Allocating vehicle registration permits

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Abstract

We compare social welfare, consumer surplus and profits in two different institutional settings in which an item whose quantity is fixed and controlled (vehicle registration permit) is allocated to the buyers of a complementary good (car). In the first setting, which resembles the way in which vehicle registration permits are allocated in Singapore, the central planner runs a uniform price auction for permits in which the consumers who bid the highest receive the permits and pay the highest losing bid. Then each winning consumer purchases a car from a seller. In the alternative setting, the central planner first allocates the permits to sellers and then sellers offer to consumers bundles, each consisting of a car and a permit. For two different models of product differentiation, we find that social welfare is greater when permits are auctioned to consumers, but consumers and sellers generally prefer the alternative setting.

Keywords: Auctions, Driving permits, Posted prices

1. Introduction

In this paper we look at the welfare implications associated with two alternative ways of allocating an item (vehicle registration permit) that is instrumental to the use of another item (car). Consumers need to acquire a vehicle registration permit before they can purchase and drive a car. In the standard setting, which we call *CA* (for *Consumer Allocation* of permits), first the vehicle registration permits are auctioned by the central planner to consumers, then each consumer who has won a permit buys a car from a dealer. In the alternative setting, which we call *SA* (for *Seller Allocation* of permits), first the central planner allocates the permits to sellers, then each seller *i* who has obtained $k_i \geq 1$ permits offers to consumers k_i bundles – each consisting of a permit and a car *i*.

The car and permit example is inspired by the way in which cars (whose number is regulated through the emission of the vehicle registration permits) are allocated in Singapore. A designated public authority auctions, twice a month, a number of vehicle registration permits (called certificate of entitlement, or COE), through a mechanism that is, essentially, a uniform price auction without reserve price in which the winning bidders pay the highest losing bid.[1] Each (male) consumer can only get one permit, and that permit is matched to the car he purchases.[2] Thus, each consumer needs to acquire a vehicle registration permit in this auction before he can purchase a car from a (female) car dealer.[3]

[1] There is a minimum bid of one Singapore dollar (SGD) in the auction, as each bid must be an integer multiple of one dollar, but that cannot be considered a meaningful reserve price as the price each winner pays is much higher, as we report below in this introduction.
[2] If a person wants to own two cars, then this person needs to win another permit at another auction. Permits last for ten years, after which they are either renewed at a prevailing rate or scrapped and the driver rebids for a new one, which allows the driver to purchase a new car too.
[3] The COE auction is more elaborated as cars and motorized vehicles are divided into categories and a number of permits is determined for each category. So buyers need to identify the car category in which to bid, as they can only get one permit. The interested reader can refer to https://onemotoring.lta.gov.sg/content/

Between 2015 and 2021, the cost of the COE for cars in the category A has oscillated between a high SGD 68,589 and a low SGD 23,568.[4] This can be a substantial part of the basic purchase cost of a car in the category. For example, in the first auction of June 2021, the COE for category A was SGD 48,510 and the cost of a car in that category, without the COE, oscillated between SGD 23,579 and SGD 94,647. Thus, the COE was twice as expensive as a car at the lowest end, and half as expensive as a car at the highest end.[5]

The CA setting essentially corresponds to the way of allocating permits in Singapore. The alternative SA is inspired by the fact that sellers seem to have a stock of permits that can offer in a bundle with a car to consumers who do not want to wait for the next auction.[6]

In short, our analysis shows that CA implements the socially efficient allocation of permits because the auction allocates the permits to the consumers with the highest valuations. Hence, social welfare is maximal under CA, whereas some inefficiencies arise under SA, for instance because it may occur that no consumer visits a seller who offers a bundle and that bundle remains unsold. But under CA the competition is very fierce both among consumers in the auction and between sellers – especially when there are many consumers. Very often this makes consumer surplus and profits under CA smaller than in SA.

In detail, we suppose that there are two permits to allocate and we examine competition between two sellers, A and B, who offer differentiated cars – we begin our analysis, in Section 3, with differentiation à la Hotelling. Under CA, each seller sets a price for her cars and consumers know these prices when they bid in the auction for permits; then each consumer who wins a permit chooses whether to buy car A or car B. We find that in this setting the equilibrium car prices tend to the marginal cost when the number n of consumers is large, that is the Bertrand outcome is approached asymptotically. This occurs because if seller A charges a price higher than seller B, the consumers who plan to buy car A upon winning a permit are disadvantaged in the auction with respect to the consumers who plan to buy car B. This effect, which we call the *burden effect*, makes the demand function for each seller very elastic and induces tough competition between sellers, especially when n is large, as then it is very likely that the two permits are won by consumers who plan to buy the car with the lowest price. We also find that for some parameter values, each consumer type participates in the auction – that is the market is fully covered. In particular, this occurs if marginal cost is low, and/or the cars of different sellers are scarcely differentiated, and/or n is large, since that leads to low prices. When the market is fully covered, the reduction in equilibrium prices as n increases does not affect the utility of consumers as each consumer discounts his bid in the auction by an amount equal to the price of the car he plans to buy if he wins a permit; hence, his utility does not depend on the price of cars. But we remark that cars and permits are allocated efficiently as they are received by the two consumers with the highest valuations.

In setting SA, first the government allocates the permits to the sellers, then each seller i who has received at least one permit offers to the consumers, via a posted-price mechanism, as many bundles – each consisting of a car i and a permit – as the number of permits she has obtained from the government. Each consumer chooses the seller i to visit (if any) and wins bundle i with probability inversely proportional to the number of consumers visiting seller i. When a single seller i obtains both permits from the government, she offers two bundles i (whereas the other seller is inactive); then we say that seller i is a two-bundle monopolist, and the optimal bundle price for seller i always deters some consumer types from visiting seller i. On the other hand, when each seller has obtained one permit, the two sellers compete with posted prices and depending on the parameters the market may be fully covered or not, but partial coverage arises if n is sufficiently large. This occurs because for each seller i what matters is to be visited by at least one consumer, and when there are many consumers the probability of this event decreases by a tiny amount if seller i increases the price of bundle i a bit above the highest price which produces full market coverage; that is, at such price the demand for each seller i as a (one-bundle) local monopolist is rigid.

The government must choose whether to allocate both permits to a same seller – the monopoly allocation of permits – or one permit to each seller – the duopoly allocation – and since in our environment there is complete information about the sellers' profits, the government can implement either allocation by using a simple fixed price mechanism in which the price determines the seller(s)' net profit.[7] We compare the monopoly allocation with the duopoly allocation from the point of view of social welfare,[8] consumer surplus and gross seller profit, that is gross of the sellers' payments in the fixed price mechanism. We find that when differentiation

onemotoring/home/buying/upfront-vehicle-costs/certificate-of-entitlement--coe-.html for more details. Koh and Lee (1994) offer a detailed description of the system and interesting insights of the early implications of its adoption. Chin and Smith (1997) assess the effectiveness of the vehicle quota system in Singapore via an estimated car demand model.

4 Category A includes cars up to 1600cc and 97kW. The outcomes of the auction are available at https://www.lta.gov.sg/content/ltagov/en/who_we_are/statistics_and_publications/statistics.html.

5 Over the years other big cities, from China, have adopted a vehicle quota system to curb road congestion and pollution. The mechanisms adopted to allocate the cars range from auctions (Shaghai), to lotteries (Bejiing) and, more recently, a mixed auction-lottery (Guangzhou, Hangzhou, Shenzhen and Tianjin). See Wang and Zhao (2017) and Zheng et al. (2021) for a study of policy implications.

6 In earlier implementations of the auction, sellers could bid for certain number of permits. With the current system they cannot, but they can use proxi bidders to get the permit and transfer it to a willing buyer (see, e.g. Koh and Lee, 1994).

7 In Section 3.2.2 we examine the alternative of the Vickrey-Clarke-Groves auction to allocate the permits, but this mechanism reduces the government's flexibility with respect to a fixed price mechanism by imposing, for instance, that the allocation of permits is determined by the sellers' total profits under the monopoly allocation and under the duopoly allocation.

8 We consider social welfare as the difference between the values of the buyers who receive cars and permits and the costs of the sellers. We may think that social welfare also depends on costs from traffic congestion and pollution, and these costs are a key part in the government decisions to control the number of cars and, therefore, in their choice of the number k of permits. We omit these terms from our analysis as we do not address the policy question of the socially optimal number of cars. We note, however, that the mechanisms may differ in the expected congestion and pollution costs only when markets are not fully covered. Otherwise, these costs are the same and so irrelevant to our comparisons.

between cars is small, social welfare and gross seller profit are higher under monopoly than under duopoly, but consumer surplus is lower. When instead differentiation is large, then the above inequalities are reversed. Although this suggests that the comparison between monopoly and duopoly is somewhat ambiguous, we consider the monopoly allocation as undesirable. Precisely, to fix the ideas suppose that seller A is the two-bundle monopolist. This leaves seller B out of business, and even though monopoly sometimes increases consumer surplus, it does so by increasing significantly the utility of consumer types who have a preference for car A over car B (now there are two bundles A on sale rather than just one) but reducing to zero the utility of types who have a preference for car B (now bundle B is unavailable and bundle A is too expensive for them). Therefore, monopoly has a very asymmetric effect on the distribution of utility across consumers which makes us view negatively the monopoly allocation of permits. For this reason, we consider SA when each seller i owns one permit and offers one bundle i, and compare it with CA.

As we mentioned above, the permits are allocated efficiently under CA, whereas in SA some inefficiencies occur, hence social welfare is greater in CA than in SA. In contrast, gross seller profit is always higher under SA than under CA because the burden effect makes competition tough under CA, but no such effect applies under SA. Actually, under SA partial coverage is likely to occur for large n and then each seller is a one-bundle local monopolist who earns a significantly higher gross profit than when competing with the other seller under CA. Therefore, by suitably choosing the price in a fixed price mechanism the government can induce each seller to buy a permit while leaving her a net profit greater than the profit under CA.

Also consumers almost always prefer SA because in the auction for permits under CA each consumer competes with all the others and only the two consumers with the top two values for a car get a permit; this leaves a low utility to each consumer type who has not a very high value for either car. Under SA, in case of partial coverage consumer types partition in three sets, one of types visiting seller A, one of types visiting seller B, one of types visiting neither. To fix the ideas, consider the types who visit seller A. Because of the posted price mechanism, they all have equal probability to buy bundle A, which increases the utility of types with medium-high value for bundle A as it softens the negative effect of competition from consumers with high value (for either car). Although consumers with very high value for bundle A prefer CA (see Fig. 4 in Section 3.3), the utility increase for consumers with medium-high value suffices to make total consumer surplus greater under SA. In case of full coverage, the comparison with CA is even more favorable for SA because sellers compete against each other, thus reducing the equilibrium price.

Finally, we notice that a higher consumer surplus, a higher gross seller profit and a lower social welfare in SA imply a lower government revenue in SA than in CA (in SA, the government revenue is given by the sellers' payments in the fixed price mechanism).

In the product (car) differentiation model on which Section 3 relies, each consumer's values for car A and car B are perfectly negatively correlated. This is a restriction on consumers' preferences which may not be satisfied in practice. In order to obtain insights about how this restriction affects our results, in Section 4 we examine a setting of product differentiation with random utilities as in Perloff and Salop (1985), such that each consumer has a value v_A for car A, a value v_B for car B, and v_A, v_B are i.i.d.[9] As a consequence, each consumer type is bidimensional and this complicates the analysis. We obtain a complete characterization of the unique symmetric equilibrium under CA when the market is fully covered, which has the same features as the equilibrium under CA for the setting of Section 3: tough competition among consumers and between sellers makes both consumer surplus and seller profit close to zero when n is large. Under SA, in order to determine the equilibrium we specify the parameter values and then employ numerical methods. Nevertheless, our results are consistent with Section 3 regarding the features of the equilibrium when each seller offers one bundle, regarding the comparison between the monopoly allocation and the duopoly allocation of permits, and regarding the comparison between SA when each seller offers one bundle and CA. That is, social welfare is higher under CA but consumers and sellers prefer SA for reasons similar to those which apply in Section 3.

Section 5 offers some concluding remarks. The proofs of our results are in the Appendix or in Landi and Menicucci (2023a).

2. The model

Consider a setting with $n \geq 4$ (male) consumers, each of whom is interested in buying and using a car. There are two (female) sellers, A and B, which offer cars while incurring a constant marginal cost $c \geq 0$. In order to use a car, a consumer needs to buy it from a seller and also needs to hold a vehicle registration permit. There are two permits to allocate, and we consider two institutional scenarios to allocate them. In the first one, which we will refer to as *Consumers Allocation*, first the government auctions the permits to the consumers, then each consumer who won a permit buys a car from a seller.[10] In the second one, which we will refer to as *Sellers Allocation*, first the sellers are allocated the permits by the government, then each seller offers to the consumers a number of bundles, each consisting of a car and a permit, equal to the number of permits she obtained from the government.

Data on car sales in Singapore show that in 2022, multiple auto brands competed in Singapore: in particular Toyota, Mercedes Benz, BMW, and Honda all had market shares between 10.5% and 20%.[11] This suggests that we are in the presence of product differentiation and we assume that cars A and B are differentiated in a standard Hotelling fashion. In Section 4 we consider a different specification for car differentiation, according to which for each consumer, his values for car A and car B are i.i.d., rather than perfectly negatively correlated.

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[9] In Landi and Menicucci (2023b) we examine a setting in which $v_A = v_B$, thus the values are perfectly positively correlated and we obtain similar results.

[10] We illustrate in Footnote 15 that for each winner of a permit it is convenient to buy a car from a seller.

[11] These are our own computations made from sales figures retrieved at https://www.lta.gov.sg/content/dam/ltagov/who_we_are/statistics_and_publications/statistics/pdf/MVP01-6_Cars_by_make.pdf.

Each consumer is characterized by a type $\theta \in \Theta = [0, 1]$ which determines as follows his valuation $v_A(\theta)$ for a permit plus a car A,[12] and his valuation $v_B(\theta)$ for a car B:

$$v_A(\theta) = 1 - t\theta \qquad \text{and} \qquad v_B(\theta) = 1 - t(1-\theta) \qquad \text{for each } \theta \in \Theta \tag{1}$$

with $t \in (0, 1)$. For ease of expression, in the rest of the paper we will refer to $v_i(\theta)$ just as type θ's valuation for a car i, for $i = A, B$. The consumer's net utility if he obtains car i (and a permit) is $v_i(\theta)$ minus his total payment to seller i and (possibly) to the government. The sellers and the consumers are all risk neutral. Notice that each type of consumer has a positive value for either car, type $\theta = \frac{1}{2}$ has the same value $1 - \frac{t}{2} > \frac{1}{2}$ for each car, and

$$v(\theta) = \max\{v_A(\theta), v_B(\theta)\} \tag{2}$$

which we call type θ's *maximal valuation*, is greater than $1 - \frac{t}{2}$ for each $\theta \neq \frac{1}{2}$. We assume that $c \leq \frac{1}{2}$, which implies $\max\{v_A(\theta), v_B(\theta)\} > c$ for each $\theta \in [0, 1]$. Thus, in each state of the world it is socially optimal that cars and permits are allocated to the two consumers with the two highest maximal valuations.

