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Rank-guaranteed auctions

Wei HE

Jiangtao LI

Weijie ZHONG

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Wei He† Jiangtao Li‡ Weijie Zhong§

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Abstract

We propose a combinatorial ascending auction that is "approximately" optimal, requiring minimal rationality to achieve this level of optimality, and is robust to strategic and distributional uncertainties. Specifically, the auction is *rank-guaranteed*, meaning that for any menu $\mathcal{M}$ and any valuation profile, the ex-post revenue is guaranteed to be at least as high as the highest revenue achievable from feasible allocations, taking the $(|\mathcal{M}|+1)^{th}$-highest valuation for each bundle as the price. Our analysis highlights a crucial aspect of combinatorial auction design, namely, the design of menus. We provide simple and sufficient menus in various settings.

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†Department of Economics, The Chinese University of Hong Kong, hewei@cuhk.edu.hk

‡School of Economics, Singapore Management University, jtli@smu.edu.sg

§Graduate School of Business, Stanford University, weijie.zhong@stanford.edu

> "My expectation—and my hope—is that progress in mechanism design over the coming decades will come from developing more useful general models of preferences, information, and actions in complex environments, relatively free of structural assumptions; and developing conceptual tools to argue for why certain kinds of mechanisms will work well in such environments. [...] Making this progress toward more free-form models will require a shift in the criteria by which research in economic theory is evaluated. [...] But in any situation that remotely approaches the complexity of most real-world design problems, there is no hope of describing the exact optimum."
>
> Carroll (2019)

1 Introduction

The online advertising sector is a dynamic engine of economic activity, generating hundreds of billions of dollars annually through a simple mechanism: the auctioning of advertising "slots". Despite the critical role auctions play, there is a surprising lack of theoretical groundwork to navigate the intricacies of auction design. This gap in knowledge stems from a unique challenge: "*not all slots are created equal*"—bidders typically have complex, combinatorial preferences for different slots. For instance, YouTube intersperses promotional videos at regular intervals within longer content. Here, some advertisers might see value in the repetition of their ads, leveraging the complementarity, while others may fear overexposure could lead to negative perceptions akin to "spamming." Similarly, in the layout of Facebook's Marketplace, which presents content in two columns, preferences can vary widely. Some advertisers might seek to blend seamlessly with organic content by choosing single slots, whereas others might opt for an entire row to ensure their message is fully conveyed. Current methodologies oscillate between bespoke solutions designed for very specific preference frameworks—like the Generalized Second Price Auction (GSP) employed by Yahoo & Google, suited for singular demand and vertically differentiated slots—and more universal models not primarily aimed at revenue optimization, such as the Vickrey-Clarke-Groves auction (VCG) employed by Meta.

This reality starkly contrasts with traditional auction theory, which often simplifies to an extent that overlooks the nuanced realities of the markets. Iconic theories, such as Myerson (1981), typically focus on the allocation of a *single item*, assuming *independent valuations* among participants, along with both the auctioneer and bidders possessing fully *Bayesian rationality*, underpinned by accurate, *shared prior beliefs*. However, none of these assumptions hold water in the complex scenarios described earlier, highlighting a clear disconnect. The development of a comprehensive theory for maximizing revenue through auction design in these more intricate and generalized environments remains a significant, unmet challenge.

In this study, we address the quadrilemma in multi-item auction design, achieving a *near-optimal* resolution: it is possible to simultaneously attain four pivotal objectives—approximate optimality, minimal rationality on the part of the bidders, strategic robustness, and distributional robustness—through a simple auction mechanism. Our focus is on scenarios where an auctioneer sells multiple items to several (potentially a large number of) strategic bidders, each with private valuations. We introduce a multi-item variant of the open ascending auction termed the (C)ombinatorial (As)cending (A)uction (CASA). Prior to the auction, the auctioneer curates a *menu* of item bundles for allocation. With the formal game theoretic form of the auction described in Section 2, this auction model distills down to two straightforward principles:

1. Bidders are allowed to place binding bids (increase prices) on any assortment of bundles from the menu, even if these selections overlap.
2. The auction concludes when bid prices stabilize, with the winning bids being those that maximize the total selling price.

Our findings reveal that CASA meets the quadrilemma's four criteria within certain approximate bounds:

- **Minimal rationality**: For all our results, we only need to assume that bidders are rational in the sense of avoiding strategies that are obviously dominated. This is an extremely weak assumption on rationality.
- **Approximate optimality**: Any non-obviously dominated strategy profile yields an *ex-post* revenue that is *rank-guaranteed* — achieving the maximal revenue when each bundle within menu can be sold at the k^{th}-highest value among all bidders.
- **Distributional robustness**: Neither the auction format nor the revenue guarantee depends on a Bayesian prior on either the auctioneer's side or the bidders' side.
- **Strategic robustness**: CASA is rank-guaranteed even with irrational or collusive bidders.

To our knowledge, this paper is the first to systematically study the rank-guarantee approximation property. The rank-guarantee is a highly desirable feature. Clearly, when the values are independent and identically distributed, then rank-guarantee is an appealing approximation when N is large, as all order statistics converge to the upper bound of the valuation support. We also quantify the rank-guarantee using canonical robust optimality criteria, performing a worst-case analysis against distributional uncertainties, i.e., an adversarial nature choosing the joint distribution of values to minimize expected revenue against the mechanism (see for example Carroll (2017)). We show that in the worst case, rank-guarantee remains an appealing approximation. Under a menu $\mathcal{M}$, the expected k^{th}-guarantee approximates the total surplus (the 1^{st}-guarantee) at the rate of $O\left(\frac{k|\mathcal{M}|}{N}\right)$, i.e., the k^{th}-guarantee asymptotically achieves full surplus extraction when the number of bidders is large relative to the menu size $|\mathcal{M}|$. We prove this by developing a novel statistical result that bounds the k^{th} largest order statistic of a given sample using a random element.

The rank-guarantee offers the added benefit of being easy to evaluate across various Bayesian and non-Bayesian models. Unlike other familiar approximation notions, such as maxmin guarantee or constant fraction approximation, the rank-guarantee criterion is more informative in the following sense. It provides an easily computable lower bound on revenue, even when the underlying environment changes, such as when there are more bidders or when the auctioneer has additional information on the distribution of bidders' valuations. This lower bound performance is also straightforward to assess outside adversarial scenarios, which is crucial for approaches like the maxmin guarantee or fractional approximation. For instance, beyond understanding the worst-case scenario of the rank-guarantee, it can be useful to consider its potential upside in best-case scenarios.

Our framework highlights a crucial aspect of combinatorial auction design, namely, the design of menus. Crucially, the k^{th}-guarantee we derive reveals a novel trade-off between *menu sufficiency* and *approximation efficiency*: a more complete menu achieves a higher benchmark total surplus but increases k and $|\mathcal{M}|$. Therefore, to close the approximation gap towards the various goals, a key exercise is to reduce the menu size while maintaining the allocation efficiency, leveraging further knowledge about the bidders' preferences. We focus on a specific type of *sufficient* menus that improves approximation efficiency "for free"—menus that achieve the same worst-case total surplus as the complete menu. Specifically, we show that when the bidder's preference exhibits canonical preference structures, without loss of the benchmark total surplus, the size of menus can be reduced to be polynomial in the number of items being auctioned and so is the convergence rate of revenue guarantee. The result is summarized in Table 1.

