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Describing fuzzy sets using a new concept: fuzzify functor*

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This paper proposed a fuzzify functor as an extension of the concept of fuzzy sets. The fuzzify functor and the first-order operated fuzzy set are defined. From the theory analysis, it can be observed that when the fuzzify functor acts on a simple crisp set, we get the first order fuzzy set or type-1 fuzzy set. By operating the fuzzify functor on fuzzy sets, we get the higher order fuzzy sets or higher type fuzzy sets and their membership functions. Using the fuzzify functor we can exactly describe the type-1 fuzzy sets, type-2 fuzzy sets and higher type or higher order fuzzy sets. The fuzzify functor makes type-1, type-2 and any fuzzy sets much more accessible to all readers.

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Zadeh[1] introduced the concept of fuzzy sets in order to resemble human reasoning in the use of approximate information and uncertainty to generate decision in 1965. In 1975, he also originally introduced type-2 fuzzy sets[2], which can provide additional design degrees of freedom in Mamdani and TSK fuzzy logic system. So after 1975, it became common to refer to the pre-existing fuzzy set as type-1 fuzzy set [3]. Type-2 fuzzy sets can be very useful when such systems are used in situations where lots of uncertainties are presented [4,6]. There are many researchers engaged in fuzzy sets and their applications[5-8]. The application of type-1 fuzzy logic is discussed in reference[5,7], and literature[6] details the applications of the type-2 fuzzy sets. Both type-1 and type-2 fuzzy sets have been used for information processing, classification, control, decision making, forecasting, function approximation and so on.

In our previous work[8], type-1 fuzzy logic was combined with neural network to be employed for nonlinear network traffic predicting. Since both type-1 and type-2 fuzzy sets are powerful tool to deal with uncertainty[6, 7]. The motivation of this paper is to analyse the difference and relationship between them and make any fuzzy set much more accessible.

All fuzzy sets are characterized by membership functions. Type-1 fuzzy sets may be represented by a set of ordered pairs of generic element $x\in X$ and its grade of membership [6,7,9], such that

$$A=\{x,\mu(x)\,|\,x\in X\}\quad, \tag{1}$$

or

$$A=\int_{x\in X}\mu(x)/x\ . \tag{2}$$

In this equation the integral sign does not denote integration. It denotes the collection of all point $x\in X$ with the associated membership function, $\mu(x)$.

When X is discrete, A is commonly written as

$$A=\sum_{x\in X}\mu(x)/x\ \cdot \tag{3}$$

Σ sign should be interpreted in the same way as the integral sign given above, in which each element of the type-1 fuzzy set has membership grade that is a crisp number in [0, 1]. Imagine blurring the type-1 membership function depicted in Fig.1(a) by shifting the points on triangle either to the up or to the down and not necessarily by the same amounts, as in Fig.1(b), then, at a specific value of x, there is no longer a single value for the membership. And we call the union of all such allowable membership value as the footprint of uncertainty (FOU)[3,4,6]. This region can be drawn as Fig.1(c).Those values need to be weighted and be assigned a degree. Then, we get the type-2 fuzzy set, which are characterized by fuzzy membership that is three dimensions[6-10] shown in Fig.1(d).

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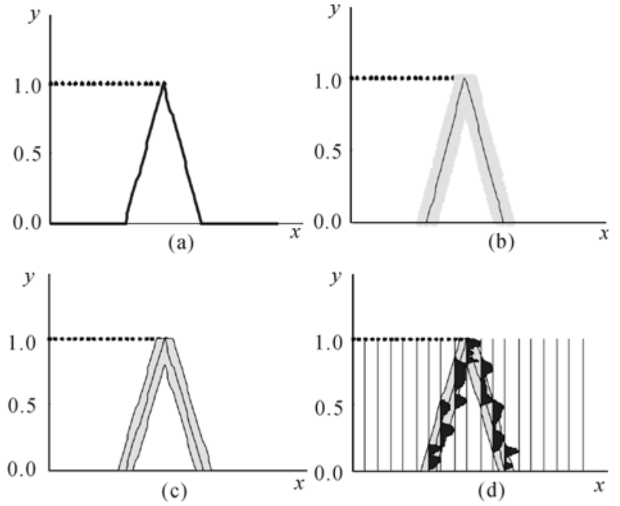

Fig.1 Type-1 fuzzy set, footprint of uncertain and type-2 fuzzy set. (a) The membership function of the fuzzy set A; (b) The membership function has been blurred; (c) Foot print of uncertainty (FOU); (d) The secondary membership functions (blue figures) that come out of the page in the third dimension.

