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ABSTRACT

We study a competition model with two sellers that auction non-identical objects, unlike most of the literature on competing auctions. Each bidder has bidimensional private information, his values for the objects, and chooses the auction in which he participates (if any) after each seller has set a reserve price for her auction. We show that in some cases the duopoly reserve price is greater than the reserve price for a monopolist auctioning a single object; thus, an increase in the number of sellers may make some bidder's types worse off. In our analysis we first characterize the unique symmetric equilibrium for the game of auction choice played by the bidders and investigate its features. Then for the game of reserve price setting played by the sellers, we show that in each symmetric equilibrium the reserve price level is determined by the interplay of two effects, a virtual value effect and a business stealing effect. The former tends to lift the equilibrium reserve price above the monopoly level, the latter may drive the equilibrium reserve price significantly below the monopoly level when the number n of bidders is small. For a large n we find that different reserve prices in the two auctions have little effect on bidders' auction choices. As a consequence, when n is large the business stealing effect weakens with respect to the virtual value effect, and for the cases we consider the equilibrium reserve price is above the monopoly level.

1. Introduction

Consider a market with a monopolist auctioneer offering a single object through a second-price auction with a reserve price. Then suppose that an additional auctioneer becomes active and offers an imperfect substitute, still through a second-price auction with a reserve price. We show that the equilibrium reserve price in the second setting may be higher than in the first setting. Therefore going from one monopoly auction to two competing auctions may be detrimental to some bidders.

In detail, we study a (technically challenging) duopolist model of auction competition with differentiated goods, in which two (female) sellers, A and B, face $n \geq 2$ (male) bidders. For $i = A, B$, seller i offers object i through a second-price auction, auction i, for which she chooses the reserve price r_i.[1] The two objects are heterogeneous and each bidder privately observes his value v_A for object A and his value v_B for object B; the values are ex ante i.i.d. across objects and across bidders, each with c.d.f. F, density f, support $[\underline{v}, \bar{v}]$. After sellers have chosen r_A, r_B, each bidder decides in which auction (if any) to participate. Then the auctions take place and the objects are allocated. Auctions for cars, houses, art pieces provide examples of the scenario we have in mind, with auction houses competing in the sale of non-identical items in the same product category.

There is a large literature on monopoly auctions, but auction competition has been much less studied, and most of the related literature focuses on auctions for identical objects with many sellers and many bidders. In this case, in equilibrium each seller charges a reserve price of zero or close to zero – see next section for a brief literature review. Conversely, we assume there are just two sellers, who offer heterogeneous objects. We prove that there exists a pure-strategy equilibrium in the continuation game of auction choice among bidders, and show that there is a pure-strategy equilibrium in the game in which sellers choose reserve prices, if the set of feasible reserve prices is a suitable discretized set – unlike in settings with identical objects and finitely many agents. Moreover, we show that under duopoly the equilibrium reserve price may be higher than under monopoly. This is somewhat surprising because it is common that a greater number of competing sellers results in a lower price. Precisely, start from a situation in which seller B is monopolist, that is she offers

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[1] In Landi et al. (2023) we show that our results extend to the case in which sellers use first-price auctions.

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0304-4068/

object B through auction B, and there is no seller A nor object A. Then it may be reasonable to expect that the entry of seller A is convenient for the bidders because it widens their choice set and reduces the equilibrium reserve price. We find that the latter conjecture is sometimes incorrect, because with competition a reserve price reduction may be less effective at winning bidders than under monopoly. This induces each seller to increase the reserve price above the monopoly level, and ultimately reduces the utility of some bidder's types.

As a first step, we characterize the unique symmetric *bidders' equilibrium* for any pair r_A, r_B, that is, the equilibrium of the game in which each bidder chooses between auction A, auction B, and non-participation. Such equilibrium is described by a partition of the type space, the square $[\underline{v}, \bar{v}] \times [\underline{v}, \bar{v}]$, into three sets: the set of the types who stay out of each auction, the set of the types who enter auction A, the set of the types who enter auction B. We find that the equilibrium partition is characterized by an integro-differential equation for which there is no closed form solution.

Next, we consider the *sellers' game*, in which each seller simultaneously chooses her reserve price, r_A or r_B, under the assumption that the hazard rate of the distribution of the bidders' values is increasing, which implies regularity in terms of virtual valuations. We focus on symmetric Nash equilibria, that is such that $r_A = r_B = r^D$ (D is from duopoly). The first order condition determining r^D is influenced by two effects, which we name the *business stealing effect* and the *virtual value effect*. Starting from $r_A = r_B = r$, a reduction of r_B below r induces the entry in auction B of some types with $v_B < v_A$ who choose auction A when $r_B = r$; this increases the revenue of seller B and we call it the business stealing effect. In addition, a reduction in r_B reduces the revenue of seller B if just one bidder enters auction B, but it also determines the entry in auction B of types who previously would find it best not to join any auction. The combination of the latter effects is the virtual value effect. This is similar to what takes place in a monopoly auction when the reserve price is reduced, as in a monopoly there is no business stealing, and the sign of the total effect on the revenue is captured by the virtual values of the newly entered types. But under duopoly the virtual value effect refers to a c.d.f. $\tilde{F}$ which is different from F and which we describe in next paragraph.

To isolate the virtual value effect, consider the optimal reserve price in the artificial setting in which any value of the reserve price for auction B is automatically matched by the reserve price for auction A. In this case there is no business stealing effect and the types who enter auction B are those with valuation for object B greater than the valuation for object A and greater than the reserve price. Thus, the corresponding c.d.f. of types entering auction B is $\tilde{F}(v) = \frac{1}{2} + \frac{1}{2}F^2(v)$, with associated virtual value $\tilde{J}(v) = v - \frac{1-\tilde{F}(v)}{\tilde{f}(v)}$. We find that $\tilde{F}(v) > F(v)$ and $\tilde{f}(v) < f(v)$, so that the comparison between $\tilde{J}(v)$ and the monopoly virtual value $J(v) = v - \frac{1-F(v)}{f(v)}$ is not ex ante immediate. But a mechanical computation shows that $\tilde{J}(v)$ is strictly smaller than $J(v)$, which automatically implies that the optimal reserve price in this artificial version of the duopoly, denoted r^V, is larger than the optimal reserve price r^M for a monopolist.

Now consider seller B's problem of determining the profitability of reducing r_B below r, given r_A fixed at r. If $r \geq r^V$, then the business stealing effect exerts a downward pressure and induces seller B to lower r_B below r; thus the equilibrium reserve price r^D is smaller than r^V. Because of technical difficulties linked to solving the above mentioned integro-differential equation, we resort to a discretization of the set of reserve prices which each seller can choose, and in this context we use numerical methods to prove the existence of a pure-strategy equilibrium under the assumption of uniformly distributed values. We find that r^D may be smaller than $\underline{v}$, whereas r^M is never below $\underline{v}$. An important feature of the bidders' equilibrium is that the equilibrium partition of $[\underline{v}, \bar{v}] \times [\underline{v}, \bar{v}]$ is not very sensible to a reserve price difference when there are many bidders, which weakens the business stealing effect when n is large. This reduces each seller i's incentive to reduce r_i to widen the set of types entering auction i. Consistently, in the settings we consider we see that r^D increases with n and is larger than r^M for large n. However, r^D is bounded away from r^V because even though $\Delta r_B < 0$ makes only a small set of types switch from auction A to auction B, this generates a non-negligible effect on the c.d.f. of the second highest bid in auction B, since the latter is very sensitive to greater participation in auction B when n is large. Therefore, even though the business stealing effect is weak when n is large, it does not disappear in the limit.

As a consequence, we learn that for each seller, say seller B, there are significant differences with respect to choosing the reserve price as a monopolist. In the latter setting, a given r_B induces participation in auction B by all bidders with value v_B greater than r_B. But under duopoly, the existence of a competing auction A induces some types with $v_B > r_B$ to enter auction A if they have high v_A. This reduces seller B's revenue, but also affects her incentives in the choice of r_B, as r_B determines the set of bidder types entering auction B in a more subtle way than under monopoly. This affects the comparison between r^D and r^M.

When $r^D > r^M$ holds, it is immediate that some bidder's types are worse off under duopoly. Precisely, consider the types with both values less than r^D, but with value for object B higher than r^M. These types are better off when there is no auction A, hence when seller B is monopolist, than under duopoly.

The remainder of the paper is organised as follows. In Section 2 we briefly review the related literature. In Section 3 we introduce the model. In Section 4 we examine the bidders' game. In Section 5 we deal with the sellers' game. In Section 6 we conclude. In the appendices we provide the proofs of our results.

2. Literature review

To the best of our knowledge, Parlane (2008) and Troncoso-Valverde (2014) are the only available papers that examine competing auctions with heterogenous objects.

Parlane (2008) assumes that the two objects are differentiated à la Hotelling, thus bidders are distributed in the interval $[0, 1]$ and have perfectly negatively correlated values for the objects. After determining a bidders' equilibrium, consisting of a partition of $[0, 1]$ into three sets – types entering auction A, types entering auction B, types entering no auction – Parlane (2008) determines a pure-strategy sellers' equilibrium. Our paper analyzes a bidimensional type space, in which each bidder's values for the objects are ex ante uncorrelated and cannot be summarized by a single parameter. Thus, in our context a bidders' equilibrium is more complex, as it consists of a partition of the square $[\underline{v}, \bar{v}] \times [\underline{v}, \bar{v}]$ determined by a suitable function which we denote g, rather than of a partition of $[0, 1]$ determined by two cutoffs.[2] Moreover, in Section 5.5 we prove a new result for Parlane's setting, showing that also in such environment duopoly may increase the equilibrium reserve price with respect to monopoly.

At the final stage of preparation of this paper, we have learnt that Troncoso-Valverde (2014) analyzes the same environment for distributions with support $[0, 1]$, and for this context provides our Proposition 1. Troncoso-Valverde (2014) does not

[2] In a sense, the relation between our paper and Parlane (2008) is analogous to the relation between an Hotelling setting and the random utility model of Perloff and Salop (1985).

identify cases in which a pure-strategy equilibrium for the sellers' game exists, nor cases in which the equilibrium reserve price is higher under duopoly than under monopoly. Specifically, Troncoso-Valverde (2014) characterizes each bidder's best replies through cutoff strategies – in which each bidder's auction selection is determined by a function analogous to g mentioned above – and characterizes symmetric bidders' equilibria through the integro-differential equation mentioned in the introduction, showing that a unique solution to that equation exists – hence a unique symmetric bidders' equilibrium exists – if the reserve prices are both positive or both zero.[3]

Our paper allows for bidders' values drawn from the interval $[\underline{v}, \bar{v}]$ with $\underline{v} \geq 0$, and we determine a bidders' equilibrium even in the event that $r_A \leq \underline{v}$ and/or $r_B \leq \underline{v}$. In fact, we prove existence and uniqueness of symmetric bidders' equilibria for any pair of reserve prices. Moreover, we provide a more detailed characterization of the bidders' equilibrium, that allows to perform comparative statics (e.g. on the number of bidders and the sellers' reserve prices) which is useful in understanding some properties of the sellers' equilibrium. Then we derive some results for the sellers' game,[4] by identifying the virtual valuation effect, the business stealing effect and showing that r^V is an upper bound for the equilibrium price. Finally, we perform a numeric analysis of settings with uniformly distributed values in which we illustrate how the interplay of the two effects determines r^D. We show that r^D can be smaller or larger than the reserve price under monopoly, but may also be smaller than $\underline{v}$.

Our paper is related also to Burguet and Sakovics (1999), who analyze a setting in which the two auctions offer identical objects and each bidder has a same (bidder-specific) value for the two objects, ex ante distributed with support [0, 1]. Hence, each bidder has perfectly positively correlated values for the objects and our paper – given uncorrelated values – can be viewed as intermediate between Parlane (2008) with perfect negative correlation, and Burguet and Sakovics (1999) with perfect positive correlation. As we illustrate in detail in Section 5.4, the bidders' equilibrium in Burguet and Sakovics (1999) is in mixed strategies, whereas in our setting we determine a pure-strategy equilibrium. Moreover, Burguet and Sakovics (1999) show that no symmetric pure-strategy equilibrium exists in their sellers' game because a seller's downward deviation is always profitable – but a symmetric mixed-strategy equilibrium exists; no comparison with the monopoly reserve price is provided. Conversely, in our environment a pure-strategy equilibrium exists, at least for the settings we consider, that is heterogeneity in bidders' values for the two objects softens enough competition between sellers that a pure-strategy equilibrium may exist in the sellers' game. In Section 5.4 we discuss the differences between the two settings.

Virag (2010) examines the same setting as Burguet and Sakovics (1999), but allows for more than two sellers and shows that in some cases a symmetric pure-strategy Nash equilibrium exists in the sellers' game if the support for the bidders' values has a positive lower bound. Virag (2010) also shows that if both the number of sellers and the number of bidders tend to infinity, with an upper bound on the ratio between the latter and the former, then for each sequence of equilibria (in pure or mixed strategies), the equilibrium reserve price converges to zero in distribution. McAfee (1993), Peters and Severinov (1997) obtain analogous results for markets that are infinitely large. Hernando-Veciana (2005) proves that there exists an equilibrium in which each seller chooses a reserve price of zero when the number of agents is large (but finite), under the assumption that each seller can pick her reserve price from a finite set.

[3] About uniqueness when $r_A = r_B = 0$, see also Troncoso-Valverde (2018).

[4] Troncoso-Valverde (2014) establishes that a mixed-strategy equilibrium exists. Our results suggest that a pure-strategy equilibrium exists in the settings we consider.

3. The model

We consider a setting with two (female) sellers, A and B, and $n \geq 2$ (male) bidders. For $i = A, B$, seller i offers an object, object i, using a second-price auction, called auction i, for which she selects the reserve price r_i. For $j = 1, \ldots, n$, bidder j privately observes his valuations v_{Aj} for object A and v_{Bj} for object B. Each other bidder and the sellers view v_{Aj}, v_{Bj} as realizations of two i.i.d. random variables, each having support $[\underline{v}, \bar{v}]$ (with $0 \leq \underline{v} < \bar{v}$), c.d.f. F and density f which is continuous and positive in $[\underline{v}, \bar{v}]$. Therefore, each bidder's values belong to the set $V = [\underline{v}, \bar{v}] \times [\underline{v}, \bar{v}]$. Moreover, the valuations of different bidders are independently distributed. Each seller is risk neutral and wants to maximize her expected revenue (in the following we sometimes omit the word expected). Each bidder is risk neutral and wants to maximize the product between his valuation for the object offered in the auction in which he participates and his probability to win such object, minus his expected payment. For the sake of brevity, in the main text we assume $\underline{v} = 0$; in Appendix F we examine the case of $\underline{v} > 0$.

We analyze a game with the following (standard) timing:

Stage one: Seller i sets a reserve price $r_i \in [0, \bar{v})$ for auction i, for all i;[5] the sellers choose r_A, r_B simultaneously.

Stage two: After observing r_A, r_B, each bidder decides whether to participate in auction A, or in auction B, or in no auction at all; no bidder can participate in both auctions.

Stage three: After the bidders' participation decisions, the auctions take place. For $i = A, B$, the highest bidder in auction i wins object i and pays to seller i the highest between r_i and the second highest bid in auction i.

In Section 5, when we examine the sellers' game, in a setting with uniformly distributed values we determine a candidate equilibrium reserve price based on the first order condition for maximization of a seller's expected revenue. Then we consider a discretization of the set of reserve prices available to each seller, with a step of 0.001, and use numerical methods to establish that in such setting there exists a symmetric equilibrium in which each seller chooses the reserve price identified above.

4. Bidders' auction choice

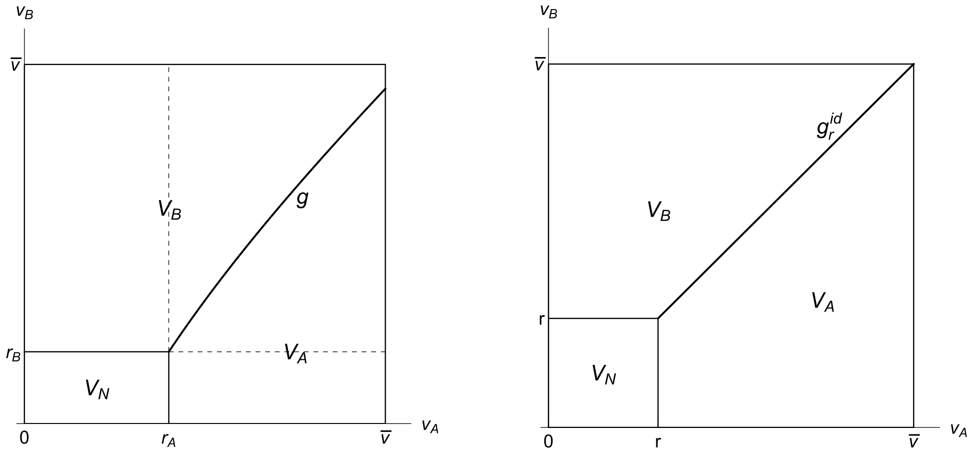
Since auctions A and B are second-price auctions, a weakly dominant action at stage three for each bidder participating in auction i is to bid his value for object i, for $i = A, B$.

At stage two, each bidder chooses among participation in auction A, participation in auction B, and non-participation. For the continuation game that begins at stage two, a Nash equilibrium is called *bidders' equilibrium* in the following. We focus on symmetric bidders' equilibria (but sometimes omit the word symmetric), in which each bidder follows the same strategy of auction choice, which depends on the bidder's valuations but not on his identity. Moreover, and without loss of generality, we suppose that $r_B \leq r_A$. Let V_i, for $i = A, B$, be the set of types (v_A, v_B) of bidders attending auction i, and let V_N be the set of types (v_A, v_B) of bidders who do not attend any auction:

$$V_N = \{(v_A, v_B) \in V : v_A < r_A \quad \text{and} \quad v_B < r_B\} \tag{1}$$

It is clear that $(v_A, v_B) \in V_i$ if $v_i \geq r_i$ and $v_j < r_j$ because then the bidder earns zero utility in auction j but may earn a positive utility in auction i: see in Fig. 1(a) the rectangle with dashed upper horizontal edge (in this case $i = A, j = B$) and the rectangle

[5] If seller i sets $r_i = \bar{v}$, then the set of types choosing to participate in auction i has zero measure; hence the revenue of seller i is zero.



(a) The sets V_A, V_B, V_N given r_A, r_B such that $0 < r_B < r_A$.

(b) The sets V_A, V_B, V_N given $0 < r_A = r_B = r$.

