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Managing Swaption Portfolio Risk under Different Interest Rate Regimes

Poh Ling Neo*, Chyng Wen Tee†

Abstract

Efficient risk managing of swaption portfolios is crucial in the hedging of interest rate exposure. This paper formulates a portfolio risk management framework under stochastic volatility models. The implication of using the right volatility backbone in the stochastic-alpha-beta-rho (SABR) model is analyzed. In order to handle negative interest rates, we derive a displaced-diffusion stochastic volatility (DDSV) model with closed-form analytical expression for swaption pricing. We demonstrate that the dynamics naturally allow for negative rates, and is also able to fit the market well. Finally, we show that choosing the right backbone in the DDSV model results in optimal hedging performance and P&L explanation.

Keywords: *derivatives valuation, stochastic volatility models, interest rate markets, swaptions, risk management, portfolio management, pricing and hedging.*

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1 Introduction

Swaptions are the main interest rate volatility instrument in the fixed income market, and are traded in high volume between interdealers and institutional investors to hedge interest rate exposure, or to take on positions on swap curve movements. In addition to being the main instrument for interest rate risk management, they also form the basis for all volatility-sensitive interest rate product valuations, including Bermudan swaptions, callable LIBOR exotics, and constant maturity swap (CMS) payoffs, to name a few. Therefore, efficient risk management of swaption portfolio plays a crucial role across the whole spectrum of interest rate volatility products.

A key question swaption portfolio managers face is whether the forward swap rates follow a “normal” or a “lognormal” model. Several recent studies investigated this subject extensively. Levin (2004) explores the swaption market and demonstrates that swaptions with low strikes are traded with a close-to-normal volatility, while swaptions with higher strikes are traded with a square root volatility. CEV analysis unambiguously rejects lognormality and reveals a more suitable model. Deguillaume, Rebonato and Pogudin (2013) look at the dependence of the magnitude of rate moves on the level of rates, and find a universal relationship that holds across currencies and over a very extended period of time (almost 50 years). Interestingly, they found that volatilities of very low and very high rates behave in a lognormal fashion, while intermediate rates exhibit normal behavior. The results are robust across currencies, tenors and time periods. More recently, Meucci and Loregian (2016) show that US Treasury (UST) yields and Japanese Government Bond (JGP) yields are neither normal nor lognormal. Using the “shadow rate” concept introduced by Black (1995), they develop an “inverse-call” method to convert observable interest rates into shadow rates. They then show that these shadow rates has superior quality from a risk management perspective, in that the behavior is consistent whether rates are low or high.

In the fixed-income market, the stochastic alpha-beta-rho (SABR) model proposed by Hagan et al. (2002) is the de facto model used for swaptions pricing. Compared to other stochastic volatility models (say, for instance, the Heston (1993) model), the main advantage of SABR model lies in its ability to express implied volatility as a closed-form analytical formula, allowing swaptions to be priced in a quick and efficient manner. Being able to value swaption portfolio efficiently using analytical formula is important, as swaptions are used as the basis to price more exotic products. For instance, pricing CMS payoffs involve a 1-d integral across a continuum of weighted swaptions (see Brigo and Mercurio (2006) and Andersen and Piterbarg (2010) for more information). Having an

analytical expression for the swaption prices significantly speed up the pricing speed of exotic payoffs.

The performance of SABR model has been investigated extensively in the literature. Wu (2012) explore the application of SABR to the interest rate cap market. The study concludes that SABR model exhibits excellent pricing accuracy and captures the dynamics of the volatility smile over time very well. Separately, Yang, Fabozzi and Bianchi (2015) apply SABR model to the foreign exchange market. They use empirical study to show that SABR model can fit market option prices and predict volatility well. SABR model is also useful in analysis involving volatility risk premia. Duyvesteyn and de Zwart (2015) use SABR model to test and analyze the maturity effect in the volatility risk premium in swaption markets by looking at the returns of two long-short straddle strategies.

It has long been established that the swaption market follows neither normal or lognormal backbone, but something in between. Swaption portfolio managers heuristically set the value of β in their SABR model based on subjective perception of the prevailing backbone behavior, and calibrate the rest of the model parameters (α , ρ , and ν) to match the swaption prices observed in the market. In high interest rate environments, traders tend to assume the rate is closer to lognormally distributed ($\beta \rightarrow 1$), while in low interest rate environments, the rate is closer to be normally distributed ($\beta \rightarrow 0$). Fixing a constant beta, or equivalently fixing a constant assumption on the underlying distribution, such as a normal or lognormal, cannot fully capture market risk.

Given the wide-spread use of SABR model in the risk management of interest rate derivatives, Zhang and Fabozzi (2016) investigate the issue of choosing the right backbone under SABR model, and how an optimal choice of beta leads to superior hedging performance by minimizing pricing error. The key to the proposed method is that the option pricing model parameters not only can be estimated by calibrating the model to the cross-sectional data, such as the implied volatility smile, but also can be estimated by choosing the set of parameters that minimize the hedging error. The proposed method meets the no-arbitrage condition, delivering better hedging performance than the existing fixed-beta style calibration method. The most important discovery reported in this article is the often-overlooked fact that although the beta parameter in the SABR model does not have a major impact on the fit of the model to market data, it does play a critical role in controlling the model's hedging performance.

Building on the insights and findings of Zhang and Fabozzi (2016), we further develop the concept of optimal hedging performance. SABR model is known to be able to fit market prices extremely well. If the model parameters (α , ρ , and ν) are calibrated daily to the market, the fitting error is expected to be negligible. From a risk management perspective, the daily profit and loss (P&L) of holding a

swaption portfolio from one day to the next can be explained by contribution from interest rate delta, vega, and skew/smile sensitivity (ρ and ν). In the context of hedging, optimal performance is attained when the majority of the daily P&L movement can be explained by delta, followed by vega, with significantly smaller contribution from skew and smile movement, barring genuine movement in the volatility market. The underlying assumption is that given the right backbone choice, the bulk of the P&L should be attributed to rates movement, followed by volatility movement. Changes in skew and smile should be slowly varying.

It is important to note that the process in SABR model specified for the forward swap rate follows a constant elasticity of variance (CEV) process first proposed by Cox and Ross (1976) (see Cox (1996) for further information). However, unless we are explicitly setting $\beta = 0$, the model cannot support negative rates. Practitioners circumvent this problem by either using a normal SABR model with $\beta = 0$, or a shifted SABR model that moves the rates (and strikes) up by a pre-determined fixed positive amount. Apart from these off-the-cuff fixes, more sophisticated solutions have also been recently proposed. For instance, Anthonov, Konikov and Spector (2015a) formulated a free boundary SABR model by providing a structure to remove the negative rates boundary, making it flexible in terms of calibration to market data. Anthonov, Konikov and Spector (2015b) also propose a method to handle negative rates by mixing 0-correlation free boundary SABR model with a normal SABR. Nevertheless, these models are known to be unstable in the calibration process.

A good alternative model to use is the displaced-diffusion model first proposed by Rubinstein (1983). This parameterization can be interpreted as a simple linearization of the CEV dynamics around the initial value of the underlying. Similar to the CEV model, a displaced-diffusion model implies that the forward rate behaves more like a normal distribution when rates are low, and vice versa. Unlike CEV model, negative rates are admissible in a displaced-diffusion model. This coincides with the recent observation that interest rates have not only been negative but distributed more like a normal distribution. In fact, Marris (1999) shows that there exists a close correspondence between the CEV and the displaced-diffusion dynamics, and that, once the two models are suitably calibrated, the resulting interest rate caplet prices are virtually indistinguishable over a wide range of strikes and maturities. Joshi and Rebonato (2003) therefore use the displaced diffusion setting, which, unlike the CEV case, allows simple closed-form solutions for the realization of the forward rates after a finite period of time, as a computationally simple and efficient substitute for the theoretically more pleasing CEV framework, which does not allow negative forward rates. In fact, Svoboda-Greenwood (2009) posited displaced-diffusion processes as suitable alternatives to a lognormal process in modelling the

dynamics of market variables such as stock prices and interest rates. The mathematical properties of a displaced diffusion model is rigorously investigated further in Lee and Wang (2012).

Observation in the recent negative interest rate regime in EUR shows us that zero rate did not become an absorbing barrier, contrary to the behavior of a CEV process with $\beta \in (0, 1)$. On the other hand, rates did become negative, but there appears to be a lower bound as to how negative it can be, which is controlled by the European Central Bank (ECB). These are consistent with the behavior of a displaced-diffusion dynamics, as opposed to a normal model which a lower bound. Recent use cases of displaced-diffusion model include Chen, Hsieh and Huang (2018) to resolve severe problems of the existing Libor Market Model (LMM) that has failed since 2008 crisis.

In this paper, we formulate a displaced-diffusion stochastic volatility model for efficient swaption valuation, which is able to match market quotes well in both positive and negative interest rate regimes alike. We also introduce the concept of optimal hedging performance, measured by the “concentration” of P&L breakdown. We show that choosing the right volatility backbone yields the best hedging performance. This paper is organized as follows: Section II presents the data used in this study, and documents the empirical analyses performed on the data set. To handle negative interest rate regime, a displaced-diffusion stochastic volatility model is derived in Section III. Next, a P&L explanation framework and hedging performance benchmark are formulated in Section IV, followed by our results on the hedging performance of the models. Finally, conclusions are drawn in Section V.

