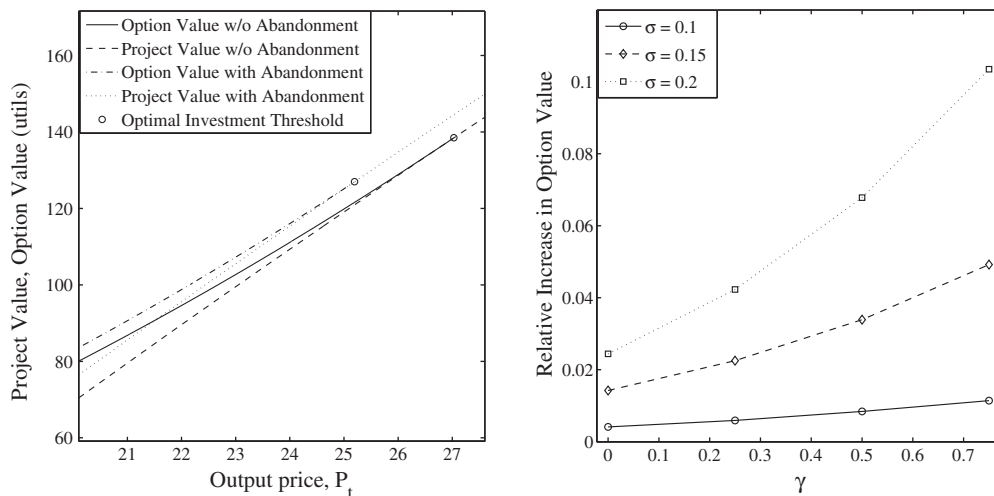


Fig. 7. Effect of the abandonment option on optimal investment threshold and option value.

Although both risk aversion and uncertainty increase the option value of abandonment, the impact of each factor on $P_{t_2}^{0*}$ is different. While risk aversion precipitates abandonment due to a decrease in project value, uncertainty delays abandonment because it increases its opportunity cost. In particular, Fig. 8 indicates that as risk aversion increases, for a fixed level of volatility, the decision maker becomes more willing to abandon the project in order to avoid potential losses. An increase in uncertainty, however, leads to a decrease in the optimal abandonment threshold.

For large price values, the MB of accelerating abandonment is less than the MC, and, therefore, it is optimal to continue, as Fig. 9 illustrates. As risk aversion increases, both the MB and MC of accelerating abandonment decrease. However, the MC, which consists entirely of risky cash flows and therefore gets affected more by risk aversion, decreases relatively more. As a result, the marginal utility of the payoff from accelerating abandonment increases, which, in turn, increases the incentive to abandon the project and results in an increased optimal abandonment threshold.

Using the same parameter values as in Section 5.1, we plot the MB and MC of waiting to invest versus P_t . The embedded abandonment option causes the marginal utility of the payoff from delaying investment to decrease, which, in turn, increases the incentive to invest. This happens because the MC increases relatively more than the MB, and as a result, the MB and MC curves intersect at a lower level of $P_{t_1}^*$, as Fig. 10 illustrates.

5.3. Investment with a single suspension and a single resumption option

Having the option to suspend operations combined with an option to resume them permanently increases the value of the investment opportunity further and decreases the optimal investment threshold as Fig. 11 illustrates. Moreover, the percentage increase in option value due to the subsequent resumption option is greater compared to the case of investment with a single abandonment option.

In Fig. 12, we illustrate the impact of the additional resumption option on the MB and MC of waiting to invest. The embedded resumption option increases the MC relatively more than the MB, and, as a result, the marginal utility of the payoff decreases further, thereby

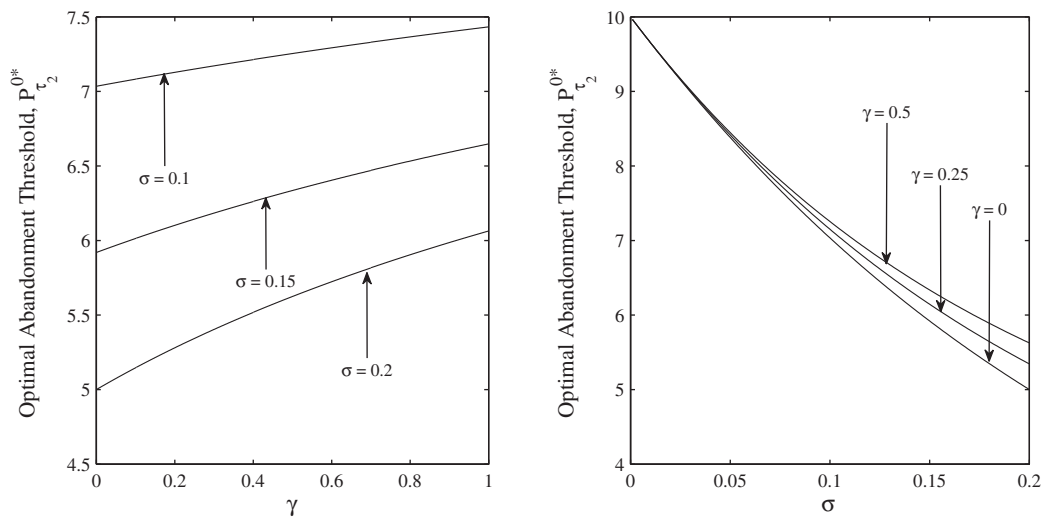


Fig. 8. Optimal abandonment threshold versus γ for $\sigma = 0.1, 0.15, 0.2$ (left), optimal abandonment threshold versus σ for $\gamma = 0, 0.25, 0.5$ (right).

