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Abstract

This paper proposes estimators for the parameters of an explosive fractional Ornstein-Uhlenbeck process. The asymptotic properties for the diffusion estimators are developed under the in-fill asymptotic scheme, while the asymptotic properties for the drift estimators are developed under the double asymptotic scheme for the full range of the Hurst parameter. Simulation results demonstrate the effectiveness of the proposed estimators, and the asymptotic distributions provide a good approximation in finite samples. Empirical applications are presented to demonstrate the model's usefulness and the practical value of the asymptotic theory.

JEL Classification: C02, C13, C15.

Keywords: Explosive process, Hurst parameter, Long memory, Anti-persistency, Double asymptotics, In-fill asymptotics

1 Introduction

In recent years, mildly explosive discrete-time models have been utilized to capture the dynamic behavior of economic and financial time series. This approach has been explored in various studies such as Phillips and Yu (2011), Phillips et al. (2011), Phillips et al. (2015a, 2015b), Harvey et al. (2016, 2017), Chen et al. (2017), Lui et al. (2021, 2022), and Astill et al. (2018).

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The mildly explosive model and the asymptotic theory for the least squares (LS) estimator were first introduced in the seminal paper by Phillips and Magdalinos (2007a), where the error term is assumed to be independent and identically distributed (iid). Phillips and Magdalinos (2007b) extended the model and asymptotic theory to include weakly dependent errors, while Magdalinos (2012) further extended it to strongly dependent errors. Lui et al. (2021) expanded the model and asymptotic theory to incorporate anti-persistent errors. In all of these studies, an initial condition was selected to ensure that it becomes negligible in the asymptotic distribution.

Wang and Yu (2016) demonstrated that mild explosiveness can be achieved from an explosive Ornstein-Uhlenbeck (OU) process under the double asymptotic scheme when the sampling interval approaches zero and the time span becomes infinite. In this scenario, since the randomness is governed by the standard Brownian motion,[1] the error term in the exact discrete-time representation of the model is iid. Wang and Yu (2016) obtained the double asymptotic distribution of the LS estimator of the persistency parameter and showed that it explicitly depends on the initial condition. The reason that the initial condition is given a prominence in the continuous-time setup is because a bigger initial condition than what is typically imposed in the discrete-time literature is allowed in the exact discrete-time representation when the sampling interval shrinks to zero.

In this paper, we extend the OU model of Wang and Yu (2016) by replacing the standard Brownian motion with the fractional Brownian motion (fBm), that is, an explosive fractional OU process (fOUp). The exact discrete-time representation of fOUp extends the models considered in Magdalinos (2012) and Lui et al. (2021) in four aspects. First, our model allows for the full range of the Hurst parameter. Second, we permit a larger initial condition in the exact discrete-time representation of fOUp than that considered in Magdalinos (2012) and Lui et al. (2021). Third, we estimate and examine the asymptotic properties of all four parameters in the model, not just the persistency parameter. Finally, although the error term in our model shares the same covariance structure as those in Magdalinos (2012) and Lui et al. (2021), it cannot be expressed as a linear combination of martingale difference sequences. This distinction leads to completely different technical proof procedures.

[1] In the most general case, Wang and Yu (2016) considered the Lévy process instead of the standard Brownian motion.

We adopt the same estimators of the two diffusion parameters, including the Hurst parameter, as those proposed in Wang et al. (2023), where a stationary fOUp process is considered. For the drift parameters, including the persistency parameter, we obtain the estimators via LS, which have analytical expressions and are easy to implement. The asymptotic theory for the diffusion parameters is established under the in-fill asymptotic scheme, while the asymptotic theory for the drift parameters is established under the double asymptotic scheme.

The remainder of the paper is organized as follows. Section 2 introduces the model and estimators and develops the asymptotic properties of the estimators. Section 3 conducts Monte Carlo studies to check the finite sample performance of the proposed estimators and asymptotic distributions. Section 4 provides an empirical study to illustrate the usefulness of our estimators and the asymptotic theory. Section 5 concludes the paper. All proofs of the theorems are collected in the Appendix. The proofs of all the lemmas, which are useful to prove the theorems, are collected in the Online Supplement. Throughout the paper, we use $\overset{p}{\to}$, $\overset{a.s.}{\to}$, $\overset{\mathcal{L}}{\to}$, $\overset{d}{=}$, $\sim$, and $\succ$ to denote convergence in probability, convergence almost surely, convergence in distribution, equivalence in distribution, asymptotic equivalence, and asymptotic dominance, respectively. We denote C, C_1, C_2, which may change from line to line, positive constants that depend only on the parameters of fOUp.

2 Model, Estimators, and Asymptotics

The fOUp is given by the following stochastic differential equation:

$$dX_t = (\theta X_t + \mu)\, dt + \sigma dB_t^H\,, \tag{2.1}$$

where $X_0 = O_p(1)$ is independent of B_t^H, $\sigma \in \mathbb{R}^+$, $\mu \in \mathbb{R}$, $\theta > 0$, and B_t^H is an fBm with the Hurst parameter, $H \in (0, 1)$, with mean zero and the following covariance

$$R(t,s) = \mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \quad \forall\, t, s \geq 0\,. \tag{2.2}$$

For $t > 0$, Mandelbrot and van Ness (1968) present the following integral representation for B_t^H:

$$B_t^H = \frac{1}{c_H}\left\{\int_{-\infty}^0 \left[(t-u)^{H-1/2} - (-u)^{H-1/2}\right] dW_u + \int_0^t (t-u)^{H-1/2} dW_u\right\}, \tag{2.3}$$

where W_u is a standard Brownian motion, $c_H = \frac{\Gamma(H+1/2)}{\sqrt{\Gamma(2H+1)\sin(\pi H)}}$, $B_0^H = 0$ and $\Gamma(\cdot)$ denotes the Gamma function.

Obviously, the fBm becomes the standard Brownian motion W_t when $H = 1/2$. Moreover, the fBm is self-similar in the sense that for any $a \in \mathbb{R}$, $B_{at}^H \overset{d}{=} |a|^H B_t^H$. Let $L_t^H = B_t^H - B_{t-1}^H$ be the so-called the fractional Gaussian noise (fGn) which is always stationary. The autocovariance function of fGn is

$$\gamma(k) = \text{Cov}\left(L_t^H, L_{t+k}^H\right) = \frac{1}{2}\left[(k+1)^{2H} - 2k^{2H} + (k-1)^{2H}\right], \tag{2.4}$$

for $k \geq 0$ and $\gamma(k) = \gamma(-k)$ for $k < 0$.

Applying the Taylor expansion to the right-hand side of (2.4), we can see that if $H \in (0, 1/2) \cup (1/2, 1)$, $\gamma(k) \sim H(2H-1)k^{2H-2}$ for large k. Hence, for $\frac{1}{2} < H < 1$, it has $\gamma(k) > 0$ for all k and $\sum_{k=-\infty}^{\infty} \gamma(k) = \infty$. In this case, fGn has the long memory property and positive (negative) increments are likely to be followed by positive (negative) increments. For $0 < H < \frac{1}{2}$, it can be verified that $\gamma(k) < 0$ for all $k \neq 0$ and $\sum_{k=-\infty}^{\infty} \gamma(k) = 0$. Therefore, the process is anti-persistent.

When $\theta < 0$, X_t is asymptotically stationary and ergodic with the long-run mean $-\mu/\theta$. In this case, the coefficient θ is the speed of adjustment of X_t towards its long-run mean. When $\theta = 0$ and $\mu = 0$, $X_t = B_t^H$ that is non-stationary and null recurrent. When $\theta > 0$, $|\mathbb{E}(X_t|\mathcal{F}_{t-1})| > |X_{t-1}|$, implying X_t is non-stationary and explosive, where $\mathcal{F}_t$ is a sigma-algebra generated by B_s^H with $s \in [0, t]$.

In practice, we often have access to discretely sampled data only. Let $\{X_{i\Delta}\}_{i=1}^n$ denote the discretely sampled data, where n is the sample size and Δ is the sampling interval. Let $T (:= n\Delta)$ be the time span. When X_t is annualized and observed monthly (weekly or daily), then $\Delta = 1/12$ ($1/52$ or $1/252$) for assets that are traded five days in a week. The in-fill asymptotics assume $\Delta \to 0$ with T being fixed while the double asymptotics assume $\Delta \to 0, T \to \infty$. In both cases, $n \to \infty$. In model (2.1), there are four parameters, two diffusion parameters, H and σ, and two drift parameters, θ and μ. We would like to estimate these four parameters based on $\{X_{i\Delta}\}_{i=0}^n$ generated from model (2.1) with $\theta > 0$, that is, an explosive fOUp.

When $\theta \neq 0$, the strong solution of fOUp is given by

$$X_t = X_0 e^{\theta t} + \frac{\mu}{\theta}(e^{\theta t} - 1) + \sigma e^{\theta t} \int_0^t e^{-\theta s} dB_s^H, \tag{2.5}$$

where the stochastic integral in (2.5) is interpreted as a Young integral.

Therefore, the exact discrete-time representation of model (2.1) is

$$X_{i\Delta} = \beta_\Delta X_{(i-1)\Delta} + \frac{\mu}{\theta}(e^{\theta\Delta} - 1) + \sigma\epsilon_{i\Delta}\,,\, \beta_\Delta = e^{\theta\Delta}, \tag{2.6}$$

where

$$\epsilon_{i\Delta} = \int_{(i-1)\Delta}^{i\Delta} e^{\theta(i\Delta - s)} dB_s^H = \left(B_{i\Delta}^H - B_{(i-1)\Delta}^H\right) + O_p\left(\Delta^{1+H}\right) \sim O_p(\Delta^H), \text{ as } \Delta \to 0.$$

When $\theta > 0$, $\beta_\Delta = e^{\theta\Delta} > 1$ since $\Delta > 0$. If $\Delta \to 0$, $\beta_\Delta \searrow 1$. The speed that β_Δ approaches unity depends on whether T is fixed or goes to infinity.

As shown in Wang and Yu (2016), under the in-fill asymptotic scheme, model (2.6) with $\theta > 0$ corresponds to a local-to-unity model with the AR(1) parameter larger than unity but approaching to unity as $\Delta \to 0$. It can be seen that with a fixed T and $\Delta \to 0$, we have

$$(1 - \beta_\Delta)\, n = \left(1 - e^{\theta\Delta}\right) n = (-\theta\Delta + o(\Delta))\, n \to -\theta T,$$

where $-\theta T$ is the scale parameter. Whereas, under the double asymptotic scheme, the exact discrete-time representation of model (2.1) with $\theta > 0$ is an explosive model with the AR(1) coefficient larger than but approaching to unity slower than $1/n$ as $\Delta \to 0$. It can be seen that with $\Delta \to 0, T \to \infty$, we have

$$(1 - \beta_\Delta)\, n = \left(1 - e^{\theta\Delta}\right) n = (-\theta\Delta + o(\Delta))\, n = -\theta T + o(T) \to -\infty.$$

Using the terminology of Phillips and Magdalinos (2007a), the model is mildly explosive.

Since $\epsilon_{i\Delta} \sim O_p(\Delta^H)$, to ensure the error term is $O_p(1)$, dividing both sides of equation (2.6) by Δ^H, we have

$$Y_{i\Delta} = \beta_\Delta Y_{(i-1)\Delta} + \frac{\mu}{\Delta^H \theta}(e^{\theta\Delta} - 1) + \sigma e_{i\Delta}, \tag{2.7}$$

where $Y_{i\Delta} = X_{i\Delta}/\Delta^H$, $e_{i\Delta} = \epsilon_{i\Delta}/\Delta^H$. Clearly, as $\Delta \to 0$ with a fixed T, $e_{i\Delta} \sim O_p(1)$ and $Y_0 = X_0/\Delta^H \sim O_p(n^H)$.

Magdalinos (2012) considered the following AR(1) model,

$$Y_t = \rho_n Y_{t-1} + \sigma u_t, \rho_n = 1 + \frac{c}{n^\alpha}, \alpha \in (0,1), c > 0, Y_0 \sim o_p(n^{\alpha(0.5+d)}), \tag{2.8}$$

where $u_t = \sum_{j=0}^\infty c_j v_{t-j}$ with $c_j \sim \gamma j^{d-1}$ for some $d \in (0, 1/2)$ and v_t being a martingale difference sequence and v_t^2 is a uniformly integrable sequence.[2]

[2]The exact assumption in Magdalinos (2012) is $c_j = L(j)j^{-k}$ for some $k \in (1/2, 1)$ where L is a slowly varying function at infinity in Assumption LP(ii).