Preference	Simple and Sufficient Menu	k
Weak substitutability	Individual items	$O(M)$
Weak complementarity	Grand bundle	2
Partitional complementarity	Partitional bundles	$O(M)$
Homogeneous goods	Menu of quantities	$O(M^2)$

Table 1: Simple and sufficient menus

The remainder of the introduction reviews related literature. Section 2 introduces the auction format of CASA and the notion of rank-guarantee, and shows that CASA achieves the rank-guarantee. Section 3 bounds the worst-case performance of rank-guaranteed auctions under distributional uncertainties. Section 4 explores specific preference structures where CASA with simple menus performs as well as the complete menu.

1.1 Related literature

(Approximately) optimal auction design Beyond the simple environment studied in Myerson (1981) and Bulow and Klemperer (1996), solving for the exact optimal mechanism with confounding factors like multiple heterogeneous items, bounded distributional knowledge or bounded rationality is generally intractable. Various alternative optimality notions have been proposed to make progress (see surveys by Roughgarden (2015) and Hartline (2013)). Aggarwal and Hartline (2006) and Goldberg and Hartline (2001) obtained the "constant fraction" approximation in the auction of sponsored search and digital goods. Following a broader literature on robust mechanism design pioneered by Carroll (2017), various authors have studied "robustly" optimal auctions that maximize the distributional worst-case revenue.[1] Particularly, He and Li (2022), Zhang (2021), and Suzdaltsev (2022) study robust versions of the single-unit auction problem in the distribution robust framework where the auctioneer has non-Bayesian uncertainty about the joint distribution of the bidders' valuations. In this paper, we propose the notion of rank-guarantee, a distinct notion of approximate optimality. In Section 3, we apply the distributional robustness analysis similar to that of Carroll (2017) and show that the rank-guarantee has an appealing worst-case performance.

[1] This research direction complements the large body of papers that focus on the case in which the designer does not have reliable information about the agents' hierarchies of beliefs about each other while assuming the knowledge of the payoff environment; see, for example, Bergemann and Morris (2005), Chung and Ely (2007), Chen and Li (2018), Du (2018), Brooks and Du (2021), Yamashita and Zhu (2022), and Brooks and Du (2023).

Multi-item auctions Beyond the efficient Vickrey auction, few results have been established regrading multi-item auctions with combinatorial preferences. Jehiel and Moldovanu (2001a) point out the vulnerability of efficiency under multidimensional bidder information. Ausubel and Milgrom (2002) point out the poor revenue performance and strategic vulnerability of the Vickrey auction and propose simultaneous ascending auctions with package bidding (SAAPB). The multi-item auction design problem has also been extensively studied in the field of combinatorial auctions (see Cramton et al. (2006) for a survey). This literature mainly focuses on (approximately) efficient auction design and their computational complexity, which is orthogonal to our focus on revenue performance and strategic simplicity. In comparison to the simultaneous ascending auction, allowing for bidding on bundles (package bidding) has the advantage of mitigating the demand reduction problem or the exposure problem. Compared to other proposals like SAAPB (Ausubel and Milgrom (2002)) and CCA (Ausubel et al. (2006)), CASA uses a simpler "pay-as-bid" rule. Importantly, we do not assume that the bidders are single-minded. CASA can be viewed as a simpler variant of the SAAPB and the *Combinatorial Clock Auction* (CAA, see Ausubel et al. (2006) and Levin and Skrzypacz (2016)) in that bidders simply raise the prices of the bundles, as opposed to personalized prices in SAAPB and demand reporting in CCA.

Implementation in strategies that are not obviously dominated We study outcomes when agents are rational in the sense of avoiding obviously dominated strategies. This solution concept draws from the idea of obvious strategy-proof mechanisms (see Li (2017)) and is systematically studied in Li and Dworczak (2021). As in Li and Dworczak (2021), we assume that agents avoid obviously dominated strategies, but refrain from making assumptions regarding how agents select among strategies that are not obviously dominated. This methodology aligns with the spirit of implementation in undominated strategies; see for example Carroll (2014), Börgers (1991), Jackson (1992), and Yamashita (2015).

2 CASA and rank-guarantee

2.1 The auction environment

There is a set S of M items to be sold to N bidders. Let $\mathcal{N} = \{1, 2, \ldots, N\}$. We write $b \subseteq S$ to denote a generic bundle of items. Let $\boldsymbol{v}^n = \{v_b^n\}_{b \subseteq S}$ denote the valuation vector of bidder n, where v_b^n is bidder n's valuation of bundle b. Valuations are normalized so that $v^n_\emptyset = 0$ and $v^n_b \in [\underline{v}, \bar{v}]$ ($\underline{v} \geq 0$) for all $b \neq \emptyset$. A generic valuation profile is denoted by $\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2, \ldots, \boldsymbol{v}^N)$. Let $\mathcal{M} \subseteq 2^S$ denote a *menu* of bundles chosen by the auctioneer. While $\mathcal{M}$ is a choice variable of the auctioneer, for now we take it as exogenously given; we defer the discussion of menu design to Section 4. Assume that $N \geq |\mathcal{M}| + 1$. Let

$$\mathcal{B}(\mathcal{M}) = \{X \subseteq \mathcal{M} \,|\, \forall b, b' \in X,\, b \cap b' = \emptyset\}$$

denote the set of *feasible* allocations of bundles within the menu $\mathcal{M}$, i.e., all collections consisting of non-overlapping bundles.

2.2 The Combinatorial Ascending Auction

We define the *Combinatorial Ascending Auction* (CASA) as follows. The auction has an iterative structure, with the "state of the auction" characterized by the identity of the leading bidder and the leading price for each bundle. Initially, the leading price for each bundle is zero and none of the bidders is a leading bidder for any bundle. Bidders take turns raising the bids on the bundles, which determines new leading bidders and leading prices. The process repeats itself until when there are no new bids on any bundle. At that point, the auction stops. The auctioneer chooses a feasible allocation to maximize revenue, taking the leading prices as the prices for the bundles. There is also an activity rule designed to ensure that bidding activity starts out high and declines during the auction as prices rise far enough to discourage some bidders from continuing.

Formally, let $P \subset \mathbb{R}^+$ be a finite grid of feasible bids with grid size ϵ and $\max P > \bar{v}$.

(1) **Initialization stage** $t = 0$. Define

- the *leading bidder vector* at stage 0: $\boldsymbol{\phi}^0 = (\phi^0_b)_{b \in \mathcal{M}} = \mathbf{0}$,
- the *leading price vector* at stage 0: $\boldsymbol{p}^0 = (p^0_b)_{b \in \mathcal{M}} = \mathbf{0}$,
- the set of *active bidders* at stage 0: $\mathcal{N}^0 = \{1, 2, \ldots, N\}$.