Each consumer privately observes his own θ, which is viewed by the other consumers, by the sellers and by the government as a realization of a uniform random variable with support Θ; let G, g denote the c.d.f. and the density, respectively, for this distribution. Moreover, the types of different consumers are independently distributed.

3. Consumers allocation vs. sellers allocation

In this section we examine the two institutional scenarios we mentioned in Section 2, and compare the results.

3.1. Consumer allocation of permits, CA

We denote with CA the setting in which the government auctions the permits to the consumers. The timing under CA is as follows:

Stage 1 Seller i sets a price $p_i < 1$ for each of the cars she offers, for $i = A, B$. Sellers choose prices simultaneously.
Stage 2 After observing p_A and p_B, each consumer decides whether to participate in the government auction for the permits.
Stage 3 After the consumers' participation decisions, the government holds a sealed bid auction in which each consumer participating in the auction submits a bid for one permit, and the permits are awarded to the two highest bidders, each of whom pays the same price, which is the highest losing bid, that is the third highest bid.[13] Each consumer who wins a permit chooses the car i he buys and pays p_i to seller i.[14]

In next section we describe the consumers' behavior, and from it we derive the sellers demand and profit functions.

3.1.1. Consumers' auction participation, car choice, and demand functions

A consumer with type θ knows that he can buy a car i at price p_i if he wins a permit, which leaves him a net utility of $v_i(\theta) - p_i$ minus his payment in the auction. Therefore his value from winning a permit is $\max\{v_A(\theta) - p_A, v_B(\theta) - p_B\}$ and he participates in the auction if and only if $\max\{v_A(\theta) - p_A, v_B(\theta) - p_B\} \geq 0$. In such a case, he bids $\max\{v_A(\theta) - p_A, v_B(\theta) - p_B\}$.[15] This reveals that the sellers exert an externality on the auction, as p_A, p_B determine consumers' participation in the auction – in particular, an increase in p_A and in p_B reduces participation and reduces the bids of the participating consumer.

In order to derive the sellers demand functions it is necessary to determine the car a consumer chooses upon winning a permit. A consumer with type θ prefers car A if and only if $\theta < \frac{1}{2} + \frac{p_B - p_A}{2t}$, and prefers car B if and only if the opposite inequality holds. We let $\tilde{\theta}^c = \frac{1}{2} + \frac{p_B - p_A}{2t}$ denote the indifferent type.[16] If the utility of the indifferent type, $v_A(\tilde{\theta}^c) - p_A = v_B(\tilde{\theta}^c) - p_B = 1 - \frac{1}{2}(p_A + p_B + t)$, is non-negative, then each consumer with $\theta \in [0, \tilde{\theta}^c]$ joins the auction, bids $v_A(\theta) - p_A$ and buys car A if he wins a permit. Likewise, each type $\theta \in (\tilde{\theta}^c, 1]$ joins the auction, bids $v_B(\theta) - p_B$ and buys car B if he wins a permit. In this case we say that the market is *fully covered*.

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[12] A car without permit has zero value, as a permit without a car.

[13] Ties are resolved randomly and fairly. Precisely, if there is just one highest bid and at least two second highest bids, then each second highest bidder gets a permit with equal probability. If there are more than two highest bids, then each highest bidder gets one of the two permits with equal probability. Note that in equilibrium (see the first paragraph of next subsection) ties occur with zero probability.

[14] If just two or fewer than two consumers participate in the auction, then each active consumer wins a permit for free. As we explained in Footnote 1, there is no reserve price in this auction.

[15] This is a weakly dominant bid for him since he has a unit demand and the price each winner pays is the highest losing bid, as in a second-price auction for a single object (see e.g. chapter 13 in Krishna (2010)). Also notice that if a consumer with type θ wins a permit and (to fix the ideas) $v_A(\theta) - p_A > v_B(\theta) - p_B$, then $v_A(\theta)$ is no less than p_A and hence it is sequentially rational for him to buy car A after winning a permit.

[16] We consider p_A, p_B such that $\frac{1}{2} + \frac{p_B - p_A}{2t}$ belongs to the interval $(0, 1)$ because otherwise the demand for one seller is zero, which is not consistent with equilibrium.

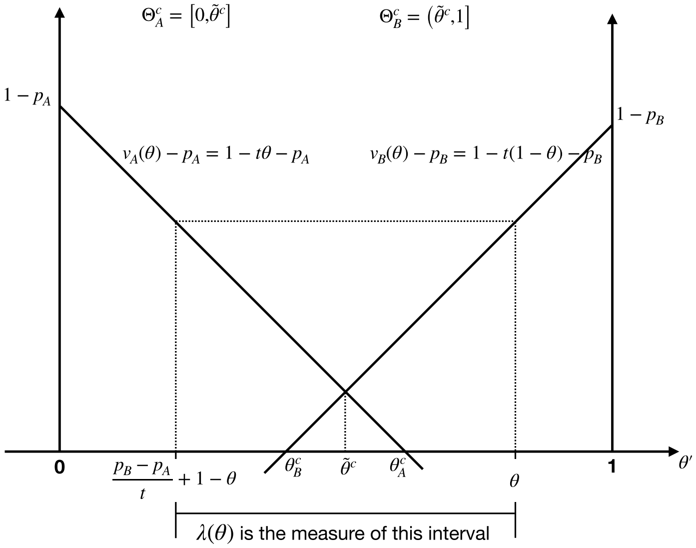

Fig. 1. Graphical description of one step in computing the probability of winning a permit for a buyer with type $\theta \in \Theta_B^c$. This consumer bids more than any consumer with type $\theta' \in \left(\frac{p_B - p_A}{t} + 1 - \theta, \theta\right)$, a set with measure $\lambda(\theta) = G(\theta) - G\left(\frac{p_B - p_A}{t} + 1 - \theta\right)$.

If, instead, the utility of the indifferent type is negative, then there is an interval (θ_A^c, θ_B^c), with $\theta_A^c = \frac{1}{t}(1 - p_A)$ and $\theta_B^c = 1 - \frac{1}{t}(1 - p_B)$, such that no type in (θ_A^c, θ_B^c) participates in the auction; each type $\theta \in [0, \theta_A^c]$ joins the auction, bids $v_A(\theta) - p_A$ and buys car A if he wins a permit; each type $\theta \in [\theta_B^c, 1]$ joins the auction, bids $v_B(\theta) - p_B$ and buys car B if he wins a permit. Remark that θ_A^c and θ_B^c are the solutions to $v_A(\theta) - p_A = 0$ and to $v_B(\theta) - p_B = 0$, respectively. We then say that the market is *not fully covered*; such case occurs if $\theta_A^c < \theta_B^c$, that is if $p_A + p_B > 2 - t$, which is equivalent to the fact that the utility of the indifferent type $\tilde{\theta}^c$, $1 - \frac{1}{2}(p_A + p_B + t)$, is negative.

To summarize, we have determined a partition of Θ, depending on whether $\theta_A^c < \theta_B^c$ or $\theta_B^c \le \theta_A^c$:

$$\Theta_A^c = \left[0, \theta_A^c\right], \qquad \Theta_N^c = (\theta_A^c, \theta_B^c), \qquad \Theta_B^c = \left[\theta_B^c, 1\right] \qquad \text{if } \theta_A^c < \theta_B^c$$
$$\Theta_A^c = \left[0, \tilde{\theta}^c\right], \qquad \Theta_N^c = \emptyset, \qquad \Theta_B^c = \left(\tilde{\theta}^c, 1\right] \qquad \text{if } \theta_B^c \le \theta_A^c$$

in which Θ_N^c is the set of types who do not participate in the auction, whereas for $i = A, B$, each type in Θ_i^c participates in the auction, bids $v_i(\theta) - p_i$, and buys a car i if he wins a permit.

Now we proceed with determining the sellers demand functions. To this purpose, and without loss of generality, we suppose that $p_A \le p_B$. We denote with $D_i(p_A, p_B)$ the expected number of cars sold by seller i, for $i = A, B$, and define η as $\Pr\{\theta \in \Theta_N^c\}$, that is $\eta = \max\{G(\theta_B^c) - G(\theta_A^c), 0\}$ is the probability that a consumer does not participate in the auction. It follows that $2\eta^n + n\eta^{n-1}(1-\eta)$ is the expected quantity of permits that are not sold in the auction. Hence, $D_A(p_A, p_B)$ and $D_B(p_A, p_B)$ satisfy

$$D_A(p_A, p_B) + D_B(p_A, p_B) + 2\eta^n + n\eta^{n-1}(1-\eta) = 2 \tag{3}$$

Now we derive $D_B(p_A, p_B)$ and then we use (3) to obtain $D_A(p_A, p_B)$. To derive $D_B(p_A, p_B)$ we evaluate, for each consumer, the probability that he wins a permit and buys car B.

Consider consumer 1 with type $\theta \in \Theta_B^c$. Consumer 1 bids $v_B(\theta) - p_B$ in the auction. Now consider a different consumer, say consumer 2 to fix the ideas, with type θ'. If θ' belongs to Θ_B^c, then consumer 2 bids more than consumer 1 if and only if $v_B(\theta') - p_B > v_B(\theta) - p_B$, which is equivalent to $\theta' > \theta$. If θ' belongs to Θ_N^c, then consumer 2 stays out of the auction. If θ' belongs to Θ_A^c, then consumer 2 bids more than consumer 1 if and only if $v_A(\theta') - p_A > v_B(\theta) - p_B$, which is equivalent to $\theta' \in \left[0, \frac{p_B - p_A}{t} + 1 - \theta\right)$.[17] Therefore consumer 1 with type θ bids more than consumer 2 with probability

$$\lambda(\theta) = G(\theta) - G\left(\frac{p_B - p_A}{t} + 1 - \theta\right) \tag{4}$$

See Fig. 1 for an illustration of how $\lambda(\theta)$ is determined, and notice that $\Theta_N^c = \emptyset$ in Fig. 1. Since there are two permits on sale, we need to determine the probability that at least $n-2$ of the other $n-1$ consumers bid less than consumer 1. This is given by $\lambda^{n-1}(\theta) + (n-1)\lambda^{n-2}(\theta)(1 - \lambda(\theta))$. Hence, the probability that consumer 1 wins a permit and buys car B is given by

[17] The inequality $p_B \ge p_A$ implies that $\frac{p_B - p_A}{t} + 1 - \theta \ge 0$ since $\theta \le 1$.

Θ_B^c $\left(\lambda^{n-1}(\theta)+(n-1)\lambda^{n-2}(\theta)(1-\lambda(\theta))\right) f(\theta)d\theta$. Finally, $D_B(p_A, p_B)$ is the sum of the analogous probabilities over all the n consumers, that is[18]

$$D_B(p_A, p_B) = n \int_{\Theta_B^c} \left(\lambda^{n-1}(\theta)+(n-1)\lambda^{n-2}(\theta)(1-\lambda(\theta))\right) g(\theta)d\theta \tag{5}$$

To conclude, from (3) it follows that[19]

$$D_A(p_A, p_B) = 2 - 2\eta^n - n\eta^{n-1}(1-\eta) - D_B(p_A, p_B) \tag{6}$$

The profit for seller $i = A, B$ is $\pi_i(p_A, p_B) = (p_i - c)D_i(p_A, p_B)$.

Before moving to determine the equilibrium prices, we notice an important feature of this competition setting. Starting from $p_A = p_B = p$, with fully covered market to fix the ideas, suppose that seller B increases p_B by $\Delta p_B > 0$. This shifts $\tilde{\theta}^c$ to the right, hence Θ_B^c shrinks and Θ_A^c widens as in a standard Hotelling model. But $\Delta p_B > 0$ has the additional effect of reducing the bids of all consumers with θ in (the new) Θ_B^c since each such consumer will bid $v_B(\theta) - p - \Delta p_B$ in the auction. Thus, it becomes less likely that the winners of the permits buy a car B. This reduces $\lambda(\theta)$, hence D_B in (5) decreases not only because the integration occurs over a smaller set Θ_B^c, but also because the integrand function is reduced. Since $\Delta p_i > 0$ puts a burden on each consumer who plans to buy a car i (if he wins a permit), we call this the *burden effect* of a price increase in CA. This effect makes D_i more elastic, which suggests that competition is harsh in CA.

3.1.2. Equilibrium prices

Since sellers are ex ante symmetric, we focus on symmetric equilibria, in which each seller charges the same price p^c. Proposition 1 identifies a symmetric equilibrium for each $n \geq 4$, $t \in (0,1)$, $c \leq 1/2$.

Proposition 1. *In setting CA*

(i) *there exists a symmetric equilibrium with fully covered market (i.e., $\Theta_N^c = \emptyset$) if and only if either $c \leq \frac{n-4}{2n}$, or $c > \frac{n-4}{2n}$ and $t \leq \frac{2n(1-c)}{n+4}$. In this case the equilibrium prices are $(p_A^c, p_B^c) = (p^c, p^c)$ with $p^c = c + \frac{2}{n}t$.*

(ii) *There exists a symmetric equilibrium with non-fully covered market (i.e., $\Theta_N^c = (\theta_A^c, \theta_B^c) \neq \emptyset$) if and only if $c > \frac{n-4}{2n}$ and $t > \frac{2n(1-c)}{n+4}$. In this case the equilibrium prices are $(p_A^c, p_B^c) = (p^c, p^c)$ in which p^c is the unique solution to equation (29) in Appendix A, and $1 - t/2 < p^c < \min\{1, c + \frac{2}{n}t\}$.*

We notice that the equilibrium prices fail to deliver full market coverage only when c and t are sufficiently large, as specified in Proposition 1*(ii)*. In this case, partial coverage occurs because the larger is t, the lower are $v_A(\theta)$ and $v_B(\theta)$ and then a relatively large marginal cost makes it not worthwhile for sellers to lower prices enough to attract types with θ close to $\frac{1}{2}$ – these are the types with the lowest values. The sellers would rather exclude some of them and increase the profit margin from the remaining types.

We also notice that when the market is not fully covered, we do not have a situation of local monopolies. Indeed, a small increase in, say, p_B makes the set Θ_B^c shrink, but because of the burden effect it also makes it more difficult for buyers in Θ_B^c to win a permit in the auction. This increases the demand for cars A even though Θ_A^c does not change. However, as the number of buyers increases, the equilibrium prices decrease and the set of pairs (c, t) that support an equilibrium without full coverage becomes smaller. (See Fig. 2 for the cases of $n = 4$ and $n = 10$.) Precisely, for each $(c, t) \in [0, 1/2] \times (0, 1)$, full coverage emerges if n is large enough, and the equilibrium prices converge to the marginal cost as n diverges. Although this might seem counterintuitive, it is actually quite plausible in light of the burden effect, as we now illustrate for the case of fully covered market.