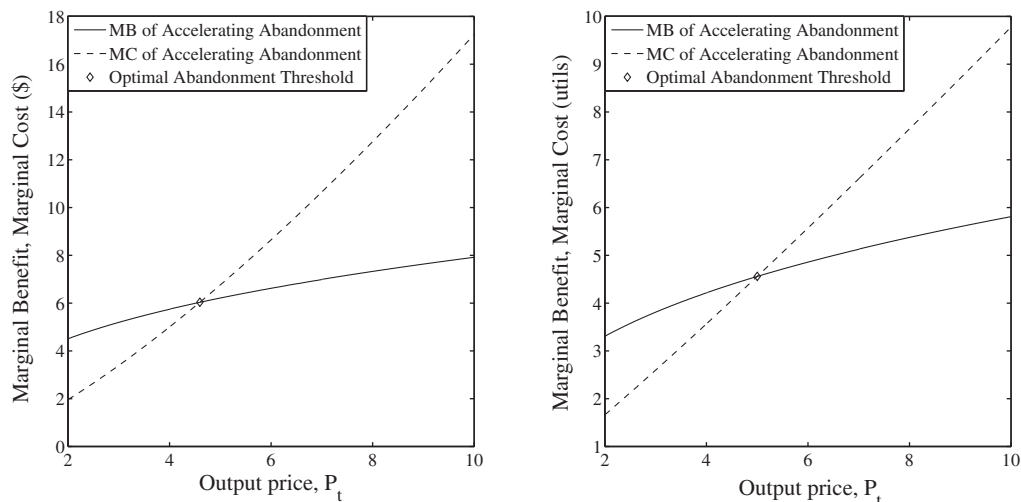


Fig. 9. Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$, (right) for an irreversible abandonment opportunity.

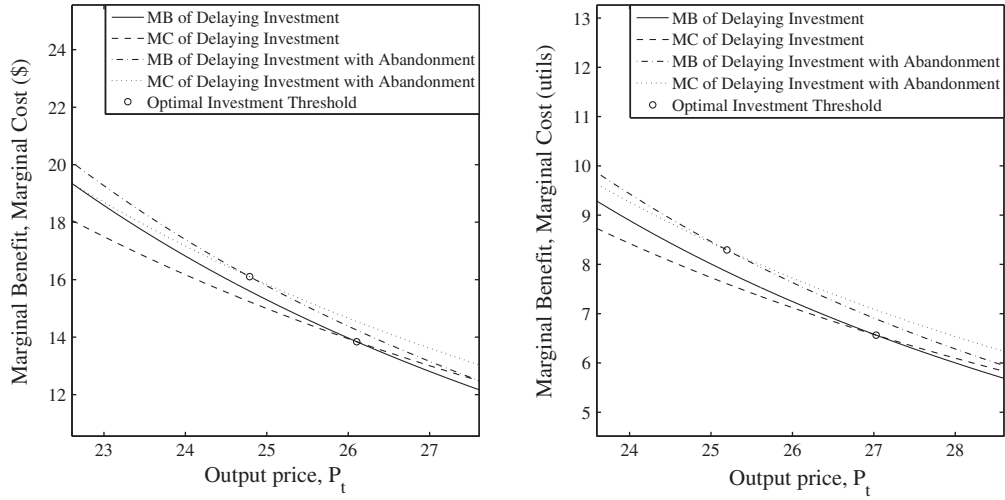


Fig. 10. Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$ (right) for an investment opportunity with an embedded abandonment option.

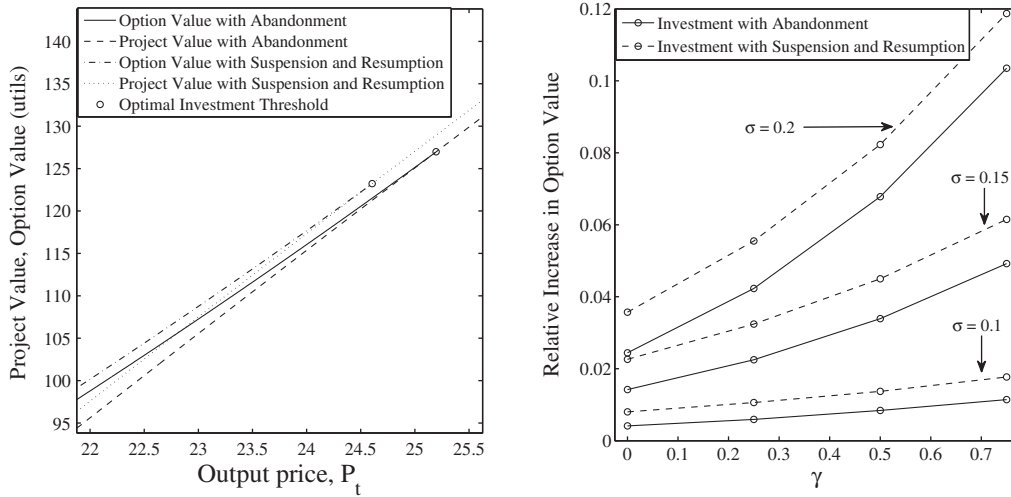


Fig. 11. Effect of the resumption option on optimal investment threshold and option value.