(2) **Bidding stage** $t \geq 1$. An active bidder $n \in \mathcal{N}^{t-1}$ observes $(\boldsymbol{p}^{t-1}, \{b \,|\, \phi^{t-1}_b = n\})$, and decides whether to quit, which bundles to bid on, and how much to bid.[2]

- Bidder n may choose to quit only if $\{b \,|\, \phi^{t-1}_b = n\} = \emptyset$, i.e., bidder n is not a leading bidder for any bundle in stage $t - 1$. Quitting is irreversible, that is, if

[2] The bidder selection rule and the observability of history is inconsequential for our analysis. For concreteness, we consider the selection rule that active bidders are cycled in ascending order according to their indices, and the observability of history is minimized to maximally protect privacy.

bidder n chooses to quit, then bidder n becomes an inactive bidder and does not participate in future bidding rounds. Update:

  - $\boldsymbol{\phi}^t = \boldsymbol{\phi}^{t-1}$, $\boldsymbol{p}^t = \boldsymbol{p}^{t-1}$, $\mathcal{N}^t = \mathcal{N}^{-1} \setminus \{n\}$.

- If bidder n chooses not to quit, then she could bid on multiple bundles $\{(b, p_b)\} \subset \mathcal{M} \times P$, subject to the requirements that (1) Leading bids are binding: if bidder n is the leading bidder at some bundle in stage $t-1$, then she must include that bundle in her bid with a bid that is *weakly* higher than the current leading price for that bundle, and (2) Minimum bid increment: if bidder n would like to bid on some bundle for which she is not the leading bidder, then her bid for that bundle must be *strictly* higher than the current leading price for that bundle. Update:
  - $\phi_b^t = n$ and $p_b^t = p_b$ for any bundle b included in her bid,
  - $\phi_{b'}^t = \phi_{b'}^{t-1}$ and $p_{b'}^t = p_{b'}^{t-1}$ for any bundle b' not included in her bid,
  - $\mathcal{N}^t = \mathcal{N}^{t-1}$.

Then, move on to the bidding stage $t+1$.

(3) **Allocation**. The auction ends (in stage T) when the leading prices stay constant for N consecutive periods. The auctioneer chooses a feasible allocation to maximize

$$\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} p_b^T.$$

Denote the maximizer by $\boldsymbol{b}^*$. Each bundle $b \in \boldsymbol{b}^*$ is allocated to ϕ_b^T at the price p_b^T.

In words, the auction format of CASA runs parallel ascending auctions for each bundle $b \in \mathcal{M}$. Then, the items are allocated to maximize the total price. The similarities to the standard ascending auction make the CASA auction format easier to understand and implement. At the same time, CASA has important differences from some of the well-known combinatorial auction formats. In comparison to the simultaneous ascending auction, allowing for bidding on bundles (package bidding) has the advantage of mitigating the demand reduction problem or the exposure problem. Compared to other proposals like SAAPB (Ausubel and Milgrom (2002)) and CCA (Ausubel et al. (2006)), CASA uses a simpler "pay-as-bid" rule. Additionally, as we formally demonstrate in the next subsection, we do not assume that the bidders are single-minded.

The auction format of CASA can be tailored to the auctioneer's needs and practical considerations without losing its various desirable properties. Notably, the menu $\mathcal{M}$ is a design variable of the auctioneer, which we examine in detail in Section 4. Additionally, instead of allowing bidders to bid on multiple bundles, we could restrict them to bid on a single bundle when it is their turn to move (if a bidder is already leading for some bundle when it is her turn to move, she could not bid on any other bundle but she would remain an active bidder). Moreover, we could also allow the bidders to observe the entire history if that is considered desirable for transparency purposes.

2.3 The rank-guarantee of CASA

In this subsection, we study the strategic behavior of the bidders and establish the rank-guarantee property of CASA. We only assume minimal rationality on the part of the bidders—bidders are rational in the sense of not playing obviously dominated strategies. Formally we adopt the solution concept of implementation in strategies that are not obviously dominated (see Li (2017) and Li and Dworczak (2021)). We first sketch the intuition, and then provide the formal arguments.

At any history, consider a non-leading bidder's choice as to whether to quit. Obviously, as quitting is irreversible, quitting the auction leads to a best possible outcome of a zero payoff. Suppose that there is some bundle for which the bidder's valuation is higher than the current leading price for that bundle. Consider the following strategy where the bidder raises the price for this particular bundle and never revises her bid afterwards. Clearly, this continuing strategy guarantees a non-negative payoff for the bidder. Thus, at least for the purpose of deciding whether to quit, it is "obviously optimal" not to quit.

More formally, let $h = (t, (\mathcal{N}^0, \ldots, \mathcal{N}^{t-1}), (\boldsymbol{p}^0 \ldots, \boldsymbol{p}^{t-1}), (\boldsymbol{\phi}^0, \ldots, \boldsymbol{\phi}^{t-1}))$ denote a history of the game in stage t, H_t the set of such histories in stage t, and $H = \cup_{t\geq 0} H_t$. Suppose that bidder n is the active bidder in some stage t and I_n is bidder n's information set. Then the observed prices $\boldsymbol{p}$ and n's leading bundles $\boldsymbol{b}$ are the same for all $h \in I_n$. Let $\mathcal{I}_n$ denote all information sets of n. Let $s_n : \mathcal{I}_n \to 2^{\mathcal{M}\times\mathbb{R}^+}$ denote bidder n's (pure behavioral) strategy and $u_n(\boldsymbol{s}, \boldsymbol{v}^n|h)$ the payoff to bidder n given valuation vector $\boldsymbol{v}^n$, strategy profile $\boldsymbol{s}$, conditional on the current history h and n bidding in period t.

Definition 1. *A bidding strategy $s_n : \mathcal{I}_n \to 2^{\mathcal{M}\times P}$ is* obviously dominated *if there exists s'_n such that at any earliest point of departure I_n between s_n and s'_n,*

$$\sup_{s_{-n}, h\in I_n} u_n(\boldsymbol{s}, \boldsymbol{v}^n|h) \leq \inf_{s_{-n}, h\in I_n} u_n(s'_n, \boldsymbol{s}_{-n}, \boldsymbol{v}^n|h);$$

$$\inf_{s_{-n}, h\in I_n} u_n(\boldsymbol{s}, \boldsymbol{v}^n|h) < \sup_{s_{-n}, h\in I_n} u_n(s'_n, \boldsymbol{s}_{-n}, \boldsymbol{v}^n|h).$$

The first inequality is identical to the definition of the obvious dominance relation in Li (2017), i.e., the best outcome under s_n is weakly worse than the worst outcome under s'_n. In addition, we require the dominated strategy to be non-equivalent in terms of the induced outcome to the strategy that dominates it. The second requirement guarantees that the set of non-obviously dominated strategies is non-empty. The earlier intuition then translates to:

Lemma 1. *If there exists an information set $I_n \in \mathcal{I}_n$ (with observed prices $\boldsymbol{p}$) such that $s_n(I_n) = \emptyset$ (i.e., bidder n quits) and*

- $\exists \boldsymbol{p}' \in P$ *and* $b \in \arg\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b' \in \boldsymbol{b}} p'_{b'}$ *such that* $\boldsymbol{p}' \geq \boldsymbol{p}$ *and* $v_b^n > p'_b > p_b$,

then s_n is obviously dominated.

Proof. To show that s_n is obviously dominated, we explicitly construct another strategy s'_n that obviously dominates it. Consider the information set $I_n \in \mathcal{I}_n$ (with observed prices $\boldsymbol{p}$) such that $s_n(I_n) = \emptyset$ and

- $\exists \boldsymbol{p}' \in P$ and $b \in \arg\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b' \in \boldsymbol{b}} p'_{b'}$ such that $\boldsymbol{p}' \geq \boldsymbol{p}$ and $v_b^n > p'_b > p_b$.