Recall that for a consumer with type θ, the maximal valuation is $\max\{v_A(\theta), v_B(\theta)\}$. Since $p_A = p_B$ in equilibrium, the auction is won by the two consumers with the two highest maximal valuations. In expectation, one of the winning consumers buys a car A (i.e., the consumer's type belongs to Θ_A^c), the other buys a car B (i.e., the consumer's type is in Θ_B^c); thus the demand for each seller is 1. Now consider a small increase in p_B, which reduces B's demand. The key remark is that the higher is n, the greater is the resulting demand decrease for seller B.

As n increases, it becomes more likely that there are at least two consumers with θ close to 0, that is two consumers with $v_A(\theta)$ close to 1, and at least two consumers with θ close to 1, hence with $v_B(\theta)$ close to 1. Since $p_A < p_B$, the consumers with v_B close to 1 have a disadvantage with respect to the consumers with v_A close to 1, and bid lower in the auction, that is they bid about $1 - p_B$ rather than about $1 - p_A$. For a large n, it is very likely that there are at least two consumers with v_A close to 1 and therefore is it very likely that no car B will be sold. Therefore, competition between the sellers gets close to Bertrand competition because even a

[18] For the purpose of comparison, it may be useful to notice that in the standard Hotelling model, $\lambda(\theta) = 1$ for each $\theta \in \Theta_B^c$ because there is no issue of scarcity (here due to the limited number of permits). Hence, $D_B(p_A, p_B) = n\int_{\Theta_B^c} g(\theta)d\theta = n[1 - G(\max\{\theta_B^c, \tilde{\theta}^c\})]$. Then n plays no role in determining the equilibrium prices, and indeed often it is set equal to 1.

[19] Remark that $D_B(p_A, p_B)$ in (5) and $D_A(p_A, p_B)$ in (6) are the demand functions of seller B and seller A when $p_A \leq p_B$. If instead $p_A > p_B$, the demands are swapped, that is the one of seller B is given by $D_A(p_B, p_A)$ and the one for seller A is given by $D_B(p_B, p_A)$.

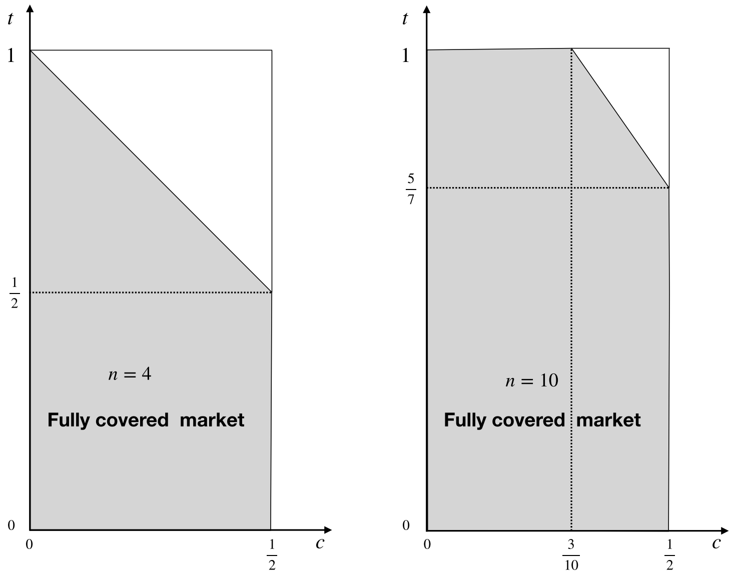

Fig. 2. Set of parameter values (c, t) under which an equilibrium with fully covered market exists (gray regions) for $n = 4$ and $n = 10$.

small difference in prices very likely allows the more aggressive seller to sell two cars. Indeed, p^c in Proposition 1*(i)* tends to c as n tends to infinity.

Finally, we notice that even though an increase in n reduces each car's equilibrium price and the seller profit, it actually makes competition among consumers in the auction more intense. This increases the government's expected revenue from the auction, which is given by 2 times the expected third highest maximal valuation minus p^c, and which tends to the whole surplus generated in this market as n diverges.

3.2. Sellers allocation of permits, SA

In this section we examine setting SA, in which first the government allocates the permits to the sellers through a mechanism described in Section 3.2.2, and then each seller i offers to consumers each permit she obtained from the government bundled with a car i. For consistency with setting CA, we assume that each seller offers her bundle(s) through a posted price mechanism.[20]

Precisely, the timing for SA is as follows:

Stage 1 The government allocates the permits to sellers using an auction or a fixed price.

Stage 2 For $i = A, B$, if seller i obtained $k_i = 1$ or $k_i = 2$ permits at stage 1, then she offers k_i bundles, each consisting of a car i and a permit, at a posted price p_i set by seller i.

Stage 3 After observing p_A, p_B, each consumer decides to visit seller A, or seller B, or no seller at all.

Stage 4 For $i = A.B$, if the number n_i of consumers visiting seller i is not greater than k_i then each consumer visiting seller i receives a bundle i and pays p_i to seller i. If instead $n_i > k_i$, then each bundle i is allocated randomly among the consumers visiting seller i: each consumer visiting seller i wins a bundle i with probability $\frac{k_i}{n_i}$.

We need to distinguish the case in which each seller obtained one permit at stage 1, thus each seller offers one bundle to consumers, from the case in which a single seller – say seller A – obtained both permits at stage 1 and offers two bundles while seller B is inactive. We use *duopoly allocation* of permits, or just *duopoly*, to refer to the first case, and *monopoly allocation*, or just *monopoly*, to refer to the second case. Under the monopoly allocation, seller A chooses a price p and the consumers visiting seller A are those with $\theta \in [0, \frac{1}{t}(1-p)]$. This determines the expected number of bundles A which are sold. We denote with p^s_{M2} the optimal price for seller A given this demand function; in Appendix B we provide more details about p^s_{M2}. Next subsection is about competition between sellers under the duopoly allocation.

3.2.1. Competition between sellers in SA

When each seller offers one bundle, the prices p_A, p_B induce a partition of the set Θ into three subsets, analogously to what happens in setting CA (see Section 3.1): Θ^s_A (Θ^s_B) is the set of types who visit seller A (seller B) and Θ^s_N is the set of types who do not visit either seller. From Θ^s_A, Θ^s_B we define

[20] In Landi and Menicucci (2023b) we examine a setting in which each seller uses a second-price auction to offer her bundle(s), and sets the reserve price for the auction.

$$\mu_A = \Pr\{\theta \in \Theta^s_A\}, \qquad \mu_B = \Pr\{\theta \in \Theta^s_B\}$$

hence seller i's expected profit is given by $(p_A - c)(1-(1-\mu_i)^n)$ since seller i earns $p_i - c$ if at least one consumer visits her, but earns 0 otherwise, and the former event has probability $1-(1-\mu_i)^n$.

Also in this context the cutoffs $\theta^s_A = \frac{1}{t}(1-p_A)$ and $\theta^s_B = 1 - \frac{1}{t}(1-p_B)$ are relevant, as a consumer with $\theta > \theta^s_A$ does not consider visiting seller A since $v_A(\theta) < p_A$, and likewise a consumer with $\theta < \theta^s_B$ does not consider visiting seller B. When p_A, p_B are such that $\theta^s_A < \theta^s_B$, we have that the market is not fully covered and $\Theta^s_A = [0, \theta^s_A]$, $\Theta^s_B = [\theta^s_B, 1]$, $\Theta^s_N = (\theta^s_A, \theta^s_B) \neq \emptyset$. In this case the profit of seller A (B) is equal to the profit of a one-bundle monopolist seller who charges price p_A (p_B).[21] It follows that the market is not fully covered in equilibrium if the optimal price for a one-bundle monopolist, which we denote p^s_{M1}, is such that $\theta^s_A = \frac{1}{t}(1-p^s_{M1}) < \theta^s_B = 1 - \frac{1}{t}(1-p^s_{M1})$, that is if p^s_{M1} is greater than $1-\frac{t}{2}$, the value for either car of type $\theta = \frac{1}{2}$.

Proposition 2 below is expressed in terms of the index $\rho = \frac{1-c}{t}$, a measure of the profitability of the market for a monopolist: ρ is greater the lower is the marginal cost c, and the lower is t – that is the higher are the valuations. Proposition 2 establishes that an equilibrium with non-fully covered market arises if and only if $\rho \leq \frac{1}{2} + \frac{2^n-1}{2n}$, a condition which is more likely to be satisfied as n increases. In particular, for a given ρ, local monopolies arise provided that n is large enough. This occurs because, as we mentioned above, all that matters for seller i is that at least one consumer visits her. Given $p_A = p_B = 1-\frac{t}{2}$, if n is large then the probability that at least one consumer visits seller i is close to 1 and, more importantly, a small increase in p_i above $1-\frac{t}{2}$ produces just a small decrease in demand for seller i – unless t is very close to zero.[22] That is, the demand for seller i is quite inelastic and it is profitable to increase p_i above $1-\frac{t}{2}$, so that the equilibrium price under SA is equal to p^s_{M1}. As n tends to $+\infty$, we find that p^s_{M1} tends to 1, and so does the equilibrium price under SA.

Proposition 2. *For setting SA, suppose that each seller has received one permit at stage 1. Then there exists a unique symmetric equilibrium $(p^*_A, p^*_B) = (p^s, p^s)$, in which p^s is as follows, with $\rho = \frac{1-c}{t}$:*

$$p^s = \begin{cases} p^s_{M1} & \text{if } \rho < \frac{1}{2} + \frac{2^n-1}{2n} \\ 1 - \frac{t}{2} & \frac{1}{2} + \frac{2^n-1}{2n} \leq \rho \leq \frac{1}{2} + \frac{2^n-1}{n} \\ 1 - \frac{n}{2^{n+1}-n-2}(\rho-1)t & \frac{1}{2} + \frac{2^n-1}{n} < \rho \end{cases} \tag{7}$$

When the market is fully covered, the inequality $\theta^s_B \leq \theta^s_A$ holds and there exists an indifferent type $\tilde{\theta}^s$ such that $\Theta^s_A = [0, \tilde{\theta}^s]$, $\Theta^s_B = (\tilde{\theta}^s, 1]$, hence $\mu_A = G(\tilde{\theta}^s)$, $\mu_B = 1 - G(\tilde{\theta}^s)$. In order to determine $\tilde{\theta}^s$, we denote with γ_i the probability that a consumer visiting seller i wins bundle i. Hence,

$$u^s_i(\theta) = (v_i(\theta) - p_i)\gamma_i \qquad \text{for } i = A, B \tag{8}$$

is type θ's expected utility from visiting seller i. The probability γ_i is constant with respect to the consumer's type and is given by

$$\gamma_i = \sum_{j=0}^{n-1} \frac{1}{j+1}\binom{n-1}{j}\mu_i^j(1-\mu_i)^{n-1-j} = \frac{1}{n\mu_i}\sum_{m=1}^{n}\binom{n}{m}\mu_i^m(1-\mu_i)^{n-m} = \frac{1-(1-\mu_i)^n}{n\mu_i} \tag{9}$$

From (8) it follows that $\tilde{\theta}^s$ satisfies

$$(v_A(\theta) - p_A)\gamma_A = (v_B(\theta) - p_B)\gamma_B \tag{10}$$

As it is intuitive, $\tilde{\theta}^s$ is increasing in p_B and decreasing in p_A. Each seller i chooses p_i taking into account that p_i affects $\tilde{\theta}^s$ as determined by (10). This allows to derive the equilibrium price p^s in Proposition 2 when $\rho \geq \frac{1}{2} + \frac{2^n-1}{2n}$. Notice that $p^s < 1 - \frac{t}{2}$ (and then each consumer type earns a positive expected utility) if ρ is sufficiently large, that is if t is close enough to 0, because then the cars of the two sellers are almost homogeneous and, so, the consumers are very responsive to differences in prices. This intensifies competition between sellers and leads, in equilibrium, to $p^s < 1 - \frac{t}{2}$. More in detail, (7) reveals that

$$\lim_{t\to 0} p^s = 1 - \frac{n}{2^{n+1}-n-2}(1-c) \tag{11}$$

that is the limit of p^s as t tends to 0 is greater than the marginal cost c. In particular, even though t is close to 0, no Bertrand-like competition with infinitely elastic demands occurs, that is if p_A is slightly smaller than p_B, then it is not true that all consumer types visit seller A. Indeed, if each consumer type visits seller A, then each consumer wins bundle A with probability $\frac{1}{n}$ but a deviating consumer who visits seller B wins bundle B with probability 1. If $p_A < p_B$ but $p_B - p_A$ is small, then this deviation is profitable.

We also remark that in setting CA, as n increases it becomes more likely that the market is fully covered, and the equilibrium price p^c tends to the marginal cost because a large n makes the demand for each seller very elastic (see Proposition 1). Conversely,

[21] We are referring here to a monopolist offering a single bundle and not to a monopolist who offers two bundles as in the case mentioned in the last paragraph before Section 3.2.1.

[22] If t is about zero, then cars are almost homogeneous and each seller loses a significant set of types by raising her price.

in SA, as n increases it becomes more likely that the market is not fully covered, and the equilibrium price p^s tends to 1 – the highest value a consumer may have for either car – as n tends to $+\infty$. This occurs because when n is large, the demand for each seller i is rigid, as we mentioned above, unless p_i is close to 1. But notice that even when full market coverage occurs under SA, some inefficiencies arise because the posted price mechanism does not necessarily select the consumer with the highest value for car i among the consumers visiting seller i. Moreover, there is a positive probability that all consumers visit a same seller, say seller A, which leaves bundle B unsold even though each consumer not winning bundle A has value for bundle B above c.

3.2.2. The allocation stage

In this subsection we examine stage 1 for setting SA, in which the government allocates the permits to the sellers.

Allocation through auction Suppose first that the government auctions the permits to the sellers. In such auction, each seller is a bidder with a two-unit demand and two valuations. One valuation for a single permit, which we denote π^s_D and coincides with the seller's profit in the duopoly equilibrium of Proposition 2; and one valuation for the package of both permits, which we denote by π^s_{M2} and corresponds to the profit of a monopolist seller i offering two bundles, each consisting of a car i and a permit.[23] We notice that the sellers are symmetric, and so π^s_D and π^s_{M2} are the same for both sellers. Moreover, the auction occurs with complete information, that is π^s_D and π^s_{M2} are commonly known as no seller has private information on her own profit function.

