3-2024

The interplay of interdependence and correlation in bilateral trade

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STATISTICS

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March 2024

Paper No. 04-2024

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This Version: March 2024

Abstract

Crémer and McLean (1988) show that the seller can extract full surplus almost always by an incentive compatible, individually rational mechanism in a single-unit auction model with a finite type space in which agents' beliefs are correlated and their valuations can be interdependent. We first show that this paradoxically positive result can be extended to a model of bilateral trades. To make it more realistic, we investigate when ex-post efficiency and ex-post budget balance in bilateral trades can also be achieved by an incentive compatible, individually rational mechanism. We identify a necessary condition for the existence of such mechanisms and show that it is also sufficient for a two-type model. We next show that the identified condition is not sufficient in general. Through a series of examples, we show that the imposition of ex post budget balance in a bilateral trade model induces a delicate interaction between interdependent values and correlated beliefs, so that the existence of incentive compatible, individually rational mechanisms becomes a very subtle problem. Finally, focusing on a model with linear valuations, we give the precise sense in which a possibility result under interdependent values is more fragile than that under private values.

JEL Classification: C72, D78, D82.

Keywords: bilateral trade, interdependence, correlation.

*We thank Shurojit Chatterji and Takuro Yamashita for useful comments.

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1 Introduction

Crémer and McLean (1985, 88) show that the auctioneer can almost always extract the full surplus through a mechanism satisfying Bayesian incentive compatibility (BIC) and interim individual rationality (IIR) in a single-unit auction model with a finite type space in which agents' beliefs are correlated and their valuations can be interdependent. This is an overly permissive result. Our first result is to extend this permissive result to a model of bilateral trade, describing a simple trading problem in which two individuals, one of whom (a seller) has a single indivisible object to sell to the other (a buyer), attempt to agree to exchange the object for money.[1] We show in our Proposition 1 that, if what we call the Crémer-McLean condition is satisfied, there exists a mechanism that satisfies BIC, IIR, ex post efficiency (EFF), and ex ante budget balance (EABB), where EFF means that the good is traded if and only if the buyer's expected value for the good is strictly higher than the seller's counterpart, and EABB means that the expected payment made by the buyer is exactly the same as that received by the seller "before agents' types are realized."[2] Furthermore, the seller can extract the full surplus in the mechanism and the Crémer-McLean condition is almost always satisfied.

To make this permissive result less extreme, we find it natural to strengthen EABB to ex post budget balance (EPBB), which means that the budget balance is rather satisfied "for each realization of agents' types." We show in our Theorem 1 that what we call "the Ex-Post condition" is necessary for the existence of mechanisms that satisfy BIC, IIR, EFF, and EPBB. We also show in Proposition 3 that the Ex-Post condition is also sufficient for the existence of desired mechanisms "when both agents have two types." This result generalizes Gresik (1991a) and Matsuo (1989), both of which focus on a model with private values (i.e., each agent is certain about the value of the object at the time of trade). However, when each agent has three possible types, we also show by example that the Ex-Post condition is satisfied, while there are no desired mechanisms (Example 1).

Thus, an important question remains to find a sufficient condition for the existence of desired mechanisms when each agent has more than two types. In his Proposition 3, Matsushima (2007) provides a sufficient condition under which there are no mechanisms that satisfy BIC, IIR, and EPBB. If we further impose EFF on mechanisms, we show that Matsushima (2007)'s sufficient condition has no bites

[1] McAfee and Reny (1992) obtain a similar result in a bilateral trade model with continuous types.

[2] We follow Börgers (2015) and Kosenok and Severinov (2008) for the term of the Crémer-McLean condition.

in all nontrivial cases of bilateral trade. This implies that Matsushima (2007) does not help us obtain an impossibility result in our bilateral trade model. In their Theorem 1, Kosenok and Severinov (2008) establish that, in general mechanism design problems, there exists a mechanism that satisfies BIC, IIR, EFF, and EPBB when the Crémer-McLean condition and what they call the *Identifiability* condition are satisfied. However, when there are only two agents, as in our bilateral trade model, Kosenok and Severinov (2008) show that the Identifiability condition is satisfied only if the agents' beliefs are independent. Since the Identifiability condition is clearly violated in a model with correlated beliefs, Theorem 1 of Kosenok and Severinov (2008) does not help us get a possibility result in our bilateral trade model. This leads us to conclude that previous attempts in the literature have not yet found a sufficient condition for the existence of mechanisms that satisfy BIC, IIR, EFF, and EPBB in a bilateral trade model with interdependent values and correlated beliefs.

What we accomplish in the rest of the paper is not to provide such a sufficient condition but to illustrate the complexity of obtaining it. Through a series of examples, we show that the imposition of EPBB in a bilateral trade model induces a delicate interaction between interdependent values and correlated beliefs, so that finding such a sufficient condition for desirable mechanisms becomes a very subtle problem. Recall that there are no desired mechanisms in Example 1. We only change the agents' beliefs, but retain the rest of the model in Example 1. This modified example (Example 2) now admits a desired mechanism. We "slightly" perturb the agents' beliefs, but keep the rest of the model in Example 2. In this modified example (Example 3), however, we show that there are no desired mechanisms. Next, we modify Example 3 only by changing the interdependent values model to a private values one and keeping the rest of the model. In this modified example (Example 4), we show that there is a desired mechanism in Example 4, and that it persists even under small perturbations of the agents' beliefs, just as Example 3 is a small perturbation of Example 2. We construct Example 5 in the same way as Example 4, except that we increase the size of the perturbations of the agents' beliefs much more. In Example 5, we show that there are no desired mechanisms.

All of these examples suggest that a possibility result under interdependent values is more fragile than that under private values, in the sense that the size of perturbations of the agents' beliefs needed to break down the possibility result under private values models is much larger than that needed to break down that under interdependent values models. We generalize this insight by focusing on linear valuation functions. Recall that the Ex-Post condition is a necessary condition for the existence of desired mechanisms. In particular, we show that the greater the degree of interdependence, the more likely it is that the Ex-Post condition will be violated. Finally, we discuss the general difficulty of obtaining a sufficient condition for the existence of desired mechanisms.

Our paper is also related to Myerson and Satterthwaite (1983), who established a celebrated impossibility result: in a bilateral trade model with (i) private values, (ii) independent beliefs, and (iii) continuous types, there are generally no mechanisms that satisfy BIC, IIR, EFF, and EPBB. When extending a model of private values to that of interdependent values but keeping (ii) and (iii), Fieseler, Kittsteiner, and Moldovanu (2003) and Gresik (1991b) restore the Myerson and Satterthwaite impossibility theorem. This is perfectly consistent with the fact that we consider a finite type space with correlated beliefs to obtain a possibility result.

The rest of the paper is organized as follows. In Section 2, we introduce the general notation and basic concepts of the paper. In Section 3, we show that when the Crémer-McLean condition is satisfied, there are mechanisms that satisfy BIC, IIR, EFF, and EABB. To make the permissive result in the previous section more realistic, we strengthen EABB to EPBB in the rest of the paper. In Section 4, we introduce the Ex-Post condition, show that it is a necessary condition, it is also a sufficient condition for the case of two possible types, but it is not sufficient in general. We also discuss prior attempts and conclude that they do not help us find a sufficient condition for the existence of desired mechanisms. Section 5 provides a series of examples to illustrate the delicate interplay of interdependence and correlated beliefs. In Section 6, we generalize the insight obtained in Section 5 by focusing on environments in which agents' have linear valuation functions. Section 7 concludes the paper by discussing the general difficulty of obtaining a sufficient condition for the existence of desired mechanisms.

2 Preliminaries

A seller (agent 1) has one indivisible object for sale and there is one potential buyer (agent 2). Each agent has some private information concerning the value of the good, which is summarized as their own type θ_i. For each agent $i \in \{1, 2\}$, their type space is $\Theta_i = \{\theta_i^1, \cdots, \theta_i^{m_i}\}$ which is a finite subset of a Euclidean space and $m_i \geq 2$ denotes the number of types for agent i. Throughout the paper, we use the notation convention that $\Theta = \Theta_1 \times \Theta_2$ and $\Theta_{-i} = \Theta_j$ where $j \neq i$ with a generic element θ_{-i}.

Types are drawn from a full-support probability distribution μ over Θ: $\mu(\theta_1, \theta_2) > 0$ for each $(\theta_1, \theta_2) \in \Theta$ and $\sum_{\theta_1 \in \Theta_1} \sum_{\theta_2 \in \Theta_2} \mu(\theta_1, \theta_2) = 1$. For every θ_i, we denote by $\mu_i(\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_i, \theta_{-i})$ the marginal probability that agent i is of type θ_i. We further define the following notation for conditional probabilities: for each $(\theta_1, \theta_2) \in \Theta$, $\mu(\theta_2|\theta_1) \equiv \mu(\theta_1, \theta_2)/\mu(\theta_1)$ and $\mu(\theta_1|\theta_2) \equiv \mu(\theta_1, \theta_2)/\mu(\theta_2)$. (There is some abuse of notation in using the same symbol μ for probability distributions over different spaces, but this lead to no confusion.)

Each agent's valuation for the good depends not only on his own types, but also on the other's types. For each agent $i \in \{1, 2\}$ and type profile $\theta \in \Theta$, we denote by $v_i(\theta) \geq 0$ their valuation function of the good. Hence, we consider an interdependent values environment.

Let $q_i \in Q = [0, 1]$ denote the probability that agent i is allocated the good. Preferences of each agent depends on the probability of being allocated the good, the type profile and their monetary transfer. Each agents i's utility function $u_i : Q \times \Theta \times \mathbb{R} \to \mathbb{R}$ is represented by

$$u_i(q_i, \theta, p_i) = q_i v_i(\theta_i, \theta_{-i}) + p_i$$

where $q_1 + q_2 = 1$ and p_i denotes the monetary transfer received by agent i.

A *direct mechanism* is defined as a triplet $\Gamma = (\Theta, x, t)$ where Θ_1 and Θ_2 are the set of actions available to the seller and buyer, respectively; $x : \Theta \to [0, 1]$ is the *decision rule* which specifies the probability that trade occurs; and $(t_1, t_2) : \Theta \to \mathbb{R}^2$ is the *transfer rule* which describes the monetary transfers to both agents. By the revelation principle, we lose nothing to focus on direct mechanisms. In what follows, we simply call them mechanisms. We shall now discuss the properties we want a mechanism to satisfy.

Definition 1. A mechanism (x, t) satisfies *Bayesian incentive compatibility* (BIC) if, for each agent $i \in \{1, 2\}$ and $\theta_i, \theta_i' \in \Theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_{-i}|\theta_i) \left[x_i(\theta_i, \theta_{-i}) v_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i})\right] \geq \sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_{-i}|\theta_i) \left[x_i(\theta_i', \theta_{-i}) v_i(\theta_i, \theta_{-i}) + t_i(\theta_i', \theta_{-i})\right],$$

where $x_i(\theta_i', \theta_{-i})$ denotes the probability that agent i is allocated the good given the type report (θ_i', θ_{-i}).

To ensure every agent's voluntary participation in the mechanism, we assume that, if they do not participate in the mechanism, the buyer obtains a utility of zero and the seller obtains an expected utility equal to her expected valuation of the good from their outside options, respectively. We denote by $U_i^O(\theta_i)$ the expected utility of agent i of type θ_i from the outside option utility and we have $U_1^O(\theta_1) = \sum_{\theta_2 \in \Theta_2} \mu(\theta_2|\theta_1) v_1(\theta_1, \theta_2)$ for all $\theta_1 \in \Theta_1$ and $U_2^O(\theta_2) = 0$ for all $\theta_2 \in \Theta_2$.

Definition 2. A mechanism (x, t) satisfies the *interim individual rationality* (IIR) if, for each agent i and each type θ_i,

$$\sum_{\theta_{-i} \in \Theta_{-i}} \mu(\theta_{-i}|\theta_i) \left[x_i(\theta_i, \theta_{-i}) v_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i})\right] \geq U_i^O(\theta_i).$$

See, for example, Myerson and Satterthwaite (1983), Matsuo (1989), and Gresik (1991a, 1991b) for the same treatment of IIR.

Throughout the paper, we also impose the following ex post efficiency of bilateral trade.

Definition 3. A mechanism (x, t) satisfies *decision efficiency* (EFF) if, for each $(\theta_1, \theta_2) \in \Theta$,

$$x(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } v_2(\theta_1, \theta_2) > v_1(\theta_1, \theta_2) \\ 0 & \text{if } v_2(\theta_1, \theta_2) \leq v_1(\theta_1, \theta_2), \end{cases}$$

where $x(\theta_1, \theta_2)$ denotes the probability that trade occurs given the type report (θ_1, θ_2).

We denote by $x^*(\cdot)$ the efficient decision rule. In our paper, we consider two kinds of budget balance constraints: *ex ante* budget balance and *ex post* budget balance.

Definition 4. A mechanism (x, t) satisfies *ex ante budget balance* (EABB) if,

$$\sum_{\theta_1 \in \Theta_1} \sum_{\theta_2 \in \Theta_2} \mu(\theta_1, \theta_2) \left(t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2)\right) = 0.$$

Definition 5. A mechanism (x, t) satisfies *ex post budget balance* (EPBB) if, for each $(\theta_1, \theta_2) \in \Theta$,

$$t_1(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) = 0.$$

Clearly, EPBB implies EABB. Börgers (2015, Proposition 3.6) show that if the types are *independently* distributed, for every direct mechanism that is ex ante budget balanced, there is an equivalent direct mechanism that is ex post budget balanced.[3] As we will show in this paper, when agents' beliefs are correlated, EABB can be satisfied almost always by a BIC, IIR, and EFF mechanism (Section 3), whereas we argue that the existence of mechanisms satisfying EPBB as well as other desirable properties becomes a very subtle problem (Section 4).

[3] The reader is referred to Börgers (2015, Definition 3.7) for the definition of equivalent direct mechanisms.

3 The Existence of Mechanisms Satisfying EABB, BIC, IIR, and EFF

In their Theorem 2, Crémer and McLean (1988) identify a condition (henceforth, we call it the *Crémer-McLean condition*) as necessary and sufficient for full surplus extraction to be achieved by a mechanism satisfying BIC and IIR in a single-unit auction setup.

Definition 6. The distribution μ satisfies the *Crémer-McLean condition* if, for each agent $i \in \{1,2\}$, there do not exist $\theta_i \in \Theta_i$ and a family of real numbers $\{\lambda_i(\theta_i')\}_{\theta_i' \in \Theta_i \backslash \{\theta_i\}}$ such that

1. $\lambda_i(\theta_i') \geq 0$ for all $\theta_i' \in \Theta_i \backslash \{\theta_i\}$, and
2. $\mu(\theta_{-i}|\theta_i) = \sum_{\theta_i' \neq \theta_i} \lambda_i(\theta_i')\mu(\theta_{-i}|\theta_i')$ for all $\theta_{-i} \in \Theta_{-i}$.

We interpret the Crémer-McLean condition as follows: $\mu(\cdot \mid \theta_i)$ is considered a vector with as many entries as Θ_{-i} has elements. Agent i's conditional beliefs are described by a collection of vectors of this form, one for each of agent i's type. The Crémer-McLean condition requires that none of these vectors can be written as a convex combination of all the other vectors where the weights are denoted by $\lambda(\theta_i')$.[4]

In auctions, the seller plays the role of an outsider whose valuation is normalized to zero and the seller makes no monetary transfer other than collecting payments from the buyers. In the bilateral trade environment, however, the seller has private information which should be elicited within the mechanism and she is asked to make monetary transfers based on the reports. Thus, unlike the proof of Theorem 2 of Crémer and McLean (1988), we also need to propose a transfer scheme for the seller in our bilateral trade setup. Despite the above mentioned differences, we still find it natural to conjecture that we can extend Theorem 2 of Crémer and McLean (1988) to a bilateral trade setup.[5] The result below is such an extension.

Proposition 1. There exists a transfer rule t such that the mechanism (x^*, t) satisfies BIC, IIR, EFF and EABB and gives all of the rents to the seller if the distribution μ satisfies the Crémer-McLean condition.

Proof. Assume that for each $i \in \{1,2\}$ and $\theta_i \in \Theta_i$, there is no family of real numbers $\{\lambda_i(\theta_i')\}_{\theta_i' \in \Theta_i \backslash \{\theta_i\}}$ which satisfies Properties 1 and 2 of the Crémer-McLean

[4]See Börgers (2015, pp.120-121) for the interpretation adopted in this paper.
[5]See Börgers (2015, p.127) for a similar view.

condition. Fix $i \in \{1,2\}$ and $\theta_i \in \Theta_i$. By Farkas' Lemma, there exists a family of real numbers $\{f_i(\theta_i,\theta_{-i})\}_{(\theta_i,\theta_{-i})\in\Theta}$ such that

$$\sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i) f_i(\theta_i,\theta_{-i}) > 0$$

and

$$\sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i') f_i(\theta_i,\theta_{-i}) \leq 0$$

for all $\theta_i' \in \Theta_i \backslash \{\theta_i\}$. Let $\varepsilon \equiv \sum_{\theta_{-i}\in\Theta_{-i}} \mu_i(\theta_{-i}|\theta_i) f_i(\theta_i,\theta_{-i})$ and, for each $\theta_{-i} \in \Theta_{-i}$, let $h_i(\theta_i,\theta_{-i}) \equiv f_i(\theta_i,\theta_{-i}) - \varepsilon$. We obtain

$$\begin{aligned}
\sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i) h_i(\theta_i,\theta_{-i}) &= \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i)\left(f_i(\theta_i,\theta_{-i}) - \varepsilon\right) \\
&= \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i) f_i(\theta_i,\theta_{-i}) - \varepsilon \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i) \\
&= \varepsilon - \varepsilon = 0,
\end{aligned}$$

and for all $\theta_i' \in \Theta_i \backslash \{\theta_i\}$,

$$\begin{aligned}
\sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i') h_i(\theta_i,\theta_{-i}) &= \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i')\left(f_i(\theta_i,\theta_{-i}) - \varepsilon\right) \\
&= \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i') f_i(\theta_i,\theta_{-i}) - \varepsilon \sum_{\theta_{-i}\in\Theta_{-i}} \mu(\theta_{-i}|\theta_i') \\
&\leq 0 - \varepsilon < 0.
\end{aligned}$$

Consider the following transfer rule: for each $(\theta_1,\theta_2) \in \Theta$,

$$\begin{aligned}
t_1(\theta_1,\theta_2) &= -\sum_{\tilde{\theta}_2\in\Theta_2} \mu(\tilde{\theta}_2|\theta_1)\left(1 - x^*(\theta_1,\tilde{\theta}_2)\right) v_1(\theta_1,\tilde{\theta}_2) + \gamma_1(\theta_1) h_1(\theta_1,\theta_2) + v_1(\theta_1,\theta_2) \\
&\quad + \sum_{\tilde{\theta}_1\in\Theta_1}\sum_{\tilde{\theta}_2\in\Theta_2} \mu(\tilde{\theta}_1,\tilde{\theta}_2) x^*(\tilde{\theta}_1,\tilde{\theta}_2)\left(v_2(\tilde{\theta}_1,\tilde{\theta}_2) - v_1(\tilde{\theta}_1,\tilde{\theta}_2)\right)
\end{aligned}$$

for some collection of real numbers $\{\gamma_1(\theta_1)\}_{\theta_1\in\Theta_1}$ and

$$t_2(\theta_1,\theta_2) = -\sum_{\tilde{\theta}_1\in\Theta_1} \mu(\tilde{\theta}_1|\theta_2) x^*(\tilde{\theta}_1,\theta_2) v_2(\tilde{\theta}_1,\theta_2) + \gamma_2(\theta_2) h_2(\theta_1,\theta_2)$$

for some collection of real numbers $\{\gamma_2(\theta_2)\}_{\theta_2\in\Theta_2}$. In the rest of the proof, we must show that the mechanism (x^*,t) satisfies EABB, IIR, BIC and distributes the full surplus to the seller. This portion of the proof is in the Appendix. □

McAfee and Reny (1992) establish in their Theorem 2 that the continuum analogue of the Crémer-Mclean condition is necessary and sufficient for (almost) full surplus extraction. They also show that if the continuum analogue of the Crémer-Mclean condition is satisfied, the seller can extract "almost" full surplus in a bilateral trade model. This is in a contrast with our Proposition 1 where the Crémer-McLean condition is sufficient for "exact" full surplus extraction.[6]

4 The Existence of Mechanisms Satisfying EPBB, BIC, IIR, and EFF

In the rest of the paper, we illustrate the implication of Proposition 1 by imposing the *full rank* condition, which requires that the rank of the collection of vectors that describe agent i's conditional beliefs, $\{\mu(\cdot|\theta_i)\}_{\theta_i \in \Theta_i}$ be equal to the number of agent i's types. The Crémer-McLean condition is obviously satisfied if the full rank condition is satisfied, and hence the vectors are linearly independent. Therefore, the Crémer-McLean condition will be satisfied by "nearly any" distribution μ. This is an extremely permissive implementation result. To make this result less paradoxical, we strengthen ex ante budget balance (EABB) into ex post budget balance (EPBB) and investigate the condition under which there exist mechanisms satisfying EPBB as well as BIC, IIR, and EFF.

In what follows, we assume that each agent's type space is one dimensional and that both agents have the same number of types denoted by m.[7] For each agent $i \in \{1, 2\}$, their type space is $\Theta_i = \{\theta_i^1, \cdots, \theta_i^m\}$ with $0 < \theta_i^1 < \cdots < \theta_i^m$ and we denote a generic element by θ_i. We also assume that for each agent i, $v_i(\theta_i, \theta_{-i})$ is strictly increasing in agent i's own type θ_i and nondecreasing in the other's type θ_{-i}. We further impose the single crossing condition on the valuation functions.

Definition 7. The valuation functions satisfy the *single crossing condition* if the following statement holds: for each agent $i \in \{1, 2\}$, if there exists some type profile $(\theta_i, \theta_{-i}) \in \Theta$ such that $v_i(\theta_i, \theta_{-i}) > v_{-i}(\theta_i, \theta_{-i})$, then $v_i(\theta_i', \theta_{-i}) > v_{-i}(\theta_i', \theta_{-i})$ must hold for all $\theta_i' > \theta_i$.

[6] To replace "almost" with "exact," McAfee and Reny (1992) identify an additional condition together with the Crémer-McLean condition under which exact full surplus can be extracted. More importantly, the mechanism which achieves full surplus extractions can be explicitly constructed in their paper whereas we rely on Farkas' Lemma to establish the existence of such a mechanism.

[7] This assumption is made without loss of generality under the full rank condition. In a bilateral trade setup where there are only two agents, the full rank condition implicitly requires that both agents have the same number of types.

Intuitively, the single crossing condition requires that if agent i already has a higher valuation at some type profile and we keep increasing agent i's type while keeping the other's type fixed, then agent i should continue to have a higher valuation.

4.1 The Efficient Decision Rule

There are many different efficient decision rules we can consider. To make our analysis tractable, we aim to restrict the set of efficient decision rules to a small class of efficient decision rules. In particular, we focus on what we call *responsive* decision rules.[8]

Definition 8. An efficient decision rule $x^*(\cdot)$ is *responsive* if the following two requirements are satisfied:

1. for any two distinct types of the seller, $\theta_1^k, \theta_1^{k'} \in \Theta_1$, there exists some $\theta_2^l \in \Theta_2$ such that
$$x^*(\theta_1^k, \theta_2^l) \neq x^*(\theta_1^{k'}, \theta_2^l),$$
where we denote by θ_1^k and θ_2^l a generic element in the seller and buyer's type space, respectively, where $k, l \in \{1, \cdots, m\}$.

2. for any two distinct types of the buyer, $\theta_2^l, \theta_2^{l'} \in \Theta_2$, there exists some $\theta_1^k \in \Theta_1$ such that
$$x^*(\theta_1^k, \theta_2^l) \neq x^*(\theta_1^k, \theta_2^{l'}).$$

We show below that responsiveness together with the single crossing condition pins down the unique efficient decision rule in any finite type space.

Proposition 2. Suppose that the single crossing condition holds; the efficient decision rule is responsive; and $x^*(\theta_1^1, \theta_2^1) = 1$, i.e., it is efficient to trade at the lowest type profile. Then, the efficient decision rule is uniquely determined as follows: for each $(\theta_1^k, \theta_2^l) \in \Theta$,
$$x^*(\theta_1^k, \theta_2^l) = \begin{cases} 1 & \text{if } k \leq l \\ 0 & \text{otherwise.} \end{cases}$$

Remark: If $x^*(\theta_1^1, \theta_2^1) = 0$, we can similarly show that the efficient decision rule is uniquely determined: for each $(\theta_1^k, \theta_2^l) \in \Theta$, $x^*(\theta_1^k, \theta_2^l) = 1$ if and only if $k \leq l - 1$. Thus, it is essentially without loss of generality to assume $x^*(\theta_1^1, \theta_2^1) = 1$.