Obviously, the payoff from s_n conditional on any $h \in I_n$ is zero. Let s'_n be the same as s_n before the information set I_n and let bidder n bid $s'_n(I_n) = (b, p'_b)$ and never revise her bid afterwards.

We first discuss a special case in which $|\mathcal{N}^{t-1}| = 1$ and n is not a current leading bidder (otherwise quitting the auction is not feasible for bidder n). Then, all the current prices must be 0 and all the other bidders have quit (as this is the unique consistent history). Following the strategy s'_n, the auction ends with bidder n bidding (b, p'_b), leading to a positive payoff of $v_b^n - p'_b > 0$ for bidder n.

Next, we consider the case in which $|\mathcal{N}^{t-1}| > 1$. It is clear that bidder n will get a nonnegative payoff regardless of the value profiles and bidding strategies of other bidders. We show that the best possible payoff for bidder n following the strategy s'_n is positive. It suffices to consider the case in which any other subsequent active bidder bids $p'_{b'}$ for some bundle b' whenever possible, the auction ends with prices $\boldsymbol{p}'$, leading to a positive payoff of $v_b^n - p'_b > 0$ for bidder n. *Q.E.D.*

Note that Lemma 1 suggests that our intuition is incomplete as we cannot fully rule out strategies that quit when the bidder's value for some bundle is above the leading price for that bundle. To rule out all such strategies, it further requires the existence of some scenario under which bidder n may be pivotal: a strictly profitable bid of n may ever be selected in the end. Let $S^n_{NOD}(P)$ denote the set of non-obviously dominated strategies of n given price grid P. Let $R(\boldsymbol{s}, \boldsymbol{v})$ denote the revenue to seller given value profile $\boldsymbol{v}$ and strategy profile $\boldsymbol{s}$. Define

$$\underline{R}_{CASA}(\boldsymbol{v}) := \varliminf_{\epsilon \to 0} \inf_{s_n \in S^n_{NOD}(P)} R(\boldsymbol{s}, \boldsymbol{v}).$$

That is, $\underline{R}_{CASA}(\boldsymbol{v})$ is the worst-case ex-post revenue from CASA under non-obviously dominated strategies in the limit where grid P becomes dense.

Given the valuation profile $\boldsymbol{v}$ and menu $\mathcal{M}$, the k^{th}-guarantee is defined as the maximal revenue from feasible allocations within $\mathcal{M}$, taking the k^{th}-highest valuations for each bundle as the price.

Definition 2. *The k^{th}-guarantee given the menu $\mathcal{M}$ and the value profile $\boldsymbol{v}$ is*

$$R^k_{\mathcal{M}}(\boldsymbol{v}) := \max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b^{(k)},$$

where $v_b^{(k)}$ denotes the k^{th}-highest value of bundle b.

Our key observation is that CASA achieves the k^{th}-guarantee as long as bidders avoid strategies that are obviously dominated. In other words, for CASA to achieve the k^{th}-guarantee, we only need minimal rationality on the part of the bidders.

Theorem 1. $\underline{R}_{CASA}(\boldsymbol{v}) \geq R^k_{\mathcal{M}}(\boldsymbol{v})$ *for $k = |\mathcal{M}| + 1$.*

Proof. Consider a grid P with any grid size $\epsilon > 0$. We show that as long as bidders avoid obviously dominated strategies, for any value profile $\boldsymbol{v}$,

$$\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} p_b^T \geq \max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} (v_b^{(k)} - \epsilon).$$

Suppose to the contrary, this is not true. Then, there exists strategy profile s (where s_n is not obviously dominated for each bidder n), value profile $\boldsymbol{v}$, and $\delta > 0$ such that

$$\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} p_b^T < \max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} (v_b^{(k)} - \epsilon - \delta).$$

Evidently, there exists some bundle b such that $p_b^T < v_b^{(k)} - \epsilon - \delta$, or equivalently, $p_b^T + \epsilon < v_b^{(k)} - \delta$. For each such bundle b, raise the price of bundle b to (the closest price below) $v_b^{(k)} - \delta$ sequentially until we find the first pivotal bundle $\tilde{b}$ when $\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} p_b$ is strictly improved by such increase in prices.

Since $k = |\mathcal{M}| + 1$ and $p^T_{\tilde{b}} < v^{(k)}_{\tilde{b}} - \epsilon - \delta$, there exists a bidder n with $v^n_{\tilde{b}} \geq v^{(k)}_{\tilde{b}}$ that quits the auction before the auction ends. Let h be the (on-path) history at which n quits (note that at this history, n must not be a leading bidder and $p_{\tilde{b}}(h) \leq p^t_{\tilde{b}} \leq v^n_{\tilde{b}} - \delta - \epsilon$, where $p_{\tilde{b}}(h)$ is the price of bundle $\tilde{b}$ at the history h). Let $h \in I_n$. Then, $s_n(I_n) = \emptyset$. Let $\boldsymbol{p}'$ be the raised prices. Then $\tilde{b} \in \arg\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b' \in \boldsymbol{b}} p'_{b'}$, and $\boldsymbol{p}'$ satisfies the condition in Lemma 1. Therefore, s_n is obviously dominated. We arrive at a contradiction. *Q.E.D.*

As we have pointed out following Lemma 1, elimination of obviously dominated strategies does *not* guarantee the ex-post prices to be above the k^{th}-highest values. To establish Theorem 1, we prove in addition that the behaviors of non-pivotal bidders are inconsequential. With Theorem 1, we say that CASA is a rank-guaranteed auction format as long as we are comfortable assuming that bidders avoid strategies that are obviously dominated.

2.4 Discussions

Strategic simplicity: There are two important features of CASA that make it strategically simple to decide whether to quit—the bidder chooses to continue as long as if there is some bundle for which the bidder's valuation is higher than the leading price for that bundle. The ascending format makes the "prices" transparent and allowing for package bidding makes the "allocation" transparent; in its essentials, the bidder is committed to purchasing a certain bundle at a certain price by continuing. As the same time, we note that the decision of which bundles to bid on and how much to bid remains complex for the bidders. While the bidder can fully avoid the "exposure problem" by bidding only on the bundles he wants, he might strategically chooses to expose himself so that to manipulate the prices of the complementary bundles.

Robustness to collusion / irrationality: Since the argument of Theorem 1 solely relies on the existence of one strategic bidder with a value v^n_b above $v^{(k)}_b$ who quits the auction. The analysis extends to the case with non-strategic bidders easily. Firstly, when the number of non-strategic bidders is bounded, then Theorem 1 still holds when k is relaxed by the number of non-strategic bidders.

Proposition 1. *Suppose there are j non-strategic bidders, then $\underline{R}_{CASA}(\boldsymbol{v}) \geq R^k_{\mathcal{M}}(\boldsymbol{v})$ for $k = |\mathcal{M}| + 1 + j$.*

Proof. Observe that in the proof of Theorem 1, since $k = |\mathcal{M}| + 1 + j$, there exists at least one *strategic* player n that quits the auction before period t and $v^n_{\tilde{b}} \geq v^{(k)}_{\tilde{b}}$. The rest of the proof follows. *Q.E.D.*

Secondly, when the bidders form coalitions and they strategically maximize group payoffs[3], then Theorem 1 still holds when k is scaled by the coalition sizes.