2 Data and Empirical Analyses

The swaption data used in this study is acquired from IHS Markit. The swaptions are denominated in EUR. IHS Markit collects market data quotes from all data vendors and subject the data to specifically designed checks before cleaning and collated them into aggregated data in daily frequency. The data used in this paper covers 5 full calendar years from 1-Oct-2012 through to 30-Sep-2017, with 1,305 trading days. The data on each day comprises of 20 expiries and 14 tenors, with 14 strikes available for each swaption chain (tenor-expiry pair), defined by their respective moneyness. Table I provides a quick summary of the market data used in our study.

Standard convention in the fixed-income market is to quote implied lognormal volatility based on the Black (1976) model in forward space (as opposed to Black and Scholes (1973)). However, as swap rates become lower, and eventually enter the negative regime, swaptions with negative strikes and

forward rates can no longer be quoted using the Black (1976) lognormal model. As a workaround, IHS Markit data also provide implied normal volatility quotes based on a Black (1976) normal mode, which support negative rates. On the other hand, ICAP, a major interest rate derivatives broker, continue showing implied lognormal volatility quotes, but started shifting the forward rates and strikes up by a pre-determined fixed amount in December 2012. Today, the shift amount is 3% for EUR.

Figure 1 plots the forward swap rates across the 5-year period included in this study. For economy of representation, only 4 liquid tenor-expiry pairs are plotted, though the same trend and behavior are observed across the entire data set. The important economic landmark events are also labeled in the figure. The ECB cut EUR rates to negative on 25-June-2014¹, and swap rate levels started falling after that. Although short expiries forward swap rates only became negative on 10-March-2015, strikes of short maturity swaptions have already become negative prior to that. From the figure, it is also obvious that the 5y period included in the study can be split into a “high” rate regime (prior to March 2015) and a “low” rate regime (post March 2015).

We measure volatility by plotting annualized standard deviation of daily increments vs the rate level. We can collect all daily rate increments and group them into quintiles, with each one corresponding to a specific range of rate level. After the data are collected, we calculate the standard deviation of each bucket, and then annualize them ($\times 10000 \times \sqrt{252}$). Figure 2 plots the standard deviation against forward swap rate quintiles, along with the number of observations in each bucket. From the figure, it is clear that as the swap rate levels increase, the standard deviations decrease. This observation is fully consistent with swaption market convention, where portfolio managers use a SABR model with β closer to 1 under high rates regime, but β closer to 0 under low rates regime. Again, for the sake of economy in presentation, we only plot 4 commonly traded expiry-tenor pairs, namely 5y10y, 10y10y, 20y20y, and 30y30y, but similar results is obtained for all expiry-tenor pairs in our data set.

Next, we also calculate skewness to explore asymmetry in daily swap rate movement. The forward swap rate levels are grouped into quartile, and Figure 3 plots the skewness against forward swap rate quartiles, along with the number of observations in each bucket. Skewness is negative under low rates regime, but positive under high rates regime. This shows that the rates movement distribution has a heavier right tail when rates are higher, but a heavier left tail when rates are lower. We also calculate

¹ECB first cut rate to negative to -0.10% to spur inflation, and further cut to -0.20% in September 2014. Deposit facility rate was cut to -0.30% in December 2015 to boost low inflation, and further cut to -0.40% in March 2016.

the excess kurtosis of the daily swap rate movement. The forward swap rate levels are again grouped into quartile, and Figure 4 plots the excess kurtosis against the forward swap rate level. The excess kurtosis are all positive, highlighting the fact that the distribution of daily changes in swap rate has a heavier tail than normal distribution. However, the tails are heavier under low rates regime, and relatively lighter under high rates regime.

As explained earlier, prior to the negative interest rate regime, the swaption market's convention is to quote prices in terms of implied volatilities of the Black76 lognormal model. This convention has changed as negative rates are inadmissible under lognormal models. A commonly adopted convention is to quote the implied volatilities of a Black76 normal model instead. Since a normal model for rate movement allows for negative rates, this quoting convention is able to provide consistent price quotes without having to ensure that the shift amount is sufficient to guarantee positive forward swap rates and strikes. Figure 5 plots the implied lognormal volatilities (top panel) and the implied normal volatilities (bottom panel) against forward swap rates. In the top panel, it is obvious that as rates become lower, a high lognormal implied volatility is required to match the market price. On the other hand, when rates are higher, the lognormal implied volatility required to match the market swaption price is lower. This figure is a clear and visual indication that the backbone of the swaption market is not lognormal – high rates are associated with lower lognormal volatilities, while low rates are associated with higher lognormal volatilities. Compared to the lognormal volatilities, the normal implied volatilities in the bottom panel is relatively flatter. This shows that the backbone of the swaption market is closer to the normal model.

Next, we use principal component analysis (PCA) to investigate the factor structures of daily changes in implied volatility curves of the swaption chains. PCA are generally sensitive to the units in which the underlying variables are measured. It is customary, therefore, to standardize variables to unit variances, or equivalently to extract the eigenvalues and eigenvectors from the correlation matrix. Figure 6 plots the first, second and third principal components before and after the negative rate regime, using March 2015 as the split. Similar to the common case of yield curve analysis, the first principal component (PC) captures parallel implied volatility curve movement, the second PC captures the change in volatility skew (asymmetric slope movement), while the third PC accounts for the variation in implied volatility smile (symmetric curvature movement). The explanatory power of each PC is measured by the magnitude of the eigenvalues. The explained variance ratio of each PC is labeled in the figure. Prior to the negative rate regime, the first 3 PCs collectively account for 98.85% of the implied volatility curve movement. Once we entered the negative rate regime, the curve

movement is dominated by parallel movement, with the first PC alone accounting for 98.1% of the variance.

Statistical decomposition of the implied volatility curve’s daily changes demonstrates that there are three principal factors explaining a majority of the variation: level, slope, and curvature. Generally speaking, the level of the curve is typically anchored by the at-the-money volatility (ATM), the slope is the difference between highest strike (+300bps) and lowest strike (−300bps) volatility, and curvature as the ATM volatility relative to an average of highest strike and lowest strike volatilities. As a sensitivity check, we present our estimates for each of the implied volatility smile latent factor together with the corresponding empirical proxies directly computable from market quotes in Figure 7:

$$\begin{aligned} \text{Empirical Skew Proxy} &= \sigma_{\text{ATM}+300\text{bps}} - \sigma_{\text{ATM}-300\text{bps}} \\ \text{Empirical Smile Proxy} &= -2 \times \sigma_{\text{ATM}} + \sigma_{\text{ATM}-200\text{bps}} + \sigma_{\text{ATM}+200\text{bps}} \text{ Black Volatility} \end{aligned}$$

We borrow the concept of proxy “empirical slope” and “empirical curvature” from the yield curve literature to provide a model-free approach to quantify skew (asymmetric) and smile (symmetric) in the implied volatilities. We use 2%-shifted implied lognormal volatility quotes in this figure, but the same behavior is observed for other shift values. When forward swap rates are low, empirical skew is negative and empirical smile is high. When rates become higher, empirical skew becomes closer to 0 and eventually positive, while empirical smile is relatively smaller. This observation is consistent with the skew and excess kurtosis calculation presented in earlier figures.

3 Model

3.1 Volatility Backbone

Let F_t denote the forward swap rate. Under a model following normal distribution, the volatility of interest rate movements over time is independent of the interest rate level. This can be expressed as the following stochastic differential equation:

$$dF_t = \sigma_n dW_t \quad \Rightarrow \quad F_t = F_0 + \sigma_n W_t,$$

where σ_n is the normal volatility of the swap rate, and W_t is a standard Brownian motion with the distribution $W_t \sim N(0, t)$. In words, the interest rate behaves like a random walk. In contrast, under a model following lognormal distribution, the volatility of interest rate movements over time is

proportional to the interest rate level, such that high rates are associated with high volatilities, and vice versa. This can be expressed in the following stochastic differential equation:

$$dF_t = \sigma_{ln} F_t dW_t \quad \Rightarrow \quad F_t = F_0 e^{-\frac{\sigma_{ln}^2 t}{2} + \sigma_{ln} W_t}.$$

where σ_{ln} is the lognormal volatility of the swap rate. In words, the log rates behave like a random walk. Whether forward rates follow a normal model, a lognormal model, or any model in between, has important implication in risk management. Consider a series of following implied volatility curves under different forward rates. As the forward rate F_t varies, the at-the-money implied volatility σ_{ATM} also varies. The curve traced out by the this ATM volatility as a function of the forward rate is referred to as the backbone. Because ATM swaptions are by far the most liquidly traded in the market, choosing the right backbone plays an important role in guaranteeing the stability of the hedged portfolio. Suppose $\beta = 1$, market follows a lognormal backbone, and any movement in the forward rates will result in the same implied lognormal volatility. On the other hand, if $\beta = 0$, market follows a normal backbone, and any movement in the forward rates will result in identical normal volatility. Comparing between the process

$$dF_t = \sigma_n dW_t = \sigma_{cev}^\beta dW_t = \sigma_{ln} F_t dW_t,$$

it should be clear that for normal volatility to remain unchanged when rates move, the implied lognormal volatility will decrease when rate moves up, and increase when rate moves down.

In this section, we provide an example to illustrate the importance of choosing the optimal backbone (β) from a risk management perspective. Suppose the market backbone is given by β_{mkt} , and that the implied volatilities in the market is plotted in Figure 8. A swaption portfolio manager uses SABR model with the right backbone (β_{mkt}) to calibrate to these quotes, and is able to match observed swaption prices with a high degree of accuracy. Suppose another swaption portfolio manager is using an incorrect backbone of β_{model} . This portfolio manager will still be able to calibrate to the market with a close match in prices, as denoted by the dashed red line in the figure. In other words, in terms of daily mark-to-market, whether or not the right backbone (β) is used, portfolio managers will always be able to match market prices closely as long as they recalibrate the model parameters frequently.

