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Further Exploration on Relationship between Crisp Sets and Fuzzy Sets

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***Abstract*—This paper employs the concept of fuzzify upgrade operator to describe the relationship between crisp sets, fuzzy sets and higher order fuzzy sets, such as type-2 fuzzy sets. Theory analysis and case studies demonstrate that a crisp set can be represented by a type-1 fuzzy set, a type-1 fuzzy set can be represented by a type-2 fuzzy set, and a type-n (n>=2) fuzzy set can be represented by a type-(n+1) (n>=2) fuzzy set. The fuzzify upgrade operator makes crisp sets, type-1 fuzzy sets, type-2 fuzzy sets and any fuzzy sets much more accessible to all readers. The relationship between crisp sets and fuzzy sets are explored.**

Keywords-Fuzzify upgrade operator, crisp set, type-1 fuzzy set; type-2 fuzzy set.

I. INTRODUCTION

Zadeh proposed the concept of fuzzy sets [1] to resemble human reasoning in the use of approximate information and uncertainty to generate decision in 1965, known as type- 1 fuzzy set. The methodology of type-1 fuzzy set has been successfully used in many applications which include classification, control, forecasting, function approximation and so on [2]–[7]. Zadeh further proposed the type-2 fuzzy sets [8] and Mendel has established a set of terms to be used when working with type-2 fuzzy sets and, in particular, introduced a concept known as the footprint of uncertainty (FOU) which can provide a useful verbal and graphical description of the uncertainty for any given type-2 fuzzy sets [9]–[11]. Type- 2 fuzzy sets are described by three dimensions [9]–[15] and the membership functions of them are characterized by more parameters than the membership functions of type-1 fuzzy sets. Therefore, type-2 fuzzy sets has greater freedom for designing fuzzy sets and achieved greater success than type-1 fuzzy sets in various fields to handle uncertainties [9], [10], [14], [16]– [19].

In order to analyze the difference between crisp sets, type-1 fuzzy sets and type-2 fuzzy sets, our previous work proposed a fuzzify upgrade operator, which is called fuzzify functor, to describe the relationship between crisp sets, type-1 fuzzy sets and type-2 fuzzy sets [12]. Fuzzify upgrade operator can describe the relationship between crisp sets, type-1 fuzzy sets, type-2 fuzzy sets and higher type or higher order fuzzy sets.

A crisp set can be upgraded to a type-1 fuzzy set, a type-1 fuzzy set can be upgraded to a type-2 fuzzy set, and a type-(n-1) (n>=2) fuzzy set can be upgraded to a type-n (n>=2) fuzzy set when a fuzzify upgrade operator acts on them [12].

This paper explores the deep relationship between crisp sets, type-1 fuzzy sets, type-2 fuzzy sets, higher type or higher order fuzzy sets. In order to make this paper as self-contained as possible, Section 2 reviews the concept of fuzzify upgrade operator. In Section 3, the deep relationship between crisp sets and fuzzy set are analyzed. The theory analysis and the case studies demonstrate the deep relationship between crisp set, type-1 fuzzy sets, type-2 fuzzy sets and higher type fuzzy sets. When the uncertainty disappears, a type-1 fuzzy set can become a crisp set. When the higher order uncertainty disappears, a type-2 fuzzy set can become a type-1 fuzzy set and a type-n (n>=2) fuzzy set can become a type-(n-1) (n>=2) fuzzy set. Section 4 concludes this paper.