Proposition 2. *Suppose bidders are partitioned into strategic coalitions $\{c_i\}_{i\in I}$, where the index is chosen such that $|c_i|$ decreases in i. Then, $\underline{R}_{CASA}(\boldsymbol{v}) \geq R^k_{\mathcal{M}}(\boldsymbol{v})$ for $k = \sum_{i\leq|\mathcal{M}|}|c_i| + 1$.*

Proof. Observe that in the proof of Theorem 1, since $k = \sum_{i\leq|\mathcal{M}|}|c_i| + 1$, there exists at least one coalition of players c that all quit the auction before period t and $\max_{n\in c}\{v^n_{\tilde{b}}\} \geq v^{(k)}_{\tilde{b}}$. Let h be the (on-path) history at which the last member n in the coalition quits (note that at this history, n must not be a leading bidder and $p_{\tilde{b}} \leq p^t_{\tilde{b}} \leq \max_n\{v^n_{\tilde{b}}\} - \delta - \epsilon$). Let $h \in I_n$. Then, $s_n(I_n) = \emptyset$. Obviously, quitting gives the entire group zero payoff while bidding $p'_{\tilde{b}}$ guarantees a non-negative payoff. Suppose all other bidders bid up to p'_b when it is their turn, the auction ends with $\boldsymbol{p}'$ and the group obtains a strictly positive payoff. Therefore, s_n is obviously dominated for coalition c. *Q.E.D.*

The intuition behind the two extensions is exactly the strategic simplicity of "how to bid". The price of each bundle must be higher than the value of any *losing strategic* bidder or any *losing coalition group* as otherwise they will outbid the price. Of course, Proposition 2 has no bite when k is large compared to N; hence, it should be interpreted as the strategic robustness of CASA only in relatively thick markets. Nevertheless, CASA is also aligned with the philosophy of anti-collusion design even in thin markets (Klemperer (2002)), for the reason that CASA permits minimum transmission of information.[4]

Efficiency: Note that it is *not* obviously dominated to bid strictly above the true valuation for a bundle.[5] Therefore, CASA might not satisfy ex-post IR; hence k^{th}-guaranteed revenue does not implie k^{th}-guaranteed surplus. Of course, CASA still satisfies ex-ante IR (assuming bidders having correct Bayesian priors) since quitting at the beginning is always an option; hence, the ex-ante bounds we derive in this paper on the revenue of CASA also apply to surplus.

[3] Each group can freely shift allocations within the group and maximize the total payoff.

[4] Anonymity prevents the reciprocity behavior documented in Cramton and Schwartz (2000). Theorem 1 holds when each bidder can only submit bid on one bundle at a time, which prevents the strategic communication documented in Jehiel and Moldovanu (2001b) and Grimm et al. (2003).

[5] Consider, for example, the case with three bidders $1, 2, 3$ and three items a, b, c . Bidder 1 only wants a, bidder 2 only wants b, and bidder 3 only wants the grand bundle $\{a, b, c\}$. The valuation of each bidder for the desired bundle is 1. By strategically bidding up item b even though bidder 1 gets zero value from it, bidder 1 can reduce the bid required for him to win item a, creating an exposure problem for 1.

VCG & SAAPB: The celebrated pivot VCG mechanism is known to underperform the k^{th}-guarantee when bidder's preferences exhibits complementarity.[6] The SAAPB mechanism is indeed k^{th}-guaranteed under "straightforward bidding" strategies (Theorem 1 of Ausubel and Milgrom (2002)). However, whether SAAPB is k^{th}-guaranteed with fully strategic bidders is yet unknown to us. Nevertheless, our result suggests that the spirit of SAAPB, when carefully executed, leads to an appealing revenue guarantee.

3 Rank-guarantee as a desideratum

Section 2 shows that CASA achieves the rank-guarantee as long as bidders avoid strategies that are obviously dominated. In this section, we explore the concept of rank-guarantee as a desideratum in auction design.

Clearly, if the auctioneer knows that the bidders' valuations are independent and identically distributed, then rank-guarantee is an appealing approximation when N is large, as all order statistics converge to the upper bound of the valuation support (at the rate of $\frac{1}{N}$). In what follows, we show that even when the auctioneer has non-Bayesian uncertainty about the joint distribution of the bidders' valuations and maximizes the revenue-guarantee (the worst-case expected revenue where the worst case is taken over all joint distributions that are perceived to be plausible), in many settings, rank-guarantee remains an appealing approximation.

Let $\mathbb{G} \subset \Delta([\underline{v}, \bar{v}]^{2^S})$ be an arbitrary subset of distributions of valuation vector. We interpret $\mathbb{G}$ as the auctioneer's estimate of a *representative bidder*'s valuation. Then, the joint distributions of the bidders' valuations that are considered possible by the auctioneer are

$$\mathbb{F} = \left\{ F \in \Delta([\underline{v}, \bar{v}]^{N \times 2^S}) \,\middle|\, \frac{1}{N} \sum F_n \in \mathbb{G} \right\},$$

where F_n is the marginal distribution of bidder n's valuation. Thus, $\frac{1}{N} \sum F_n$ is the cumulative distribution function of the valuation of a uniformly randomly selected bidder in the population. We call $\mathbb{F}$ an *ambiguity set*. Such an ambiguity set $\mathbb{F}$ could come from the statistical estimation of F based on a "sanitized" dataset about valuations, that is, past bidders' valuations with identity information removed. The ambiguity set $\mathbb{F}$ captures the type of distributional uncertainty introduced by Carroll (2017), while further generalizing it to capture realistic knowledge structures stemming from statistical

[6] Imagine the case $\mathcal{M} = \{(a), (b), (a, b)\}$. $v_{(a,b)} = 1$ for all bidders. $v_a = v_b = 0$ for all bidders except for two, whose value for a and b are 1. The VCG revenue is 0, while the k^{th}-guarantee is 1 for any $k \geq 2$.

inference.[7]

For any $F \in \Delta([\underline{v}, \bar{v}]^{N \times 2^S})$, define the ex-ante *efficient* surplus with respect to menu $\mathcal{M}$

$$V_{\mathcal{M}}(F) := \mathbb{E}_F \left[\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \max_{\iota : \boldsymbol{b} \to \mathcal{N}} \sum_{b \in \boldsymbol{b}} v_b^{\iota(b)} \right],$$

where $\iota(b) \neq \iota(b')$ for any $b \neq b'$. The ex-ante efficient surplus $V_{2^S}(F)$ with respect to the complete menu 2^S is denoted by $V^*(F)$.