That said, the disadvantage of choosing the wrong backbone manifests when the portfolio man-

agers try to use the model for risk management, and to breakdown daily P&L in terms of sensitivity and market movement. Suppose the swap rate moved up overnight, and this is the only movement in the market (volatility market remains unchanged). For the portfolio manager using the right backbone value of $\beta = 0.7$, no changes in the SABR parameters (α , ρ , and ν) is required to match the swaption prices after the move – the P&L movement can be explained entirely by interest rate delta. On the other hand, the portfolio manager using the incorrect backbone value of $\beta = 1$ will have to recalibrate to the swaption market in order to match the market prices closely. Under the wrong backbone, the same amount of P&L movement will now have to be explained by delta, vega, and a combination of skew and smile sensitivity.

As a numerical example, suppose $\beta_{mkt} = 0.7$. Consider an out-of-the-money (OTM) receiver struck at 1.5%, and an out-of-the-money payer struck at 4%. As illustrated in Figure 8, suppose the swap rate moves up, without other changes in the volatility market, the portfolio manager risk managing this receiver swaption with the right backbone of $\beta = 0.7$ will be able to explain the P&L of the position by interest rate (IR) delta. On the other hand, the portfolio manager using the incorrect backbone of $\beta = 1$ will have to explain the same P&L movement via offsetting components in IR delta, IR vega, and skew sensitivities (ρ and ν).

In this simple example, it should be clear that also SABR model is always able to match market quotes well by frequent recalibration, the advantage of choosing the right volatility backbone becomes apparent in that efficient risk management and P&L explanation can be done in a more economical fashion.

3.2 Displaced-diffusion Stochastic Volatility Model

In this section, we propose a displaced-diffusion stochastic volatility model for swaption pricing. The key strength of the displaced-diffusion process lies in its ability to accommodate negative interest rates. We derive a closed-form analytical expression for swaption pricing, and show that it can also match market prices with a high degree of accuracy.

Consider the displaced-diffusion forward swap rate process as follows

$$dF_t = \sigma[\beta F_t + (1 - \beta)F_0]dW_t, \tag{1}$$

σ is the volatility, the β is the displaced-diffusion model parameter, and W_t is a standard Brownian motion with $W_t \sim N(0, t)$. For now, let us assume that σ is a deterministic constant. We will

generalize this to a stochastic volatility model in later part of this section. The process can also be written as

$$d\left(F_t + \frac{1-\beta}{\beta}F_0\right) = \sigma\beta\left(F_t + \frac{1-\beta}{\beta}F_0\right) dW_t.$$

Written in this way, it should be clear that with $\left(F_t + \frac{1-\beta}{\beta}F_0\right)$ modeled as a geometric Brownian process, it is strictly positive. In this case, as long as the β parameter is positive, the forward rate process F_t is now allowed to take on negative value, since the process is well-defined as long as $F_t + \frac{1-\beta}{\beta}F_0 > 0$. In other words, under the displaced-diffusion model, the forward rate process is allowed to be negative, so long as $F_t > \frac{\beta-1}{\beta}F_0$. When $F_0 > 0$, any choice of $0 < \beta < 1$ will provide a negative value as the lowerbound to the forward rate process. If $F_0 < 0$ however, then we can choose $\beta < 0$, which corresponds to a super-normal process, and still support negative rates. Solving the stochastic differential equation in Equation (1), we obtain

$$F_T = \frac{F_0}{\beta} \exp\left[-\frac{\beta^2\sigma^2T}{2} + \beta\sigma W_T\right] - \frac{1-\beta}{\beta}F_0. \quad (2)$$

To value a payer swaption with payoff $(F_T - K)^+$, we note that

$$\begin{aligned} \frac{F_0}{\beta} \exp\left[-\frac{\beta^2\sigma^2T}{2} + \beta\sigma\sqrt{T}x\right] - \frac{1-\beta}{\beta}F_0 &> K \\ \Rightarrow x &> \frac{\log\left(\frac{K + \frac{1-\beta}{\beta}F_0}{F_0/\beta}\right) + \frac{\beta^2\sigma^2T}{2}}{\beta\sigma\sqrt{T}} = x^* \end{aligned}$$

and so

$$\begin{aligned} V_p(0) &= A(0)\mathbb{E}[(F_T - K)^+] = \frac{A(0)}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(\frac{F_0}{\beta} \exp\left[-\frac{\beta^2\sigma^2T}{2} + \beta\sigma\sqrt{T}x\right] - \frac{1-\beta}{\beta}F_0 - K\right) e^{-\frac{x^2}{2}} dx \\ &= \frac{A(0)}{\sqrt{2\pi}} \int_{x^*}^{\infty} \left(F'_0 e^{-\frac{\sigma'^2T}{2} + \sigma'\sqrt{T}x} - K'\right) e^{-\frac{x^2}{2}} dx \\ &= A(0)\text{Black76LognormalCall}(F'_0, K', \sigma', T, \beta), \end{aligned}$$

where

$$K' = K + \frac{1-\beta}{\beta}F_0, \quad F'_0 = \frac{F_0}{\beta}, \quad \sigma' = \beta\sigma,$$

and $A(0) = \sum_{i=1}^N D(0, T_i)$ is the swap annuity, N is the total number of swap cashflows, and

$D(0, T)$ is a discount factor discounting cashflow from T to 0. In other words, if the volatility σ is deterministic, the displaced-diffusion model can be expressed in the closed-form expression of Black (1976) lognormal model by simply adjusting the parameters. Obviously, with a deterministic σ and a β parameter, we will only be able to fit to the implied volatility skew. A stochastic volatility extension is therefore required so that the volatility-of-volatility parameter can be used to calibrate to the volatility smile observed in the swaption market.

To this end, we propose the following stochastic variance model:

$$\begin{cases} dF_t = \sigma_t [\beta F_t + (1 - \beta)F_0] dW_t \\ dV_t = \nu V_t dZ_t \end{cases} \quad (3)$$

where W_t and Z_t are independent Brownian motions ($W_t \perp Z_t$), and $\sigma_t = \sqrt{V_t}$. Under this formulation, we model the stochastic variance as a lognormal process². Solving the displaced-diffusion process for F_t in Equation (3), we obtain

$$\begin{aligned} \log \left[\frac{\beta F_T - (1 - \beta)F_0}{F_0} \right] &= -\frac{\beta^2}{2} \int_0^T V_t dt + \beta \int_0^T \sqrt{V_t} dZ_t \\ \Rightarrow F_T &= \frac{F_0}{\beta} \exp \left[-\frac{\beta^2}{2} \int_0^T V_t dt + \beta \int_0^T \sqrt{V_t} dZ_t \right] - \frac{1 - \beta}{\beta} F_0, \end{aligned}$$

Next, we define the mean integrated variance (\bar{V}) as

$$\bar{V} = \frac{1}{T} \int_0^T V_t dt. \quad (4)$$

Conditional on this integrated variance \bar{V} , we have the distribution

$$\log \left[\frac{\beta F_T - (1 - \beta)F_0}{F_0} \right] \sim N \left(-\frac{\beta^2 \bar{V} T}{2}, \bar{V} T \right).$$

Let ψ denote the probability density function of the mean integrated variance \bar{V} in Equation (4), the swaption can be priced as

$$V_p = A(0) \int_0^\infty \int_0^\infty (F_T - K)^+ f(F_T | \bar{V}) dF_T \psi(\bar{V}) d\bar{V}$$

²A quick application of Itô's formula to $\sigma_t = f(V_t) = \sqrt{V_t}$ shows that the stochastic volatility σ_t follows the process $d\sigma_t = -\frac{1}{2}\nu^2\sigma_t dt + \frac{1}{2}\nu\sigma_t dZ_t$.

Under the assumptions that the forward rate movements are uncorrelated with the variance, then the probability density $f(F_T, \bar{V})$ can be written as

$$f(F_T, \bar{V}) = \psi(\bar{V})f(F_T|\bar{V}),$$

Now the expected value of the sum of the swaption payoffs over all forward rates contingent on a fixed mean integrated variance is equal to the displaced-diffusion formula, which has a closed-form expression Displaced-Diffusion($F_0, K, \bar{V}, T, \beta$), so that we can write

$$V_p = A(0) \int_0^\infty \text{Displaced-Diffusion}(S_0, K, \bar{V}, T, \beta) \psi(\bar{V}) d\bar{V}$$

In other words, since $\log \left[\frac{\beta F_T - (1-\beta)F_0}{F_0} \right]$ conditional on \bar{V} is normally distributed with known mean and variance (under the assumption that F_t and V_t are uncorrelated), the inner integral becomes the closed-form displaced diffusion formula. In words, the DDSV option price is the weighted sum over the displaced-diffusion formula for different integrated variance. This intuitively pleasing result is often called the “mixing” theorem and was first derived by Hull and White (1987).