His model with $n \to \infty$ is closely linked to model (2.7). To see the connection, in model (2.7), if $\Delta \to 0, T \to \infty$, we have

$$\beta_\Delta \to 1, (1 - \beta_\Delta) n \to -\infty, \mathbb{E}\left(e_{i\Delta} e_{(i+j)\Delta}\right) \sim Cj^{2H-2} \text{ for large } j,$$

where the last part is due to Lemma 2.1 of Cheridito et al. (2003). In model (2.8), if $n \to \infty$, we have

$$\rho_n \to 1, (1 - \rho_n) n \to -\infty, \mathbb{E}(u_i u_{i+j}) \sim Cj^{2d-1} \text{ for large } j.$$

If $H = 1/2 + d$, model (2.7) and model (2.8) share the same covariance structure for large j.

Lui et al. (2021) considered the following AR(1) model,

$$Y_t = \rho_{n.m} Y_{t-1} + \sigma u_t, \rho_{n,m} = 1 + \frac{cm}{n}, c > 0, Y_0 \sim o_p(n^{0.5+d}), \tag{2.9}$$

where $u_t = \sum_{j=0}^{\infty} c_j v_{t-j}$ with $c_j \sim \gamma j^{d-1}$ for some $d \in (-1/2, 0)$ and v_t being an iid sequence.

Model (2.9) with $n \to \infty$ followed by $m \to \infty$ is also closely linked to model (2.7). In model (2.9), if $n \to \infty$ followed by $m \to \infty$, we have

$$\rho_{n,m} \to 1, (1 - \rho_{n,m}) n \to -\infty, \mathbb{E}(u_i u_{i+j}) \sim Cj^{2d-1} \text{ for large } j.$$

If $H = 1/2 + d$, model (2.8) and model (2.9) also share the same covariance structure for large j.

However, there are three important differences between the two existing models and model (2.7). First, they have different initial conditions. In particular, since $O_p(n^H) \succ o_p(n^H) \succ o_p(n^{\alpha H})$, the initial condition in model (2.7) larger than that in (2.9), which is in turn larger than that in (2.8). It turns out the initial condition enters the asymptotic distribution in our model but not in the asymptotic distributions obtained in Magdalinos (2012) and Lui et al. (2021). Since the finite sample distribution should depend on the initial condition, naturally it is expected our asymptotic distribution delivers more accurate finite sample approximations. Second, in (2.8) it is assumed that $d \in (0, 1/2)$ which is equivalent to $H \in (1/2, 1)$, and hence, a long memory error term is assumed. In (2.9) it is assumed that $d \in (-1/2, 0)$ which is equivalent to $H \in (0, 1/2)$, and hence, an anti-persistent error term is assumed. In model (2.7), a full range of $H \in (0, 1)$ is allowed. That is, both long memory error terms and anti-persistent error terms are allowed in our

Table 1: Comparison of model (2.7) and the models considered in Magdalinos (2012) and Lui et al. (2021)

Model (2.7) (with $\Delta \to 0, T \to \infty$)	Model in (2.8) (with $n \to \infty$)	Model in (2.9) (with $n \to \infty, m \to \infty$)
$\beta_\Delta = e^{\theta\Delta} = 1 + \theta\frac{T}{n} + o(\Delta)$	$\rho_n = 1 + \frac{c}{n^\alpha}$	$\rho_{n,m} = 1 + \frac{cm}{n}$
$1 - \beta_\Delta \nearrow 0, (1 - \beta_\Delta)\, n \to -\infty$	$1 - \rho_n \nearrow 0, (1 - \rho_n)\, n \to -\infty$	$1 - \rho_{n,m} \nearrow 0, (1 - \rho_{n,m})\, n \to -\infty$
$Y_0 \sim O_p(n^H)$ with a fixed T	$Y_0 \sim o_p(n^{\alpha H})$	$Y_0 \sim o_p(n^H)$
$H \in (0, 1)$	$H \in (1/2, 1)$	$H \in (0, 1/2)$

model. While some empirical evidence has been reported to support long memory error terms in the context of the mildly explosive model for equity prices in the literature (see, for example, Lui et al., 2022), some other empirical evidence that supports anti-persistent error terms has also been reported in the literature (see, for example, Gatheral et al., 2018, Lui et al., 2021, Bennedsen et al., 2022, Shi and Yu, 2022, Wang et al., 2023, Bolk et al., 2023). In practice it is often impossible to have a knowledge about a restricted range of H *ex ante*. Table 1 compares the two existing models with model (2.7). Third, although our model shares the same covariance structure as model (2.7) and model (2.9), unlike the error terms in their models, our $e_{i\Delta}$ in (2.7) cannot be written as a linear combination of martingale difference sequence. As a result, our proof strategy is remarkably different from those in Magdalinos (2012) and Lui et al. (2021).

2.1 Estimators

Our model is the same as that of Wang et al. (2023). The only difference between the two models is that we assume $\theta > 0$ while Wang et al. (2023) assume $\theta < 0$ in fOUp. Following Wang et al. (2023), we also consider a two-stage estimation method. Our first stage estimation focuses on estimating the two parameters in the diffusion term and takes the same expressions as those proposed by Wang et al. (2023). In particular, we estimate the Hurst parameter H based on the second-order differences of X_t at two different frequencies:[3]

$$\widehat{H}_\Delta = \frac{1}{2}\log_2\left(\frac{\sum_{i=1}^{n-4}\left(X_{(i+4)\Delta} - 2X_{(i+2)\Delta} + X_{i\Delta}\right)^2}{\sum_{i=1}^{n-2}\left(X_{(i+2)\Delta} - 2X_{(i+1)\Delta} + X_{i\Delta}\right)^2}\right), \tag{2.10}$$

[3] If H is known to be less than $3/4$, a more efficient estimator of H may be obtained from first-order differences.

where $\log_2(\cdot)$ is the base-2 logarithm. We estimate the volatility coefficient σ using

$$\widehat{\sigma}_\Delta = \sqrt{\frac{\sum_{i=1}^{n-2}\left(X_{(i+2)\Delta} - 2X_{(i+1)\Delta} + X_{i\Delta}\right)^2}{n\left(4 - 2^{2\widehat{H}}\right)\Delta^{2\widehat{H}}}}. \tag{2.11}$$

In the second stage, we consider the estimators of the two drift parameters in (2.1) based on LS. Let $\alpha_\Delta = \frac{\mu}{\theta}(e^{\theta\Delta} - 1)$. Then, (2.6) can be rewritten as:

$$X_{i\Delta} = \beta_\Delta X_{(i-1)\Delta} + \alpha_\Delta + \sigma\epsilon_{i\Delta}\,, X_0 = O_p(1)\,.$$

The LS estimators of α_Δ and β_Δ are

$$\widehat{\beta}_\Delta = \frac{n\sum_{i=1}^n X_{i\Delta}X_{(i-1)\Delta} - \sum_{i=1}^n X_{i\Delta}\sum_{i=1}^n X_{(i-1)\Delta}}{n\sum_{i=1}^n X_{(i-1)\Delta}^2 - \left(\sum_{i=1}^n X_{(i-1)\Delta}\right)^2}\,, \tag{2.12}$$

$$\widehat{\alpha}_\Delta = \frac{\sum_{i=1}^n X_{i\Delta}\sum_{i=1}^n X_{(i-1)\Delta}^2 - \sum_{i=1}^n X_{(i-1)\Delta}\sum_{i=1}^n X_{i\Delta}X_{(i-1)\Delta}}{n\sum_{i=1}^n X_{(i-1)\Delta}^2 - \left(\sum_{i=1}^n X_{(i-1)\Delta}\right)^2}\,. \tag{2.13}$$

Based on $\widehat{\alpha}_\Delta$ and $\widehat{\beta}_\Delta$, we can propose the LS estimators of θ and μ as

$$\widehat{\theta}_\Delta = \frac{1}{\Delta}\log\frac{\sum_{i=1}^n X_{i\Delta}X_{(i-1)\Delta} - \frac{1}{n}\sum_{i=1}^n X_{i\Delta}\sum_{i=1}^n X_{(i-1)\Delta}}{\sum_{i=1}^n X_{(i-1)\Delta}^2 - \frac{1}{n}\left(\sum_{i=1}^n X_{(i-1)\Delta}\right)^2}\,, \tag{2.14}$$

$$\widehat{\mu}_\Delta = \widehat{\theta}_\Delta\frac{\widehat{\alpha}_\Delta}{\widehat{\beta}_\Delta - 1}\,. \tag{2.15}$$

Remark 2.1. *Wang et al. (2023) use the ergodic property of X_t to construct the method-of-moment estimators of θ and μ when $\theta < 0$. With $\theta > 0$, the fOUp is explosive and hence, non-ergodic. Consequently, the estimators for the drift term of Wang et al. (2023) are not applicable when $\theta > 0$.*

Remark 2.2. *The proposed LS estimators of θ and μ ignore the dependence structure in the error term and are independent of the two diffusion parameters. Later we will examine the efficiency loss in the LS estimators relative to the maximum likelihood estimators (MLE) that take account of the dependence structure in the error term.*

2.2 Asymptotic properties

In this subsection, we develop the in-fill asymptotic theory for $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$ and the double asymptotic theory for $\widehat{\mu}_\Delta$ and $\widehat{\theta}_\Delta$. For $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$, Theorem 4.1 of Wang et al. (2023) is directly applicable to fOUp with $\theta > 0$. Hence, we state it here with slightly re-phrasing but without proof.

Theorem 2.1. *Let $\widehat{H}_\Delta$ and $\widehat{\sigma}_\Delta$ be the estimators defined in (2.10) and (2.11) for model (2.1) with $\theta > 0$. For any $H \in (0,1)$, when $\Delta \to 0$ with a fixed $T > 0$,*
(a) $\widehat{H}_\Delta \overset{a.s.}{\to} H$ *and*

$$\sqrt{n}\left(\widehat{H}_\Delta - H\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{\Sigma_{11} + \Sigma_{22} - 2\Sigma_{12}}{(2\log 2)^2}\right); \tag{2.16}$$

(b) $\widehat{\sigma}_\Delta \overset{a.s.}{\to} \sigma$ *and*

$$\frac{\sqrt{n}}{\log(\Delta)}\left(\widehat{\sigma}_\Delta - \sigma\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{\Sigma_{11} + \Sigma_{22} - 2\Sigma_{12}}{(2\log 2)^2}\sigma^2\right), \tag{2.17}$$

where

$$\Sigma_{11} = 2 + 2^{2-4H}\sum_{j=1}^{\infty}\left(\rho_{j+2} + 4\rho_{j+1} + 6\rho_j + 4\rho_{|j-1|} + \rho_{|j-2|}\right)^2,$$

$$\Sigma_{12} = 2^{1-2H}\left(4(\rho_1+1)^2 + 2\sum_{j=0}^{\infty}(\rho_{j+2} + 2\rho_{j+1} + \rho_j)^2\right),$$

$$\Sigma_{22} = 2 + 4\sum_{j=1}^{\infty}\rho_j^2,$$

with

$$\rho_j = \frac{-|j+2|^{2H} + 4|j+1|^{2H} - 6|j|^{2H} + 4|j-1|^{2H} - |j-2|^{2H}}{2(4-2^{2H})}.$$

In the following, we shall state the main results concerning the strong consistency and the asymptotic distributions of $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$. First, we give the strong consistency of $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$, as well as the asymptotic theory for $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$.

Theorem 2.2. *Let $\Delta \to 0$ and $\frac{\log\Delta}{T} \to 0$. If either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have $\widehat{\theta}_\Delta \overset{a.s.}{\to} \theta$ and*

$$\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta - \theta\right) \overset{\mathcal{L}}{\to} \frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu}{X_0 + \frac{\mu}{\theta} + \sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega}, \tag{2.18}$$

where ν and ω are two independent standard normal variables.

Theorem 2.3. *Let $\Delta \to 0$ and $\frac{(\log\Delta)^3}{T^2} \to 0$. If either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have $\widehat{\mu}_\Delta \overset{a.s.}{\to} \mu$.*

Second, based on Theorem 2.2, we can develop the following joint distribution for $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$ in the explosive fOUp.

Theorem 2.4. *Let $\Delta \to 0$ and $\frac{\log \Delta}{T} \to 0$. If either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have*

$$\left(\frac{e^{\theta T}}{2\theta} \left(\widehat{\theta}_\Delta - \theta \right), T^{1-H} \left(\widehat{\mu}_\Delta - \mu \right) \right) \xrightarrow{\mathcal{L}} \left(\frac{\sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \nu}{X_0 + \frac{\mu}{\theta} + \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \omega}, \sigma\eta \right),$$

where ν, ω and η are independent standard normal variables.

From Theorem 2.4, the following result follows immediately.