Fig. 1. Sets V_A, V_B, V_N for $0 < r_B < r_A$ (left) and $0 < r_B = r_A = r$ (right).

with dashed right vertical edge (in this case $i = B, j = A$). Finally, bidders with values $v_A \geq r_A$ and $v_B \geq r_B$ will be partitioned between V_A and V_B as we illustrate below.

It looks intuitive that a bidder selects auction B if and only if v_B is large relative to v_A, and we express this intuition by postulating the existence of a strictly increasing and differentiable function $g : [r_A, \bar{v}] \to [r_B, \bar{v}]$ such that the bidder enters auction B (auction A) if and only if $v_B > g(v_A)$ (if and only if $v_B \leq g(v_A)$).[6] Precisely, we suppose that g determines the equilibrium choices of bidder's types between the auctions as described in (2)–(3):[7]

$$V_A = \{(v_A, v_B) \in V : v_A \geq r_A \text{ and } v_B \leq g(v_A)\} \tag{2}$$

$$\begin{aligned} V_B = \{(v_A, v_B) \in V : v_A < r_A \text{ and } v_B \geq r_B,\\ \text{or } v_A \geq r_A \text{ and } v_B > g(v_A)\} \end{aligned} \tag{3}$$

and then we prove that such a function g indeed exists. Fig. 1(a) represents the sets V (i.e., the whole square), V_A, V_B, V_N for a case with $0 < r_B < r_A$.

The function g determines the boundary between the sets V_A and V_B and is such that each bidder with values v_A, $v_B = g(v_A)$ is indifferent between participating in auction A and participating in auction B. In Appendix A we show that this implies $g(r_A) = r_B$ and that g solves the integro-differential equation

$$\begin{aligned} &\left(1 - \int_{v_A}^{\bar{v}} F(g(v))f(v)dv\right)^{n-1} \\ &= g'(v_A)\left(F(g(v_A))F(v_A) + \int_{v_A}^{\bar{v}} F(g(v))f(v)dv\right)^{n-1} \end{aligned} \tag{4}$$

for each $v_A \in [r_A, \bar{v}]$.

The equality $g(r_A) = r_B$ is intuitive, since a bidder with values $(v_A, v_B) = (r_A, r_B)$ earns zero utility from participating in either auction, hence he is indifferent between the auctions. Eq. (4) identifies g such that each bidder with values satisfying $v_B = g(v_A)$ is indifferent between the auctions.

In order to gain some insights about (4), given $r \in [0, \bar{v})$ we define g_r^{id} as the identity function on $[r, \bar{v}]$, that is $g_r^{id} : [r, \bar{v}] \to [r, \bar{v}]$ satisfies $g_r^{id}(v) = v$ for each $v \in [r, \bar{v}]$. The subscript r is the lower extreme of the interval in which g_r^{id} is defined. It follows that when $r_A = r_B = r$, the function g_r^{id} solves (4), thus when the sellers set the same reserve price r, each bidder participates in the auction for the object for which he has the higher valuation, as long as such value is not less than r. Fig. 1(b) represents the resulting sets V_A, V_B, V_N. If instead $r_B < r_A$, then we cannot determine analytically a solution to (4), $g(r_A) = r_B$, but we prove in Appendix B that a unique solution exists.

In this way we establish the existence of a bidders' equilibrium, but in fact Troncoso-Valverde (2014) shows that each bidders' equilibrium (recall that we are considering symmetric equilibria) is characterized by the sets V_N, V_A, V_B in (1)–(3) with a function g satisfying (4), $g(r_A) = r_B$. Since we prove that there exists a unique solution to (4), $g(r_A) = r_B$, it follows that a unique bidders' equilibrium exists.

As the initial condition $g(r_A) = r_B$ for (4) implies that (r_A, r_B) affect g, in the following we write $g(v_A; r_A, r_B)$, or $g(v_A; \mathbf{r})$ with $\mathbf{r} = (r_A, r_B)$, to represent the unique solution to (4), $g(r_A) = r_B$.

Proposition 1. *Consider r_A, r_B such that $r_B \leq r_A < \bar{v}$. There exists a unique symmetric bidders' equilibrium, and it is described by the sets V_N, V_A, V_B in (1)–(3), in which g is defined in $[r_A, \bar{v}]$ and is the unique solution to (4), $g(r_A; \mathbf{r}) = r_B$. Moreover, for each $v_A \in (r_A, \bar{v}]$, $g(v_A; \mathbf{r})$ is strictly increasing with respect to r_B and strictly decreasing with respect to r_A. In particular, if $r_B < r_A$ then $g(v_A; \mathbf{r}) < v_A$ for each $v_A \in [r_A, \bar{v}]$.*[8]

By Proposition 1, if $r_B < r_A$ then $g(v_A; \mathbf{r}) < v_A$ holds for each $v_A \in [r_A, \bar{v}]$. In particular, starting from $r_B = r_A$, a reduction in r_B makes auction B more attractive by reducing the price a bidder pays in auction B if no other bidder enters auction B. This attracts into auction B some types with a lower value for object B than for object A. Precisely, given $r_B < r_A$, each type with $v_B \geq v_A \geq r_A$ participates in auction B – these are types above the graph of $g_{r_A}^{id}$ in Fig. 2 – but also the types with $v_A > r_A$ and $v_A > v_B$, as long as v_B is close enough to v_A, enter auction B to benefit from a lower reserve price: see the shaded region in Fig. 2 between the graph of g and the graph of $g_{r_A}^{id}$.

As we mentioned in Section 2, Burguet and Sakovics (1999) examine a setting in which the two auctions offer identical objects and each bidder j has a same value v_j for each of the two objects. Burguet and Sakovics (1999) prove that a unique symmetric bidders' equilibrium exists, and it is in mixed strategies. We explain in Section 5.4 why we obtain a different result, and how it affects the sellers' game.

[6] This intuition bas been formalized by Troncoso-Valverde (2014), and indeed equation (4) below is the same equation he obtains for the case in which $\bar{v} = 1$.

[7] In case a bidder is indifferent between the two auctions, we assume the bidder enters auction A. Since indifference involves a set of types with zero measure, this assumption has no practical consequences.

[8] As we mentioned in Section 2, Troncoso-Valverde (2014) proves Proposition 1 for the case of $[\underline{v}, \bar{v}] = [0, 1]$.

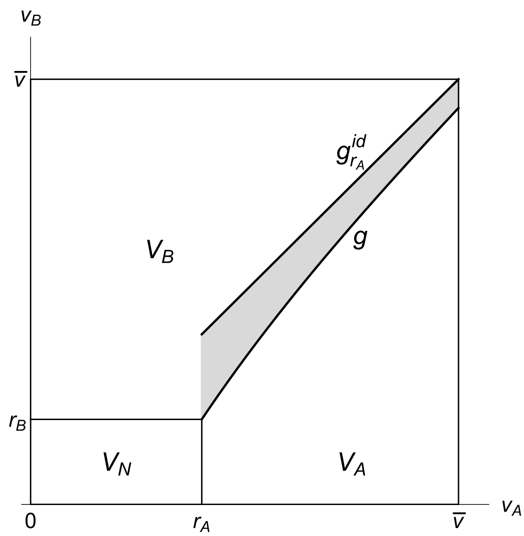


Fig. 2. Given $0 < r_B < r_A$, the types in the shaded region enter auction B even though $v_A > r_A$ and $v_A > v_B$.

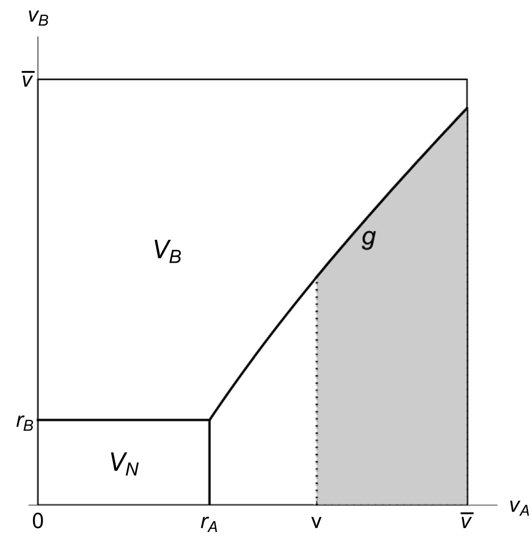


Fig. 3. Given r_A, r_B such that $0 < r_B < r_A$ and given $v \in (r_A, \bar{v})$, $F_A(v; r_A, r_B)$ is the measure of the non-shaded region.

5. The sellers' game

In this section we examine the game in which seller i selects a reserve price r_i, anticipating that given r_A, r_B, the bidders will choose among auction A, auction B, and non-participation according to the bidders' equilibrium described by Proposition 1. We suppose that the hazard rate $\frac{f(v)}{1-F(v)}$ is increasing, which implies that the virtual valuation function $J(v) = v - \frac{1-F(v)}{f(v)}$ is strictly increasing. In the next subsection we derive the expected revenue functions.

5.1. The revenue functions

Let $R_i(r_A, r_B)$ denote the expected revenue of seller i. In this subsection we derive an expression for $R_A(r_A, r_B)$ and for $R_B(r_A, r_B)$. Without loss of generality, we still suppose that $r_B \leq r_A$.

For each $v \in [r_A, \bar{v}]$ we denote by $F_A(v; \mathbf{r})$ the probability that a bidder stays out of auction A, or enters auction A but has a value for object A less than the given v, hence bids less than v in auction A. Therefore

$$F_A(v; \mathbf{r}) = 1 - \int_v^{\bar{v}} F(g(x); \mathbf{r})f(x)dx \qquad \text{for each } v \in [r_A, \bar{v}] \tag{5}$$

In Fig. 3, $F_A(v; \mathbf{r})$ is the measure of the non-shaded region. In order to interpret $F_A(v; \mathbf{r})$, consider for instance bidder 1, assume that he participates in auction A and that his value for object A is $v \geq r_A$. Then consider another bidder, say bidder 2. From the point of view of bidder 1, $F_A(v; \mathbf{r})$ is the probability he beats bidder 2 in auction A, either because bidder 2 does not enter auction A, or because bidder 2 participates in auction A but bids less than v.[9] We employ α to represent the probability that a bidder has values in V_A and then we see that $F_A(r_A; \mathbf{r}) = 1 - \alpha$. We extend F_A for $v < r_A$ in such a way that $F_A(v; \mathbf{r}) = 0$ for each $v < 0$, and $F_A(v; \mathbf{r}) = 1 - \alpha$ for each $v \in [0, r_A)$. Then F_A can be viewed as a c.d.f. with support $[0, \bar{v}]$,

In a similar way we define F_B. We first denote with g^{-1} the inverse function of g with respect to its first variable: for each $v \in [r_B, g(\bar{v}; \mathbf{r})]$, $g^{-1}(v; \mathbf{r})$ is defined as w such that $g(w; \mathbf{r}) = v$. Then, for each $v \in [r_B, \bar{v}]$, we let $F_B(v; \mathbf{r})$ represent the probability that a given bidder stays out of auction B, or enters auction B but has value for object B less than v, hence bids less than v in auction B:

$$F_B(v; \mathbf{r}) = \begin{cases} F(v)F(g^{-1}(v; \mathbf{r})) & \\ \quad + \int_{g^{-1}(v;\mathbf{r})}^{\bar{v}} F(g(x; \mathbf{r}))f(x)dx & \text{if } v \in [r_B, g(\bar{v}; \mathbf{r})] \\ F(v) & \text{if } v \in (g(\bar{v}; \mathbf{r}), \bar{v}] \end{cases} \tag{6}$$

and we extend F_B for $v < r_B$ as we described for F_A.

The functions F_A, F_B determine the sellers' revenues because for $i = A, B$, the c.d.f. F_i can be viewed, for each bidder, as the c.d.f. of the bid that the considered bidder makes in auction i, with the interpretation that a bid less than r_i is equivalent to the bidder not entering auction i. This reveals that seller i earns the same expected revenue as a monopolist seller setting reserve price r_i and facing n bidders, each with value distributed according to F_i. Therefore, for $v \geq r_i$ we let

$$J_i(v; \mathbf{r}) = v - \frac{1 - F_i(v; \mathbf{r})}{f_i(v; \mathbf{r})} \qquad \text{for } i = A, B$$

denote the virtual value function for F_i, in which $f_i(v; \mathbf{r})$ is the partial derivative of F_i with respect to its first variable, and we use J_i to evaluate $R_i(r_A, r_B)$ as

$$R_i(r_A, r_B) = \int_{r_i}^{\bar{v}} J_i(v; \mathbf{r})dF_i^n(v; \mathbf{r}) \tag{7}$$

That is, $R_i(r_A, r_B)$ is the expectation of the virtual value of the bidder with the highest value (as long as it is greater than r_i), out of n draws from the c.d.f. F_i.[10] It is important to notice that r_i affects both the interval of integration in the right hand side in (7), as in a monopoly auction, and the integrand function.[11] In next subsection, from (7) we derive a first order condition for a

[9] This approach is analogous to the one in McAfee (1993), and indeed the function F_A here is analogous to the function z introduced in (12) in McAfee (1993). But in our setting bidders have different values for the two objects, thus we introduce below a function F_B for auction B, and also F_B is analogous to the function z.

[10] Since we focus on the case of r_A, r_B such that $r_A \geq r_B$, it follows that $R_A(r_A, r_B)$, $R_B(r_A, r_B)$ introduced in (7) are defined only when $r_A \geq r_B$. In case that $r_B > r_A$, the revenues can be obtained by using (7) as follows: the revenue of seller A is equal to $R_B(r_B, r_A)$, the revenue of seller B is equal to $R_A(r_B, r_A)$.

[11] Notice that how F_i has been extended for values less than r_i does not affect $R_i(r_A, r_B)$ in (7).

symmetric equilibrium in the sellers' game. Before that, we make two remarks about F_A, F_B.

Remark 1. As it is intuitive, under duopoly each seller faces a worse distribution of bids in her auction compared to when she is monopolist. For instance, suppose that seller B is a monopolist as there is no seller A nor auction A. Then seller B faces n bidders, each with a value distributed according to F. Moreover, suppose that in such a setting seller B sets a reserve price equal to r_B. From (6) it follows that $F(v) \le F_B(v;\mathbf{r})$ for each $v \in [r_B, \bar{v}]$, with strict inequality for $v \in [r_B, g(\bar{v};\mathbf{r}))$, that is F first order stochastically dominates F_B. In this sense, for each $r_B \le r_A$ the existence of seller A makes seller B face a worse distribution of bids relative to a monopolist, thus lowering seller B's expected revenue. This occurs because each bidder has the option to enter auction A, and there are some types with $v_B > r_B$ who do not enter auction B as they prefer auction A. (These are the types with (v_A, v_B) below the graph of g.) Similarly, F first order stochastically dominates F_A in (5). Hence, each seller – independently of whether she plays the higher or the lower reserve price – faces a worse c.d.f. of bids with respect to a monopolist.[12] □

Remark 2. Here we introduce a c.d.f. $\tilde{F}$ which plays an important role in the following. Remember that when $r_A = r_B = r$, the solution to (4) is g_r^{id}. In this case the bidder's types who enter auction B (auction A) are those with $v_B > v_A$ and $v_B \ge r$ (with $v_A \ge v_B$ and $v_A > r$): see Fig. 1(b). Then F_B in (6) and F_A in (5) both coincide with $\tilde{F}$ such that[13]

$$\tilde{F}(v) = \frac{1}{2} + \frac{1}{2}F^2(v) \quad \text{for each } v \in [0,\bar{v}], \qquad \text{and}$$
$$\tilde{f}(v) = \tilde{F}'(v) = f(v)F(v) \tag{8}$$

Consistently with Remark 1, F first order stochastically dominates $F_B = \tilde{F}$. Since we consider equilibria such that $r_A = r_B = r$, the c.d.f. $\tilde{F}$ will appear in the first order condition determining the equilibrium reserve price and in the evaluation of the equilibrium revenue. □

5.2. First order condition for the equilibrium reserve price

We focus on symmetric Nash Equilibria, such that $r_A = r_B = r$, and without loss of generality we consider the point of view of seller B. From (7) with $i = B$, given $r_A = r$, we derive a first order condition for the equilibrium reserve price. Precisely, in (9) below we use $\frac{\partial F_B(v;\mathbf{r})}{\partial r_B}$ to denote the partial derivative of F_B with respect to its third variable r_B, and in particular $\frac{\partial F_B(r_B;\mathbf{r})}{\partial r_B}$ denotes such partial derivative evaluated at $v = r_B$. We obtain (the derivation steps are in Appendix C)

$$\begin{aligned}\frac{\partial R_B(r, r_B)}{\partial r_B} &= -nF_B^{n-1}(r_B;\mathbf{r})f_B(r_B;\mathbf{r})J_B(r_B;\mathbf{r}) \\ &\quad -nr_BF_B^{n-1}(r_B;\mathbf{r})\frac{\partial F_B(r_B;\mathbf{r})}{\partial r_B} - n(n-1) \\ &\quad \times \int_{r_B}^{\bar{v}} F_B^{n-2}(v;\mathbf{r})[1-F_B(v;\mathbf{r})]\frac{\partial F_B(v;\mathbf{r})}{\partial r_B}dv\end{aligned} \tag{9}$$

The term in the right hand side of the first line in (9) is the derivative of R_B with respect to the lower extreme of the integration interval in (7), and the terms in the second line in (9) represent the effect on the integrand function in (7) of a change in F_B determined by a change in r_B through g. Precisely, a change in r_B induces a shift of g which affects F_B through (6). Indeed, from (6) we find[14]

$$\begin{aligned}&\frac{\partial F_B(v,\mathbf{r})}{\partial r_B} \\ &= \int_{g^{-1}(v;\mathbf{r})}^{\bar{v}} f(g(x;\mathbf{r}))f(x)\frac{\partial g(x;\mathbf{r})}{\partial r_B}dx \quad \text{for each } v \in [r_B, g(\bar{v};\mathbf{r})]\end{aligned} \tag{10}$$

Since $r_A = r_B = r$ in a symmetric equilibrium, we evaluate (9) at $r_B = r$. Then g and g^{-1} coincide with g_r^{id}, F_B reduces to $\tilde{F}$ in (8), and (9) yields

$$\begin{aligned}\frac{\partial R_B(r,r)}{\partial r_B} &= -n\tilde{F}^{n-1}(r)\tilde{f}(r)\tilde{J}(r) \\ &\quad - nr\tilde{F}^{n-1}(r)\int_r^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv - n(n-1) \\ &\quad \times \int_r^{\bar{v}} \tilde{F}^{n-2}(v)(1-\tilde{F}(v))\int_v^{\bar{v}} f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}dudv\end{aligned} \tag{11}$$

in which

$$\tilde{J}(r) = r - \frac{1-\tilde{F}(r)}{\tilde{f}(r)}$$

is the virtual value function of $\tilde{F}$.[15]

In order to interpret (11), we notice that a reduction in r_B below r shifts g downward from g_r^{id} by Proposition 1, that is induces some types of bidder with $v_B < v_A$ to move from auction A to auction B – see the lighter shaded region in Fig. 4 – which increases R_B. We call this the *business stealing effect* (from auction A to auction B) of a reduction in r_B and it is represented by the terms in the second line in (11), in which $\frac{\partial g(v;r,r)}{\partial r_B}$ is the derivative of g with respect to r_B evaluated at $r_A = r$, $r_B = r$. But a decrease in r_B also induces the entry in auction B of some types with $v_B > v_A$ who would stay out of either auction without the reduction in r_B: see the darker shaded region in Fig. 4. Finally, $\Delta r_B < 0$ reduces the revenue of seller B if just one bidder enters auction B. The combination of the two latter effects is captured by the first term in the right hand side of (11). Because of the function $\tilde{J}$, we call this combination the *virtual value effect*, and notice that $-n\tilde{F}^{n-1}(r)\tilde{f}(r)\tilde{J}(r)$ can be written as $n\tilde{F}^{n-1}(r)(1-\tilde{F}(r)-r\tilde{f}(r))$, perhaps a more familiar expression. The virtual value effect is the unique effect at work in a monopoly context, in which a change in the reserve price does not affect the distribution of values of the bidders participating in the auction, but affects only the marginal type of bidder who participates. The virtual value of that type captures the effect on revenue of participation of such type.