It is impossible to obtain an analytical form of the distribution for \bar{V} . However, following Hull and White (1987), while the distribution of the integrated variance \bar{V} is unknown, its moments can be readily evaluated. The first three moments are given by:

$$\mathbb{E}[\bar{V}] = V_0, \quad \mathbb{E}[\bar{V}^2] = \frac{2(e^{\nu^2 T} - \nu^2 T - 1)}{\nu^4 T^2} V_0^2, \quad \mathbb{E}[\bar{V}^3] = \frac{e^{3\nu^2 T} - 9e^{\nu^2 T} + 6\nu^2 T + 8}{3\nu^6 T^3} V_0^3.$$

Using Taylor expansion, we expand the displaced diffusion formula around its expected value to obtain

$$\begin{aligned} V &= \int_0^\infty \text{Displaced-Diffusion}(F_0, K, \bar{V}, T, \beta) \psi(\bar{V}) d\bar{V} \\ &= \text{Displaced-Diffusion}(F_0, K, \sigma_0^2, T, \beta) \\ &\quad + \frac{1}{2} \frac{\partial^2 \text{Displaced-Diffusion}(F_0, K, \sigma_0^2, T, \beta)}{\partial \bar{V}^2} \left(\mathbb{E}[\bar{V}^2] - \mathbb{E}[\bar{V}]^2 \right) \\ &\quad + \frac{1}{6} \frac{\partial^3 \text{Displaced-Diffusion}(F_0, K, \sigma_0^2, T, \beta)}{\partial \bar{V}^3} \left(\mathbb{E}[\bar{V}^3] - 3\mathbb{E}[\bar{V}]\mathbb{E}[\bar{V}^2] - \mathbb{E}[\bar{V}]^2 - \mathbb{E}[\bar{V}]^3 \right) + \dots \end{aligned} \tag{5}$$

For sufficiently small values of ν , which is the case for most cases, the series converges very quickly. Higher accuracy can be attained by adding higher order corrections to the expansion series. Once calibrated to swaption market quotes, Equation (5) provides an alternative way for us to evaluate swaption prices using closed-form expression. The main advantage of our proposed model over SABR

model is that it can incorporate negative rates without any further tweak or adjustment, allowing it to be used consistently in both positive and negative interest rate regimes.

Figure 9 provides a comparison of SABR model and the DDSV model formulated in this paper. Both models are able to match observed swaption market quotes closely. For the sake of comparison, two dates are shown in this figure: the left panel show that during positive interest rate regime, both modes fit the market implied lognormal volatility quotes well. However, the right panel show that as we enter negative interest rate regime, SABR model is no longer able to calibrate due to negative rates and strikes. Practitioners get around this issue by shifting all rates and strikes up by 3% before calibrating the SABR model. On the other hand, the DDSV model can be directly calibrated to market prices without any further adjustment.

4 Analysis of Hedging Performance

This section provides an exposition on the hedging performance of the swaption pricing models in risk managing swaption portfolio. First, we describe how sensitivities to market movement (Greeks) are quantified, and how the daily dollar P&L can be expressed in a risk-related P&L explanation framework.

SABR model provide a closed-form expression for the Black volatility as a function of market and model parameters, i.e. $\sigma_{\text{SABR}}(\alpha(\sigma_{\text{ATM}}), F, K, \beta, \rho, \nu, T)$. At-the-money swaptions are very liquid, and must be repriced exactly. It is therefore common among practitioners for the α parameter to be fitted on-the-fly via a root solver to match the ATM volatility, rather than merely assigning more weights to the ATM swaption in the calibration process. Here, σ_{ATM} is the at-the-money volatility, marked according to a specific backbone (CEV beta). The value of a swaption is valued as

$$V(F, K, \sigma_{\text{SABR}}, T) = \text{Black76Formula}(F, K, \sigma_{\text{SABR}}, T)$$

As explained in previous sections, if the right volatility backbone is chosen, the bulk of the daily P&L movement can be captured by interest rate delta, with vega capturing actual changes in the volatility market. Further, skew and smile (ρ and ν sensitivities) are expected to be slowly varying.

The sensitivities of the SABR swaption prices are given by

$$\begin{aligned}
\text{IR Delta} = \Delta &= \frac{dV}{dF} = \frac{\partial V}{\partial F} + \frac{\partial V}{\partial \sigma_{\text{SABR}}} \cdot \frac{\partial \sigma_{\text{SABR}}}{\partial F} \\
\text{IR Vega} &= \frac{dV}{\sigma_{\text{ATM}}} = \frac{\partial V}{\partial \sigma_{\text{SABR}}} \cdot \frac{\partial \sigma_{\text{SABR}}}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \sigma_{\text{ATM}}} \\
\text{IR Skew} &= \frac{dV}{d\rho} = \frac{\partial V}{\partial \sigma_{\text{SABR}}} \cdot \frac{\partial \sigma_{\text{SABR}}}{\partial \rho} \\
\text{IR Smile} &= \frac{dV}{d\nu} = \frac{\partial V}{\partial \sigma_{\text{SABR}}} \cdot \frac{\partial \sigma_{\text{SABR}}}{\partial \nu}
\end{aligned}$$

Moving from one day to the next, suppose the SABR model parameters (α , ρ , and ν) are calibrated on both days, the dollar P&L of a swaption position from one day ($t - 1$) to the next (t) can be explained as

$$\begin{aligned}
\text{SABR P\&L Explanation}_t &= \frac{dV}{dF} \times (F_t - F_{t-1}) + \frac{dV}{d\sigma_{\text{ATM}}} \times (\sigma_{\text{ATM},t} - \sigma_{\text{ATM},t-1}) \\
&\quad + \frac{dV}{d\rho} \times (\rho_t - \rho_{t-1}) + \frac{dV}{d\nu} \times (\nu_t - \nu_{t-1})
\end{aligned} \tag{6}$$

The actual P&L, which can be readily calculated as the dollar price difference between the 2 days, is given by

$$V_t = V_{t-1} + \text{SABR P\&L Explanation}_t + \epsilon_t,$$

where ϵ_t is the residual difference that cannot be captured by the hedging and P&L explanation framework, which is expected to be negligible. Note that we can also include the theta (1-day time decay) in the framework, though the contribution of this is generally minimal.

On the other hand, for the DDSV model, given that the pricing formula provides prices directly, the derivatives (sensitivities) can be directly calculated:

$$\text{IR Delta} = \Delta = \frac{dV}{dF}, \quad \text{IR Vega} = \frac{dV}{d\sigma_{\text{ATM}}}, \quad \text{IR Skew} = \frac{dV}{d\beta}, \quad \text{IR Smile} = \frac{dV}{d\nu}.$$

And the daily P&L can be explained as

$$\begin{aligned}
\text{DDSV P\&L Explanation} &= \frac{dV}{dF} \times (F_t - F_{t-1}) + \frac{dV}{d\sigma_{\text{ATM}}} \times (\sigma_t - \sigma_{t-1}) \\
&\quad + \frac{dV}{d\beta} \times (\beta_t - \beta_{t-1}) + \frac{dV}{d\nu} \times (\nu_t - \nu_{t-1}).
\end{aligned} \tag{7}$$

Note that in the DDSV model, the β parameter is used to capture both volatility skew and backbone. In both Equation (6) and (7), the explanation is not expected to match exactly the actual dollar P&L. The residual is typically quantified as “unexplained” P&L, though an efficient model for risk management should be able to provide an accurate P&L breakdown with negligible ϵ_t . Figure 10 provides a comparison of hedging performance across different swaption pricing models. The top panel shows the contributions of each risk to the overall P&L breakdown, calculated as the absolute mean of each category. The bottom panel shows the dollar hedging error, defined as

$$\text{Dollar Hedging Err} = \left| \sum_k |ex_k| - |\text{P\&L}| \right|,$$

where ex_k are delta, vega, skew (ρ or β), and smile (ν) contribution to daily P&L movements. The optimal model should have the smallest hedging error. Our analysis show that the DDSV model with the right backbone provides the smallest hedging error among all models investigated.

In order to provide a metric to quantify the “concentration” or “fragmentation” of the hedging performance of the swaption pricing model in terms of P&L explanation, we borrow the concept of the Herfindahl-Hirschman index. Originally designed as a measure commonly used in the industrial organization literature to measure market concentration, this metric has since been adapted in other fields for similar measures. For instance, Madhavan (2012) uses a volume Herfindahl-Hirschman index definition to measure market fragmentation. Here, we define the hedging performance Herfindahl-Hirschman index as a measure of concentration in P&L breakdown:

$$H^h = \sum_k \left(\frac{|ex_k|}{\sum_k |ex_k|} \right)^2,$$

where ex_k carries similar meaning as the equation above. Note that unlike common definition, in the context of P&L explanation it is necessary to take the absolute value in order to prevent ignoring offsetting values of opposite signs. The Herfindahl-Hirschman index in our definition ranges from 0 to 1, which higher figures indicating higher concentration (less fragmentation) in P&L explanation, which is a more desirable characteristics. Figure 11 compares the Herfindahl-Hirschman index across different swaption pricing models. Again, we use a 3% shifted SABR model to handle negative rate regime, while no further adjustment is necessary for the DDSV model. Given the right choice of β , our calculations reveal that othe DDSV model is able to provide optimal hedging performance with highest amount of concentration in P&L breakdown.

5 Conclusions

The interest rate markets use OTC swaptions as the main interest rate volatility instrument. In addition to hedging interest rate risk, traders also use swaptions to gain exposure, or to structure more exotics products such as CMS payoffs, Bermudan swaptions, callable Libor exotics. Therefore, efficient risk management of swaptions portfolio impacts the whole spectrum of interest rate volatility products.

This paper focuses on the recent transition of interest rate regime from intermediate to negative, and the behavior of volatility of daily rate movement. PCA analysis reveals that before the negative interest rate regime (prior to March 2015), the first 3 PCs collectively account for 98.85% of the daily changes in implied volatility curve. After moving into the negative interest rate regime (post March 2015), the first PC alone (parallel shift) accounts for in excess for 98% of the daily implied volatility curve movement.