There are many formats for multi-unit auctions that the government may use, but a useful reference point, at least in a theoretical sense, is the Vickrey-Clarke-Groves (VCG) auction. In a VCG auction each seller i submits a bid $b_i(1)$ for a single permit and a bid $b_i(2)$ for the package of both permits (see, e.g., Chapter 16 in Krishna (2010)). The VCG auction determines the following allocation and payments, τ_A for seller A and τ_B for seller B, depending on the bids $b_A(1)$, $b_A(2)$, $b_B(1)$, $b_B(2)$[24]:

$$\begin{cases} \text{both permits are allocated to seller } A \text{ and } \tau_A = b_B(2),\ \tau_B = 0 & \text{if } b_A(2) > \max\{b_B(2), b_A(1) + b_B(1)\} \\ \text{both permits are allocated to seller } B \text{ and } \tau_A = 0,\ \tau_B = b_A(2) & \text{if } b_B(2) > \max\{b_A(2), b_A(1) + b_B(1)\} \\ \text{each seller receives one permit and} & \\ \tau_A = b_B(2) - b_B(1),\ \tau_B = b_A(2) - b_A(1) & \text{if } b_A(1) + b_B(1) > \max\{b_A(2), b_B(2)\} \end{cases}$$

It is well known that in a VCG auction, for each bidder it is a weakly dominant strategy to bid the own values for the objects, and so each seller i bids $b_i(1) = \pi^s_D$ and $b_i(2) = \pi^s_{M2}$. As a result, the allocation of permits maximizes the sellers' total profits gross of their payments in this auction, that is a monopoly allocation results if $\pi^s_{M2} > 2\pi^s_D$, whereas a duopoly allocation emerges if $\pi^s_{M2} < 2\pi^s_D$. Proposition 3 below establishes the existence of a threshold for ρ such that $\pi^s_{M2} < 2\pi^s_D$ holds if and only if ρ is smaller than the threshold.

Proposition 3. *There exists $\rho^* > \frac{1}{2} + \frac{2^n-1}{2n}$ such that the VCG auction allocates one permit to each seller if $\rho < \rho^*$, but allocates both permits to a single seller if $\rho > \rho^*$.*

In fact, it is simple to see that $2\pi^s_D > \pi^s_{M2}$ if ρ is small, and $2\pi^s_D < \pi^s_{M2}$ if ρ is large. Consider first a small ρ. Then Proposition 2 establishes that competition for consumers in SA leads to local monopolies, hence π^s_D coincides with the profit of a one-bundle monopolist, which we denote π^s_{M1}, and $2\pi^s_D = 2\pi^s_{M1}$. The inequality $2\pi^s_{M1} > \pi^s_{M2}$ holds because twice the expected demand for a one-bundle monopolist is greater than the expected demand for a two-bundle monopolist, as we prove in Appendix B. Now consider a large ρ and suppose the two-bundle monopolist is seller A. A large ρ is equivalent to t close to 0, which makes the value of each consumer type for bundle A about equal to 1 and makes it possible for seller A to extract almost all consumer surplus, earning a profit close to $2 - 2c$. Conversely, we know from (11) that when t is close to 0, under duopoly the equilibrium price p^s is bounded away from 1 (from below), hence the profit of each seller is less than $1 - c$ and $\pi^s_{M2} > 2\pi^s_D$.

By relying on Proposition 3 we can evaluate the outcome of the VCG auction from the point of view of social welfare, consumer surplus, sellers profits. However, here we notice that the VCG auction generates a particular allocation of permits, a particular distribution of utilities across consumers, and a particular distribution of profits across sellers. Yet, the government may want to induce a duopoly/monopoly without being constrained by the allocation rule and the payments of the VCG auction, and in fact an auction may not even be necessary in this context. A rationale for adopting an auction mechanism is to allocate goods to the bidders who value them the most, in the presence of incomplete information. In our model, however, the bidders are the sellers and there is no incomplete information about their values, which are given by profits in the continuation games. There are essentially just two alternatives for the allocation of permits: the duopoly allocation and the monopoly allocation, and we now describe how the government may implement either allocation, and with more flexibility than under the VCG auction, by resorting to a different mechanism based on a fixed price.

[23] This is because the sellers' payments in the auction are sunk when each seller i sets the price p_i at stage 2, either as a duopolist or a monopolist. Therefore, the results in Proposition 2 and Appendix B still hold.

[24] In case of a tie (which we neglect here for the sake of brevity), any tiebreaking rule can be used.

Allocation through a fixed price Suppose the government offers one permit to each seller at a given price τ_D such that $\pi^s_D - \tau_D > \frac{2}{n}t$,[25] with the following conditions: If both sellers accept, then each seller receives one permit and pays τ_D to the government, earning a net profit of $\pi^s_D - \tau_D$. If both sellers reject, then each seller receives no permit, pays zero to the government, and the permits are auctioned directly to consumers as in setting CA. If seller i accepts and seller j rejects, then seller i receives both permits paying τ_D, thus becomes a two-bundle monopolist.[26] The normal form for this game is

		Seller B	
		accept	reject
Seller A	accept	$\pi^s_D - \tau_D, \pi^s_D - \tau_D$	$\pi^s_{M2} - \tau_D, 0$
	reject	$0, \pi^s_{M2} - \tau_D$	$\frac{2}{n}t, \frac{2}{n}t$

Since $\pi^s_D - \tau_D > \frac{2}{n}t$, for each seller the action accept strictly dominates reject in this game, and in the equilibrium (accept, accept) each seller is better off relative to CA. The precise value of τ_D determines the profit distribution among sellers and government.

Likewise, the government may allocate both permits to a same seller by requiring the designated monopolist to pay τ_{M2} such that $\pi^s_{M2} - \tau_{M2} > \frac{2}{n}t$, so the monopolist's net profit is not smaller than her profit under CA.

Hence, through a fixed price the government can implement the duopoly allocation or the monopoly allocation without being constrained by the allocation rule and the payments of the VCG auction, and determine the profits left to sellers by suitably choosing τ_D or τ_{M2}.[27] In next subsection we compare the two allocations.

3.2.3. Monopoly vs. duopoly in setting SA

In this subsection we compare the duopoly allocation with the monopoly allocation in setting SA and we conclude that monopoly is less appealing than duopoly.

Proposition 3 compares the sellers' total gross profits under the two allocations – sellers' net profits are determined by their payments to win the permits at stage 1. Here we investigate social welfare and consumer surplus. Under duopoly, let $\bar{\theta}^s_A$ denote the highest consumer type who visits seller A, that is $\bar{\theta}^s_A$ is the supremum of the set Θ^s_A; then (7) implies $\bar{\theta}^s_A = \frac{1}{t}(1 - p^s_{M1})$ when $\rho < \frac{1}{2} + \frac{2^n-1}{2n}$, but $\bar{\theta}^s_A = \frac{1}{2}$ when $\rho \geq \frac{1}{2} + \frac{2^n-1}{2n}$; furthermore, $1 - (1 - G(\bar{\theta}^s_A))^n$ is the probability that bundle A is sold. In such case the expected social welfare is $\int_0^{\bar{\theta}^s_A} (v_A(\theta) - c)\frac{g(\theta)}{G(\bar{\theta}^s_A)}d\theta$. This yields (12) below, in which the factor 2 takes into account the social welfare generated by the sale of bundle B. About consumer surplus, (8) reveals that CS^s_D is given by (13), after taking into account the sale of bundle B and that there are n consumers.

Similar arguments apply when seller A is a two-bundle monopolist. Given the optimal price p^s_{M2}, the set of consumer types visiting seller A is $[0, \theta^s_{M2}]$ with $\theta^s_{M2} = \frac{1}{t}(1 - p^s_{M2})$ and the expected number of bundles A sold is $2 - 2(1 - G(\theta^s_{M2}))^n - nG(\theta^s_{M2})(1 - G(\theta^s_{M2}))^{n-1}$; this yields (14). About consumer surplus, a slight modification of the argument which delivers (9) reveals that for each type $\theta \in [0, \theta^s_{M2}]$, the probability γ^s_{M2} to win a bundle A is $2\frac{1-(1-G(\theta^s_{M2}))^n}{nG(\theta^s_{M2})} - (1 - G(\theta^s_{M2}))^{n-1}$,[28] and (15) follows.

Corollary 1. *(i) When each seller offers one bundle under SA, under the equilibrium described by Proposition 2, social welfare W^s_D and consumer surplus CS^s_D are as follows:*

$$W^s_D = 2\int_0^{\bar{\theta}^s_A} (v_A(\theta) - c)\frac{g(\theta)}{G(\bar{\theta}^s_A)}d\theta \left[1 - (1 - G(\bar{\theta}^s_A))^n\right] \tag{12}$$

$$CS^s_D = 2n\int_0^{\bar{\theta}^s_A} (v_A(\theta) - p^s)\gamma_A g(\theta)d\theta \tag{13}$$

(ii) When seller A offers two bundles and seller B is inactive, social welfare W^s_{M2} and consumer surplus CS^s_{M2} are as follows:

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[25] Proposition 5(iii) below establishes that $\pi^s_D > \frac{2}{n}t$. Hence, there exists $\tau_D > 0$ such that $\pi^s_D - \tau_D > \frac{2}{n}t$.

[26] This mechanism is similar in spirit to Kamien et al. (1992), who solve the problem that an innovating firm faces in optimally allocating a cost-reducing innovation among competing firms.

[27] Once an allocation of permits has been determined, for instance one permit to each seller, the sellers' payments to the government do not affect downstream competition between sellers for consumers, hence do not affect consumer surplus nor social welfare.

[28] Given that seller A offers two bundles, for each consumer visiting seller A the probability to win a bundle is determined by multiplying by 2 the probabilities in (9) for the cases in which at least one other consumer visits seller A, but the probability remains the same for the case in which if no other consumer visits seller A, an event with probability $(1 - G(\theta^s_{M2}))^{n-1}$.

$$W^s_{M2} = \int_0^{\theta^s_{M2}} (v_A(\theta) - c)\frac{g(\theta)}{G(\theta^s_{M2})}d\theta\left(2 - 2(1 - G(\theta^s_{M2}))^n - nG(\theta^s_{M2})(1 - G(\theta^s_{M2}))^{n-1}\right) \tag{14}$$

$$CS^s_{M2} = n\int_0^{\theta^s_{M2}} (v_A(\theta) - p^s_{M2})\gamma^s_{M2} g(\theta)d\theta \tag{15}$$

The comparison between duopoly and monopoly is straightforward when ρ is large, but less so when ρ is small. Precisely, if ρ is large then t is about 0 and each consumer attaches value about 1 to either car. Then under monopoly of seller A, both bundles are offered at a price only slightly higher than $1 - t$ (where $1 - t$ is the lowest possible valuation for bundle A), which makes the probability of sale of both bundles nearly 1, so that the expected social welfare is very close to its maximum value. However, consumers earn about zero surplus, as mentioned in Section 3.2.2, whereas under duopoly consumers earn a positive surplus because each seller charges a price bounded away from 1 – see (11). But under duopoly social welfare is lower than under monopoly. Precisely, we just remarked that monopoly leads to a nearly efficient allocation when t is about zero. Under duopoly, we know that bundle i may be allocated inefficiently among the consumers visiting seller i, but this is negligible since when t is close to zero each consumer type has about the same value for bundle i (that is, value 1). However, it is not negligible that each consumer may have θ smaller than $\frac{1}{2}$ (say) and hence visit seller A, an event with probability $\frac{1}{2^n}$. In this case no consumer visits seller B and bundle B is unsold, which is inefficient (likewise, bundle A goes unsold with probability $\frac{1}{2^n}$). Hence, $W^s_D < W^s_{M2}$ and $CS^s_D > C^s_{M2}$ when ρ is large. For smaller ρ the comparison is less immediate because of analytical difficulties linked to determining p^s_{M2}, but at least for some cases we can prove that the previous inequalities are reversed.

Proposition 4. *Consider setting SA.*
(i) If ρ is large, then $W^s_D < W^s_{M2}$ and $CS^s_D > CS^s_{M2}$;
(ii) If $\rho = 1$ and $n \leq 10$, then $W^s_D > W^s_{M2}$ and $CS^s_D < CS^s_{M2}$.

In particular, Proposition 4(ii) establishes that in some cases consumer surplus is higher with monopoly than with duopoly, and here we illustrate why. We consider $\rho = 1$, which implies $p^s = p^s_{M1}$ by Proposition 2. Then the comparison between CS^s_D and CS^s_{M2} boils down to comparing local monopolies against a two-bundle monopoly by seller A. A consumer with θ close to 0 has a preference for bundle A and is much better off under monopoly of seller A than with local monopolies because two bundles A are offered instead of one, and moreover $p^s_{M2} < p^s_{M1}$ holds, so that his payment in case he wins a bundle A is smaller. Although the inequality $p^s_{M2} < p^s_{M1}$ implies that more consumer types visit seller A under monopoly than under local monopolies, such negative effect turns out to be dominated by the previous ones, and as a result each type θ close to 0 has a significantly higher utility under two-bundle monopoly. Conversely, types with θ close to 1 are worse off with monopoly of seller A because their preferred bundle B is not offered at all, and the price of bundle A is too high for them to visit seller A. Hence, they are excluded from the market and have zero utility. However, the utility gains of types with θ close to 0 are large enough to compensate for the losses of types located near 1 and to make CS^s_{M2} greater than CS^s_D.

In order to see one example of the effect of monopoly on the utility distribution among consumers, suppose that $c = \frac{1}{4}$, $t = \frac{3}{4}$, and $n = 10$. Then $p^s_{M1} = 0.8401$,[29] hence $\Theta^s_A = [0, 0.2132]$, $\Theta^s_B = [0.7868, 1]$ and $\Theta^s_N = (0.2132, 0.7868)$, that is the utility of type θ is zero for each $\theta \in (0.2132, 0.7868)$ as the maximal valuation of type θ is below p^s_{M1}. We can use (8) to derive the utility of each consumer type under duopoly: $u^s_i(\theta)$ for $i = A, B$: see the thick curve in Fig. 3.

Under monopoly of seller A, the utility of type θ is equal to $(1 - \frac{3}{4}\theta - p^s_{M2})\gamma^s_{M2}$ if $\theta \in [0, \theta^s_{M2}]$, is equal to zero if $\theta > \theta^s_{M2}$. We find that $p^s_{M2} = 0.7936$, hence $\theta^s_{M2} = 0.2752$, $\gamma^s_{M2} = 0.6425$ and the utility of each type θ is described by the thin curve in Fig. 3. The two curves illustrate how, for the considered parameter values, monopoly benefits consumers with small θ and damages consumers with high θ.

Although in some cases the monopoly allocation is superior to the duopoly allocation from the point of view of average consumer surplus or of social welfare, we do not see the monopoly allocation as an attractive alternative: Monopoly leaves one seller out of business (without a good reason to select seller A or B), and although it produces a higher consumer surplus on average, it leaves a significant set of types significantly worse off with respect to duopoly. That is its consequences on the distribution of utility across consumers are very asymmetric and undesirable, as they are skewed in favor of (against) types who have a preference for the cars offered by the monopolist (offered by the seller who is out of business). This makes us view negatively the monopoly allocation of permits.

3.3. *Welfare evaluation: CA vs. SA with duopoly*

In this subsection we consider SA with duopoly – in which the duopoly allocation has been implemented by a fixed price mechanism – and perform a comparison between CA and SA with duopoly in terms of social welfare, consumer surplus, seller

[29] From $c = \frac{1}{4}$ and $t = \frac{3}{4}$ it follows that $\rho = 1$, and $\rho = 1$ makes it possible to write p^s_{M1} in closed form: $p^s_{M1} = c + t\frac{1}{(n+1)^{1/n}}$.