In order to get a better intuition about the virtual value effect, consider the equation

$$\tilde{J}(r) = 0 \tag{12}$$

12 We thank one referee for suggesting this remark.

13 Precisely, from (6) we obtain $F_B(v) = \frac{1}{2} + \frac{1}{2}F^2(v)$ for each v in the interval $[r, \bar{v}]$. For $v \in [0, r)$ we extend F_B, that is $\tilde{F}$, as specified in (8), that is by using the same expression $\frac{1}{2}+\frac{1}{2}F^2(v)$. This allows to view $\tilde{F}$ as independent of r, which facilitates the interpretation in the following but does not affect revenues.

14 In the proof of Proposition 1 we show that g is a continuously differentiable function of (r_A, r_B).

15 In fact, as Footnote 10 explains, the revenue of seller B is given by $R_B(r_A, r_B)$ defined in (7) if and only if $r_A \ge r_B$. Thus $\frac{\partial R_B(r,r)}{\partial r_B}$ in (11) is the left derivative of R_B with respect to r_B, at $r_B = r$. We can obtain the right derivative recalling that when $r_B > r$ (here r_A is fixed at r) the revenue of seller B is $R_A(r_B, r)$, and taking the derivative of this function with respect to its first variable when that variable is equal to r. In Landi et al. (2023) we show that when $r_A = r_B$, the left and the right derivative coincide. Hence, $\frac{\partial R_B(r,r)}{\partial r_B} = 0$ from (11) is indeed the "bilateral" first order condition for maximization of the revenue of seller B with respect to r_B.

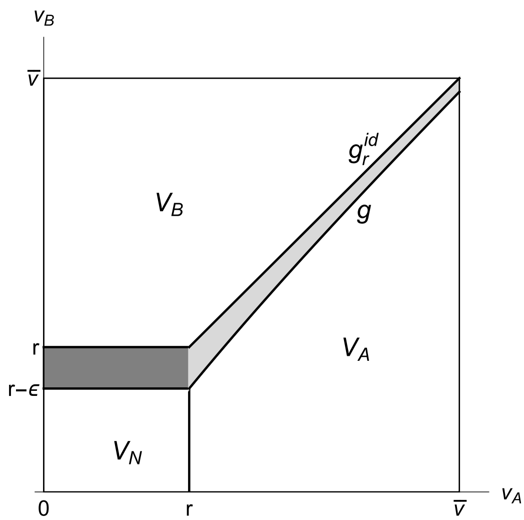


Fig. 4. The effect on V_B of a reduction in r_B from r to $r-\varepsilon$, given $r_A = r$.

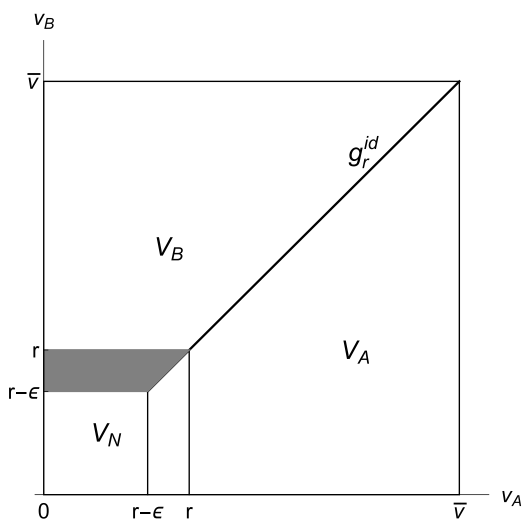


Fig. 5. Increased participation in auction B when $r_A = r_B$ and the common reserve price is reduced from r to $r-\varepsilon$.

for which there exists a unique solution in $(0, \bar{v})$,[16] which we denote with r^V (V is from *virtual value* effect). We can see why (12) and its solution r^V are relevant to us by considering a setting in which for each r_B chosen by seller B, r_A is set equal to r_B; we denote with r this common reserve price. Then a reduction in the common reserve price from r to $r-\varepsilon$ (with $\varepsilon > 0$) reduces the revenue in auction B if just one bidder enters auction B but also induces the participation in auction B of some types with $v_B > v_A$: see the shaded region in Fig. 5. However, $\Delta r < 0$ does not generate business stealing because r_A and r_B vary by the same amount, hence g does not shift and the set V_B widens only by the shaded region in Fig. 5. Therefore, here we have the same virtual value effect captured by the right hand side of the first line in (11). In fact, the darker shaded region in Fig. 4 does not coincide precisely with the shaded region in Fig. 5 because of the set of types in the triangle $\{(v_A, v_B) : v_A \in [r-\varepsilon, r]$ and $v_B \in [r-\varepsilon, v_A]\}$. But when ε is close to zero, the measure of this triangle is an infinitesimal of order greater than the measures of the other sets, hence it does not affect the derivative in (11).

In the context with equal reserve prices, $R_B(r,r)$ is equal to $\int_r^{\bar{v}} \tilde{J}(v_B) d\tilde{F}^n(v_B)$ and it is interesting to maximize $R_B(r,r)$. This is a standard monopoly problem in which the c.d.f. for each bidder's value is $\tilde{F}$. The optimal r solves (12), that is it eliminates the bidder's types with negative virtual valuation (or negative marginal revenues: see Bulow and Roberts (1989). Hence, the optimal r is equal to r^V.

Remark 3. On the interpretation of r^V. It may look unintuitive to maximize R_B with respect to r, given $r_A = r_B = r$, since r_A is not chosen by seller B. Alternatively, we may think of a seller running both auctions, and setting the same reserve price $r_A = r_B = r$. Then the seller's objective is $R_A(r,r) + R_B(r,r)$, which is equal to $2R_B(r,r)$ since $R_A(r,r) = R_B(r,r)$, and the optimal r is r^V also in this case. Thus we can view r^V as the optimal reserve price for a seller running both auctions. □

Now we let r^M denote the optimal reserve price that, say, seller B would set in the absence of auction A – in this case seller B is monopolist – and we show that $r^V > r^M$. For a monopolist facing bidders with values distributed according to the c.d.f. F, it is well known that r^M solves $J(r) = 0$, that is $r - \frac{1-F(r)}{f(r)} = 0$. From (8) it follows that the inequality $\frac{1-\tilde{F}(r)}{\tilde{f}(r)} > \frac{1-F(r)}{f(r)}$ holds for each $r \in (0, \bar{v})$, hence $\tilde{J}(r) < J(r)$. Therefore the virtual value $\tilde{J}$ for a seller running both auctions is lower than the virtual value for a single-object monopolist auction, therefore $r^V > r^M$. In other words, maximizing $R_B(r,r)$ leads to a reserve price greater than r^M, a result we record in Proposition 2 below. We remark that the inequality $\frac{1-\tilde{F}(v)}{\tilde{f}(v)} > \frac{1-F(v)}{f(v)}$, which implies $r^V > r^M$, is determined by the inequality $\tilde{f}(v) = f(v)F(v) < f(v)$. The latter inequality means that a reserve price reduction given the density $\tilde{f}$ is less effective in attracting new bidders into the auction than given the density f.

We now use (11) to show that if $(r_A, r_B) = (r^D, r^D)$ is an equilibrium, then $r^D < r^V$. In fact, the first term in the right hand side of (11), $-n\tilde{F}^{n-1}(r)\tilde{f}(r)\tilde{J}(r)$, is zero at $r = r^V$, since $\tilde{J}(r^V) = 0$, and is negative for $r > r^V$, since $\tilde{J}$ is strictly increasing (see Footnote 16). Furthermore, Proposition 1 reveals that $\frac{\partial g(v;r,r)}{\partial r_B} \geq 0$ for each $v \in [r, \bar{v}]$, and $g(r;r,r_B) = r_B$ implies $\frac{\partial g(r;r,r)}{\partial r_B} = 1$ and that $\frac{\partial g(v;r,r)}{\partial r_B} > 0$ for v close to r. Therefore, both $nr\tilde{F}^{n-1}(r)\int_r^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv$ and $n(n-1)\int_r^{\bar{v}} \tilde{F}^{n-2}(v)(1-\tilde{F}(v))\int_v^{\bar{v}} f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}du dv$ are positive for each $r \in [0, \bar{v})$. This implies $\frac{\partial R_B(r,r)}{\partial r_B} < 0$ for each $r \in [r^V, \bar{v})$ and $r^D < r^V$ – that is the business stealing effect makes r^D smaller than r^V. In other words, if seller A chooses $r_A \geq r^V$ then seller B is unwilling to set $r_B = r^V$ because a slightly smaller r_B induces a few types with $v_A > v_B$ to choose auction B, which widens the set V_B in a profitable way for seller B.

Proposition 2. *The expected revenue $R_B(r,r)$ is maximized at $r = r^V$ which solves (12), and r^V is greater than r^M, the optimal reserve price for a monopolist. If $(r_A, r_B) = (r^D, r^D)$ is an equilibrium when the two sellers compete, then $r^D < r^V$.*

From the perspective described in Remark 3, we conclude that a seller running both auctions – and choosing $r_A = r_B$ – sets a reserve price r^V higher than the reserve price r^M chosen by a monopolist who auctions a single object and faces no competition (and thus is more harmful in terms of social welfare). Duopoly competition leads to a reserve price lower than r^V, but the comparison between r^D and r^M is not immediate and we show in next subsection that it depends on the virtual value effect and the business stealing effect. We remark that the comparison between r^V and r^D involves settings with two objects on sale – in this case competition leads to lower reserve prices – whereas the

16 The property that the hazard rate of F is increasing implies that $\tilde{J}$ is strictly increasing.

comparison between r^M and r^D involves one setting with a single object and one with two objects – in this case competition (and a higher number of objects) may lead to higher reserve prices.

5.3. Nash equilibrium for the uniform environment

From (11) we see that evaluating $\frac{\partial g(v;r,r)}{\partial r_B}$ is necessary in order to evaluate $\frac{\partial R_B(r,r)}{\partial r_B}$. In Appendix C we show that it is possible to represent $\frac{\partial g(v;r,r)}{\partial r_B}$ through the solution of a variational matrix differential equation derived from (4). Unfortunately, no analytic solution is available for this equation, but once F is specified, numerical methods can be applied to solve the equation. Thus we focus on the case in which F is the c.d.f. of the uniform distribution with support $[0, 1]$ and we consider n in the following set $\mathcal{N}$:

$$\mathcal{N} = \{2, 5, 10, 15, 20\} \tag{13}$$

For each $n \in \mathcal{N}$, given $r < \bar{v}$ we evaluate (11) and determine r which satisfies $\frac{\partial R_B(r,r)}{\partial r_B} = 0$; we denote such r with r_n^D. In (14) below we report r_n^D for each $n \in \mathcal{N}$ – in addition to r^M, r^V which do not depend on n – after rounding to the third decimal digit (see Landi et al. (2023). We conjecture that $(r_A, r_B) = (r_n^D, r_n^D)$ is an equilibrium of the sellers' game, and in order to prove this conjecture we need to show that $R_B(r_n^D, r_B)$, a function of r_B, is maximized at $r_B = r_n^D$. The lack of an explicit solution to (4) prevents us from proving this result analytically. However, for each given r_B we can use Wolfram Mathematica to solve (4) numerically, which allows to determine the c.d.f. F_B and to evaluate $R_B(r_n^D, r_B)$. Then we consider the finite set S_n consisting of each number in $[0, 1)$ that differs from r_n^D by a multiple (negative or positive) of $\frac{1}{1000}$ – see (15) below – and suppose that each seller's reserve price must be chosen from this set. By evaluating $R_B(r_n^D, r_B)$ numerically we find that $R_B(r_n^D, r_n^D) > R_B(r_n^D, r_B)$ for each $r_B \in S_n \setminus \{r_n^D\}$, and for each $n \in \mathcal{N}$.[17] Therefore, for each $n \in \mathcal{N}$ we conclude that $(r_A, r_B) = (r_n^D, r_n^D)$ is indeed an equilibrium of the sellers' game when S_n is the set of feasible reserve prices for each seller. We have also verified that $r_B = r_n^D$ is a local maximum point for $R_B(r_n^D, r_B)$ as we find that $\frac{\partial^2 R_B(r_n^D, r_n^D)}{\partial r_B^2}$ is negative.[18] Fig. 6 shows the graph we obtain for $R_B(r_2^D, r_B)$ when $n = 2$.

Proposition 3. *Suppose that each valuation is uniformly distributed with support* $[0, 1]$. *For each n in the set $\mathcal{N}$ in (13), consider r_n^D as follows:*

$$r_2^D = 0.445, \quad r_5^D = 0.479, \quad r_{10}^D = 0.496, \quad r_{15}^D = 0.503, \quad r_{20}^D = 0.507; \quad r^M = 0.5, \quad r^V = 0.577 \tag{14}$$

Then $(r_A, r_B) = (r_n^D, r_n^D)$ is an equilibrium of the sellers' game in which the set of feasible reserve prices for each seller is

$$S_n = \{r_B \in [0, 1) : r_B = r_n^D + \frac{1}{1000} z \quad \textit{for some integer number } z\} \tag{15}$$

Proposition 3 establishes that a pure-strategy equilibrium exists in the sellers' game, at least for the environments it covers, unlike the sellers' game examined by Burguet and Sakovics (1999) with homogenous objects. In Section 5.4 we discuss the relationship between the two results.

Moreover, Proposition 3 shows that the equilibrium reserve price under duopoly can be smaller or higher than under monopoly. Precisely, (14) shows that $r_n^D < r^M$ when n is relatively small, but $r_n^D > r^M$ holds for large n. We can understand this result if we recall the virtual value effect and the business stealing effect introduced in Section 5.2. The former effect tends to favor a reserve price higher than r^M – recall from Proposition 2 that $r^V > r^M$ – but if n is small business stealing provides an incentive for each seller to be aggressive because g is relatively sensible to differences in reserve prices when n is small. This pushes the equilibrium reserve price below r^M.

However, the next proposition describes a feature of the function g which makes the business stealing effect weak for a large n.

Proposition 4.

Given r_A, r_B such that $r_B \le r_A$, let g be the unique solution to (4), $g(r_A; \boldsymbol{r}) = r_B$. Then:

(i) As n tends to $+\infty$, g converges uniformly to $g^{id}_{\hat{v}_A}$ in the interval $[\hat{v}_A, \bar{v}]$ for each $\hat{v}_A \in (r_A, \bar{v})$.

(ii) Given an arbitrary $r \in (0, \bar{v})$, let $\theta = \max_{w \in [r, \bar{v}]} \frac{f(w)}{F(w)} > 0$. Then we have

$$\left(\frac{1+F^2(r)}{1+F^2(v)}\right)^{n-1}\left(1 - 2\theta\frac{n-1}{n-2}(v - r)\right) < \frac{\partial g(v; r, r)}{\partial r_B} < \left(\frac{1+F^2(r)}{1+F^2(v)}\right)^{n-1} \quad \textit{for each } v \in (r, \bar{v}] \tag{16}$$

Proposition 4(i) implies that for a large n, a difference between reserve prices has a small effect on bidders' auction choices since for given $r_B < r_A$, g becomes close to $g^{id}_{r_A}$ and the set of types with $v_A > v_B$ who participate in auction B is small. In fact, if a bidder with value v_i participates in auction i, he (wins and) pays r_i if and only if no other bidder enters auction i. But when n is large, such event is very unlikely. It is much more likely that a substantial number of bidders participate in either auction, and so it is more important for a bidder to be in the auction for the object for which he has a higher value, even though the reserve price in that auction is higher. Hence, r_B smaller than r_A induces only a small set of types with $v_A > v_B$ to enter auction B.

By Proposition 4(i), for a large n a reduction in r_B below $r_A = r$ shifts g only slightly below the identity function, and indeed (16) in Proposition 4(ii) reveals that $\frac{\partial g(v_A;r,r)}{\partial r_B}$ converges to 0 for each $v_A \in (r, \bar{v}]$ as n tends to infinity. This suggests that the business stealing effect is weak when n is large and provides a reduced incentive to sellers to be aggressive. In fact, also the virtual value effect weakens as n tends to infinity, since $-n\tilde{F}^{n-1}(r)\tilde{f}(r)\tilde{J}(r)$ in (11) tends to 0. This is linked to the fact that $n\tilde{F}^{n-1}(r)\tilde{f}(r)$ is the density of the highest out of n draws from the c.d.f. $\tilde{F}$, and when n is large almost all the probability is concentrated near $\bar{v}$; thus the probability that the highest draw is close to $r < \bar{v}$ is very small. However, our numerical results indicate that the business stealing effect weakens with respect to the virtual value effect when n increases, and this makes r^D greater than r^M.

From a different perspective, suppose we are at $r_A = r_B = r^M$. We notice that for, say, seller B the virtual value effect favors an increase in r_B above r^M, since $\tilde{J}(r^M) < 0$. For a large n, the business stealing effect is weak with respect to the virtual value effect. Thus the bidder's types who move from auction B to auction A because of $\Delta r_B > 0$ constitute a set with measure close to 0, which does not discourage seller B from increasing r_B above r^M. This suggests that r^D is greater than r^M when n is large,

[17] Numerical methods involve approximation errors, but (i) $R_B(r_n^D, r_n^D)$ can be evaluated analytically without any approximation error; (ii) Mathematica allows to set an upper bound for approximation errors, and the upper bound on the approximation error when evaluating $R_B(r_n^D, r_B)$ is smaller, for each $r_B \in S_n \setminus \{r_n^D\}$, than the difference between $R_B(r_n^D, r_n^D)$ and the evaluated $R_B(r_n^D, r_B)$: see Landi et al. (2023).