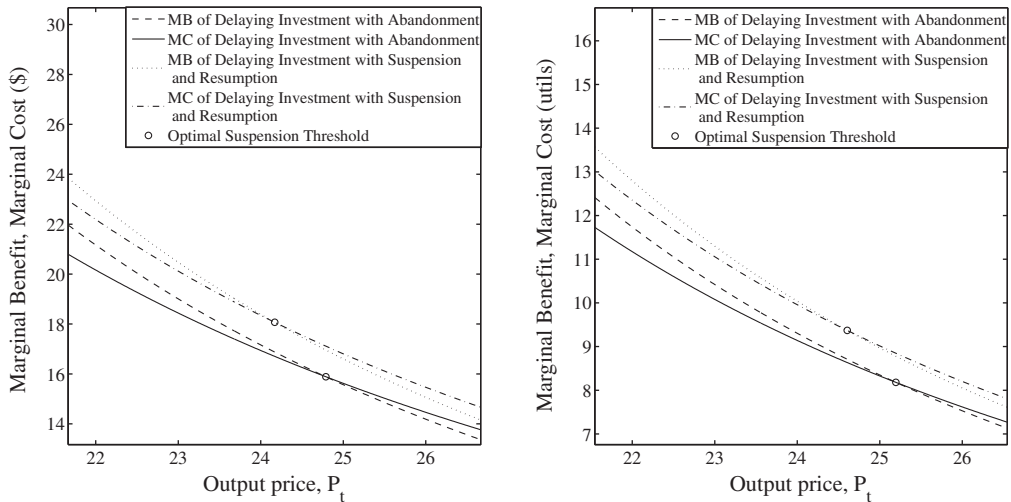


Fig. 12. Marginal benefit versus marginal cost under risk neutrality (left) and risk aversion, $\gamma = 0.25$, (right) for an investment opportunity with a suspension and resumption option.

increasing the incentive to invest. Thus, the MB and MC curves intersect at a lower level of the output price, compared to the case of investment with abandonment.

Interestingly, the results also indicate that the decision maker is less willing to suspend operations as the level of risk aversion increases. This outcome may seem counterintuitive, but it can be explained by the fact that as risk aversion increases, the following two opposing effects take place. First, the marginal utility of accelerating suspension increases with risk aversion, thereby increasing the likelihood of suspension. This happens because the MC of the abandonment option decreases faster with risk aversion than the MB. Second, the marginal utility of delaying resumption from a suspended state increases with risk aversion, thereby decreasing the likelihood of resumption. Here, the MC of the embedded resumption option decreases faster than the MB. Thus, higher risk aversion reduces the marginal value of the payoff from the resumption option, which makes suspension less attractive. Under the assumption of costless suspension and resumption and for the values of the parameters used here, we observe that the impact of risk aversion on the embedded resumption option dominates and postpones the suspension of the project. Fig. 13 illustrates the impact of risk aversion and uncertainty on the optimal suspension threshold. The graph on the left indicates that as risk aversion increases, the wedge between the MB of suspension and the MB of abandonment decreases, thereby indicating that the impact of risk aversion on the embedded resumption option is more profound and results in the decreased likelihood of suspension. On the other hand, as in the previous section, the likelihood that the project will be suspended decreases with uncertainty since the decision maker is inclined to wait for uncertainty to be resolved before exercising the suspension option.

Also interesting is that by allowing a further abandonment option after resumption, the aforementioned counterintuitive result is no longer observed. Due to the additional abandonment option, the marginal utility of the payoff from the option to suspend operations increases faster with risk aversion than in the case of suspension with a subsequent option of permanent resumption. In particular, the rate of this increase is greater than the rate at which the marginal utility of the payoff from the embedded call option increases. Hence, the impact of risk aversion on the embedded suspension and abandonment options is now greater than that on the single resumption option and, thus, causes the likelihood of suspension to increase with risk aversion. In fact, we observe that the impact of risk aversion on an optimal suspension threshold dominates when the number of subsequent options to suspend operations exceeds the number of the options to resume them.

Fig. 14 summarizes the impact of operational flexibility and risk aversion on the optimal decision thresholds. The direction of the arrows indicates greater operational flexibility. Here, additional flexibility facilitates all operational decisions and causes the optimal investment and resumption thresholds to decrease and the optimal suspension threshold to increase. Meanwhile, the impact of risk aversion on the optimal investment and operational thresholds diminishes as additional flexibility becomes available.