A closed-form analytical swaption pricing model capable handling negative rates in a consistent manner is essential for swaption portfolio managers. In this work, we propose a displaced-diffusion stochastic volatility model with closed-form expression. The displaced-diffusion dynamic is able to handle negative rates with a lower bound. We show that the model is able to fit the market quotes well, and is able to fit prices in negative interest rate regime without any further adjustment.

Building on the insights of Zhang and Fabozzi (2016), we set out a swaption portfolio risk management framework that accounts for variation in forward rates, implied volatilities, as well as the shape of the implied volatility curve (skew and smile). SABR model is widely known to be able to fit to market quotes extremely well, as long as the model parameters are calibrated frequently. When the right backbone is chosen, the bulk of the daily P&L should be explained by IR delta, followed by IR vega. Changes in skew and smile are expected to be slowly varying compared to rates movement. If the incorrect backbone is chosen, daily calibration of SABR parameters will still ensure that we fit the market well, and are able to capture the daily dollar P&L. However, the P&L breakdown will have offsetting contribution from IR delta and IR vega, and also contribution from the changes in ρ and ν in order to fit market prices. Given the right choice of volatility backbone, we show that the DDSV model has optimal P&L breakdown performance. Our results provide important insights for swaption portfolio manager in choosing the optimal model for risk management.

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Table I: Summary of Data Set

Expiries	20	1m, 2m, 3m, 6m, 9m, 1y, 18m, 2y, 3y, 4y, 5y, 6y, 7y, 8y, 9y, 10y, 15y, 20y, 25y, 30y
Tenors	15	1y, 2y, 3y, 4y, 5y, 6y, 7y, 8y, 9y, 10y, 15y, 20y, 25y, 30y
Moneyiness	15	ATM, $\pm 25bp$, $\pm 50bp$, $\pm 75bp$, $\pm 100bp$, $\pm 150bp$, $\pm 200bp$, $\pm 300bp$

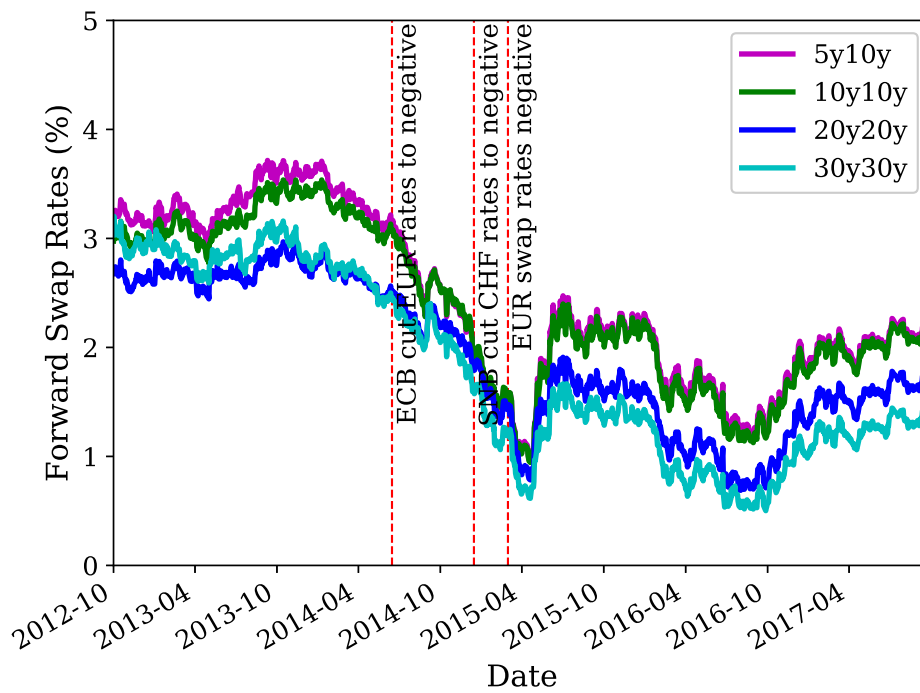


Figure 1: EUR forward swap rates.

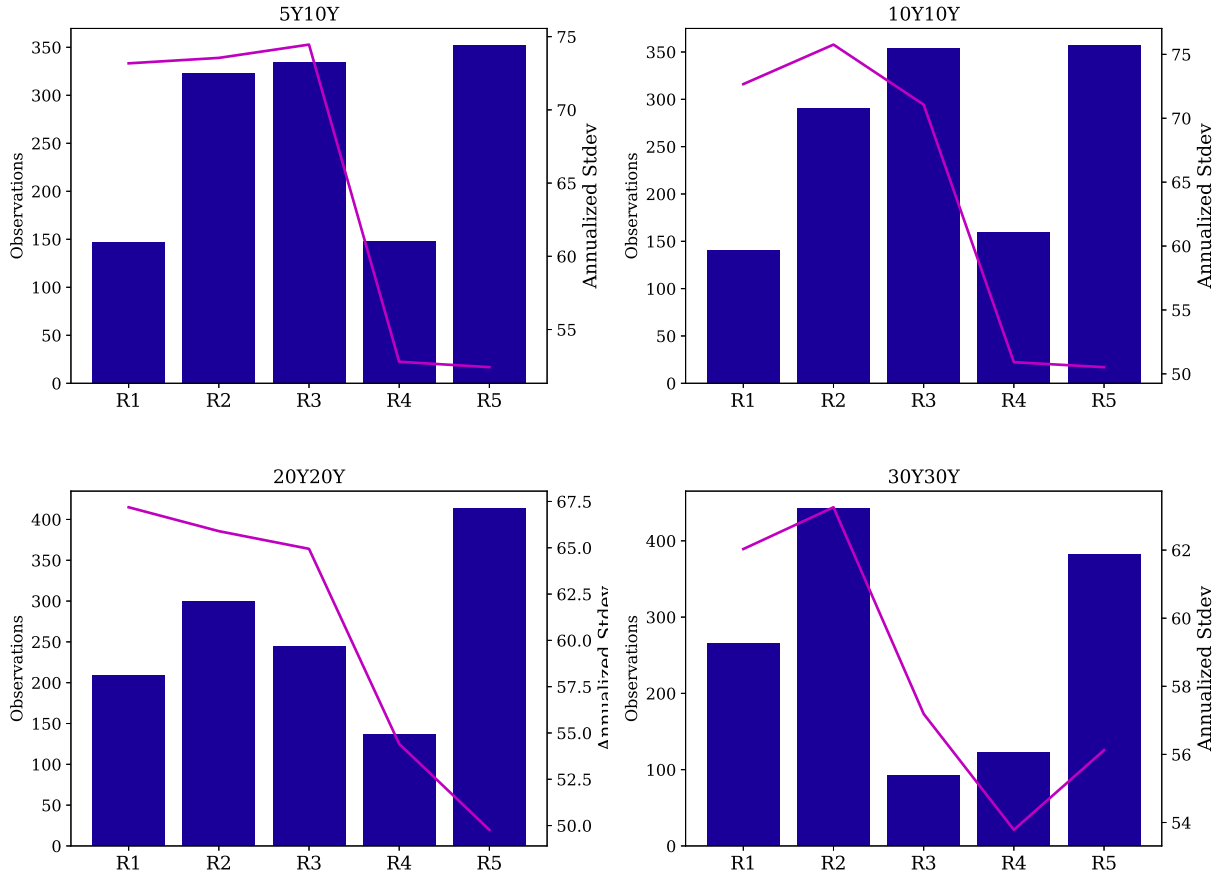


Figure 2: Standard deviation *vs* forward swap rate level. As rates increase, the standard deviations (volatilities) decrease. This is consistent with market observation — that $\beta = 1$ under high rates regime, and $\beta = 0$ under low rates regime.