Corollary 2.5. *Let $\Delta \to 0$ and $\frac{\log \Delta}{T} \to 0$. If either (i) $H = 1/2$, or (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have*

$$T^{1-H} \left(\widehat{\mu}_\Delta - \mu \right) \xrightarrow{\mathcal{L}} \sigma\eta\,, \tag{2.19}$$

where η is a standard normal variable.

Remark 2.3. *According to Theorem 2.2, the same asymptotic law holds for the LS estimator of θ regardless of H in the explosive fOUp. That is, the rate of convergence is $e^{\theta T}$ and, if $X_0 = \mu = 0$, the limit distribution is a standard Cauchy. However, from the technical proofs in the Appendix and Online Supplement, it can be seen that we need to deal with the cases of $H \in (1/2, 1)$, $H = 1/2$ and $H \in (0, 1/2)$ separately. The result in Theorem 2.2 is in sharp contrast with that of the method-of-moments estimator of θ for the stationary fOUp. Theorem 4.4 in Wang et al. (2023) shows that the asymptotic law for the method-of-moments estimator of θ changes as H passes 3/4. In particular, when $H \in (0, 3/4)$, the rate of convergence is $\sqrt{T}$ and the limit distribution is normal; when $H = 3/4$, the rate of convergence is $\sqrt{T}/\log T$ and the limit distribution is different normal; when $H \in (3/4, 1)$, the rate of convergence is T^{2-2H} and the limit distribution is the Rosenblatt random variable.*

Remark 2.4. *From Corollary 2.5, we can see that the asymptotic law of $\widehat{\mu}_\Delta$ is normal, where the rate of convergence is T^{1-H}. Theorem 5 in Tanaka et al. (2020) states that the MLE of μ (denoted by $\widehat{\mu}_{MLE}$) based on a continuous-time record is*

$$T^{1-H} \left(\widehat{\mu}_{MLE} - \mu \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left(0, \sigma^2 \frac{2H\Gamma(3-2H)\Gamma\left(H + 1/2\right)}{\Gamma\left(3/2 - H\right)} \right).$$

Comparing the above asymptotic theory with Corollary 2.5, we can see that, the rate of convergence of the LS estimator of μ based on the discrete-sampled data is identical to *that of the MLE of μ based on a continuous-time record. However, the LS estimator of μ is less efficient than the MLE of μ since the variance of MLE is smaller when $H \in (0, 1/2) \cup (1/2, 1)$ (i.e., $\frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)} < 1$). This efficiency loss is expected as the LS estimator ignores the dependence in the error term. When $H = 1/2$, the two variances are the same (i.e., $\frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)} = 1$). This is also expected because, when $H = 1/2$, the error term becomes iid.*

Remark 2.5. *From Theorem 2.2, we can see that the limiting distribution of $\widehat{\theta}_\Delta - \theta$ depends explicitly on the initial condition X_0 (as well as μ/θ). This dependence is the same as that in Wang and Yu (2016). The reason is that when $\Delta \to 0$, the initial condition in model (2.7) is larger than those assumed in Magdalinos (2012) and in Lui et al. (2021). If $X_0 = -\frac{\mu}{\theta}$ in the fOUp, then the limiting distribution of $\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta - \theta\right)$ is a standard Cauchy distribution, which is the same as that obtained in Magdalinos (2012) and Lui et al. (2021).*

Remark 2.6. *Since all four parameters are consistently estimated, the developed asymptotic distributions can be readily used to make statistical inference. For example, the 95% confidence interval for H is*

$$\left(\widehat{H}_\Delta - 1.96\frac{\sqrt{\widehat{\Sigma}_{11} + \widehat{\Sigma}_{22} - 2\widehat{\Sigma}_{12}}}{2\sqrt{n}\log 2}, \widehat{H}_\Delta + 1.96\frac{\sqrt{\widehat{\Sigma}_{11} + \widehat{\Sigma}_{22} - 2\widehat{\Sigma}_{12}}}{2\sqrt{n}\log 2}\right);$$

the 95% confidence interval for σ is

$$\left(\widehat{\sigma}_\Delta - 1.96\frac{\widehat{\sigma}_\Delta \log(\Delta)\sqrt{\widehat{\Sigma}_{11} + \widehat{\Sigma}_{22} - 2\widehat{\Sigma}_{12}}}{2\sqrt{n}\log 2}, \widehat{\sigma}_\Delta + 1.96\frac{\widehat{\sigma}_\Delta \log(\Delta)\sqrt{\widehat{\Sigma}_{11} + \widehat{\Sigma}_{22} - 2\widehat{\Sigma}_{12}}}{2\sqrt{n}\log 2}\right);$$

the 95% confidence interval for μ is

$$\left(\widehat{\mu}_\Delta - 1.96\widehat{\sigma}_\Delta (T)^{\widehat{H}_\Delta - 1}, \widehat{\mu}_\Delta + 1.96\widehat{\sigma}_\Delta (T)^{\widehat{H}_\Delta - 1}\right),$$

where $\widehat{\Sigma}_{ij}$ is obtained from Σ_{ij} by replacing H with $\widehat{H}_\Delta$. The 95% confidence interval for θ may be obtained from first drawing random samples from $\frac{\widehat{\sigma}_\Delta \frac{\sqrt{\widehat{H}_\Delta \Gamma(2\widehat{H}_\Delta)}}{\widehat{\theta}_\Delta^{\widehat{H}}}\nu}{X_0 + \frac{\widehat{\mu}_\Delta}{\widehat{\theta}_\Delta} + \widehat{\sigma}_\Delta \frac{\sqrt{\widehat{H}_\Delta \Gamma(2\widehat{H}_\Delta)}}{\widehat{\theta}_\Delta^{\widehat{H}}}\omega}$ and then obtaining the 2.5 and 97.5 percentiles from the random samples, as in Wang and Yu (2016).

Remark 2.7. *From (2.18) and (2.19), if* $\Delta \to 0$ *and* $\frac{\log \Delta}{T} \to 0$*, under either (i)* $H = 1/2$*, or (ii)* $H \in (1/2, 1)$ *and* $T^{2H}\Delta \to 0$*, or (iii)* $H \in (0, 1/2)$ *and* $T^{2H+1}\Delta \to 0$*, we can easily get*

$$\frac{e^{\theta T}}{2\theta\Delta}\left(\widehat{\beta}_\Delta - \beta_\Delta\right) \xrightarrow{\mathcal{L}} \frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu}{X_0 + \frac{\mu}{\theta} + \sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega}\,, \tag{2.20}$$

$$\frac{T^{1-H}}{\Delta}\left(\widehat{\alpha}_\Delta - \alpha_\Delta\right) \xrightarrow{\mathcal{L}} \sigma\eta\,, \tag{2.21}$$

where ν*,* ω *and* η *are defined by Theorem 2.4. If* $H = 1/2$*, the asymptotic theory given in (2.20) and (2.21) becomes that given in Theorem 3.3 (a)-(b) in Wang and Yu (2016).*

Remark 2.8. *When* $H < 3/4$*, based on first-order differences, we can provide a more efficient estimator of* H *as*

$$\widetilde{H}_\Delta = \frac{1}{2}\log_2\left(\frac{\sum_{i=1}^{n-2}\left(X_{(i+2)\Delta} - X_{i\Delta}\right)^2}{\sum_{i=1}^{n-1}\left(X_{(i+1)\Delta} - X_{i\Delta}\right)^2}\right). \tag{2.22}$$

Using similar arguments as Theorem 4.1 (a) in Wang et al. (2023), we can obtain $\widetilde{H}_\Delta \xrightarrow{a.s.} H$ *for* $H \in (0, 1)$*. Moreover, for* $0 < H < 3/4$*, when* $\Delta \to 0$ *with a fixed* T*, we can get*

$$\sqrt{n}\left(\widetilde{H}_\Delta - H\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\Omega_{11} - 2^{1+2H}\Omega_{12} + 2^{4H}\Omega_{22}}{2^{4H+2}\log^2(2)}\right), \tag{2.23}$$

where

$$\Omega_{11} = 2^{4H+1} + \sum_{j=1}^{\infty} 2^{4H+2}\widetilde{\rho}_{j,*}^2,\ \Omega_{22} = 2 + \sum_{j=1}^{\infty} 4\widetilde{\rho}_j^2,$$
$$\Omega_{12} = \Omega_{21} = 2^{4H-1} + \sum_{j=1}^{\infty}\left[2\left(\widetilde{\rho}_{j+1} + \widetilde{\rho}_j\right)^2 + 2\left(\widetilde{\rho}_{|j-1|} + \widetilde{\rho}_j\right)^2\right],$$

with

$$\widetilde{\rho}_{j,*} = \frac{1}{2^{2H+1}}\left[|j-2|^{2H} + (j+2)^{2H} - 2j^{2H}\right],\ \widetilde{\rho}_j = \frac{1}{2}\left[|j-1|^{2H} + (j+1)^{2H} - 2j^{2H}\right].$$

When $H = 1/2$*, a standard calculation shows that*

$$\Omega_{11} - 2^{1+2H}\Omega_{12} + 2^{4H}\Omega_{22} = 4.$$

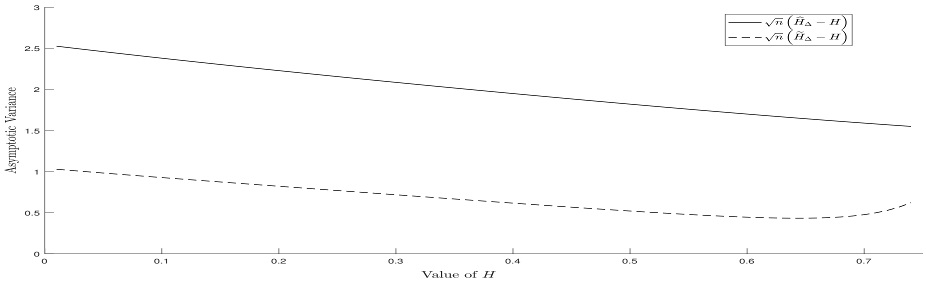

Figure 1: Asymptotic variance of $\sqrt{n}\left(\widehat{H}_\Delta - H\right)$ and $\sqrt{n}\left(\widetilde{H}_\Delta - H\right)$ as functions of $H \in (0, 3/4)$.

Consequently, for $H = 1/2$, when $\Delta \to 0$ with a fixed T, we can obtain

$$\sqrt{n}\left(\widetilde{H}_\Delta - H\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{4\log^2(2)}\right). \tag{2.24}$$

Comparing Corollary 4.2 in Wang et al. (2023) with (2.24), we can see that $\widetilde{H}_\Delta$ is more efficient than $\widehat{H}_\Delta$ for $H = 1/2$. Indeed, this conclusion holds true for $0 < H < 3/4$. Figure 1 compares the asymptotic variance of $\sqrt{n}\left(\widehat{H}_\Delta - H\right)$ and that of $\sqrt{n}\left(\widetilde{H}_\Delta - H\right)$ for $0 < H < 3/4$. When $0 < H < 3/4$, it is more efficient to estimate H via the first-order differences than via the second-order differences. However, when $H > 3/4$, the central limit theorem of the first-order differences does not hold. Whereas, we always have the central limit theorem for the second-order differences.

3 Simulation Studies

In this section, we conduct Monte Carlo simulations to evaluate the finite sample performance of the proposed estimator and the derived asymptotic limit theory. Following Wang and Yu (2016) and Chen et al. (2017), we first examine the sensitivity of the finite sample distribution of $\widehat{\theta}_\Delta$ and $\widehat{\beta}_\Delta$ with respect to the initial condition and to μ. We then check the finite sample performance of the derived asymptotic theory of (2.18) and (2.20).