Theorem 2. *For any menu $\mathcal{M}$,*

$$\inf_{F \in \mathbb{F}} \mathbb{E}_F[R^k_{\mathcal{M}}(\boldsymbol{v})] \geq \inf_{F \in \mathbb{F}} V_{\mathcal{M}}(F) - \frac{k|\mathcal{M}|\bar{v}}{N}.$$

Proof. Let ® be a uniform random element of $\mathcal{N}$.

$$\begin{aligned}
R^k_{\mathcal{M}}(\boldsymbol{v}) &= \max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b^{(k)} \\
&\geq \max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \left(\sum_{b \in \boldsymbol{b}} v_b^{®} - \sum_{b \in \boldsymbol{b}} \max\{v_b^{®} - v_b^{(k)}, 0\} \right) \\
&\geq \max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \left(\sum_{b \in \boldsymbol{b}} v_b^{®} \right) - \sum_{b \in \mathcal{M}} \max\{v_b^{®} - v_b^{(k)}, 0\} \\
\implies \mathbb{E}_F[R^k_{\mathcal{M}}(\boldsymbol{v})] &\geq \mathbb{E}_F \left[\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b^{®} \right] - \sum_{b \in \mathcal{M}} \mathbb{E}_F \left[v_b^{®} - v_b^{(k)} | v_b^{®} > v_b^{(k)} \right] \text{Prob}(v_b^{®} > v_b^{(k)}) \\
&\geq \mathbb{E}_F \left[\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b^{®} \right] - \sum_{b \in \mathcal{M}} \bar{v} \, \text{Prob}(v_b^{®} > v_b^{(k)}) \\
&\geq \mathbb{E}_F \left[\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b^{®} \right] - \frac{k|\mathcal{M}|\bar{v}}{N}.
\end{aligned}$$

This further implies that

$$\begin{aligned}
\inf_{F \in \mathbb{F}} \mathbb{E}_F \left[R^k_{\mathcal{M}}(\boldsymbol{v}) \right] &\geq \inf_{F \in \mathbb{F}} \mathbb{E}_F \left[\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b^{®} \right] - \frac{k|\mathcal{M}|\bar{v}}{N} \\
&= \inf_{G \in \mathbb{G}} \mathbb{E}_G \left[\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b \right] - \frac{k|\mathcal{M}|\bar{v}}{N} \\
&\geq \inf_{F \in \mathbb{F}} V_{\mathcal{M}}(F) - \frac{k|\mathcal{M}|\bar{v}}{N},
\end{aligned}$$

[7] This setup covers a wide range of scenarios, as $\mathbb{G}$ is completely general. $\mathbb{G}$ could be a singleton set capturing the case in which the auctioneer has no uncertainty about the distribution of a representative bidder's valuation. $\mathbb{G}$ could also be the set of distributions satisfying certain statistical properties (say moment conditions), capturing scenarios in which the auctioneer also has some non-Bayesian uncertainty about the distribution of a representative bidder's valuation.

where the equality follows from the definition of the set $\mathbb{F}$. For the last inequality, observe that for any $G \in \mathbb{G}$, the joint distribution where each bidder's value is distributed according to G and all bidders' values are maximally positively correlated is contained in $\mathbb{F}$. Therefore,

$$\inf_{F \in \mathbb{F}} V_{\mathcal{M}}(F) \leq \inf_{G \in \mathbb{G}} \mathbb{E}_G \left[\max_{\boldsymbol{b} \in \mathcal{B}(\mathcal{M})} \sum_{b \in \boldsymbol{b}} v_b \right].$$

Q.E.D.

Theorem 2 highlights a key trade-off between *menu sufficiency* and *approximation efficiency*. Evidently, presenting the bidders with a larger menu has the potentially of increasing allocation efficiency. Particularly, $\mathcal{M}$ can be naively chosen to be the complete menu 2^S to guarantee full menu sufficiency, i.e., $V_{\mathcal{M}}(F) = V^*(F)$. However, this leads to $|\mathcal{M}|$ growing exponentially in M, causing both complex auction process and slow convergence. On the other hand, choosing a small menu achieves approximation efficiency but sacrifices allocation efficiency. Although such trade-off is generally non-trivial under general combinatorial preferences, we show in the next section that under canonical preference structures, menu sufficiency and approximation efficiency can often be achieved simultaneously.

Tightness of the bound: The coefficient $k|\mathcal{M}|$ in Theorem 2 consists of two parts. The coefficient k comes from the k^{th} highest value approximation. The coefficient $|\mathcal{M}|$ comes from the total number of bundles in the menu $\mathcal{M}$. Proposition 3 below shows that the dependence on k is tight.

Proposition 3. *For any $M, N, \mathcal{M}, k$, there exists some $\mathbb{G}$ such that*

$$\inf_{F \in \mathbb{F}} \mathbb{E}_F[R^k_{\mathcal{M}}(\boldsymbol{v})] \leq \inf_{F \in \mathbb{F}} V_{\mathcal{M}}(F) - O\left(\tfrac{k}{N}\right).$$

Proof. See Appendix A.1. *Q.E.D.*

4 Menu design and simple menus

In this section, we examine several canonical classes of preference structures where there exist menus that are both *sufficient*, ensuring full allocation efficiency, and *small*, with menu size growing at a polynomial rate as M increases.

Definition 3. *Menu $\mathcal{M}$ is $\mathbb{G}$-sufficient if:*

$$\inf_{G\in\mathbb{G}} \mathbb{E}_G\left[\max_{\boldsymbol{b}\in\mathcal{B}(\mathcal{M})}\sum_{b\in\boldsymbol{b}} v_b\right] = \inf_{G\in\mathbb{G}} \mathbb{E}_G\left[\max_{\boldsymbol{b}\in\mathcal{B}(2^S)}\sum_{b\in\boldsymbol{b}} v_b\right].$$

In words, a menu $\mathcal{M}$ is $\mathbb{G}$-sufficient if the worst-case surplus from allocating to (hypothetically) identical bidders with valuation distribution from $\mathbb{G}$ is the same as that under the complete menu 2^S. Importantly, $\mathbb{G}$-sufficiency is defined with respect to the preference of a single bidder instead of all bidders. It is much weaker than assuming that restricting to allocations within $\mathcal{M}$ is without loss for ex-post efficiency.[8] Nevertheless, sufficiency guarantees full allocation efficiency:

Theorem 3. *If menu $\mathcal{M}$ is $\mathbb{G}$-sufficient, then*

$$\inf_{F\in\mathbb{F}} \mathbb{E}_F\left[R^k_{\mathcal{M}}(\boldsymbol{v})\right] \geq \inf_{F\in\mathbb{F}} V^*(F) - \frac{k|\mathcal{M}|\bar{v}}{N}.$$

Proof.

$$\begin{aligned}\inf_{F\in\mathbb{F}} \mathbb{E}_F\left[R^k_{\mathcal{M}}(\boldsymbol{v})\right] &\geq \inf_{G\in\mathbb{G}} \mathbb{E}_G\left[\max_{\boldsymbol{b}\in\mathcal{B}(\mathcal{M})}\sum_{b\in\boldsymbol{b}} v_b\right] - \frac{k|\mathcal{M}|\bar{v}}{N}\\ &= \inf_{G\in\mathbb{G}} \mathbb{E}_G\left[\max_{\boldsymbol{b}\in\mathcal{B}(2^S)}\sum_{b\in\boldsymbol{b}} v_b\right] - \frac{k|\mathcal{M}|\bar{v}}{N}\\ &\geq \inf_{F\in\mathbb{F}} V^*(F) - \frac{k|\mathcal{M}|\bar{v}}{N},\end{aligned}$$

where the equality follows from the $\mathbb{G}$-sufficiency of menu $\mathcal{M}$, and the two inequalities have been established in the proof of Theorem 2. *Q.E.D.*

A simple sufficient condition for the $\mathbb{G}$-sufficiency of menu $\mathcal{M}$ is that

$$\max_{\boldsymbol{b}\in\mathcal{B}(2^S)}\sum_{b\in\boldsymbol{b}} v_b = \max_{\boldsymbol{b}\in\mathcal{B}(\mathcal{M})}\sum_{b\in\boldsymbol{b}} v_b$$

holds ex-post. This condition allows us to convert combinatorial preferences into sufficiency. To simplify notation, let $\text{Supp}(\mathbb{G}) := \cup_{G\in\mathbb{G}}\text{Supp}(G)$.