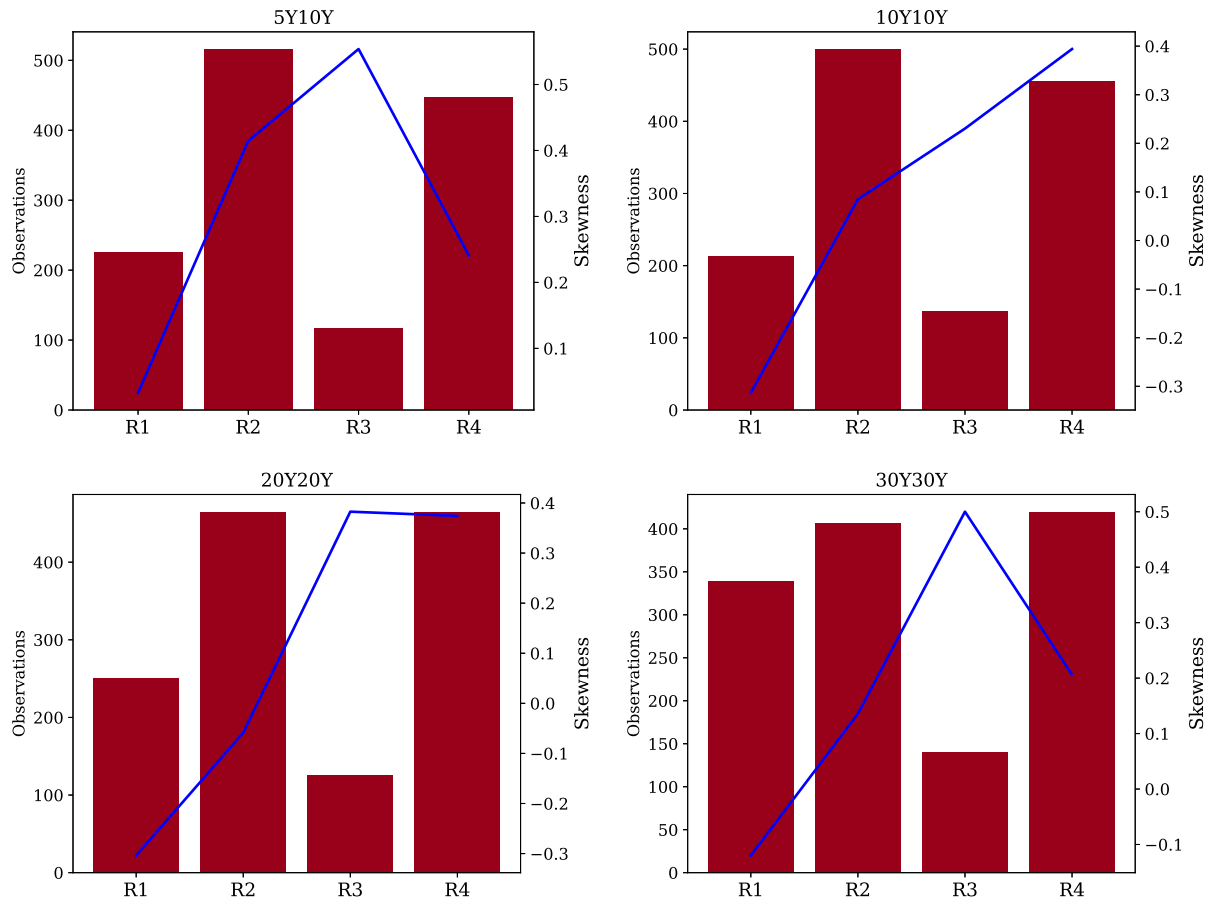


Figure 3: Skewness *vs* forward swap rate level. Empirical results show that as rates increase, the skewness also increases.

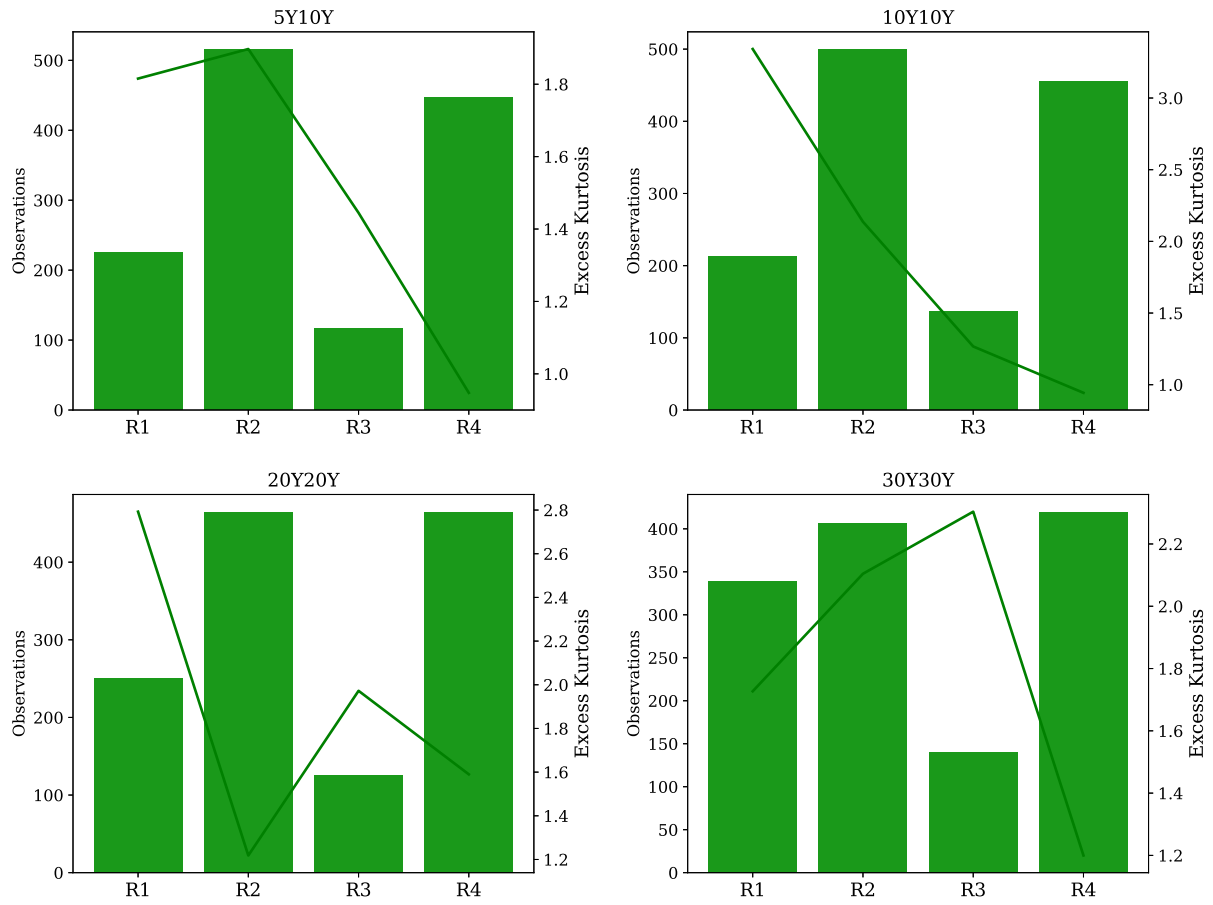


Figure 4: Excess kurtosis *vs* forward swap rate level. Empirical results show that as rates increase, the kurtosis decreases.

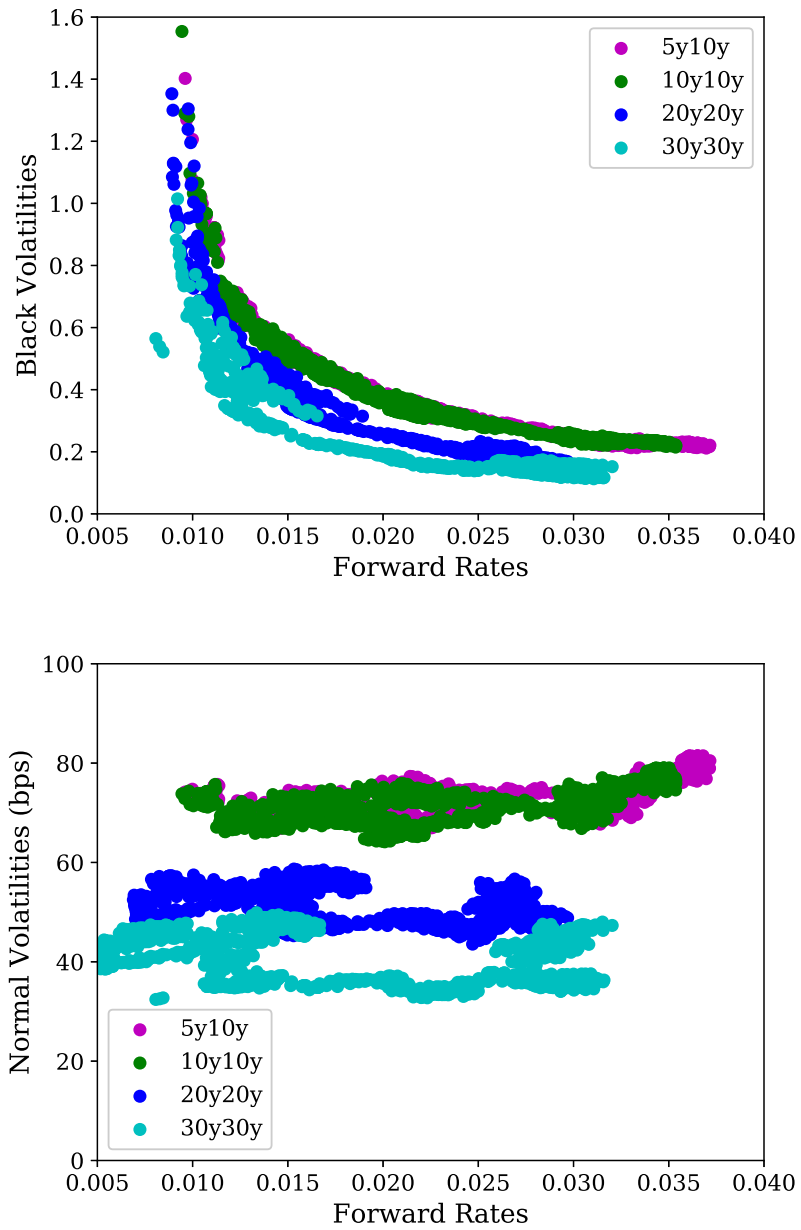


Figure 5: Comparison of Black implied volatility *vs* normal implied volatility, plotted against forward swap rate level.