For this purpose, we simulate 10,000 sample paths from model (2.1) with $\theta = 2$, $\sigma = 1$ and $\mu = 0$. However, we allow H to take different values, $0.15, 0.35, 0.55, 0.75$. The first two values imply anti-persistent errors while the last two values imply long-memory errors. We set the sampling interval $\Delta = 1/252, 1/52, 1/12$, the time span

Table 2: This table reports six percentiles of the Cauchy distribution and the finite sample distribution of $\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_{\Delta}-\theta\right)$ and $\frac{e^{\theta T}}{2\theta\Delta}\left(\widehat{\beta}_{\Delta}-\beta\right)$ when $\mu=0$ and $X_0=0$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	-31.8205	-12.7062	-3.0777	3.0777	12.7062	31.8205
$H=0.15$	$\Delta=1/252$	Finite Sample$^{\theta}$	-39.9078	-13.9255	-3.0534	3.0426	11.6275	28.0315
		Finite Sample$^{\beta}$	-39.9073	-13.9253	-3.0533	3.0426	11.6273	28.0312
	$\Delta=1/52$	Finite Sample$^{\theta}$	-30.3777	-12.4841	-2.9167	3.1683	12.8068	36.0481
		Finite Sample$^{\beta}$	-30.3702	-12.4810	-2.9160	3.1675	12.8037	36.0392
	$\Delta=1/12$	Finite Sample$^{\theta}$	-34.8777	-15.4491	-3.6273	3.6117	15.5121	39.6442
		Finite Sample$^{\beta}$	-34.7167	-15.3778	-3.6106	3.5951	15.4405	39.4613
$H=0.35$	$\Delta=1/252$	Finite Sample$^{\theta}$	-24.0441	-12.2667	-3.0488	3.0094	12.8997	32.8611
		Finite Sample$^{\beta}$	-24.0438	-12.2666	-3.0487	3.0093	12.8996	32.8607
	$\Delta=1/52$	Finite Sample$^{\theta}$	-28.2827	-11.7502	-2.9669	3.2007	12.2671	29.8610
		Finite Sample$^{\beta}$	-28.2757	-11.7474	-2.9662	3.1999	12.2641	29.8536
	$\Delta=1/12$	Finite Sample$^{\theta}$	-35.1576	-14.5487	-3.5612	3.6065	14.4238	37.9829
		Finite Sample$^{\beta}$	-34.9954	-14.4816	-3.5448	3.5898	14.3572	37.8076
$H=0.55$	$\Delta=1/252$	Finite Sample$^{\theta}$	-31.6969	-13.8355	-3.0380	2.9457	12.8903	31.5700
		Finite Sample$^{\beta}$	-31.6966	-13.8353	-3.0379	2.9456	12.8902	31.5696
	$\Delta=1/52$	Finite Sample$^{\theta}$	-30.6575	-13.2005	-3.1697	3.2053	12.6823	31.7275
		Finite Sample$^{\beta}$	-30.6500	-13.1973	-3.1689	3.2045	12.6792	31.7197
	$\Delta=1/12$	Finite Sample$^{\theta}$	-33.2604	-13.7933	-3.4648	3.5866	17.0857	40.2489
		Finite Sample$^{\beta}$	-33.1069	-13.7297	-3.4488	3.5700	17.0069	40.0631
$H=0.75$	$\Delta=1/252$	Finite Sample$^{\theta}$	-29.8430	-12.2791	-2.8160	3.1137	13.7555	35.4082
		Finite Sample$^{\beta}$	-29.8427	-12.2790	-2.8160	3.1137	13.7553	35.4078
	$\Delta=1/52$	Finite Sample$^{\theta}$	-31.0889	-13.5346	-3.1458	3.3624	13.4632	33.0715
		Finite Sample$^{\beta}$	-31.0812	-13.5313	-3.1450	3.3616	13.4599	33.0634
	$\Delta=1/12$	Finite Sample$^{\theta}$	-35.3672	-13.8410	-3.3564	3.7404	14.9353	40.5611
		Finite Sample$^{\beta}$	-35.2040	-13.7772	-3.3409	3.7231	14.8664	40.3739

$T=10$, the initial value $X_0=(0,3.5,10)$. For each simulated path, we estimate θ by (2.14) and calculate $\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_{\Delta}-\theta\right)$. Moreover, we also estimate β by (2.12) and calculate $\frac{e^{\theta T}}{2\theta\Delta}\left(\widehat{\beta}_{\Delta}-\beta\right)$. We report percentiles at levels $\{1\%, 2.5\%, 10\%, 90\%, 97.5\%, 99\%\}$ in the limit distributions of (2.18) and (2.20). Tables 2–4 report the percentiles of the Cauchy asymptotic distribution, the newly derived asymptotic distributions, and the finite sample distribution when $X_0=0,3.5,10$, respectively.

When $X_0=0$, since $\mu=0$, the newly derived asymptotic distribution becomes the Cauchy asymptotic distribution. Table 2 only report the percentiles of the Cauchy asymptotic distribution and the finite sample distributions. It is clear that the finite sample distributions are close to the Cauchy asymptotic distribution.

When $X_0\neq 0$, the newly derived asymptotic distribution is different from the Cauchy asymptotic distribution. From Table 3, when $X_0=3.5$, it is clear that the finite sample distributions are sensitive to the change of the initial condition and very far away to the

Table 3: The Cauchy distribution, the new asymptotic distribution and the finite sample distribution of $\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_{\Delta}-\theta\right)$ and $\frac{e^{\theta T}}{2\theta\Delta}\left(\widehat{\beta}_{\Delta}-\beta\right)$ when $\mu=0$ and $X_0=3.5$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	-31.8205	-12.7062	-3.0777	3.0777	12.7062	31.8205
$H=0.15$		New Asym.	-0.4388	-0.3595	-0.2269	0.2271	0.3593	0.4374
	$\Delta=1/252$	Finite Sample$^{\theta}$	-0.4129	-0.3442	-0.2178	0.2089	0.3237	0.3918
		Finite Sample$^{\beta}$	-0.4129	-0.3442	-0.2178	0.2089	0.3237	0.3918
	$\Delta=1/52$	Finite Sample$^{\theta}$	-0.4013	-0.3335	-0.2138	0.2088	0.3272	0.3977
		Finite Sample$^{\beta}$	-0.4012	-0.3335	-0.2138	0.2088	0.3271	0.3976
	$\Delta=1/12$	Finite Sample$^{\theta}$	-0.5114	-0.4090	-0.2602	0.2608	0.4093	0.5026
		Finite Sample$^{\beta}$	-0.5090	-0.4071	-0.2590	0.2596	0.4074	0.5003
$H=0.35$		New Asym.	-0.3761	-0.3106	-0.1977	0.1976	0.3100	0.3750
	$\Delta=1/252$	Finite Sample$^{\theta}$	-0.3240	-0.2667	-0.1697	0.1645	0.2538	0.3060
		Finite Sample$^{\beta}$	-0.3240	-0.2667	-0.1697	0.1645	0.2538	0.3060
	$\Delta=1/52$	Finite Sample$^{\theta}$	-0.2958	-0.2502	-0.1620	0.1605	0.2454	0.3014
		Finite Sample$^{\beta}$	-0.2957	-0.2501	-0.1619	0.1604	0.2453	0.3013
	$\Delta=1/12$	Finite Sample$^{\theta}$	-0.4283	-0.3529	-0.2254	0.2240	0.3526	0.4463
		Finite Sample$^{\beta}$	-0.4263	-0.3513	-0.2243	0.2229	0.3510	0.4442
$H=0.55$		New Asym.	-0.3481	-0.2882	-0.1841	0.1843	0.2881	0.3473
	$\Delta=1/252$	Finite Sample$^{\theta}$	-0.2715	-0.2251	-0.1448	0.1370	0.2123	0.2507
		Finite Sample$^{\beta}$	-0.2715	-0.2251	-0.1448	0.1370	0.2123	0.2507
	$\Delta=1/52$	Finite Sample$^{\theta}$	-0.2417	-0.2044	-0.1328	0.1302	0.1993	0.2387
		Finite Sample$^{\beta}$	-0.2416	-0.2043	-0.1328	0.1301	0.1992	0.2386
	$\Delta=1/12$	Finite Sample$^{\theta}$	-0.3958	-0.3259	-0.2091	0.2072	0.3232	0.3983
		Finite Sample$^{\beta}$	-0.3940	-0.3244	-0.2081	0.2062	0.3217	0.3965
$H=0.75$		New Asym.	-0.3407	-0.2822	-0.1805	0.1807	0.2822	0.3400
	$\Delta=1/252$	Finite Sample$^{\theta}$	-0.2447	-0.2016	-0.1293	0.1204	0.1844	0.2206
		Finite Sample$^{\beta}$	-0.2447	-0.2016	-0.1293	0.1204	0.1844	0.2206
	$\Delta=1/52$	Finite Sample$^{\theta}$	-0.2159	-0.1792	-0.1154	0.1100	0.1649	0.2025
		Finite Sample$^{\beta}$	-0.2159	-0.1792	-0.1154	0.1100	0.1649	0.2024
	$\Delta=1/12$	Finite Sample$^{\theta}$	-0.4022	-0.3268	-0.2068	0.1980	0.3044	0.3595
		Finite Sample$^{\beta}$	-0.4003	-0.3253	-0.2058	0.1970	0.3030	0.3579

Cauchy asymptotic distribution. For example, the 1 percentile of the Cauchy asymptotic distribution is -31.8205 while the 1 percentiles of the finite sample distribution move around -0.4. In sharp contrast, the 1 percentile of the newly derived asymptotic distribution is -0.4388, suggesting the newly derived asymptotic distribution yields good approximations to the finite sample distributions. From Table 4, when $X_0 = 10$, the finite sample distributions are even further away to the Cauchy asymptotic distribution. Whereas, the newly derived asymptotic distribution yields good approximations to the finite sample distributions.

Next, we investigate the sensitivity of the finite sample distribution of $\widehat{\theta}$ and $\widehat{\beta}$ with respect to the value of μ. For this purpose, we set $\theta = 1, \sigma = 0.2, X_0 = 0$, $\mu = -0.7$. The choice of μ is based on the empirical results of the log monthly Nasdaq real price

Table 4: The Cauchy distribution, the new asymptotic distribution and the finite sample distribution of $\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta-\theta\right)$ and $\frac{e^{\theta T}}{2\theta\Delta}\left(\widehat{\beta}_\Delta-\beta\right)$ when $\mu=0$ and $X_0=10$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	-31.8205	-12.7062	-3.0777	3.0777	12.7062	31.8205
$H=0.15$		New Asym.	-0.1421	-0.1192	-0.0777	0.0777	0.1192	0.1417
	$\Delta=1/252$	Finite Sample$^\theta$	-0.1379	-0.1129	-0.0745	0.0717	0.1088	0.1291
		Finite Sample$^\beta$	-0.1378	-0.1129	-0.0745	0.0717	0.1088	0.1291
	$\Delta=1/52$	Finite Sample$^\theta$	-0.1330	-0.1127	-0.0733	0.0717	0.1084	0.1325
		Finite Sample$^\beta$	-0.1329	-0.1127	-0.0733	0.0717	0.1084	0.1324
	$\Delta=1/12$	Finite Sample$^\theta$	-0.1642	-0.1383	-0.0896	0.0899	0.1374	0.1621
		Finite Sample$^\beta$	-0.1634	-0.1377	-0.0892	0.0895	0.1367	0.1613
$H=0.35$		New Asym.	-0.1243	-0.1043	-0.0681	0.0680	0.1043	0.1240
	$\Delta=1/252$	Finite Sample$^\theta$	-0.1087	-0.0902	-0.0587	0.0565	0.0863	0.1018
		Finite Sample$^\beta$	-0.1087	-0.0902	-0.0587	0.0565	0.0863	0.1018
	$\Delta=1/52$	Finite Sample$^\theta$	-0.1002	-0.0853	-0.0560	0.0557	0.0839	0.1010
		Finite Sample$^\beta$	-0.1001	-0.0853	-0.0560	0.0557	0.0838	0.1009
	$\Delta=1/12$	Finite Sample$^\theta$	-0.1411	-0.1209	-0.0778	0.0771	0.1192	0.1421
		Finite Sample$^\beta$	-0.1405	-0.1203	-0.0774	0.0767	0.1186	0.1415
$H=0.55$		New Asym.	-0.1160	-0.0974	-0.0636	0.0635	0.0974	0.1157
	$\Delta=1/252$	Finite Sample$^\theta$	-0.0907	-0.0768	-0.0498	0.0478	0.0728	0.0863
		Finite Sample$^\beta$	-0.0907	-0.0768	-0.0498	0.0478	0.0728	0.0863
	$\Delta=1/52$	Finite Sample$^\theta$	-0.0817	-0.0712	-0.0458	0.0455	0.0690	0.0816
		Finite Sample$^\beta$	-0.0817	-0.0712	-0.0458	0.0455	0.0690	0.0816
	$\Delta=1/12$	Finite Sample$^\theta$	-0.1310	-0.1103	-0.0729	0.0712	0.1094	0.1309
		Finite Sample$^\beta$	-0.1304	-0.1098	-0.0725	0.0708	0.1089	0.1303
$H=0.75$		New Asym.	-0.1138	-0.0955	-0.0624	0.0623	0.0955	0.1135
	$\Delta=1/252$	Finite Sample$^\theta$	-0.0807	-0.0690	-0.0444	0.0425	0.0648	0.0753
		Finite Sample$^\beta$	-0.0807	-0.0690	-0.0443	0.0425	0.0648	0.0753
	$\Delta=1/52$	Finite Sample$^\theta$	-0.0728	-0.0607	-0.0399	0.0386	0.0583	0.0713
		Finite Sample$^\beta$	-0.0727	-0.0607	-0.0398	0.0386	0.0583	0.0713
	$\Delta=1/12$	Finite Sample$^\theta$	-0.1303	-0.1089	-0.0702	0.0698	0.1064	0.1265
		Finite Sample$^\beta$	-0.1297	-0.1084	-0.0698	0.0695	0.1059	0.1259

between January 1990 to June 2000. Table 5 reports the percentiles of the finite sample distribution, the Cauchy asymptotic distribution, and the new asymptotic distribution. Compared with Table 2, Table 5 suggests that the finite sample distributions are sensitive to the change of μ and far away to the Cauchy asymptotic distribution. Whereas, the newly derived asymptotic distribution yields good approximations to the finite sample distributions.