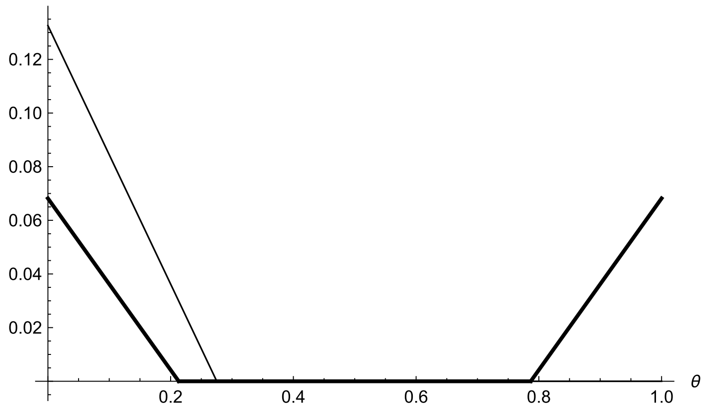

Fig. 3. The thick curve is the graph of the utility of type θ under SA with duopoly. The thin curve is the utility of type θ under SA with monopoly of seller A. Both curves are plotted by setting $c = \frac{1}{4}$, $t = \frac{3}{4}$, and $n = 10$.

profit. In this comparison we restrict to the case in which (c, t) are such that $\frac{1-c}{t} \geq 1$, that is $\rho \geq 1$, as the case of $\rho < 1$ involves more complex arguments but produces different results only for small values of n. When $\rho \geq 1$, by Proposition 1*(i)* the market is fully covered under CA.

We show that there exists a price $\tau_D > 0$ such that the fixed price allocation mechanism described near the end of Section 3.2.2 leads to an outcome for SA which sellers and consumers on average prefer to CA (except when $n = 4$ and ρ is small), although social welfare and government revenue are lower.

In order to derive social welfare W^c, consumer surplus CS^c and seller profit PS^c under CA, we notice that the two permits are won by the two consumers with the top two maximal values. Precisely, the maximal value of a consumer with type θ is $v(\theta)$ from (2) and the range of values for this function v is the interval $[1 - \frac{t}{2}, 1]$. We use F^c to denote the c.d.f. for each consumer's maximal value, which is derived from the c.d.f. G as follows:

$$F^c(z) = G\left(1 - \frac{1-z}{t}\right) - G\left(\frac{1-z}{t}\right) \qquad \text{for each } z \in \left[1 - \frac{t}{2}, 1\right] \tag{16}$$

Denoting with $F_j^{c,(n)}$ the c.d.f. of the j-th largest value out of n independent draws from the c.d.f. F^c, for $j = 1, 2$, we can write W^c as in Corollary 2 below.

For consumer surplus, we consider a particular consumer, say consumer 1, with maximal value z and let y denote the second highest maximal value among the other consumers. Then consumer 1 wins a permit if and only if $z > y$, and in such case his utility is $z - y$ because the consumer with value y bids $y - p^c$ in the auction and consumer 1's utility is given by his value z minus his payment $y - p^c$ in the auction, minus the price p^c of the car he will buy; hence, the consumer's utility is independent of p^c. Since the c.d.f. of y is $F_2^{c,(n-1)}$, the expected utility $u^c(z)$ is

$$u^c(z) = \int_{1-t/2}^{z} (z - y) d F_2^{c,(n-1)}(y) \tag{17}$$

which is the utility of a bidder with type z in a third-price auction with two objects. Taking the expectation of $u^c(z)$ according to F^c and multiplying by the number n of consumers yields CS^c in Corollary 2.

Regarding seller profit, it is immediate that PS^c is equal to $p^c - c$ times the total demand – which is 2 – with $p^c = \frac{2}{n}t$ from Proposition 1*(i)*. Then we obtain PS^c in Corollary 2.

Corollary 2. *Given the equilibrium under CA described by Proposition 1(i), the social welfare W^c, the consumer surplus CS^c, the seller profit PS^c are as follows:*

$$W^c = \int_{1-t/2}^{1} (z - c) d\left(F_1^{c,(n)}(z) + F_2^{c,(n)}(z)\right)$$

$$CS^c = n \int_{1-t/2}^{1} [1 - F^c(z)] F_2^{c,(n-1)}(z) dz = \frac{3}{2(n+1)} t$$

$$PS^c = \frac{4}{n} t$$

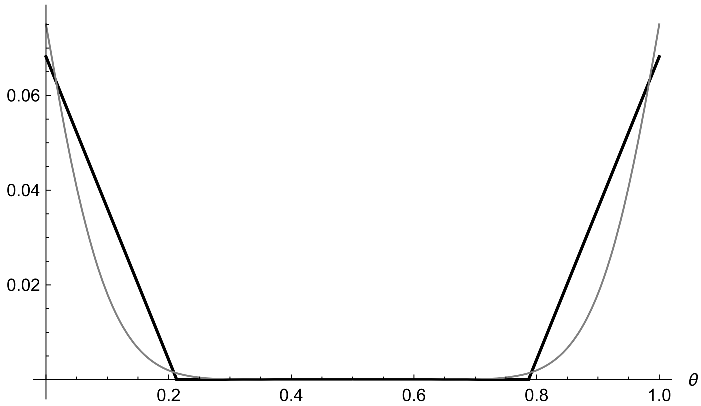

Fig. 4. The thick curve is the graph of the utility of type θ under SA. The thin curve is the utility of type θ under CA. Both curves are plotted by setting $c = \frac{1}{4}$, $t = \frac{3}{4}$, and $n = 10$.

From Corollaries 1(i), 2 and Proposition 2 we obtain the following result, in which PS_D^s is the sellers' gross total profit in SA, that is the total profit before each seller's payment of τ_D to the government to obtain a permit.

Proposition 5. *In the comparison between CA and SA with duopoly we find that*

(i) $W^c > W_D^s$ *for each* $n \geq 4$ *and* $\rho \geq 1$.
(ii) $CS_D^s > CS^c$ *for each* $n \geq 5$ *and* $\rho \geq 1$. *When* $n = 4$, *the inequality* $CS_D^s > CS^c$ *holds if and only if* $\rho > 1.156$.
(iii) $PS_D^s \equiv 2\pi_D^s > PS^c$ *for each* $n \geq 4$ *and* $\rho \geq 1$.

The social welfare comparison It is immediate that social welfare W^c under CA is greater than W_D^s because the final allocation of permits under CA maximizes social welfare: cars and permits are received by the two consumers with the two highest maximal values, and this is the socially optimal allocation as we illustrated in Section 3.1.2. Conversely, we have illustrated at the end of Section 3.2.1 that some inefficiencies occur under SA. Therefore SA generates a social welfare lower than the social welfare in CA.

The consumer surplus comparison Proposition 5(ii) establishes that $CS_D^s > CS^c$, except when $n = 4$ and ρ is small. The inequality $CS_D^s > CS^c$ may be somewhat surprising, especially when SA leads to local monopolies with limited consumer participation and a relatively high p^s. In order to see how this may lead to $CS_D^s > CS^c$, consider the parameters we have used in Section 3.2.3 to derive Fig. 3, that is $c = \frac{1}{4}$, $t = \frac{3}{4}$, $n = 10$. Then from (8) we derive the expected utility of each type θ under SA, represented by the thick curve in Fig. 4.

Using (17) we derive the expected utility of each type θ under CA, which yields the thin curve in Fig. 4. A significant difference with respect to SA is that under CA, each consumer type participates in the auction and earns positive utility, but types close to $\theta = \frac{1}{2}$ have a relatively low (maximal) value, hence very low probability to win and very low utility. Conversely, in setting SA there is a sizable set Θ_N^s of types who are excluded from the market, but the types who are not excluded have all the same probability to win a bundle, which makes a consumer's utility grow quickly as his type gets close to 0 or to 1. As a consequence, a substantial portion of types with relatively medium-high valuations receive higher utility under SA, and this more than compensates on average for the zero utility of types close to $\frac{1}{2}$. As a result, consumer surplus is greater in SA.

Although $\rho = 1$ in the particular case above, we prove in Appendix C that $CS_D^s > CS^c$ holds for each $\rho > 1$ if $n \geq 5$, as an increase in ρ does not change the expression of CS^c, but makes sellers choose prices which determine greater consumer participation when local monopolies arise, and induces tougher competition between sellers when the market is fully covered.

The seller profit comparison Proposition 5(iii), which considers the sellers' gross total profits, establishes that $PS^s > PS^g$. In order to explain this inequality, recall that under CA the burden effect toughens competition, and as a result each seller's profit is $\frac{2}{n}t$, which is small especially when n is large. Conversely, under SA there is no burden effect and seller A (say) is certain to sell bundle A as long as she receives the visit of at least one consumer. Specifically, consider the case in which sellers act as local monopolists. Given that seller B sets $p_B = p_{M1}^s > 1 - \frac{t}{2}$, if seller A chooses $p_A > 1 - \frac{t}{2}$ then her profit is $(p_A - c)[1 - (1 - \frac{1-p_A}{t})^n]$, and pick in particular $p_A = 1 - \frac{1}{4}t$. This p_A is not necessarily optimal but it shows that π_{M1}^s is no less than $(\rho - \frac{1}{4})t(1 - (\frac{3}{4})^n)$. Since the latter expression is increasing in ρ, we suppose here that ρ is equal to 1, the smallest value among those we consider. When $\rho = 1$ we have that $c = 1 - t$, hence $p_A = 1 - \frac{1}{4}t$ can be written as $p_A = c + \frac{3}{4}t$; hence, seller A's profit margin is $\frac{3}{4}t$, greater than the profit margin in CA, $\frac{2}{n}t$, for each $n \geq 4$. The profit of seller A under SA with $p_A = 1 - \frac{1}{4}t$ is $\frac{3}{4}t(1 - (\frac{3}{4})^n)$ and is greater than $\frac{2}{n}t$ for each $n \geq 4$. This suffices to conclude that sellers' total gross profits are greater in SA than in CA for each $\rho \quad 1$,[30] but in fact for each $p_A < 1$, seller A's probability of sale is close to 1 when n is large, hence seller A's profit tends to 1 as n tends to $+\infty$. This is in strong contrast with the outcome under CA, where a large n makes competition tough because of the burden effect. Under SA, a large n makes more likely that local monopolies arise and makes each seller better off since a monopolist facing a high number of consumers is almost certain to sell her bundle even upon setting a high price.

A fixed price mechanism which is superior to CA for consumers and sellers By relying on Proposition 5, we can identify a fixed price mechanism leading to an outcome which both sellers and consumers (on average) prefer to the outcome of CA:

Proposition 6. *Suppose that $\rho \geq 1$, $n \geq 5$, and consider a fixed price mechanism in which the government offers a permit to each seller, with $\tau_D = \pi_D^s - \frac{2}{n}t - \epsilon$ and $\epsilon > 0$. Then each seller decides to buy one permit and ends up with a net profit by ϵ greater than her profit under CA. Also consumer surplus is greater than in CA, but social welfare and government revenue are lower than under CA.*

The precise value of ϵ in the mechanism described by Proposition 6 determines how much profit the government leaves to each seller above the profit level under CA. Conversely, CS_D^s does not depend on ϵ but is greater than CS^c by Proposition 5. In particular, CS_D^s is greater than $\frac{\ln(0.9\rho n)}{n}$ when n is large.[31] Hence, in relative terms CS_D^s is much greater than $CS^c = \frac{3}{2(n+1)}$ when n is large in the sense that $\lim_{n\to+\infty} \frac{CS_D^s}{CS^c} = +\infty$. However, we know from Proposition 5(i) that social welfare is lower under SA. Since consumers and sellers are better off under SA than under CA, it follows that the government revenue under SA, which is equal to $2\tau_D$, is lower than in CA.

4. An extension with uncorrelated values for each consumer

The results in Section 3 are obtained under the assumption that each consumer's values for car A and car B are perfectly negatively correlated: see (1). In this section we remove this restriction and explore a random utility model, from Perloff and Salop (1985), in which a consumer's utilities from different brands are i.i.d. random variables.

In detail, we suppose that a consumer's value v_A for car A is uniformly distributed over an interval $[\underline{v}, \bar{v}]$ with $0 < \underline{v} < \bar{v}$, and likewise his value v_B for car B has uniform distribution on $[\underline{v}, \bar{v}]$, but v_A and v_B are stochastically independent.[32] We use F, f to denote, respectively, the c.d.f. and the density of the uniform distribution on $[\underline{v}, \bar{v}]$. Moreover, also the values across different consumers are stochastically independent. In order to simplify notation we suppose that $\bar{v} = \underline{v} + 1$ and use $V = [\underline{v}, \underline{v}+1] \times [\underline{v}, \underline{v}+1]$ to denote the set of possible pairs (v_A, v_B). The purpose of this section is to analyze and compare settings CA and SA under the above assumptions. The results we obtain are qualitatively similar to those of Section 3.

4.1. Analysis of CA

The timing for CA is like in Section 3.1, and arguing like in Section 3.1 we deduce that for a consumer with values v_A, v_B, the value from obtaining a license in the government auction is $\max\{v_A - p_A, v_B - p_B\}$. Hence, this consumer participates in the auction if and only if $\max\{v_A - p_A, v_B - p_B\} \geq 0$, and in such case he bids $\max\{v_A - p_A, v_B - p_B\}$. We focus on parameters such that the equilibrium prices are less than $\underline{v}$, so that each consumer type participates in the auction (see Proposition 7 below), hence, for prices close to the equilibrium prices, the set V of consumer types partitions into two regions V_A, V_B as follows:

$$V_A = \{(v_A, v_B) \in V : v_B - p_B \leq v_A - p_A\} \qquad V_B = \{(v_A, v_B) \in V : v_B - p_B > v_A - p_A\}$$

A consumer with (v_A, v_B) in V_A bids $v_A - p_A$ in the auction, and buys a car from seller A if he wins a license. Likewise, a consumer with values in V_B bids $v_B - p_B$ in the auction, and buys a car from seller B if he wins a license. We suppose without loss of generality that $p_A \geq p_B$ and that $p_A \leq \min\{\bar{v}, p_B + \bar{v} - \underline{v}\}$. The latter inequality is relevant because $V_A = \emptyset$ if $p_A > \bar{v}$ or if $p_A > p_B + \bar{v} - \underline{v}$; hence, in such case no winner of the auction buys car A. Fig. 5 represents the regions V_A, V_B for a case with $p_B < \underline{v}$ and $p_B < p_A < p_B + \bar{v} - \underline{v}$.

We now derive the demand for seller A, denoted $D_A(p_A, p_B)$.[33] To this purpose we consider a consumer, say consumer 1, with value v_A for car A and determine the probability that he wins a license and then buys a car from seller A. Consumer 1 buys a car A upon winning a permit if and only if his value v_B for car B is less than $v_A - p_A + p_B$, which occurs with probability $F(v_A - p_A + p_B)$. Given $v_B < v_A - p_A + p_B$, consider another consumer, say consumer 2, with values v'_A, v'_B, and notice that consumer 1 bids more than consumer 2 in the auction if and only if $v'_A < v_A$ and $v'_B < v_A - p_A + p_B$; these conditions are both satisfied with probability $F_A(v_A) = F(v_A - p_A + p_B)F(v_A)$ (this is analogous to the function λ introduced in (4)). Hence, $(n-1)F_A^{n-2}(v_A) - (n-2)F_A^{n-1}(v_A)$ is the probability that the second highest bid of consumers $2, \ldots, n$ is less than 1's bid, and $\int_{\underline{v}}^{\bar{v}} f(v_A)F(v_A - p_A + p_B)\left((n-1)F_A^{n-2}(v_A) - (n-2)F_A^{n-1}(v_A)\right) dv_A$ is the ex ante probability that consumer 1 wins a license and buys a car from seller A.