This paper proposes a new concept, namely fuzzify functor, as an extension of the concepts of fuzzy sets. It can enrich the fuzzy sets theory. The fuzzify functor and the first order fuzzify functor operated fuzzy set, simply called first order operated fuzzy set, are defined. By the fuzzify functor on the fuzzy sets and the membership functions of the fuzzy sets, we get fuzzify operated fuzzy sets, which is higher order or higher type-fuzzy sets. Using the fuzzify functor we can certainly describe the type-1 fuzzy set, type-2 fuzzy set and any higher type fuzzy set. The concept of fuzzy functor supports the idea that a fuzzy set is a generalization of a crisp set[6, 7].

Considering when something is uncertainty, we have trouble in determining its exact value, so we use fuzzy sets. But then, even in the fuzzy sets, we specify the membership function exactly. When the membership function is fuzzy itself, we can use type-2 fuzzy sets. If we continue thinking along these lines, we can get type-n fuzzy sets and type-8 fuzzy sets[6].

However, we can consider there is a fuzzify functor acting on the fuzzy sets and make them become operated fuzzy sets or higher order fuzzy sets.

A fuzzify functor, denoted as $\widetilde{B}$, operates on constant variable or crisp number, which can be represented by a crisp set $A^0=\{v_0\}$, and it will produce first-order fuzzify functor operated fuzzy set A , and the membership function of the fuzzy set A, $A(x)$, which is a term of membership function defined for each x in a term of new domains in [0,1]. i.e,

$$A=\widetilde{B}A^0=\widetilde{B}\{v_0\}=\int_{x\in X}A(x)/x \ , \tag{4}$$

where the integral sign denotes the collection of all point, x , $x \in X$ is the primary variable, $A(x)$ is the primary membership function, which represent the grade of membership of each $x \in X$ in A.

$$A(x)=e^{\frac{-(x-v_0)}{c^2}} \ . \tag{5}$$

Although there are many methods to select the distribution of the blurred points, normal distribution, also called the Gaussian distribution, is employed here in this paper to explain how a crisp set becomes a type-1 fuzzy set when the fuzzy functor acts on it.

The definition of fuzzify functor is described above. The fuzzify functor has two functions: making fuzzy and assigning degree. Making fuzzy is to blur or to disturb the constant variables to be in range. Assigning degree is to give an amplitude distribution to all of those points. $A(x)$ represents the amplitude distribution.

The fuzzify functor is left-distributive over union or collection of crisp sets. It can be also said that fuzzify functor acts on crisp sets or crisp numbers left distributes over the union or collection of the crisp sets or crisp numbers.

$$A^0=\{v_0,v_2\}=\{v_0\}+\{v_1\}=A_1^0+A_2^0 \ , \tag{6}$$

$$\widetilde{B}(A^0)=\widetilde{B}(A_1^0+A_2^0)= \widetilde{B}(A_1^0)+\widetilde{B}(A_2^0)=\widetilde{B}\{v_0\}+\widetilde{B}\{v_1\} \ . \tag{7}$$

Example1: fuzzify functor operated on crisp number, $x = v_1$=5, which can be represented by crisp set $A^0= \{v_1\}=\{5\}$ shown in Fig.2 (a1) . First step, making fuzzy is to blur $x = v_1$=5 to be in range, such as x: *xnearto* 5 , which can be represented by crisp set $A^0= \{x|xnearto\ 5\}$ as shown in Fig.2 (a2). The second step, assigning degree is to give amplitude distribution, we can get the fuzzy number shown in Fig.2 (a3), which is a quantity whose value is imprecise, rather than exact as is the case with “ordinary” (single-valued) numbers[7].