Finally, we conduct Monte Carlo simulations to evaluate the finite sample performance of the derived asymptotic distributions of $\widehat{H}_\Delta$, $\widehat{\mu}_\Delta$ and $\widehat{\sigma}_\Delta$. In particular, we obtain the finite sample distributions of the following statistics:

$$\Phi_{\widehat{H}_\Delta}=\sqrt{n}\left(\widehat{H}_\Delta-H\right),\ \Phi_{\widehat{\sigma}_\Delta}=\frac{\sqrt{n}}{\log(\Delta)}\left(\widehat{\sigma}_\Delta-\sigma\right),\ \Phi_{\widehat{\mu}_\Delta}=T^{1-H}\left(\widehat{\mu}_\Delta-\mu\right). \tag{3.1}$$

Table 5: The Cauchy asymptotic distribution, the new asymptotic distribution and the finite sample distribution of $\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta - \theta\right)$ and $\frac{e^{\theta T}}{2\theta\Delta}\left(\widehat{\beta}_\Delta - \beta\right)$ when $\mu = -0.7$ and $X_0 = 0$.

		Percentiles	1%	2.5%	10%	90%	97.5%	99%
		Cauchy Asym.	-31.8205	-12.7062	-3.0777	3.0777	12.7062	31.8205
$H = 0.15$		New Asym.	-0.4981	-0.4052	-0.2529	0.2540	0.4065	0.4980
	$\Delta = 1/252$	Finite Sample$^\theta$	-0.4972	-0.4154	-0.2692	0.2730	0.4379	0.5303
		Finite Sample$^\beta$	-0.4972	-0.4154	-0.2692	0.2730	0.4379	0.5303
	$\Delta = 1/52$	Finite Sample$^\theta$	-0.4679	-0.3919	-0.2484	0.2575	0.4309	0.5121
		Finite Sample$^\beta$	-0.4679	-0.3918	-0.2484	0.2575	0.4309	0.5120
	$\Delta = 1/12$	Finite Sample$^\theta$	-0.5212	-0.4326	-0.2814	0.2945	0.4784	0.5960
		Finite Sample$^\beta$	-0.5206	-0.4321	-0.2811	0.2942	0.4778	0.5954
$H = 0.35$		New Asym.	-0.5021	-0.4082	-0.2546	0.2557	0.4094	0.5020
	$\Delta = 1/252$	Finite Sample$^\theta$	-0.4407	-0.3700	-0.2385	0.2486	0.4001	0.4907
		Finite Sample$^\beta$	-0.4407	-0.3700	-0.2385	0.2486	0.4001	0.4907
	$\Delta = 1/52$	Finite Sample$^\theta$	-0.4011	-0.3414	-0.2206	0.2322	0.3753	0.4605
		Finite Sample$^\beta$	-0.4011	-0.3413	-0.2206	0.2321	0.3752	0.4605
	$\Delta = 1/12$	Finite Sample$^\theta$	-0.5097	-0.4187	-0.2731	0.3011	0.4910	0.6077
		Finite Sample$^\beta$	-0.5091	-0.4183	-0.2728	0.3007	0.4904	0.6070
$H = 0.55$		New Asym.	-0.5495	-0.4438	-0.2746	0.2757	0.4451	0.5495
	$\Delta = 1/252$	Finite Sample$^\theta$	-0.4051	-0.3355	-0.2192	0.2414	0.3822	0.4775
		Finite Sample$^\beta$	-0.4051	-0.3355	-0.2192	0.2414	0.3822	0.4775
	$\Delta = 1/52$	Finite Sample$^\theta$	-0.3635	-0.3015	-0.2001	0.2102	0.3467	0.4197
		Finite Sample$^\beta$	-0.3634	-0.3015	-0.2001	0.2102	0.3467	0.4197
	$\Delta = 1/12$	Finite Sample$^\theta$	-0.5142	-0.4193	-0.2746	0.3147	0.5312	0.6669
		Finite Sample$^\beta$	-0.5136	-0.4188	-0.2743	0.3144	0.5306	0.6661
$H = 0.75$		New Asym.	-0.6460	-0.5142	-0.3126	0.3139	0.5155	0.6462
	$\Delta = 1/252$	Finite Sample$^\theta$	-0.3563	-0.2932	-0.1950	0.2209	0.3552	0.4257
		Finite Sample$^\beta$	-0.3563	-0.2932	-0.1950	0.2209	0.3552	0.4257
	$\Delta = 1/52$	Finite Sample$^\theta$	-0.3037	-0.2620	-0.1710	0.1842	0.3025	0.3641
		Finite Sample$^\beta$	-0.3037	-0.2620	-0.1710	0.1842	0.3025	0.3640
	$\Delta = 1/12$	Finite Sample$^\theta$	-0.4819	-0.4023	-0.2627	0.3191	0.5614	0.7115
		Finite Sample$^\beta$	-0.4814	-0.4018	-0.2624	0.3187	0.5608	0.7106

To simulate data, we set $\theta = 0.2$, $\sigma = 0.2$ and $\mu = -1$ and allow H to take different values in the range of $(0, 1)$. For convenience, we choose the sampling interval $\Delta = 1/252$ and the time span $T = 10$ with 10,000 simulated sample paths from model (2.1). We then report the mean, variance, skewness and kurtosis of the finite sample distributions of $\Phi_{\widehat{H}_\Delta}$, $\Phi_{\widehat{\sigma}_\Delta}$ and $\Phi_{\widehat{\mu}_\Delta}$ and those of the asymptotic normal distributions (i.e., $\mathcal{N}(0, 1)$) in Table 6. As we can see from Table 6, the derived asymptotic distributions well approximate the finite sample distributions for all three parameters.

Table 6: Mean, variance, skewness and kurtosis of $\Phi_{\widehat{H}_\Delta}$, $\Phi_{\widehat{\sigma}_\Delta}$, $\Phi_{\widehat{\mu}_\Delta}$ in (3.1) and the standard normal limiting distribution.

Value of H	Statistics	Mean	Variance	Skewness	Kurtosis
	$\mathcal{N}(0,1)$	0	1	0	3
$H=0.1$	$\Phi_{\widehat{H}_\Delta}$	-0.017311	0.923305	-0.013949	3.024885
	$\Phi_{\widehat{\sigma}_\Delta}$	0.022569	0.888150	-0.091405	3.176769
	$\Phi_{\widehat{\mu}_\Delta}$	0.0315831	1.062469	-0.163131	3.570116
$H=0.2$	$\Phi_{\widehat{H}_\Delta}$	-0.019234	0.953352	-0.007881	2.991773
	$\Phi_{\widehat{\sigma}_\Delta}$	0.064560	0.927148	-0.087279	3.172227
	$\Phi_{\widehat{\mu}_\Delta}$	0.044777	0.938544	0.108009	3.096121
$H=0.3$	$\Phi_{\widehat{H}_\Delta}$	-0.020780	0.925791	-0.006025	2.978431
	$\Phi_{\widehat{\sigma}_\Delta}$	0.029536	0.933122	-0.086998	3.168624
	$\Phi_{\widehat{\mu}_\Delta}$	0.033007	0.952455	-0.096756	3.131023
$H=0.4$	$\Phi_{\widehat{H}_\Delta}$	-0.022140	0.929721	-0.007054	2.979255
	$\Phi_{\widehat{\sigma}_\Delta}$	0.030560	1.085842	-0.089672	3.268379
	$\Phi_{\widehat{\mu}_\Delta}$	-0.015380	0.915075	0.058212	2.962478
$H=0.5$	$\Phi_{\widehat{H}_\Delta}$	-0.011831	0.934033	-0.029452	3.108228
	$\Phi_{\widehat{\sigma}_\Delta}$	-0.065678	1.022217	-0.035021	3.207613
	$\Phi_{\widehat{\mu}_\Delta}$	-0.140621	0.922738	-0.084704	3.231211
$H=0.6$	$\Phi_{\widehat{H}_\Delta}$	-0.022225	0.942600	-0.014839	2.991179
	$\Phi_{\widehat{\sigma}_\Delta}$	0.075118	0.911895	-0.053187	3.217130
	$\Phi_{\widehat{\mu}_\Delta}$	-0.029949	1.036335	-0.083410	3.278550
$H=0.7$	$\Phi_{\widehat{H}_\Delta}$	-0.013036	0.952611	-0.021022	2.991612
	$\Phi_{\widehat{\sigma}_\Delta}$	0.081108	0.942304	-0.051543	3.299545
	$\Phi_{\widehat{\mu}_\Delta}$	-0.018987	1.039631	-0.070227	2.896728
$H=0.8$	$\Phi_{\widehat{H}_\Delta}$	0.039712	0.967824	-0.027516	2.986774
	$\Phi_{\widehat{\sigma}_\Delta}$	0.061376	0.945947	-0.167950	3.281394
	$\Phi_{\widehat{\mu}_\Delta}$	-0.016915	1.056022	0.040067	2.856668

4 Empirical Studies

To illustrate the usefulness of the proposed model and the derived limit distribution in practice, in this section, we consider two empirical studies. In the first empirical study, we use a dataset of the log monthly Nasdaq real price between January 1990 to June 2000. Model (2.1) is fitted to the data with $\Delta = 1/12$, $T = 10.5$, $n = 126$, and $X_0 = 5.0628$, which is the log Nasdaq real price in December 1989. In the second empirical study, we use a dataset of the monthly price-dividend ratio of Nasdaq between January 1990 to June 2000. Model (2.1) is fitted to the data with $\Delta = 1/12$, $T = 10.5$, $n = 126$, and $X_0 = 1.7753$, which is the price-dividend ratio of Nasdaq in December 1989. Figure 2 presents time series plots of the log monthly Nasdaq real price (left), the monthly price-dividend ratio of Nasdaq (right). Both time series exhibit a non-stationary feature.

Using these two time series, we estimate H, σ, θ, μ using (2.10), (2.11), (2.14) and (2.15), respectively. The point estimates and their corresponding 90% confidence in-

Table 7: Empirical results for the real Nasdaq index and the monthly price-dividend ratio of Nasdaq from January 1990 to June 2000

Names	$\widehat{H}$	$\widehat{\sigma}$	$\widehat{\mu}$	$\widehat{\theta}$
log real price	0.5031 (0.3055, 0.7006)	0.2155 (0.1983, 0.2326)	-0.7016 (-0.8122, -0.5910)	0.1591 (0.0528,0.2648)
price-dividend ratio	0.1088 (-0.1131, 0.3307)	0.9560 (0.8706, 1.0414)	0.8759 (0.6811, 1.0707)	0.0521 (0.0473, 0.0569)

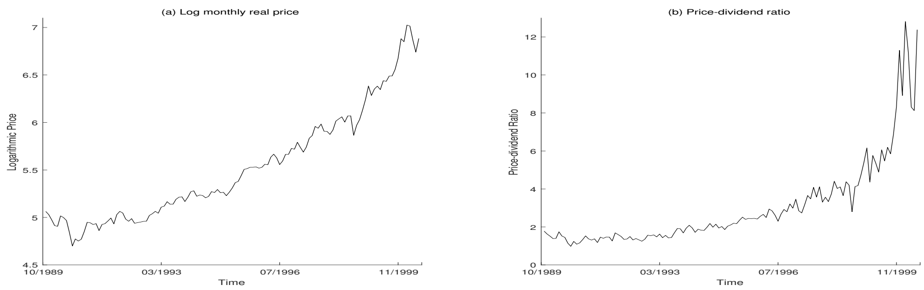

Figure 2: Time series plot of the log monthly Nasdaq real price (left), the monthly price-dividend ratio of Nasdaq (right).

tervals based on the derived asymptotic distributions are reported in Table 7. Since the estimated θ is greater than zero in both studies, model (2.1) is relevant and the asymptotic theory developed in this paper is applicable. From Table 7, we can see that the 90% confidence interval of θ excludes zero, which implies explosiveness in both time series. Interestingly, in the first empirical example, the point estimate of H is slightly larger than 0.5, implying long memory in the error term, albeit not statistically significantly different from 0.5. In the second empirical example, the point estimate of H is much smaller than 0.5, implying anti-persistence in the error term. The evidence of anti-persistence is statistically significant. These two studies suggest that in practice it is indeed difficult to know the range of H *ex ante*.