[18] In Landi et al. (2023) we derive variational equations to evaluate $\frac{\partial^2 g(v;r_n^D,r_n^D)}{\partial r_B^2}$ and $\frac{\partial^2 R_B(r_n^D,r_n^D)}{\partial r_B^2}$.

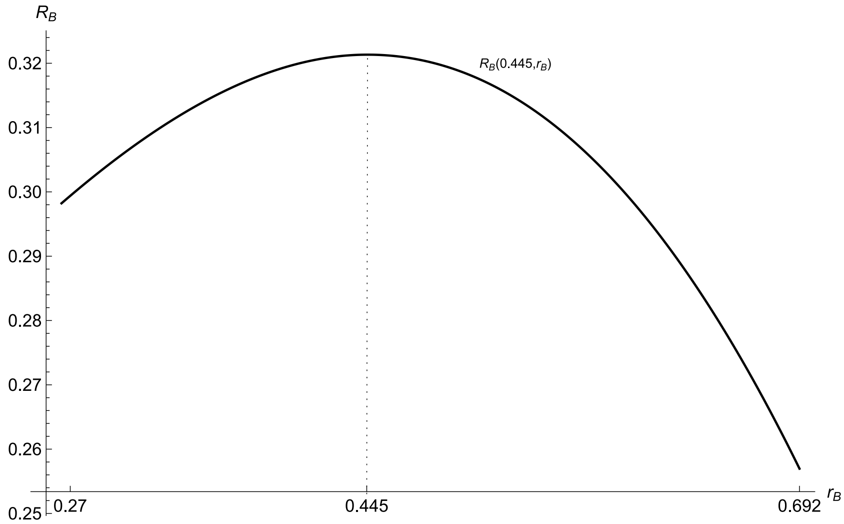


Fig. 6. The graph of $R_B(r_2^D, r_B)$ when $n = 2$, with $r_2^D = 0.445$.

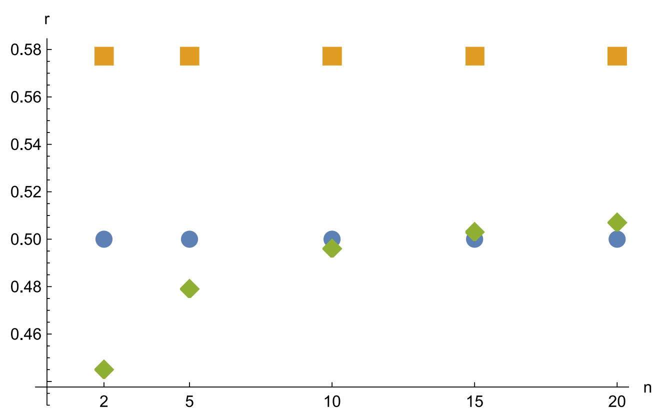


Fig. 7. Equilibrium reserve prices under duopoly (diamonds) given $n = 2, 5, 10, 15, 20$, and r^M (circles), r^V (squares) when F is uniform on $[0, 1]$.

as Proposition 3 indicates for a specific environment.[19] Fig. 7 represents graphically r^M, r^V, and r_n^D for $n \in \mathcal{N}$.[20]

Corollary 5. *There exist duopoly settings with a discretized set of feasible reserve prices (with step of 0.001), in which the equilibrium reserve price is greater than the reserve price under monopoly.*

When $r^M < r^D$, some bidder's types have zero utility under duopoly even though they would earn positive utility under monopoly. In order to see this, suppose that seller B is monopolist as there is no auction A and, therefore, only the value v_B is relevant. Then consider a type with values (v_A, v_B) such that $v_A < r^D$ and $r^M < v_B < r^D$. Under monopoly of seller B, this type has a positive utility from participating in auction B since $v_B > r^M$. But since v_A and v_B are both smaller than r^D, under duopoly this type does not participate in any auction; hence he is harmed by the existence of a second auction.

Although (14) suggests that r_n^D increases with n, we remark that as n tends to $+\infty$ it is not the case that the equilibrium reserve price converges to r^V. Proposition 6 below establishes that there exists an upper bound r' for r^D, and that r' is smaller than r^V. Such result does not rely on uniformly distributed values, nor on discretizations or numeric analysis, but relies on the lower bound for $\frac{\partial g(v_A;r,r)}{\partial r_B}$ provided by (16).

Proposition 6. *There exists $r' < r^V$ such that for each n, if r satisfies $\frac{\partial R_B(r,r)}{\partial r_B} = 0$ then $r < r'$.*

We mentioned above that the business stealing effect weakens with respect to the virtual value effect as n increases, but Proposition 6 establishes that it does not disappear in the limit as n tends to $+\infty$, and rather makes r^D not only smaller than r^V, but also bounded away from r^V for each n. In the following we provide a few details about how this result is established, which clarify the root of the result. First notice that $n\tilde{F}^{n-1}(r) > 0$ for each $r \in [0, \bar{v})$, then observe that the left hand side in (17) is equal to the right hand side in (11) divided by $n\tilde{F}^{n-1}(r)$, since $-\tilde{f}(r)\tilde{J}(r) = 1 - \tilde{F}(r) - r\tilde{f}(r)$. Hence, we can write $\frac{\partial R_B(r,r)}{\partial r_B} = 0$ as

$$1 - \tilde{F}(r) - r\tilde{f}(r) - r\int_r^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv - \int_r^{\bar{v}} \frac{\tilde{F}^{n-2}(v)(1-\tilde{F}(v))}{\tilde{F}^{n-1}(r)}\left((n-1)\int_v^{\bar{v}} f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}du\right)dv = 0 \tag{17}$$

From our discussion just before Proposition 2, we see that the left hand side in (17) is negative for each $r \geq r^V$. Considering $r < r^V$, the upper bound for $\frac{\partial g(v;r,r)}{\partial r_B}$ in (16) implies that $\lim_{n\to+\infty} \frac{\partial g(v;r,r)}{\partial r_B} = 0$ and $\lim_{n\to+\infty} r\int_r^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv = 0$. However, using the lower bound for $\frac{\partial g(v;r,r)}{\partial r_B}$ in (16) we prove in Appendix E that the term $\int_r^{\bar{v}} \frac{\tilde{F}^{n-2}(v)(1-\tilde{F}(v))}{\tilde{F}^{n-1}(r)}\left((n-1)\int_v^{\bar{v}} f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}du\right)dv$ is positive and bounded away from zero. As a consequence, there exists a left neighborhood of r^V in which the left hand side in (17) is negative for each n and each solution to (17) is smaller than an r' which is less than r^V.

We notice that $\int_r^{\bar{v}} \frac{\tilde{F}^{n-2}(v)(1-\tilde{F}(v))}{\tilde{F}^{n-1}(r)}\left((n-1)\int_v^{\bar{v}} f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}du\right)dv$ is linked to the effect of a change in g on the c.d.f. of the

19 Proposition 3 also reveals that the equilibrium reserve price under duopoly depends on the number of bidders, as n affects both $\frac{\partial g(v;\mathbf{r})}{\partial r_B}$ and the effect of a change in r_B on $F_B^{(n)}$ in (7) when $i = B$. Under monopoly, the optimal reserve price is constant with respect to n when the virtual valuation function is increasing, but may depend on n otherwise: see Menicucci (2021).

20 In Appendix F we allow for $\underline{v} > 0$ and provide an example in which $r^D < \underline{v}$, although under monopoly the seller never sets the reserve price below $\underline{v}$.

Table 1
r_{ab}, r^M and r^V for $a, b \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ and $n = 2$.

a/b	1			2			3			4		
1	0.445	0.500	0.577	0.318	$0.\overline{333}$	0.393	0.243	0.250	0.297	0.196	0.200	0.239
2	0.485	0.577	0.669	0.385	0.422	0.500	0.312	$0.\overline{333}$	0.400	0.262	0.276	0.333
3	0.506	0.630	0.723	0.428	0.486	0.571	0.361	0.398	0.474	0.311	0.338	0.405
4	0.515	0.669	0.760	0.456	0.536	0.623	0.397	0.450	0.530	0.349	0.389	0.462

second highest of n draws from the c.d.f. F_B, denoted $F_B^{(2)}$ (see the derivation of $\frac{\partial R_B(r,r)}{\partial r_B}$ in Appendix C). When n is large, a small change in r_B generates a tiny change in g, which leads to a tiny change in F_B. But this generates a non-negligible change in $F_B^{(2)}$ as a large n makes $F_B^{(2)}$ very sensible to a change in F_B; this explains why the last term in (17) is bounded away from 0.

The result that the equilibrium reserve price under duopoly may be greater than under monopoly turns out to hold also in the model of Parlane (2008), although such result is not mentioned in Parlane (2008). In Section 5.5 we describe this result, provide an intuition for it, and compare it to our result.

Remark 4. Business stealing effect and virtual value effect for some Beta distributions. The Beta distribution has support [0, 1] and density $f(x) = x^{a-1}(1-x)^{b-1}/B(a, b)$, with $B(a, b) = \int_0^1 x^{a-1}$ $(1-x)^{b-1}dx$. After setting $n = 2$, we have used numerical methods to solve $\frac{\partial R_B(r,r)}{\partial r_B} = 0$ for $a = 1, 2, 3, 4$ and $b = 1, 2, 3, 4$, denoting the solution with r_{ab}.[21] We also computed the corresponding values of r^M and r^V. The results are reported in Table 1.

We find that as a (b) increases, r_{ab}, r^M, r^V all increase (decrease), as it is intuitive since the Beta distribution with a higher a (b) first order stochastically dominates (is first order stochastically dominated by) the Beta distribution with a lower a (b). Whereas we find $r_{ab} < r^M$ for each a, b we consider, we see that the difference $r^M - r_{ab}$ increases (decreases) as a (b) increases. In particular, the virtual valuation effect – which favors an increase in r_{ab} above r^M – becomes stronger as a increases, but also the business stealing effect – which favors a decrease in r_{ab} below r^M – becomes stronger as a increases and the strengthening of the latter is greater than the strengthening of the former. The result is that r_{ab} is increasingly distant from r^M as a increases. The reverse occurs if b increases, as then both the virtual value effect and the business stealing effect becomes weaker, but the weakening of the latter is more significant than the weakening of the former, and r_{ab} is increasingly close to r^M as b increases. □

5.4. Comparison with Burguet and Sakovics (1999)

In the setting with homogenous objects of Burguet and Sakovics (1999), each bidder j has a same (bidder-specific) value v_j for each of the two objects; v_j is ex ante i.i.d. across bidders, with support [0, 1] and c.d.f. F. We here illustrate and explain the main differences between the results in our paper and in Burguet and Sakovics (1999).

5.4.1. Differences in the bidders' game

Given $r_B \leq r_A$, Burguet and Sakovics (1999) prove that a unique symmetric bidders' equilibrium exists, it is in mixed strategies and is characterized by a cutoff $w \geq r_A$ as follows:

$$\begin{cases} \text{each bidder with value } v < r_B \text{ stays out of each auction} \\ \text{each bidder with value } v \in [r_B, w) \text{ enters auction } B \\ \text{each bidder with } v \in [w, 1] \text{ randomizes,} \\ \qquad \text{entering either auction with probability } \frac{1}{2} \end{cases} \tag{18}$$

and type w is indifferent between the auctions, a condition that reduces to

$$r_A = w - \frac{\int_{r_B}^{w}(F(z) + \frac{1}{2} - \frac{1}{2}F(w))^{n-1}dz}{\left(\frac{1}{2} + \frac{1}{2}F(w)\right)^{n-1}} \tag{19}$$

In (19), the left hand side is the expected payment of type w in case he participates in auction A and wins – in view of (18), he wins if and only if no other bidder enters auction A, and then pays r_A. The right hand side is the expected payment of type w if he participates in auction B and wins.[22] Since type w has the same probability $\left(\frac{1}{2} + \frac{1}{2}F(w)\right)^{n-1}$ to win in either auction, he is indifferent between the auctions if and only if the two expected payments are the same, which (19) establishes. In particular, $w = r_A$ if and only if $r_B = r_A$. Moreover, $w = 1$ if r_A is quite larger than r_B, as then the right hand side in (19) is less than the left hand side for each $w \in [r_B, 1]$.

In order to explain the randomized entry choices of each type in the interval $[w, 1]$, Burguet and Sakovics (1999) show that in equilibrium these types need to be indifferent between the auctions.[23] Since indifference holds for type w, the utility of type $v > w$ from participating in auction A must grow at the same rate, as v grows, as the utility from entering auction B; this occurs if and only if each type in $[w, 1]$ selects each auction with probability $\frac{1}{2}$. Hence, the entry choice of a bidder with value above w does not depend on r_A, r_B, although w does. For instance, suppose that $r_A = r_B = r$, which implies $w = r$. Then consider a reduction of r_B to $r - \varepsilon$, with $\varepsilon > 0$. This creates an interval $[r - \varepsilon, w)$ of types, with $w > r$, who enter auction B with probability 1. But $\Delta r_B < 0$ does not affect the entry choices of bidders with value above w, that is it has only a local effect, affecting bidders with value close to r but not bidders with high value.

Conversely, in our setting with heterogeneous objects no mixed strategies emerge in equilibrium because there is only a zero measure set of bidders which are indifferent between the auctions, that is the types with values on the graph of g. More in detail, recall that for $i = A, B$, $F_i(v_i; \mathbf{r})$ can be interpreted as the

[21] We have not analyzed a seller's profitability from deviating from r_{ab} given that the other seller chooses r_{ab}.

[22] Precisely, the right hand side is determined by deriving a c.d.f. F_B for the bid submitted by any bidder in auction B (as we have done in (6) for our setting), which is $F_B(v; \mathbf{r}) = F(v) + \frac{1}{2} - \frac{1}{2}F(w)$ for each $v \in [r_B, w]$. Then we evaluate the expectation of the highest between r_B and $n-1$ realizations from F_B, given that each realization is less than w.

[23] There cannot be an interval I included in $[w, 1]$ such that all types in I prefer auction B to A (say), because if such interval existed then all types in I would enter auction B and there would be more competition in auction B than in A. Thus the types in I would earn a lower utility in auction B than in A, which is a contradiction.

probability that a bidder with value v_i for object i participating in auction i beats another given bidder relative to auction i. Then we apply Lemma 2 in Myerson (1981), which implies that for a bidder who participates in auction A and has valuation v_A for object A, the utility is the integral of the probability to win the auction, over the values smaller than v_A, plus the utility of the lowest type – which here is zero. Hence, for a bidder with values (v_A, v_B), and $v_i > r_i$, the expected utility from participating in auction i is

$$u_i(v_i; \mathbf{r}) = \int_{r_i}^{v_i} F_i^{n-1}(z; \mathbf{r})dz \tag{20}$$

Then, as Troncoso-Valverde (2014) remarks, given $v_A \geq r_A$ and $v_B \geq r_B$, a bidder is indifferent between the two auctions if and only if $u_A(v_A; \mathbf{r}) = u_B(v_B; \mathbf{r})$. Since u_A, u_B are strictly increasing functions, it follows that the equality $u_A(v_A; \mathbf{r}) = u_B(v_B; \mathbf{r})$ holds only in a set of (v_A, v_B) with zero measure. For the types in this set it is optimal to choose randomly or deterministically the auction to participate in (about this, see Footnote 7), but this has no effect on the sellers' revenues.

Another difference with respect to the case of homogenous objects is revealed by Proposition 1: Given $r_A = r_B = r$, a reduction in r_B below r reduces $g(v_A; \mathbf{r})$, with respect to $g_r^{id}(v_A) = v_A$, for each $v_A \in [r, \bar{v}]$. Therefore, a reduction in r_B determines a downward shift of the whole graph of g, affecting the bidders' entry decisions not only locally, but also of bidders with values significantly different from (r, r).

5.4.2. Differences in the sellers' game

In the context of Burguet and Sakovics (1999), no symmetric pure-strategy equilibrium exists in the sellers' game, unlike for the settings we have considered in Section 5.3. In order to explain this difference, fix an arbitrary $r \in (0, 1)$ in the model with homogenous objects and suppose that $r_A = r_B = r$; thus $w = r$. As we described above, a reduction in r_B to $r - \varepsilon$, with $\varepsilon > 0$ and small, generates an interval $[r - \varepsilon, w)$ of types, with $w > r$, who participate in auction B for sure. Before $\Delta r_B = -\varepsilon$, the types in the interval $[r - \varepsilon, r)$ do not participate in any auction whereas the types in $[r, w)$ choose either auction with probability $\frac{1}{2}$. Thus the latter types' entry in auction B with probability 1 is the business stealing effect of $\Delta r_B = -\varepsilon$. Even though $\Delta r_B < 0$ reduces seller B's revenue when just one bidder enters auction B, the key aspect is that from (19) it follows that $\frac{dw}{dr_B} = -\infty$, that is a slight reduction of r_B below r generates a proportionally huge increase in w, hence a very significant entry in auction B due to business stealing. This dominates the negative effect mentioned above, hence a small $\Delta r_B < 0$ is a profitable deviation for seller B and no equilibrium exists such that $r_A = r_B = r \in (0, 1)$ because of a strong business stealing effect.[24] In order to see why $\frac{dw}{dr_B} = -\infty$, notice that the right hand side in (19) coincides with the bid of type w in a first-price auction with n bidders, reserve price r_B and c.d.f. for each value equal to $F(v) + \frac{1}{2} - \frac{1}{2}F(w)$ for each $v \in [r_B, w]$. Such bid is strictly increasing in r_B and in w, hence $\Delta r_B < 0$ implies $\Delta w > 0$ in order to keep (19) satisfied. But at $r_A = r_B = r$ we have that $w = r$ and the equilibrium bidding function in a first-price auction is flat at the value r [see for instance Hu et al. (2010). Hence, starting from $w = r$, a small $\Delta w > 0$ has a zero first order effect on the right hand side in (19) and a small $\Delta r_B < 0$ requires a proportionally very large $\Delta w > 0$ to satisfy (19); therefore $\frac{dw}{dr_B} = -\infty$.