According to the above, we get first order operated fuzzy set A shown in Fig.2 (a3), which is also a typical fuzzy set[6,7].

$$A=\widetilde{B}A^0=\widetilde{B}\{5\}=\int_{x\in X}A(x)/x= \int_{x\in X} e^{\frac{-(x-5)}{c^2}}/x \ . \tag{8}$$

Although there are many functions that can be selected as the membership function of fuzzy number and fuzzy set, Gaussian function is used here in the case analysis in this paper. Fig. 2 (a1-a3) shows the process that crisp number became fuzzy number and Fig.2 (b1-b3) shows the process that crisp set became fuzzy set. It is clearly, that crisp number can be represented by a specific crisp set, which have only one element and fuzzy number can be also represented by a specific fuzzy set, which have only one membership function.

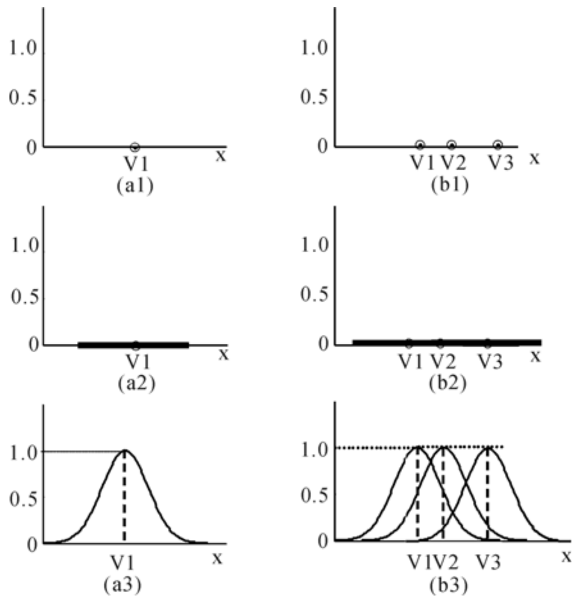

Fig.2 Fuzzy number and fuzzy sets produced when fuzziy functor acted on crisp number and crisp set.

A fuzzify functor, which operates on a fuzzy set A and the membership function of the fuzzy set A, $A(x)$, will produce first-order operated fuzzy set A^1, the membership function of the fuzzy set A^1, $\widetilde{B}(A(x))$, which is a term of new membership functions defined for each x in a term of new domains in [0,1], and produce an additional dimension. i.e.,

$$\begin{aligned} A^1 &= \widetilde{B}(\underline{A}) \\ &= \widetilde{B}\int_{x\in X}\underline{A}(x)/x \\ &= \int_{x\in X}\widetilde{B}(\underline{A}(x))/x \\ &= \int_{x\in X}\left[\int_{u\in D_x\subseteq[0,1]}A^1(x)(u)/u\right]/x \\ &= \int_{x\in X}\int_{u\in D_x\subseteq[0,1]}A^1(x)(u)/u/x \quad , \end{aligned} \tag{9}$$

where $x\in X$ is the primary variable and $A(x)$ is the primary membership function, which represent the grade of membership of each $x\in X$ in A; $u\in D_x\subseteq[0,1]$ is the secondary variable, $A^1(x)(u)\subseteq[0,1]$ is a new membership function, we call it the secondary membership function[4, 6].

$$A^1(x)(u) = e^{\frac{-(u-A(x))}{c_1^2}} \quad . \tag{10}$$

Eq.(8) represents the grade of membership of each $u\in D_x\subseteq[0,1]$ in $A(x)(u)$ for each $x\in X$. And the FOU[4,6] can be expressed as $FOU(A^1)=\bigcup_{x\in X}D_x$. A^1 is the first-order fuzzify operated fuzzy set of fuzzy set A, and $A^1(x)(u)$ is the secondary membership function of the first-order fuzzify operated fuzzy set A^1.