II. FUZZIFY UPGRADE OPERATOR

Definition 1: A fuzzify upgrade operator [12], denoted as $\tilde{B}$,operates on constant variable or crisp number, which can be represented by a crisp set $A^0 = \{v_0\}$ shown in Fig. 1 (a1), and it will produce type-1 fuzzy set, A^1 :

$$\begin{aligned} & A^0 \\ & = \tilde{B} A^0 \\ & = \tilde{B}\{v_0\} \\ & = \int_{x\in X} A(x)/x \end{aligned} \tag{1}$$

where the integral sign denotes the collection of all point, x, $x \in X$ is the variable, $A^1(x)$ is the membership function, which represents the grade of membership of each $x \in X$ in A^1 . Using Gauss fuzzy model as case study, the process of fuzzify upgrade operator acting on a crisp set is shown in Fig. (a1) – (a3). Fig. 1 (a1) shows the crisp number, which also a crisp set that has one element in it. Fig. 1 (a2) shows that the crisp value of the crisp number has been blurred by making fuzzy step of fuzzify upgrade operator. Fig. 1 (a3) shows a type-1 fuzzy set and also a Gauss fuzzy number, which is produced after fuzzify upgrade operator

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acting on the crisp set when selecting Gaussian distribution at the assigning amplitude distribution step [12].

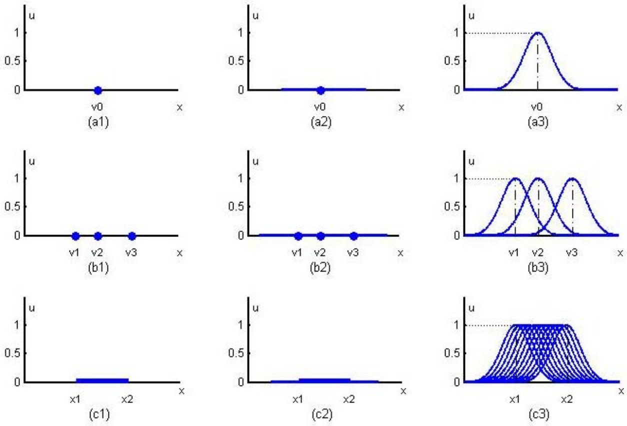


Figure 1. The process of the fuzzify upgrade operator acting on a crisp number or the crisp sets: (a1) a crisp number or a crisp set, which has only one element; (a2) the crisp value is blurred; (a3) assign degree to all of those points and a type-1 fuzzy set produced; (b1) a crisp set, which have more than one elements; (b2) all the crisp values are blurred; (b3) assign degree to all of those blurred points and a set of fuzzy set are produced. (c1) a crisp set, which have continue elements; (c2) all the crisp values are blurred; (c3) assign degree to all of those blurred points and fuzzy sets are produced.

Definition 2: A Gauss fuzzify upgrade operator, denoted as $\tilde{B}_G$, is a fuzzify upgrade operator, in which the Gauss fuzzy model is used to capture the uncertainty. Similarly, the Triangular fuzzify upgrade operator, Sigmoid fuzzify upgrade operator, bell-shaped fuzzify upgrade operator can be obtained when selecting the corresponding distribution model to capture the uncertainty at the assigning degree step of the fuzzify upgrade operator

Theorem 1: fuzzify upgrade operator is left-distributive over union or collection of crisp sets. It acts on crisp sets or crisp numbers left distributes over the union or collection of the crisp sets or crisp numbers as the following equation [12].

$$\begin{aligned} &A^0 \\ &= \{v_1\} + \{v_2\} \\ &= A_a^0 + A_b^0 \end{aligned} \tag{2}$$

where the sum symbol denotes union or collection. When the fuzzify upgrade operator acts on the crisp set $A^0 = \{v_1, v_2\}$, according to the Definition 1 and Theorem 1:

$$\begin{aligned} &\tilde{B}(A^0) \\ &= \tilde{B}(A_a^0 + A_b^0) \\ &= \tilde{B}(A_a^0) + \tilde{B}(A_b^0) \\ &= \tilde{B}\{v_1\} + \tilde{B}\{v_2\} \end{aligned} \tag{3}$$

Fig. 1 (b1) and (c1) show the crisp A_2^0 set and A_3^0, which have more than one element. $A_2^0 = \{v_1, v_2, v_3\}$ and $A_3^0 = \{x \mid x \in [x_1, x_2]\}$. Fig. 1 (b2) and (c2) show that the crisp values in the crisp sets have been blurred by making fuzzy step of fuzzify upgrade operator. Fig. 1 (b3) and (c3) show the produced type-1 fuzzy sets after capture the uncertainty in the assigning degree step of fuzzify upgrade operator [12].