Conversely, with heterogenous objects a small reduction in r_B never generates a large change in a cutoff. We consider $r_A = r_B = r > 0$, so that $g = g_r^{id}$, $F_B = \tilde{F}$, and pick values (v_A, v_B) such that $v_A = \hat{v}_A \in (r, \bar{v})$, $v_B = g_r^{id}(\hat{v}_A) = \hat{v}_A$. Here indifference is not equivalent to equal expected payment in the two auctions, hence we argue in terms of utility rather than in terms of payments. We notice that a small reduction in r_B below r has the effect of increasing the utility from participating in auction B: see (20) with $i = B$ and with $F_B = \tilde{F}$. This implies that the cutoff $g(\hat{v}_A)$ (given $v_A = \hat{v}_A$) decreases below $\hat{v}_A$ in order for type $(v_A, v_B) = (\hat{v}_A, g(\hat{v}_A; \mathbf{r}))$ to be indifferent between the auctions.[25] But the derivative of $u_B(v_B; \mathbf{r})$ with respect to v_B at $v_B = g(\hat{v}_A; \mathbf{r})$ is strictly positive, hence it follows that $\frac{\partial g(\hat{v}_A; \mathbf{r})}{\partial r_B}$ is bounded, that is it is not equal to $-\infty$. Thus, it never occurs that a small $\Delta r_B < 0$ increases participation in auction B in a proportionally huge measure like with homogenous objects. Hence, it is not the case that a small $\Delta r_B < 0$ is necessarily a profitable deviation for seller B, which is consistent with our results in Proposition 3.

In a sense, the difference between auction competition with homogenous goods and auction competition with heterogenous goods is similar to the difference between the standard Bertrand duopoly and a duopoly with differentiated products like the Hotelling duopoly, for instance. In the former, there exists no equilibrium in which the equilibrium price is greater than the marginal cost because even a tiny downward deviation of a firm makes the demand for the firm's product jump up discontinuously, thus the deviation is profitable. In the auction competition setting of Burguet and Sakovics (1999) there is no discontinuity but the proportionally huge increase in w following a small $\Delta r_B < 0$ has the same effect in terms of profitable deviation.

Conversely, in a duopoly with differentiated products like the Hotelling model the demand for each firm is continuous and smooth, so that a small downward deviation is not necessarily profitable, and indeed often an equilibrium exists in which the equilibrium price is greater than the marginal cost. Likewise, in our auction competition setting the function g is a continuous and smooth function of r_B; thus a small $\Delta r_B < 0$ is not automatically profitable for seller B. Therefore, as suggested by one referee, a way to interpret our result is that heterogeneity in bidders' values for the objects softens competition among sellers to the point that a pure-strategy equilibrium may exist in the sellers' game.

5.5. Comparison with Parlane (2008)

In Parlane (2008) there are two sellers, A and B, and products are differented à la Hotelling, with bidders distributed on the interval $[0, 1]$; H denotes the c.d.f. for the bidders' locations in $[0, 1]$. Seller A (B) is located at $\theta = 0$ (at $\theta = 1$). A bidder with location $\theta \in [0, 1]$ has value $w_A(\theta) = 1 - t\theta$ for object A and value $w_B(\theta) = 1 - t(1 - \theta)$ for object B, with $t \in (0, 1)$. When r_A, r_B are such that the market is fully covered ($V_N = \varnothing$ in our notation), Parlane (2008) shows that there exists a marginal type – we denote its location with $\hat{\theta}_D$ – such that the types in $[0, \hat{\theta}_D)$ enter auction A, the types in $(\hat{\theta}_D, 1]$ enter auction B, and type $\hat{\theta}_D$ is indifferent between the auctions. Hence, $\hat{\theta}_D$ satisfies

$$[1 - H(\hat{\theta}_D)]^{n-1}[w_A(\hat{\theta}_D) - r_A] = H^{n-1}(\hat{\theta}_D)[w_B(\hat{\theta}_D) - r_B] \tag{21}$$

In order to fix the ideas and to simplify the exposition, we suppose that H is the uniform c.d.f. over $[0, 1]$, and we focus on

[24] A different argument shows that $(r_A, r_B) = (0, 0)$ is not an equilibrium since seller A gains from increasing r_A above zero. This is less relevant for us, as the main difference between the two settings is that with heterogenous objects the derivative of $g(v; \mathbf{r})$ with respect to r_B is bounded, as we illustrate below. But when the support $[\underline{v}, \bar{v}]$ for the values is such that $\underline{v} > 0$, it is possible that an equilibrium with $r_A = r_B = r < \underline{v}$ exists (see Virag (2010)) because the equation that determines w is different from (19) and is such that dw/dr_B is negative but is not $-\infty$. Thus a small $\Delta r_B < 0$ is not necessarily a profitable deviation for seller B.

[25] Actually, there is also an effect generated by the fact that the whole graph of g shifts downward after $\Delta r_B < 0$. This increases F_A, reduces F_B and favors auction A over B. But this effect is relatively weak and is not crucial for our argument.

symmetric equilibria.[26] Then the results in Parlane (2008) imply that in equilibrium the market is fully covered, with $\hat{\theta}_D = \frac{1}{2}$, and the symmetric equilibrium reserve price r^D is

$$r^D = \begin{cases} 1 - \frac{1}{2}t - \frac{1}{n}(1 - \frac{3}{2}t) & \text{if } t \in (0, \frac{2}{3}) \\ 1 - \frac{1}{2}t & \text{if } t \in [\frac{2}{3}, 1) \end{cases} \tag{22}$$

The bidder located at $\hat{\theta}_D = \frac{1}{2}$ is the one for whom max $\{w_A(\theta), w_B(\theta)\}$ is minimal, and this bidder has value $1 - \frac{1}{2}t$ for either object. His utility is 0 if $t \in [\frac{2}{3}, 1)$ because then $r^D = 1 - \frac{1}{2}t$; his utility is positive if $t < \frac{2}{3}$ because then $r^D < 1 - \frac{1}{2}t$.

Now suppose for one moment that seller B is a monopolist – i.e. there is no auction A. Then a bidder located at θ enters auction B if and only if $w_B(\theta) \geq r_B$, that is if and only if $\theta \geq \hat{\theta}_M$, with $\hat{\theta}_M = 1 - \frac{1-r_B}{t}$. The optimal reserve price for the monopolist seller B is

$$r^M = \begin{cases} 1 - t & \text{if } t \in (0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } t \in [\frac{1}{2}, 1) \end{cases} \tag{23}$$

From (22), (23) it follows that $r^M < r^D$ for each $t \in (\frac{2}{n+3}, 1)$. Hence, except for small t, duopoly increases the reserve price above r^M.

In order to see why r^D may be greater than r^M, suppose $t > \frac{1}{2}$, hence $r^M = \frac{1}{2}$, and consider duopoly with $r_A = r_B = \frac{1}{2}$. Then (21) implies that bidder's types split between the auctions with $\hat{\theta}_D = \frac{1}{2}$ and each type earns a positive rent. As we illustrate below, in this context either seller, for instance seller B, has an incentive to increase her reserve price above $\frac{1}{2}$, which explains why $r^D > r^M$. Precisely, both under monopoly and under duopoly, $\Delta r_B > 0$ has the positive effect of increasing seller B's revenue when just one bidder enters auction B, but has also the effect of moving to the right the marginal type, which reduces entry in auction B; under duopoly, this is the opposite of business stealing. Under monopoly, starting from $r_B = r^M$, a $\Delta r_B > 0$ necessarily has a negative net effect as we know that $r_B = r^M$ is optimal. But under duopoly, a key difference is that the entry-reduction effect of $\Delta r_B > 0$ is *smaller* than under monopoly, that is $\frac{d\hat{\theta}_D}{dr_B} < \frac{d\hat{\theta}_M}{dr_B}$, hence seller B loses entrants in auction B at a lower rate than under monopoly. Coupled with the positive effect of a higher revenue when just one bidder enters auction B, this makes a small $\Delta r_B > 0$ profitable for seller B.

In order to see why $\frac{d\hat{\theta}_D}{dr_B} < \frac{d\hat{\theta}_M}{dr_B}$, notice that under duopoly, given $r_A = r_B = \frac{1}{2}$ and $\hat{\theta}_D = \frac{1}{2}$, a small $\Delta r_B > 0$ decreases $w_B(\hat{\theta}_D) - r_B$ in (21), which suggests that $\hat{\theta}_D$ needs to increase to keep (21) satisfied. But after writing (21) as $\left(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)}\right)^{n-1} [w_A(\hat{\theta}_D) - r_A] = w_B(\hat{\theta}_D) - r_B$, we notice that as $\hat{\theta}_D$ increases, the difference $w_A(\hat{\theta}_D) - r_A$ decreases, that is the value of object A minus r_A for the marginal type decreases because of the negative correlation between the objects' values. Moreover, an increase in $\hat{\theta}_D$ increases for each bidder the probability $H(\hat{\theta}_D)$ to enter auction A and decreases the probability $1 - H(\hat{\theta}_D)$ to enter auction B; thus the term $\left(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)}\right)^{n-1}$ decreases and this effect is stronger the greater is n.[27] Both the decrease in $w_A(\hat{\theta}_D) - r_A$ and the decrease in $\left(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)}\right)^{n-1}$ temper the increase in $\hat{\theta}_D$ needed to satisfy (21), and make the marginal type grow significantly less under duopoly than under monopoly. Indeed, from $\hat{\theta}_M = 1 - \frac{1-r_B}{t}$ we see that $\frac{d\hat{\theta}_M}{dr_B} = \frac{1}{t}$, whereas from (21) – given $r_A = r_B = r$ – it follows that

$$\frac{d\hat{\theta}_D}{dr_B} = \frac{1}{2(n-1)(2-t-2r)h(1/2) + 2t} \tag{24}$$

in which h is the density of H. At $r = \frac{1}{2}$ we have $\frac{d\hat{\theta}_D}{dr_B} = \frac{1}{2(n-1)(1-t)h(1/2)+2t}$, which is less than $\frac{d\hat{\theta}_M}{dr_B}$.

The key difference between duopoly and monopoly is that in the latter setting, a small $\Delta r_B > 0$ generates a change in $\hat{\theta}_M$ determined by the fact that for each bidder, not entering auction B yields a constant utility of zero. Conversely, under duopoly each bidder not entering auction B has the opportunity to enter auction A, which is a better alternative to staying out for types with θ close to $\frac{1}{2}$. But as $\hat{\theta}_D$ increases, auction A becomes relatively less attractive to type $\hat{\theta}_D$ since $w_A(\hat{\theta}_D) - r_A$ decreases, as well as $\left(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)}\right)^{n-1}$. This makes the marginal type increase less than under monopoly after $\Delta r_B > 0$, and ultimately leads to $r^D > r^M$.

The more standard inequality $r^M > r^D$ holds when t is small, because in such case $r^M = 1-t$ and starting from $r_A = r_B = 1-t$, $\hat{\theta}_D$ is significantly more sensible to a change in r_B than when t is large and $r_A = r_B = \frac{1}{2}$.[28] This makes it profitable for seller B to decrease, rather than increase, r_B below r^M and explains why $r^D < r^M$.

As a final remark, we notice that for a given $t > 0$, even if very small, a large n implies $\frac{2}{n+3} < t$ and therefore $r^D > r^M$. As (24) shows, a large n makes $\frac{d\hat{\theta}_D}{dr_B}$ close to zero because $\left(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)}\right)^{n-1}$ decreases quickly as $\hat{\theta}_D$ increases above $\frac{1}{2}$. Hence, starting from $r_A = r_B = r^M$, a small $\Delta r_B > 0$ moves the marginal type only slightly rightward, so that the entry-reduction effect for auction B is dominated by the higher revenue when a single bidder enters auction B. Thus it is profitable for seller B to increase r_B above r^M.

The arguments above suggest that a small $\frac{d\hat{\theta}_D}{dr_B}$ is important for the inequality $r^D > r^M$ to hold. A small $\frac{d\hat{\theta}_D}{dr_B}$ can be seen as a weak business stealing effect, and in Section 5.3 we have remarked that a weak business stealing effect plays an important role for our result in Corollary 5. In the setting of Parlane (2008), negatively correlated values contribute to weak business stealing, but in both settings a large number of bidders weakens business stealing. A difference between the two settings is that in Parlane (2008), $r^D > r^M$ occurs only if the market is fully covered (which is true under the assumptions in this subsection), hence a bidder type who leaves auction B because of $\Delta r_B > 0$ switches to auction A. Instead, in our setting $\Delta r_B > 0$ induces both some types to move from auction B to A, and some types to exit auction B to remain out of each auction. The size of this set of types, in addition to business stealing, affects the profitability of $\Delta r_B > 0$ for seller B.

6. Conclusions

In this paper we examine competing auctions when each bidder has uncorrelated values for the objects on sale, whereas most of the literature deals with valuations that are perfectly correlated, positively or negatively. The latter assumption implies that each bidder's private information is one-dimensional. In

26 Parlane (2008) allows for more general distributions and allows for asymmetric equilibria.

27 The decrease in $(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)})^{n-1}$ means that auction A becomes more crowded – i.e., more competitive – than auction B.

28 A small t makes the objects almost homogeneous, which magnifies the importance of a change in a reserve price. Moreover, a decrease in $\left(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)}\right)^{n-1}$ has a reduced effect on the left hand side of $\left(\frac{1-H(\hat{\theta}_D)}{H(\hat{\theta}_D)}\right)^{n-1} [w_A(\hat{\theta}_D) - r_A] = w_B(\hat{\theta}_D) - r_B$ because $[w_A(\hat{\theta}_D) - r_A]$ is smaller with respect to when $r_A = r_B = \frac{1}{2}$. Thus a higher increase in $\hat{\theta}_D$ is necessary, that is $\hat{\theta}_D$ reacts more to $\Delta r_B > 0$.

our setting, a bidimensional private information for each bidder complicates the analysis, but it is still possible to prove the existence of a unique symmetric equilibrium in the game of bidders' auction choice – as in Troncoso-Valverde (2014) – and to determine some of its features, which provide useful insights for the study of the sellers' game of reserve price setting. We also employ numeric analysis to reach some conclusions about equilibria in environments with uniformly distributed values and a discretized action set.

Precisely, for (say) seller B a reduction in r_B has a more subtle effect on bidders' entry in auction B with respect to when seller B is monopolist, as under duopoly each bidder has the option of entering auction A, and the value of such option depends on r_A and on the bidder's values. Precisely, $\Delta r_B < 0$ induces the entry in auction B of some bidder's types who would otherwise stay out of each auction. The existence of auction A makes this set of marginal types for seller B smaller than under monopoly, and this tends to lift the equilibrium reserve price above the optimal reserve price under monopoly. The reduction in r_B attracts also types who would otherwise enter auction A, and this business stealing effect tends to reduce the equilibrium reserve price, in some cases significantly below the monopoly level. But the business stealing effect is weak when the number of bidders is large because then a difference in reserve prices affects the entry decisions of a set of types with small measure. Indeed, in the setting we consider a higher n increases the equilibrium reserve price, eventually above the monopoly level.

It would be useful if further research could extend our Proposition 3 in terms of allowing for a continuum of feasible reserve prices, and more general value distributions. In particular, we have assumed that for each bidder, his values for the two objects are identically distributed. It may be interesting to allow for asymmetries in the value distributions for objects of different sellers in order to study – for instance – competition under vertical differentiation. It would also be interesting to extend the analysis in this paper to the case of arbitrary numbers of sellers and bidders, and eventually study the limit as the market becomes very large. However, it appears difficult even to determine a bidders' equilibrium when there are $k \geq 3$ sellers and each bidder has k-dimensional private information.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Derivation of (4)

Using F_A and F_B from (5) and (6), we can derive each bidder's utility from participating in either auction as described in (20). More in detail,

$$u_A(v_A; \mathbf{r}) = \int_{r_A}^{v_A} F_A^{n-1}(z; \mathbf{r})dz \text{ for each } v_A \in [r_A, \bar{v}] \quad \text{and}$$
$$u_B(v_B; \mathbf{r}) = \int_{r_B}^{v_B} F_B^{n-1}(z; \mathbf{r})dz \text{ for each } v_B \in [r_B, \bar{v}] \tag{25}$$

A bidder with values $v_A \geq r_A$, $v_B \geq r_B$ is indifferent between auction A and auction B if and only if $u_A(v_A; \mathbf{r}) = u_B(v_B; \mathbf{r})$. Therefore, the property that indifference holds when $v_B = g(v_A; \mathbf{r})$ is equivalent to

$$u_A(v_A; \mathbf{r}) = u_B(g(v_A; \mathbf{r}); \mathbf{r}) \tag{26}$$

From (25) we notice that $u_A(r_A; \mathbf{r}) = u_B(r_B; \mathbf{r}) = 0$, hence $g(r_A; \mathbf{r}) = r_B$. Moreover, differentiating the two sides of (26) with respect to v_A yields

$$F_A^{n-1}(v_A; \mathbf{r}) = \frac{\partial g(v_A; \mathbf{r})}{\partial r_A} F_B^{n-1}(g(v_A; \mathbf{r}); \mathbf{r}) \quad \text{for each } v_A \in [r_A, \bar{v}] \tag{27}$$

Substituting $v_B = g(v_A; \mathbf{r})$ in (6) we obtain $F_B(g(v_A; \mathbf{r}); \mathbf{r}) = F(g(v_A; \mathbf{r}))F(v_A) + \int_{v_A}^{\bar{v}} F(g(v; \mathbf{r}))f(v)dv$. This and (5) reveal that (27) reduces to (4).

Appendix B. Proof of Proposition 1

Proof of existence of bidders' equilibrium if g satisfies (4) and $g(r_A) = r_B$ We begin by proving that if g satisfies (26), that is if g solves (4) and $g(r_A; \mathbf{r}) = r_B$, then there exists a bidders' equilibrium in which each bidder enters auction A (auction B) if his values (v_A, v_B) belong to V_A (belong to V_B), but stays out of each auction if $(v_A, v_B) \in V_N$. Precisely, if $n-1$ bidders follow this participation decision based on V_N, V_A, V_B, then we show here that it is optimal for the remaining bidder to do the same. As we explained in Section 4, it is immediate that a bidder does not participate in any auction if $(v_A, v_B) \in V_N$ but participates in auction A (in auction B) if $v_A \geq r_A$ and $v_B < r_B$ (if $v_A < r_A$ and $v_B \geq r_B$). In case that the remaining bidder's values are such that $v_A \geq r_A$, $v_B \geq r_B$ and $v_B < g(v_A; \mathbf{r})$, then $(v_A, v_B) \in V_A$ and $u_A(v_A; \mathbf{r}) = u_B(g(v_A; \mathbf{r}); \mathbf{r}) > u_B(v_B; \mathbf{r})$, where the equality comes from (26), the inequality comes from the fact that u_B in (25) is strictly increasing. Hence, this type with $(v_A, v_B) \in V_A$ prefers auction A to B. Arguing likewise, we see that if $v_B > g(v_A; \mathbf{r})$, then $(v_A, v_B) \in V_B$ and $u_A(v_A; \mathbf{r}) = u_B(g(v_A; \mathbf{r}); \mathbf{r}) < u_B(v_B; \mathbf{r})$. Hence, this type with $(v_A, v_B) \in V_B$ prefers auction B to A.