[8]The terminology is adopted from Bergemann and Morris (2009, p.1187), who propose responsiveness a one-to-one property of social choice functions in their analysis of robust implementation.

Proof. The proof consists of four steps. In Step 1, we show that $x^*(\theta_1^k, \theta_2^l)$ is nondecreasing in θ_2^l. In Step 2, we show that $x^*(\theta_1^k, \theta_2^l)$ is nonincreasing in θ_1^k. Finally, Steps 3 and 4 pin down the unique efficient decision rule $x^*(\cdot)$.

Step 1: $x^*(\theta_1^k, \theta_2^l)$ is nondecreasing in θ_2^l for every $k \in \{1, \ldots, m\}$.

Proof. Fix an arbitrary $\theta_1^k \in \Theta_1$. Suppose, on the contrary, that there exist $\theta_2^l, \theta_2^{l'} \in \Theta_2$ with $l < l'$ such that $x^*(\theta_1^k, \theta_2^l) = 1$ and $x^*(\theta_1^k, \theta_2^{l'}) = 0$. Notice that $x^*(\theta_1^k, \theta_2^l) = 1$ implies $v_2(\theta_1^k, \theta_2^l) > v_1(\theta_1^k, \theta_2^l)$. If the buyer's type increases to $\theta_2^{l'}$, by the single crossing condition, the buyer's valuation must continue to be higher than the seller's, implying $x^*(\theta_1^k, \theta_2^{l'}) = 1$, which contradicts our hypothesis that $x^*(\theta_1^k, \theta_2^{l'}) = 0$. □

Step 2: $x^*(\theta_1^k, \theta_2^l)$ is nonincreasing in θ_1^k for every $l \in \{1, \ldots, m\}$.

Proof. Fix an arbitrary $\theta_2^l \in \Theta_2$. Suppose, on the contrary, that there exist $\theta_1^k, \theta_1^{k'} \in \Theta_1$ with $k < k'$ such that $x^*(\theta_1^k, \theta_2^l) = 0$ and $x^*(\theta_1^{k'}, \theta_2^l) = 1$. Notice that $x^*(\theta_1^k, \theta_2^l) = 0$ implies $v_1(\theta_1^k, \theta_2^l) \geq v_2(\theta_1^k, \theta_2^l)$. If the seller's type increases to $\theta_1^{k'}$, by the single crossing condition, the seller's valuation must continue to be higher than the buyer's, implying $x^*(\theta_1^{k'}, \theta_2^l) = 0$, which contradicts our hypothesis $x^*(\theta_1^{k'}, \theta_2^l) = 1$. □

Step 3: $x^*(\theta_1^k, \theta_2^k) = 1$ and $x^*(\theta_1^k, \theta_2^{k-1}) = 0$ for all $k \in \{2, \cdots, m\}$.

Proof. The proof is in the Appendix.[9] □

Step 4: For each $(\theta_1^k, \theta_2^l) \in \Theta$, $x^*(\theta_1^k, \theta_2^l) = 1$ if and only if $k \leq l$.

Proof. The following table describes the allocation decisions implied by Step 3.

Table 1

$x^*(\cdot)$	θ_2^1	θ_2^2	θ_2^3	$\cdots$	θ_2^{m-1}	θ_2^m
θ_1^1						
θ_1^2	0	1				
θ_1^3		0	1			
$\vdots$			$\ddots$	$\ddots$		
θ_1^{m-1}				$\ddots$	$\ddots$	
θ_1^m					0	1

[9]This is the only place where responsiveness plays a role.

For all $k \in \{2, \cdots, m\}$, since $x^*(\theta_1^k, \theta_2^k) = 1$ (by Step 3) and $x^*(\theta_1^k, \theta_2^l)$ is nondecreasing in θ_2^l (by Step 1), it follows that $x^*(\theta_1^k, \theta_2^l) = 1$ for all $l \geq k$. Moreover, for all $l \in \{1, \cdots, m\}$, since $x^*(\theta_1^{l+1}, \theta_2^l) = 0$ (by Step 3) and $x^*(\theta_1^k, \theta_2^l)$ is nonincreasing in θ_1^k (by Step 2), it follows that $x^*(\theta_1^k, \theta_2^l) = 0$ for all $k \geq l+1$. Finally, since $x^*(\theta_1^1, \theta_2^1) = 1$ (by our assumption) and $x^*(\theta_1^1, \theta_2^l)$ is nondecreasing in θ_2^l (by Step 1), we obtain $x^*(\theta_1^1, \theta_2^l) = 1$ for all $l \geq 1$. To combine all the implications, we conclude that $x^*(\theta_1^k, \theta_2^l) = 1$ if and only if $k \leq l$. □

We complete the proof of the proposition. □

In the rest of the paper, we focus on the following efficient decision rule depicted in the table below.

Table 2: The Efficient Decision Rule $x^*(\cdot)$

$x^*(\cdot)$	θ_2^1	θ_2^2	θ_2^3	$\cdots$	θ_2^m
θ_1^1	1	1	1	$\cdots$	1
θ_1^2	0	1	1	$\cdots$	1
θ_1^3	0	0	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
θ_1^m	0	0	0	$\cdots$	1

4.2 The Ex-Post Condition is Necessary

In this section, we further impose the following property on the class of mechanisms:

Definition 9. A transfer rule $t : \Theta \to \mathbb{R}^2$ satisfies the *no-trade-then-no-payments* (NTNP) property if, for any $(\theta_1^k, \theta_2^l) \in \Theta$,

$$t_1(\theta_1^k, \theta_2^l) = t_2(\theta_1^k, \theta_2^l) = 0 \text{ whenever } x(\theta_1^k, \theta_2^l) = 0.$$

This property says that if trade does not occur, no monetary transfers are made.[10] In what follows, we call such a transfer rule a no-trade-then-no-payments (NTNP) transfer rule.

Here we propose the Ex-Post condition as the key condition for the existence of mechanisms satisfying BIC, IIR, EFF, and EPBB.

[10]Theorem 1 of Gresik (1991a) also impose this property.

Definition 10. The distribution μ and the collection of valuations $(v_1(\theta), v_2(\theta))_{\theta\in\Theta}$ satisfy *the Ex-Post condition* if the following inequality holds:

$$\begin{aligned} & \sum_{l=1}^{m} \mu(\theta_1^1, \theta_2^l) v_2(\theta_1^1, \theta_2^1) + \sum_{k=2}^{m} \sum_{l=k}^{m} \mu(\theta_1^k, \theta_2^l) v_2(\theta_1^k, \theta_2^l) \\ \geq \ & \sum_{k=1}^{m} \mu(\theta_1^k, \theta_2^m) v_1(\theta_1^m, \theta_2^m) + \sum_{k=1}^{m-1} \sum_{l=k}^{m-1} \mu(\theta_1^k, \theta_2^l) v_1(\theta_1^k, \theta_2^l). \end{aligned} \tag{1}$$

We shall show that the Ex-Post condition is necessary for the existence of an NTNP transfer rule $t(\cdot)$ such that, together with the efficient decision rule x^* described in Table 2, the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB.

Theorem 1. If there exists an NTNP and EPBB transfer rule $t(\cdot)$ such that, together with the efficient decision rule x^* described in Table 2, the resulting mechanism (x^*, t) satisfies BIC, IIR and EFF, then the Ex-Post condition holds.

Proof. The proof is in the Appendix. □

4.3 The Ex-Post Condition is Sufficient When There Are Only Two Types

When each agent has only two types, the efficient decision rule in Table 2 reduces to the following.

$x^*(\cdot)$	θ_2^1	θ_2^2
θ_1^1	1	1
θ_1^2	0	1

The above efficient decision rule $x^*(\cdot)$ is exactly identical to the one considered by Matsuo (1989) and Gresik (1991a) in a two-type space model with private values.

In two-type space, the Ex-Post condition, i.e., inequality (1), can be simplified to the following:

$$\sum_{l=1}^{2} \mu(\theta_1^1, \theta_2^l) v_2(\theta_1^1, \theta_2^1) + \mu(\theta_1^2, \theta_2^2) v_2(\theta_1^2, \theta_2^2) \geq \sum_{k=1}^{2} \mu(\theta_1^k, \theta_2^2) v_1(\theta_1^2, \theta_2^2) + \mu(\theta_1^1, \theta_2^1) v_1(\theta_1^1, \theta_2^1). \tag{2}$$

We establish below that in two-type space, inequality (1) is also sufficient for the existence of BIC, IIR, EFF, EPBB and NTNP mechanisms.

Proposition 3. Assume that each agent has only two types, i.e., $m = 2$. Then, there exists an NTNP transfer rule t such that, together with the efficient decision rule described in Table 2, the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB if and only if the Ex-Post condition, i.e., inequality (2), holds.

Proof. The proof is in the Appendix. □

Matsuo (1989) also establishes a necessary and sufficient condition for the existence of BIC, IIR, EFF and EPBB mechanism in two-type space under private values and independent distribution, which is inequality (9) in his paper. We argue below that our necessary and sufficient condition, inequality (2), generalizes Matsuo (1989)'s inequality (9) to a setup with interdependent values and correlated distribution.

Remark 1. Consider a private-value setup where $v_1(\theta_1^k, \theta_2^l) = \theta_1^k$ and $v_2(\theta_1^k, \theta_2^l) = \theta_2^l$ for each $(\theta_1^k, \theta_2^l) \in \Theta$. Suppose that the agents' types are independently distributed and for each agent $i \in \{1, 2\}$, let μ_i be the probability distribution over $\Theta_i = \{\theta_i^1, \theta_i^2\}$. We follow the notation of Matsuo (1989) and set $\mu_1(\theta_1^2) = \varepsilon$ and $\mu_2(\theta_2^1) = \delta$ where $\varepsilon, \delta \in (0, 1)$. Then, the Ex-Post condition, i.e., inequality (2), becomes

$$\sum_{l=1}^{2} \mu_2(\theta_2^l)\mu_1(\theta_1^1)\theta_2^1 + \mu_1(\theta_1^2)\mu_2(\theta_2^2)\theta_2^2 \geq \sum_{k=1}^{2} \mu_1(\theta_1^k)\mu_2(\theta_2^2)\theta_1^2 + \mu_1(\theta_1^1)\mu_2(\theta_2^1)\theta_1^1.$$

Plugging the probability distributions into the above condition, we can rewrite it as

$$(1 - \varepsilon)\theta_2^1 + \varepsilon(1 - \delta)\theta_2^2 \geq (1 - \delta)\theta_1^2 + (1 - \varepsilon)\delta\theta_1^1,$$

which is the same as inequality (9) in the main theorem of Matsuo (1989).

4.4 The Ex-Post Condition is not Sufficient When Each Agent Has More Than Two Types

In this subsection, we start our discussion from introducing the following example.

Example 1. Suppose that each agent has three possible types, i.e., $m = 3$. The seller's type space is $\Theta_1 = \{\theta_1^1, \theta_1^2, \theta_1^3\} = \{1, 3, 5\}$ with $\theta_1^1 < \theta_1^2 < \theta_1^3$. The buyer's type space is $\Theta_2 = \{\theta_2^1, \theta_2^2, \theta_2^3\} = \{2, 4, 6\}$ with $\theta_2^1 < \theta_2^2 < \theta_2^3$. Moreover, we assume that the agents' valuation functions are $v_1(\theta_1^k, \theta_2^l) = \theta_1^k + 0.5\theta_2^l$ and $v_2(\theta_1^k, \theta_2^l) = \theta_2^l + 0.5\theta_1^k$ for all $(\theta_1^k, \theta_2^l) \in \Theta$. We compute

$$v_2(\theta_1^k, \theta_2^l) - v_1(\theta_1^k, \theta_2^l) = \theta_2^l + 0.5\theta_1^k - \theta_1^k - 0.5\theta_2^l = 0.5(\theta_2^l - \theta_1^k),$$

Table 3

$x^*(\cdot)$	θ_2^1	θ_2^2	θ_2^3
θ_1^1	1	1	1
θ_1^2	0	1	1
θ_1^3	0	0	1

implying that the efficient decision rule x^* is as follows:

Moreover, the types are drawn from the following joint probability distribution:

$$\mu \equiv \begin{pmatrix} \mu(\theta_1^1, \theta_2^1) & \mu(\theta_1^1, \theta_2^2) & \mu(\theta_1^1, \theta_2^3) \\ \mu(\theta_1^2, \theta_2^1) & \mu(\theta_1^2, \theta_2^2) & \mu(\theta_1^2, \theta_2^3) \\ \mu(\theta_1^3, \theta_2^1) & \mu(\theta_1^3, \theta_2^2) & \mu(\theta_1^3, \theta_2^3) \end{pmatrix} = \begin{pmatrix} 1/21 & 1/21 & 1/21 \\ 1/21 & 6/21 & 1/21 \\ 1/21 & 3/21 & 6/21 \end{pmatrix}.$$

It is easy to check that the above joint probability matrix Γ has full rank. Since the full rank condition implies the Crémer-McLean condition, we know from Proposition 1 that EABB is satisfied in Example 1. However, the claim below shows that it is impossible to satisfy EPBB in Example 1. This implies that the Ex-Post condition is not sufficient for the existence of mechanisms satisfying BIC, IIR, EFF, EPBB, and NTNP.

Claim 1. In Example 1, the Ex-Post condition is satisfied, whereas there exists no NTNP transfer rule t such that the mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB.

Proof. The proof is in the Appendix. □

4.5 Previous Attempts in the Literature

We discuss some previous attempts in the literature that address the existence of mechanisms satisfying BIC, IIR, EFF, and EPBB under interdependent values and correlated beliefs. We argue that the two prominent previous works we mention here do not help us obtain the kind of results we try to obtain in the later sections.

In his Proposition 3, Matsushima (2007) establishes a sufficient condition under which there exist no BIC, IIR and EPBB mechanisms when there are two agents. If we further impose EFF on mechanisms, Proposition 3 of Matsushima (2007) becomes the following: if

$$\sum_{k=1}^{m} \sum_{l=1}^{m} x^*(\theta_1^k, \theta_2^l) \left(v_2(\theta_1^k, \theta_2^l) - v_1(\theta_1^k, \theta_2^l) \right) \mu(\theta_1^k, \theta_2^l) = 0, \tag{3}$$

and there exists a type profile $(\theta_1^{\hat{k}}, \theta_2^{\hat{l}})$ such that

$$\begin{aligned}&\sum_{k=1}^{m}\sum_{l=1}^{m}\left[\left(1-x^*(\theta_1^k,\theta_2^l)\right)v_1(\theta_1^{\hat{k}},\theta_2^l)+x^*(\theta_1^k,\theta_2^l)v_2(\theta_1^k,\theta_2^{\hat{l}})\right]\mu(\theta_2^l|\theta_1^{\hat{k}})\mu(\theta_1^k|\theta_2^{\hat{l}})\\ > &\sum_{l=1}^{m}v_1(\theta_1^{\hat{k}},\theta_2^l)\mu(\theta_2^l|\theta_1^{\hat{k}}),\end{aligned} \tag{4}$$

then there exists no transfer rule t such that (x^*, t) satisfies BIC, IIR, EFF and EPBB. Fix $(\theta_1^k, \theta_2^l) \in \Theta$ arbitrarily. If $x^*(\theta_1^k, \theta_2^l) = 1$, we have

$$x^*(\theta_1^k,\theta_2^l)\left(v_2(\theta_1^k,\theta_2^l)-v_1(\theta_1^k,\theta_2^l)\right)\mu(\theta_1^k,\theta_2^l)=\left(v_2(\theta_1^k,\theta_2^l)-v_1(\theta_1^k,\theta_2^l)\right)\mu(\theta_1^k,\theta_2^l)>0,$$

because $x^*(\theta_1^k, \theta_2^l) = 1$ implies $v_2(\theta_1^k, \theta_2^l) - v_1(\theta_1^k, \theta_2^l) > 0$. If $x^*(\theta_1^k, \theta_2^l) = 0$, we trivially have

$$x^*(\theta_1^k,\theta_2^l)\left(v_2(\theta_1^k,\theta_2^l)-v_1(\theta_1^k,\theta_2^l)\right)\mu(\theta_1^k,\theta_2^l)=0.$$

This implies that the equation (3) is satisfied if and only if $x^*(\theta_1^k, \theta_2^l) = 0$, implying that it is always efficient not to trade. Therefore, if we further impose EFF, Proposition 3 of Matsushima (2007) becomes vacuous in all "nontrivial" cases of bilateral trade.[11] If, however, we allow for non-efficient decision rules, then Proposition 3 of Matsushima (2007) is not vacuous.

In their Theorem 1, Kosenok and Severinov (2008) establish that, in general mechanism design problems, there exists a BIC, IIR, EFF, and EPBB mechanism if the distribution μ satisfies both the Crémer and Mclean condition and the *Identifiability* condition.[12] They also show in their Theorem 2 that when there are at least three agents, the identifiability condition holds generically. However, when there are only two agents, they argue that only independent probability distributions are identifiable.[13] Therefore, their result becomes vacuous in our bilateral trade setup when the agents have correlated beliefs. In the next section, we focus on this particular environment with correlated beliefs where the identifiability condition is clearly violated and investigate the condition under which there exist BIC, IIR, EFF, and EPBB mechanisms.

[11] We say that an efficient bilateral trade model is *nontrivial* if there exist $\theta, \theta' \in \Theta$ such that $x^*(\theta) = 0$ and $x^*(\theta') = 1$.

[12] The reader is referred to Kosenok and Severinov (2008, Definition 1) for their definition of the Identifiability condition. Moreover, they establish the result for any ex-ante socially rational decision rule, including the ex-post efficient ones. The reader is referred to p.131 of their paper for the definition of ex-ante social rationality.

[13] See Kosenok and Severinov (2008, p.135) for this.

5 Interactions of Interdependence and Correlation in Bilateral Trade

Through a series of examples, we illustrate how interdependent values and correlated beliefs are intertwined in bilateral trade, so that it becomes a delicate problem to see when there exist BIC, IIR, EFF, and EPBB mechanisms when each agent has at least three types. Let us start from the following example.

Example 2. This example is exactly the same as Example 1 except that the joint probability distribution is changed from

$$\mu \equiv \begin{pmatrix} \mu(\theta_1^1,\theta_2^1) & \mu(\theta_1^1,\theta_2^2) & \mu(\theta_1^1,\theta_2^3) \\ \mu(\theta_1^2,\theta_2^1) & \mu(\theta_1^2,\theta_2^2) & \mu(\theta_1^2,\theta_2^3) \\ \mu(\theta_1^3,\theta_2^1) & \mu(\theta_1^3,\theta_2^2) & \mu(\theta_1^3,\theta_2^3) \end{pmatrix} = \begin{pmatrix} 1/21 & 1/21 & 1/21 \\ 1/21 & 6/21 & 1/21 \\ 1/21 & 3/21 & 6/21 \end{pmatrix}$$

to

$$\mu' \equiv \begin{pmatrix} \mu'(\theta_1^1,\theta_2^1) & \mu'(\theta_1^1,\theta_2^2) & \mu'(\theta_1^1,\theta_2^3) \\ \mu'(\theta_1^2,\theta_2^1) & \mu'(\theta_1^2,\theta_2^2) & \mu'(\theta_1^2,\theta_2^3) \\ \mu'(\theta_1^3,\theta_2^1) & \mu'(\theta_1^3,\theta_2^2) & \mu'(\theta_1^3,\theta_2^3) \end{pmatrix} = \begin{pmatrix} 0.3 & 0.05 & 0.03 \\ 0.21 & 0.1 & 0.01 \\ 0.05 & 0.21 & 0.04 \end{pmatrix}$$

In the claim below, we show that in Example 2, there exists a BIC, IIR, EFF, EPBB, and NTNP mechanism.

Claim 2. In Example 2, there exists an NTNP transfer rule t such that the mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB.

Proof. In order for the mechanism (x^*, t) to satisfy EPBB, we let $t_1(\theta_1^k,\theta_2^l) = -t_2(\theta_1^k,\theta_2^l) = t(\theta_1^k,\theta_2^l)$ for all $(\theta_1^k,\theta_2^l) \in \Theta$. For any $(\theta_1^k,\theta_2^l) \in \Theta$, we define the following transfer rule:

$$t(\theta_1^k,\theta_2^l) = \begin{cases} v_2(\theta_1^1,\theta_2^1) & \text{if either } (\theta_1^k,\theta_2^l) = (\theta_1^1,\theta_2^1) \text{ or } (\theta_1^k,\theta_2^l) = (\theta_1^1,\theta_2^2); \\ v_2(\theta_1^2,\theta_2^2) & \text{if } (\theta_1^k,\theta_2^l) = (\theta_1^2,\theta_2^2); \\ v_1(\theta_1^1,\theta_2^3) & \text{if } (\theta_1^k,\theta_2^l) = (\theta_1^1,\theta_2^3); \\ v_1(\theta_1^1,\theta_2^2) & \text{if } (\theta_1^k,\theta_2^l) = (\theta_1^2,\theta_2^3); \\ v_1(\theta_1^3,\theta_2^3) & \text{if } (\theta_1^k,\theta_2^l) = (\theta_1^3,\theta_2^3); \\ 0 & \text{otherwise.} \end{cases}$$

Note that NTNP is satisfied in this transfer rule. The rest of the proof for the claim is relegated to the Appendix, where we need to show that the mechanism (x^*, t) satisfies BIC and IIR. □

In the next example, we maintain the structure of Example 2 except that we slightly perturb the joint probability matrix $\mu^{'}$ in Example 2.

Example 3. The example is exactly the same as Example 2 except that the joint probability matrix is changed from

$$\mu^{'} \equiv \begin{pmatrix} \mu^{'}(\theta_1^1,\theta_2^1) & \mu^{'}(\theta_1^1,\theta_2^2) & \mu^{'}(\theta_1^1,\theta_2^3) \\ \mu^{'}(\theta_1^2,\theta_2^1) & \mu^{'}(\theta_1^2,\theta_2^2) & \mu^{'}(\theta_1^2,\theta_2^3) \\ \mu^{'}(\theta_1^3,\theta_2^1) & \mu^{'}(\theta_1^3,\theta_2^2) & \mu^{'}(\theta_1^3,\theta_2^3) \end{pmatrix} = \begin{pmatrix} 0.3 & 0.05 & 0.03 \\ 0.21 & 0.1 & 0.01 \\ 0.05 & 0.21 & 0.04 \end{pmatrix}$$

to the following:

$$\mu^{''} \equiv \begin{pmatrix} \mu^{''}(\theta_1^1,\theta_2^1) & \mu^{''}(\theta_1^1,\theta_2^2) & \mu^{''}(\theta_1^1,\theta_2^3) \\ \mu^{''}(\theta_1^2,\theta_2^1) & \mu^{''}(\theta_1^2,\theta_2^2) & \mu^{''}(\theta_1^2,\theta_2^3) \\ \mu^{''}(\theta_1^3,\theta_2^1) & \mu^{''}(\theta_1^3,\theta_2^2) & \mu^{''}(\theta_1^3,\theta_2^3) \end{pmatrix} = \begin{pmatrix} 0.31 & 0.04 & 0.04 \\ 0.2 & 0.1 & 0.01 \\ 0.06 & 0.2 & 0.04 \end{pmatrix}$$

We define the distance between any two joint probability matrices to be equal to the maximal pointwise distance, that is, for any two $m \times m$ joint probability matrices $\mu, \hat{\mu}$, their distance is defined by

$$d(\mu,\hat{\mu}) = \max_{1\le k\le m} \max_{1\le l\le m} \left|\mu(\theta_1^k,\theta_2^l) - \hat{\mu}(\theta_1^k,\theta_2^l)\right|.$$

It is easy to observe that $d(\mu^{'},\mu^{''}) = 0.01$.

In the claim below, we show that the Ex-Post condition is violated in Example 3, which implies that there are no mechanisms satisfying BIC, IIR, EFF, EPBB, and NTNP. In other words, the possibility result under interdependent values could be very fragile.

Claim 3. In Example 3, there exists no NTNP transfer rule t such that the resulting mechanism (x^*,t) satisfies BIC, IIR, EFF, and EPBB.

Proof. It suffices to verify that the Ex-Post condition is violated under $\mu^{''}$ in Example 3. It follows from the proof of Claim 1 (found in the Appendix) that the Ex-Post condition becomes the following:

$$\begin{aligned} & 2.5\sum_{l=1}^{3}\mu^{''}(\theta_1^1,\theta_2^l) + 5.5\mu^{''}(\theta_1^2,\theta_2^2) + 7.5\mu^{''}(\theta_1^2,\theta_2^3) + 8.5\mu^{''}(\theta_1^3,\theta_2^3) \\ \ge\ & 8\sum_{k=1}^{3}\mu^{''}(\theta_1^k,\theta_2^3) + 2\mu^{''}(\theta_1^1,\theta_2^1) + 3\mu^{''}(\theta_1^1,\theta_2^2) + 5\mu^{''}(\theta_1^2,\theta_2^2) \\ \Rightarrow\ & 2.5\times 0.39 + 5.5\times 0.1 + 7.5\times 0.01 + 8.5\times 0.04 \ge 8\times 0.09 + 2\times 0.31 + 3\times 0.04 + 5\times 0.1 \\ \Rightarrow\ & 1.94 \ge 1.96, \end{aligned}$$

which is a contradiction. It follows from Theorem 1 that there exists no NTNP transfer rule t, together with the efficient decision rule x^*, such that the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB. This completes the proof of Claim 3. □

Now we turn our attention to a model with *private values* and consider the counterparts of Examples 2 and 3. We show that there exists a NTNP transfer rule t such that the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB under private values, as long as the distribution μ' is perturbed by a very small amount. This is in a stark contrast with Claim 3 where the possibility result under interdependent values will be gone after we introduce the same amount of perturbation of the distribution μ'.