5.4. Investment with complete operational flexibility

In Fig. 15, the left figure compares the case of investment with complete flexibility to that of investment with a single suspension and a single resumption option for $\sigma = 0.2$ and $\gamma = 0.25$. Now, the ability to suspend and resume operations at any time increases the value of the investment opportunity, which reduces further the investment threshold price. Also, the proportional increase in option value is greater than that in the case of investment with a single suspension and a single resumption option as the figure on the right illustrates. Finally, according to the numerical results, the optimal suspension and optimal resumption thresholds under complete flexibility are equal to the operating cost, c . Intuitively, additional flexibility facilitates the suspension and resumption of the investment project and, as a result, causes the optimal suspension threshold to increase and the optimal resumption threshold to decrease. Assuming that no rational decision maker would exercise a suspension option at $P_{t_{2j}}^i > c$ and a resumption option at $P_{t_{2j+1}}^i < c$, we can expect both of these thresholds to converge toward the operating cost as additional flexibility becomes available. Thus, as $i \rightarrow \infty$, we expect that $P_{t_{2j}}^i \rightarrow c$ and $P_{t_{2j+1}}^i \rightarrow c$. Hence, the ability to suspend and resume operations costlessly at any time completely mitigates the impact of risk aversion and volatility on the optimal operational thresholds and drives them to the same level as in the risk-neutral case.

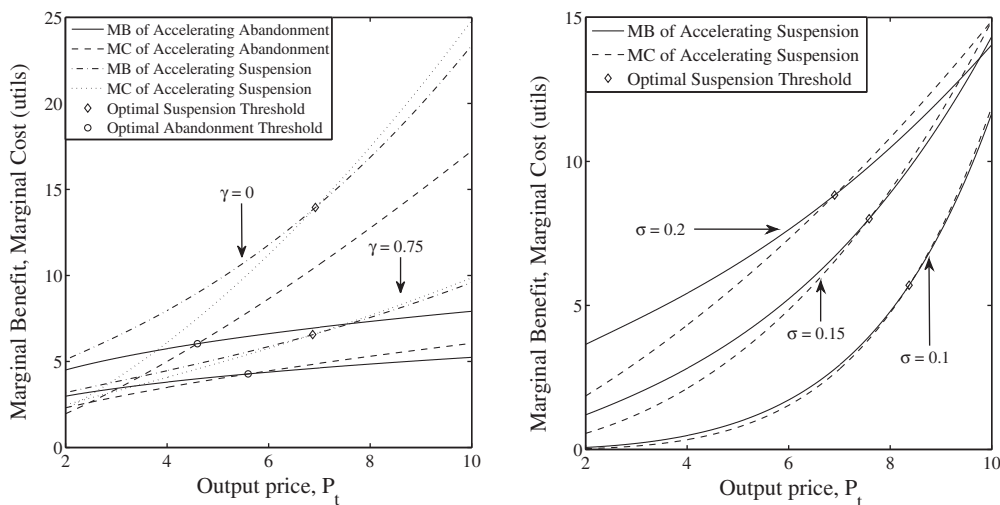


Fig. 13. MB and MC of abandonment versus MB and MC of suspension for $\sigma = 0.2$ (left), and MB of suspension versus MC of suspension for $\gamma = 0.25$ and $\sigma = 0.1, 0.2, 0.15$ (right).

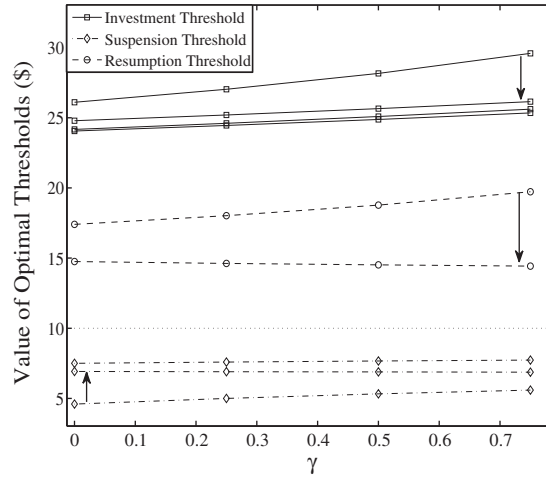


Fig. 14. Impact of operational flexibility and risk aversion on optimal decision thresholds.

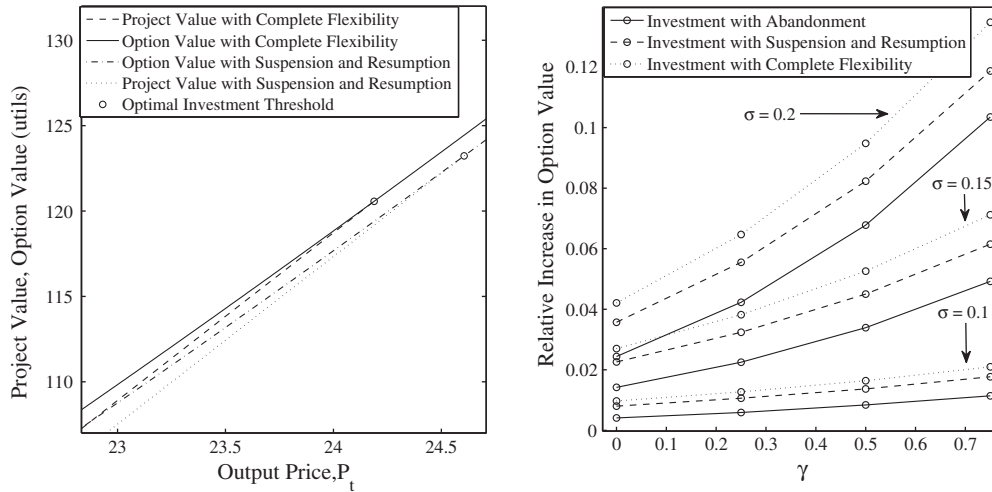


Fig. 15. Impact of complete flexibility on the optimal investment threshold and option value.