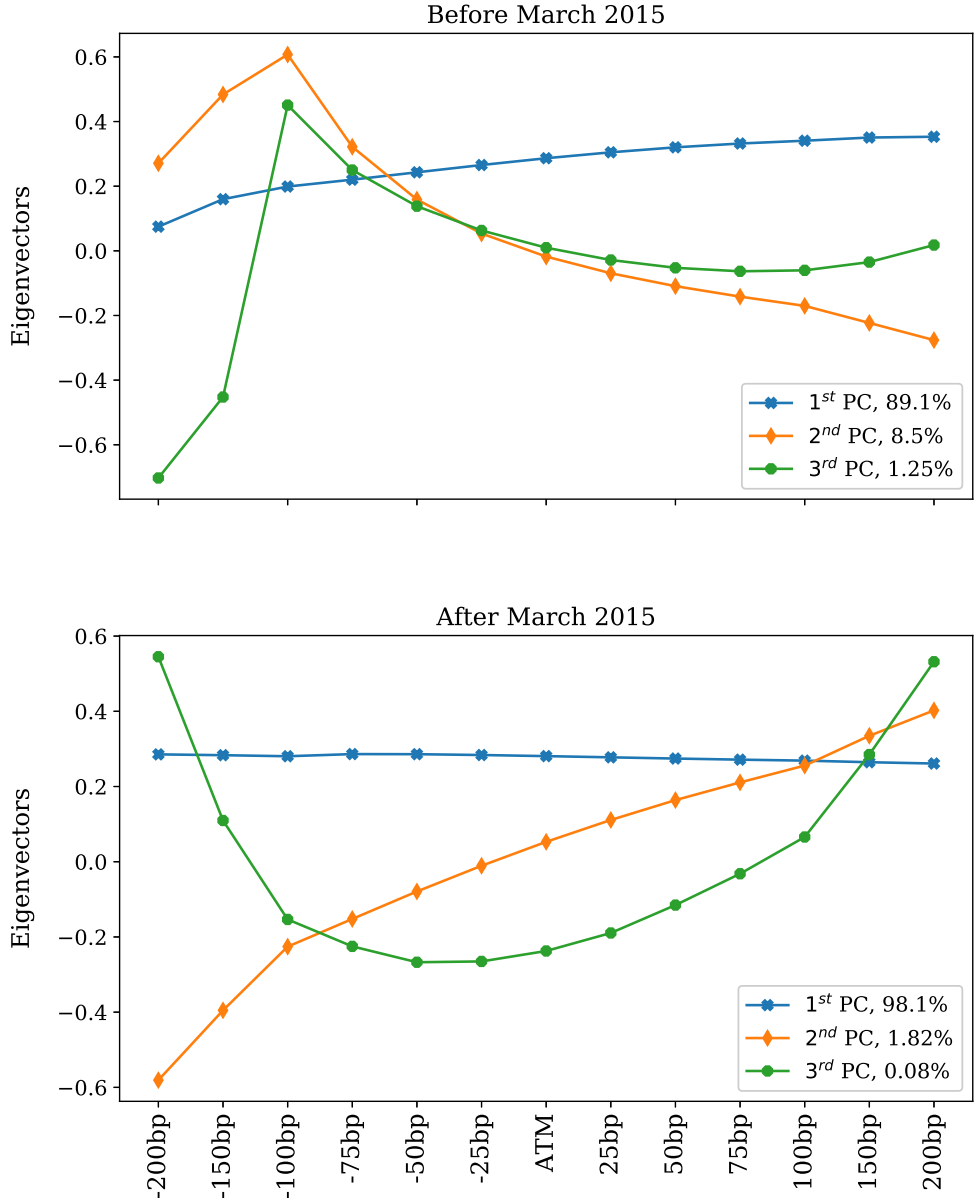


Figure 6: Principal components of daily changes in shifted (3%) lognormal implied volatility.

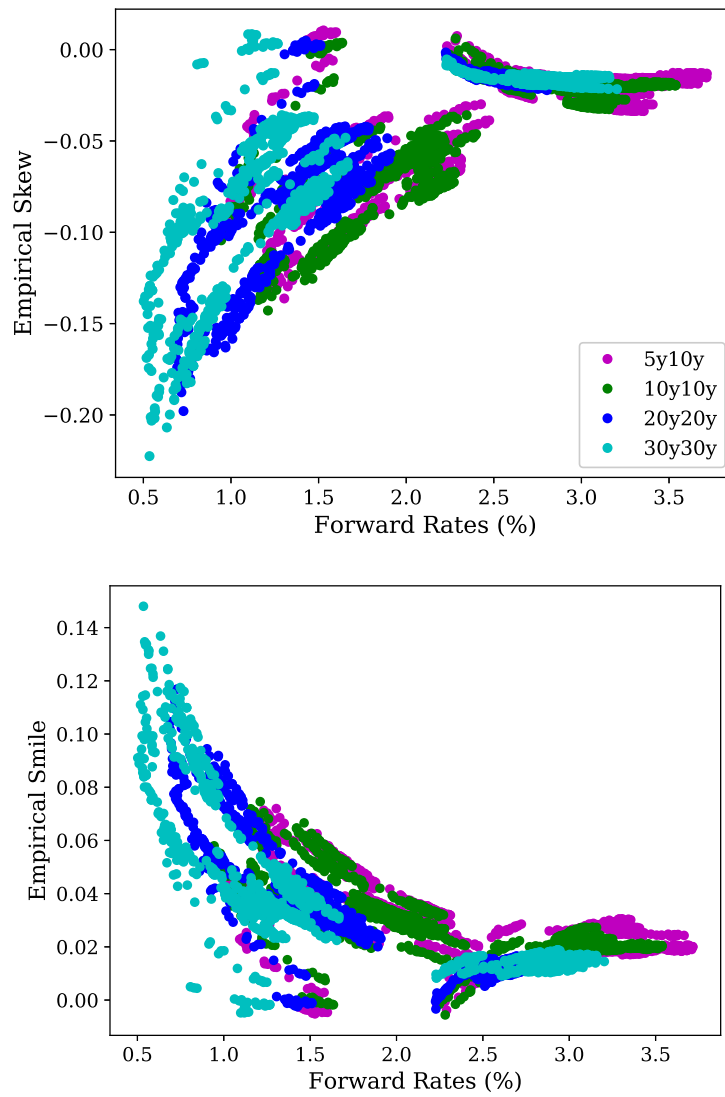


Figure 7: Empirical slope and curvature in implied volatility, plotted against forward swap rates.

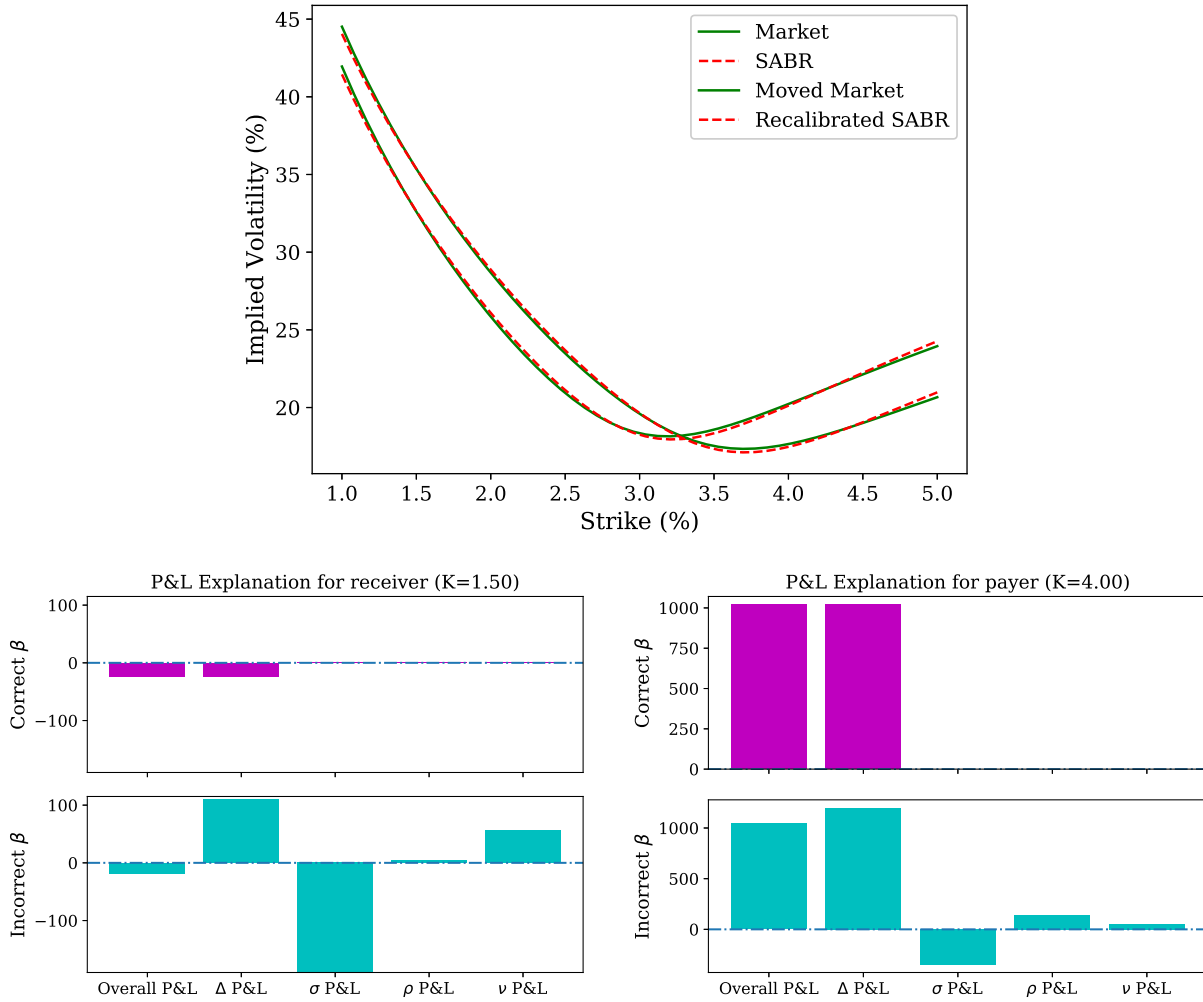


Figure 8: Assume $\beta_{mkt} = 0.7$. Suppose swap rate increases overnight, but the volatility outlook is the same. Using the right backbone, the P&L of swaption holding can be explained entirely by interest rate delta, and no recalibration of model parameters is required. If the wrong backbone is used, the portfolio manager will need to recalibrate α , ρ , and ν to match market prices again. The same P&L movement will also comprise of contribution from vega, ρ and ν sensitivity in addition to delta.