Example 4. The example is exactly the same as Example 2 except that we consider a private values model in which the agent's valuation depends only on their own type, that is, $v_1(\theta_1^k) = \theta_1^k$ for all $\theta_1^k \in \Theta_1$ and $v_2(\theta_2^l) = \theta_2^l$ for all $\theta_2^l \in \Theta_2$.

Claim 4. In Example 4, for any perturbed distribution $\hat{\mu}$ such that $d(\hat{\mu}, \mu') \leq 0.01$, there always exists an NTNP and EPBB transfer rule t, such that, together with the efficient decision rule x^*, the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB under $\hat{\mu}$.

Proof. Fix $\hat{\mu}$ as an arbitrary full-support distribution such that $d(\hat{\mu}, \mu') \leq 0.01$:

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}(\theta_1^1, \theta_2^1) & \hat{\mu}(\theta_1^1, \theta_2^2) & \hat{\mu}(\theta_1^1, \theta_2^3) \\ \hat{\mu}(\theta_1^2, \theta_2^1) & \hat{\mu}(\theta_1^2, \theta_2^2) & \hat{\mu}(\theta_1^2, \theta_2^3) \\ \hat{\mu}(\theta_1^3, \theta_2^1) & \hat{\mu}(\theta_1^3, \theta_2^2) & \hat{\mu}(\theta_1^3, \theta_2^3) \end{pmatrix} = \begin{pmatrix} 0.3 + \varepsilon_{11} & 0.05 + \varepsilon_{12} & 0.03 + \varepsilon_{13} \\ 0.21 + \varepsilon_{21} & 0.1 + \varepsilon_{22} & 0.01 + \varepsilon_{23} \\ 0.05 + \varepsilon_{31} & 0.21 + \varepsilon_{32} & 0.04 + \varepsilon_{33} \end{pmatrix},$$

where $\varepsilon_{k,l} \in [-0.01, 0.01]$ for all $k, l \in \{1, \cdots, 3\}$; $\varepsilon_{23} > -0.01$; and $\sum_{k=1}^{3} \sum_{l=1}^{3} \varepsilon_{kl} = 0$. This implies that $\hat{\mu}(\theta_1^k, \theta_2^l) > 0$ for each $(\theta_1^k, \theta_2^l) \in \Theta$ and $\sum_{k=1}^{m} \sum_{l=1}^{m} \hat{\mu}(\theta_1^k, \theta_2^l) = 1$.

Note that the efficient decision rule x^* is identical to the one under interdependent values, which is described in Table 3. In order for the mechanism (x^*, t) to satisfy EPBB, we let $t_1(\theta_1^k, \theta_2^l) = -t_2(\theta_1^k, \theta_2^l) = t(\theta_1^k, \theta_2^l)$ for all $(\theta_1^k, \theta_2^l) \in \Theta$, i.e., $t(\theta_1^k, \theta_2^l)$ is the amount of money the buyer pays the seller when the type report is

(θ_1^k, θ_2^l). We propose the following transfer rule:

$$t(\theta_1^k, \theta_2^l) = \begin{cases} v_2(\theta_2^1) = \theta_2^1 & \text{if either } (\theta_1^k, \theta_2^l) = (\theta_1^1, \theta_2^1) \text{ or } (\theta_1^k, \theta_2^l) = (\theta_1^1, \theta_2^2) \text{ holds} \\ v_2(\theta_2^2) = \theta_2^2 & \text{if } (\theta_1^k, \theta_2^l) = (\theta_1^2, \theta_2^2) \\ v_1(\theta_1^1) = \theta_1^1 & \text{if } (\theta_1^k, \theta_2^l) = (\theta_1^1, \theta_2^3) \\ v_1(\theta_1^3) = \theta_1^3 & \text{if either } (\theta_1^k, \theta_2^l) = (\theta_1^2, \theta_2^3) \text{ or } (\theta_1^k, \theta_2^l) = (\theta_1^3, \theta_2^3) \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$$

This transfer rule can be seen as adapted from the one in Claim 2 under interdependent values except that the monetary payment made by the buyer under the type report (θ_1^2, θ_2^3) is changed from $v_1(\theta_1^1, \theta_2^2) = v_1(\theta_1)$ to $v_1(\theta_1^3)$. Note that NTNP is satisfied in this transfer rule. The rest of the proof is relegated to the Appendix where we need to show that the mechanism (x^*, t) satisfies BIC and IIR. □

Indeed, we can show that if the joint probability matrix $\mu^{'}$ is perturbed by the size of 0.1, rather than that of 0.01, there are no mechanisms satisfying BIC, IIR, EFF, and EPBB in the model with private values.

Example 5. The example is exactly the same as Example 4 except that the joint probability matrix is changed from

$$\mu^{'} \equiv \begin{pmatrix} \mu^{'}(\theta_1^1, \theta_2^1) & \mu^{'}(\theta_1^1, \theta_2^2) & \mu^{'}(\theta_1^1, \theta_2^3) \\ \mu^{'}(\theta_1^2, \theta_2^1) & \mu^{'}(\theta_1^2, \theta_2^2) & \mu^{'}(\theta_1^2, \theta_2^3) \\ \mu^{'}(\theta_1^3, \theta_2^1) & \mu^{'}(\theta_1^3, \theta_2^2) & \mu^{'}(\theta_1^3, \theta_2^3) \end{pmatrix} = \begin{pmatrix} 0.3 & 0.05 & 0.03 \\ 0.21 & 0.1 & 0.01 \\ 0.05 & 0.21 & 0.04 \end{pmatrix}$$

to the following:

$$\tilde{\mu} \equiv \begin{pmatrix} \tilde{\mu}(\theta_1^1, \theta_2^1) & \tilde{\mu}(\theta_1^1, \theta_2^2) & \tilde{\mu}(\theta_1^1, \theta_2^3) \\ \tilde{\mu}(\theta_1^2, \theta_2^1) & \tilde{\mu}(\theta_1^2, \theta_2^2) & \tilde{\mu}(\theta_1^2, \theta_2^3) \\ \tilde{\mu}'(\theta_1^3, \theta_2^1) & \tilde{\mu}(\theta_1^3, \theta_2^2) & \tilde{\mu}(\theta_1^3, \theta_2^3) \end{pmatrix} = \begin{pmatrix} 0.2 & 0.05 & 0.13 \\ 0.21 & 0.1 & 0.01 \\ 0.15 & 0.14 & 0.01 \end{pmatrix}$$

It is easy to observe that $d(\mu^{'}, \tilde{\mu}) = 0.1$ in this example.

Claim 5. In Example 5, there exists no NTNP transfer rule t such that, together with the efficient decision rule x^*, the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB.

Proof. Rearranging the terms, we can rewrite the Ex-Post condition as follows:

$$\begin{aligned}
&\sum_{l=1}^{m-1} \tilde{\mu}(\theta_1^1, \theta_2^l) \left[v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^l)\right] + \sum_{k=2}^{m} \tilde{\mu}(\theta_1^k, \theta_2^m) \left[v_2(\theta_1^k, \theta_2^m) - v_1(\theta_1^m, \theta_2^m)\right] \\
+ \; & \tilde{\mu}(\theta_1^1, \theta_2^m) \left[v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^m, \theta_2^m)\right] + \sum_{k=2}^{m-1} \sum_{l=k}^{m-1} \tilde{\mu}(\theta_1^k, \theta_2^l) \left[v_2(\theta_1^k, \theta_2^l) - v_1(\theta_1^k, \theta_2^l)\right] \geq 0.
\end{aligned} \tag{5}$$

Since $v_1(\theta_1^k, \theta_2^l) = \theta_1^k$ and $v_2(\theta_1^k, \theta_2^l) = \theta_2^l$, the above inequality becomes

$$\begin{aligned}
&\sum_{l=1}^{m-1} \tilde{\mu}(\theta_1^1, \theta_2^l)(\theta_2^1 - \theta_1^1) + \sum_{k=2}^{m} \tilde{\mu}(\theta_1^k, \theta_2^m)(\theta_2^m - \theta_1^m) + \tilde{\mu}(\theta_1^1, \theta_2^m)(\theta_2^1 - \theta_1^m) \\
&+ \sum_{k=2}^{m-1} \sum_{l=k}^{m-1} \tilde{\mu}(\theta_1^k, \theta_2^l)(\theta_2^l - \theta_1^k) \geq 0.
\end{aligned} \tag{6}$$

Since $m = 3$, the left-hand side of the inequality (6) becomes

$$\begin{aligned}
& \tilde{\mu}(\theta_1^1, \theta_2^1) + \tilde{\mu}(\theta_1^1, \theta_2^2) + \tilde{\mu}(\theta_1^2, \theta_2^3) + \tilde{\mu}(\theta_1^3, \theta_2^3) - 3\tilde{\mu}(\theta_1^1, \theta_2^3) + \tilde{\mu}(\theta_1^2, \theta_2^2) \\
= \; & 0.2 + 0.05 + 0.01 + 0.01 - 3 \times 0.13 + 0.1 = -0.02,
\end{aligned}$$

which contradicts the inequality (6). Therefore, the Ex-Post condition is violated. Then, it follows from Theorem 1 that there exists no NTNP transfer rule t, together with the efficient decision rule x^*, such that the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB under $\tilde{\mu}$. This completes the proof of Claim 5. □

6 The Interplay of Degree of Interdependence and the Ex-Post Condition

All of examples and claims we have discussed in the previous section seem to conclude that the size of perturbations of the agents' beliefs needed to break down the possibility result under private values models is much larger than that needed to break down that under interdependent values models. In other words, a possibility result under interdependent values is more fragile than that under private values.

In this section, we generalize this insight. More specifically, we can argue that as long as we focus on linear valuation functions, the greater the degree of interdependence, the more likely it is that the Ex-Post condition will be violated. To make a meaningful comparative statics, we need to maintain the same efficient decision rule throughout the exercise. This requirement is translated into the restrictions on the permissive class of degrees of interdependence.

Lemma 1. Consider the linear valuation functions $v_1(\theta_1^k, \theta_2^l) = \theta_1^k + \gamma_1 \theta_2^l$ and $v_2(\theta_1^k, \theta_2^l) = \theta_2^l + \gamma_2 \theta_1^k$ where $\gamma_1, \gamma_2 > 0$ for all $k, l \in \{1, \cdots, m\}$. Suppose that the single crossing condition holds. The efficient decision rule is identical to the one in Table 2 if and only if (γ_1, γ_2) satisfies

$$\max_{k \in \{1, \cdots, m\}} \left(1 - (1-\gamma_1)\frac{\theta_2^k}{\theta_1^k}\right) < \gamma_2 \leq \min_{k \in \{2, \cdots, m\}} \left(1 - (1-\gamma_1)\frac{\theta_2^{k-1}}{\theta_1^k}\right). \tag{7}$$

Proof. Fix $(\theta_1^k, \theta_2^l) \in \Theta$. Under linear valuation functions, we have

$$v_2(\theta_1^k, \theta_2^l) - v_1(\theta_1^k, \theta_2^l) = \theta_2^l + \gamma_2 \theta_1^k - \theta_1^k - \gamma_1 \theta_2^l = (1-\gamma_1)\theta_2^l - (1-\gamma_2)\theta_1^k.$$

The single crossing condition requires $\gamma_1 < 1$ and $\gamma_2 < 1$. Moreover, we require $(1-\gamma_1)\theta_2^l - (1-\gamma_2)\theta_1^k > 0$ if and only if $l \geq k$ so that the efficient decision rule is identical to the one described in Table 2. Since the agents' types are already fixed in the model setup, we consider this as a restriction on γ_1 and γ_2 only.

The efficient decision rule dictates that, for all $k, l \in \{1, \ldots, m\}$ with $k \leq l$,

$$(1-\gamma_1)\theta_2^l - (1-\gamma_2)\theta_1^k > 0 \quad \Rightarrow \quad \gamma_2 > 1 - (1-\gamma_1)\frac{\theta_2^l}{\theta_1^k}.$$

For every $\theta_1^k \in \Theta_1$, the right-hand side of the above inequality is strictly decreasing in θ_2^l. Since $l \geq k$, the right-hand side of it attains its maximum at $\theta_2^l = \theta_2^k$. As a result, for each $\theta_1^k \in \Theta_1$, γ_2 must be larger than the greatest lower bound, which is

$$\gamma_2 \quad > \quad 1 - (1-\gamma_1)\frac{\theta_2^k}{\theta_1^k}.$$

Besides, the efficient decision rule requires, that for all $k, l \in \{1, \ldots, m\}$ with $k > l$,

$$(1-\gamma_1)\theta_2^l - (1-\gamma_2)\theta_1^k \leq 0 \quad \Rightarrow \quad \gamma_2 \leq 1 - (1-\gamma_1)\frac{\theta_2^l}{\theta_1^k}.$$

For every $\theta_1^k \in \Theta_1$, the right-hand side of the above inequality is strictly decreasing in θ_2^l. Since $l < k$, the right-hand side of it attains its minimum at θ_2^{k-1}. As a result, for each $\theta_1^k \in \Theta_1$, γ_2 cannot exceed the least upper bound, which is

$$\gamma_2 \quad \leq \quad 1 - (1-\gamma_1)\frac{\theta_2^{k-1}}{\theta_1^k}.$$

To summarize, we have

$$\max_{k \in \{1, \cdots, m\}} \left(1 - (1-\gamma_1)\frac{\theta_2^k}{\theta_1^k}\right) < \gamma_2 \leq \min_{k \in \{2, \cdots, m\}} \left(1 - (1-\gamma_1)\frac{\theta_2^{k-1}}{\theta_1^k}\right),$$

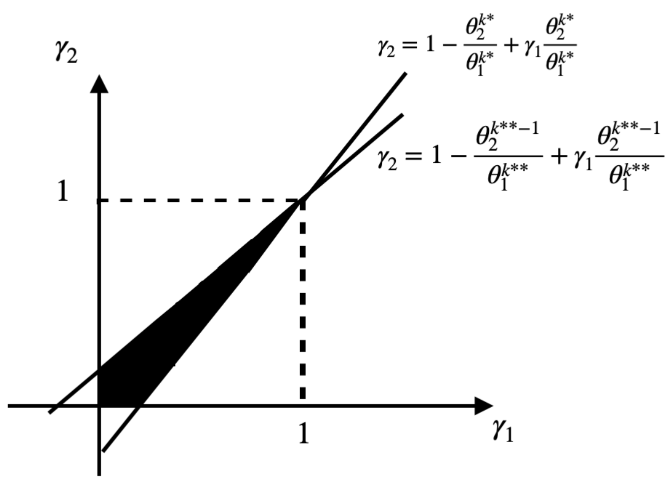
as desired. □

To simplify the notation in the rest of the argument, we let $k^* \equiv \arg\min_{k\in\{1,\cdots,m\}} \theta_2^k/\theta_1^k$ and $k^{**} \equiv \arg\max_{k\in\{2,\cdots,m\}} \theta_2^{k-1}/\theta_1^k$. Then, the inequality (7) can be rewritten as

$$1-(1-\gamma_1)\frac{\theta_2^{k^*}}{\theta_1^{k^*}} \quad < \gamma_2 \leq \quad 1-(1-\gamma_1)\frac{\theta_2^{k^{**}-1}}{\theta_1^{k^{**}}}. \tag{8}$$

Since both agents' types are strictly positive, both the upper and lower bounds of the inequality (8) are linear, increasing functions of γ_1. Moreover, since the efficient decision rule remains the same under private values where $\gamma_1 = \gamma_2 = 0$, we have that $\theta_2^k > \theta_1^k$ for all $k \in \{1,\dots,m\}$ and $\theta_2^{k-1} \leq \theta_1^k$ for all $k \in \{2,\dots,m\}$. Then, the lower bound of γ_2 has a slope larger than one and a negative intercept, whereas the upper bound has a slope smaller than one and a positive intercept. The shaded region of Figure 1 describes the set of (γ_1, γ_2) satisfying the inequality (8).

Figure 1: The Feasible Region of (γ_1, γ_2)



Furthermore, if we subtract γ_1 from both sides of inequality (8), it follows from the fact that $1-\gamma_1 > 0$ that

$$\begin{aligned} 1-\gamma_1-(1-\gamma_1)\frac{\theta_2^{k^*}}{\theta_1^{k^*}} < \quad \gamma_2-\gamma_1 \quad &\leq 1-\gamma_1-(1-\gamma_1)\frac{\theta_2^{k^{**}-1}}{\theta_1^{k^{**}}} \\ \Rightarrow 1-\frac{\theta_2^{k^*}}{\theta_1^{k^*}} < \quad \tfrac{\gamma_2-\gamma_1}{1-\gamma_1} \quad &\leq 1-\frac{\theta_2^{k^{**}-1}}{\theta_1^{k^{**}}}. \end{aligned} \tag{9}$$

Since $1-(\theta_2^k/\theta_1^k) < 0$ for all $k \in \{1,\cdots,m\}$ and $1-(\theta_2^{k-1}/\theta_1^k) \geq 0$ for all $k \in \{2,\cdots,m\}$, it follows from the inequality (9) that $\gamma_1 = \gamma_2$ is always possible. In fact, we can interpret $(\gamma_2-\gamma_1)/(1-\gamma_1)$ as a normalized difference between γ_2 and γ_1, and the inequality (9) requires that this normalized difference should be bounded from both above and below.

Remark 2. Specializing in the linear valuation functions in the inequality (5), we rewrite the Ex-Post condition as

$$\sum_{l=1}^{m-1}\mu(\theta_1^1,\theta_2^l)\left[(\theta_2^1-\theta_1^1)+(\gamma_2\theta_1^1-\gamma_1\theta_2^l)\right]+\sum_{k=2}^{m}\mu(\theta_1^k,\theta_2^m)\left[(\theta_2^m-\theta_1^m)+(\gamma_2\theta_1^k-\gamma_1\theta_2^m)\right]$$
$$+\quad\mu(\theta_1^1,\theta_2^m)\left[(\theta_2^1-\theta_1^m)+(\gamma_2\theta_1^1-\gamma_1\theta_2^m)\right]+\sum_{k=2}^{m-1}\sum_{l=k}^{m-1}\mu(\theta_1^k,\theta_2^l)\left[(\theta_2^l-\theta_1^k)+(\gamma_2\theta_1^k-\gamma_1\theta_2^l)\right]\geq 0. \tag{10}$$

Observe that the left-hand side of the inequality (10) is strictly increasing in γ_2 and strictly decreasing in γ_1, implying that the lower γ_2 and the higher γ_1, the more likely it is that the Ex-Post condition will be violated. However, it follows from the inequality (8) that both the upper and lower bounds of γ_2 are in general strictly increasing in γ_1, so we will not be able to keep increasing γ_1 and decreasing γ_2 at the same time. This can also be seen clearly in Figure 1.

To avoid this problem, we restrict our attention to the following set of configurations of γ_1 and γ_2. We choose some δ from the interval $\left(1-(\theta_2^{k^*}/\theta_1^{k^*}),1-(\theta_2^{k^{**}-1}/\theta_1^{k^{**}})\right]$ and let

$$\frac{\gamma_2-\gamma_1}{1-\gamma_1}=\delta.$$

We know from the inequality (9) that any $(\gamma_1,\gamma_2)\in[0,1)^2$ satisfying the above expression will ensure that the efficient decision rule is identical to the one described in Table 2. We can further rewrite γ_2 as a function of γ_1, which is

$$\gamma_2=\gamma_1+\delta(1-\gamma_1)=(1-\delta)\gamma_1+\delta.$$

In words, (γ_1,γ_2) moves along the line $\gamma_2=(1-\delta)\gamma_1+\delta$ within the unit square $[0,1]^2$. We will show in our Proposition 4 that under this configuration of γ_1 and γ_2, the higher γ_1, the more likely that the Ex-Post condition will be violated.

We are ready to establish the following comparative statics result.

Proposition 4. Consider the linear valuation functions $v_1(\theta_1^k,\theta_2^l)=\theta_1^k+\gamma_1\theta_2^l$ and $v_2(\theta_1^k,\theta_2^l)=\theta_2^l+\gamma_2\theta_1^k$ where $\gamma_1,\gamma_2>0$ for all $k,l\in\{1,\cdots,m\}$. Suppose that the single crossing condition holds. Assume further that there exists $\delta\in\left(1-(\theta_2^{k^*}/\theta_1^{k^*}),1-(\theta_2^{k^{**}-1}/\theta_1^{k^{**}})\right]$ such that γ_2 can be expressed as $\gamma_2=(1-\delta)\gamma_1+\delta$. Then, the left-hand side of the inequality (10) is strictly decreasing in γ_1. That is, the higher γ_1, the more likely that the Ex-Post condition will be violated.

Proof. Plugging $\gamma_2 = (1-\delta)\gamma_1 + \delta$ into the inequality (10), we can further rewrite the Ex-Post condition as

$$\begin{aligned} & \sum_{l=1}^{m-1} \mu(\theta_1^1, \theta_2^l) \left[(\theta_2^l - \theta_1^1) + \gamma_1 \left((1-\delta)\theta_1^1 - \theta_2^l \right) + \delta\theta_1^1 \right] \\ + & \sum_{k=2}^{m} \mu(\theta_1^k, \theta_2^m) \left[(\theta_2^m - \theta_1^m) + \gamma_1 \left((1-\delta)\theta_1^k - \theta_2^m \right) + \delta\theta_1^k \right] \\ + & \mu(\theta_1^1, \theta_2^m) \left[(\theta_2^1 - \theta_1^m) + \gamma_1 \left((1-\delta)\theta_1^1 - \theta_2^m \right) + \delta\theta_1^1 \right] \\ + & \sum_{k=2}^{m-1} \sum_{l=k}^{m-1} \mu(\theta_1^k, \theta_2^l) \left[(\theta_2^l - \theta_1^k) + \gamma_1 \left((1-\delta)\theta_1^k - \theta_2^l \right) + \delta\theta_1^k \right] \geq 0. \end{aligned}$$

We now argue that the left-hand side of the above inequality is strictly decreasing in γ_1. Then, it suffices to show that each coefficient associated with γ_1 on the left-hand side of the above inequality is strictly negative:

1. for each $l \in \{1, \cdots, m\}$, we have
$$(1-\delta)\theta_1^1 - \theta_2^l = \theta_1^1 \left(1 - \delta - \frac{\theta_2^l}{\theta_1^1} \right) < 0$$
because $\delta > 1 - (\theta_2^{k^*}/\theta_1^{k^*}) = \max_{k \in \{1,\cdots,m\}} \left(1 - (\theta_2^k/\theta_1^k) \right)$ implies
$$\delta > 1 - \frac{\theta_2^1}{\theta_1^1} \geq 1 - \frac{\theta_2^l}{\theta_1^1}$$
for all $l \in \{1, \cdots, m\}$.

2. for each $k \in \{2, \cdots, m\}$ and each $l \in \{k, \cdots, m\}$, we have
$$(1-\delta)\theta_1^k - \theta_2^l = \theta_1^k \left(1 - \delta - \frac{\theta_2^l}{\theta_1^k} \right) < 0$$
because $\delta > 1 - (\theta_2^{k^*}/\theta_1^{k^*}) = \max_{k \in \{1,\cdots,m\}} \left(1 - (\theta_2^k/\theta_1^k) \right)$ implies
$$\delta > 1 - \frac{\theta_2^k}{\theta_1^k} \geq 1 - \frac{\theta_2^l}{\theta_1^k}$$
for all $k \in \{2, \cdots, m\}$ and all $l \in \{k, \cdots, m\}$.

As a result, given the configuration of (γ_1, γ_2), the left-hand side of the inequality (10) is strictly decreasing in γ_1. Since the Ex-Post condition requires that the left-hand side of the inequality (10) be at least as large as zero, we conclude that the higher γ_1, the more likely it is that the Ex-Post condition will be violated. □

In conclusion, interdependent values make it more difficult to find desired mechanisms than private values. This is consistent with our findings in terms of the series of examples.

7 Final Remarks

Finally, we would like to conclude the paper by illustrating the general difficulty of establishing a sufficient condition for the existence of mechanisms satisfying BIC, IIR, EFF, and EPBB.