6. Conclusions

In a world of increasing economic uncertainty, the need to examine the interaction between risk aversion and operational flexibility, so as to provide optimal investment and operational decisions, is of great essence. In this paper, an effort is made to extend the results of McDonald and Siegel (1985, 1986) and Hugonnier and Morellec (2007) to examine how investment and operational decisions are affected by situations of uncertainty encountered by risk-averse decision makers. Although the impact of risk aversion has already been demonstrated in Hugonnier and Morellec (2007), its implications when combined with operational flexibility have not been thoroughly examined yet. Here, we develop the results regarding the problem of optimal investment under the assumption of risk aversion and operational flexibility assuming that the decision maker faces incomplete markets. We demonstrate how operational flexibility facilitates investment and operational decisions by increasing the likelihood of investment, suspension, and resumption of the investment project. We show that risk aversion provides an incentive for decision makers to delay the investment and resumption of the investment project and speed up their decision to abandon it. Moreover, we describe how an environment of increasing uncertainty may affect the optimal investment policy and lead to hysteresis. Also, we provide insights regarding the behavior of the optimal suspension threshold when the level of risk aversion changes. Finally, we demonstrate how operational flexibility becomes more valuable as risk aversion increases and the economic environment becomes more volatile.

Directions for further research could include the application of an alternative stochastic process such as an arithmetic Brownian motion or a mean-reverting process, providing information regarding the robustness of the numerical, theoretical, and intuitive results. Relaxing the assumption of costless suspension and resumption may also provide further insights. In addition to this, a different class of utility functions could be applied in order to obtain further insight regarding the impact of risk aversion on the optimal investment policy and allow for comparisons with the approach presented in this paper. Other aspects of the real options literature, e.g. dealing with endogenous capacity (Dangl, 1999) and the time-to-build problem, may also be investigated under the framework outlined in this paper.

Acknowledgment

The authors thank the participants of the MSOM Special Interest Group Meeting on the Interface of Finance, Operations, and Risk Management (Boston, June 2010), as well as the participants of the 2010 INFORMS Annual Meeting in San Diego, CA, USA. Feedback from attendees of seminars at University College London, Manchester Business School, and Tokyo Metropolitan University has helped to improve the paper. All remaining errors are the authors' own. Siddiqui acknowledges the support of the ELDEV project in Trondheim, Norway.

Appendix A

Proposition 4.1: *The objective function is strictly concave at $P_{\tau_1}^{0*}$ iff $\gamma < 1$.*

Proof. The objective function evaluated at the critical value, $P_{\tau_1}^{0*}$, is the following:

$$F_1^0(P_{\tau_1}^{0*}) = \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \left[\frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_1}^{0*1-\gamma}}{1-\gamma} - \frac{(c+rK)^{1-\gamma}}{\rho(1-\gamma)} \right] \quad (37)$$

Differentiating the objective function with respect to $P_{\tau_1}^{0*}$ twice yields the following result:

$$\begin{aligned} \frac{\partial^2 F_1^0(P_{\tau_1}^{0*})}{\partial P_{\tau_1}^{0*2}} &= (1+\beta_1)\beta_1 \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \left(\frac{1}{P_{\tau_1}^{0*}}\right)^2 \left[\frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_1}^{0*1-\gamma}}{1-\gamma} - \frac{(c+rK)^{1-\gamma}}{\rho(1-\gamma)} \right] \\ &\quad + \beta_1 \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \left(-\frac{1}{P_{\tau_1}^{0*}}\right) \left[\frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_1}^{0*-\gamma} \right] + \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \left[-\frac{2}{\sigma^2} \frac{-\gamma-\beta_1}{(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_1}^{0*-\gamma-1} \right] \end{aligned} \quad (38)$$

The SOSC requires that $\frac{\partial^2 F_1^0(P_{\tau_1}^{0*})}{\partial P_{\tau_1}^{0*2}} < 0$. Simplifying the above expression yields:

$$\frac{\partial^2 F_1^0(P_{\tau_1}^{0*})}{\partial P_{\tau_1}^{0*2}} < 0 \iff \frac{2\beta_1 + \gamma + 1}{1-\gamma} > 0 \quad (39)$$

Notice that the numerator is positive, which implies that for the inequality to hold the denominator needs to be positive as well. Hence, the SOSC is satisfied if and only if $0 \leq \gamma < 1$. \square

Corollary 4.1: *The MB curve is steeper than the MC curve at $P_{\tau_1}^{0*}$.*

Proof. We will show that $\left| \frac{\partial MB}{\partial P_{\tau_1}^{0*}} \right| > \left| \frac{\partial MC}{\partial P_{\tau_1}^{0*}} \right|_{P_{\tau_1}^{0*} = P_{\tau_1}^{0*}}$.