Proof of existence of a unique solution to (4), $g(r_A) = r_B$ To shorten a bit the notation, in this proof we write $g(v)$ rather than $g(v; \mathbf{r})$; notice that r_A, r_B are fixed throughout this proof. Here we show that there exists a unique solution to (4), $g(r_A) = r_B$.

We define

$$G(v_A) = 1 - \int_{v_A}^{\bar{v}} F(g(v))f(v)dv \tag{28}$$

and show that a unique solution exists for the system, derived from (4) and (28), which consists of

$$\begin{cases} G'(v_A) = F(g(v_A))f(v_A) \\ g'(v_A) = \left(\frac{G(v_A)}{F(g(v_A))F(v_A)+1-G(v_A)}\right)^{n-1} \end{cases}, \qquad v_A \in [r_A, \bar{v}]. \tag{29}$$

with $g(r_A) = r_B$. Besides, by the definition of $g(r_A)$ and according to (28),

$$G(\bar{v}) = 1, \qquad g(r_A) = r_B. \tag{30}$$

It is evident that any solution of (29), (30) which has positive components with g strictly increasing provides a solution to (4), $g(r_A) = r_B$. We show below that there exists a unique solution to (29), (30). Precisely, we consider the Cauchy problem given by (29) and the initial conditions

$$G(r_A) = \gamma, \qquad g(r_A) = r_B \tag{31}$$

in which $\gamma \in [0, 1)$, and prove that for it there exists a unique solution. Then we show that there exists a unique γ in $(0, 1)$ such that the associated solution satisfies $G(\bar{v}) = 1$, consistently with (30).

For (29), (31), local existence and uniqueness of a solution follow from standard results [see for instance Theorem 2 in Pontryagin (1962)] as long as the denominator $\Delta(v_A) = F(g(v_A))F(v_A) + 1 - G(v_A)$ at the right-hand side of the second equation of (29) is positive, and hence the right-hand side is C^1-smooth.

We consider the set

$$\{(v_A, g, G) \in \mathbb{R}^3 : v_A \in [r_A, \bar{v}], g \geq r_B, G \geq 0, F(g)F(v_A)+1-G > 0\} \tag{32}$$

and now show that it is invariant for the system (29). We have that $\Delta(r_A) = F(r_B)F(r_A) + 1 - \gamma > 0$. Differentiating $\Delta(v_A)$, by virtue of (29) we get

$$\Delta'(v_A) = F(v_A)f(g(v_A))g'(v_A) = F(v_A)f(g(v_A))\frac{(G(v_A))^{n-1}}{(\Delta(v_A))^{n-1}}$$

and hence

$$\frac{d\Delta^n(v_A)}{dv_A} = nG^{n-1}(v_A)f(g(v_A))F(v_A) \geq 0.$$

Therefore, $0 < \Delta(r_A) \leq \Delta(v_A)$ for each $v_A \in (r_A, \bar{v}]$.

To prove global existence and uniqueness of the solution on the interval $[r_A, \bar{v}]$ it suffices to demonstrate that the trajectory $(G(v_A), g(v_A))$ of (29) is bounded on the interval $[r_A, \bar{v}]$ [see for instance Proposition (B) in Section 24 in Pontryagin (1962)]. It is immediate from (29) that $G'(v_A) \leq M_f$ for each $v_A \in [r_A, \bar{v}]$, with $M_f = \max_{v\in[0,\bar{v}]} f(v) > 0$, and hence $G(v_A)$ is bounded. Regarding g, we recall that $0 < \Delta(r_A) \leq \Delta(v_A)$ and notice that $G(v_A) < 1 + M_f(v_A - r_A)$. Then

$$g'(v_A) < \left(\frac{1 + M_f(v_A - r_A)}{\Delta(v_A)}\right)^{n-1} < \left(\frac{1 + M_f\bar{v}}{\Delta(r_A)}\right)^{n-1} = \mu$$

and thus $g(v_A) \leq r_B + \mu(\bar{v} - r_A)$. Note that also $\Delta(v_A)$ is bounded: there exists M_Δ such that $0 < \Delta(v_A) \leq M_\Delta$ for each $v_A \in [r_A, \bar{v}]$.

As far as the right-hand side of (29) is C^1-smooth in the domain (32), the solution (G, g) to (29) depends C^1-smoothly on its initial data.

Now we prove that there exists a unique γ such that the unique solution to (29), (31) satisfies $G(\bar{v}) = 1$.

Since $G(v_A)$ is continuous with respect to the initial data γ, we first demonstrate that there exist γ_1, γ_2 such that the associated solutions (G_1, g_1), (G_2, g_2) satisfy $G_1(\bar{v}) < 1 < G_2(\bar{v})$, and then that $G(\bar{v})$ is strictly increasing with respect to γ.

We take $\gamma_1 = 0$ and prove that $G_1(\bar{v}) < 1$. Indeed

$$G_1(\bar{v}) = \int_{r_A}^{\bar{v}} F(g_1(v))f(v)dv < F(g_1(\bar{v}))\int_{r_A}^{\bar{v}} f(v)dv$$
$$= F(g_1(\bar{v}))(1 - F(r_A)) \leq F(g_1(\bar{v})) \leq 1.$$

Now we take $\gamma_2 = 1 - \delta$ with $\delta > 0$. Then the function Δ_2 is such that $\Delta_2(r_A) = F(g_2(r_A))F(r_A) + \delta \geq \delta > 0$. For the corresponding solution of (29), (31) we obtain

$$G_2(\bar{v}) = 1 - \delta + \int_{r_A}^{\bar{v}} F(g_2(v))f(v)dv \geq 1 - \delta + m_f\int_{r_A}^{\bar{v}} F(g_2(v))dv,$$

where $m_f = \min_{v\in[0,\bar{v}]} f(v) > 0$. By virtue of (29), $(F(g_2(v)))' = f(g_2(v))\frac{(G_2(v))^{n-1}}{(\Delta_2(v))^{n-1}} \geq m_f\frac{(1-\delta)^{n-1}}{M_{\Delta_2}^{n-1}}$ and $F(g_2(v)) \geq m_f\frac{(1-\delta)^{n-1}}{M_{\Delta_2}^{n-1}}(v-r_A)$. Substituting it into the inequality for $G_2(\bar{v})$ we conclude that

$$G_2(\bar{v}) \geq 1 - \delta + m_f^2\frac{(1-\delta)^{n-1}}{M_{\Delta_2}^{n-1}}\frac{(\bar{v} - r_A)^2}{2}$$

which is greater than 1, provided that $\delta > 0$ is chosen sufficiently small.

Finally, we prove that the initial data $G(r_A) = \gamma \in [0, 1)$ for which $G(\bar{v}) = 1$ is uniquely defined and also the solution to (4) is unique. More generally, we prove below that if $\gamma_1 < \gamma_2$ and the associated solutions are (G_1, g_1), (G_2, g_2), then

$$g_1(v_A) < g_2(v_A) \quad \text{and} \quad G_1(v_A) < G_2(v_A) \quad \text{for each} \quad v_A \in [r_A, \bar{v}] \tag{33}$$

Therefore, in particular, $G(\bar{v})$ is strictly increasing with respect to γ and hence there exists a unique γ such that $G(\bar{v}) = 1$.

By the continuity of the solutions of (29), (31), the inequalities (33) hold for each $v_A \in [r_A, r_A + \delta]$ for some sufficiently small $\delta > 0$. Assuming that $G_1(v_A) \geq G_2(v_A)$ for some $v_A \in (r_A + \delta, \bar{v}]$, we take $\hat{v}_A = \min\{v_A \in [r_A + \delta, \bar{v}] : G_1(v_A) \geq G_2(v_A)\}$. Then

$$\hat{v}_A > r_A + \delta,\ G_1(\hat{v}_A) = G_2(\hat{v}_A),\ G_1(v_A) < G_2(v_A)\ \forall v_A \in [r_A, \hat{v}_A).$$

If $g_1(v_A) \leq g_2(v_A)$ for each $v_A \in [r_A, \hat{v}_A)$, then

$$G_1(\hat{v}_A) = \gamma_1 + \int_{r_A}^{\hat{v}_A} F(g_1(v_A))f(v_A)dv_A < \gamma_2$$
$$+ \int_{r_A}^{\hat{v}_A} F(g_2(v_A))f(v_A)dv_A = G_2(\hat{v}_A)$$

contradicting $G_1(\hat{v}_A) = G_2(\hat{v}_A)$. This implies that $g_1(v_A) > g_2(v_A)$ for some $v_A \in [r_A, \hat{v}_A)$. Let $\tilde{v}_A = \min\{v_A \in [r_A + \delta, \hat{v}_A) : g_1(v_A) \geq g_2(v_A)\}$, and note that $\tilde{v}_A > r_A + \delta$, $g_1(\tilde{v}_A) = g_2(\tilde{v}_A)$, $G_1(\tilde{v}_A) < G_2(\tilde{v}_A)$. From the second equation in (29) it follows that $g_1'(\tilde{v}_A) < g_2'(\tilde{v}_A)$, which together with $g_1(\tilde{v}_A) = g_2(\tilde{v}_A)$ implies $g_1(v_A) > g_2(v_A)$ for v_A slightly smaller than $\tilde{v}_A$, in contradiction to the definition of $\tilde{v}_A$. This establishes the second inequality in (33).

The first inequality in (33) is proved similarly. Given $G_1(v_A) < G_2(v_A)$ in $[r_A, \bar{v}]$, if $g_1(v_A) \geq g_2(v_A)$ holds for some v_A, then take $\tilde{v}_A = \min\{v_A \in [r_A + \delta, \bar{v}] : g_1(v_A) \geq g_2(v_A)\}$. Again, $\tilde{v}_A > r_A + \delta$, $g_1(\tilde{v}_A) = g_2(\tilde{v}_A)$, $G_1(\tilde{v}_A) < G_2(\tilde{v}_A)$. Therefore $g_1'(\tilde{v}_A) < g_2'(\tilde{v}_A)$ follows, which yields a contradiction.

Proof that $g(v_A)$ is strictly increasing with respect to r_B for each $v_A \in [r_A, \bar{v}]$ Consider $r_A, r_B, \hat{r}_B$ such that $r_B < \hat{r}_B < r_A$. Let g be the solution to (4) with $g(r_A) = r_B$ and $\hat{g}$ be the solution to (4) with $\hat{g}(r_A) = \hat{r}_B$. We show that $g(v_A) < \hat{g}(v_A)$ for each $v_A \in [r_A, \bar{v}]$.

Assuming the converse, suppose there exists $v_A \in (r_A, \bar{v}]$ such that $g(v_A) = \hat{g}(v_A)$. Let $\hat{v}_A = \max\{v_A \in (r_A, \bar{v}] : g(v_A) = \hat{g}(v_A)\}$. If $\hat{v}_A = \bar{v}$, then $g(\bar{v}) = \hat{g}(\bar{v})$. Since $G(\bar{v}) = \hat{G}(\bar{v}) = 1$, by the uniqueness theorem we conclude that $g(v_A) = \hat{g}(v_A)$ for each $v_A \in [r_A, \bar{v}]$, contradicting $g(r_A) = r_B < \hat{g}(r_A) = \hat{r}_B$.

If $\hat{v}_A < \bar{v}$ and, say,

$$g(v_A) < \hat{g}(v_A) \text{ for each } v_A \in (\hat{v}_A, \bar{v}] \tag{34}$$

then $\int_{\hat{v}_A}^{\bar{v}} F(g(v))f(v)dv < \int_{\hat{v}_A}^{\bar{v}} F(\hat{g}(v))f(v)dv$. Using this inequality together with $g(\hat{v}_A) = \hat{g}(\hat{v}_A)$, and (4), we conclude that $g'(\hat{v}_A) > \hat{g}'(\hat{v}_A)$, and hence $g(v_A) > \hat{g}(v_A)$ in some right semi-neighborhood of $\hat{v}_A$, which contradicts (34). The case in which $g(v_A) > \hat{g}(v_A)$ for each $v_A \in (\hat{v}_A, \bar{v}]$ is treated similarly.

Proof that $g(v)$ is strictly decreasing with respect to r_A Consider $r_A, \hat{r}_A, r_B$ such that $r_B < r_A < \hat{r}_A$. Let g be the solution to (4) with $g(r_A) = r_B$ and $\hat{g}$ be the solution to (4) with $\hat{g}(\hat{r}_A) = r_B$. We show that $\hat{g}(v_A) < g(v_A)$ for each $v_A \in [\hat{r}_A, \bar{v}]$.

Notice that $g(\hat{r}_A) > r_B$ since $g(r_A) = r_B$, $r_A < \hat{r}_A$, and g is strictly increasing in v; let $\mu = g(\hat{r}_A)$. Then, in the interval $[\hat{r}_A, \bar{v}]$, g solves (4) with the initial condition $g(\hat{r}_A) = \mu$, whereas $\hat{g}$ solves (4) with the initial condition $\hat{g}(\hat{r}_A) = r_B$. Since $\mu > r_B$, it follows from the result proved above that $g(v_A) > \hat{g}(v_A)$ in the interval $[\hat{r}_A, \bar{v}]$.

Proof that $g(v_A) < v_A$ for each $v_A \in [r_A, \bar{v}]$ This is a consequence of the fact that if $r_B = r_A$, then $g(v_A) = v_A$ for each $v_A \in [r_A, \bar{v}]$, and g is strictly increasing with respect to r_B. Thus if r_B is reduced below r_A, then we obtain $g(v_A) < v_A$ for each $v_A \in [r_A, \bar{v}]$.

Appendix C. Derivation of (9), of $\frac{\partial g(v;r,r)}{\partial r_B}$ in (11)

To derive $\frac{\partial R_B(r,r_B)}{\partial r_B}$, consider $r_B \le r_A$ and notice that $R_B(r, r_B)$ can be written as $\int_{r_B}^{\bar{v}} [vnF_B^{n-1}(v;\mathbf{r})f_B(v;\mathbf{r}) - n(1-F_B(v;\mathbf{r}))F_B^{n-1}(v;\mathbf{r})]dv$. Then integration by parts of the first term in the integrand function yields $R_B(r, r_B) = \bar{v} - r_B F_B^n(r_B;\mathbf{r}) - \int_{r_B}^{\bar{v}} F_B^{(2)}(v;\mathbf{r})dv$, in which $F_B^{(2)}$ is the c.d.f. of the second highest valuation out of n draws from the c.d.f. F_B, that is $F_B^{(2)}(v;\mathbf{r}) = nF_B^{n-1}(v;\mathbf{r}) - (n-1)F_B^n(v;\mathbf{r})$. Therefore

$$\begin{aligned}\frac{\partial R_B(r,r_B)}{\partial r_B} &= -F_B^n(r_B;\mathbf{r}) - nr_B F_B^{n-1}(r_B;\mathbf{r})\left(f_B(r_B;\mathbf{r}) + \frac{\partial F_B(r_B;\mathbf{r})}{\partial r_B}\right)\\ &\quad + F_B^{(2)}(r_B;\mathbf{r}) - n(n-1)\int_{r_B}^{\bar{v}} F_B^{n-2}(v;\mathbf{r})[1-F_B(v;\mathbf{r})]\\ &\quad \times \frac{\partial F_B(v;\mathbf{r})}{\partial r_B}dv\\ &= nF_B^{n-1}(r_B;\mathbf{r})[1-F_B(r_B;\mathbf{r}) - r_B f_B(r_B;\mathbf{r})]\\ &\quad - nr_B F_B^{n-1}(r_B;\mathbf{r})\frac{\partial F_B(r_B;\mathbf{r})}{\partial r_B}\\ &\quad - n(n-1)\int_{r_B}^{\bar{v}} F_B^{n-2}(v;\mathbf{r})[1-F_B(v;\mathbf{r})]\frac{\partial F_B(v;\mathbf{r})}{\partial r_B}dv\end{aligned}$$

which coincides with (9).

From (11) we see that evaluating $\frac{\partial R_B(r,r)}{\partial r_B}$ requires $\frac{\partial g(v;r,r)}{\partial r_B}$, and to this purpose we use the notation $G(v;\gamma,r,r_B)$ and $g(v;\gamma,r,r_B)$ (r between γ and r_B is the value of r_A) to emphasize the dependence of the solution of (29) on the initial data in (31). We need to take into account that r_B affects g both directly through the second initial condition in (31), and also indirectly through γ – in the first initial condition – which needs to satisfy

$$G(\bar{v};\gamma,r,r_B) = 1 \tag{35}$$

as we set in (30). Below, we apply the Implicit Function Theorem to (35) to conclude that there exists a differentiable function $\gamma(r_B)$, defined in a neighborhood of r, such that $G(\bar{v};\gamma(r_B),r,r_B) = 1$ for each r_B in such neighborhood of r. Therefore, g is written as $g(v;\gamma(r_B),r,r_B)$ and the derivative of this function with respect to r_B at $r_B = r$ is denoted with $\frac{\partial g(v;r,r)}{\partial r_B}$ in the main text and in Lemma 1 below.