In general, if each agent has m types, then the number of BIC and IIR constraints equals $2m(m-1)+2m = 2m^2$, and the number of unknown transfers equals m^2. After imposing NTNP on the mechanisms, the number of unknown transfers decreases to $m(m+1)/2$. Observe that as m increases, the number of BIC and IIR constraints increases rapidly. In the following claim, we propose a way to significantly reduce the number of constraints.

In order for the mechanism (x^*, t) to satisfy EPBB, we let $t_1(\theta_1^k, \theta_2^l) = -t_2(\theta_1^k, \theta_2^l) = t(\theta_1^k, \theta_2^l)$ for all $(\theta_1^k, \theta_2^l) \in \Theta$, i.e., $t(\theta_1^k, \theta_2^l)$ is the amount of money the buyer pays the seller when the type report is (θ_1^k, θ_2^l).

Claim 6. Suppose that the mechanism (x^*, t) satisfies EPBB and NTNP and that $t(\theta_1^1, \theta_2^1) = v_2(\theta_1^1, \theta_2^1)$ and $t(\theta_1^m, \theta_2^m) = v_1(\theta_1^m, \theta_2^m)$. The following statements hold:

1. If the seller of type θ_1^k where $k \neq m$ has no incentive to deviate to θ_1^m, then the IIR constraints for the seller of these types are also satisfied.

2. If the buyer of type θ_2^l where $l \neq 1$ has no incentive to deviate to θ_2^1, then the IIR constraints for the buyer of these types are also satisfied.

Proof. Suppose that the seller of type θ_1^k where $k \neq m$ has no incentive to deviate to θ_1^m, that is, $IC_{\theta_1^k \to \theta_1^m}$ is satisfied for each $k \neq m$. The seller's BIC constraint $IC_{\theta_1^k \to \theta_1^m}$ with $k \neq m$ is given as follows:

$$\sum_{l=k}^{m} \mu(\theta_2^l|\theta_1^k) t(\theta_1^k, \theta_2^l) + \sum_{l=1}^{k-1} \mu(\theta_2^l|\theta_1^k) v_1(\theta_1^k, \theta_2^l) \geq \mu(\theta_2^m|\theta_1^k) t(\theta_1^m, \theta_2^m) + \sum_{l=1}^{m-1} \mu(\theta_2^l|\theta_1^k) v_1(\theta_1^k, \theta_2^l).$$

By our hypothesis and the single crossing condition, we have $t(\theta_1^m, \theta_2^m) = v_1(\theta_1^m, \theta_2^m) > v_1(\theta_1^k, \theta_2^m)$ for all $k \in \{1, \cdots, m-1\}$. Then, the seller's expected utility under truth-telling must be strictly higher than $\sum_{l=1}^{m} \mu(\theta_2^l|\theta_1^k) v_1(\theta_1^k, \theta_2^l)$. Since the seller's outside option utility equals $\sum_{l=1}^{m} \mu(\theta_2^l|\theta_1^k) v_1(\theta_1^k, \theta_2^l)$, we conclude that the IIR is satisfied for the seller of type θ_1^k.

Suppose that the buyer of type θ_2^l where $l \neq 1$ has no incentive to deviate to θ_2^1, that is, $IC_{\theta_2^l \to \theta_2^1}$ is satisfied for each $l \neq 1$. The buyer's BIC constraint $IC_{\theta_2^l \to \theta_2^1}$ with $l \neq 1$ is given as follows:

$$\sum_{k=1}^{l} \mu(\theta_1^k|\theta_2^l) \left(v_2(\theta_1^k, \theta_2^l) - t(\theta_1^k, \theta_2^l) \right) \geq \mu(\theta_1^1|\theta_2^l) \left(v_2(\theta_1^1, \theta_2^l) - t(\theta_1^1, \theta_2^1) \right).$$

By our hypothesis and the single crossing condition, we have $t(\theta_1^1, \theta_2^1) = v_2(\theta_1^1, \theta_2^1) < v_2(\theta_1^1, \theta_2^l)$ for all $l \in \{2, \cdots, m\}$. Then, the buyer's expected utility under truth-telling must be strictly positive. Since the buyer's outside option utility is always zero, we conclude that the IIR is satisfied for the buyer of type θ_2^l. This completes the proof of the claim. □

According to Claim 6, if the mechanism (x^*, t) satisfies EPBB and NTNP and that $t(\theta_1^1, \theta_2^1) = v_2(\theta_1^1, \theta_2^1)$ and $t(\theta_1^m, \theta_2^m) = v_1(\theta_1^m, \theta_2^m)$, it suffices to focus only on the system of BIC constraints. This allows us to reduce the number of constraints from $2m^2$ to $2m(m-1)$.

We can also reduce the number of unknown transfers. To see this, we fix the transfer $t(\theta_1^k, \theta_2^l)$ for all (θ_1^k, θ_2^l) with $l \neq k+1$. Then, the remaining unknown transfers are $\{t(\theta_1^k, \theta_2^{k+1})\}_{k \in \{1,\dots,m-1\}}$. To prevent the buyer of type θ_2^l from deviating to $\theta_2^{l'}$, we can adjust the values of $t(\theta_1^{l-1}, \theta_1^l)$ and $t(\theta_1^{l'-1}, \theta_1^{l'})$. Similarly, to prevent the seller of type θ_1^k from deviating to $\theta_1^{k'}$, we can adjust the values of $t(\theta_1^k, \theta_1^{k+1})$ and $t(\theta_1^{k'}, \theta_1^{k'+1})$. In this way, we could significantly reduce the number of adjustable transfers from $m(m+1)/2$ to $m-1$.

The BIC constraints will then impose the upper and lower bounds on these adjustable transfers. Since their upper and lower bounds depend on both the distribution μ and the valuation functions $(v_1(\cdot), v_2(\cdot))$, we need to impose constraints on μ and $(v_1(\cdot), v_2(\cdot)$ such that the upper bound of each adjustable transfer is compatible with its lower bound, just as we can do in the two-type case (See Section 4.3 for this).

Despite the significant reduction in the number of unknown transfers, there are multiple upper and lower bounds to check. It is difficult to provide one single condition under which all these upper and lower bounds are compatible with each other. Moreover, as the number of types increases, the number of adjustable transfers is still increasing. To ensure that the boundaries on these unknown transfers imposed by the BIC and IIR constraints are compatible with each other, we probably need to impose more and more restrictions, which means that it is less likely that the mechanism (x^*, t) satisfies BIC, IIR, EFF and EPBB.

8 Appendix

In the Appendix, we provide all the proofs omitted from the main text of the paper.

8.1 Proof of Proposition 1

The proof is divided into four steps.

8.1.1 Step 1: EABB is satisfied in the proposed mechanism.

Proof. We first compute the interim expected transfer of each agent. Suppose that the buyer reports his type truthfully and that the seller's true type is θ_1 but she reports θ_1'. Then the seller's interim expected revenue is computed as follows:

$$\begin{aligned}
&\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)t_1(\theta_1',\theta_2)\\
=\;&\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)\left[-\sum_{\tilde\theta_2\in\Theta_2}\mu(\tilde\theta_2|\theta_1')\left(1-x^*(\theta_1',\tilde\theta_2)\right)v_1(\theta_1',\tilde\theta_2)+\gamma_1(\theta_1')h_1(\theta_1',\theta_2)+v_1(\theta_1',\theta_2)\right]\\
&+\sum_{\tilde\theta_1\in\Theta_1}\sum_{\tilde\theta_2\in\Theta_2}\mu(\tilde\theta_1,\tilde\theta_2)x^*(\tilde\theta_1,\tilde\theta_2)\left(v_2(\tilde\theta_1,\tilde\theta_2)-v_1(\tilde\theta_1,\tilde\theta_2)\right).
\end{aligned}$$

After rearrangement, the seller's interim expected revenue becomes

$$\begin{aligned}
&\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)t_1(\theta_1',\theta_2)\\
=\;&-\sum_{\tilde\theta_2\in\Theta_2}\mu(\tilde\theta_2|\theta_1')\left(1-x^*(\theta_1',\tilde\theta_2)\right)v_1(\theta_1',\tilde\theta_2)+\gamma_1(\theta_1')\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)h_1(\theta_1',\theta_2)+\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)v_1(\theta_1',\theta_2)\\
&+\sum_{\tilde\theta_1\in\Theta_1}\sum_{\tilde\theta_2\in\Theta_2}\mu(\tilde\theta_1,\tilde\theta_2)x^*(\tilde\theta_1,\tilde\theta_2)\left(v_2(\tilde\theta_1,\tilde\theta_2)-v_1(\tilde\theta_1,\tilde\theta_2)\right).
\end{aligned}\tag{11}$$

On the other hand, suppose that the seller reports her type truthfully and that the buyer's true type is θ_2 but he reports θ_2'. Then the buyer's interim expected income is computed as follows:

$$\begin{aligned}
&\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)t_2(\theta_1,\theta_2')\\
=\;&\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)\left[-\sum_{\tilde\theta_1\in\Theta_1}\mu(\tilde\theta_1|\theta_2')x^*(\tilde\theta_1,\theta_2')v_2(\tilde\theta_1,\theta_2')+\gamma_2(\theta_2')h_2(\theta_1,\theta_2')\right]\\
=\;&-\sum_{\tilde\theta_1\in\Theta_1}\mu(\tilde\theta_1|\theta_2')x^*(\tilde\theta_1,\theta_2')v_2(\tilde\theta_1,\theta_2')+\gamma_2(\theta_2')\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)h_2(\theta_1,\theta_2').
\end{aligned}\tag{12}$$

Now, we verify that EABB is satisfied. Suppose that both agents report their types truthfully. The ex ante budget deficit from the mechanism (x^*,t) is computed below:

$$\begin{aligned}&\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)\left[t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\right]\\=&\sum_{\theta_1\in\Theta_1}\mu(\theta_1)\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)t_1(\theta_1,\theta_2)-\sum_{\theta_2\in\Theta_2}\mu(\theta_2)\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)t_2(\theta_1,\theta_2).\end{aligned}$$

After substituting the formulas of interim expected transfers, expressions (11) and (12), into the above expression, we rewrite the ex ante budget deficit as follows:

$$\begin{aligned}&\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)\left[t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\right]\\=&-\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)\left(1-x^*(\theta_1,\theta_2)\right)v_1(\theta_1,\theta_2)+\sum_{\theta_1\in\Theta_1}\gamma_1(\theta_1)\mu(\theta_1)\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)h_1(\theta_1,\theta_2)\\&+\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)v_1(\theta_1,\theta_2)+\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)x^*(\theta_1,\theta_2)\left(v_2(\theta_1,\theta_2)-v_1(\theta_1,\theta_2)\right)\\&-\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)x^*(\theta_1,\theta_2)v_2(\theta_1,\theta_2)+\sum_{\theta_2\in\Theta_2}\gamma_2(\theta_2)\mu(\theta_2)\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)h_2(\theta_1,\theta_2).\end{aligned}$$

Since $\sum_{\theta_{-i}\in\Theta_{-i}}\mu(\theta_{-i}|\theta_i)h_i(\theta_i,\theta_{-i})=0$ for each agent $i\in\{1,2\}$ and each type $\theta_i\in\Theta_i$, we can simplify the ex ante budget deficit as follows:

$$\begin{aligned}&\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)\left[t_1(\theta_1,\theta_2)+t_2(\theta_1,\theta_2)\right]\\=&-\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)\left(1-x^*(\theta_1,\theta_2)\right)v_1(\theta_1,\theta_2)+\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)v_1(\theta_1,\theta_2)\\&+\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)x^*(\theta_1,\theta_2)\left(v_2(\theta_1,\theta_2)-v_1(\theta_1,\theta_2)\right)-\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)x^*(\theta_1,\theta_2)v_2(\theta_1,\theta_2)\\=&\ 0,\end{aligned}$$

implying that EABB is satisfied. This completes the proof. □

8.1.2 Step 2: The seller extracts the full surplus in the proposed mechanism.

Proof. Suppose that both agents report their types truthfully. The ex ante expected payment made by the buyer is

$$\begin{aligned} & -\sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)t_2(\theta_1,\theta_2) \\ = & -\sum_{\theta_2\in\Theta_2}\mu(\theta_2)\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)t_2(\theta_1,\theta_2) \\ = & -\sum_{\theta_2\in\Theta_2}\mu(\theta_2)\left[-\sum_{\tilde{\theta}_1\in\Theta_1}\mu(\tilde{\theta}_1|\theta_2)x^*(\tilde{\theta}_1,\theta_2)v_2(\tilde{\theta}_1,\theta_2)+\gamma_2(\theta_2)\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)h_2(\theta_1,\theta_2)\right] \\ = & \sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)x^*(\theta_1,\theta_2)v_2(\theta_1,\theta_2)-\sum_{\theta_2\in\Theta_2}\gamma_2(\theta_2)\mu(\theta_2)\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)h_2(\theta_1,\theta_2) \\ = & \sum_{\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\theta_1,\theta_2)x^*(\theta_1,\theta_2)v_2(\theta_1,\theta_2), \end{aligned}$$

where the last equality holds because $\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)h_2(\theta_1,\theta_2)=0$ for every $\theta_2\in\Theta_2$. Hence, the ex ante expected amount paid by the buyer equals his ex ante expected valuation. Since EABB is satisfied, the seller's ex ante expected revenue equals the buyer's ex ante expected valuation as well, implying that the seller extracts the full surplus. This completes the proof. □

8.1.3 Step 3: IIR is satisfied in the proposed mechanism.

Proof. If everyone reports truthfully and the seller's true type is θ_1, then the seller's expected utility after participation, denoted by $U_1(\theta_1;\theta_1)$, is computed as follows:

$$\begin{aligned} U_1(\theta_1;\theta_1) &= \sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)\left[(1-x^*(\theta_1,\theta_2))\,v_1(\theta_1,\theta_2)+t_1(\theta_1,\theta_2)\right] \\ &= \sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)\,(1-x^*(\theta_1,\theta_2))\,v_1(\theta_1,\theta_2)+\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)t_1(\theta_1,\theta_2). \end{aligned}$$

After substituting the formula of the seller's interim expected transfer, expression (11), into the above expression, we can rewrite the seller's expected utility as follows:

$$\begin{aligned} U_1(\theta_1;\theta_1) = & \sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)\,(1-x^*(\theta_1,\theta_2))\,v_1(\theta_1,\theta_2)-\sum_{\tilde{\theta}_2\in\Theta_2}\mu(\tilde{\theta}_2|\theta_1)\left(1-x^*(\theta_1,\tilde{\theta}_2)\right)v_1(\theta_1,\tilde{\theta}_2) \\ & +\gamma_1(\theta_1)\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)h_1(\theta_1,\theta_2)+\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)v_1(\theta_1,\theta_2) \\ & +\sum_{\tilde{\theta}_1\in\Theta_1}\sum_{\tilde{\theta}_2\in\Theta_2}\mu(\tilde{\theta}_1,\tilde{\theta}_2)x^*(\tilde{\theta}_1,\tilde{\theta}_2)\left(v_2(\tilde{\theta}_1,\tilde{\theta}_2)-v_1(\tilde{\theta}_1,\tilde{\theta}_2)\right) \end{aligned}$$

Since $\sum_{\theta_2 \in \Theta_2} \mu(\theta_2|\theta_1) h_1(\theta_1, \theta_2) = 0$ for all $\theta_1 \in \Theta_1$, we can simplify the seller's expected utility as follows:

$$U_1(\theta_1; \theta_1) = \sum_{\theta_2 \in \Theta_2} \mu(\theta_2|\theta_1) v_1(\theta_1, \theta_2) + \sum_{\tilde{\theta}_1 \in \Theta_1} \sum_{\tilde{\theta}_2 \in \Theta_2} \mu(\tilde{\theta}_1, \tilde{\theta}_2) x^*(\tilde{\theta}_1, \tilde{\theta}_2) \left(v_2(\tilde{\theta}_1, \tilde{\theta}_2) - v_1(\tilde{\theta}_1, \tilde{\theta}_2) \right).$$

Observe that the second term is the ex ante expected gains from trade and thus nonnegative. We conclude that the seller's expected utility from participating in the mechanism is larger than or equal to $\sum_{\theta_2 \in \Theta_2} \mu(\theta_2|\theta_1) v_1(\theta_1, \theta_2)$, which equals the seller's expected utility from the outside option. Hence, IIR is satisfied for the seller.

If everyone reports truthfully and the buyer's true type is θ_2, then the buyer's expected utility after participation, denoted by $U_2(\theta_2; \theta_2)$, is computed as follows:

$$\begin{aligned} U_2(\theta_2; \theta_2) &= \sum_{\theta_1 \in \Theta_1} \mu(\theta_1|\theta_2) \left[x^*(\theta_1, \theta_2) v_2(\theta_1, \theta_2) + t_2(\theta_1, \theta_2) \right] \\ &= \sum_{\theta_1 \in \Theta_1} \mu(\theta_1|\theta_2) x^*(\theta_1, \theta_2) v_2(\theta_1, \theta_2) + \sum_{\theta_1 \in \Theta_1} \mu(\theta_1|\theta_2) t_2(\theta_1, \theta_2). \end{aligned}$$

After substituting the formula of the buyer's interim expected transfer, expression (12), into the above expression, we can rewrite the buyer's expected utility as follows:

$$\begin{aligned} U_2(\theta_2; \theta_2) &= \sum_{\theta_1 \in \Theta_1} \mu(\theta_1|\theta_2) x^*(\theta_1, \theta_2) v_2(\theta_1, \theta_2) \\ &\quad - \sum_{\tilde{\theta}_1 \in \Theta_1} \mu(\tilde{\theta}_1|\theta_2) x^*(\tilde{\theta}_1, \theta_2) v_2(\tilde{\theta}_1, \theta_2) + \gamma_2(\theta_2) \sum_{\theta_1 \in \Theta_1} \mu(\theta_1|\theta_2) h_2(\theta_1, \theta_2) \\ &= 0, \end{aligned}$$

where the last equality follows because $\sum_{\theta_1 \in \Theta_1} \mu(\theta_1|\theta_2) h_2(\theta_1, \theta_2) = 0$ for all $\theta_2 \in \Theta_2$. Since the buyer's outside option utility is always zero, we conclude that IIR is satisfied for the buyer. This completes the proof. □

8.1.4 Step 4: BIC is satisfied in the proposed mechanism.

Proof. If the seller of type θ_1 deviates to $\theta_1^r \neq \theta_1$, then her expected utility, denoted by $U_1(\theta_1; \theta_1^r)$, becomes

$$\begin{aligned} U_1(\theta_1; \theta_1^r) &= \sum_{\theta_2 \in \Theta_2} \mu(\theta_2|\theta_1) \left[(1 - x^*(\theta_1^r, \theta_2)) v_1(\theta_1, \theta_2) + t_1(\theta_1^r, \theta_2) \right] \\ &= \sum_{\theta_2 \in \Theta_2} \mu(\theta_2|\theta_1) (1 - x^*(\theta_1^r, \theta_2)) v_1(\theta_1, \theta_2) + \sum_{\theta_2 \in \Theta_2} \mu(\theta_2|\theta_1) t_1(\theta_1^r, \theta_2). \end{aligned}$$

After substituting the formula of the seller's interim expected transfer, expression (11), into the above expression, we rewrite the seller's expected utility of reporting θ_1^r as follows:

$$
\begin{aligned}
U_1(\theta_1;\theta_1^r) &= \sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)\left(1-x^*(\theta_1^r,\theta_2)\right)v_1(\theta_1,\theta_2)-\sum_{\tilde\theta_2\in\Theta_2}\mu(\tilde\theta_2|\theta_1^r)\left(1-x^*(\theta_1^r,\tilde\theta_2)\right)v_1(\theta_1^r,\tilde\theta_2)\\
&\quad+\gamma_1(\theta_1^r)\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)h_1(\theta_1^r,\theta_2)+\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)v_1(\theta_1^r,\theta_2)\\
&\quad+\sum_{\tilde\theta_1\in\Theta_1}\sum_{\theta_2\in\Theta_2}\mu(\tilde\theta_1,\theta_2)x^*(\tilde\theta_1,\theta_2)\left(v_2(\tilde\theta_1,\theta_2)-v_1(\tilde\theta_1,\theta_2)\right).
\end{aligned}
$$

Since $\sum_{\theta_2\in\Theta_2}\mu(\theta_2|\theta_1)h_1(\theta_1^r,\theta_2)<0$ for all $\theta_1^r\neq\theta_1$, because of the finiteness of Θ_1, we can choose $\gamma_1(\theta_1^r)$ large enough such that, for all $\theta_1,\theta_1^r\in\Theta_1$,

$$U_1(\theta_1;\theta_1)\geq U_1(\theta_1;\theta_1^r).$$

Therefore, BIC is satisfied for the seller.

If the buyer of type θ_2 deviates to $\theta_2^r\neq\theta_2$, then his expected utility, denoted by $U_2(\theta_2;\theta_2^r)$, becomes

$$
\begin{aligned}
U_2(\theta_2;\theta_2^r) &= \sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)\left[x^*(\theta_1,\theta_2^r)v_2(\theta_1,\theta_2)+t_2(\theta_1,\theta_2^r)\right]\\
&= \sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)x^*(\theta_1,\theta_2^r)v_2(\theta_1,\theta_2)+\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)t_2(\theta_1,\theta_2^r).
\end{aligned}
$$

After substituting the formula of the buyer's interim expected transfer, expression (12), into the above expression, we rewrite the buyer's expected utility after deviation as follows:

$$
\begin{aligned}
U_2(\theta_2;\theta_2^r) &= \sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)x^*(\theta_1,\theta_2^r)v_2(\theta_1,\theta_2)-\sum_{\tilde\theta_1\in\Theta_1}\mu(\tilde\theta_1|\theta_2^r)x^*(\tilde\theta_1,\theta_2^r)v_2(\tilde\theta_1,\theta_2^r)\\
&\quad+\gamma_2(\theta_2^r)\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)h_2(\theta_1,\theta_2^r).
\end{aligned}
$$

Since $\sum_{\theta_1\in\Theta_1}\mu(\theta_1|\theta_2)h_2(\theta_1,\theta_2^r)<0$ for all $\theta_2^r\neq\theta_2$, because of the finiteness of Θ_2, we can choose $\gamma_2(\theta_2^r)$ large enough such that, for all $\theta_2,\theta_2^r\in\Theta_2$,

$$U_2(\theta_2;\theta_2)\geq U_2(\theta_2;\theta_2^r).$$

Therefore, BIC is satisfied for the buyer. This completes the proof. □

8.2 Proof of Step 3 in Proposition 2

Proof. We will prove by induction that $x^*(\theta_1^k, \theta_2^k) = 1$ and $x^*(\theta_1^k, \theta_2^{k-1}) = 0$ for all $k \in \{2, \cdots, m\}$. Consider the case of $k = 2$. We first claim that $x^*(\theta_1^2, \theta_2^1) = 0$. Suppose, on the contrary, that $x^*(\theta_1^2, \theta_2^1) = 1$. Since $x^*(\theta_1^1, \theta_2^1) = x^*(\theta_1^2, \theta_2^1) = 1$ and $x^*(\theta_1^k, \theta_2^l)$ is nondecreasing in θ_2^l for each θ_1^k (by Step 1), we have $x^*(\theta_1^1, \theta_2^l) = x^*(\theta_1^2, \theta_2^l) = 1$ for all θ_2^l, violating the responsiveness of the efficient decision rule. Hence, $x^*(\theta_1^2, \theta_2^1) = 0$.

We next claim $x^*(\theta_1^2, \theta_2^2) = 1$. Suppose, on the contrary, that $x^*(\theta_1^2, \theta_2^2) = 0$. Since $x^*(\theta_1^1, \theta_2^1) = 1$ and $x^*(\theta_1^1, \theta_2^l)$ is nondecreasing in θ_2^l, we have $x^*(\theta_1^1, \theta_2^2) = x^*(\theta_1^1, \theta_2^1) = 1$. Moreover, since $x^*(\theta_1^2, \theta_2^1) = x^*(\theta_1^2, \theta_2^2) = 0$ and $x^*(\theta_1^k, \theta_2^l)$ is nonincreasing in θ_1^k for each θ_2^l, we have $x^*(\theta_1^k, \theta_2^1) = x^*(\theta_1^k, \theta_2^2) = 0$ for all $k \geq 2$. To summarize, we have

$$x^*(\theta_1^{k'}, \theta_2^1) = x^*(\theta_1^{k'}, \theta_2^2), \forall k' \in \{1, \ldots, m\},$$

which violates the responsiveness of the efficient decision rule $x^*(\cdot)$. Hence, $x^*(\theta_1^2, \theta_2^2) = 1$.