$$\begin{aligned} &\beta_1 \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \frac{1}{P_{\tau_1}^{0*}} \left[\frac{\beta_1 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_1}^{0*-\gamma} + \frac{\beta_1}{P_{\tau_1}^{0*}} \frac{U(c+rK)}{\rho} \right] \\ &\quad + \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \left[\frac{\beta_1 \beta_2 \gamma}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_1}^{0*-\gamma-1} + \frac{\beta_1}{P_{\tau_1}^{0*2}} \frac{U(c+rK)}{\rho} \right] \\ &> \beta_1 \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \frac{1}{P_{\tau_1}^{0*}} \frac{\beta_1^2 \beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_1}^{0*-\gamma}}{1-\gamma} + \left(\frac{P_0}{P_{\tau_1}^{0*}}\right)^{\beta_1} \frac{\beta_1^2 \beta_2 \gamma}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_1}^{0*-\gamma-1}}{1-\gamma} \end{aligned} \quad (40)$$

Simplifying (40) and substituting for $P_{\tau_1}^{0*}$ leads to the following result:

$$2\beta_1 + \gamma + 1 > 0 \quad (41)$$

This is true since $\beta_1 > 1$ and $0 \leq \gamma < 1$ \square

Proposition 4.2: *The optimal investment threshold is increasing with risk aversion.*

Proof. Differentiating the optimal investment threshold, $P_{\tau_1}^{0*}$, with respect to γ yields:

$$P_{\tau_1}^{0*} = (c+rK) \left(\frac{\beta_2 + \gamma - 1}{\beta_2} \right)^{\frac{1}{1-\gamma}} \Rightarrow \frac{\partial P_{\tau_1}^{0*}}{\partial \gamma} = P_{\tau_1}^{0*} \frac{\partial}{\partial \gamma} \ln \left[(c+rK) \left(\frac{\beta_2 + \gamma - 1}{\beta_2} \right)^{\frac{1}{1-\gamma}} \right] \quad (42)$$

Since $P_{\tau_1}^{0*} > 0$, we only need to determine the sign of $\frac{\partial}{\partial \gamma} \ln \left[(c+rK) \left(\frac{\beta_2 + \gamma - 1}{\beta_2} \right)^{\frac{1}{1-\gamma}} \right]$. Hence,

$$\frac{\partial \ln P_{\tau_1}^{0*}}{\partial \gamma} > 0 \iff \ln \left[\frac{1-\beta_2-\gamma}{-\beta_2} \right] > 1 - \frac{-\beta_2}{1-\beta_2-\gamma} \quad (43)$$

We now set $x = \frac{-\beta_2}{1-\beta_2-\gamma} > 0 \Rightarrow \frac{1}{x} = \frac{1-\beta_2-\gamma}{-\beta_2}$. Hence, we now need to show that:

$$-\ln x > 1 - x \iff \ln x < x - 1 \quad (44)$$

The equality $\ln x = x - 1$ holds for $\gamma = 1$, which is not considered in this paper. To show that inequality (44) holds, we first need to show that:

$$e^x \geq 1 + x, \quad \forall x \in \mathbb{R} \quad (45)$$

For all $x \in \mathbb{R}$, we assume a $v \in \mathbb{N}$ such that $v > -x$, i.e. $v + x > 0$. Then, $1 + \frac{x}{v} > 0$, and so we have $(1 + \frac{x}{v})^v \geq 1 + v\frac{x}{v} = 1 + x$ from Bernoulli's inequality. Finally, we have:

$$e^x = \lim_{v \rightarrow \infty} \left(1 + \frac{x}{v}\right)^v \geq \lim_{v \rightarrow \infty} (1 + x) = 1 + x \Rightarrow e^x \geq 1 + x \quad \forall x \in \mathbb{R} \quad (46)$$

Thus, we have shown that $e^x \geq 1 + x$, $\forall x \in \mathbb{R}$. Hence, assuming that $x > 0$ and using $\ln x$ instead of x , we have:

$$e^{\ln x} = x \geq 1 + \ln x \Rightarrow \ln x \leq x - 1 \quad \square \quad (47)$$

Proposition 4.3: *The optimal investment threshold is increasing with volatility.*

Proof. Since $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}$, substituting into the expression of the optimal investment threshold we have:

$$P_{\tau_1}^{0*} = (c + rK) \left(1 + \frac{\gamma - 1}{\frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}}\right)^{\frac{1}{1-\gamma}} \quad (48)$$

Differentiating with respect to σ^2 yields:

$$\frac{\partial P_{\tau_1}^{0*}}{\partial \sigma^2} = c \left(\frac{1}{1-\gamma}\right) \left[1 + \frac{\gamma - 1}{\frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}}\right]^{\frac{1}{1-\gamma}-1} \times \left[\frac{(\gamma - 1) \left(\frac{\mu}{\sigma^4} - \frac{1}{2\sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}} \right) \left\{ 2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \left(-\frac{\mu}{\sigma^4}\right) - \frac{2\rho}{\sigma^4} \right\}}{\left(\frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}\right)^2} \right] \quad (49)$$