5 Conclusions

The fOUp has been used to model the realized volatility in recent years. Moreover, the discrete-time representation of fOUp has been used to model equity price (Lui, et al. 2021, 2022). In this paper, we introduce estimators for all four parameters in fOUp. The estimators of two diffusion parameters are the same as those in Wang et al. (2023). The estimators of two drift parameters are based on the LS method. The asymptotic theory for the diffusion estimators are established under the in-fill asymptotic scheme. The asymptotic theory for the drift estimators are established under the double asymptotic scheme for explosive fOUp with a full range of the Hurst parameter.

Our double asymptotic theory contributes to the literature in two aspects. First, our theory permits explicit consideration of the effects from the initial condition. Monte Carlo evidence suggests that the new asymptotic theory provides a better approximation to the finite sample distribution than the limit theory that is independent of the initial condition. Second, our theory works for the full range value of $H \in (0,1)$. Our asymptotic distribution for θ is the same whether $H < 1/2$ or $H > 1/2$.

Our simulation studies show that the finite sample distribution of θ is indeed very sensitive to the change of the initial condition and that our asymptotic distribution can well approximate the finite sample distribution not only for θ but also for other parameters in the model. Our empirical studies based on the real equity price and the price-dividend ratio suggest that explosiveness in fOUp can be empirically relevant and that in practice it is difficult to know the range of H *ex ante*.

6 Appendix: Proofs of Theorems 2.2-2.4

6.1 Useful Lemmas and Propositions

Before we prove Theorems 2.2-2.4, we first establish some useful propositions as well as some lemmas whose proofs may be found the Online Supplement.

To introduce the propositions and lemmas, we rewrite $\widehat{\theta}_\Delta$ and $\widehat{\mu}_\Delta$ as

$$\widehat{\theta}_\Delta = \theta + \frac{1}{\Delta}\log\left(1 + e^{-\theta\Delta}\frac{U_T}{V_T}\right), \tag{6.1}$$

$$\widehat{\mu}_\Delta = \mu + \left(\widehat{\theta}_\Delta - \theta\right)\left(\frac{\mu}{\theta} + \frac{M_T}{N_T}\right) + \theta\frac{M_T}{N_T}, \tag{6.2}$$

where

$$U_T = \sigma\sum_{i=1}^n \epsilon_{i\Delta} X_{(i-1)\Delta} - \sigma\frac{1}{n}\sum_{i=1}^n \epsilon_{i\Delta}\sum_{i=1}^n X_{(i-1)\Delta}, \tag{6.3}$$

$$V_T = \sum_{i=1}^n X_{(i-1)\Delta}^2 - \frac{1}{n}\left(\sum_{i=1}^n X_{(i-1)\Delta}\right)^2, \tag{6.4}$$

$$
\begin{aligned}
M_T = &\sigma \sum_{i=1}^n \epsilon_{i\Delta} \sum_{i=1}^n X^2_{(i-1)\Delta} + \sigma \frac{\mu}{\theta} \sum_{i=1}^n \epsilon_{i\Delta} \sum_{i=1}^n X_{(i-1)\Delta} \\
&- \sigma \frac{\mu}{\theta} n \sum_{i=1}^n \epsilon_{i\Delta} X_{(i-1)\Delta} - \sigma \sum_{i=1}^n X_{(i-1)\Delta} \sum_{i=1}^n \epsilon_{i\Delta} X_{(i-1)\Delta} \,, \qquad (6.5) \\
N_T = &n \left(e^{\theta\Delta} - 1\right) \sum_{i=1}^n X^2_{(i-1)\Delta} - \left(e^{\theta\Delta} - 1\right) \left(\sum_{i=1}^n X_{(i-1)\Delta} \right)^2 \\
&- \sigma \sum_{i=1}^n \epsilon_{i\Delta} \sum_{i=1}^n X_{(i-1)\Delta} + \sigma n \sum_{i=1}^n \epsilon_{i\Delta} X_{(i-1)\Delta}. \qquad (6.6)
\end{aligned}
$$

Then, we present further expansions for the terms U_T, V_T, M_T and N_T, which play crucial roles in our analysis. Let

$$
\begin{aligned}
\widetilde{U}_T = &\sigma \sum_{i=1}^n e^{-\theta(n-i)\Delta} \epsilon_{i\Delta} = \sigma \sum_{i=1}^n e^{-\theta T} e^{2\theta i \Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB^H_s \,, \qquad (6.7) \\
\widetilde{X}_T = &e^{-\theta T} X_T = X_0 + \frac{\mu}{\theta} \left(1 - e^{-\theta T}\right) + \sigma \int_0^T e^{-\theta s} dB^H_s \,, \qquad (6.8) \\
\Xi_T = &T^{-H} \sum_{i=1}^n \epsilon_{i\Delta} \,. \qquad (6.9)
\end{aligned}
$$

Proposition 6.1. *Let U_T, V_T, M_T, N_T, $\widetilde{U}_T$, $\widetilde{X}_T$ and Ξ_T be defined by (6.3)-(6.9), respectively. Then, we have*

$$
\begin{aligned}
e^{-\theta(T+\Delta)} U_T &= e^{-2\theta\Delta} \widetilde{U}_T \widetilde{X}_T + R_{1n} \,, \qquad (6.10) \\
\left(1 - e^{-2\theta\Delta}\right) e^{-2\theta T} V_T &= e^{-2\theta\Delta} \widetilde{X}^2_{(n-1)\Delta} - R_{2n} \,, \qquad (6.11) \\
T^{-H} \left(1 - e^{-2\theta\Delta}\right) e^{-2\theta T} M_T &= \sigma e^{-2\theta\Delta} \Xi_T \widetilde{X}^2_{(n-1)\Delta} - R_{3n} \,, \qquad (6.12) \\
\left(n(e^{\theta\Delta} - 1)\right)^{-1} \left(1 - e^{-2\theta\Delta}\right) e^{-2\theta T} N_T &= e^{-2\theta\Delta} \widetilde{X}^2_{(n-1)\Delta} - R_{4n} \,, \qquad (6.13)
\end{aligned}
$$

where the remainder terms R_{1n}, R_{2n}, R_{3n} and R_{4n} are defined as

$$
R_{1n} = \sigma e^{-2\theta\Delta} \sum_{i=1}^n e^{-\theta(T-i\Delta)} \epsilon_{i\Delta} \left(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T\right) \quad - \sigma \frac{e^{-\theta(T+\Delta)}}{n} \sum_{i=1}^n \epsilon_{i\Delta} \sum_{i=1}^n X_{(i-1)\Delta}, \qquad (6.14)
$$

$$
\begin{aligned}
R_{2n} = &e^{-2\theta\Delta} \sum_{i=2}^n e^{-2\theta(T-i\Delta+\Delta)} \left(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta}\right) + e^{-2\theta\Delta} e^{-2\theta T} \widetilde{X}^2_0 \\
&+ \frac{(1 - e^{-2\theta\Delta}) e^{-2\theta T}}{n} \Big(\sum_{i=1}^n X_{(i-1)\Delta} \Big)^2 , \qquad (6.15)
\end{aligned}
$$

$$
\begin{aligned}
R_{3n} =& \sigma \Xi_T \left[e^{-2\theta\Delta} \sum_{i=2}^{n} e^{-2\theta(T-i\Delta+\Delta)} \left(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta} \right) + e^{-2\theta\Delta} e^{-2\theta T} \widetilde{X}^2_0 \right] \\
&- \sigma \frac{\mu}{\theta} \left(1 - e^{-2\theta\Delta} \right) e^{-2\theta T} \Xi_T \sum_{i=1}^{n} X_{(i-1)\Delta} + e^{-2\theta\Delta} e^{-2\theta T} \widetilde{X}^2_0 \\
&+ \sigma \frac{\mu}{\theta} n T^{-H} \left(1 - e^{-2\theta\Delta} \right) e^{-2\theta T} \sum_{i=1}^{n} \epsilon_{i\Delta} X_{(i-1)\Delta} \\
&+ \sigma T^{-H} \left(1 - e^{-2\theta\Delta} \right) e^{-2\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \sum_{i=1}^{n} \epsilon_{i\Delta} X_{(i-1)\Delta} \,, \qquad (6.16) \\
R_{4n} =& e^{-2\theta\Delta} \sum_{i=2}^{n} e^{-2\theta(T-i\Delta+\Delta)} \left(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta} \right) + e^{-2\theta\Delta} e^{-2\theta T} \widetilde{X}^2_0 \\
&+ \frac{1}{n} \left(1 - e^{-2\theta\Delta} \right) e^{-2\theta T} \left(\sum_{i=1}^{n} X_{(i-1)\Delta} \right)^2 \\
&+ \sigma \frac{\left(1 - e^{-2\theta\Delta} \right) e^{-2\theta T}}{n \left(e^{\theta\Delta} - 1 \right)} \sum_{i=1}^{n} \epsilon_{i\Delta} \sum_{i=1}^{n} X_{(i-1)\Delta} - \sigma \frac{\left(1 - e^{-2\theta\Delta} \right) e^{-2\theta T}}{e^{\theta\Delta} - 1} \sum_{i=1}^{n} \epsilon_{i\Delta} X_{(i-1)\Delta}.
\end{aligned}
\tag{6.17}
$$

Proof. Using (2.6), we can write

$$
\sigma e^{-\theta(T+\Delta)} \sum_{i=1}^{n} \epsilon_{i\Delta} X_{(i-1)\Delta} = e^{-2\theta\Delta} \widetilde{U}_T \widetilde{X}_T + \sigma e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta(T-i\Delta)} \epsilon_{i\Delta} \left(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T \right), \tag{6.18}
$$

where $\widetilde{U}_T$ and $\widetilde{X}_T$ are defined by (6.7) and (6.8), respectively. Then, together with (6.3) and (6.18), we can easily obtain (6.10). Using (2.6) again, we can see that

$$
\left(1 - e^{-2\theta\Delta} \right) e^{-2\theta T} \sum_{i=1}^{n} X^2_{(i-1)\Delta} = e^{-2\theta\Delta} \widetilde{X}^2_{(n-1)\Delta} - e^{-2\theta\Delta} \sum_{i=2}^{n} e^{-2\theta(n-i+1)\Delta} \left(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta} \right).
$$

Together with (6.4), we have (6.11). Furthermore, combining (6.5), (6.6) and (6.11), we can obtain (6.12) and (6.13) easily.

6.1.1 Lemmas

Let $\alpha_H = H(2H-1)$. The first five lemmas hold when $H \in (1/2, 1)$.

Lemma 6.1. *Let $\Delta \to 0$ and $T \to \infty$. Then we have*

$$
\alpha_H \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta j\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dt ds \to \frac{H\Gamma(2H)}{\theta^{2H}}. \tag{6.19}
$$

Lemma 6.2. *Let $\Delta \to 0$ and $T \to \infty$. Then we obtain*

$$\begin{aligned}
&\alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&\sim \theta T^{2H}\Delta + Te^{-\theta T}\Delta^{2H-1} + o(1). \qquad (6.20)
\end{aligned}$$

Lemma 6.3. *Let $\Delta \to 0$ and $T \to \infty$. Then we obtain*

$$\alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i+j)\Delta} \int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(t+s)} |t-s|^{2H-2} dtds \sim T^{2H} + o\left(T^{2H}\right). \qquad (6.21)$$

Lemma 6.4. *Let $\Delta \to 0$ and $T \to \infty$. We obtain*

$$\alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(t+s)} |t-s|^{2H-2} dtds \sim \frac{\theta}{2} T^{2H}\Delta + \theta H T^{2H-1} + O(1). \qquad (6.22)$$

Lemma 6.5. *Let $\Delta \to 0$ and $T \to \infty$. We can have*

$$\alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta(i+2j)\Delta} \int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(t+s)} |t-s|^{2H-2} dtds \sim \frac{\theta}{2} T^{2H}\Delta + \theta H T^{2H-1} + O(1). \qquad (6.23)$$

The next five lemmas hold when $H \in (0, 1/2)$.

Lemma 6.6. *Let $\Delta \to 0$ and $T \to \infty$. Then we have*

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta} \Big(\theta \int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dtds \\
&\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Big) \\
&\to \frac{H\Gamma(2H)}{\theta^{2H}}, \qquad (6.24)
\end{aligned}$$

where $R(t,s)$ is defined by (2.2).