We begin by introducing the variational system

$$\begin{bmatrix} y'(v)\\ z'(v)\end{bmatrix} = M(v)\begin{bmatrix} y(v)\\ z(v)\end{bmatrix} \quad\text{for}\quad v\in[r,\bar{v}] \tag{36}$$

Here $M(v)$ is the Jacobi matrix of the right hand side of (29) in the variables G, g evaluated at the solution when $r_A = r_B = r$. In such case, the solution is known to be $g(v;\gamma(r),r,r) = v$ for $v \in [r,\bar{v}]$, hence $G(v;\gamma(r),r,r) = \frac{1}{2} + \frac{1}{2}F^2(v) = \tilde{F}(v)$ from (29) (and $\gamma(r) = \tilde{F}(r)$). Therefore

$$M(v) = \begin{bmatrix} 0 & f^2(v)\\ \frac{2(n-1)}{\tilde{F}(v)} & -\frac{(n-1)\tilde{f}(v)}{\tilde{F}(v)}\end{bmatrix}$$

and (36) reduces to

$$\begin{cases} y'(v) = f^2(v)z(v)\\ z'(v) = \frac{2(n-1)}{\tilde{F}(v)}y(v) - \frac{(n-1)\tilde{f}(v)}{\tilde{F}(v)}z(v)\end{cases} \quad\text{for}\quad v\in[r,\bar{v}] \tag{37}$$

Now we rely on a classical result (see Theorem 18 in Pontryagin (1962), which establishes that G, g which solve (29) are C^1 functions of γ, r_B, and the vector of partial derivatives $\begin{bmatrix}\frac{\partial G(v;\gamma(r),r,r)}{\partial\gamma}\\ \frac{\partial g(v;\gamma(r),r,r)}{\partial\gamma}\end{bmatrix}$ satisfies the variational system (37) and the initial condition

$$\begin{bmatrix} y(r)\\ z(r)\end{bmatrix} = \begin{bmatrix} 1\\ 0\end{bmatrix}. \tag{38}$$

We denote with y_1, z_1 the solution to (37) with the initial condition (38). Similarly, $\begin{bmatrix}\frac{\partial G(v;\gamma(r),r,r)}{\partial r_B}\\ \frac{\partial g(v;\gamma(r),r,r)}{\partial r_B}\end{bmatrix}$ solves the variational system (37), with the initial condition

$$\begin{bmatrix} y(r)\\ z(r)\end{bmatrix} = \begin{bmatrix} 0\\ 1\end{bmatrix} \tag{39}$$

We denote with y_2, z_2 the solution to (37) with the initial condition (39). As we demonstrate in Lemma 1 below, the solution to (37), (38) satisfies $y_1(\bar{v}) > 0$. Hence, $\frac{\partial G(\bar{v};\gamma(r),r,r)}{\partial\gamma} > 0$ and by the Implicit Function Theorem there exists a differentiable function $\gamma(r_B)$ such that $G(\bar{v};\gamma(r_B),r,r_B) = 1$ for each r_B close to r, with $\gamma'(r) = -\frac{\frac{\partial G(\bar{v};\gamma(r),r,r)}{\partial r_B}}{\frac{\partial G(\bar{v};\gamma(r),r,r)}{\partial\gamma}} = -\frac{y_2(\bar{v})}{y_1(\bar{v})}$.

The derivative of $g(v;\gamma(r_B),r,r_B)$ with respect to r_B, at $r_B = r$, is $\frac{\partial g(v;\gamma(r),r,r)}{\partial\gamma}\gamma'(r) + \frac{\partial g(v;\gamma(r),r,r)}{\partial r_B}$ and is denoted as $\frac{\partial g(v;r,r)}{\partial r_B}$ in the main text. Lemma 1 expresses $\frac{\partial g(v;r,r)}{\partial r_B}$ in terms of the solutions to (37), (38) and to (37), (39).

Lemma 1. *For any $r \in [0,\bar{v})$,*

$$\frac{\partial g(v;r,r)}{\partial r_B} = z_2(v) - \frac{y_2(\bar{v})}{y_1(\bar{v})}z_1(v) \quad \textit{for each } v\in[r,\bar{v}]$$

where y_1, z_1 solve (37), (38) and y_2, z_2 solve (37), (39).

Proof. We only need to show that $y_1(\bar{v}) > 0$, as the rest of the proof has been accomplished before the statement of the lemma. We show that $y_1(v) > 1$ for each $v \in [r,\bar{v}]$. Since $z_1(r) = 0$ and $z_1'(r) = \frac{2(n-1)}{\tilde{F}(r)} > 0$, it follows that $z_1(v) > 0$ in the interval $[r, r+\delta]$ for a small $\delta > 0$. Suppose there exists $v > r+\delta$ such that $z_1(v) \le 0$, and let $v^* = \min\{v \in [r+\delta,\bar{v}] : z_1(v) \le 0\}$; then $z_1(v^*) = 0$ and $z_1(v) > 0$ in (r, v^*). Moreover, $y_1(v) > 0$ in $(r, v^*]$ since $y_1(r) = 1$ and $y_1'(v) = f^2(v)z_1(v) > 0$ in (r, v^*). Finally, notice that $z_1'(v^*) = \frac{2(n-1)}{\tilde{F}(v^*)}y_1(v^*) > 0$. Since $z_1(v^*) = 0$, this implies $z_1(v) < 0$ for v slightly smaller than v^*, a contradiction with the definition of v^*. Hence $z_1(v) > 0$ and $y_1(v) > 1$ in $(r,\bar{v}]$.

Appendix D. Proof of Proposition 4

D.1. Proof of part (i)

In this proof we denote with g_n the solution to (4), $g(r_A) = r_B$. We show that given any $\hat{v}_A \in (r_A,\bar{v})$ and any $\varepsilon > 0$, there exists N such that for $n > N$: $g_n(\hat{v}_A) > \hat{v}_A - \varepsilon$. Since $g_n'(v_A) > 1$ for each $v_A \in [r_A,\bar{v}]$,[29] it follows that $g_n(v_A) > v_A - \varepsilon$ for each $v_A \in (\hat{v}_A,\bar{v}]$; thus g_n converges uniformly to $g_{\hat{v}_A}^{id}$ in $[\hat{v}_A,\bar{v}]$.

Assuming the converse, suppose there exists $\hat{v}_A \in (r_A,\bar{v})$ and $\varepsilon > 0$ such that $g_n(\hat{v}_A) \le \hat{v}_A - \varepsilon$ for infinitely many n; such inequality and $g_n'(v_A) > 1$ imply $g_n(v_A) < v_A - \varepsilon$ for each $v_A \in [r_A,\hat{v}_A)$. As a consequence, for all $v_A \in [r_A,\hat{v}_A)$ we have $F(g_n(v_A))F(v_A) \le F(v_A-\varepsilon)F(v_A) \le F^2(v_A) - \eta F(v_A)$, where $\eta = \varepsilon\min_{v\in[0,\bar{v}]} f(v) > 0$. Without loss of generality we can assume $\eta < 1$.

[29] Inserting in (4) the inquality $g_n(v_A) < v_A$ (for each $v_A \in [r_A,\bar{v}]$) yields $g_n'(v_A) > 1$.

From the inequality $g_n(v) < v$ for each $v \in [r_A, \bar{v}]$ we deduce $\int_{v_A}^{\bar{v}} F(g_n(v))f(v)dv < \frac{1}{2} - \frac{1}{2}F^2(v_A)$ and applying these estimates to (4) we conclude that

$$g_n'(v_A) > \left(1 + \frac{2\eta F(v_A)}{1 + F^2(v_A) - 2\eta F(v_A)}\right)^{n-1}, \quad \text{for each } v_A \in [r_A, \hat{v}_A].$$

The quotient $\frac{2\eta F(v_A)}{1+F^2(v_A)-2\eta F(v_A)}$ is strictly increasing with respect to v_A. Hence it is greater than $\frac{2\eta F(r_A)}{1+F^2(r_A)-2\eta F(r_A)} > \eta F(r_A)$. Therefore, for infinitely many n and for $\rho = 1 + \eta F(r_A) > 1$ we have $g_n'(v_A) > \rho^{n-1}$ for each $v_A \in [r_A, \hat{v}_A]$, which contradicts $g_n(\hat{v}_A) \le \hat{v}_A - \varepsilon$ for sufficiently large n.

D.2. Proof of part (ii)

The functions y_1, z_1 and y_2, z_2 which solve (37), (38) and (37), (39), respectively, are linearly independent solutions of (37). Consider now (37) with the initial condition $y(r) = a, z(r) = b$. Then we obtain the solution $y(v) = ay_1(v) + by_2(v), z(v) = az_1(v)+bz_2(v)$. In particular, if we pick $a = -\frac{y_2(\bar{v})}{y_1(\bar{v})}$ and $b = 1$, then $y(v) = y_2(v) - \frac{y_2(\bar{v})}{y_1(\bar{v})}y_1(v)$ and $z(v) = z_2(v) - \frac{y_2(\bar{v})}{y_1(\bar{v})}z_1(v)$; hence $z(v)$ coincides with $\frac{\partial g(v;r,r)}{\partial r_B}$ from Lemma 1. Since $y(\bar{v}) = 0$, it follows that y, z solve (37) with the boundary conditions $y(\bar{v}) = 0, z(r) = 1$. We know from Proposition 1 that $z(v) \ge 0$, hence the first equation in (37) reveals that y is increasing; thus $y(v) \le 0$ for each $v \in [r, \bar{v}]$. Now we write the second equation in (37) as

$$z'(v) + p(v)z(v) = q(v)y(v) \quad \text{with} \quad p(v) = \frac{(n-1)\tilde{f}(v)}{\tilde{F}(v)} \quad \text{and}$$
$$q(v) = \frac{2(n-1)}{\tilde{F}(v)}$$

and we consider it as a linear first-order non-homogenous equation in the unknown z, for given y. For such equation, given $z(r) = 1$, the solution is well known to be

$$z(v) = e^{-\int_r^v p(\tau)d\tau} + \int_r^v e^{-\int_x^v p(\tau)d\tau} q(x)y(x)dx \tag{40}$$

We obtain the upper bound in (16) by observing that $q(x)y(x) \le 0$ in $[r, \bar{v}]$. Hence, $z(v) \le e^{-\int_r^v p(\tau)d\tau} = \left(\frac{\tilde{F}(r)}{\tilde{F}(v)}\right)^{n-1} = \left(\frac{1+F^2(r)}{1+F^2(v)}\right)^{n-1}$. Notice that for each $v > r$ we have $0 < \frac{1+F^2(r)}{1+F^2(v)} < 1$, hence $\lim_{n\to+\infty}\left(\frac{1+F^2(r)}{1+F^2(v)}\right)^{n-1} = 0$ and $\lim_{n\to+\infty}\frac{\partial g(v;r,r)}{\partial r_B} = 0$ for each $v \in (r, \bar{v}]$. From this it is possible to prove that also $\int_r^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv$ in (17) tends to 0.

Now we prove the lower bound in (16). We start from $y(\bar{v}) - y(v) = \int_v^{\bar{v}} y'(w)dw$ and use $y(\bar{v}) = 0$ plus the first equation in (37) to obtain $y(v) = -\int_v^{\bar{v}} y'(w)dw = -\int_v^{\bar{v}} f^2(w)z(w)dw$. Now we rely on the upper bound for $z(w)$ in (16), that is $z(w) \le \left(\frac{\tilde{F}(r)}{\tilde{F}(w)}\right)^{n-1}$, to see that $y(v) \ge -\int_v^{\bar{v}} f^2(w)\left(\frac{\tilde{F}(r)}{\tilde{F}(w)}\right)^{n-1}dw$. Then (40) yields

$$z(v) \ge \left(\frac{\tilde{F}(r)}{\tilde{F}(v)}\right)^{n-1}\left(1 - \int_r^v 2(n-1)\tilde{F}^{n-2}(x)\left(\int_x^{\bar{v}}\frac{f^2(w)}{\tilde{F}^{n-1}(w)}dw\right)dx\right)$$

Now we focus on $\int_x^{\bar{v}}\frac{f^2(w)}{\tilde{F}^{n-1}(w)}dw$ and notice that

$$\int_x^{\bar{v}}\frac{f^2(w)}{\tilde{F}^{n-1}(w)}dw = \int_x^{\bar{v}}\frac{f(w)}{F(w)}\frac{f(w)F(w)}{\tilde{F}^{n-1}(w)}dw$$
$$\le \int_x^{\bar{v}}\theta\frac{f(w)F(w)}{\tilde{F}^{n-1}(w)}dw = \int_x^{\bar{v}}\theta\frac{\tilde{f}(w)}{\tilde{F}^{n-1}(w)}dw$$

by definition of θ. We can evaluate precisely the rightmost term and obtain

$$\int_x^{\bar{v}}\theta\frac{\tilde{f}(w)}{\tilde{F}^{n-1}(w)}dw = \left[-\frac{\theta}{n-2}\tilde{F}^{2-n}(w)\right]_x^{\bar{v}} = \frac{\theta}{n-2}\left(\frac{1}{\tilde{F}^{n-2}(x)} - 1\right)$$

Hence, $z(v) \ge \left(\frac{\tilde{F}(r)}{\tilde{F}(v)}\right)^{n-1}\left(1 - 2\theta\frac{n-1}{n-2}\int_r^v(1 - \tilde{F}^{n-2}(x))dx\right)$ and the lower bound in (16) follows as $\tilde{F}^{n-2}(x) > 0$.

Appendix E. Proof of Proposition 6

We write the right hand side in (11) as

$$n\tilde{F}^{n-1}(r)\left(1 - \tilde{F}(r) - r\tilde{f}(r)\right.$$
$$\left.-r\int_r^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv - \int_r^{\bar{v}}\Lambda(v)dv\right) = 0 \tag{41}$$

with $\Lambda(v) = (n-1)\frac{\tilde{F}^{n-2}(v)(1-\tilde{F}(v))}{\tilde{F}^{n-1}(r)}\int_v^{\bar{v}} f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}du$. We know from Footnote 16 that $1 - \tilde{F}(r) - r\tilde{f}(r) > 0$ for $r \in (0, r^V)$ and $1-\tilde{F}(r)-r\tilde{f}(r) < 0$ for $r \in (r^V, \bar{v})$. As we remarked in Section 5.2, $r\int_r^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv$ and $\int_r^{\bar{v}}\Lambda(v)dv$ are both positive for each $r \in (0, \bar{v})$, hence if r^D solves (41) then $r^D < r^V$. Now consider a left neighborhood of r^V, for instance the interval $[\frac{1}{2}r^V, r^V]$. We show that there exists a continuous function of r, $\lambda(r)$, such that $\int_r^{\bar{v}}\Lambda(v)dv > \lambda(r) > 0$ for each large n, for each $r \in [\frac{1}{2}r^V, r^V]$. Hence, there exists a $r' < r^V$ such that the left hand side in (41) is negative for each $r > r'$, for each n, and each solution to (41) is smaller than r' for each n.

Step 1. Given $r \in [\frac{1}{2}r^V, r^V]$ and given a small $\varepsilon > 0$, let $b = r + \varepsilon$ and let $\delta = (\min f)^2\left(1 - 2\theta\frac{n-1}{n-2}\varepsilon\right) > 0$ with $\theta = \max_{w\in[r^V/2,\bar{v}]}\frac{f(w)}{F(w)} > 0$. Then

$$\int_r^{\bar{v}}\Lambda(v)dv > \delta\int_r^b\int_v^b(n-1)\frac{\tilde{F}^{n-2}(v)(1-\tilde{F}(v))}{\tilde{F}^{n-1}(u)}dudv \tag{42}$$

Proof of Step 1. For each $v \in [r, \bar{v}]$ we have $\frac{\partial g(v;r,r)}{\partial r_B} \ge 0$ and $\Lambda(v) \ge 0$; thus if in $\int_r^{\bar{v}}\Lambda(v)dv$ we reduce the upper extreme of the interval of integration from $\bar{v}$ to $b > r$, then we obtain a lower bound for $\int_r^{\bar{v}}\Lambda(v)dv$:

$$\int_r^{\bar{v}}\Lambda(v)dv$$
$$\ge \int_r^b(n-1)\frac{\tilde{F}^{n-2}(v)(1-\tilde{F}(v))}{\tilde{F}^{n-1}(r)}\left(\int_v^b f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}du\right)dv \tag{43}$$

In particular, we consider $b = r + \varepsilon$ with $\varepsilon > 0$ and small and now prove that the right hand side in (43) is greater than the right hand side in (42). From the lower bound in (16) we see that the right hand side in (43) is greater than

$$\int_r^b(n-1)\frac{\tilde{F}^{n-2}(v)(1-\tilde{F}(v))}{\tilde{F}^{n-1}(r)}$$
$$\times\left(\int_v^b f^2(u)\left(\frac{\tilde{F}(r)}{\tilde{F}(u)}\right)^{n-1}\left(1 - 2\theta\frac{n-1}{n-2}(u-r)\right)du\right)dv \tag{44}$$

and $f^2(u)(1-2\theta\frac{n-1}{n-2}(u-r)) \ge \delta$ for each $u \in [r, b]$. Therefore (44) is greater than the right hand side in (42).

Step 2. Given $r \in [\frac{1}{2}r^V, r^V]$, let $\rho = \min_{x\in[r^V/2,\bar{v}]}\frac{1}{\tilde{f}(x)} > 0$ and let $\lambda(r) = \frac{1}{3}\delta\rho^2\left(\tilde{F}(b) - \tilde{F}(r)\right)\left(2 - \tilde{F}(b) - \tilde{F}(r)\right) > 0$. Then the right hand side in (42), and therefore also $\int_r^{\bar{v}} \Lambda(v)dv$, is greater than $\lambda(r)$ for each large n.

Proof of Step 2. We apply to the right hand side of (42) the change of variables

$$s = \tilde{F}(v), \qquad t = \tilde{F}(u)$$

Hence, we write the right hand side in (42) as

$$\delta \int_{\tilde{F}(r)}^{\tilde{F}(b)} \int_{s}^{\tilde{F}(b)} (n-1)\frac{s^{n-2}(1-s)}{t^{n-1}} \frac{1}{\tilde{f}(\tilde{F}^{-1}(s))} \frac{1}{\tilde{f}(\tilde{F}^{-1}(t))} dtds \tag{45}$$

This is no less than $\delta\rho^2 \int_{\tilde{F}(r)}^{\tilde{F}(b)} \int_s^{\tilde{F}(b)} (n-1)\frac{s^{n-2}(1-s)}{t^{n-1}} dtds$, which is equal to

$$\delta\rho^2 \frac{n-1}{n-2} \int_{\tilde{F}(r)}^{\tilde{F}(b)} \left(1 - s - \frac{s^{n-2}-s^{n-1}}{\tilde{F}^{n-2}(b)}\right) ds$$
$$= \delta\rho^2 \frac{n-1}{n-2} \left(\frac{1}{2}\left(\tilde{F}(b)-\tilde{F}(r)\right)\left(2-\tilde{F}(b)-\tilde{F}(r)\right) + \frac{1}{\tilde{F}^{n-2}(b)}\left(\frac{\tilde{F}^n(b)-\tilde{F}^n(r)}{n} - \frac{\tilde{F}^{n-1}(b)-\tilde{F}^{n-1}(r)}{n-1}\right)\right)$$

The limit of the last expression as $n \to +\infty$ is $\frac{1}{2}\delta\rho^2\left(\tilde{F}(b)-\tilde{F}(r)\right)\left(2-\tilde{F}(b)-\tilde{F}(r)\right) > 0$, which is greater than $\lambda(r)$. Hence (45) is greater than $\lambda(r)$ if n is large enough.

Appendix F. The case of $\underline{v} > 0$

The analysis in the main text relies on the assumption that $\underline{v} = 0$. When $\underline{v} > 0$, such analysis can be extended in a straightforward way to cover the case of $\underline{v} \leq r_B \leq r_A < \bar{v}$. But $\underline{v} > 0$ makes it feasible for each seller i to set r_i below $\underline{v}$. Therefore, in Appendix F.1 we characterize the unique symmetric bidders' equilibrium when at least one reserve price is smaller than $\underline{v}$. Using such characterization, in Appendix F.2 we derive a first order condition for a symmetric equilibrium in which the equilibrium reserve price is less than $\underline{v}$, and then consider an example.