Note that:

$$2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \left(-\frac{\mu}{\sigma^4}\right) - \frac{2\rho}{\sigma^4} < 0 \iff -\frac{\mu^2}{\sigma^2} - \frac{2\rho - \mu}{2} < 0 \quad (50)$$

which is true. Hence, the last term in (49) is positive. Since the rest of the factors in (49) are positive, we conclude that $\frac{\partial P_{\tau_1}^{0*}}{\partial \sigma^2} > 0$. \square

Proposition 4.4: *The objective function is strictly concave at $P_{\tau_2}^{0*}$ iff $\gamma < 1$.*

Proof. The objective function evaluated at $P_{\tau_2}^{0*}$ takes the following expression,

$$F_{\tau_2}^0(P_{\tau_1}^1) = \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^{0*}}\right)^{\beta_2} \left[\frac{c^{1-\gamma}}{\rho(1-\gamma)} - \frac{\beta_1\beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_2}^{0*1-\gamma}}{1-\gamma} \right] \quad (51)$$

Differentiating twice with respect to $P_{\tau_2}^{0*}$ yields the following,

$$\begin{aligned} \frac{\partial^2 F_{\tau_2}^0(P_{\tau_1}^1)}{\partial P_{\tau_2}^{0*2}} &= \beta_2(\beta_2 + 1) \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^{0*}}\right)^{\beta_2} \left(\frac{1}{P_{\tau_2}^{0*}}\right)^2 \left[\frac{c^{1-\gamma}}{\rho(1-\gamma)} - \frac{\beta_1\beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} \frac{P_{\tau_2}^{0*1-\gamma}}{1-\gamma} \right] \\ &+ \beta_2 \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^{0*}}\right)^{\beta_2} \left(\frac{1}{P_{\tau_2}^{0*}}\right) \left[\frac{\beta_1\beta_2}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_2}^{0*-\gamma} \right] + \left(\frac{P_{\tau_1}^1}{P_{\tau_2}^{0*}}\right)^{\beta_2} \left[\frac{\beta_1\beta_2(\beta_2 + \gamma)}{\rho(1-\beta_1-\gamma)(1-\beta_2-\gamma)} P_{\tau_2}^{0*-\gamma-1} \right] \end{aligned} \quad (52)$$

Simplifying the above expression, we have the following result:

$$\frac{\partial^2 F_{\tau_2}^0(P_{\tau_1}^1)}{\partial P_{\tau_2}^{0*2}} < 0 \iff \gamma < 1 \quad (53)$$

Hence, the objective function is concave at $P_{\tau_2}^{0*}$ if and only if $\gamma < 1$. \square

Proposition 4.5: *The optimal abandonment threshold is increasing with risk aversion.*

Proof. Following the same steps as in Proposition 4.2 we have:

$$\frac{\partial P_{\tau_2}^{0*}}{\partial \gamma} > 0 \iff \ln \left[\frac{\beta_1 + \gamma - 1}{\beta_1} \right] > 1 - \frac{\beta_1}{\beta_1 + \gamma - 1} \quad (54)$$

We now set $x = \frac{\beta_1}{\beta_1 + \gamma - 1}$. Thus, we need to show that $\ln x < x - 1$ which we have already shown in Proposition 4.2 that holds. \square

Proposition 4.6: *The optimal abandonment threshold is decreasing in volatility.*

Proof. The optimal abandonment threshold is given by the following equation:

$$P_{\tau_2}^{0*} = c \left[\frac{\beta_1 + \gamma - 1}{\beta_1} \right]^{\frac{1}{1-\gamma}} \quad (55)$$

Substituting for β_1 , (55) takes the following expression:

$$P_{\tau_2}^{0*} = c \left[1 + \frac{\gamma - 1}{\frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}} \right]^{\frac{1}{1-\gamma}} \quad (56)$$

Differentiating (56) with respect to σ^2 we have:

$$\frac{\partial P_{\tau_2}^{0*}}{\partial \sigma^2} = c \left(\frac{1}{1-\gamma} \right) \left[1 + \frac{\gamma - 1}{\frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}} \right]^{\frac{1}{1-\gamma}-1} \times \left[\frac{(\gamma - 1) \left(\frac{\mu}{\sigma^4} + \left(\frac{1}{2\sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}} \right) \left\{ 2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \left(-\frac{\mu}{\sigma^4}\right) - \frac{2\rho}{\sigma^4} \right\} \right)}{\left(\frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left[\frac{\mu}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} \right)^2} \right] \quad (57)$$

Notice that in (57):

$$2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \left(-\frac{\mu}{\sigma^4}\right) - \frac{2\rho}{\sigma^4} < 0 \iff -\frac{\mu^2}{\sigma^2} - \frac{2\rho - \mu}{2} < 0 \quad (58)$$

The latter is true, and, therefore, the last term of (57) is negative. Hence, since the first three factors are positive, we conclude that $\frac{\partial P_{\tau_2}^{0*}}{\partial \sigma^2} < 0$. \square

Proposition 4.7: *The optimal investment threshold price when a single abandonment option is available is less than that with an irreversible investment opportunity, ceteris paribus.*