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[30] We are focusing here on the case of local monopolies, but we remark that $PS^s > PS^c$ holds also when the market is fully covered.

[31] Precisely, for each $\alpha \in (0, \rho)$, we can prove that CS_D^s is greater than $\frac{\ln(\alpha n)}{n}$ when n is large.

[32] In this section v_A and v_B are random variables, not functions of θ as in Section 3.

[33] Given $D_A(p_A, p_B)$, we can obtain $D_B(p_B, p_A)$ as $2 - D_A(p_A, p_B)$.

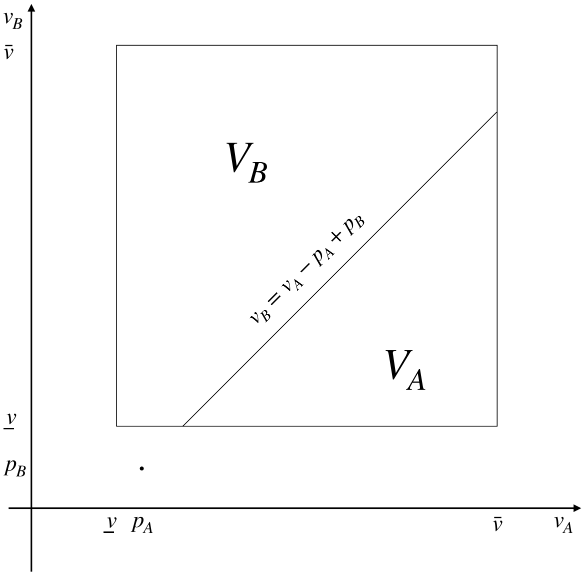

Fig. 5. The partition of the set of types V into V_A and V_B.

Finally, $D_A(p_A, p_B)$ is the sum of the analogous probabilities over the n consumers:

$$D_A(p_A, p_B) = n \int_{\underline{v}}^{\bar{v}} f(v_A)F(v_A - p_A + p_B)\left((n-1)F_A^{n-2}(v_A) - (n-2)F_A^{n-1}(v_A)\right) dv_A \tag{18}$$

Next proposition identifies the circumstances in which the unique symmetric equilibrium is such that the market is fully covered:

Proposition 7. *In setting CA, suppose that $\frac{(2n-1)(2n-3)}{2n^3-2n^2-n} \leq \underline{v} - c$. Then the unique symmetric equilibrium is $(p_A^c, p_B^c) = (p^c, p^c)$ with $p^c = c + \frac{(2n-1)(2n-3)}{2n^3-2n^2-n}$.*

Since $p_A^c = p_B^c \leq \underline{v}$, in this equilibrium the permits are allocated efficiently, that is to the two bidders with the two highest maximal valuations (here, the maximal valuation for a bidder is $\max\{v_A, v_B\}$). In the inequality $\frac{(2n-1)(2n-3)}{2n^3-2n^2-n} \leq \underline{v} - c$, the left hand side is decreasing in n; hence, the inequality is more likely to be satisfied the larger is n and the equilibrium price p^c is decreasing in n, just as in the Hotelling setting of Proposition 1, and for the same reason. That is, the burden effect applies under CA also in the current environment and it makes each seller's demand very elastic when n is large, so that the Bertrand outcome emerges when $n \to +\infty$. An increase in n makes competition in the auction more intense and increases the government revenue.

4.2. Analysis of SA

The timing for SA is like in Section 3.2, and we analyze the scenario in which each seller is allocated one permit.[34] Hence, each seller offers one bundle. Following the notation introduced in Section 3.2, we let μ_i denote the probability that a consumer visits seller i, and we let η denote the probability that a consumer visits no seller. We begin with the case of market partially covered and we assume, without loss of generality, that $p_A \geq p_B > \underline{v}$.

It is immediate that any consumer with values $v_A \geq p_A$ and $v_B < p_B$ visits seller A because the consumer earns negative utility from visiting seller B and non-negative utility from visiting seller A; this is the set of types in the rectangle in Fig. 6 with a dashed horizontal edge. Likewise, any consumer with values $v_A < p_A$ and $v_B \geq p_B$ visits seller B; any consumer with values $v_A < p_A$ and $v_B < p_B$ visits neither seller. To understand the behavior of a consumer with values $v_A \geq p_A$ and $v_B \geq p_B$ we need to identify his most preferred alternative, which depends on the (endogenous) choices made by the other consumers. Below we show that there is a linear and strictly increasing function $h : [p_A, \bar{v}] \to [p_B, \bar{v}]$, with $h(p_A) = p_B$, such that any consumer with values $v_A \geq p_A$ and $v_B = h(v_A)$ is indifferent between the two sellers, whereas if $v_A \geq p_A$ and $v_B < h(v_A)$, then the consumer visits seller A; if $v_A \geq p_A$ and $v_B > h(v_A)$, then the consumer visits seller B. Therefore, we partition the set V of types as follows: $V_A = \{(v_A, v_B) \in V : v_A \geq p_A \text{ and } v_B \leq h(v_A)\}$ denotes the set of types who visit seller A; $V_B = \{(v_A, v_B) \in V : v_A < p_A \text{ and } v_B \geq p_B, \text{ or } v_A \geq p_A \text{ and } v_B > h(v_A)\}$ denotes the set

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[34] The scenario where one seller is allocated both permits and the proofs of the results in this part are available in Landi and Menicucci (2023a).

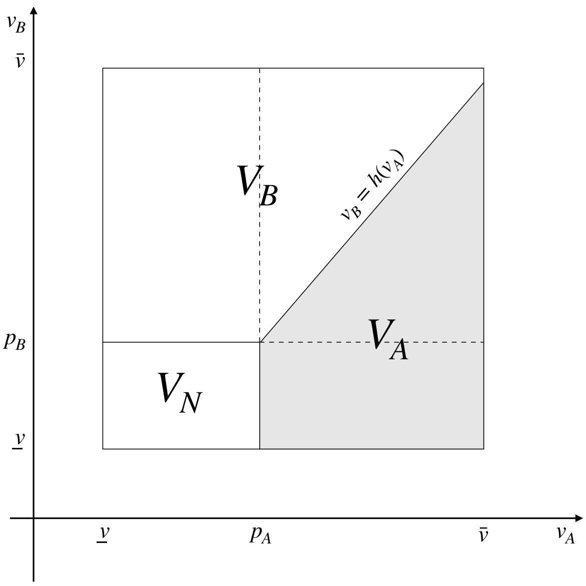

Fig. 6. The partition of the set V of types into V_A, V_B, V_N. The measure of the shaded region is μ_A.

of types visiting seller B[35]; $V_N = \{(v_A, v_B) \in V : v_A < p_A \text{ and } v_B < p_B\}$ denotes the set of types visiting neither seller: See Fig. 6, in which V_A is the shaded region.

Since a consumer type with values such that $v_B = h(v_A)$ is indifferent between visiting either seller, it follows that

$$(v_A - p_A)\frac{1-(1-\mu_A)^n}{n\mu_A} = (h(v_A) - p_B)\frac{1-(1-\mu_B)^n}{n\mu_B} \tag{19}$$

where $\mu_A = \int_{p_A}^{\bar{v}} \int_{\underline{v}}^{h(v_A)} f(v_B) f(v_A) dv_B dv_A = \int_{p_A}^{\bar{v}} F(h(v_A)) f(v_A) dv_A$ is the measure of V_A, that is the measure of the shaded region in Fig. 6, and μ_B is the measure of V_B. Defining $\eta = F(p_A)F(p_B)$ as the measure of V_N, we find that $\mu_B = 1 - \mu_A - \eta$ and from (19) we get that h is linear and is given by

$$h(v_A) = p_B + m(\mu_A, \eta) \cdot (v_A - p_A) \tag{20}$$

where $m(\mu_A, \eta) = \frac{1-(1-\mu_A)^n}{1-(\mu_A+\eta)^n} \frac{1-\mu_A-\eta}{\mu_A}$. Therefore μ_A solves the equation

$$\mu_A - \int_{p_A}^{\bar{v}} F\left(p_B + m(\mu_A, \eta) \cdot (x - p_A)\right) f(x) dx = 0 \tag{21}$$

Since the sale of bundle A occurs when at least one consumer visits seller A, an event that has probability $1 - (1 - \mu_A)^n$, the expected profit for seller A is given by

$$\Pi_A = (1 - (1 - \mu_A)^n)(p_A - c) \tag{22}$$

The first order necessary condition for a symmetric pure strategy equilibrium $(p_A, p_B) = (p^s, p^s)$ in the sellers' game is that p^s solves $\frac{d\Pi_A}{dp_A} = 1 - (1-\mu_A)^n + n(1-\mu_A)^{n-1} \frac{d\mu_A}{dp_A}(p_A - c) = 0$ at $p_A = p_B = p^s$,[36] that is

$$\tilde{F}^n(p) - 2\left[1 - \tilde{F}^n(p)\right] \frac{1-F(p)}{1+F(p)} \Omega(p) + n(p-c)\tilde{F}^{n-1}(p) \left[1 + F(p)\frac{1-F(p)}{1+F(p)}\Omega(p)\right] = 1 \tag{23}$$

where $F(p) = p - \underline{v}$, $\tilde{F}(p) = \frac{1+F(p)^2}{2}$, and $\Omega(p) = \frac{1-\tilde{F}^n(p) - n(1-\tilde{F}(p))\tilde{F}^{n-1}(p)}{1-\tilde{F}^n(p)}$. We remark that a sufficient condition for (23) to have a solution in the interval $(\underline{v}, \underline{v}+1)$ is $\underline{v} - c < \frac{3}{2n}(2^n - 1) - 1$.

[35] Precisely, the inequalities $v_A < p_A$ and $v_B \geq p_B$ in the definition of V_B identify the rectangle in Fig. 6 with a vertical dashed edge. The inequalities $v_A \geq p_A$ and $v_B > h(v_A)$ identify the trapezoid to the right of such rectangle. Conversely, in order to identify the set V_A a single pair of inequalities suffices, $v_A \geq p_A$ and $v_B \leq h(v_A)$, because the function h is defined over the interval $[p_A, \bar{v}]$ and then $v_B \leq h(v_A)$ catches both the rectangle with a dashed horizontal edge and the triangle above it.

[36] This is the right derivative of Π_A. However, in Landi and Menicucci (2023a) we show that Π_A is differentiable at $p_A = p_B = p$.

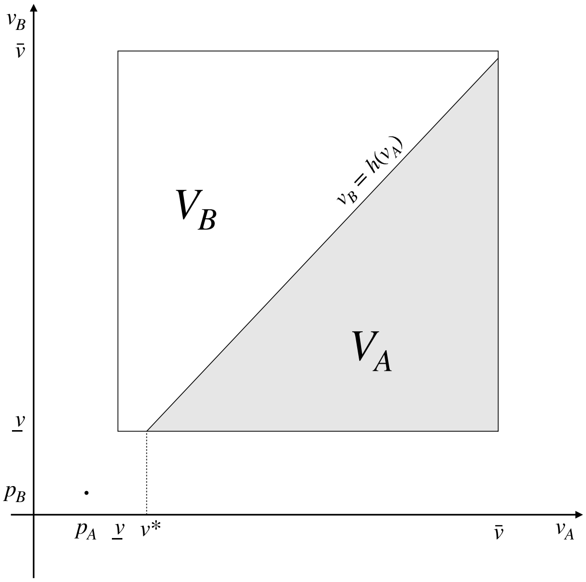

Fig. 7. The partition of the set V of types into V_A, V_B when the market is fully covered.

Table 1
The equilibrium price p^s under SA for $n \in \mathcal{N}$, $\underline{v} \in \mathcal{V}$. Values are rounded up to the fourth decimal.

$n/\underline{v}$	1	5	10	50
4	1.4284	4.9423	9.1731	43.0192
10	1.6921	5.5637	10.4891	50.2695
25	1.8423	5.7955	10.7698	50.7045
100	1.9470	5.9358	10.9296	50.9141

We now consider the case in which the market is fully covered. We assume again $p_A \geq p_B$, but now $\underline{v} \geq p_B$ holds. Hence, $V_N = \emptyset$, $\eta = 0$ and the line separating V_A from V_B has the following equation, still derived from (19), with $\mu_B = 1 - \mu_A$:

$$h(v_A) = p_B + m(\mu_A)(v_A - p_A) = p_B + \frac{1-\mu_A}{\mu_A}\frac{1-(1-\mu_A)^n}{1-\mu_A^n}(v_A - p_A)$$

There exists a unique $v^* \geq \max\{\underline{v}, p_A\}$ such that $h(v^*) = \underline{v}$ (see Fig. 7), and it is $v^* = p_A + \frac{1}{m(\mu_A)}(\underline{v} - p_B)$. As a result, μ_A solves the equation

$$\mu_A - \int_{v^*}^{\bar{v}} F\left(p_B + \frac{1-\mu_A}{\mu_A}\frac{1-(1-\mu_A)^n}{1-\mu_A^n}(x - p_A)\right) f(x)dx = 0 \tag{24}$$

From the first order condition for profit maximization we now obtain

$$p^s = \frac{nc + (2^n - 1 - n)\left(2\underline{v} + 1\right) + 2^{n-1} - 1/2}{2(2^n - 1) - n} \tag{25}$$

The condition that p^s in (25) is not larger than $\underline{v}$ is equivalent to $\underline{v} - c \geq \frac{3}{2n}(2^n - 1) - 1$. Proposition 8 summarizes our results about determining a candidate equilibrium price p^s. However, in order to solve (23) when $\underline{v} - c < \frac{3}{2n}(2^n - 1) - 1$ and to prove that $(p_A, p_B) = (p^s, p^s)$ is indeed an equilibrium, we need to fix the values of n, $\underline{v}$, c and employ numerical methods. To this end we consider $n \in \mathcal{N} = \{4, 10, 25, 100\}$, $\underline{v} \in \mathcal{V} = \{1, 5, 10, 50\}$, $c = 0$ and Proposition 8 establishes that p^s in Table 1 is an equilibrium price.

Proposition 8. *Suppose values are uniformly distributed in $[\underline{v}, \underline{v} + 1]$. If a symmetric equilibrium $(p_A, p_B) = (p^s, p^s)$ exists when $\underline{v} - c < \frac{3}{2n}(2^n - 1) - 1$, then p^s is a solution to (23). In this case, $p^s > \underline{v}$ and the market is not fully covered. Otherwise, p^s is given by (25) when $\underline{v} - c \geq \frac{3}{2n}(2^n - 1) - 1$. In this case $p^s \leq \underline{v}$ and the market is fully covered.*

When, in particular, $n \in \mathcal{N}$, $\underline{v} \in \mathcal{V}$, $c = 0$, then p^s from Table 1 is such that $(p_A, p_B) = (p^s, p^s)$ is an equilibrium.