Fix $k \in \{2, \ldots, m\}$ arbitrarily. By our induction hypothesis, we assume that $x^*(\theta_1^{\tilde{k}}, \theta_2^{\tilde{k}}) = 1$ and $x^*(\theta_1^{\tilde{k}}, \theta_2^{\tilde{k}-1}) = 0$ for each $\tilde{k} \in \{2, \ldots, k\}$. First, we claim that $x^*(\theta_1^{k+1}, \theta_2^k) = 0$. Suppose, on the contrary, that $x^*(\theta_1^{k+1}, \theta_2^k) = 1$. Since $x^*(\theta_1^k, \theta_2^{k-1})$ is nonincreasing in θ_1^k (by Step 2) and $x^*(\theta_1^k, \theta_2^{k-1}) = 0$ (induction hypothesis), we have $x^*(\theta_1^{k+1}, \theta_2^{k-1}) = 0$.

Since $x^*(\theta_1^k, \theta_2^{k-1}) = x^*(\theta_1^{k+1}, \theta_2^{k-1}) = 0$ and $x^*(\theta_1^{k'}, \theta_2^l)$ is nondecreasing in θ_2^l for each $\theta_1^{k'}$ (by Step 1), we also have

$$x^*(\theta_1^k, \theta_2^l) = x^*(\theta_1^{k+1}, \theta_2^l) = 0$$

for all $l \leq k - 1$.

Finally, since $x^*(\theta_1^k, \theta_2^k) = x^*(\theta_1^{k+1}, \theta_2^k) = 1$ and $x^*(\theta_1^{k'}, \theta_2^l)$ is nondecreasing in θ_2^l for each $\theta_1^{k'}$ (by Step 1), we have

$$x^*(\theta_1^k, \theta_2^l) = x^*(\theta_1^{k+1}, \theta_2^l) = 1$$

for all $l \geq k$.

To summarize, we obtain

$$x^*(\theta_1^k, \theta_2^l) = x^*(\theta_1^{k+1}, \theta_2^l), \ \forall l \in \{1, \ldots, m\},$$

which violates the responsiveness of the efficient decision rule $x^*(\cdot)$. Hence, $x^*(\theta_1^{k+1}, \theta_2^k) = 0$.

We next claim that $x^*(\theta_1^{k+1}, \theta_2^{k+1}) = 1$. Suppose, on the contrary, that $x^*(\theta_1^{k+1}, \theta_2^{k+1}) = 0$. Since $x^*(\theta_1^k, \theta_2^l)$ is nondecreasing in θ_2^l (by Step 1) and $x^*(\theta_1^k, \theta_2^k) = 1$ (inductive hypothesis), we have $x^*(\theta_1^k, \theta_2^{k+1}) = 1$.

Since $x^*(\theta_1^k, \theta_2^k) = x^*(\theta_1^k, \theta_2^{k+1}) = 1$ and $x^*(\theta_1^{k'}, \theta_2^l)$ is nonincreasing in $\theta_1^{k'}$ for each θ_2^l (by Step 2), we also have

$$x^*(\theta_1^{k'}, \theta_2^k) = x^*(\theta_1^{k'}, \theta_2^{k+1}) = 1$$

for all $k' \leq k$.

Finally, since $x^*(\theta_1^{k+1}, \theta_2^k) = x^*(\theta_1^{k+1}, \theta_2^{k+1}) = 0$ and $x^*(\theta_1^{k'}, \theta_2^k)$ is nonincreasing in $\theta_1^{k'}$ for each θ_2^k (by Step 2), we have

$$x^*(\theta_1^{k'}, \theta_2^k) = x^*(\theta_1^{k'}, \theta_2^{k+1}) = 0$$

for all $k' \geq k+1$.

To summarize, we obtain

$$x^*(\theta_1^{k'}, \theta_2^k) = x^*(\theta_1^{k'}, \theta_2^{k+1}), \ \forall k^{'} \in \{1, \ldots, m\},$$

which violates the responsiveness of the efficient decision rule $x^*(\cdot)$. Hence, $x^*(\theta_1^{k+1}, \theta_2^{k+1}) = 1$.

Therefore, we conclude that $x^*(\theta_1^k, \theta_2^k) = 1$ and $x^*(\theta_1^k, \theta_2^{k-1}) = 0$ for all $k \in \{2, \cdots, m\}$. □

8.3 Proof of Theorem 1

Proof. In order for the mechanism (x^*, t) to satisfy EPBB, we let $t_1(\theta_1^k, \theta_2^l) = -t_2(\theta_1^k, \theta_2^l) = t(\theta_1^k, \theta_2^l)$ for all $(\theta_1^k, \theta_2^l) \in \Theta$, i.e., $t(\theta_1^k, \theta_2^l)$ is the amount of money the buyer pays the seller when the type report is (θ_1^k, θ_2^l). Besides, the efficient decision rule x^* requires that, for all $(\theta_1^k, \theta_2^l) \in \Theta$, $x^*(\theta_1^k, \theta_2^l) = 0$ if and only if $k > l$. By NTNP, we have that, for all $(\theta_1^k, \theta_2^l) \in \Theta$, $t(\theta_1^k, \theta_2^l) = 0$ whenever $k > l$. We write down the following IIR constraints:

$$\begin{aligned} IR_{\theta_1^m} : \quad & \mu(\theta_2^m|\theta_1^m) t(\theta_1^m, \theta_2^m) + \sum_{l=1}^{m-1} \mu(\theta_2^l|\theta_1^m) v_1(\theta_1^m, \theta_2^l) \geq \sum_{l=1}^{m} \mu(\theta_2^l|\theta_1^m) v_1(\theta_1^m, \theta_2^l); \\ IR_{\theta_2^1} : \quad & \mu(\theta_1^1|\theta_2^1) \left(v_2(\theta_1^1, \theta_2^1) - t(\theta_1^1, \theta_2^1) \right) \geq 0, \end{aligned}$$

where $IR_{\theta_i^k}$ denotes the IIR constraint for agent i of type θ_i^k. The IIR constraints can be rewritten as follows:

$$\begin{aligned} IR_{\theta_1^m} : \quad & \mu(\theta_1^m, \theta_2^m) \left(t(\theta_1^m, \theta_2^m) - v_1(\theta_1^m, \theta_2^m) \right) \geq 0; \\ IR_{\theta_2^1} : \quad & \mu(\theta_1^1, \theta_2^1) \left(v_2(\theta_1^1, \theta_2^1) - t(\theta_1^1, \theta_2^1) \right) \geq 0. \end{aligned}$$

Consider the seller of type θ_1^k with $k \neq m$. To stop the seller from deviating to θ_1^m, the following BIC constraints must be satisfied: for any $k \neq m$,

$$
\begin{aligned}
IC_{\theta_1^k \to \theta_1^m}: \quad & \sum_{l=k}^{m} \mu(\theta_2^l|\theta_1^k) t(\theta_1^k, \theta_2^l) + \sum_{l=1}^{k-1} \mu(\theta_2^l|\theta_1^k) v_1(\theta_1^k, \theta_2^l) \\
& \geq \mu(\theta_2^m|\theta_1^k) t(\theta_1^m, \theta_2^m) + \sum_{l=1}^{m-1} \mu(\theta_2^l|\theta_1^k) v_1(\theta_1^k, \theta_2^l) \\
\Rightarrow \quad & \sum_{l=k}^{m} \mu(\theta_1^k, \theta_2^l) t(\theta_1^k, \theta_2^l) - \mu(\theta_1^k, \theta_2^m) t(\theta_1^m, \theta_2^m) \geq \sum_{l=k}^{m-1} \mu(\theta_1^k, \theta_2^l) v_1(\theta_1^k, \theta_2^l).
\end{aligned}
$$

Consider the buyer of type θ_2^l with $l \neq 1$. To stop the buyer from deviating to θ_2^1, the following BIC constraints must be satisfied: for any $l \neq 1$,

$$
\begin{aligned}
IC_{\theta_2^l \to \theta_2^1}: \quad & \sum_{k=1}^{l} \mu(\theta_1^k|\theta_2^l) \left(v_2(\theta_1^k, \theta_2^l) - t(\theta_1^k, \theta_2^l)\right) \geq \mu(\theta_1^1|\theta_2^l) \left(v_2(\theta_1^1, \theta_2^l) - t(\theta_1^1, \theta_2^1)\right) \\
\Rightarrow \quad & \sum_{k=2}^{l} \mu(\theta_1^k, \theta_2^l) v_2(\theta_1^k, \theta_2^l) - \sum_{k=1}^{l} \mu(\theta_1^k, \theta_2^l) t(\theta_1^k, \theta_2^l) + \mu(\theta_1^1, \theta_2^l) t(\theta_1^1, \theta_2^1) \geq 0.
\end{aligned}
$$

In the rest of the proof, using the above BIC and IIR constraints, we verify that the Ex-Post condition holds. First, we multiply $IR_{\theta_1^m}$ by $\sum_{k=1}^{m} \mu(\theta_1^k, \theta_2^m)/\mu(\theta_1^m, \theta_2^m)$ to obtain

$$\sum_{k=1}^{m} \mu(\theta_1^k, \theta_2^m) \left(t(\theta_1^m, \theta_2^m) - v_1(\theta_1^m, \theta_2^m)\right) \geq 0. \tag{13}$$

Second, we multiply $IR_{\theta_2^1}$ by $\sum_{l=1}^{m} \mu(\theta_1^1, \theta_2^l)/\mu(\theta_1^1, \theta_2^1)$ to obtain

$$\sum_{l=1}^{m} \mu(\theta_1^1, \theta_2^l) \left(v_2(\theta_1^1, \theta_2^1) - t(\theta_1^1, \theta_2^1)\right) \geq 0. \tag{14}$$

Third, summing up $IC_{\theta_1^1 \to \theta_1^m}, IC_{\theta_1^2 \to \theta_1^m}, \cdots, IC_{\theta_1^{m-1} \to \theta_1^m}$, we have

$$\sum_{k=1}^{m-1} \sum_{l=k}^{m} \mu(\theta_1^k, \theta_2^l) t(\theta_1^k, \theta_2^l) - \sum_{k=1}^{m-1} \mu(\theta_1^k, \theta_2^m) t(\theta_1^m, \theta_2^m) - \sum_{k=1}^{m-1} \sum_{l=k}^{m-1} \mu(\theta_1^k, \theta_2^l) v_1(\theta_1^k, \theta_2^l) \geq 0. \tag{15}$$

Notice that the first term of the left-hand-side of the inequality (15) can be decomposed into the following:

$$\sum_{k=1}^{m-1} \sum_{l=k}^{m} \mu(\theta_1^k, \theta_2^l) t(\theta_1^k, \theta_2^l) \;=\; \sum_{k=2}^{m-1} \sum_{l=k}^{m} \mu(\theta_1^k, \theta_2^l) t(\theta_1^k, \theta_2^l) + \sum_{l=2}^{m} \mu(\theta_1^1, \theta_2^l) t(\theta_1^1, \theta_2^l) + \mu(\theta_1^1, \theta_2^1) t(\theta_1^1, \theta_2^1).$$

Substituting it back into the inequality (15), we obtain

$$\sum_{k=2}^{m-1}\sum_{l=k}^{m}\mu(\theta_1^k,\theta_2^l)t(\theta_1^k,\theta_2^l)+\sum_{l=2}^{m}\mu(\theta_1^1,\theta_2^l)t(\theta_1^1,\theta_2^l)+\mu(\theta_1^1,\theta_2^1)t(\theta_1^1,\theta_2^1)$$
$$-\sum_{k=1}^{m-1}\mu(\theta_1^k,\theta_2^m)t(\theta_1^m,\theta_2^m)-\sum_{k=1}^{m-1}\sum_{l=k}^{m-1}\mu(\theta_1^k,\theta_2^l)v_1(\theta_1^k,\theta_2^l)\geq 0. \tag{16}$$

Fourth, summing up $IC_{\theta_2^2\to\theta_2^1}, IC_{\theta_2^3\to\theta_2^1},\cdots,IC_{\theta_2^m\to\theta_2^1}$, we have

$$\sum_{l=2}^{m}\sum_{k=2}^{l}\mu(\theta_1^k,\theta_2^l)v_2(\theta_1^k,\theta_2^l)-\sum_{l=2}^{m}\sum_{k=1}^{l}\mu(\theta_1^k,\theta_2^l)t(\theta_1^k,\theta_2^l)+\sum_{l=2}^{m}\mu(\theta_1^1,\theta_2^l)t(\theta_1^1,\theta_2^1)\geq 0. \tag{17}$$

Notice that the second term of the left-hand-side in the inequality (17) can be decomposed into the following:

$$-\sum_{l=2}^{m}\sum_{k=1}^{l}\mu(\theta_1^k,\theta_2^l)t(\theta_1^k,\theta_2^l) \;=\; -\sum_{l=2}^{m-1}\sum_{k=1}^{l}\mu(\theta_1^k,\theta_2^l)t(\theta_1^k,\theta_2^l)-\sum_{k=1}^{m-1}\mu(\theta_1^k,\theta_2^m)t(\theta_1^k,\theta_2^m)-\mu(\theta_1^m,\theta_2^m)t(\theta_1^m,\theta_2^m).$$

Substituting it back into the inequality (17), we obtain

$$\sum_{l=2}^{m}\sum_{k=2}^{l}\mu(\theta_1^k,\theta_2^l)v_2(\theta_1^k,\theta_2^l)-\sum_{l=2}^{m-1}\sum_{k=1}^{l}\mu(\theta_1^k,\theta_2^l)t(\theta_1^k,\theta_2^l)-\sum_{k=1}^{m-1}\mu(\theta_1^k,\theta_2^m)t(\theta_1^k,\theta_2^m)$$
$$-\mu(\theta_1^m,\theta_2^m)t(\theta_1^m,\theta_2^m)+\sum_{l=2}^{m}\mu(\theta_1^1,\theta_2^l)t(\theta_1^1,\theta_2^1)\geq 0. \tag{18}$$

Fifth, we add up inequalities (13), (14), (16) and (18). After adding up these inequalities, we show that all terms involving the transfers $\{t(\theta_1^k,\theta_2^l)\}_{(k,l):k\leq l}$ are cancelled out. To see this, we divide the payment into three groups: $t(\theta_1^1,\theta_2^1)$, $t(\theta_1^m,\theta_2^m)$, and the transfers at all the other type profiles.

Observe that $t(\theta_1^1,\theta_2^1)$ appears in (i) the inequality (14) with its coefficient $-\sum_{l=1}^{m}\mu(\theta_1^1,\theta_2^l)$; (ii) the inequality (16) with its coefficient $\mu(\theta_1^1,\theta_2^1)$; and (iii) the inequality (18) with its coefficient $\sum_{l=2}^{m}\mu(\theta_1^1,\theta_2^l)$. The summation of the coefficients equals

$$-\sum_{l=1}^{m}\mu(\theta_1^1,\theta_2^l)+\mu(\theta_1^1,\theta_2^1)+\sum_{l=2}^{m}\mu(\theta_1^1,\theta_2^l)=0.$$

Hence, the terms involving $t(\theta_1^1,\theta_2^1)$ are cancelled out.

Observe that $t(\theta_1^m,\theta_2^m)$ appears in (i) the inequality (13) with its coefficient $\sum_{k=1}^{m}\mu(\theta_1^k,\theta_2^m)$; (ii) the inequality (16) with its coefficient $-\sum_{k=1}^{m-1}\mu(\theta_1^k,\theta_2^m)$; and (iii) the inequality (18) with its coefficient $-\mu(\theta_1^m,\theta_2^m)$. The summation of the coefficients equals

$$\sum_{k=1}^{m}\mu(\theta_1^k,\theta_2^m)-\sum_{k=1}^{m-1}\mu(\theta_1^k,\theta_2^m)-\mu(\theta_1^m,\theta_2^m)=0.$$

Hence, the terms involving $t(\theta_1^m, \theta_2^m)$ are cancelled out.

Observe that any other arbitrary payment $t(\theta_1^k, \theta_2^l)$ appears in (i) the inequality (16) with its coefficient $\mu(\theta_1^k, \theta_2^l)$; and (ii) the inequality (18) with its coefficient $-\mu(\theta_1^k, \theta_2^l)$. Since the summation of the coefficients equals

$$\mu(\theta_1^k, \theta_2^l) - \mu(\theta_1^k, \theta_2^l) = 0,$$

the terms involving all the other payments $t(\theta_1^k, \theta_2^l)$ are cancelled out.

Therefore, the summation of the left-hand side of these added inequalities (13), (14), (16) and (18) leads to

$$\begin{aligned} & \sum_{l=1}^{m} \mu(\theta_1^1, \theta_2^l) v_2(\theta_1^1, \theta_2^1) + \sum_{l=2}^{m} \sum_{k=2}^{l} \mu(\theta_1^k, \theta_2^l) v_2(\theta_1^k, \theta_2^l) \\ \geq \ & \sum_{k=1}^{m} \mu(\theta_1^k, \theta_2^m) v_1(\theta_1^m, \theta_2^m) + \sum_{k=1}^{m-1} \sum_{l=k}^{m-1} \mu(\theta_1^k, \theta_2^l) v_1(\theta_1^k, \theta_2^l). \end{aligned}$$

This can be rewriten as

$$\begin{aligned} & \sum_{l=1}^{m} \mu(\theta_1^1, \theta_2^l) v_2(\theta_1^1, \theta_2^1) + \sum_{k=2}^{m} \sum_{l=k}^{m} \mu(\theta_1^k, \theta_2^l) v_2(\theta_1^k, \theta_2^l) \\ \geq \ & \sum_{k=1}^{m} \mu(\theta_1^k, \theta_2^m) v_1(\theta_1^m, \theta_2^m) + \sum_{k=1}^{m-1} \sum_{l=k}^{m-1} \mu(\theta_1^k, \theta_2^l) v_1(\theta_1^k, \theta_2^l), \end{aligned}$$

which is identical to the Ex-Post condition. This completes the proof. □

8.4 Proof of Proposition 3

Proof. Dividing both hand sides of inequality (2) by $\mu(\theta_1^1, \theta_2^2)$, we obtain

$$\begin{aligned} & \left(\frac{\mu(\theta_1^1, \theta_2^1)}{\mu(\theta_1^1, \theta_2^2)} + 1 \right) v_2(\theta_1^1, \theta_2^1) + \frac{\mu(\theta_1^2, \theta_2^2)}{\mu(\theta_1^1, \theta_2^2)} v_2(\theta_1^2, \theta_2^2) \\ \geq \ & \left(1 + \frac{\mu(\theta_1^2, \theta_2^2)}{\mu(\theta_1^1, \theta_2^2)} \right) v_1(\theta_1^2, \theta_2^2) + \frac{\mu(\theta_1^1, \theta_2^1)}{\mu(\theta_1^1, \theta_2^2)} v_1(\theta_1^1, \theta_2^1). \end{aligned}$$

To simplify the notation, we set $\alpha_1 \equiv \mu(\theta_1^1, \theta_2^1)/\mu(\theta_1^1, \theta_2^2)$ and $\alpha_2 \equiv \mu(\theta_1^2, \theta_2^2)/\mu(\theta_1^1, \theta_2^2)$. Then, the inequality (2) can further be rewritten as

$$(1 + \alpha_1) v_2(\theta_1^1, \theta_2^1) + \alpha_2 v_2(\theta_1^2, \theta_2^2) \geq (1 + \alpha_2) v_1(\theta_1^2, \theta_2^2) + \alpha_1 v_1(\theta_1^1, \theta_2^1). \qquad (19)$$

It suffices to show that the inequality (19) is also a sufficient condition for the existence of mechanisms satisfying BIC, IIR, EFF, EPBB and NTNP. In order for the mechanism (x^*, t) to satisfy EPBB, we set $t_1(\theta_1^k, \theta_2^l) = -t_2(\theta_1^k, \theta_2^l) = t(\theta_1^k, \theta_2^l)$

for all $(\theta_1^k, \theta_2^l) \in \Theta$, i.e., $t(\theta_1^k, \theta_2^l)$ is the amount of money the buyer pays the seller when the type report is (θ_1^k, θ_2^l). Moreover, since we require the mechanism (x^*, t) to satisfy NTNP, $x^*(\theta_1^2, \theta_2^1) = 0$ implies $t(\theta_1^2, \theta_2^1) = 0$. Let t be the proposed transfer rule such that $t(\theta_1^1, \theta_2^1) = v_2(\theta_1^1, \theta_2^1)$; $t(\theta_1^2, \theta_2^2) = v_1(\theta_1^2, \theta_2^2)$; and $t(\theta_1^1, \theta_2^2)$ satisfy the following inequality:

$$\begin{aligned} &\max\left\{v_2(\theta_1^1, \theta_2^1), v_1(\theta_1^2, \theta_2^2) - \alpha_1\left(v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1)\right)\right\} \\ \leq &t(\theta_1^1, \theta_2^2) \leq \min\left\{v_1(\theta_1^2, \theta_2^2), v_2(\theta_1^1, \theta_2^1) + \alpha_2\left(v_2(\theta_1^2, \theta_2^2) - v_1(\theta_1^2, \theta_2^2)\right)\right\} \end{aligned}$$

To simplify the notation, we let $A_1 \equiv v_1(\theta_1^2, \theta_2^2) - \alpha_1 (v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1))$ and $A_2 \equiv v_2(\theta_1^1, \theta_2^1) + \alpha_2 (v_2(\theta_1^2, \theta_2^2) - v_1(\theta_1^2, \theta_2^2))$. The rest of the proof is completed by the following three steps: In Step 1, we show that the proposed transfer $t(\theta_1^1, \theta_2^2)$ is well-defined. In Step 2, we verify that the proposed mechanism (x^*, t) satisfies IIR. In Step 3, we verify that it satisfies BIC as well.

Step 1: The proposed transfer $t(\theta_1^1, \theta_2^2)$ is well-defined.

Given the new notation, it suffices to show

$$\max\left\{v_2(\theta_1^1, \theta_2^1), A_1\right\} \leq \min\left\{v_1(\theta_1^2, \theta_2^2), A_2\right\}.$$

This will be shown in the following substeps:

Step 1-1: $v_2(\theta_1^1, \theta_2^1) \leq v_1(\theta_1^2, \theta_2^2)$.

This follows because $v_2(\theta_1^1, \theta_2^1) \leq v_2(\theta_1^2, \theta_2^1) \leq v_1(\theta_1^2, \theta_2^1) \leq v_1(\theta_1^2, \theta_2^2)$.

Step 1-2: $v_2(\theta_1^1, \theta_2^1) \leq A_2$.

This follows because

$$\begin{aligned} v_2(\theta_1^1, \theta_2^1) - A_2 &= v_2(\theta_1^1, \theta_2^1) - v_2(\theta_1^1, \theta_2^1) - \alpha_2\left(v_2(\theta_1^2, \theta_2^2) - v_1(\theta_1^2, \theta_2^2)\right) \\ &= -\alpha_2\left(v_2(\theta_1^2, \theta_2^2) - v_1(\theta_1^2, \theta_2^2)\right) \\ &< 0 \ (\because\ v_2(\theta_1^2, \theta_2^2) - v_1(\theta_1^2, \theta_2^2) > 0). \end{aligned}$$

Step 1-3: $A_1 \leq v_1(\theta_1^2, \theta_2^2)$.

This follows because

$$\begin{aligned} A_1 - v_1(\theta_1^2, \theta_2^2) &= v_1(\theta_1^2, \theta_2^2) - \alpha_1\left(v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1)\right) - v_1(\theta_1^2, \theta_2^2) \\ &= -\alpha_1\left(v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1)\right) \\ &< 0 \ (\because\ v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1) > 0). \end{aligned}$$

Step 1-4: $A_1 \leq A_2$.

This follows because

$$\begin{aligned}
A_1 - A_2 &= v_1(\theta_1^2, \theta_2^2) - \alpha_1 \left(v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1)\right) - v_2(\theta_1^1, \theta_2^1) - \alpha_2 \left(v_2(\theta_1^2, \theta_2^2) - v_1(\theta_1^2, \theta_2^2)\right) \\
&= -(1+\alpha_1) v_2(\theta_1^1, \theta_2^1) - \alpha_2 v_2(\theta_1^2, \theta_2^2) + \alpha_1 v_1(\theta_1^1, \theta_2^1) + (1+\alpha_2) v_1(\theta_1^2, \theta_2^2) \\
&\leq 0 \ (\because \text{ the inequality (19)}).
\end{aligned}$$

Step 2: The proposed mechanism (x^*, t) satisfies IIR.

We show this by considering the following four cases.