It is clearly, in eq.(3,4) if $A=A^0$ is a crisp set, first-order operated fuzzy set A^1 will be type-1 fuzzy set. If A is type-1 fuzzy set, first-order operated fuzzy set A^1 will be type-2 fuzzy set from equation (4) and (9).

A fuzzify functor is left-distributive over union or collection of fuzzy sets. It can be also said that fuzzify functor acts on fuzzy sets or fuzzy number left distributes over the union or collection of the fuzzy sets or fuzzy numbers.

Example 2: fuzzify functor operated on fuzzy set $A=\int_{x\in X}A(x)/x=\int_{x\in X}e^{\frac{-(x-5)}{c^2}}/x$, which is shown in Fig.3(a).

From the definition 2,

$$\begin{aligned} \widetilde{B}(\underline{A}) &= \widetilde{B}\int_{x\in X}\underline{A}(x)/x \\ &= \int_{x\in X}\widetilde{B}(\underline{A}(x))/x \\ &= \int_{x\in X}\left[\int_{u\in D_x\subseteq[0,1]}A^1(x)(u)/u\right]/x \\ &= \int_{x\in X}\int_{u\in D_x\subseteq[0,1]}A^1(x)(u)/u/x \\ &= \int_{x\in X}\int_{u\in D_x\subseteq[0,1]} e^{\frac{-(u-A(x))}{c_2}}/u/x \, . \end{aligned} \tag{11}$$

Fig.3 (a)-(d) and equation (11) show the process that type-1 fuzzy set became type-2 fuzzy set according to definition 2. If the value of the second membership function is to give the same degree, $A^1(x)(u)\equiv 1$, we can get the interval type-2 fuzzy set[6].

The content of the fuzzy set of fuzzy sets and details of this fuzzy functor will be discussed in another paper.The fuzziy functor can be proved that it can act on first-order operated fuzzy set and higher-order operated fuzzy set. Clearly, more works need to be done about the fuzzify functor. The theory and testify of fuzzy functor as well as applications of the higher order fuzzy set will be our future work too.

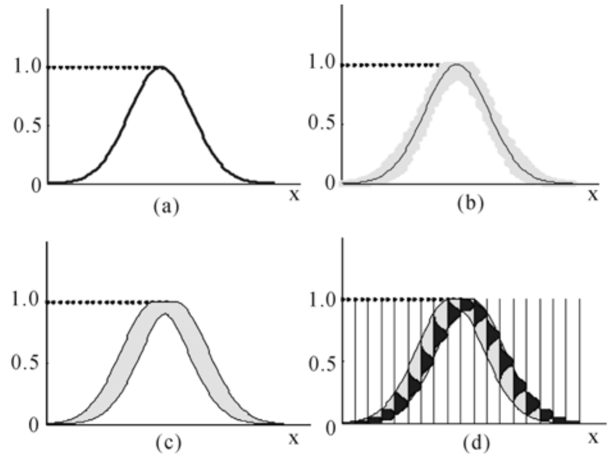

Fig.3 Type-2 fuzzy sets produced when fuzzify functor acted on type-1 fuzzy sets

This paper proposes the concept of the fuzzify functor. It can express higher order fuzzy sets simply and clearly, and make the theory fuzzy sets complete.

$\widetilde{B}$ is the simplest form of fuzzify functor. The fuzzify functor has two functions, which can be represented by two operators. It can be represented as $\widetilde{B} \Rightarrow \hat{B}^a \hat{B}^b$. $\hat{B}^b$ represents the blur operator or fuzzy operator and $\hat{B}^a$ denotes promotion operator or promote order operator, which assign an amplitude distribution to all of those blurred point. The use of fuzzify functor may enable fuzzy sets more accessible. The concepts of the fuzzify functor can describe the relationship of crisp sets, type-1 fuzzy sets and type-2 fuzzy sets and higher type fuzzy sets.

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