We note that the condition $\underline{v} - c < \frac{3}{2n}(2^n - 1) - 1$ for an equilibrium with partial coverage is more likely to be satisfied the greater is n and indeed it is violated when $n = 4$ and $\underline{v} = 5, 10, 50$ but it is satisfied if $n \geq 10$, for each $\underline{v} \in \mathcal{V}$. This result is consistent with Proposition 2.

The equilibrium price increases with n, as an increase in the number of consumers softens the competition between sellers. Precisely, with an increase in n it becomes more likely that each seller will be visited by at least one consumer with a high value for the seller's bundle. This makes each seller's demand less price sensitive, which in turn leads each seller to charge a bigger price.

As in Section 3.2.2, the government can use a fixed price mechanism to implement a monopoly allocation of permits or a duopoly allocation. We find that social welfare, consumer surplus, seller profit may be higher or lower under monopoly than under duopoly depending on the parameters, and for the parameters we consider, an increase in $\underline{v}$ tends to favor duopoly over monopoly from the point of view of consumer surplus, but the result is reversed from the point of view of social welfare and seller profit. This is similar to what happens in Section 3, if we view an increase in $\underline{v}$ – which increases consumers' values – as analogous to an increase in ρ in Section 3. We also remark that often consumers have a preference for the alternative which produces a lower social welfare and lower profits. But even in the current environment, our negative point of view about the monopoly allocation applies, as it leaves one seller inactive and even when it produces a higher consumer surplus than duopoly, it does so by increasing the utility of consumer types with high values for the cars offered by the monopolist but hurting the types who are more interested in the cars of the seller who is left out of business (with no good reason to select one seller or the other). For this reason in next subsection we compare setting CA with setting SA when each seller owns one permit and offers one bundle.

4.3. Welfare evaluation: CA vs SA with duopoly

We compare social welfare, consumer surplus and seller profit under CA and under SA when each seller is allocated one permit.

Corollary 3. *Given $c = 0$ and given the equilibrium described in Proposition 8, when each seller offers one bundle under SA, social welfare W_D^s, consumer surplus CS_D^s and seller profit PS_D^s are as follows:*

$$W_D^s = 2\frac{1-(1-\mu_A)^n}{\mu_A} \int_{\max\{\underline{v}, p^s\}}^{\overline{v}} vF(v)f(v)dv$$

$$CS_D^s = 2\frac{1-(1-\mu_A)^n}{\mu_A} \int_{\max\{\underline{v}, p^s\}}^{\overline{v}} (v-p^s)F(v)f(v)dv$$

$$PS_D^s = 2(1-(1-\mu_A)^n)p^s$$

Given $c = 0$ and given the equilibrium under CA described in Proposition 7, social welfare W^c, consumer surplus CS^c and seller profit PS^c are as follows:

$$W^c = \int_{\underline{v}}^{\overline{v}} vd\hat{F}_1^{c,(n)}(v) + \int_{\underline{v}}^{\overline{v}} vd\hat{F}_2^{c,(n)}(v)$$

$$CS^c = n\int_{\underline{v}}^{\overline{v}} (1-\hat{F}^c(v))\hat{F}_2^{c,(n-1)}(v)dv$$

$$PS^c = 2p^c$$

where $\hat{F}^c$ is the c.d.f. of each consumer's maximal value, that is $\hat{F}^c(v) = F^2(v)$, and $\hat{F}_j^{c,(n)}$ is the c.d.f. of the j-th largest value out of n independent draws from $\hat{F}^c$, for $j = 1, 2$.

Proposition 9. *In the comparison between CA and SA with duopoly we find that*

(i) $W^c > W_D^s$ for each $n \in \mathcal{N}$ and $\underline{v} \in \mathcal{V}$.
(ii) $CS_D^s > CS^c$ for each $n \in \mathcal{N}$ and $\underline{v} \in \mathcal{V}$.
(iii) $PS_D^s > PS^c$ for each $n \in \mathcal{N}$ and $\underline{v} \in \mathcal{V}$.

Proposition 9 establishes that for each parameter values we consider, under SA with the duopoly allocation, social welfare is lower, consumer surplus is higher, gross seller profit is higher than under CA. These are essentially the same results provided by Proposition 5 for Hotelling differentiation, and indeed also the sources of these results are the same.

First, it is immediate that $W^c > W_D^s$ because CA allocates permits efficiently, whereas in SA some inefficiencies may occur.

Second, regarding consumer surplus, consider the case in which the market is not fully covered under SA – which often occurs for the parameters we consider. Then there is a set of consumer types who have zero utility under SA because their maximal valuation is lower than p^s. These types earn a positive utility under CA, but the latter utility is very low because the low maximal valuation implies that the probability to win is low. Conversely, the types with maximal valuation greater than p^s have a relatively high probability to win under SA, equal to the winning probability of the types with the highest maximal valuation. This makes their utilities higher than in CA and more than compensates for the lower utility of types with low maximal valuation.

Third, regarding seller profit, we focus again on the case in which the market is not fully covered under SA and consider seller A. Given $p^s > \underline{v}$ played by seller B, by playing $p_A = \underline{v}$ seller A sells her own bundle with probability greater than $1 - \frac{1}{2^n}$, and with a profit margin of $\underline{v}$. Under CA, seller A expects to sell one car with a profit margin of $\frac{(2n-1)(2n-3)}{2n^3-2n^2-n}$, which is less than $\underline{v}$ otherwise the market would not be fully covered under CA. Hence the profit under SA is greater than under CA if n is large enough to make $1 - \frac{1}{2^n}$ close enough to 1, but in fact for the parameters we consider we find that $n \geq 4$ suffices.

By the virtue of Proposition 9, it is possible to design a suitable fixed price mechanism to allocate permits between sellers (analogous to the one described in Proposition 6), and then sellers competition results in a greater consumer surplus and seller profit (by an amount determined by the government) with respect to CA, although social welfare and the government revenue are reduced.

5. Discussion and concluding remarks

This paper offers a comparison of social welfare, profits and consumer surplus between two alternative ways of allocating an item (e.g. vehicle registration permit) that is instrumental to the use of another item (e.g. car), when the quantity of the first item is controlled. Our results show that social welfare is higher when first the government auctions the permits to consumers and then each consumer who has obtained a permit chooses the seller from whom he buys a car. However, consumers and sellers are typically better off if first the government allocates one permit to each seller – in exchange for a suitable payment – and then each seller offers one bundle to consumers. However, in this case government revenue is lower than the revenue under the mechanism mentioned above.

Our analysis is carried out with only two permits. We conjecture that the insights obtained here apply even with an arbitrary number of permits, as long as there are sufficiently many consumers. Future work may be devoted to extending our analysis to this case, and presumably this line of work could also allow for a richer set of possible outcomes in the study of permits allocation to sellers.

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Declaration of competing interest

The authors have no competing interest. We report that in July 2013, Massimiliano Landi joined a team of academics for a one-day *Engagement with academics on COE auction system* run by the Land Transport Authority in Singapore. Landi did not receive any monetary compensation for his participation and has not interacted with the Land Authority of Singapore since then.

Data availability

No data was used for the research described in the article.

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Appendix A. Proof of Proposition 1

A.1. *Symmetric equilibrium with fully covered market*

We use $x = \theta^c_A$ and $y = \theta^c_B$ as the sellers' choice variables. Therefore, $\tilde{\theta}^c = \frac{1}{2} + \frac{p_B - p_A}{2t} = \frac{x+y}{2}$ and $\lambda(\theta) = 2\theta - x - y$; remark that $\lambda(\frac{x+y}{2}) = 0$.

In each symmetric equilibrium, $x^c + y^c = 1$ and the market is fully covered if and only if $y^c \leq \frac{1}{2} \leq x^c$, whereas it is not fully covered whenever $x^c < \frac{1}{2} < y^c$. The profit for firm B with fully covered market is given by (we are using (5) and $\Theta^c_B = [\tilde{\theta}^c, 1]$)

$$\begin{aligned}\pi_B(x, y) &= n(1 - t - c + ty) \int_{(x+y)/2}^{1} \left[(n-1)\lambda(\theta)^{n-2} - (n-2)\lambda(\theta)^{n-1}\right] d\theta \\ &= \frac{n}{2}(1 - t - c + ty) \left[\lambda(1)^{n-1} - \frac{n-2}{n}\lambda(1)^n\right]\end{aligned} \tag{26}$$

because $\lambda(\frac{x+y}{2}) = 0$; notice that $\lambda(1) = 2 - x - y$ and, so, $\frac{\partial \lambda(1)}{\partial y} = -1$.

The partial derivative of $\frac{2}{n}\pi_B(x, y)$ with respect to y is

$$\frac{2}{n}\frac{\partial \pi_B(x, y)}{\partial y} = t\left[\lambda(1)^{n-1} - \frac{n-2}{n}\lambda(1)^n\right] - (1 - t - c + ty)\left[(n-1)\lambda(1)^{n-2} - (n-2)\lambda(1)^{n-1}\right] \quad (27)$$

At each symmetric equilibrium, $x^c + y^c = 1$, and so $\lambda(1) = 1$. Hence, $\frac{\partial \pi_B(x^c, y^c)}{\partial y} = 0$ yields $y^c = \frac{n+2}{n} - \frac{1-c}{t}$. This, in turn, gives $x^c = \frac{1-c}{t} - \frac{2}{n}$ and $p^c = \frac{2t}{n} + c$. The market is fully covered if and only if $y^c \leq x^c$, which is equivalent to $t \leq \frac{2n(1-c)}{4+n}$. Since $t < 1$ by assumption, the condition is necessarily satisfied if $\frac{2n(1-c)}{4+n} \geq 1$, which is equivalent to $c \leq \frac{n-4}{2n}$.

No upward deviation is profitable Here we check that no profitable upward deviation exists. Since we are considering p_A, p_B such that $p_A \leq p_B$, we prove that no $p_B > p^c$ (i.e., no $y > y^c$) is a profitable deviation for seller B if seller A plays $p_A = p^c$ (i.e., $x = x^c$). We begin by considering deviations of B that keep the market fully covered; then we examine deviations that induce some buyer type not to participate. The first class of deviations is such that $y^c < y \leq \min\{x^c, 2 - x^c\}$. Precisely, if $x^c < 1$ (which holds if $t > \frac{n(1-c)}{2+n}$) then $y > x^c$ leads to non-fully covered market (with $\Theta^c_N = (x^c, y)$), which we examine below. If $x^c \geq 1$, then the market is fully covered for each y, but the set Θ^c_B is empty when $y > 2 - x^c$; thus, the demand for seller B is zero.

We show that π_B is decreasing with respect to y for $y \in (y^c, \min\{x^c, 2 - x^c\}]$. We derive $\frac{2}{n}\frac{\partial \pi_B(x^c, y)}{\partial y}$ from (27), with $\lambda(1) = 2 - x^c - y$, and we note that $1 - t - c + ty = t(\frac{n+2}{n} - \lambda(1))$. Hence,

$$\frac{2}{n}\frac{\partial \pi_B(x^c, y)}{\partial y} = -\frac{t}{n}\lambda(1)^{n-2}\left[(n-2)(n+1)\lambda(1)^2 - 2(n^2-2)\lambda(1) + (n+2)(n-1)\right]$$

and we remark that the quadratic expression in the square brackets is positive for all $\lambda(1) \in (0, 1)$. Hence, $\frac{2}{n}\frac{\partial \pi_B(x^c, y)}{\partial y} < 0$ for each $y \in (y^c, \min\{x^c, 2 - x^c\}]$.

Now we consider deviations from B that lead to a non-fully covered market. This occurs when $x^c < 1$ and $y \in (x^c, 1)$. In this case the profit of B is given by

$$\begin{aligned}\overline{\pi}_B(x^c, y) &= n(1 - t - c + ty)\int_y^1 ((n-1)\lambda(\theta)^{n-2} - (n-2)\lambda(\theta)^{n-1})d\theta \\ &= \pi_B(x^c, y) - \frac{n}{2}(1 - t - c + ty)\left[\lambda(y)^{n-1} - \frac{n-2}{n}\lambda(y)^n\right]\end{aligned}$$

with $\lambda(y) = y - x^c \in (0, 1)$. From the previous analysis we know that $\pi_B(x^c, y^c) > \pi_B(x^c, x^c)$, and $\pi_B(x^c, x^c) = \lim_{y\to(x^c)^+} \overline{\pi}_B(x^c, y)$. The partial derivative of $\frac{2}{n}\overline{\pi}_B(x^c, y)$ with respect to y, for $y \in (x^c, 1)$, is

$$\frac{2}{n}\frac{\partial \overline{\pi}_B(x^c, y)}{\partial y} = \frac{2}{n}\frac{\partial \pi_B(x^c, y)}{\partial y} - t\left[\lambda(y)^{n-1} - \frac{n-2}{n}\lambda(y)^n\right] - (1 - t - c + ty)\left[(n-1)\lambda(y)^{n-2} - (n-2)\lambda(y)^{n-1}\right]$$

and is negative because the first term is negative on the basis of the earlier analysis, and the other two terms are negative since $\lambda(y)^{n-1} - \frac{n-2}{n}\lambda(y)^n > 0$ and $(n-1)\lambda(y)^{n-2} - (n-2)\lambda(y)^{n-1} > 0$. Thus, no $y \in (x^c, 1)$ is a profitable deviation for seller B.

No downward deviation is profitable Here we show that no profitable downward deviation exists. Since we are considering p_A, p_B such that $p_A \leq p_B$, we prove that no $p_A < p^c$ (i.e., no $x > x^c$) is a profitable deviation for seller A if seller B adopts $p_B = p^c$. Clearly, no $p_A < c$ is chosen, as otherwise A's profit is negative. Therefore, the range of values for x we consider is given by $(x^c, (1-c)/t]$.[37] Demand for A is given by $2 - n\int_{(x+y)/2}^1 ((n-1)\lambda(\theta)^{n-2} - (n-2)\lambda(\theta)^{n-1})d\theta$ and, therefore, her profit is

$$\begin{aligned}\pi_A(x, y^c) &= (1 - tx - c)\left(2 - n\int_{(x+y)/2}^1 ((n-1)\lambda(\theta)^{n-2} - (n-2)\lambda(\theta)^{n-1})d\theta\right) \\ &= (1 - tx - c)\left(2 - \frac{n}{2}\lambda(1)^{n-1} + \frac{n-2}{2}\lambda(1)^n\right)\end{aligned}$$

where we remind that $\lambda(1) = 2 - x - y^c$ so that, in particular, $\frac{\partial \lambda(1)}{\partial x} = -1$. We have that

$$\frac{\partial^2 \pi_A(x, y^c)}{\partial x^2} = -nt\left[(n-1)\lambda(1)^{n-2} - (n-2)\lambda(1)^{n-1}\right] - \frac{n(n-1)(n-2)}{2}(1 - tx - c)\left(\lambda(1)^{n-3} - \lambda(1)^{n-2}\right)$$

[37] Notice that $\frac{1-c}{t} < 2 - y^c$, therefore, for no x in $(x^c, (1-c)/t]$, firm A wins the whole market.

is negative for each $\lambda(1) \in (0,1)$, therefore, π_A is concave with respect to x. Jointly with $\frac{\partial \pi_A(x^c, y^c)}{\partial x} = 0$, this implies $\frac{\partial \pi_A(x, y^c)}{\partial x} < 0$ for $x > x^c$.