Definition 3: A fuzzify upgrade operator, denoted as $\tilde{B}$, operates on a fuzzy set $A = A^1$ shown in Fig. 2 (1), and it will produce a type-2 fuzzy set A^2 as following equation [12]

$$\begin{aligned} &A^2 \\ &= \tilde{B} \int_{x\in X} A^1(x)/x \\ &= \int_{x\in X} \tilde{B}(A^1(x))/x \\ &= \int_{x X\in} [\int_{u\in D_x \subseteq[0,1]} A^2(x)(u)/u]/x \\ &= \int_{x\in X} \int_{u\in D_x \subseteq[0,1]} A^2(x)(u)/u/x \end{aligned} \tag{4}$$

Where

$$\tilde{B}(A(x)) = \int_{u\in D_x \subseteq[0,1]} A^2(x)(u)/u \tag{5}$$

$x \in X$ is the primary variable. $A^1(x)$ is the primary membership function, which represents the grade of membership of each $x \in X$ in A^1. $u \in D_x \subseteq [0,1]$ is the secondary variable. $A^2(x)(u) \subseteq [0,1]$ is a new membership function, which we call the secondary membership function. It represents the grade of membership of each $u \in D_x \subseteq [0,1]$ in $A^2(x)$ for each $x \in X$. The FOU can be expressed as $FOU(A^2) = \bigcup_{x\in X} D_x$

A^2 is the first-order fuzzify operated fuzzy set of fuzzy set A^1, and $A^2(x)(u)$ is the secondary membership function of the first-order fuzzify operated fuzzy set A^2.

It is obvious that when fuzzify upgrade operator acts on a crisp set, the produced first-order operated fuzzy set is the type-1 fuzzy set. When fuzzify upgrade operator acts on a type-1 fuzzy set, the produced fuzzy set is a type-2 fuzzy set [12].

The process of fuzzify upgrade operator acting on type-1 fuzzy set are shown in Fig. 2 (1) - (3). Fig. 2 (1) shows the type-1 fuzzy set A^1; Fig. 2 (2) shows that the crisp values of the membership function has been blurred by making fuzzy step of the fuzzify upgrade operator; Fig. 2 (3) shows the type-2 fuzzy set, which is produced after the fuzzify upgrade operator acting on the type-1 fuzzy set A^1.

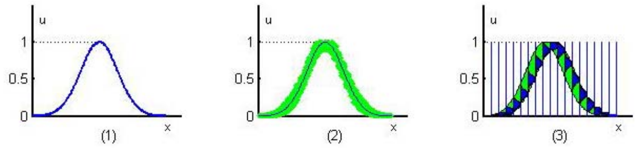


Figure 2. The process of the fuzzify upgrade operator acting on a type-1 fuzzy set or a fuzzy number: (a1) a type-1 fuzzy set or a Gauss fuzzy number; (a2) the crisp values of the membership are blurred; (a3) assign degree to all of those points and a type-2 fuzzy set producedelement.

Theorem2: The fuzzify upgrade operator is left-distributive over union or collection of fuzzy sets [12]. It can also be said that the fuzzify functor acting on fuzzy sets or fuzzy numbers left distributes over the union or collection of the fuzzy sets or fuzzy numbers. A^1 and A^2 are two fuzzy sets or two fuzzy numbers.

$$\tilde{B}(A_1^1 + A_2^1) = \tilde{B}(A_1^1) + \tilde{B}(A_2^1) \qquad (6)$$

where the sum symbol is to denote union or collection of the sets. There are many methods to blur the exact values and many methods to assign amplitude distribution to all of those points. In this paper the normal distribution, also called the Gaussian distribution is used in the case analysis in this paper.