Proof. The FONCs that provide the optimal investment thresholds for the case of irreversible investment and investment with a single abandonment option are:

$$\frac{\beta_2}{1 - \beta_2 - \gamma} P_{\tau_1}^{0*1-\gamma} + (c + rK)^{1-\gamma} = 0 \quad (59)$$

$$\frac{\beta_2}{1 - \beta_2 - \gamma} P_{\tau_1}^{1*1-\gamma} + (c + rK)^{1-\gamma} + \frac{\beta_2 - \beta_1}{\beta_1} \rho (1 - \gamma) F_{\tau_2}^0 \left(P_{\tau_1}^{1*} \right) = 0 \quad (60)$$

Subtracting the two equations, we have:

$$P_{\tau_1}^{0*1-\gamma} - P_{\tau_1}^{1*1-\gamma} = (1 - \beta_2 - \gamma) F_{\tau_2}^0 \left(P_{\tau_1}^{1*} \right) \quad (61)$$

Since $\beta_2 < 0$ and $F_{\tau_2}^0 \left(P_{\tau_1}^{1*} \right) > 0$ the quantity on the right-hand side is positive. Hence:

$$P_{\tau_1}^{0*1-\gamma} - P_{\tau_1}^{1*1-\gamma} > 0 \Rightarrow P_{\tau_1}^{0*} > P_{\tau_1}^{1*} \quad \square \quad (62)$$

Proposition 4.8: *The optimal resumption threshold is increasing with risk aversion.*

Proof. Similar to Proposition 4.2. \square

Proposition 4.9: *The optimal suspension threshold is higher than the optimal abandonment one.*

Proof. Comparing the two FONCs that provide the optimal abandonment and optimal suspension thresholds, we have:

$$\frac{1}{1 - \beta_1 - \gamma} P_{\tau_2}^{0*1-\gamma} + \frac{c^{1-\gamma}}{\beta_1} = 0 \quad (63)$$

$$\frac{1}{1 - \beta_1 - \gamma} P_{\tau_2}^{1*1-\gamma} + \frac{c^{1-\gamma}}{\beta_1} - \frac{\rho(\beta_1 - \beta_2)(1 - \gamma)}{\beta_1 \beta_2} F_{\tau_3}^0 \left(P_{\tau_2}^{1*} \right) = 0 \quad (64)$$

By subtracting the two equations we have:

$$P_{\tau_2}^{1*1-\gamma} - P_{\tau_2}^{0*1-\gamma} = \frac{\rho(\beta_1 - \beta_2)(1 - \beta_1 - \gamma)(1 - \gamma)}{\beta_1 \beta_2} F_{\tau_3}^0 \left(P_{\tau_2}^{1*} \right) \quad (65)$$

Since $\beta_1 > 1$, $\beta_2 < 0$ and $F_{\tau_3}^0(P_{\tau_2}^{1*}) \geq 0$ quantity on the right-hand side is positive. Therefore,

$$P_{\tau_2}^{1*1-\gamma} - P_{\tau_2}^{0*1-\gamma} > 0 \Rightarrow P_{\tau_2}^{1*} > P_{\tau_2}^{0*} \quad \square \quad (66)$$

Proposition 4.10: *The optimal investment threshold price when a single suspension and a single resumption option is available is lower than that with an investment opportunity with a single abandonment option.*

Proof. The FONC that yield the optimal investment thresholds are:

$$\frac{\beta_2}{1 - \beta_2 - \gamma} P_{\tau_1}^{1*1-\gamma} + (c + rK)^{1-\gamma} + \frac{\beta_2 - \beta_1}{\beta_1} \rho(1 - \gamma) F_{\tau_2}^0(P_{\tau_1}^{1*}) = 0 \quad (67)$$

$$\frac{\beta_2}{1 - \beta_2 - \gamma} P_{\tau_1}^{2*1-\gamma} + (c + rK)^{1-\gamma} + \frac{\beta_2 - \beta_1}{\beta_1} \rho(1 - \gamma) F_{\tau_2}^1(P_{\tau_1}^{2*}) = 0 \quad (68)$$

Subtracting these two equations we have:

$$\begin{aligned} \frac{\beta_2}{1 - \beta_2 - \gamma} [P_{\tau_1}^{1*1-\gamma} - P_{\tau_1}^{2*1-\gamma}] + \frac{\beta_2 - \beta_1}{\beta_1} \rho(1 - \gamma) [F_{\tau_2}^0(P_{\tau_1}^{1*}) - F_{\tau_2}^1(P_{\tau_1}^{2*})] &= 0 \\ \Rightarrow P_{\tau_1}^{1*1-\gamma} - P_{\tau_1}^{2*1-\gamma} &= \frac{(1 - \beta_2 - \gamma)(\beta_1 - \beta_2)}{\beta_1 \beta_2} \rho(1 - \gamma) [F_{\tau_2}^0(P_{\tau_1}^{1*}) - F_{\tau_2}^1(P_{\tau_1}^{2*})] \end{aligned} \quad (69)$$

Since the option to suspend operations with the embedded option to resume them permanently, $F_{\tau_2}^1(P_{\tau_1}^{2*})$, is greater than the abandonment option, $F_{\tau_2}^0(P_{\tau_1}^{1*})$, the right-hand side of (68) is negative indicating that:

$$P_{\tau_1}^{1*1-\gamma} - P_{\tau_1}^{2*1-\gamma} > 0 \Rightarrow P_{\tau_1}^{1*} > P_{\tau_1}^{2*} \quad \square \quad (70)$$

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