Case 1: IIR for the seller of type θ_1^1

If both agents report their type truthfully, the expected utility for the seller of type θ_1^1 becomes

$$\begin{aligned}
U_1(\theta_1^1; \theta_1^1) &= \mu(\theta_2^1|\theta_1^1) t(\theta_1^1, \theta_2^1) + \mu(\theta_2^2|\theta_1^1) t(\theta_1^1, \theta_2^2) \\
&\geq \mu(\theta_2^1|\theta_1^1) v_2(\theta_1^1, \theta_2^1) + \mu(\theta_2^2|\theta_1^1) A_1 \\
&= \mu(\theta_2^1|\theta_1^1) v_2(\theta_1^1, \theta_2^1) + \mu(\theta_2^2|\theta_1^1) \left[v_1(\theta_1^2, \theta_2^2) - \alpha_1 \left(v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1)\right)\right] \\
&= \mu(\theta_2^1|\theta_1^1) v_2(\theta_1^1, \theta_2^1) + \mu(\theta_2^2|\theta_1^1) v_1(\theta_1^2, \theta_2^2) - \mu(\theta_2^1|\theta_1^1) \left(v_2(\theta_1^1, \theta_2^1) - v_1(\theta_1^1, \theta_2^1)\right) \\
&= \mu(\theta_2^2|\theta_1^1) v_1(\theta_1^2, \theta_2^2) + \mu(\theta_2^1|\theta_1^1) v_1(\theta_1^1, \theta_2^1),
\end{aligned}$$

where the first weak inequality follows because $t(\theta_1^1, \theta_2^1) = v_2(\theta_1^1, \theta_2^1)$ and $t(\theta_1^1, \theta_2^2) \geq A_1$; the second equality follows from the expression of A_1; and the third equality follows because $\alpha_1 = \mu(\theta_1^1, \theta_2^1)/\mu(\theta_1^1, \theta_2^2)$.

With the seller's outside option utility $U_1^O(\theta_1^1)$, we obtain

$$\begin{aligned}
U_1(\theta_1^1; \theta_1^1) - U_1^O(\theta_1^1) &\geq \mu(\theta_2^2|\theta_1^1) v_1(\theta_1^2, \theta_2^2) + \mu(\theta_2^1|\theta_1^1) v_1(\theta_1^1, \theta_2^1) - \sum_{l=1}^{2} \mu(\theta_2^l|\theta_1^1) v_1(\theta_1^1, \theta_2^l) \\
&= \mu(\theta_2^2|\theta_1^1) \left(v_1(\theta_1^2, \theta_2^2) - v_1(\theta_1^1, \theta_2^2)\right) \\
&> 0,
\end{aligned}$$

where the last inequality follows because the seller's valuation is strictly increasing in her own type. Therefore, IIR is satisfied for the seller of type θ_1^1.

Case 2: IIR for the seller of type θ_1^2

If both agents report their type truthfully, the expected payoff for the seller of type θ_1^2 becomes

$$U_1(\theta_1^2; \theta_1^2) = \mu(\theta_2^1|\theta_1^2) v_1(\theta_1^2, \theta_2^1) + \mu(\theta_2^2|\theta_1^2) t(\theta_1^2, \theta_2^2) \;=\; \mu(\theta_2^1|\theta_1^2) v_1(\theta_1^2, \theta_2^1) + \mu(\theta_2^2|\theta_1^2) v_1(\theta_1^2, \theta_2^2),$$

where the second equality follows because $t(\theta_1^2, \theta_2^2) = v_1(\theta_1^2, \theta_2^2)$.

With the seller's outside option utility $U_1^O(\theta_1^2)$, we obtain

$$\begin{aligned} U_1(\theta_1^2;\theta_1^2) - U_1^O(\theta_1^2) &= \mu(\theta_2^1|\theta_1^2)v_1(\theta_1^2,\theta_2^1) + \mu(\theta_2^2|\theta_1^2)v_1(\theta_1^2,\theta_2^2) - \sum_{l=1}^{2}\mu(\theta_2^l|\theta_1^2)v_1(\theta_1^2,\theta_2^l) \\ &= 0, \end{aligned}$$

implying that IIR is satisfied for the seller of type θ_1^2.

Case 3: IIR for the buyer of type θ_2^1

If both agents report their type truthfully, the expected utility for the buyer of type θ_2^1 becomes

$$U_2(\theta_2^1;\theta_2^1) = \mu(\theta_1^1|\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - t(\theta_1^1,\theta_2^1)\right) = \mu(\theta_1^1|\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - v_2(\theta_1^1,\theta_2^1)\right) = 0,$$

where the second equality follows because $t(\theta_1^1,\theta_2^1) = v_2(\theta_1^1,\theta_2^1)$. Since the buyer's outside option utility is zero, IIR is satisfied for the buyer of type θ_2^1.

Case 4: IIR for the buyer of type θ_2^2

If both agents report their type truthfully, the expected utility for the buyer of type θ_2^2 becomes

$$\begin{aligned} U_2(\theta_2^2;\theta_2^2) &= \mu(\theta_1^1|\theta_2^2)\left(v_2(\theta_1^1,\theta_2^2) - t(\theta_1^1,\theta_2^2)\right) + \mu(\theta_1^2|\theta_2^2)\left(v_2(\theta_1^2,\theta_2^2) - t(\theta_1^2,\theta_2^2)\right) \\ &= \mu(\theta_1^1|\theta_2^2)\left[v_2(\theta_1^1,\theta_2^2) - t(\theta_1^1,\theta_2^2) + \alpha_2\left(v_2(\theta_1^2,\theta_2^2) - v_1(\theta_1^2,\theta_2^2)\right)\right] \\ &\geq \mu(\theta_1^1|\theta_2^2)\left[v_2(\theta_1^1,\theta_2^2) - A_2 + \alpha_2\left(v_2(\theta_1^2,\theta_2^2) - v_1(\theta_1^2,\theta_2^2)\right)\right] \\ &= \mu(\theta_1^1|\theta_2^2)\big[v_2(\theta_1^1,\theta_2^2) - v_2(\theta_1^1,\theta_2^1) - \alpha_2\left(v_2(\theta_1^2,\theta_2^2) - v_1(\theta_1^2,\theta_2^2)\right) \\ &\qquad\qquad +\alpha_2\left(v_2(\theta_1^2,\theta_2^2) - v_1(\theta_1^2,\theta_2^2)\right)\big] \\ &= \mu(\theta_1^1|\theta_2^2)\left(v_2(\theta_1^1,\theta_2^2) - v_2(\theta_1^1,\theta_2^1)\right) \\ &> 0, \end{aligned}$$

where the second equality follows because $\alpha_2 = \mu(\theta_1^2,\theta_2^2)/\mu(\theta_1^1,\theta_2^2)$ and $t(\theta_1^2,\theta_2^2) = v_1(\theta_1^2,\theta_2^2)$; the first weak inequality follows because $t(\theta_1^1,\theta_2^2) \leq A_2$; the third equality follows from the expression of A_2, and the last strict inequality follows because the buyer's valuation is strictly increasing in his own type.

Since the buyer's outside option utility is zero, IIR is satisfied for the buyer of type θ_2^2.

Step 3: The proposed mechanism (x^*, t) satisfies BIC.

We show this by considering the following four cases.

Case I: BIC for the seller of type θ_1^1

According to the previous argument in Case 1 of Step 2, we have the following inequality:

$$U_1(\theta_1^1;\theta_1^1) \geq \mu(\theta_2^2|\theta_1^1)v_1(\theta_1^2,\theta_2^2) + \mu(\theta_2^1|\theta_1^1)v_1(\theta_1^1,\theta_2^1).$$

If she deviates to θ_1^2, her expected utility, denoted by $U_1(\theta_1^1;\theta_1^2)$, becomes

$$\begin{aligned} U_1(\theta_1^1;\theta_1^2) &= \mu(\theta_2^2|\theta_1^1)t(\theta_1^2,\theta_2^2) + \mu(\theta_2^1|\theta_1^1)v_1(\theta_1^1,\theta_2^1) \\ &= \mu(\theta_2^2|\theta_1^1)v_1(\theta_1^2,\theta_2^2) + \mu(\theta_2^1|\theta_1^1)v_1(\theta_1^1,\theta_2^1) \leq U_1(\theta_1^1;\theta_1^1), \end{aligned}$$

where the second equality follows because $t(\theta_1^2,\theta_2^2) = v_1(\theta_1^2,\theta_2^2)$. Therefore, BIC is satisfied for the seller of type θ_1^1.

Case II: BIC for the seller of type θ_1^2

According to the previous argument in Case 2 of Step 2, we have the following:

$$U_1(\theta_1^2;\theta_1^2) = \mu(\theta_2^1|\theta_1^2)v_1(\theta_1^2,\theta_2^1) + \mu(\theta_2^2|\theta_1^2)v_1(\theta_1^2,\theta_2^2).$$

If she deviates to θ_1^1, her expected utility, denoted by $U_1(\theta_1^2;\theta_1^1)$, becomes

$$\begin{aligned} U_1(\theta_1^2;\theta_1^1) &= \mu(\theta_2^1|\theta_1^2)t(\theta_1^1,\theta_2^1) + \mu(\theta_2^2|\theta_1^2)t(\theta_1^1,\theta_2^2) \\ &= \mu(\theta_2^1|\theta_1^2)v_2(\theta_1^1,\theta_2^1) + \mu(\theta_2^2|\theta_1^2)t(\theta_1^1,\theta_2^2), \end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^1) = v_2(\theta_1^1,\theta_2^1)$. Since $v_2(\theta_1^1,\theta_2^1) \leq v_2(\theta_1^2,\theta_2^1) \leq v_1(\theta_1^2,\theta_2^1)$ and $t(\theta_1^1,\theta_2^2) \leq v_1(\theta_1^2,\theta_2^2)$, we have

$$U_1(\theta_1^2;\theta_1^1) \leq \mu(\theta_2^1|\theta_1^2)v_1(\theta_1^2,\theta_2^1) + \mu(\theta_2^2|\theta_1^2)v_1(\theta_1^2,\theta_2^2) = U_1(\theta_1^2;\theta_1^2).$$

Therefore, BIC is satisfied for the seller of type θ_1^2.

Case III: BIC for the buyer of type θ_2^1

According to the previous argument in Case 3 of Step 2, if both agents report their type truthfully, the expected utility for the buyer of type θ_2^1 is zero. If he deviates to θ_2^2, his expected utility, denoted by $U_2(\theta_2^1;\theta_2^2)$, becomes

$$\begin{aligned} U_2(\theta_2^1;\theta_2^2) &= \mu(\theta_1^1|\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - t(\theta_1^1,\theta_2^2)\right) + \mu(\theta_1^2|\theta_2^1)\left(v_2(\theta_1^2,\theta_2^1) - t(\theta_1^2,\theta_2^2)\right) \\ &= \mu(\theta_1^1|\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - t(\theta_1^1,\theta_2^2)\right) + \mu(\theta_1^2|\theta_2^1)\left(v_2(\theta_1^2,\theta_2^1) - v_1(\theta_1^2,\theta_2^2)\right) \\ &\leq \mu(\theta_1^1|\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - v_2(\theta_1^1,\theta_2^1)\right) + \mu(\theta_1^2|\theta_2^1)\left(v_2(\theta_1^2,\theta_2^1) - v_1(\theta_1^2,\theta_2^2)\right) \\ &= \mu(\theta_1^2|\theta_2^1)\left(v_2(\theta_1^2,\theta_2^1) - v_1(\theta_1^2,\theta_2^2)\right) \\ &\leq 0 = U_2(\theta_2^1;\theta_2^1), \end{aligned}$$

where the second equality follows because $t(\theta_1^2,\theta_2^2) = v_1(\theta_1^2,\theta_2^2)$; the first weak inequality follows because $t(\theta_1^1,\theta_2^2) \geq v_2(\theta_1^1,\theta_2^1)$; and the second weak inequality follows because $v_2(\theta_1^2, \theta_2^1) \le v_1(\theta_1^2, \theta_2^1) \le v_1(\theta_1^2, \theta_2^2)$. Therefore, BIC is satisfied for the buyer of type θ_2^1.

Case IV: BIC for the buyer of type θ_2^2

According to the previous argument in Case 4 of Step 2, we have the following inequality:

$$U_2(\theta_2^2; \theta_2^2) \ge \mu(\theta_1^1|\theta_2^2)\left(v_2(\theta_1^1, \theta_2^2) - v_2(\theta_1^1, \theta_2^1)\right).$$

If he deviates to θ_2^1, his expected utility, denoted by $U_2(\theta_2^2; \theta_2^1)$, becomes

$$\begin{aligned} U_2(\theta_2^2; \theta_2^1) &= \mu(\theta_1^1|\theta_2^2)\left(v_2(\theta_1^1, \theta_2^2) - t(\theta_1^1, \theta_2^1)\right) \\ &= \mu(\theta_1^1|\theta_2^2)\left(v_2(\theta_1^1, \theta_2^2) - v_2(\theta_1^1, \theta_2^1)\right) \le U_2(\theta_2^2; \theta_2^2), \end{aligned}$$

where the second equality follows because $t(\theta_1^1, \theta_2^1) = v_2(\theta_1^1, \theta_2^1)$. Therefore, BIC is satisfied for the buyer of type θ_2^2.

Taking into account that both EFF and EPBB are already built in the mechanism (x^*, t), we complete the proof of Proposition 3. □

8.5 Proof of Claim 1

Proof. We first verify that the Ex-Post condition is satisfied in Example 1. In this example, the Ex-Post condition becomes the following:

$$\begin{aligned} &\sum_{l=1}^{3} \mu(\theta_1^1, \theta_2^l) v_2(\theta_1^1, \theta_2^1) + \mu(\theta_1^2, \theta_2^2) v_2(\theta_1^2, \theta_2^2) + \sum_{k=2}^{3} \mu(\theta_1^k, \theta_2^3) v_2(\theta_1^k, \theta_2^3) \\ \ge\ &\sum_{k=1}^{3} \mu(\theta_1^k, \theta_2^3) v_1(\theta_1^3, \theta_2^3) + \sum_{l=1}^{2} \mu(\theta_1^1, \theta_2^l) v_1(\theta_1^1, \theta_2^l) + \mu(\theta_1^2, \theta_2^2) v_1(\theta_1^2, \theta_2^2). \end{aligned}$$

After plugging the valuations and the values of types into the above inequality, we can rewrite the Ex-Post condition:

$$\begin{aligned} &2.5 \sum_{l=1}^{3} \mu(\theta_1^1, \theta_2^l) + 5.5\mu(\theta_1^2, \theta_2^2) + 7.5\mu(\theta_1^2, \theta_2^3) + 8.5\mu(\theta_1^3, \theta_2^3) \\ \ge\ &8 \sum_{k=1}^{3} \mu(\theta_1^k, \theta_2^3) + 2\mu(\theta_1^1, \theta_2^1) + 3\mu(\theta_1^1, \theta_2^2) + 5\mu(\theta_1^2, \theta_2^2). \end{aligned}$$

Finally, if we plug the joint distribution μ in Example 1 into the above inequality, the Ex-Post condition becomes the following:

$$2.5 \times \frac{3}{21} + 5.5 \times \frac{6}{21} + 7.5 \times \frac{1}{21} + 8.5 \times \frac{6}{21} \ \ge\ 8 \times \frac{8}{21} + 2 \times \frac{1}{21} + 3 \times \frac{1}{21} + 5 \times \frac{6}{21},$$

which results in $99/21 \geq 99/21$, implying that the Ex-Post condition is satisfied.

Next we argue below that there exists no NTNP transfer rule t such that the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB.

In order for a mechanism (x^*, t) to satisfy EPBB, we let $t_1(\theta_1^k, \theta_2^l) = -t_2(\theta_1^k, \theta_2^l) = t(\theta_1^k, \theta_2^l)$ for all $(\theta_1^k, \theta_2^l) \in \Theta$. After imposing NTNP and EPBB on a mechanism (x^*, t), we write down the BIC and IIR constraints, $IC_{\theta_1^1 \to \theta_1^2}, IC_{\theta_1^2 \to \theta_1^3}, IC_{\theta_2^3 \to \theta_2^2}, IC_{\theta_2^2 \to \theta_2^1}, IR_{\theta_1^3}$ and $IR_{\theta_2^1}$ and plug the valuation functions and joint probabilities into those constraints:

$$\begin{aligned}
IC_{\theta_1^1 \to \theta_1^2}: &\quad -t(\theta_1^1, \theta_2^1) - t(\theta_1^1, \theta_2^2) - t(\theta_1^1, \theta_2^3) + t(\theta_1^2, \theta_2^2) + t(\theta_1^2, \theta_2^3) \leq -2; \\
IC_{\theta_1^2 \to \theta_1^3}: &\quad -6t(\theta_1^2, \theta_2^2) - t(\theta_1^2, \theta_2^3) + t(\theta_1^3, \theta_2^3) \leq -30; \\
IC_{\theta_2^3 \to \theta_2^2}: &\quad -t(\theta_1^1, \theta_2^2) - t(\theta_1^2, \theta_2^2) + t(\theta_1^1, \theta_2^3) + t(\theta_1^2, \theta_2^3) + 6t(\theta_1^3, \theta_2^3) \leq 51; \\
IC_{\theta_2^2 \to \theta_2^1}: &\quad -t(\theta_1^1, \theta_2^1) + t(\theta_1^1, \theta_2^2) + 6t(\theta_1^2, \theta_2^2) \leq 33; \\
IR_{\theta_1^3}: &\quad -t(\theta_1^3, \theta_2^3) \leq -8; \\
IR_{\theta_2^1}: &\quad t(\theta_1^1, \theta_2^1) \leq 2.5.
\end{aligned}$$

Summing up $IC_{\theta_2^2 \to \theta_2^1}$ and $IC_{\theta_1^2 \to \theta_1^3}$, we obtain

$$\begin{aligned}
&-t(\theta_1^1, \theta_2^1) + t(\theta_1^1, \theta_2^2) - t(\theta_1^2, \theta_2^3) + t(\theta_1^3, \theta_2^3) \leq 3 \\
\Rightarrow\quad &-t(\theta_1^3, \theta_2^3) \geq -t(\theta_1^1, \theta_2^1) + t(\theta_1^1, \theta_2^2) - t(\theta_1^2, \theta_2^3) - 3. \qquad (20)
\end{aligned}$$

Summing up $IC_{\theta_2^3 \to \theta_2^2}$ and $IC_{\theta_1^1 \to \theta_1^2}$, we obtain

$$\begin{aligned}
&-2t(\theta_1^1, \theta_2^2) + 2t(\theta_1^2, \theta_2^3) + 6t(\theta_1^3, \theta_2^3) - t(\theta_1^1, \theta_2^1) \leq 49 \\
\Rightarrow\quad &-6t(\theta_1^3, \theta_2^3) \geq -t(\theta_1^1, \theta_2^1) - 2t(\theta_1^1, \theta_2^2) + 2t(\theta_1^2, \theta_2^3) - 49. \qquad (21)
\end{aligned}$$

Computing (20) + (21) $\times(1/2)$, we obtain

$$-4t(\theta_1^3, \theta_2^3) \geq -\frac{3}{2}t(\theta_1^1, \theta_2^1) - \frac{55}{2} \Rightarrow -t(\theta_1^3, \theta_2^3) \geq -\frac{3}{8}t(\theta_1^1, \theta_2^1) - \frac{55}{8}.$$

Since $IR_{\theta_1^2}$ requires $t(\theta_1^1, \theta_2^1) \leq 2.5$, we further obtain

$$-t(\theta_1^3, \theta_2^3) \quad \geq \quad -\frac{3}{8} \times 2.5 - \frac{55}{8} = -\frac{62.5}{8} > -8,$$

contradicting $IR_{\theta_1^3}$ which requires $-t(\theta_1^3, \theta_2^3) \leq -8$. Therefore, there exists no NTNP transfer rule t such that the resulting mechanism (x^*, t) satisfies BIC, IIR, EFF, and EPBB. This completes the proof. □

8.6 Proof of Claim 2

Let (x^*, t) be the proposed mechanism. The proof consists of two steps.

8.6.1 Step 1: IIR is satisfied in the proposed mechanism.

Proof. In Step 1-1, we verify that IIR is satisfied for the seller. In Step 1-2, we verify that IIR is satisfied for the buyer.

Step 1-1: IIR is satisfied for the seller.

Proof. If the seller's true type is θ_1^1 and both agents report their type truthfully, then the seller's expected utility after participation, denoted by $U_1(\theta_1^1;\theta_1^1)$, is computed as follows:

$$\begin{aligned} U_1(\theta_1^1;\theta_1^1) &= \sum_{l=1}^{3}\mu^{'}(\theta_2^l|\theta_1^1)t(\theta_1^1,\theta_2^l) \\ &= \frac{1}{\mu_1^{'}(\theta_1^1)}\left[\mu^{'}(\theta_1^1,\theta_2^1)v_2(\theta_1^1,\theta_2^1)+\mu^{'}(\theta_1^1,\theta_2^2)v_2(\theta_1^1,\theta_2^1)+\mu^{'}(\theta_1^1,\theta_2^3)v_1(\theta_1^1,\theta_2^3)\right] \\ &= \frac{0.995}{0.38}, \end{aligned}$$

where $\mu_1^{'}(\theta_1^1)$ denotes the marginal distribution of $\mu^{'}$ that the seller's type is θ_1^1 and the second equality follows because $t(\theta_1^1,\theta_2^1)=t(\theta_1^1,\theta_2^2)=v_2(\theta_1^1,\theta_2^1)$ and $t(\theta_1^1,\theta_2^3)=v_1(\theta_1^1,\theta_2^3)$.

In contrast, the seller's expected utility from her outside option when her true type is θ_1^1 is computed as follows:

$$\begin{aligned} U_1^O(\theta_1^1) &= \sum_{l=1}^{3}\mu^{'}(\theta_2^l|\theta_1^1)v_1(\theta_1^1,\theta_2^l) \\ &= \frac{1}{\mu_1^{'}(\theta_1^1)}\left[\mu^{'}(\theta_1^1,\theta_2^1)v_1(\theta_1^1,\theta_2^1)+\mu^{'}(\theta_1^1,\theta_2^2)v_1(\theta_1^1,\theta_2^2)+\mu^{'}(\theta_1^1,\theta_2^3)v_1(\theta_1^1,\theta_2^3)\right] \\ &= \frac{0.87}{0.38}. \end{aligned}$$

This implies that the seller's expected utility after participation is higher than her outside option utility.

If the seller's true type is θ_1^2 and both agents report their type truthfully, then the seller's expected utility after participation, denoted by $U_1(\theta_1^2;\theta_1^2)$, is computed as follows:

$$
\begin{aligned}
U_1(\theta_1^2;\theta_1^2) &= \sum_{l=2}^{3}\mu^{'}(\theta_2^l|\theta_1^2)t(\theta_1^2,\theta_2^l)+\mu^{'}(\theta_2^1|\theta_1^2)v_1(\theta_1^2,\theta_2^1)\\
&= \frac{1}{\mu_1^{'}(\theta_1^2)}\left[\mu^{'}(\theta_1^2,\theta_2^2)v_2(\theta_1^2,\theta_2^2)+\mu^{'}(\theta_1^2,\theta_2^3)v_1(\theta_1^1,\theta_2^2)+\mu^{'}(\theta_1^2,\theta_2^1)v_1(\theta_1^2,\theta_2^1)\right]\\
&= \frac{1.42}{0.32},
\end{aligned}
$$

where the second equality follows because $t(\theta_1^2,\theta_2^2) = v_2(\theta_1^2,\theta_2^2)$ and $t(\theta_1^2,\theta_2^3) = v_1(\theta_1^1,\theta_2^2)$.

The seller's expected utility from her outside option when her true type is θ_1^2 is computed as follows:

$$
\begin{aligned}
U_1^O(\theta_1^2) &= \sum_{l=1}^{3}\mu^{'}(\theta_2^l|\theta_1^2)v_1(\theta_1^2,\theta_2^l)\\
&= \frac{1}{\mu_1^{'}(\theta_1^2)}\left[\mu^{'}(\theta_1^2,\theta_2^1)v_1(\theta_1^2,\theta_2^1)+\mu^{'}(\theta_1^2,\theta_2^2)v_1(\theta_1^2,\theta_2^2)+\mu^{'}(\theta_1^2,\theta_2^3)v_1(\theta_1^2,\theta_2^3)\right]\\
&= \frac{1.40}{0.32}.
\end{aligned}
$$

Therefore, the seller's expected utility after participation is higher than her outside option utility.

Lastly, if the seller's true type is θ_1^3 and both agents report truthfully in both stages, then the seller's expected utility after participation, denoted by $U_1(\theta_1^3;\theta_1^3)$, is computed as follows:

$$
U_1(\theta_1^3;\theta_1^3) \;=\; \mu^{'}\theta_2^3|\theta_1^3)t(\theta_1^3,\theta_2^3)+\sum_{l=1}^{2}\mu^{'}(\theta_2^l|\theta_1^3)v_1(\theta_1^3,\theta_2^l) = \sum_{l=1}^{3}\mu^{'}(\theta_2^l|\theta_1^3)v_1(\theta_1^3,\theta_2^l),
$$

where the second equality follows because $t(\theta_1^3,\theta_2^3) = v_1(\theta_1^3,\theta_2^3)$. In contrast, the seller's expected utility from her outside option when her true type is θ_1^3 is computed as follows:

$$
U_1^O(\theta_1^3) \;=\; \sum_{l=1}^{3}\mu^{'}(\theta_2^l|\theta_1^3)v_1(\theta_1^3,\theta_2^l) = U_1(\theta_1^3;\theta_1^3),
$$

which implies that the seller is indifferent between participating in the mechanism and her outside option. We conclude that the seller's IIR constraints are satisfied.