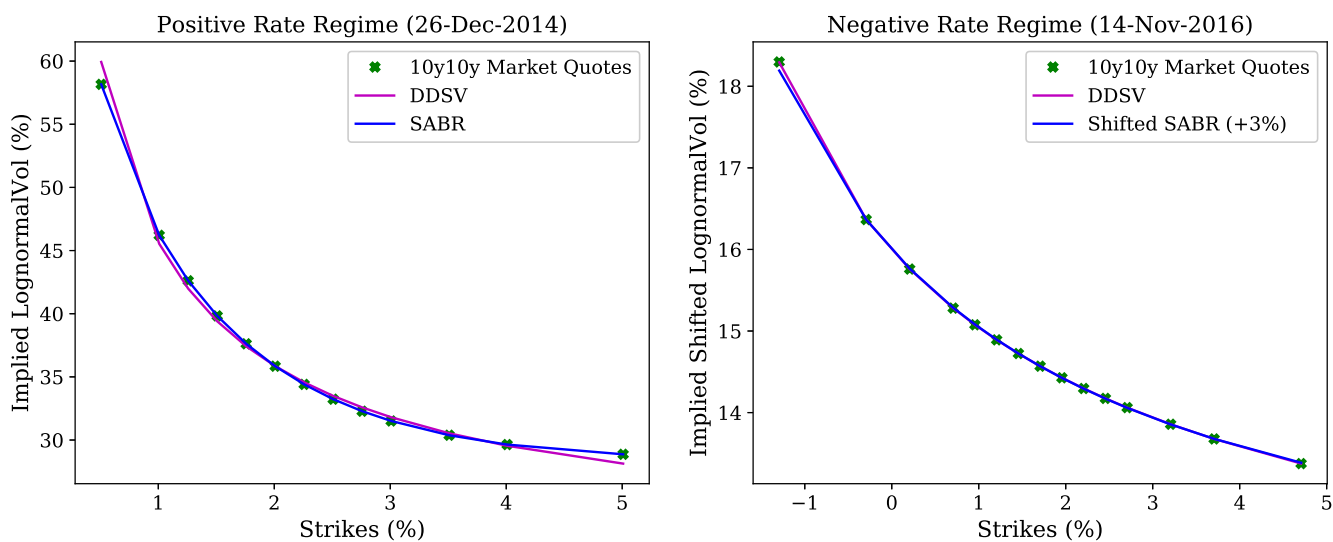


Figure 9: A comparison of SABR and DDSV models - both models are able to match observed swaption prices well once calibrated. We compare the fitting capability on a random date chosen from positive rate regime (left panel) and negative rate regime (right panel). SABR model can no longer be used once rates or strikes become negative. Market convention is to shift the rates (and strikes) up by a fixed amount before calibration. On the other hand, negative rates are admissible in the DDSV model, and it can be calibrated without any further tweaks or adjustments.

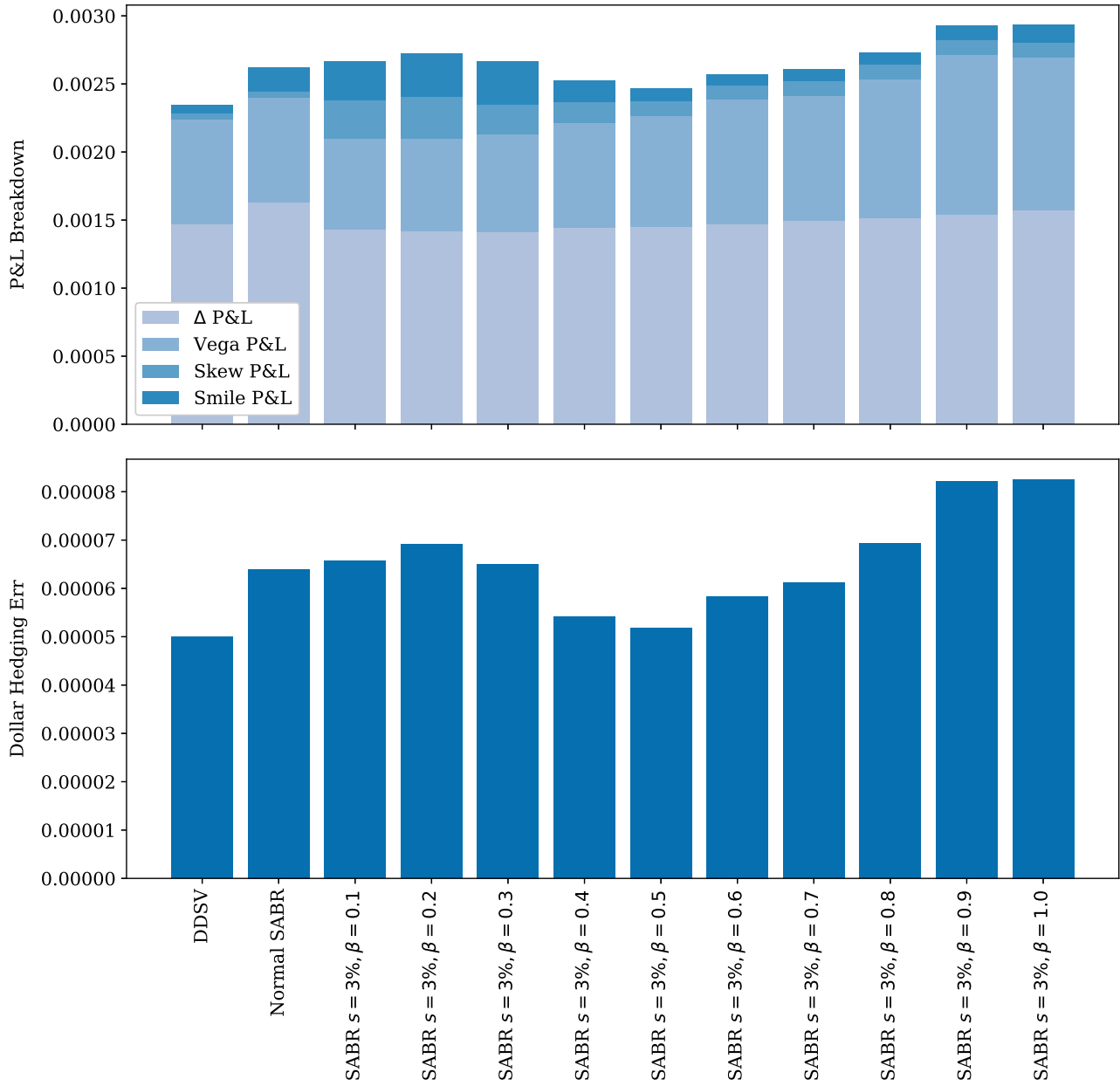


Figure 10: Comparison of hedging performance across different swaption pricing models. Given that the period studied includes negative interest rate regime, for SABR model we use a 3% shifted model in order to calibrate to swaption market prices. The DDSV model, with the correct backbone, yields the smallest hedging error and the highest concentration in P&L breakdown.

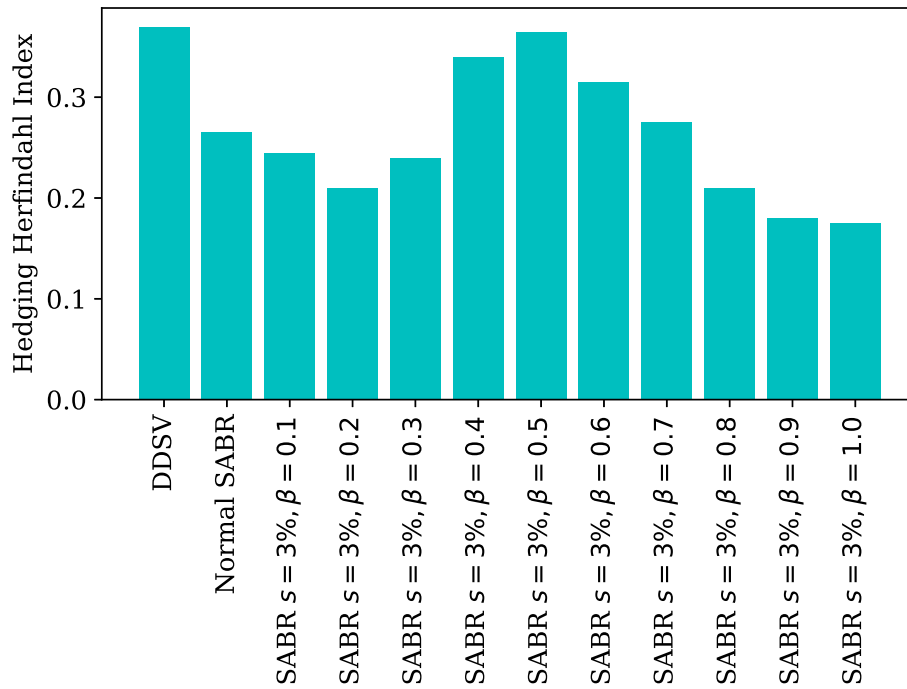


Figure 11: The hedging Herfindahl-Hirschman index of different swaption pricing models. The index measures “concentration” of risk in P&L explanation. The index ranges from 0 to 1, with higher values indicating higher concentration in P&L breakdown, which is more desirable. The DDSV with the right backbone is again shown to demonstrate superior hedging performance.