III. Deep Relationship Between Crisp Sets And Fuzzy Sets

When the fuzzify upgrade operator acts on the crisp set A^0 ,firstly, it makes the exact value of the point to be fuzzy shown in Fig. 1 (a2) according to the uncertainty principle and then it assigns the degrees to all the blurred points according to the distribution principle [12].

The fuzzify upgrade operators have two functions: making fuzzy and assigning degree [12]. Making fuzzy is to blur or to disturb the crisp value to be in a range. Assigning degree is to give an amplitude distribution to all of those points.

When the fuzzify upgrade operator acts on the crisp number shown in Fig. 1(a1) and Fig. 3 (a1), it makes the exact value of the point to be a range according to the first function: making fuzzy. After blurring the crisp value, there will be arrange in the discourse as shown in Fig. 1 (a2).

Assume ξ is a constant variable and the blurred range is assumed to be $[(v_0 - \xi),(v_0 + \xi)]$. Different blurred range can be obtained by selecting different values of ξ . However, when ξ = 0, the making fuzzy step is not really to make fuzzy, the value will be the original crisp value of the point after the first step: making fuzzy. A type-1 fuzzy set is produced shown in Fig. 3 (a2) after the second step: assigning the membership degree. The membership degree is set unit 1 since there is only one crisp point. In this situation, after the fuzzify upgrade operator acts on crisp sets shown in Fig. 3 (a1), (b1) and (c1),the obtained type-1 fuzzy sets are shown in Fig. 3 (a2), (b2)and (c2).

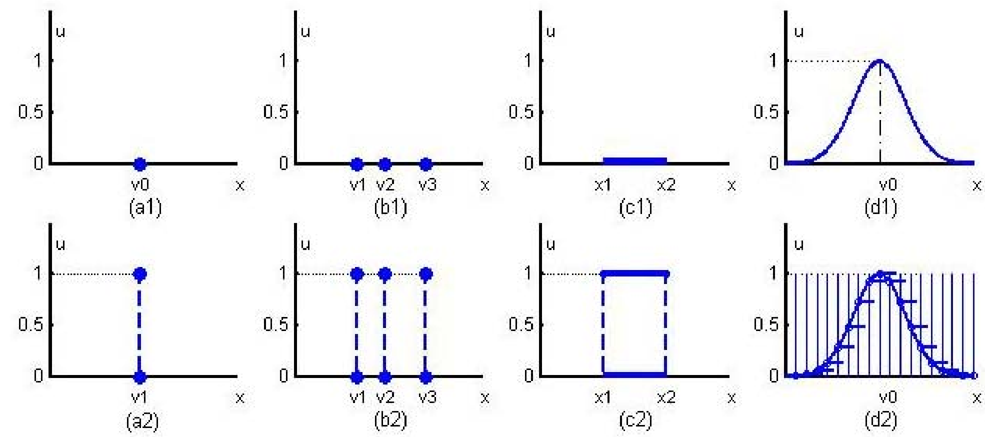


Figure 3. The deep relationship between crisp sets, type-1 fuzzy sets and type-2 fuzzy sets.

Similarly, when set the blurred range is $[(v_0 - \xi),(v_0 + \xi)]$ and $\xi = 0$, after the fuzzify upgrade operator acts on a type-1 fuzzy set shown in Fig. 3 (d1), a type-2 fuzzy set will be obtained shown in Fig. 3 (d2). That is why we also named the fuzzify functor as fuzzify upgrade operator [12]. When the fuzzify upgrade operator act on type-(n- 1) (n = 1, 2.....) fuzzy sets, we can get type-n (n = 1, 2,……) fuzzy sets. Here a crisp set is a type-0 fuzzy set. Even the process of making fuzzy is select to blur the point as the same of the original ξ = 0 , after the process of assigning degree, the crisp set become the type-1 fuzzy set and the type-1 fuzzy set become the type-2 fuzzy set [12].