Weak substitutability and itemized ascending auction

Definition 4. *We say that bidder preferences exhibit* ***weak substitutability*** *if for any*

[8] Consider for instance two items and two bidders, where for each bidder the sum of the value for each individual item is more than her value for the grand bundle. Then, menu of individual items is "sufficient" per Definition 3, but not necessarily ex-post efficient when the two bidder's values are highly asymmetric.

$\boldsymbol{v} \in \mathrm{Supp}(\mathbb{G})$ *and* $b \subseteq S$,

$$\sum_{s \in b} v_{\{s\}} \geq v_b.$$

In words, a representative bidder finds the value of any bundle weakly lower than the sum of her value for each item in the bundle. Weak substitutability is a necessary condition for various substitutability notions studied in the literature.

Proposition 4. *If bidder preferences exhibit weak substitutability, then the menu* $\mathcal{M} = S$ *is* $\mathbb{G}$*-sufficient and* $k = M + 1$.

When $\mathcal{M} = S$, CASA reduces to a simple itemized ascending auction, where each item is auctioned individually and exclusively. Weak substitutability is one of the most widely studied preference assumptions in the literature as it captures a natural diminishing return to scale. Our analysis shows that under such preference structures, CASA exhibits extreme simplicity while achieving both allocation and approximation efficiency. Intriguingly, under weak substitutability, the canonical Vickery auction performs as well as CASA, despite its much worse performance under more general preference structures.

Proposition 5. *If bidder preferences exhibit weak substitutability, then the Vickery auction achieves a revenue guarantee of* $R_S^{M+1}(\boldsymbol{v})$.

Proof. See Appendix A.2. *Q.E.D.*

An even more special case of weak substitutability is the sponsored search auction, where valuations of items are constant (and common) ratios of a one-dimensional private type. As shown in Edelman et al. (2007), the clock auction version of *generalized second price* (GSP) auction is outcome equivalent to the Vickery auction; hence achieving the same rank-guarantee.

Weak complementarity and the second-price auction

Definition 5. *Bidder preferences exhibit* ***weak complementarity*** *if for any* $\boldsymbol{v} \in \mathrm{Supp}(\mathbb{G})$ *and* $\boldsymbol{b} \in \mathcal{B}(2^S)$,

$$\sum_{b \in \boldsymbol{b}} v_b \leq v_S.$$

In words, a representative bidder finds the value of the grand bundle weakly higher than the total value of any feasible collection of bundles. Weak complementarity is a necessary condition for various complementarity notions studied in the literature.

Proposition 6. *If bidder preferences exhibit weak substitutability, then the menu* $\mathcal{M} = \{S\}$ *is* $\mathbb{G}$*-sufficient and* $k = 2$*.*

When $\mathcal{M} = \{S\}$, CASA reduces to a simple ascending auction for only the grand bundle. Evidently, in this case, the standard *second-price auction* is second-guaranteed and outcome-equivalent to CASA.

"Partitional" complementarity A hybrid case of substitutability and complementarity is the partitional complementarity which we define below, described by a partition $\mathcal{K}$ of S.

Definition 6. *Let* $\mathcal{K}$ *be a partition of* S*. Bidder preferences exhibit* $\mathcal{K}$***-partitioned complementarity*** *if for any* $\boldsymbol{v} \in \text{Supp}(\mathbb{G})$*,*

$$\text{for any } b \in \mathcal{K} \text{ and partition } \kappa \text{ of } b, \sum_{b' \in \kappa} v_{b'} \leq v_b;$$
$$\text{for any } b' \subseteq S, \sum_{b \in \mathcal{K}} v_{b \cap b'} \geq v_{b'}.$$

In words, $\mathcal{K}$-partitioned complementarity structure means there is weak complementarity within each $b \in \mathcal{K}$ and weak substitutability across each $b \in \mathcal{K}$.

Proposition 7. *If bidder preferences exhibit* $\mathcal{K}$*-partitioned complementarity, then the menu* $\mathcal{M} = \mathcal{K}$ *is* $\mathbb{G}$*-sufficient and* $k = |\mathcal{K}| + 1$*.*

In some cases, the auctioneer may understand that bidder preferences exhibits partitional complementarity, but does not know the exact partition. Proposition 7 can be easily extended to the case with multiple possible partitions $\{\mathcal{K}_i\}_{i=1}^{I}$, where I is bounded. In this case $\mathcal{M} = \cup_{i \in I} \mathcal{K}_i$ and $k \sim Poly(M)$. Such partitional complementarity preference structure arises when there is clear synergy between "nearby" bundles. Think about land auctions, for example. There are finitely many possible partitions that are determined by the major divisions of lands by rivers, highways, or railroads. If two distinct lands are segregated by those divisions, then there is substitutability among them. In such cases, our theory guarantees the performance of CASA with the partitional menu.

Homogeneous goods and quantity-CASA

Definition 7. *The goods are* ***homogeneous*** *if there exists* $u : \mathbb{N} \to [\underline{v}, \bar{v}]$ *such that for any* $\boldsymbol{v} \in \text{Supp}(\mathbb{G})$ *and* $b \in S$*,*

$$v_b = u(|b|).$$

With homogeneous goods, a representative bidder's valuation for any bundle only depends on the size of the bundle. Note that the dependence of u on $|b|$ is arbitrary. We do not even require monotonicity. In this case, we redefine the notion of feasible allocations to $\mathcal{B} : (\mathcal{M}) = \{X \subset \mathcal{M} | \sum_{b \in X} |b| \leq M\}$, i.e., an allocation is feasible as long as the total number of items being allocated is below M.

Proposition 8. *If goods are homogeneous, then the menu* $\mathcal{M} = \cup_{l \in \{1,\dots,M\}} \{b_l^1, \dots, b_l^{\lfloor \frac{M}{l} \rfloor}\}$ *is* $\mathbb{G}$*-sufficient and* $k \leq \frac{M^2+M}{2}$*, where* $\{b_l^j\}$ *are distinct bundles of size* l*.*

In this case, CASA simply auctions $\lfloor \frac{M}{l} \rfloor$ copies of each quantity level $l \leq M$ via individual ascending auctions. Like the discussion in partitional complementarity, there may be finitely many types of homogeneous goods. As long as the number of types I is bounded, the menu consists of all combinations of $\lfloor \frac{M}{l} \rfloor$ copies of each type is sufficient and of size $Poly(M)$. Such preference structure is typical in examples like the spectrum auctions. Different frequencies are almost physically homogeneous, except that "middle" frequencies might be of different value from "boundary" frequencies.

5 Concluding remarks

In this paper, we design an auction format of CASA that guarantees an approximately optimal ex-post revenue. To achieve this, we only need to assume minimal rationality on the part of the bidders. In addition, we show that CASA is robust to distributional and strategic uncertainties under certain approximations. In practice, these approximation gaps may become non-negligible, rendering the deployment of CASA challenging.