A.2. Symmetric equilibrium with non-fully covered market

We now look at the case where no symmetric equilibrium involves a fully covered market; this occurs if $t > \frac{2n(1-c)}{4+n}$. In each symmetric equilibrium (p^c, p^c) where the market is not fully covered, p^c belongs to the interval $(1 - t/2, 1)$ and $x^c < 1/2 < y^c$, $x^c + y^c = 1$. The profit for B is given by

$$\overline{\pi}_B(x, y) = n(1 - t - c + ty) \int_y^1 ((n-1)\lambda(\theta)^{n-2} - (n-2)\lambda(\theta)^{n-1}) d\theta$$
$$= \frac{n}{2}(1 - t - c + ty) \left[\lambda(1)^{n-1} - \frac{n-2}{n}\lambda(1)^n\right] - \frac{n}{2}(1 - t - c + ty) \left[\lambda(y)^{n-1} - \frac{n-2}{n}\lambda(y)^n\right]$$

with $\lambda(y) = y - x$, and $\lambda(1) = 2 - x - y$. Then

$$\begin{aligned} \frac{2}{n}\frac{\partial \overline{\pi}_B(x, y)}{\partial y} &= t\left[\lambda(1)^{n-1} - \frac{n-2}{n}\lambda(1)^n\right] - (1 - t - c + ty)\left[(n-1)\lambda(1)^{n-2} - (n-2)\lambda(1)^{n-1}\right] \\ &- t\left[\lambda(y)^{n-1} - \frac{n-2}{n}\lambda(y)^n\right] - (1 - t - c + ty)\left[(n-1)\lambda(y)^{n-2} - (n-2)\lambda(y)^{n-1}\right] \end{aligned} \tag{28}$$

Since $x^c + y^c = 1$ implies $\lambda(1) = 1$ and $\lambda(y^c) = 2y^c - 1$, we obtain the following first order condition:

$$\frac{t}{n}\left[2 - n\lambda(y^c)^{n-1} + (n-2)\lambda(y^c)^n\right] - (1 - t - c + ty^c)\left[1 + (n-1)\lambda(y^c)^{n-2} - (n-2)\lambda(y^c)^{n-1}\right] = 0 \tag{29}$$

hence $p^c = 1 - t + ty^c$. The rest of this proof is mechanical and is found in Landi and Menicucci (2023a).

Appendix B. Proof of Proposition 3

Before proving Proposition 3 we provide some remarks about the optimal price for a one-bundle monopolist and for a two-bundle monopolist.

Some remarks about the optimal price for a one-bundle or two-bundle monopolist Suppose that seller A is a one-bundle monopolist, choosing price $p \geq c$. Then the consumer types visiting seller A are those in the interval $[0, \min\{1, \frac{1}{t}(1-p)\}]$. Since it is suboptimal for seller A to set p such that $\frac{1}{t}(1-p) > 1$, we consider $p \geq \max\{1-t, c\}$, and $(p-c)\left(1 - (1 - \frac{1-p}{t})^n\right)$ is the profit. We use the change of variable $y = 1 - \frac{1-p}{t}$ (that is $p = 1 - t + ty$), hence $y \in [\max\{0, 1-\rho\}, 1]$ and the profit is written as $(\rho - 1 + y)(1 - y^n)t$. Neglecting the multiplicative constant $t > 0$, the profit's derivative is

$$1 - y^n - ny^{n-1}(\rho - 1 + y) \tag{30}$$

Let $\phi_1(y)$ denote the expression in (30). When $\rho < 1$ we find $\phi_1(1-\rho) = 1 - (1-\rho)^n > 0$; when $\rho \geq 1$ we find $\phi_1(0) = 1 > 0$. Moreover, $\phi_1(1) = -n\rho < 0$ and ϕ_1 is strictly decreasing in the interval $[\max\{0, 1-\rho\}, 1]$. Therefore $\phi_1(y) = 0$ is necessary and sufficient for a profit maximizing point. Letting y_{M1} be the unique solution to (30) in $(\max\{1-\rho, 0\}, 1)$, it follows that $\frac{dy_{M1}}{d\rho} = -\frac{ny_{M1}^{n-1}}{2ny_{M1}^{n-1} + n(n-1)y_{M1}^{n-2}(\rho - 1 + y_{M1})} < 0$ and that $y_{M1} = \frac{1}{(n+1)^{1/n}}$ when $\rho = 1$.

Now suppose that seller A is a two-bundle monopolist. Like for a one-bundle monopolist, seller A sets $p \geq \max\{1-t, c\}$ and the profit is

$$(p-c)\left(n(1 - \frac{1-p}{t})^{n-1}\frac{1-p}{t} + 2\left(1 - n(1 - \frac{1-p}{t})^{n-1}\frac{1-p}{t} - (1 - \frac{1-p}{t})^n\right)\right)$$

in which $n(1 - \frac{1-p}{t})^{n-1}\frac{1-p}{t}$ is the probability that exactly one consumer visits seller A and $1 - n(1 - \frac{1-p}{t})^{n-1}\frac{1-p}{t} - (1 - \frac{1-p}{t})^n$ is the probability that at least two consumers visit seller A. We use the change of variable $y = 1 - \frac{1-p}{t}$ (that is $p = 1 - t + ty$), hence $y \in [\max\{0, 1-\rho\}, 1]$ and the profit is written as $(\rho - 1 + y)\left(2 - ny^{n-1} + (n-2)y^n\right)t$, with derivative (neglecting t)

$$2 - ny^{n-1} + (n-2)y^n - n(\rho - 1 + y)\left((n-1)y^{n-2} - (n-2)y^{n-1}\right) \tag{31}$$

Let $\phi_2(y)$ denote the left hand side of (31). When $\rho < 1$ we find $\phi_2(1-\rho) = 2 - n(1-\rho)^{n-1} + (n-2)(1-\rho)^n > 0$; when $\rho \geq 1$ we find $\phi_2(0) = 2 > 0$. Moreover, $\phi_2(1) = -n\rho < 0$ and ϕ_2 is strictly decreasing in the interval $[\max\{0, 1-\rho\}, 1]$. Therefore $\phi_2(y) = 0$ is necessary and sufficient for a profit maximizing point. Letting y_{M2} be the unique solution to (31) in $(\max\{1-\rho, 0\}, 1)$, it follows that $\frac{dy_{M2}}{d\rho} < 0$.

Proof that $y_{M2} < y_{M1}$, that is $p_{M2} < p_{M1}$. We prove that (31) evaluated at $y = y_{M1}$ is negative, which implies $y_{M2} < y_{M1}$. From (30) we know that $\frac{1-y_{M1}^n}{ny_{M1}^{n-1}} = \rho - 1 + y_{M1}$. Then we replace $\rho - 1 + y_{M1}$ in (31) with $\frac{1-y_{M1}^n}{ny_{M1}^{n-1}}$ and obtain $2 - ny_{M1}^{n-1} + (n-2)y_{M1}^n - n\frac{1-y_{M1}^n}{ny_{M1}^{n-1}}\left((n-1)y_{M1}^{n-2} - (n-2)y_{M1}^{n-1}\right) = n - \frac{n-1}{y_{M1}} - y_{M1}^{n-1}$. The function $\phi_3(y) = n - \frac{n-1}{y} - y^{n-1}$ is such that $\phi_3(1) = 0$ and $\phi_3'(y) = \frac{n-1}{y^2}(1-y^n) > 0$. Hence, $\phi_3(y) < 0$ for each $y \in (0,1)$, and in particular $n - \frac{n-1}{y_{M1}} - y_{M1}^{n-1} < 0$.

Proof of Proposition 3 Let $\rho_1 = \frac{1}{2} + \frac{2^n-1}{2n}$, $\rho_2 = \frac{1}{2} + \frac{2^n-1}{n}$. We begin by showing that $2\pi_D^s > \pi_{M2}^s$ when $\rho < \rho_1$. We have seen above that the demand for a one-bundle monopolist is $D_{M1}(p) = 1 - (1 - \frac{1-p}{t})^n$ and the demand for a two-bundle monopolist is $D_{M2}(p) = n(1 - \frac{1-p}{t})^{n-1}\frac{1-p}{t} + 2\left(1 - n(1-\frac{1-p}{t})^{n-1}\frac{1-p}{t} - (1-\frac{1-p}{t})^n\right)$. Therefore the demand for a one-bundle monopolist is greater than half the demand for a two-bundle monopolist, that is $D_{M1}(p) > \frac{1}{2}D_{M2}(p)$. Since p_{M2}^s denotes the optimal price for a two-bundle monopolist, it follows that $\pi_{M2}^s = (p_{M2}^s - c)D_{M2}(p_{M2}^s)$. But if a one-bundle monopolist chooses $p = p_{M2}^s$, then she earns profit $(p_{M2}^s - c)D_{M1}(p_{M2}^s)$ which is greater than $(p_{M2}^s - c)\frac{1}{2}D_{M2}(p_{M2}^s) = \frac{1}{2}\pi_{M2}^s$ as $D_{M1}(p_{M2}^s) > \frac{1}{2}D_{M2}(p_{M2}^s)$. Hence, $\pi_{M1}^s \geq (p_{M2}^s - c)D_{M1}(p_{M2}^s) > \frac{1}{2}\pi_{M2}^s$, that is $2\pi_{M1}^s > \pi_{M2}^s$. Since $\pi_D^s = \pi_{M1}^s$ when $\rho < \rho_1$, it follows that $2\pi_D^s > \pi_{M2}^s$.

Now we consider $\rho \geq \rho_1$ and use Proposition 2 to obtain

$$\pi_D^s = \begin{cases} t(\rho - 1 + y_{M1})\left(1 - y_{M1}^n\right) & \text{if } \rho < \rho_1 \\ t(\rho - \frac{1}{2})(1 - \frac{1}{2^n}) & \text{if } \rho_1 \leq \rho \leq \rho_2 \\ t(\rho - \frac{n}{2^{n+1}-n-2}(\rho - 1))(1 - \frac{1}{2^n}) & \text{if } \rho_2 < \rho \end{cases} \tag{32}$$

Hence, when $\rho \geq \rho_1$ we can write π_D^s as tb_D^s, with $b_D^s = (\rho - \frac{1}{2})(1 - \frac{1}{2^n})$ if $\rho \in [\rho_1, \rho_2]$ and $b_D^s = (\rho - \frac{n}{2^{n+1}-n-2}(\rho-1))(1-\frac{1}{2^n})$ if $\rho > \rho_2$. Moreover, π_{M2}^s is equal to tb_{M2}^s with $b_{M2}^s = (\rho - 1 + y)\left(2 - ny_{M2}^{n-1} + (n-2)y_{M2}^n\right)$. It is immediate that b_D^s is concave in ρ because $1 - \frac{1}{2^n} > (1 - \frac{n}{2^{n+1}-n-2})(1 - \frac{1}{2^n})$. Moreover, b_{M2}^s is convex in ρ since the Envelope Theorem implies $\frac{db_{M2}^s}{d\rho} = 2 - ny_{M2}^{n-1} + (n-2)y_{M2}^n$, which is decreasing in y_{M2} and we know from above that y_{M2} is decreasing in ρ. Hence, $\frac{db_{M2}^s}{d\rho}$ is increasing in ρ, thus b_{M2}^s is convex. Now we write $2\pi_D^s - \pi_{M2}^s$ as $t\delta(\rho)$, with $\delta(\rho) = 2b_D^s - b_{M2}^s$, and δ is a concave function. We know that $\delta(\rho_1) > 0$ and $\delta(\rho) < 0$ if ρ is large. Hence, there exists a $\rho^* > \rho_1$ such that $\delta(\rho^*) = 0$, and the concavity of δ implies that ρ^* is unique.

Appendix C. Proof of Proposition 5

The proof of Proposition 5(i) is given in the text, just after the statement of Proposition 5.

For the proof of Proposition 5(ii) we use (13) to find

$$CS_D^s = \begin{cases} t(1 - y_{M1})\left(1 - y_{M1}^n\right) & \text{if } \rho < \rho_1 \\ t\frac{1}{2}(1 - \frac{1}{2^n}) & \text{if } \rho_1 \leq \rho \leq \rho_2 \\ t(1 - \frac{1}{2^n})\left(\frac{2n}{2^{n+1}-n-2}(\rho - 1) - \frac{1}{2}\right) & \text{if } \rho_2 < \rho \end{cases} \tag{33}$$

in which $\rho_1 = \frac{1}{2} + \frac{2^n-1}{2n}$, $\rho_2 = \frac{1}{2} + \frac{2^n-1}{n}$. When $n = 4$, numeric analysis shows that $CS_D^s > CS^c$ if and only if $\rho > 1.156$. When $n \geq 5$, consider first $\rho = 1$, which implies $y_{M1} = \frac{1}{(n+1)^{1/n}}$. Hence, $CS_D^s = (1 - \frac{1}{(n+1)^{1/n}})\frac{n}{n+1}t$, which is greater than $\frac{3}{2(n+1)}t$ for each $n \geq 5$. For $\rho \geq \rho_1$, notice from (33) that CS_D^s can be written as $th_D^s(\rho)$, with h_D^s weakly increasing in ρ. Hence, $CS_D^s - CS^c = t\left(h_D^s(\rho) - \frac{3}{2(n+1)}\right)$ and $h_D^s(\rho) - \frac{3}{2(n+1)}$ is positive at $\rho = 1$, is increasing in ρ. This implies $CS_D^s - CS^c > 0$ for each $\rho > 1$.

For the proof of Proposition 5(iii), recall from Corollary 2 that $PS^c = \frac{4}{n}t$. Consider first $\rho = 1$, which implies $y_{M1} = \frac{1}{(n+1)^{1/n}}$. Thus PS^s is equal to $2\frac{n}{(n+1)^{\frac{n+1}{n}}}t$, which is greater than $\frac{4}{n}t$ for each $n \geq 4$. For $\rho \geq \rho_1$, notice from (32) that PS^s can be written as $tm^s(\rho)$, with m^s weakly increasing in ρ. Hence, $PS^s - PS^c = t\left(m^s(\rho) - \frac{4}{n}\right)$ and $m^s(\rho) - \frac{4}{n}$ is positive at $\rho = 1$, is increasing in ρ. This implies $PS^s - PS^c > 0$ for each $\rho > 1$.

Appendix D. Proofs of Propositions 7–9

The proofs of Propositions 7, 8 and 9 are available in Landi and Menicucci (2023a).

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