Compared to the crisp set A^0 shown in Fig. 3 (a1) and type-1 fuzzy set A^1 shown in Fig. 3 (a2):

$$\begin{cases} A^0 = \{v_0\} \\ A^1 = \int_{x\in X} A(x)/x = \int_{x\in X} 1/x \end{cases} \tag{7}$$

It is obvious that crisp set, which can be described as one dimension shown in Fig. 3 (a1) can be represented in two dimensions shown in Fig. 3 (a2). In equation (7), the crisp set A^0 and the fuzzy set A^1 represent same sets. Fig. 3 (a1), (b1) and (c1) represent the following crisp sets:

$$\begin{cases} A_1^0 = \{v_0\} \\ A_2^0 = \{v_1, v_2, v_3\} \\ A_3^0 = \{x \mid x \in [x_1, x_2]\} \end{cases} \tag{8}$$

Fig. 3 (a2), (b2) and (c2) represent the following type-1 fuzzy sets:

$$\begin{cases} A_1^0 = \int_{x\in X} 1/x \quad x \in \{v_0\} \\ A_2^0 = \int_{x\in X} 1/x \quad x \in \{v_1, v_2, v_3\} \\ A_3^0 = \int_{x\in X} 1/x \quad x \in [x_1, x_2] \end{cases} \tag{9}$$

It is obvious that the crisp sets, which can be described as one dimension shown in Fig. 3 (a1), (b1), (c1) and equation (8), can be represented in two dimensions as shown in Fig. 3 (a2), (b2), (c2) and equation (9). One dimension is enough to represent these crisp sets because there is no other information missing in Fig. 3 (a1), (b1) and (c1) and in equation (8). Fig. 3 (a2), (b2) and (c2) represent crisp set using generally representation of the type-1 fuzzy set. This is consistent with the fuzzy set theory that the fuzzy set is a generalization of the crisp sets. Similarly, the type-1 fuzzy set, which is two dimensions shown in Fig. 3 (d1), can be represented as three dimensions shown in Fig. 3 (d2). The Fig. 3 (d1) represents the type-1 fuzzy set [1], [8]–[10], [12], [15]:

$$A^1 = \int_{x\in X} \mu(x)/x \tag{10}$$

Type-1 fuzzy sets also can be represented by a set of ordered pairs of generic element and its grade of membership [1, 2, 4, 6, and 13]:

$$A^1 = \{x, \mu(x) \mid x \in X\} \tag{11}$$

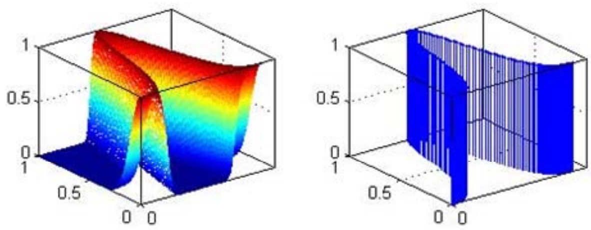
Fig. 3 (d2) also represents the type-1 fuzzy set by using generally representation of the type-2 fuzzy set:

$$A^2 = \int_{x\in X} \int_{u\in D_x = \mu(x)} 1/u/x \tag{12}$$

$$A^2 = \{((x,u),1) \mid \forall x \in X, \forall u \in D_x = \mu(x)\} \tag{13}$$

Analyzing equation (12), (13) and Fig. 3 (d2), two dimensions are enough to represent the type-1 fuzzy set because there is also no other information missing in equation (10), (11) and Fig. 3 (d1).