- *Menu design:* The revenue performance of CASA as well as its strategic robustness crucially hinges on the rank k (menu size) being small relative to the number of bidders. In the online advertising examples we introduce, the complete menu is small enough that a handful of bidders may be sufficient to make CASA an appealing design. However, other interesting auctions may suffer the large menu problem (e.g. the land auctions) or the thin market problem (e.g. the route auctions of rideshare apps) or both (e.g., the spectrum auctions), rendering the guarantee underpowered.

  In the latter cases, the menu sufficiency-approximation efficiency tradeoff becomes eminent. Our theory suggests the importance of preference estimation in those settings. Finding a simple sufficient menu still keeps the revenue guarantee appealing and CASA directly applicable. Even in settings like the spectrum auction where our theory has no bite, menu design may still be a cost-effective way to promote competition and improve the revenue performance of existing auctions.

- *Proxy bidding:* While CASA simplifies the bidding process by clarifying "whether to quit," the complexity of determining "which bundles to bid on" and "how much to bid" remains unresolved. A *truthful* and full proxy-bidding version of CASA is not yet known to us. This makes the deployment of CASA challenging in environments that require fast resolutions of auctions. Nevertheless, we propose that advancements in AI could mitigate this by introducing "copilot" features that assist bidders in decision-making. By integrating AI as the bidding proxy, bidders would only need to specify values for desired bundles, with the AI advising on bid placement. This could evolve into a hybrid model where bidders either rely fully on platform-provided AI, develop their own bidding algorithms, or use a combination of both strategies.

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A Omitted Proofs

A.1 Proof of Proposition 3

Proof. Pick an arbitrary bundle $b \in \mathcal{M}$. Let $v_{b'} = \mathbf{1}_{b'=b} \cdot U[0,1]$; that is, b is the only valuable bundle and its value is uniformly distributed on $[0,1]$. Let G denote such a distribution and $\mathbb{G} = \{G\}$. Then, $V_{\mathcal{M}}(F) \geq \frac{1}{2}$ for any $F \in \mathbb{F}$. Define F^* as follows: uniformly randomly pick $k-1$ bidders and their values for b are identical and distributed according to $U[1-\frac{k-1}{N}, 1]$. For the remaining bidders, their values for b are identical and distributed according to $U[0, 1-\frac{k-1}{N}]$. It is straightforward to verify that $F^* \in \mathbb{F}$ and

$$\mathbb{E}_{F^*}[R^k_{\mathcal{M}}(\boldsymbol{v})] = \mathbb{E}_{U[0,1-\frac{k-1}{N}]}[x] = \frac{1}{2} - \frac{k-1}{2N} \leq \inf_{F\in\mathbb{F}} V_{\mathcal{M}}(F) - O\left(\tfrac{k}{N}\right).$$

Q.E.D.

A.2 Proof of Proposition 5

Proof. We slightly abuse notation and represent an allocation by a vector of sets $\boldsymbol{b} = (b_1, b_2, \ldots, b_N)$, where $b_n \cap b_{n'} = \emptyset$ and b_n is the bundle allocated to bidder n. Let $\mathcal{B}_N$ denote the set of all feasible allocations with N bidders. Let $\boldsymbol{b}^*(\boldsymbol{v})$ denote the efficient allocation.

We establish a lower bound of the revenue-guarantee of the VCG mechanism by constructing, for each n, an allocation $\boldsymbol{b}^n \in \mathcal{B}_{N-1}$ of the objects to the bidders other than bidder n. Clearly, for any such profile $\boldsymbol{b}^n$,

$$\begin{aligned} R_{VCG}(\boldsymbol{v}) &= \sum_{n=1}^{N} \left(\sup_{\boldsymbol{b}\in\mathcal{B}_{N-1}} \sum_{n'\neq n} v^{n'}_{b_{n'}} - \sum_{n'\neq n} v^{n'}_{b^*_{n'}(\boldsymbol{v})} \right) \\ &\geq \sum_{n=1}^{N} \left(\sum_{n'\neq n} v^{n'}_{b^n_{n'}} - \sum_{n'\neq n} v^{n'}_{b^*_{n'}(\boldsymbol{v})} \right). \end{aligned} \tag{1}$$

For each n, we construct an allocation $\boldsymbol{b}^n \in \mathcal{B}_{N-1}$ via the following algorithm:

Algorithm. Bundle $b^n_{n'} = \emptyset$ for all n'. Set $O = b^*_n(\boldsymbol{v})$.

(1). For each $n' \neq n$:

If $b^*_{n'}(\boldsymbol{v}) \neq \emptyset$, set $b^n_{n'} = b^*_{n'}(\boldsymbol{v})$.

Let $\bar{N} = \{n' : b^n_{n'} = \emptyset, n' \neq n\}$.

(2). If $O \neq \emptyset$, then pick $o \in O$.

Set $b^n_{n'} = \{o\}$ for some $n' \in \arg\max_{n'' \in \bar{N}} v_{n''}(\{o\})$.

Update $O \leftarrow O \setminus \{o\}$ and $\bar{N} \leftarrow \bar{N} \setminus \{n'\}$.

(3). Repeat (2) until $O = \emptyset$.

(4). Return allocation $\boldsymbol{b}^n = (b^n_1, b^n_2, \ldots, b^n_{n-1}, b^n_{n+1}, \ldots, b^n_N)$.

In words, if an object is allocated to a bidder other than bidder n under $\boldsymbol{b}^*(\boldsymbol{v})$, then the object is still allocated to that bidder. We then iteratively pick an object o that is allocated to bidder n under $\boldsymbol{b}^*(\boldsymbol{v})$, and allocate the object to the bidder n' whose value for the object $v^{n'}_{\{o\}}$ is the highest among all the bidders who are not allocated any object yet. For each $o \in b^*_n$, define n_o to be the index n' such that $b^n_{n'} = \{o\}$.

It follows from Equation (1) that

$$\begin{aligned} R_{VCG}(\boldsymbol{v}) \geq & \sum_{n=1}^{N} \left(\sum_{n' \neq n} v^{n'}_{b^n_{n'}} - \sum_{n' \neq n} v^{n'}_{b^*_{n'}(\boldsymbol{v})} \right) \\ = & \sum_{n=1}^{N} \sum_{o \in b^*_n} v^{n_o}_{\{o\}} \\ \geq & \sum_{o=1}^{M} v^{(M+1)}_{\{o\}} \\ = & R^{M+1}_{\mathcal{M}}(\boldsymbol{v}). \end{aligned} \tag{2}$$

The first equality holds since (a) when $b^*_{n'}(\boldsymbol{v}) \neq \emptyset$, $b^n_{n'} = b^*_{n'}(\boldsymbol{v})$, and (b) when $b^*_{n'}(\boldsymbol{v}) = \emptyset$, $b^n_{n'}$ is either $\{o\}$ for some $o \in O$, or $\emptyset$ otherwise. The second inequality follows from the construction of $\boldsymbol{b}^i$: when an object $o \in b^*_i$ is being allocated, it is allocated to the bidder n' whose value for the object $v^{n'}_{\{o\}}$ is the highest among all the bidders who are not allocated any object yet. Since each iteration assigns at least one good to one bidder and there are at most M goods, we have $v^{n'_o}_{\{o\}}$ must be at least the $(M+1)^{th}$ highest value among all $v^n_{\{o\}}$. *Q.E.D.*