Lemma 6.7. *Let $\Delta \to 0$ and $T \to \infty$. Then we have*

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \Big(\theta \int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dtds \\
&\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Big) \\
&\sim -\frac{\theta^2}{2H+1} T^{2H+1}\Delta + \frac{\theta}{2} T^{2H}\Delta + o(1). \qquad (6.25)
\end{aligned}$$

Lemma 6.8. *Let $\Delta \to 0$ and $T \to \infty$. Then we have*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta(i+j)\Delta}\Big(\theta \int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Big) \sim T^{2H} + o\left(T^{2H}\right). \tag{6.26}$$

Lemma 6.9. *Let $\Delta \to 0$ and $T \to \infty$. Then, we get*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta i\Delta}\Big(\theta \int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Big) \sim -\frac{\theta}{2\left(2H+1\right)}T^{2H+1}\Delta + O(1). \tag{6.27}$$

Lemma 6.10. *Let $\Delta \to 0$ and $T \to \infty$. Then, we get*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{\theta(i+2j)\Delta}\Big(\theta \int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Big) \sim O\left(T^{2H+1}\Delta\right) + O(1). \tag{6.28}$$

Lemma 6.11. *Let $\Delta \to 0$, $T \to \infty$. If (i) $H = 1/2$, (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$, then we have*

$$\mathbb{E}\left(\widetilde{U}_T\widetilde{X}_T\right) \to 0. \tag{6.29}$$

Lemma 6.12. *Let $\Delta \to 0$ and $T \to \infty$. If (i) $H = 1/2$, (ii) $H \in (1/2, 1)$ and $T^{H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{H+1}\Delta \to 0$, then we have*

$$\mathbb{E}\left(\Xi_T\widetilde{X}_T\right) \to 0. \tag{6.30}$$

Lemma 6.13. *Let $\Delta \to 0$ and $T \to \infty$. If (i) $H = 1/2$, (ii) $H \in (1/2, 1)$ and $T^{H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{H+1}\Delta \to 0$, then we have*

$$\mathbb{E}\left(\Xi_T\widetilde{U}_T\right) \to 0. \tag{6.31}$$

Lemma 6.14. *Let $\Delta \to 0$ and $\frac{\log(\Delta)}{T} \to \infty$. Then for any $m < 1$, we have*

$$T^m\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n} X_{(i-1)\Delta} \overset{a.s.}{\to} 0\,.$$

Now, as the main results of this subsection, we give the asymptotic properties of $\widetilde{U}_T$, $\widetilde{X}_T$, R_{1n}, R_{2n}, R_{3n} and R_{4n}.

Proposition 6.2. *Let $\Delta \to 0$ and $T \to \infty$. Then, we have*

$$\widetilde{U}_T \xrightarrow{\mathcal{L}} \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu, \quad \widetilde{X}_T \xrightarrow{a.s.} \widetilde{X}_\infty, \quad \Xi_T \xrightarrow{\mathcal{L}} \eta. \tag{6.32}$$

Moreover, if (i) $H = 1/2$, (ii) $H \in (1/2, 1)$ and $T^{2H}\Delta \to 0$, or (iii) $H \in (0, 1/2)$ and $T^{2H+1}\Delta \to 0$,, then we have

$$\left(\widetilde{U}_T, \widetilde{X}_T, \Xi_T\right) \xrightarrow{\mathcal{L}} \left(\sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu, X_0 + \frac{\mu}{\theta} + \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega, \eta\right), \tag{6.33}$$

where ν, ω and η are independent standard normal variables, and $\widetilde{X}_\infty = X_0 + \frac{\mu}{\theta} + \sigma\int_0^\infty e^{-\theta s} dB_s^H$.

Proposition 6.3. *Let R_{1n} and R_{2n} be defined by (6.14) and (6.15), respectively. Then, as $\Delta \to 0$ and $\frac{\log \Delta}{T} \to 0$, we have*

$$R_{1n} \xrightarrow{\mathcal{L}} 0, \quad R_{2n} \xrightarrow{a.s.} 0.$$

Proposition 6.4. *Let R_{3n} and R_{4n} be defined by (6.16) and (6.17), respectively. Then, as $\Delta \to 0$ and $\frac{\log \Delta}{T} \to 0$, we have*

$$R_{3n} \xrightarrow{\mathcal{L}} 0, \quad R_{4n} \xrightarrow{a.s.} 0\,.$$

Moreover, as $\Delta \to 0$ and $\frac{(\log \Delta)^3}{T^2} \to 0$, we get

$$T^{H-1}R_{3n} \xrightarrow{a.s.} 0.$$

Proof of Proposition 6.2. (i). We first consider the limiting distribution for $\widetilde{U}_T$. Since $\widetilde{U}_T$ is a Gaussian process, for any n and $\Delta > 0$, we have

$$\widetilde{U}_T \stackrel{d}{=} \sigma_{\widetilde{U}_T}\mathcal{N}(0, 1)\,,$$

where $\sigma_{\widetilde{U}_T}$ denotes the standard error of $\widetilde{U}_T$.

Thus, to prove the desired result, it is sufficient to show

$$\sigma_{\widetilde{U}_T} \to \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H}, \quad \text{as } \Delta \to 0,\ T \to \infty.$$

Firstly, straightforward calculations lead to

$$\sigma^2_{\widetilde{U}_T} = \sigma^2 \sum_{i=1}^n \sum_{j=1}^n e^{-2\theta T} e^{2\theta(i+j)\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H\right)$$

$$= \sigma^2 \sum_{i=1}^{n} e^{-2\theta T} e^{4\theta i\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2$$

$$+ \sigma^2 \sum_{i,j=1}^{n} \sum_{i \neq j} e^{-2\theta T} e^{2\theta(i+j)\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right). \quad (6.34)$$

Case 1: $H \in (0, 1/2)$. Using Lemma 6.6, as $\Delta \to 0,\ T \to \infty$, we get

$$\begin{aligned}\sigma^2_{\widetilde{U}_T} =& \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right) \left(\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right) \\ =& \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta} \Big(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dt ds \\ &+ \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Big) \\ \to& \sigma^2 \frac{H\Gamma(2H)}{\theta^{2H}}. \end{aligned} \quad (6.35)$$

The first equation is from (8) in Chen and Li (2021), which is a modification of (2.3) in Hu et al. (2019); see also Hu et al. (2013).

Case 2: $H = 1/2$. The second term of (6.34) equals zero and the first term of (6.34) can be calculated by the Itô isometry formula

$$\begin{aligned}&\sigma^2 \sum_{i=1}^{n} e^{-2\theta T} e^{4\theta i\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dW_s \right)^2 = \sigma^2 \sum_{i=1}^{n} e^{-2\theta T} e^{4\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s} ds \\ &= \frac{\sigma^2}{-2\theta} \sum_{i=1}^{n} e^{-2\theta T} e^{2\theta i\Delta} \left(1 - e^{2\theta\Delta}\right) = \frac{\sigma^2}{2\theta} e^{2\theta\Delta} \left(1 - e^{2\theta T}\right) \to \frac{\sigma^2}{2\theta}, \end{aligned} \quad (6.36)$$

as $\Delta \to 0,\ T \to \infty$.

Case 3: $H \in (1/2, 1)$. Using (6.8), we can get

$$\sigma^2_{\widetilde{U}_T} \to \sigma^2 \frac{H\Gamma(2H)}{\theta^{2H}}, \quad \text{as } \Delta \to 0,\ T \to \infty,$$

which implies

$$\widetilde{U}_T \xrightarrow{\mathcal{L}} \sigma \frac{\sqrt{H\Gamma(2H)}}{\theta^H} \nu, \quad \text{as } \Delta \to 0,\ T \to \infty.$$

(ii). Now, we consider the limiting distribution for $\widetilde{X}_T$. Recall that

$$\widetilde{X}_\infty = X_0 + \frac{\mu}{\theta} + \sigma \int_0^\infty e^{-\theta s} dB_s^H.$$

Then, using (6.8), we have $\mathbb{E}\left|\widetilde{X}_T - \widetilde{X}_\infty\right| = \mathbb{E}\left|-\frac{\mu}{\theta} e^{-\theta T} - \sigma \int_T^\infty e^{-\theta s} dB_s^H\right|$.

Case 1: $H \in (0, 1/2)$. Straightforward calculations lead to

$$\begin{aligned}
\mathbb{E}\left|\widetilde{X}_T - \widetilde{X}_\infty\right| &\le \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\mathbb{E}\left|\int_T^\infty e^{-\theta s} dB_s^H\right| \\
&\le \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\mathbb{E}\left(\int_T^\infty e^{-\theta s} dB_s^H\right)^2\right)^{1/2} \\
&= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\Bigg(\theta\int_T^\infty\int_T^\infty e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds \\
&\qquad + e^{-\theta T}\int_T^\infty e^{-\theta t}\frac{\partial R(t,s)}{\partial t}dt\Bigg)^{1/2} \\
&= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\Bigg(\theta H\int_0^\infty\int_0^v e^{-2\theta T}e^{-2\theta v}e^{\theta u}u^{2H-1}dudv \\
&\qquad - \frac{H}{2}e^{-2\theta T}\int_0^\infty e^{-\theta u}u^{2H-1}du + He^{-2\theta T}\int_0^\infty e^{-\theta u}u^{2H-1}du\Bigg)^{1/2} \\
&= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(He^{-2\theta T}\int_0^\infty e^{-\theta u}u^{2H-1}du\right)^{1/2} \\
&= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\frac{H\Gamma(2H)}{\theta^{2H}}e^{-2\theta T}\right)^{1/2} \le Ce^{-\theta T}\,. \qquad (6.37)
\end{aligned}$$

Case 2: $H = 1/2$. Using the Itô isometry formula, we can see that

$$\begin{aligned}
\mathbb{E}\left|\widetilde{X}_T - \widetilde{X}_\infty\right| &\le \frac{\mu}{\theta}e^{-\theta T} + \sigma\mathbb{E}\left|\int_T^\infty e^{-\theta s}dW_s\right| \\
&\le \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\mathbb{E}\left(\int_T^\infty e^{-\theta s}dW_s\right)^2\right)^{1/2} \\
&= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\int_T^\infty e^{-2\theta s}ds\right)^{1/2} \\
&= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\frac{1}{\sqrt{2\theta}}e^{-\theta T} \le Ce^{-\theta T}. \qquad (6.38)
\end{aligned}$$

Case 3: $H \in (1/2, 1)$. It follows that

$$\begin{aligned}
\mathbb{E}\left|\widetilde{X}_T - \widetilde{X}_\infty\right| &\le \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\mathbb{E}\left|\int_T^\infty e^{-\theta s}dB_s^H\right| \\
&\le \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(\alpha_H\int_T^\infty\int_T^\infty e^{-\theta(s+r)}|s-r|^{2H-2}dsdr\right)^{1/2} \\
&= \frac{|\mu|}{\theta}e^{-\theta T} + \sigma\left(2\alpha_H\int_T^\infty\int_T^r e^{-\theta(s+r)}(r-s)^{2H-2}dsdr\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|\mu|}{\theta} e^{-\theta T} + \sigma \left(2\alpha_H \int_T^\infty \int_0^{r-T} e^{-2\theta r} e^{\theta u} u^{2H-2} du dr \right)^{1/2} \\
&= \frac{|\mu|}{\theta} e^{-\theta T} + \sigma \left(2\alpha_H e^{-2\theta T} \int_0^\infty \int_0^{v} e^{-2\theta v} e^{\theta u} u^{2H-2} du dv \right)^{1/2} \\
&= \frac{|\mu|}{\theta} e^{-\theta T} + \sigma \left(2\alpha_H e^{-2\theta T} \int_0^\infty \int_u^{\infty} e^{-2\theta v} e^{\theta u} u^{2H-2} dv du \right)^{1/2} \\
&= \frac{|\mu|}{\theta} e^{-\theta T} + \sigma \left(\alpha_H \frac{1}{\theta} e^{-2\theta T} \int_0^\infty e^{-\theta u} u^{2H-2} du \right)^{1/2} \\
&= \frac{|\mu|}{\theta} e^{-\theta T} + \sigma \left(\frac{H\Gamma(2H)}{\theta^{2H}} e^{-2\theta T} \right)^{1/2} \le C e^{-\theta T} . \qquad (6.39)
\end{aligned}
$$

Using (6.37)-(6.39), we have $\mathbb{E}\left|\widetilde{X}_T - \widetilde{X}_\infty\right| \le Ce^{-\theta T}$, which implies that

$$
\mathbb{P}\left(\left|\widetilde{X}_T - \widetilde{X}_\infty\right| \ge \epsilon\right) \le C\epsilon^{-1} e^{-\theta T} .
$$

Consequently, we can obtain

$$
\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\widetilde{X}_T - \widetilde{X}_\infty\right| \ge \epsilon\right) < \infty.
$$

Hence, by the Borel-Cantelli lemma, we have $\widetilde{X}_T \overset{a.s.}{\to} \widetilde{X}_\infty$.