F.1. Bidders' equilibrium when $\underline{v} > 0$ and $\min\{r_A, r_B\} < \underline{v}$

Without loss of generality, we suppose that $r_B \leq r_A$ and we determine a bidders' equilibrium when $r_B < \underline{v}$. First we notice that the set V_N is empty, hence $V_A \cup V_B = V$, because $r_B < \underline{v}$ allows each bidder type to earn a positive utility attending auction B. Then we still postulate the existence of a strictly increasing and differentiable function g that describes the boundary between V_A and V_B.

The fact that each bidder has positive utility from entering auction B has the consequence that a bidder with values (v_A, v_B) chooses auction A if and only if he gets at least as much utility attending A as he would attending auction B, that is $u_A(v_A; \mathbf{r}) \geq u_B(v_B; \mathbf{r})$. This inequality requires that v_A is large enough, and we denote with v_A^* the lowest value for v_A of bidders entering auction A. In other words, all bidders with values $v_A < v_A^*$ are certain to get a higher utility in auction B including those for whom $v_B = \underline{v}$, that is $u_A(v_A^*; \mathbf{r}) = u_B(\underline{v}; \mathbf{r})$, therefore $g(v_A^*; \mathbf{r}) = \underline{v}$. Hence g is defined for v_A in the interval $[v_A^*, \bar{v}]$, with v_A^* such that a bidder with values $(v_A, v_B) = (v_A^*, \underline{v})$ is indifferent between the auctions, thus $v_A^* \geq \max\{\underline{v}, r_A\}$.

We can express this property formally by noticing that the sets V_A, V_B are as follows (see Fig. 8 below for a graphical representation)

$$V_A = \{(v_A, v_B) \in V : v_A \geq v_A^* \quad \text{and} \quad v_B \leq g(v_A; \mathbf{r})\} \quad \text{and} \quad V_B = V \backslash V_A \tag{46}$$

and α denotes the probability that a bidder has values in V_A, that is

$$\alpha = \int_{v_A^*}^{\bar{v}} F(g(v; \mathbf{r})) f(v) dv \tag{47}$$

A bidder with values $(v_A, v_B) = (v_A^*, \underline{v})$ entering auction i wins object i if and only if he is the only bidder in auction i. Hence, he is indifferent between the two auctions, that is $u_A(v_A^*; \mathbf{r}) = u_B(\underline{v}; \mathbf{r})$, if and only if

$$(v_A^* - r_A)(1-\alpha)^{n-1} - (\underline{v} - r_B)\alpha^{n-1} = 0 \tag{48}$$

For each $v \in [v_A^*, \bar{v}]$ we define $F_A(v; \mathbf{r})$ like in (5), and $F_A(v_A^*; \mathbf{r}) = 1 - \alpha$. Likewise, for each $v \in [\underline{v}, \bar{v}]$ we define $F_B(v; \mathbf{r})$ as in (6), and $F_B(\underline{v}; \mathbf{r}) = \alpha$.[30] Then $u_A(v_A; \mathbf{r})$, $u_B(v_B; \mathbf{r})$, for $v_A \geq v_A^*$, $v_B \geq \underline{v}$, are

$$u_A(v_A; \mathbf{r}) = (v_A - r_A)(1-\alpha)^{n-1} + \int_{v_A^*}^{v_A} (v_A - v) dF_A^{n-1}(v; \mathbf{r})$$
$$= (v_A^* - r_A)(1-\alpha)^{n-1} + \int_{v_A^*}^{v_A} F_A^{n-1}(v; \mathbf{r}) dv \tag{49}$$

$$u_B(v_B; \mathbf{r}) = (v_B - r_B)\alpha^{n-1} + \int_{\underline{v}}^{v_B} (v_B - v) dF_B^{n-1}(v; \mathbf{r})$$
$$= (\underline{v} - r_B)\alpha^{n-1} + \int_{\underline{v}}^{v_B} F_B^{n-1}(v; \mathbf{r}) dv \tag{50}$$

For (v_A, v_B) such that $v_A > v_A^*$ and $v_B > \underline{v}$, indifference between the auctions needs to hold if $v_B = g(v_A; \mathbf{r})$, hence $u_A(v_A; \mathbf{r}) = u_B(g(v_A; \mathbf{r}); \mathbf{r})$, and differentiating this equality with respect to v_A – using (49) and (50) – yields (4). Thus we consider equation (4) with the initial condition $g(v_A^*; \mathbf{r}) = \underline{v}$, and a minor adaptation of Proposition 1 establishes that a unique solution g exists. In this way, to each v_A^* we associate g and then α from (47), that is we view α as a function of v_A^*. We show in the following that there exists a unique solution to (48), and we pick v_A^* as such unique solution. Observe that (i) the left hand side in (48) is a continuous function of v_A^*; (ii) when v_A^* is close to $\bar{v}$, we have that α is about 0, therefore the left hand side in (48) is positive; (iii) when $r_A \leq \underline{v}$ and $v_A^* = \underline{v}$, we have that $\alpha = \frac{1}{2} = 1 - \alpha$, thus the left hand side in (48) has the same sign as $r_B - r_A \leq 0$ (if $r_B = r_A$, then $v_A^* = \underline{v}$ solves (48)); (iv) when $\underline{v} < r_A$ and $v_A^* = r_A$, the left hand side in (48) has the same sign as $r_B - \underline{v} \leq 0$ (if $r_B = \underline{v}$, then $v_A^* = r_A$ solves (48)); (i)-(iv) guarantee existence of a solution. Uniqueness holds since α is decreasing with respect to v_A^*, hence the left hand side in (48) is strictly increasing with respect to v_A^*.

Proposition 7. *Given r_A, r_B such that $r_B < \underline{v}$, $r_B \leq r_A < \bar{v}$, there exists a unique symmetric bidders' equilibrium, which is described by the sets V_A, V_B in (46), in which g is defined for $v_A \in [v_A^*, \bar{v}]$ and is the unique solution to (4), $g(v_A^*; \mathbf{r}) = \underline{v}$, and $v_A^* \geq \max\{\underline{v}, r_A\}$ is the unique solution to (48).*

When $r_A = r_B \leq \underline{v}$, we obtain $v_A^* = \underline{v}$ and $g = g_{\underline{v}}^{id}$, that is each bidder participates in the auction for the object for which he has the higher valuation. When $r_B \leq \underline{v}$ and $r_B < r_A$, because of the initial condition in this context, we have that $g(v_A; r_A, r_B)$ coincides with $g(v_A; v_A^*, \underline{v})$ for each $v_A \in [v_A^*, \bar{v}]$.

[30] Moreover, we extend F_A (for $v < v_A^*$) and F_B (for $v < \underline{v}$) as described in Section 5.1.

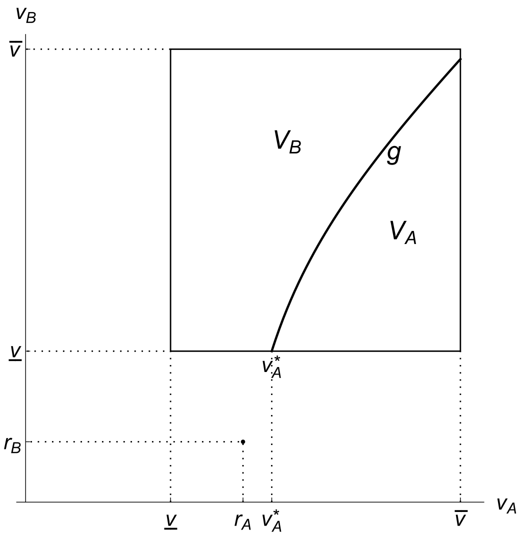

Fig. 8. The function g and the sets V_A, V_B when $n = 3$, F is the uniform distribution with support $[\underline{v}, \bar{v}] = [0.5, 1.5]$, and $r_B = 0.2$, $r_A = 0.75$. Then $v_A^* = 0.84915$.

F.2. Sellers' game when $\underline{v} > 0$ and $\min\{r_A, r_B\} < \underline{v}$

When $\underline{v} > 0$ and the equilibrium reserve price is not smaller than $\underline{v}$, then the analysis in Section 5 still applies. But it is possible that the equilibrium reserve price is smaller than $\underline{v}$, and then we need to use

$$\frac{\partial R_B(r,r)}{\partial r_B} = \frac{n}{2^n} + \frac{n}{2^{n-1}}\left[(n-2)r - (n-1)\underline{v}\right]\int_{\underline{v}}^{\bar{v}} f^2(v)\frac{\partial g(v;r,r)}{\partial r_B}dv - n(n-1)\int_{\underline{v}}^{\bar{v}}[\tilde{F}^{n-2}(v) - \tilde{F}^{n-1}(v)]\int_{v}^{\bar{v}} f^2(u)\frac{\partial g(u;r,r)}{\partial r_B}dudv \quad (51)$$

which holds for each $r \in [0, \underline{v})$ (the derivation of (51) is described in Appendix F.3). From (51) we obtain a first order condition when the equilibrium reserve price is smaller than $\underline{v}$.

Once F is specified, we can apply numerical methods as specified in Section 5.3 and here we assume that F is the c.d.f. of the uniform distribution with support $[1, 2]$ and $n = 2$. We find that in this case $\frac{\partial R_B(r,r)}{\partial r_B}$ is equal to 0 at $r = 0.90041$, which we denote with $r_{2,1}^D$. Then we use numerical methods to show that $R_B(r_{2,1}^D, r_{2,1}^D) \geq R_B(r_{2,1}^D, r_B)$ for each r_B in the set $S_{2,1}$ of reserve prices in $[0, 2)$ which differ from $r_{2,1}^D$ by an integer multiple of $\frac{1}{1000}$, so that $(r_A, r_B) = (r_{2,1}^D, r_{2,1}^D)$, is an equilibrium for the game in which $S_{2,1}$ is the set of feasible reserve prices for each seller.

We notice that in this case the equilibrium reserve prices is less than $\underline{v}$, something which cannot occur if $\underline{v} = 0$, although under monopoly the seller never sets the reserve price below $\underline{v}$. A similar result holds in Parlane (2008), who studies competing auctions when the objects are differentiated à la Hotelling. In case of low differentiation, bidders are nearly indifferent between object A and object B; then competition between sellers is fierce and leads to a low reserve price, which leaves a positive rent to each bidder type who participates in his favorite auction.

F.3. Derivation of $\frac{\partial R_B}{\partial r_B}$ in (51)

When $r_B < \underline{v}$ and $r_B \leq r_A = r$, the probability that a bidder enters auction B (that is, that his values belong to V_B) is $1 - \alpha$, with α determined by (47). Then the expected revenue of seller B can be written as

$$R_B(r, r_B) = r_B n(1-\alpha)\alpha^{n-1} + \int_{\underline{v}}^{\bar{v}} v dF_B^{(2)}(v; \mathbf{r}) \quad (52)$$

in which $n(1-\alpha)\alpha^{n-1}$ is the probability that just one bidder enters auction B, and $\int_{\underline{v}}^{\bar{v}} v dF_B^{(2)}(v; \mathbf{r})$ takes care of the case in which at least two bidders enter auction B. Applying integration by parts to (52), we obtain $R_B(r, r_B) = \bar{v} - n(\underline{v} - r_B)(1-\alpha)\alpha^{n-1} - \underline{v}\alpha^n - \int_{\underline{v}}^{\bar{v}} F_B^{(2)}(v; \mathbf{r})dv$. Hence

$$\begin{aligned}\frac{\partial R_B}{\partial r_B} &= n(1-\alpha)\alpha^{n-1} + r_B n[(n-1)\alpha^{n-2} - n\alpha^{n-1}]\frac{\partial \alpha}{\partial r_B} \\ &\quad -\underline{v}n(n-1)\alpha^{n-2}(1-\alpha)\frac{\partial \alpha}{\partial r_B} \\ &\quad -n(n-1)\int_{\underline{v}}^{\bar{v}}[F_B^{n-2}(v;\mathbf{r}) - F_B^{n-1}(v;\mathbf{r})]\frac{\partial F_B(v;\mathbf{r})}{\partial r_B}dv\end{aligned}$$

About $\frac{\partial \alpha}{\partial r_B}$, from (47) we find that

$$\begin{aligned}\frac{\partial \alpha}{\partial r_B} &= -F(g(v_A^*;\mathbf{r}))f(v_A^*)\frac{\partial v_A^*}{\partial r_B} + \int_{v_A^*}^{\bar{v}} f(g(v;\mathbf{r}))f(v)\frac{\partial g(v;\mathbf{r})}{\partial r_B}dv \\ &= \int_{v_A^*}^{\bar{v}} f(g(v;\mathbf{r}))f(v)\frac{\partial g(v;\mathbf{r})}{\partial r_B}dv \qquad (53)\end{aligned}$$

since $F(g(v_A^*;\mathbf{r})) = F(\underline{v}) = 0$. About $\frac{\partial F_B(v,\mathbf{r})}{\partial r_B}$, from (10) we see that $\frac{\partial F_B(v,\mathbf{r})}{\partial r_B} = \int_{g^{-1}(v;\mathbf{r})}^{\bar{v}} f(g(u;\mathbf{r}))f(u)\frac{\partial g(u;\mathbf{r})}{\partial r_B}du$. Therefore $\frac{\partial R_B}{\partial r_B}$ is

$$\begin{aligned}\frac{\partial R_B}{\partial r_B} &= n(1-\alpha)\alpha^{n-1} + n\left[\left((n-1)\alpha^{n-2} - n\alpha^{n-1}\right)r_B \right. \\ &\quad \left. -\underline{v}(n-1)\alpha^{n-2}(1-\alpha)\right]\int_{v_A^*}^{\bar{v}} f(g(v;\mathbf{r}))f(v)\frac{\partial g(v;\mathbf{r})}{\partial r_B}dv \\ &\quad -n(n-1)\int_{\underline{v}}^{\bar{v}}[F_B^{n-2}(v;\mathbf{r}) - F_B^{n-1}(v;\mathbf{r})] \\ &\quad \times \int_{g^{-1}(v;\mathbf{r})}^{\bar{v}} f(g(u;\mathbf{r}))f(u)\frac{\partial g(u;\mathbf{r})}{\partial r_B}dudv \qquad (54)\end{aligned}$$

At $r_B = r$ we have $v_A^* = \underline{v}$, g and g^{-1} coincide with $g_{\underline{v}}^{id}$, $\alpha = \frac{1}{2}$, F_B boils down to $\tilde{F}$. Thus (54) reduces to (51).

In order to evaluate $\frac{\partial g(v;r,r)}{\partial r_B}$, we recall from Appendix F.1 that for $r_B < \underline{v}$, $r_B \leq r_A$, the following equality holds:

$$g(v; r_A, r_B) = g(v; v_A^*, \underline{v}) \quad (55)$$

since the initial condition establishes that g at v_A^* has value $\underline{v}$, with v_A^* determined by r_A, r_B through (48). Therefore, a change in r_B determines a change in v_A^*, which has the same effect on g as a change in r_A in the setting of Section 4 and $\frac{\partial g(v;r_A,r_B)}{\partial r_B} = \frac{\partial v_A^*}{\partial r_B}\frac{\partial g(v;v_A^*,\underline{v})}{\partial r_A}$, in which $\frac{\partial g}{\partial r_A}$ at the right hand side indicates the partial derivative of g with respect to its second variable and (see Box I).

At $r_B = r$ we have $v_A^* = \underline{v}$, $g = g_{\underline{v}}^{id}$, $\alpha = \frac{1}{2}$. Hence, $\frac{\partial g(v;r,r)}{\partial r_B} = \frac{\partial v_A^*}{\partial r_B}\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_A}$ with $\frac{\partial v_A^*}{\partial r_B} = \frac{1}{4(\underline{v}-r)(n-1)\int_{\underline{v}}^{\bar{v}} f^2(v)\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_A}dv - 1}$, and by next lemma we conclude that

$$\frac{\partial g(v;r,r)}{\partial r_B} = \frac{1}{4(\underline{v}-r)(n-1)\int_{\underline{v}}^{\bar{v}} f^2(u)\frac{\partial g(u;\underline{v},\underline{v})}{\partial r_B}du + 1}\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_B}$$

in which $\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_B}$ can be evaluated as described by Lemma 1.

$$\frac{\partial v_A^*}{\partial r_B} = -\frac{\alpha^{n-1}}{(1-\alpha)^{n-1} - (n-1)\left[(v_A^* - r_A)(1-\alpha)^{n-2} + (\underline{v} - r_B)\alpha^{n-2}\right]\int_{v_A^*}^{\bar{v}} f(g(v; v_A^*, \underline{v}))f(v)\frac{\partial g(v; v_A^*, \underline{v})}{\partial r_A}dv}$$

Box I.

Lemma 2. $\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_A} = -\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_B}$.

Proof. Given a small ε, consider equation (4) with the initial condition $g(r_A-\varepsilon; r_A-\varepsilon, r_B) = r_B$ and notice that $g(r_A; r_A-\varepsilon, r_B)$ is equal to $r_B + \frac{\partial g(r_A-\varepsilon;r_A-\varepsilon,r_B)}{\partial v}\varepsilon + o(\varepsilon)$, with $\frac{\partial g(r_A-\varepsilon;r_A-\varepsilon,r_B)}{\partial v}$ determined by (4). Thus the shift $-\varepsilon$ in the initial condition $g(r_A-\varepsilon; r_A-\varepsilon, r_B) = r_B$ has the same effect in the first-order as a change by $\frac{\partial g(r_A-\varepsilon;r_A-\varepsilon,r_B)}{\partial v}\varepsilon + o(\varepsilon)$ in r_B in the initial condition $g(r_A; r_A, r_B) = r_B$. Hence

$$\begin{aligned}\frac{\partial g(v; r_A, r_B)}{\partial r_A} &= \frac{\partial g(v; r_A, r_B)}{\partial r_B}\lim_{\varepsilon\to 0}\frac{\frac{\partial g(r_A-\varepsilon;r_A-\varepsilon,r_B)}{\partial v}\varepsilon + o(\varepsilon)}{-\varepsilon} \\ &= -\frac{\partial g(v; r_A, r_B)}{\partial r_B}\frac{\partial g(r_A; r_A, r_B)}{\partial v}.\end{aligned}$$

When $r_B = r_A = \underline{v}$, we have that g is equal to $g_{\underline{v}}$. Hence $\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_A} = -\frac{\partial g(v;\underline{v},\underline{v})}{\partial r_B}$ since $\frac{\partial g(\underline{v};\underline{v},\underline{v})}{\partial v} = 1$.

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