□

Step 1-2: IIR is satisfied for the buyer.

Proof. If the buyer's true type is θ_2^1 and both agents report their type truthfully, then the buyer's expected utility after participation, denoted by $U_2(\theta_2^1;\theta_2^1)$, is computed as follows:

$$U_2(\theta_2^1;\theta_2^1) = \mu^{'}(\theta_1^1|\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - t(\theta_1^1,\theta_2^1)\right) = \mu^{'}(\theta_1^1|\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - v_2(\theta_1^1,\theta_2^1)\right) = 0,$$

where the second equality follows because $t(\theta_1^1,\theta_2^1) = v_2(\theta_1^1,\theta_2^1)$. Since the buyer's outside option utility is zero, the buyer is indifferent between participating in the mechanism and his outside option.

If the buyer's true type is θ_2^2 and both agents report their type truthfully, then the buyer's expected utility after participation, denoted by $U_2(\theta_2^2;\theta_2^2)$, is computed as follows:

$$\begin{aligned}
U_2(\theta_2^2;\theta_2^2) &= \mu^{'}(\theta_1^1|\theta_2^2)\left(v_2(\theta_1^1,\theta_2^2) - t(\theta_1^1,\theta_2^2)\right) + \mu^{'}(\theta_1^2|\theta_2^2)\left(v_2(\theta_1^2,\theta_2^2) - t(\theta_1^2,\theta_2^2)\right) \\
&= \frac{\mu^{'}(\theta_1^1,\theta_2^2)}{\mu_2^{'}(\theta_2^2)}\left(v_2(\theta_1^1,\theta_2^2) - v_2(\theta_1^1,\theta_2^1)\right) \\
&= \frac{0.1}{0.36},
\end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^2) = v_2(\theta_1^1,\theta_2^1)$ and $t(\theta_1^2,\theta_2^2) = v_2(\theta_1^2,\theta_2^2)$. Since the buyer's outside option utility is zero, the buyer's expected utility after participation is higher than his outside option.

Lastly, if the buyer's true type is θ_2^3 and both agents report their type truthfully, then the buyer's expected utility after participation, denoted by $U_2(\theta_2^3;\theta_2^3)$, is computed as follows:

$$\begin{aligned}
U_2(\theta_2^3;\theta_2^3) &= \sum_{k=1}^{3}\mu^{'}(\theta_1^k|\theta_2^3)\left(v_2(\theta_1^k,\theta_2^3) - t(\theta_1^k,\theta_2^3)\right) \\
&= \frac{1}{\mu_2^{'}(\theta_2^3)}\Big[\mu^{'}(\theta_1^1,\theta_2^3)\left(v_2(\theta_1^1,\theta_2^3) - v_1(\theta_1^1,\theta_2^3)\right) + \mu^{'}(\theta_1^2,\theta_2^3)\left(v_2(\theta_1^2,\theta_2^3) - v_1(\theta_1^1,\theta_2^2)\right) \\
&\quad + \mu^{'}(\theta_1^3,\theta_2^3)\left(v_2(\theta_1^3,\theta_2^3) - v_1(\theta_1^3,\theta_2^3)\right)\Big] \\
&= \frac{0.14}{0.08},
\end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^3) = v_1(\theta_1^1,\theta_2^3)$, $t(\theta_1^2,\theta_2^3) = v_1(\theta_1^1,\theta_2^2)$ and $t(\theta_1^3,\theta_2^3) = v_1(\theta_1^3,\theta_2^3)$. Since the buyer's outside option utility is zero, the buyer's expected utility after participation is higher than his outside option. We conclude that the buyer's IIR constraints are satisfied. □

This completes the proof of Step 1.

□

8.6.2 Step 2: BIC is satisfied in the proposed mechanism.

Proof. In Step 2-1, we verify that BIC is satisfied for the seller. In Step 2-2, we verify that BIC is satisfied for the buyer.

Step 2-1: BIC is satisfied for the seller.

Proof. The proof is divided into three cases, depending on the seller's true type.

Case 1: The seller's true type is θ_1^1.

Recall in Step 1-1 that if both agents report their type truthfully, the seller's expected utility is 0.995/0.38. First, if she deviates to θ_1^2, her expected utility, denoted by $U_1(\theta_1^1;\theta_1^2)$, becomes

$$\begin{aligned} U_1(\theta_1^1;\theta_1^2) &= \sum_{l=2}^{3} \mu^{'}(\theta_2^l|\theta_1^1)t(\theta_1^2,\theta_2^l) + \mu^{'}(\theta_2^1|\theta_1^1)v_1(\theta_1^1,\theta_2^1) \\ &= \frac{1}{\mu_1^{'}(\theta_1^1)}\left[\mu^{'}(\theta_1^1,\theta_2^2)v_2(\theta_1^2,\theta_2^2) + \mu^{'}(\theta_1^1,\theta_2^3)v_1(\theta_1^1,\theta_2^2) + \mu^{'}(\theta_1^1,\theta_2^1)v_1(\theta_1^1,\theta_2^1)\right] \\ &= \frac{0.965}{0.38}, \end{aligned}$$

where the second equality follows because $t(\theta_1^2,\theta_2^2) = v_2(\theta_1^2,\theta_2^2)$ and $t(\theta_1^2,\theta_2^3) = v_1(\theta_1^1,\theta_2^2)$. Since the seller's expected utility after deviating to θ_1^2 is lower than that under truth-telling, the seller has no incentive to deviate to θ_1^2.

Second, if the seller deviates to θ_1^3, her expected utility, denoted by $U_1(\theta_1^1;\theta_1^3)$, becomes

$$\begin{aligned} U_1(\theta_1^1;\theta_1^3) &= \mu^{'}(\theta_2^3|\theta_1^1)t(\theta_1^3,\theta_2^3) + \sum_{l=1}^{2} \mu^{'}(\theta_2^l|\theta_1^1)v_1(\theta_1^1,\theta_2^l) \\ &= \frac{1}{\mu_1^{'}(\theta_1^1)}\left[\mu^{'}(\theta_1^1,\theta_2^3)v_1(\theta_1^3,\theta_2^3) + \mu^{'}(\theta_1^1,\theta_2^1)v_1(\theta_1^1,\theta_2^1) + \mu^{'}(\theta_1^1,\theta_2^2)v_1(\theta_1^1,\theta_2^2)\right] \\ &= \frac{0.990}{0.38}, \end{aligned}$$

where the second equality follows because $t(\theta_1^3,\theta_2^3) = v_1(\theta_1^3,\theta_2^3)$. Since the seller's expected utility after deviating to θ_1^3 is lower than that under truth-telling, the seller has no incentive to deviate to θ_1^3.

Case 2: The seller's true type is θ_1^2.

Recall in Step 1-1 that if both agents report their type truthfully, the seller's expected utility is 1.42/0.32. First, if she deviates to θ_1^1, her expected utility, denoted by $U_1(\theta_1^2;\theta_1^1)$, becomes

$$\begin{aligned}U_1(\theta_1^2;\theta_1^1) &= \sum_{l=1}^{3}\mu^{'}(\theta_2^l|\theta_1^2)t(\theta_1^1,\theta_2^l)\\ &= \frac{1}{\mu_1^{'}(\theta_1^2)}\left[\mu^{'}(\theta_1^2,\theta_2^1)v_2(\theta_1^1,\theta_2^1)+\mu^{'}(\theta_1^2,\theta_2^2)v_2(\theta_1^1,\theta_2^1)+\mu^{'}(\theta_1^2,\theta_2^3)v_1(\theta_1^1,\theta_2^3)\right]\\ &= \frac{0.815}{0.32},\end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^1) = t(\theta_1^1,\theta_2^2) = v_2(\theta_1^1,\theta_2^1)$ and $t(\theta_1^1,\theta_2^3) = v_1(\theta_1^1,\theta_2^3)$. Since the seller's expected utility after deviating to θ_1^1 is lower than that under truth-telling, the seller has no incentive to deviate to θ_1^1.

Second, if the seller deviates to θ_1^3, her expected utility, denoted by $U_1(\theta_1^2;\theta_1^3)$, becomes

$$\begin{aligned}U_1(\theta_1^2;\theta_1^3) &= \mu^{'}(\theta_2^3|\theta_1^2)t(\theta_1^3,\theta_2^3)+\sum_{l=1}^{2}\mu^{'}(\theta_2^l|\theta_1^2)v_1(\theta_1^2,\theta_2^l)\\ &= \frac{1}{\mu_1^{'}(\theta_1^2)}\left[\mu^{'}(\theta_1^2,\theta_2^3)v_1(\theta_1^3,\theta_2^3)+\mu^{'}(\theta_1^2,\theta_2^1)v_1(\theta_1^2,\theta_2^1)+\mu^{'}(\theta_1^2,\theta_2^2)v_1(\theta_1^2,\theta_2^2)\right]\\ &= \frac{1.420}{0.32},\end{aligned}$$

where the second equality follows because $t(\theta_1^3,\theta_2^3) = v_1(\theta_1^3,\theta_2^3)$. Obviously, the seller's expected utility after deviating to θ_1^3 is the same as that under truth-telling, implying that the seller has no incentive to deviate to θ_1^3.

Case 3: The seller's true type is θ_1^3.

Recall in Step 1-1 that if both agents report their type truthfully, the seller's expected utility, $U_1(\theta_1^3;\theta_1^3)$, equals

$$\begin{aligned}U_1(\theta_1^3;\theta_1^3) &= \frac{1}{\mu_1^{'}(\theta_1^3)}\left[\mu^{'}(\theta_1^3,\theta_2^1)v_1(\theta_1^3,\theta_2^1)+\mu^{'}(\theta_1^3,\theta_2^2)v_1(\theta_1^3,\theta_2^2)+\mu^{'}(\theta_1^3,\theta_2^3)v_1(\theta_1^3,\theta_2^3)\right]\\ &= \frac{2.09}{0.3}.\end{aligned}$$

First, if she deviates to θ_1^1, her expected utility, denoted by $U_1(\theta_1^3;\theta_1^1)$, becomes

$$\begin{aligned}U_1(\theta_1^3;\theta_1^1) &= \sum_{l=1}^{3}\mu^{'}(\theta_2^l|\theta_1^3)t(\theta_1^1,\theta_2^l)\\ &= \frac{1}{\mu_1^{'}(\theta_1^3)}\left[\mu^{'}(\theta_1^3,\theta_2^1)v_2(\theta_1^1,\theta_2^1)+\mu^{'}(\theta_1^3,\theta_2^2)v_2(\theta_1^1,\theta_2^1)+\mu^{'}(\theta_1^3,\theta_2^3)v_1(\theta_1^1,\theta_2^3)\right]\\ &= \frac{0.81}{0.3},\end{aligned}$$

where the second equality follows because $t(\theta_1^1, \theta_2^1) = t(\theta_1^1, \theta_2^2) = v_2(\theta_1^1, \theta_2^1)$ and $t(\theta_1^1, \theta_2^3) = v_1(\theta_1^1, \theta_2^3)$. Since the seller's expected utility after deviating to θ_1^1 is lower than that under truth-telling, the seller has no incentive to deviate to θ_1^1.

Second, if the seller deviates to θ_1^2, her expected utility, denoted by $U_1(\theta_1^3; \theta_1^2)$, becomes

$$\begin{aligned}
U_1(\theta_1^3; \theta_1^2) &= \sum_{l=2}^{3} \mu'(\theta_2^l|\theta_1^3) t(\theta_1^2, \theta_2^l) + \mu'(\theta_2^1|\theta_1^3) v_1(\theta_1^3, \theta_2^1) \\
&= \frac{1}{\mu_1'(\theta_1^3)} \left[\mu'(\theta_1^3, \theta_2^2) v_2(\theta_1^2, \theta_2^2) + \mu'(\theta_1^3, \theta_2^3) v_1(\theta_1^1, \theta_2^2) + \mu'(\theta_1^3, \theta_2^1) v_1(\theta_1^3, \theta_2^1) \right] \\
&= \frac{1.575}{0.3},
\end{aligned}$$

where the second equality follows because $t(\theta_1^2, \theta_2^2) = v_2(\theta_1^2, \theta_2^2)$ and $t(\theta_1^2, \theta_2^3) = v_1(\theta_1^1, \theta_2^2)$. Since the seller's expected utility after deviating to θ_1^2 is lower than that under truth-telling, the seller has no incentive to deviate to θ_1^2.

Thus, BIC is satisfied for the seller. □

Step 2-2: BIC is satisfied for the buyer.

Proof. The proof is divided into three cases, depending on the buyer's true type.

Case I: The buyer's true type is θ_2^1.

Recall in Step 1-2 that if both agents report their type truthfully, the buyer receives the expected utility of zero. First, if he deviates to θ_2^2, his expected utility, denoted by $U_2(\theta_2^1; \theta_2^2)$, becomes

$$\begin{aligned}
U_2(\theta_2^1; \theta_2^2) &= \mu'(\theta_1^1|\theta_2^1) \left(v_2(\theta_1^1, \theta_2^1) - t(\theta_1^1, \theta_2^2) \right) + \mu'(\theta_1^2|\theta_2^1) \left(v_2(\theta_1^2, \theta_2^1) - t(\theta_1^2, \theta_2^2) \right) \\
&= \frac{1}{\mu_2'(\theta_2^1)} \left[\mu'(\theta_1^1, \theta_2^1) \left(v_2(\theta_1^1, \theta_2^1) - v_2(\theta_1^1, \theta_2^1) \right) + \mu'(\theta_1^2, \theta_2^1) \left(v_2(\theta_1^2, \theta_2^1) - v_2(\theta_1^2, \theta_2^2) \right) \right] \\
&= \frac{1}{0.56} \left[0.21 \times (-2) \right] < 0,
\end{aligned}$$

where the second equality follows because $t(\theta_1^1, \theta_2^2) = v_2(\theta_1^1, \theta_2^1)$ and $t(\theta_1^2, \theta_2^2) = v_2(\theta_1^2, \theta_2^2)$. Since the buyer's expected utility after deviating to θ_2^2 is lower than that under truth-telling, the buyer has no incentive to deviate to θ_2^2.

Second, if he deviates to θ_2^3, his expected utility, denoted by $U_2(\theta_2^1;\theta_2^3)$, becomes

$$\begin{aligned}
U_2(\theta_2^1;\theta_2^3) &= \sum_{k=1}^{3} \mu^{'}(\theta_1^k|\theta_2^1)\left(v_2(\theta_1^k,\theta_2^1) - t(\theta_1^k,\theta_2^3)\right) \\
&= \frac{1}{\mu_2^{'}(\theta_2^1)}\Big[\mu^{'}(\theta_1^1,\theta_2^1)\left(v_2(\theta_1^1,\theta_2^1) - v_1(\theta_1^1,\theta_2^3)\right) + \mu^{'}(\theta_1^2,\theta_2^1)\left(v_2(\theta_1^2,\theta_2^1) - v_1(\theta_1^1,\theta_2^2)\right) \\
&\quad + \mu^{'}(\theta_1^3,\theta_2^1)\left(v_2(\theta_1^3,\theta_2^1) - v_1(\theta_1^3,\theta_2^3)\right)\Big] \\
&= -\frac{0.52}{0.56} < 0 = U_2(\theta_2^1;\theta_2^1),
\end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^3) = v_1(\theta_1^1,\theta_2^3)$, $t(\theta_1^2,\theta_2^3) = v_1(\theta_1^1,\theta_2^2)$ and $t(\theta_1^3,\theta_2^3) = v_1(\theta_1^3,\theta_2^3)$. This implies that the buyer has no incentive to deviate to θ_2^3.

Case II: The buyer's true type is θ_2^2.

Recall in Step 1-2 that if both agents report their type truthfully, the buyer's expected utility is 0.1/0.36. First, if he deviates to θ_2^1, his expected utility, denoted by $U_2(\theta_2^2;\theta_2^1)$, becomes

$$U_2(\theta_2^2;\theta_2^1) = \mu^{'}(\theta_1^1|\theta_2^2)\left(v_2(\theta_1^1,\theta_2^2) - t(\theta_1^1,\theta_2^1)\right) = \frac{\mu^{'}(\theta_1^1,\theta_2^2)}{\mu_2^{'}(\theta_2^2)}\left(v_2(\theta_1^1,\theta_2^2) - v_2(\theta_1^1,\theta_2^1)\right) = \frac{0.1}{0.36},$$

where the second equality follows because $t(\theta_1^1,\theta_2^1) = v_2(\theta_1^1,\theta_2^1)$. Since the buyer's expected utility after deviating to θ_2^1 is the same as that under truth-telling, the buyer has no incentive to deviate to θ_2^1.

Second, if he deviates to θ_2^3, his expected utility, denoted by $U_2(\theta_2^2;\theta_2^3)$, becomes

$$\begin{aligned}
U_2(\theta_2^2;\theta_2^3) &= \sum_{k=1}^{3} \mu^{'}(\theta_1^k|\theta_2^2)\left(v_2(\theta_1^k,\theta_2^2) - t(\theta_1^k,\theta_2^3)\right) \\
&= \frac{1}{\mu_2^{'}(\theta_2^2)}\Big[\mu^{'}(\theta_1^1,\theta_2^2)\left(v_2(\theta_1^1,\theta_2^2) - v_1(\theta_1^1,\theta_2^3)\right) + \mu^{'}(\theta_1^2,\theta_2^2)\left(v_2(\theta_1^2,\theta_2^2) - v_1(\theta_1^1,\theta_2^2)\right) \\
&\quad + \mu^{'}(\theta_1^3,\theta_2^2)\left(v_2(\theta_1^3,\theta_2^2) - v_1(\theta_1^3,\theta_2^3)\right)\Big] \\
&= -\frac{0.04}{0.36},
\end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^3) = v_1(\theta_1^1,\theta_2^3)$, $t(\theta_1^2,\theta_2^3) = v_1(\theta_1^1,\theta_2^2)$ and $t(\theta_1^3,\theta_2^3) = v_1(\theta_1^3,\theta_2^3)$. Since the buyer's expected utility after deviating to θ_2^3 is lower than that under truth-telling, the buyer has no incentive to deviate to θ_2^3.

Case III: The buyer's true type is θ_2^3.

Recall in Step 1-2 that if both agents report their type truthfully, the buyer's expected utility is 0.14/0.08. First, if he deviates to θ_2^1, his expected utility, denoted by $U_2(\theta_2^3;\theta_2^1)$, becomes

$$U_2(\theta_2^3;\theta_2^1)=\mu^{'}(\theta_1^1|\theta_2^3)\left(v_2(\theta_1^1,\theta_2^3)-t(\theta_1^1,\theta_2^1)\right)=\frac{\mu^{'}(\theta_1^1,\theta_2^3)}{\mu_2^{'}(\theta_2^3)}\left(v_2(\theta_1^1,\theta_2^3)-v_2(\theta_1^1,\theta_2^1)\right)=\frac{0.12}{0.08},$$

where the second equality follows because $t(\theta_1^1,\theta_2^1)=v_2(\theta_1^1,\theta_2^1)$. Since the buyer's expected utility after deviating to θ_2^1 is lower than that under truth-telling, the buyer has no incentive to deviate to θ_2^1.

Second, if he deviates to θ_2^2, his expected utility, denoted by $U_2(\theta_2^3;\theta_2^2)$, becomes

$$\begin{aligned}U_2(\theta_2^3;\theta_2^2) &= \mu^{'}(\theta_1^1|\theta_2^3)\left(v_2(\theta_1^1,\theta_2^3)-t(\theta_1^1,\theta_2^2)\right)+\mu^{'}(\theta_1^2|\theta_2^3)\left(v_2(\theta_1^2,\theta_2^3)-t(\theta_1^2,\theta_2^2)\right)\\ &= \frac{1}{\mu_2^{'}(\theta_2^3)}\left[\mu^{'}(\theta_1^1,\theta_2^3)\left(v_2(\theta_1^1,\theta_2^3)-v_2(\theta_1^1,\theta_2^1)\right)+\mu^{'}(\theta_1^2,\theta_2^3)\left(v_2(\theta_1^2,\theta_2^3)-v_2(\theta_1^2,\theta_2^2)\right)\right]\\ &= \frac{0.14}{0.08},\end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^2)=v_2(\theta_1^1,\theta_2^1)$ and $t(\theta_1^2,\theta_2^2)=v_2(\theta_1^2,\theta_2^2)$. Since the buyer's expected utility after deviating to θ_2^2 is the same as that under truth-telling, the buyer has no incentive to deviate to θ_2^2.

Thus, the buyer's BIC constraints are satisfied. □

This completes the proof of Step 2. □

8.7 Proof of Claim 4

Let (x^*,t) be the proposed mechanism. The proof is divided into two steps.

8.7.1 Step 1: IIR is satisfied in the proposed mechanism.

Proof. In Step 1-1, we verify that IIR is always satisfied for the seller. In Step 1-2, we verify that IIR is always satisfied for the buyer.

Step 1-1: IIR is always satisfied for the seller.

Proof. If both agents report their type truthfully, the expected utility for the seller of type θ_1^1 after participation, denoted by $U_1(\theta_1^1;\theta_1^1)$, becomes

$$\begin{aligned}U_1(\theta_1^1;\theta_1^1) &= \sum_{l=1}^{3}\hat{\mu}(\theta_2^l|\theta_1^1)t(\theta_1^1,\theta_2^l)\\ &= \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)v_2(\theta_2^1)+\hat{\mu}(\theta_1^1,\theta_2^2)v_2(\theta_2^1)+\hat{\mu}(\theta_1^1,\theta_2^3)v_1(\theta_1^1)\right],\end{aligned}$$

where the second equality follows because $t(\theta_1^1, \theta_2^1) = t(\theta_1^1, \theta_2^2) = v_2(\theta_2^1)$ and $t(\theta_1^1, \theta_2^3) = v_1(\theta_1^1)$. With $U_1^O(\theta_1^1) = \sum_{l=1}^3 \hat{\mu}(\theta_2^l|\theta_1^1)v_1(\theta_1^1)$, we compute the following:

$$\begin{aligned} U_1(\theta_1^1;\theta_1^1) - U_1^O(\theta_1^1) &= \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)\left(v_2(\theta_2^1) - v_1(\theta_1^1)\right) + \hat{\mu}(\theta_1^1,\theta_2^2)\left(v_2(\theta_2^1) - v_1(\theta_1^1)\right)\right] \\ &= \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1) + \hat{\mu}(\theta_1^1,\theta_2^2)\right] > 0, \end{aligned}$$

where the strict inequality follows because $\hat{\mu}$ is a full-support distribution. Therefore, IIR is satisfied for the seller of type θ_1^1.

Next, if both agents report their type truthfully, the expected utility for the seller of type θ_1^2 after participation, denoted by $U_1(\theta_1^2;\theta_1^2)$, becomes

$$\begin{aligned} U_1(\theta_1^2;\theta_1^2) &= \sum_{l=2}^{3} \hat{\mu}(\theta_2^l|\theta_1^2)t(\theta_1^2,\theta_2^l) + \hat{\mu}(\theta_2^1|\theta_1^2)v_1(\theta_1^2) \\ &= \frac{1}{\hat{\mu}_1(\theta_1^2)}\left[\hat{\mu}(\theta_1^2,\theta_2^2)v_2(\theta_2^2) + \hat{\mu}(\theta_1^2,\theta_2^3)v_1(\theta_1^3) + \hat{\mu}(\theta_1^2,\theta_2^1)v_1(\theta_1^2)\right], \end{aligned}$$

where the second equality follows because $t(\theta_1^2, \theta_2^2) = v_2(\theta_2^2)$ and $t(\theta_1^2, \theta_2^3) = v_1(\theta_1^3)$. With $U_1^O(\theta_1^2) = \sum_{l=1}^3 \hat{\mu}(\theta_2^l|\theta_1^2)v_1(\theta_1^2)$, we compute the following:

$$\begin{aligned} U_1(\theta_1^2;\theta_1^2) - U_1^O(\theta_1^2) &= \frac{1}{\hat{\mu}_1(\theta_1^2)}\left[\hat{\mu}(\theta_1^2,\theta_2^2)\left(v_2(\theta_2^2) - v_1(\theta_1^2)\right) + \hat{\mu}(\theta_1^2,\theta_2^3)\left(v_1(\theta_1^3) - v_1(\theta_1^2)\right)\right] \\ &= \frac{1}{\hat{\mu}_1(\theta_1^2)}\left[\hat{\mu}(\theta_1^2,\theta_2^2) + 2\hat{\mu}(\theta_1^2,\theta_2^3)\right] > 0, \end{aligned}$$

implying that IIR is satisfied for the seller of type θ_1^2.