In fact, Fig. 2 (3) shows the vertical slices of the secondary membership functions of the type-2 fuzzy set. When fuzzify upgrade operator acting on type-1 fuzzy sets, it will produce type-2 fuzzy sets shown in Fig. 4 (a). Type-2 fuzzy sets are characterized by fuzzy membership functions that have three dimensions shown in Fig. 4 (a). The uncertainty in the primary membership function of a type-2 fuzzy set, called the footprint of uncertainty [9]–[12], [14], [15], which is shown in Fig. 2 (2) as the green figures, are only projection of the membership functions of the type-2 fuzzy sets, which have none zero values. The whole three dimension plot of the type- 2 fuzzy set, which is shown in Fig. 2 (3), is shown in Fig. 4 (a).



(a) (b)

Figure 4. The three dimension representation of a type-2 fuzzy set and a type-1 fuzzy set.

The type-2 fuzzy set, which is shown in Fig. 3 (d2) shows the vertical slices of the secondary membership functions of the this type-2 fuzzy set. The general three dimension representation is shown in Fig. 4 (b). The values of the primary membership functions in this type-2 fuzzy set are exact value because there is no uncertainty. Two dimensions plot shown in Fig. 3 (d1), which is a plot of a type-1 fuzzy set, is enough to represent this type-2 fuzzy set shown in Fig. 3 (d2).

A crisp set, shown in Fig. 3 (a1), which has only one element, can be upgrade to type-1 fuzzy sets shown in Fig. 3 (a2) by the fuzzify upgrade operator. Similarly, crisp set shown in Fig. 3 (b1), which has more than one element, can be upgrade to type-1 fuzzy sets shown in Fig. 3 (b2). Crisp set, which have continuous elements, can be upgrade to type-1 fuzzy sets shown in Fig. 3 (c2). Type-1 fuzzy set, shown in Fig. 3 (d1) can be upgrade to type-2 fuzzy sets shown in Fig. 3 (d2).

Above analysis demonstrates that crisp set, which can be described as one dimension shown in Fig. 3 (a1), (b1) and (c1), can be represented in two dimensions shown in Fig. 3 (a2), (b2) and (c2). A type-1 fuzzy set, which is two dimensions shown in Fig. 3 (d1), can be represented as three dimensions as shown in Fig. 3 (d2) and Fig. 4 (b).

Similarly, a type-*i* (i = 0,1,2,..., n) fuzzy set can be upgraded to type-(i + 1) (i = 0,1,2,..., n) fuzzy set. When higher order uncertainty disappeared, highest membership function of higher order fuzzy set has certain one membership degree for each variable of its domain shown in Fig 3 (a2), (b2), (c2) and (d2); then the type-*i* (i = 0, 1, 2,..., n) fuzzy set can be represented as a type-(i-1) (i = 1, 2,..., n)

fuzzy set, and this process is the approach of degrading a fuzzy set. When the higher uncertainties of the type-2 fuzzy set shown in Fig. 4 (a) disappeared, there is no uncertainty in the value of the primary membership functions. The secondary membership function of this type-2 fuzzy set has certain one membership degree for the primary membership functions as shown in Fig. 4 (b). This type-2 fuzzy set shown in Fig. 4 (b) can be degraded to type-1 fuzzy set shown in Fig. 3 (d1).

IV. CONCLUSION

In this paper fuzzify upgrade operator are employed to analyze the relationship between the crisp sets, type-1 fuzzy sets, type-2 fuzzy sets and type-n fuzzy sets or higher order fuzzy sets. Theory analysis and case studies demonstrate that crisp set is a special case of type-1 fuzzy set and type-1 fuzzy set is a special case of type-2 fuzzy set. The results of this paper are consistent with the set theory that the fuzzy set is a generalization of the crisp set. From crisp set to first order fuzzy set and higher order fuzzy set, we can call them from type-0 fuzzy set to type-1 fuzzy set and type-n (n>1) fuzzy set. The relationship between the crisp sets, type-1 fuzzy sets, type-2 fuzzy sets and type-n (n>2) fuzzy sets is explored.

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