Lastly, if both agents report their type truthfully, the expected utility for the seller of type θ_1^3 after participation, denoted by $U_1(\theta_1^3;\theta_1^3)$, becomes

$$U_1(\theta_1^3;\theta_1^3) = \hat{\mu}(\theta_2^3|\theta_1^3)t(\theta_1^3,\theta_2^3) + \sum_{l=1}^{2} \hat{\mu}(\theta_2^l|\theta_1^3)v_1(\theta_1^3) = \sum_{l=1}^{3} \hat{\mu}(\theta_2^l|\theta_1^3)v_1(\theta_1^3) = U_1^O(\theta_1^3),$$

where the second equality follows because $t(\theta_1^3, \theta_2^3) = v_1(\theta_1^3)$. Thus, the seller is indifferent between participation in the mechanism and her outside option so that IIR is satisfied for the seller of type θ_1^3. □

Step 1-2: IIR is always satisfied for the buyer.

Proof. If both agents report their type truthfully, the expected utility for the buyer of type θ_2^1, denoted by $U_2(\theta_2^1;\theta_2^1)$, becomes

$$U_2(\theta_2^1;\theta_2^1) = \hat{\mu}(\theta_1^1|\theta_2^1)\left(v_2(\theta_2^1) - t(\theta_1^1,\theta_2^1)\right) = \hat{\mu}(\theta_1^1|\theta_2^1)\left(v_2(\theta_2^1) - v_2(\theta_2^1)\right) = 0,$$

where the second equality follows because $t(\theta_1^1, \theta_2^1) = v_2(\theta_2^1)$. Since the buyer's outside option utility is zero, IIR is satisfied for the buyer of type θ_2^1.

Next, if both agents report their type truthfully, the expected utility for the buyer of type θ_2^2, denoted by $U_2(\theta_2^2; \theta_2^2)$, becomes

$$\begin{aligned}
U_2(\theta_2^2; \theta_2^2) &= \hat{\mu}(\theta_1^1|\theta_2^2)\left(v_2(\theta_2^2) - t(\theta_1^1, \theta_2^2)\right) + \hat{\mu}(\theta_1^2|\theta_2^2)\left(v_2(\theta_2^2) - t(\theta_1^2, \theta_2^2)\right) \\
&= \frac{1}{\hat{\mu}_2(\theta_2^2)}\left[\hat{\mu}(\theta_1^1, \theta_2^2)\left(v_2(\theta_2^2) - v_2(\theta_2^1)\right) + \hat{\mu}(\theta_1^2, \theta_2^2)\left(v_2(\theta_2^2) - v_2(\theta_2^2)\right)\right] \\
&= \frac{\hat{\mu}(\theta_1^1, \theta_2^2)}{\hat{\mu}_2(\theta_2^2)}\left(v_2(\theta_2^2) - v_2(\theta_2^1)\right) = \frac{2\hat{\mu}(\theta_1^1, \theta_2^2)}{\hat{\mu}_2(\theta_2^2)} > 0,
\end{aligned}$$

where the second equality follows because $t(\theta_1^1, \theta_2^2) = v_2(\theta_2^1)$ and $t(\theta_1^2, \theta_2^2) = v_2(\theta_2^2)$. Since the buyer's outside option utility is zero, IIR is satisfied for the buyer of type θ_2^2.

Lastly, if both agents report their type truthfully, the expected utility for the buyer of type θ_2^3, $U_2(\theta_2^3; \theta_2^3)$, becomes

$$\begin{aligned}
&\sum_{k=1}^{3} \hat{\mu}(\theta_1^k|\theta_2^3)\left(v_2(\theta_2^3) - t(\theta_1^k, \theta_2^3)\right) \\
= &\frac{1}{\hat{\mu}_2(\theta_2^3)}\left[\hat{\mu}(\theta_1^1, \theta_2^3)\left(v_2(\theta_2^3) - v_1(\theta_1^1)\right) + \hat{\mu}(\theta_1^2, \theta_2^3)\left(v_2(\theta_2^3) - v_1(\theta_1^3)\right) + \hat{\mu}(\theta_1^3, \theta_2^3)\left(v_2(\theta_2^3) - v_1(\theta_1^3)\right)\right] \\
= &\frac{1}{\hat{\mu}_2(\theta_2^3)}\left[5\hat{\mu}(\theta_1^1, \theta_2^3) + \hat{\mu}(\theta_1^2, \theta_2^3) + \hat{\mu}(\theta_1^3, \theta_2^3)\right] > 0,
\end{aligned}$$

where the second equality follows because $t(\theta_1^1, \theta_2^3) = v_1(\theta_1^1)$ and $t(\theta_1^2, \theta_2^3) = t(\theta_1^3, \theta_2^3) = v_1(\theta_1^3)$ and the strict inequality follows because $\hat{\mu}$ is a full-support distribution. Since the buyer's outside option utility is zero, IIR is satisfied for the buyer of type θ_2^3. □

This completes the proof of Step 1.

□

8.7.2 Step 2: BIC is satisfied in the proposed mechanism.

Proof. In Step 2-1, we verify that BIC is always satisfied for the seller. In Step 2-2, we verify that BIC is also always satisfied for the buyer.

Step 2-1: BIC is satisfied for the seller.

Proof. We complete the proof by considering three cases, depending on the seller's type.

Case 1: The seller's type is θ_1^1.

Recall from Step 1-1 that if both agents report their type truthfully, the seller's expected utility, $U_1(\theta_1^1;\theta_1^1)$, equals

$$U_1(\theta_1^1;\theta_1^1) = \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)v_2(\theta_2^1)+\hat{\mu}(\theta_1^1,\theta_2^2)v_2(\theta_2^1)+\hat{\mu}(\theta_1^1,\theta_2^3)v_1(\theta_1^1)\right].$$

First, if the seller deviates to θ_1^2, her expected utility, denoted by $U_1(\theta_1^1;\theta_1^2)$, becomes

$$\begin{aligned} U_1(\theta_1^1;\theta_1^2) &= \hat{\mu}(\theta_2^1|\theta_1^1)v_1(\theta_1^1)+\sum_{l=2}^{3}\hat{\mu}(\theta_2^l|\theta_1^1)t(\theta_1^2,\theta_2^l) \\ &= \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)v_1(\theta_1^1)+\hat{\mu}(\theta_1^1,\theta_2^2)v_2(\theta_2^2)+\hat{\mu}(\theta_1^1,\theta_2^3)v_1(\theta_1^3)\right], \end{aligned}$$

where the second equality follows because $t(\theta_1^2,\theta_2^2)=v_2(\theta_2^2)$ and $t(\theta_1^2,\theta_2^3)=v_1(\theta_1^3)$. We then compute the following:

$$\begin{aligned} U_1(\theta_1^1;\theta_1^1)-U_1(\theta_1^1;\theta_1^2) &= \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)-2\hat{\mu}(\theta_1^1,\theta_2^2)-4\hat{\mu}(\theta_1^1,\theta_2^3)\right] \\ &= \frac{1}{0.38+\sum_{l=1}^{3}\varepsilon_{1l}}(0.08+\varepsilon_{11}-2\varepsilon_{12}-4\varepsilon_{13}) \\ &\geq \frac{1}{0.38+\sum_{l=1}^{3}\varepsilon_{1l}}(0.08-0.01-2\times 0.01-4\times 0.01)>0. \end{aligned}$$

Thus, the seller has no incentive to deviate to θ_1^2.

Second, if the seller deviates to θ_1^3, her expected utility, denoted by $U_1(\theta_1^1;\theta_1^3)$, becomes

$$\begin{aligned} U_1(\theta_1^1;\theta_1^3) &= \sum_{l=1}^{2}\hat{\mu}(\theta_2^l|\theta_1^1)v_1(\theta_1^1)+\hat{\mu}(\theta_2^3|\theta_1^1)t(\theta_1^3,\theta_2^3) \\ &= \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)v_1(\theta_1^1)+\hat{\mu}(\theta_1^1,\theta_2^2)v_1(\theta_1^1)+\hat{\mu}(\theta_1^1,\theta_2^3)v_1(\theta_1^3)\right], \end{aligned}$$

where the second equality follows because $t(\theta_1^3,\theta_2^3)=v_1(\theta_1^3)$. We then compute the following:

$$\begin{aligned} U_1(\theta_1^1;\theta_1^1)-U_1(\theta_1^1;\theta_1^3) &= \frac{1}{\hat{\mu}_1(\theta_1^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)+\hat{\mu}(\theta_1^1,\theta_2^2)-4\hat{\mu}(\theta_1^1,\theta_2^3)\right] \\ &= \frac{1}{0.38+\sum_{l=1}^{3}\varepsilon_{1l}}(0.23+\varepsilon_{11}+\varepsilon_{12}-4\varepsilon_{13}) \\ &\geq \frac{1}{0.38+\sum_{l=1}^{3}\varepsilon_{1l}}(0.23-0.01-0.01-4\times 0.01)>0. \end{aligned}$$

Therefore, the seller has no incentive to deviate to θ_1^3.

Case 2: The seller's type is θ_1^2.

Recall from Step 1-1 that if both agents report their type truthfully, the seller's expected utility, $U_1(\theta_1^2;\theta_1^2)$, equals

$$U_1(\theta_1^2;\theta_1^2) = \frac{1}{\hat{\mu}_1(\theta_1^2)}\left[\hat{\mu}(\theta_1^2,\theta_2^1)v_1(\theta_1^2)+\hat{\mu}(\theta_1^2,\theta_2^2)v_2(\theta_2^2)+\hat{\mu}(\theta_1^2,\theta_2^3)v_1(\theta_1^3)\right].$$

First, if the seller deviates to θ_1^1, her expected utility, denoted by $U_1(\theta_1^2;\theta_1^1)$, becomes

$$\begin{aligned} U_1(\theta_1^2;\theta_1^1) &= \sum_{l=1}^{3}\hat{\mu}(\theta_2^l|\theta_1^2)t(\theta_1^1,\theta_2^l) \\ &= \frac{1}{\hat{\mu}_1(\theta_1^2)}\left[\hat{\mu}(\theta_1^2,\theta_2^1)v_2(\theta_2^1)+\hat{\mu}(\theta_1^2,\theta_2^2)v_2(\theta_2^1)+\hat{\mu}(\theta_1^2,\theta_2^3)v_1(\theta_1^1)\right], \end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^1)=t(\theta_1^1,\theta_2^2)=v_2(\theta_2^1)$ and $t(\theta_1^1,\theta_2^3)=v_1(\theta_1^1)$. We then compute the following:

$$U_1(\theta_1^2;\theta_1^2)-U_1(\theta_1^2;\theta_1^1) = \frac{1}{\hat{\mu}_1(\theta_1^2)}\left[\hat{\mu}(\theta_1^2,\theta_2^1)+2\hat{\mu}(\theta_1^2,\theta_2^2)+4\hat{\mu}(\theta_1^2,\theta_2^3)\right]>0,$$

where the strict inequality follows because $\hat{\mu}$ is a full-support distribution. Therefore, the seller has no incentive to deviate to θ_1^1.

Second, if the seller deviates to θ_1^3, her expected utility, denoted by $U_1(\theta_1^2;\theta_1^3)$, becomes

$$\begin{aligned} U_1(\theta_1^2;\theta_1^3) &= \sum_{l=1}^{2}\hat{\mu}(\theta_2^l|\theta_1^2)v_1(\theta_1^2)+\hat{\mu}(\theta_2^3|\theta_1^2)t(\theta_1^3,\theta_2^3) \\ &= \frac{1}{\hat{\mu}_1(\theta_1^2)}\left[\hat{\mu}(\theta_1^2,\theta_2^1)v_1(\theta_1^2)+\hat{\mu}(\theta_1^2,\theta_2^2)v_1(\theta_1^2)+\hat{\mu}(\theta_1^2,\theta_2^3)v_1(\theta_1^3)\right], \end{aligned}$$

where the second equality follows because $t(\theta_1^3,\theta_2^3)=v_1(\theta_1^3)$. We then compute the following:

$$U_1(\theta_1^2;\theta_1^2)-U_1(\theta_1^2;\theta_1^3) = \frac{\hat{\mu}(\theta_1^2,\theta_2^2)}{\hat{\mu}_1(\theta_1^2)}>0,$$

where the strict inequality follows because $\hat{\mu}$ is a full-support distribution. Thus, the seller has no incentive to deviate to θ_1^3.

Case 3: The seller's type is θ_1^3.

Recall from Step 1-1 that if both agents report their type truthfully, the seller's expected utility equals $U_1(\theta_1^3;\theta_1^3)=\sum_{l=1}^{3}\hat{\mu}(\theta_2^l|\theta_1^3)v_1(\theta_1^3)$. First, if the seller deviates to θ_1^1, her expected utility, denoted by $U_1(\theta_1^3;\theta_1^1)$, becomes

$$\begin{aligned}U_1(\theta_1^3;\theta_1^1) &= \sum_{l=1}^{3}\hat{\mu}(\theta_2^l|\theta_1^3)t(\theta_1^1,\theta_2^l)\\ &= \frac{1}{\hat{\mu}_1(\theta_1^3)}\left[\hat{\mu}(\theta_1^3,\theta_2^1)v_2(\theta_2^1)+\hat{\mu}(\theta_1^3,\theta_2^2)v_2(\theta_2^1)+\hat{\mu}(\theta_1^3,\theta_2^3)v_1(\theta_1^1)\right],\end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^1)=t(\theta_1^1,\theta_2^2)=v_2(\theta_2^1)$ and $t(\theta_1^1,\theta_2^3)=v_1(\theta_1^1)$. We then compute the following:

$$U_1(\theta_1^3;\theta_1^3)-U_1(\theta_1^3;\theta_1^1) = \frac{1}{\hat{\mu}_1(\theta_1^3)}\left[3\hat{\mu}(\theta_1^3,\theta_2^1)+3\hat{\mu}(\theta_1^3,\theta_2^2)+4\hat{\mu}(\theta_1^3,\theta_2^3)\right]>0,$$

where the strict inequality follows because $\hat{\mu}$ is a full-support distribution. Therefore, the seller has no incentive to deviate to θ_1^1.

Second, if the seller deviates to θ_1^2, her expected utility, denoted by $U_1(\theta_1^3;\theta_1^2)$, becomes

$$\begin{aligned}U_1(\theta_1^3;\theta_1^2) &= \hat{\mu}(\theta_2^1|\theta_1^3)v_1(\theta_1^3)+\sum_{l=2}^{3}\hat{\mu}(\theta_2^l|\theta_1^3)t(\theta_1^2,\theta_2^l)\\ &= \frac{1}{\hat{\mu}_1(\theta_1^3)}\left[\hat{\mu}(\theta_1^3,\theta_2^1)v_1(\theta_1^3)+\hat{\mu}(\theta_1^3,\theta_2^2)v_2(\theta_2^2)+\hat{\mu}(\theta_1^3,\theta_2^3)v_1(\theta_1^3)\right],\end{aligned}$$

where the second equality follows because $t(\theta_1^2,\theta_2^2)=v_2(\theta_2^2)$ and $t(\theta_1^2,\theta_2^3)=v_1(\theta_1^3)$. We then compute the following:

$$U_1(\theta_1^3;\theta_1^3)-U_1(\theta_1^3;\theta_1^2) = \frac{\hat{\mu}(\theta_1^3,\theta_2^2)}{\hat{\mu}_1(\theta_1^3)}>0,$$

where the strict inequality follows because $\hat{\mu}$ is a full-support distribution. Therefore, the seller has no incentive to deviate to θ_1^2.

We conclude that BIC is satisfied for the seller. □

Step 2-2: BIC is satisfied for the buyer.

Proof. We complete the proof by considering three cases, depending on the buyer's type.

Case I: The buyer's true type is θ_2^1.

Recall from Step 1-2 that if both agents report their type truthfully, the buyer's expected utility is zero. First, if the buyer deviates to θ_2^2, his expected utility, denoted by $U_2(\theta_2^1;\theta_2^2)$, becomes

$$\begin{aligned}U_2(\theta_2^1;\theta_2^2) &= \hat{\mu}(\theta_1^1|\theta_2^1)\left(v_2(\theta_2^1)-t(\theta_1^1,\theta_2^2)\right)+\hat{\mu}(\theta_1^2|\theta_2^1)\left(v_2(\theta_2^1)-t(\theta_1^2,\theta_2^2)\right)\\ &= \frac{1}{\hat{\mu}_2(\theta_2^1)}\left[\hat{\mu}(\theta_1^1,\theta_2^1)\left(v_2(\theta_2^1)-v_2(\theta_2^1)\right)+\hat{\mu}(\theta_1^2,\theta_2^1)\left(v_2(\theta_2^1)-v_2(\theta_2^2)\right)\right]\\ &= -\frac{2\hat{\mu}(\theta_1^2,\theta_2^1)}{\hat{\mu}_2(\theta_2^1)}<0,\end{aligned}$$

where the second equality follows because $t(\theta_1^1, \theta_2^2) = v_2(\theta_2^1)$ and $t(\theta_1^2, \theta_2^2) = v_2(\theta_2^2)$, and the strict inequality follows because $\hat{\mu}$ is a full-support distribution. This implies that the buyer has no incentive to deviate to θ_2^2.

Second, if the buyer deviates to θ_2^3, his expected utility equals $U_2(\theta_2^1; \theta_2^3) = \sum_{k=1}^3 \hat{\mu}(\theta_1^k|\theta_2^1) \left(v_2(\theta_2^1) - t(\theta_1^k, \theta_2^3)\right)$. Since $t(\theta_1^1, \theta_2^3) = v_1(\theta_1^1)$ and $t(\theta_1^2, \theta_2^3) = t(\theta_1^3, \theta_2^3) = v_1(\theta_1^3)$, we have

$$U_2(\theta_2^1; \theta_2^3) = \sum_{k=1}^{3} \hat{\mu}(\theta_1^k|\theta_2^1) \left(v_2(\theta_2^1) - t(\theta_1^k, \theta_2^3)\right) = \frac{1}{\hat{\mu}_2(\theta_2^1)} \left[\hat{\mu}(\theta_1^1, \theta_2^1) - 3\hat{\mu}(\theta_1^2, \theta_2^1) - 3\hat{\mu}(\theta_1^3, \theta_2^1)\right].$$

Then, we compute the following:

$$\begin{aligned} U_2(\theta_2^1; \theta_2^1) - U_2(\theta_2^1; \theta_2^3) &= \frac{1}{0.56}(0.48 - \varepsilon_{11} + 3\varepsilon_{21} - 3\varepsilon_{31}) \\ &\geq \frac{1}{0.56}(0.48 - 0.01 - 3 \times 0.01 - 3 \times 0.01) > 0, \end{aligned}$$

implying that the buyer has no incentive to deviate to θ_2^3.

Case II: The buyer's true type is θ_2^2.

Recall from Step 1-2 that if both agents report their type truthfully, the buyer's expected utility, $U_2(\theta_2^2; \theta_2^2)$, equals

$$U_2(\theta_2^2; \theta_2^2) = \frac{\hat{\mu}(\theta_1^1, \theta_2^2)}{\hat{\mu}_2(\theta_2^2)} \left(v_2(\theta_2^2) - v_2(\theta_2^1)\right) = \frac{2\hat{\mu}(\theta_1^1, \theta_2^2)}{\hat{\mu}_2(\theta_2^2)}.$$

First, if the buyer deviates to θ_2^1, his expected utility, denoted by $U_2(\theta_2^2; \theta_2^1)$, becomes the following:

$$U_2(\theta_2^2; \theta_2^1) = \hat{\mu}(\theta_1^1|\theta_2^2) \left(v_2(\theta_2^2) - t(\theta_1^1, \theta_2^1)\right) = \frac{\hat{\mu}(\theta_1^1, \theta_2^2)}{\hat{\mu}_2(\theta_2^2)} \left(v_2(\theta_2^2) - v_2(\theta_2^1)\right),$$

where the second equality follows because $t(\theta_1^1, \theta_2^1) = v_2(\theta_2^1)$. Since the buyer is indifferent between truth-telling and deviating to θ_2^1, he has no incentive to deviate to θ_2^1.

Second, if the buyer deviates to θ_2^3, his expected utility, denoted by $U_2(\theta_2^2; \theta_2^3)$, becomes

$$U_2(\theta_2^2; \theta_2^3) = \sum_{k=1}^{3} \hat{\mu}(\theta_1^k|\theta_2^2) \left(v_2(\theta_2^2) - t(\theta_1^k, \theta_2^3)\right) = \frac{1}{\hat{\mu}_2(\theta_2^2)} \left[3\hat{\mu}(\theta_1^1, \theta_2^2) - \hat{\mu}(\theta_1^2, \theta_2^2) - \hat{\mu}(\theta_1^3, \theta_2^2)\right],$$

where the second equality follows because $t(\theta_1^1, \theta_2^3) = v_1(\theta_1^1)$ and $t(\theta_1^2, \theta_2^3) = t(\theta_1^3, \theta_2^3) = v_1(\theta_1^3)$. We then compute the following:

$$\begin{aligned} U_2(\theta_2^2; \theta_2^2) - U_2(\theta_2^2; \theta_2^3) &= \frac{1}{0.36 + \sum_{k=1}^3 \varepsilon_{k2}} [0.26 - \varepsilon_{12} + \varepsilon_{22} + \varepsilon_{32}] \\ &\geq \frac{1}{0.36 + \sum_{k=1}^3 \varepsilon_{k2}} [0.26 - 0.01 - 0.01 - 0.01] > 0, \end{aligned}$$

implying that the buyer has no incentive to deviate to θ_2^3.

Case III: The buyer's type is θ_2^3.

Recall from Step 1-2 that if both agents report their type truthfully, the buyer's expected utility, $U_2(\theta_2^3;\theta_2^3)$, equals

$$U_2(\theta_2^3;\theta_2^3) = \frac{1}{\hat{\mu}_2(\theta_2^3)}\left[5\hat{\mu}(\theta_1^1,\theta_2^3)+\hat{\mu}(\theta_1^2,\theta_2^3)+\hat{\mu}(\theta_1^3,\theta_2^3)\right].$$

First, if the buyer deviates to θ_2^1, his expected utility, denoted by $U_2(\theta_2^3;\theta_2^1)$, becomes

$$U_2(\theta_2^3;\theta_2^1) = \hat{\mu}(\theta_1^1|\theta_2^3)\left(v_2(\theta_2^3)-t(\theta_1^1,\theta_2^1)\right) = \frac{\hat{\mu}(\theta_1^1,\theta_2^3)}{\hat{\mu}_2(\theta_2^3)}\left(v_2(\theta_2^3)-v_2(\theta_2^1)\right) = \frac{4\hat{\mu}(\theta_1^1,\theta_2^3)}{\hat{\mu}_2(\theta_2^3)},$$

where the second equality follows because $t(\theta_1^1,\theta_2^1) = v_2(\theta_2^1)$. We then compute the following:

$$U_2(\theta_2^3;\theta_2^3) - U_2(\theta_2^3;\theta_2^1) = \frac{1}{\hat{\mu}_2(\theta_2^3)}\left[\hat{\mu}(\theta_1^1,\theta_2^3)+\hat{\mu}(\theta_1^2,\theta_2^3)+\hat{\mu}(\theta_1^3,\theta_2^3)\right] > 0,$$

where the strict inequality follows because $\hat{\mu}$ is a full-support distribution. Therefore, the buyer has no incentive to deviate to θ_2^1.

Second, if the buyer deviates to θ_2^2, his expected utility, denoted by $U_2(\theta_2^3;\theta_2^2)$, becomes

$$\begin{aligned} U_2(\theta_2^3;\theta_2^2) &= \hat{\mu}(\theta_1^1|\theta_2^3)\left(v_2(\theta_2^3)-t(\theta_1^1,\theta_2^2)\right)+\hat{\mu}(\theta_1^2|\theta_2^3)\left(v_2(\theta_2^3)-t(\theta_1^2,\theta_2^2)\right) \\ &= \frac{1}{\hat{\mu}_2(\theta_2^3)}\left[\hat{\mu}(\theta_1^1,\theta_2^3)\left(v_2(\theta_2^3)-v_2(\theta_2^1)\right)+\hat{\mu}(\theta_1^2,\theta_2^3)\left(v_2(\theta_2^3)-v_2(\theta_2^2)\right)\right] \\ &= \frac{1}{\hat{\mu}_2(\theta_2^3)}\left[4\hat{\mu}(\theta_1^1,\theta_2^3)+2\hat{\mu}(\theta_1^2,\theta_2^3)\right], \end{aligned}$$

where the second equality follows because $t(\theta_1^1,\theta_2^2) = v_2(\theta_2^1)$ and $t(\theta_1^2,\theta_2^2) = v_2(\theta_2^2)$. We then compute the following:

$$\begin{aligned} U_2(\theta_2^3;\theta_2^3) - U_2(\theta_2^3;\theta_2^2) &= \frac{1}{0.08+\sum_{k=1}^3 \varepsilon_{k3}}[0.06+\varepsilon_{13}-\varepsilon_{23}+\varepsilon_{33}] \\ &\geq \frac{1}{0.08+\sum_{k=1}^3 \varepsilon_{k3}}[0.06-0.01-0.01-0.01] > 0, \end{aligned}$$

implying that the buyer has no incentive to deviate to θ_2^2.

We conclude that BIC is satisfied for the buyer.

□

This completes the proof of Step 2. □

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