(iii). Finally, we prove $\Xi_T = T^{-H} \sum_{i=1}^n \epsilon_{i\Delta} \overset{\mathcal{L}}{\to} \mathcal{N}(0,1)$. Using the definition of $\epsilon_{i\Delta}$, we can easily obtain

$$
\sum_{i=1}^n \epsilon_{i\Delta} \overset{d}{=} \sigma_{\sum_{i=1}^n \epsilon_{i\Delta}} \mathcal{N}(0,1) , \qquad (6.40)
$$

where $\sigma_{\sum_{i=1}^n \epsilon_{i\Delta}}$ denotes the standard error of $\sum_{i=1}^n \epsilon_{i\Delta}$.

Case 1: $H \in (0,1/2)$. By Lemma 6.8, we have

$$
\begin{aligned}
\sigma^2_{\sum_{i=1}^n \epsilon_{i\Delta}} &= \mathbb{E}\left(\sum_{i=1}^n e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2 \\
&= \sum_{i=1}^n \sum_{j=1}^n e^{\theta(i+j)\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n e^{\theta(i+j)\Delta} \Big(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dt ds \\
&\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Big) \\
&\sim T^{2H} + o(T^{2H}). \qquad (6.41)
\end{aligned}
$$

Case 2: $H = 1/2$. Using the Itô isometry formula, we can see that

$$\sigma^2_{\sum_{i=1}^n \epsilon_{i\Delta}} = \sum_{i=1}^n e^{2\theta i\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dW_s\right)^2 = \frac{1}{2\theta}\sum_{i=1}^n \left(e^{2\theta\Delta}-1\right) \sim T. \tag{6.42}$$

Case 3: $H \in (1/2, 1)$. Using Lemma 6.3, we have

$$\begin{aligned}
\sigma^2_{\sum_{i=1}^n \epsilon_{i\Delta}} =&\mathbb{E}\left(\sum_{i=1}^n e^{\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H\right)^2 \\
=&\sum_{i=1}^n\sum_{j=1}^n e^{\theta(i+j)\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H\right) \\
=&\alpha_H \sum_{i=1}^n\sum_{j=1}^n e^{\theta(i+j)\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)}|s-r|^{2H-2} ds dr \sim T^{2H}\,.
\end{aligned} \tag{6.43}$$

Together with (6.41)-(6.43), the asymptotic distribution of Ξ_T can be proved.
(iv). Now, we prove (6.33). By (6.7)-(6.9), we know that

$$\left(\widetilde{U}_T, \widetilde{X}_T, \Xi_T\right)^\top \xrightarrow{\mathcal{L}} \mathcal{N}\left(\overrightarrow{a}^*, B^*\right)\,,$$

where $^\top$ denotes the vector transposition and all the non-primed vectors are row vectors, $\overrightarrow{a}^* = \begin{pmatrix}\mathbb{E}\widetilde{U}_T \\ \mathbb{E}\widetilde{X}_T \\ \mathbb{E}\Xi_T\end{pmatrix}$ and

$$B^* = \begin{pmatrix} \sigma^2_{\widetilde{U}_T} & Cov\left(\widetilde{U}_T, \widetilde{X}_T\right) & Cov\left(\widetilde{U}_T, \Xi_T\right) \\ Cov\left(\widetilde{X}_T, \widetilde{U}_T\right) & \sigma^2_{\widetilde{X}_T} & Cov\left(\widetilde{X}_T, \Xi_T\right) \\ Cov\left(\Xi_T, \widetilde{U}_T\right) & Cov\left(\Xi_T, \widetilde{X}_T\right) & \sigma^2_{\Xi_T} \end{pmatrix}.$$

Since $\mathbb{E}\Xi_T = 0$ and $\mathbb{E}\widetilde{U}_T = 0$, then B^* can written as

$$B^* = \begin{pmatrix} \sigma^2_{\widetilde{U}_T} & \mathbb{E}\left(\widetilde{U}_T\widetilde{X}_T\right) & \mathbb{E}\left(\widetilde{U}_T\Xi_T\right) \\ \mathbb{E}\left(\widetilde{X}_T\widetilde{U}_T\right) & \sigma^2_{\widetilde{X}_T} & \mathbb{E}\left(\widetilde{X}_T\Xi_T\right) \\ \mathbb{E}\left(\Xi_T\widetilde{U}_T\right) & \mathbb{E}\left(\Xi_T\widetilde{X}_T\right) & \sigma^2_{\Xi_T} \end{pmatrix}.$$

Combining (6.31), Lemmas 6.11-6.13, we can obtain (6.33).
Proof of Proposition 6.3. (i). We first consider the asymptotic behavior for R_{1n}. Indeed, for the first term of R_{1n}, it holds that

$$\sigma e^{-2\theta\Delta}\sum_{i=1}^n e^{-\theta(n-i)\Delta}\epsilon_{i\Delta}\left(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T\right)$$

$$
\begin{aligned}
&= \sigma e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \left(\frac{\mu}{\theta} \left(e^{-\theta T} - e^{-\theta(i-1)\Delta} \right) - \sigma \int_{(i-1)\Delta}^{T} e^{-\theta s} dB_s^H \right) \\
&= \sigma \frac{\mu}{\theta} e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(e^{-\theta T} - e^{-\theta(i-1)\Delta} \right) \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \\
&\quad - \sigma^2 e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2 \\
&\quad - \sigma^2 e^{-2\theta\Delta} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \\
&:= A_{1n} + A_{2n} + A_{3n}\,, \qquad (6.44)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{1n} &= \sigma \frac{\mu}{\theta} e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(e^{-\theta T} - e^{-\theta(i-1)\Delta} \right) \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H, \\
A_{2n} &= -\sigma^2 e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2, \\
A_{3n} &= -\sigma^2 e^{-2\theta\Delta} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H.
\end{aligned}
$$

For A_{1n}, we consider it in three cases.

Case 1: $H \in (0, 1/2)$. Using Lemma 6.8 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\mathbb{E} \left| \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(e^{-\theta T} - e^{-\theta(i-1)\Delta} \right) \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right| \\
&\leq \left(\mathbb{E} \left(\sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(e^{-\theta T} - e^{-\theta(i-1)\Delta} \right) \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta} \left(e^{-\theta T} - e^{-\theta(i-1)\Delta} \right) \left(e^{-\theta T} - e^{-\theta(j-1)\Delta} \right) \right. \\
&\quad \left. \times \mathbb{E} \left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right) \right)^{1/2} \leq C T^H e^{-\theta T}. \qquad (6.45)
\end{aligned}
$$

Case 2: $H = 1/2$. As $n \to \infty$, we have

$$
\mathbb{E} \left| \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(e^{-\theta T} - e^{-\theta(i-1)\Delta} \right) \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dW_s \right|
$$

$$
\begin{aligned}
&\leq \left(\mathbb{E}\left(\sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta}\left(e^{-\theta T}-e^{-\theta(i-1)\Delta}\right)\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dW_s\right)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n} e^{-2\theta T} e^{4\theta i\Delta}\left(e^{-\theta T}-e^{-\theta(i-1)\Delta}\right)^2 \int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s} ds\right)^{1/2} \\
&= \left(\frac{e^{2\theta\Delta}}{2\theta} e^{-4\theta T}\left(e^{2\theta T}-1\right)+\frac{\left(1-e^{2\theta\Delta}\right)e^{2\theta\Delta}}{\theta\left(1-e^{\theta\Delta}\right)} e^{-3\theta T}\left(1-e^{\theta T}\right)\right. \\
&\qquad \left. +\frac{n\left(1-e^{2\theta\Delta}\right)e^{2\theta\Delta}}{-2\theta} e^{-2\theta T}\right)^{1/2} \\
&\leq C T^{1/2} e^{-\theta T}.
\end{aligned} \tag{6.46}
$$

Case 3: $H \in (1/2, 1)$. By Lemma 6.3, as $n \to \infty$, we have

$$
\begin{aligned}
&\mathbb{E}\left|\sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta}\left(e^{-\theta T}-e^{-\theta(i-1)\Delta}\right)\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H\right| \\
&\leq \left(\mathbb{E}\left(\sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta}\left(e^{-\theta T}-e^{-\theta(i-1)\Delta}\right)\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H\right)^2\right)^{1/2} \\
&= \left(\alpha_H \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{2\theta(i+j)\Delta}\left(e^{-\theta T}-e^{-\theta(i-1)\Delta}\right)\left(e^{-\theta T}-e^{-\theta(j-1)\Delta}\right)\right. \\
&\quad \left.\times \int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)}|s-r|^{2H-2} ds dr\right)^{1/2} \\
&\leq \left(\alpha_H e^{-2\theta T}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{2\theta\Delta} e^{\theta(i+j)\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)}|s-r|^{2H-2} ds dr\right)^{1/2} \\
&\leq C T^{H} e^{-\theta T}.
\end{aligned} \tag{6.47}
$$

Combining (6.45), (6.46) and (6.47), for $H \in (0, 1)$, we have $A_{1n} \stackrel{a.s.}{\to} 0$ by using the Markov inequality and the Borel-Cantelli lemma.

Similarly, for A_{2n}, we consider the following three cases.

Case 1: $H \in (0, 1/2)$. Using (6.16), it holds that

$$
\begin{aligned}
&\mathbb{E}\left|\sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H\right)^2\right| \\
&= \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2} |t|^{2H}}{\partial t} dt \\
&\quad - \sum_{i=1}^{n} e^{-\theta T} e^{\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2} |t-\Delta|^{2H}}{\partial t} dt \\
&\quad - \theta \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2} |t-s|^{2H}}{\partial t} dt ds \\
&\sim C T e^{-\theta T} \Delta^{2H-1}. \qquad (6.48)
\end{aligned}
$$

Case 2: $H = 1/2$. We can see that

$$
\begin{aligned}
\mathbb{E} \left| \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dW_s \right)^2 \right| &= \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s} ds \\
&= \frac{n\left(1 - e^{2\theta\Delta}\right)}{-2\theta} e^{-\theta T} \sim T e^{-\theta T}. \qquad (6.49)
\end{aligned}
$$

Case 3: $H \in (1/2, 1)$. By (6.8), we can obtain

$$
\begin{aligned}
&\mathbb{E} \left| \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2 \right| \\
&= \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \mathbb{E} \left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2 \\
&= \alpha_H \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr \\
&= \alpha_H \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \int_{0\Delta}^{\Delta} \int_{0}^{\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr \\
&\sim C T e^{-\theta T} \Delta^{2H-1}. \qquad (6.50)
\end{aligned}
$$

Together with (6.48) and (6.49), we have $A_{2n} \overset{a.s.}{\to} 0$ for $H \in (0,1)$ with the help of the Markov inequality and the Borel-Cantelli lemma.

For A_{3n}, we also consider three cases as follows.

Case 1: $H \in (0, 1/2)$. By (6.16), we can see that

$$
\begin{aligned}
&\mathbb{E} \left| \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{2\theta i\Delta} \left(\mathbb{E} \left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \right)^2 \right)^{1/2} \left(\mathbb{E} \left(\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right)^2 \right)^{1/2}
\end{aligned}
$$

$$\sim \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{2\theta i\Delta}\left(e^{-2\theta i\Delta}e^{\theta\Delta}\Delta^{2H}\right)^{1/2}\left(e^{-2\theta j\Delta}e^{\theta\Delta}\Delta^{2H}\right)^{1/2}$$
$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta\Delta}e^{-\theta T}e^{\theta(i-j)\Delta}\Delta^{2H}$$
$$= \frac{T^{2H}e^{-\theta T}}{1-e^{-\theta\Delta}} + \frac{\Delta^{2H}e^{-\theta T}\left(1-e^{-\theta T}\right)}{(1-e^{-\theta\Delta})^2}$$
$$\leq CTe^{-\theta T}\Delta^{2H-2}. \tag{6.51}$$

Case 2: $H = 1/2$. When $n \to \infty$, we have

$$\mathbb{E}\left|\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{2\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dW_s \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dW_s\right|$$
$$\leq \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{2\theta i\Delta}\left(\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dW_s\right)^2\right)^{1/2}\left(\mathbb{E}\left(\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dW_s\right)^2\right)^{1/2}$$
$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{2\theta i\Delta}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s}ds\right)^{1/2}\left(\int_{(j-1)\Delta}^{j\Delta} e^{-2\theta s}ds\right)^{1/2}$$
$$= \frac{e^{2\theta\Delta}-1}{2\theta}\sum_{i,j=1}^{n}\sum_{j>i} e^{-\theta T}e^{\theta(i-j)\Delta}$$
$$= \frac{e^{-\theta\Delta}\left(e^{2\theta\Delta}-1\right)}{2\theta\left(1-e^{-\theta\Delta}\right)}\left[ne^{-\theta T} - \frac{e^{-2\theta T}e^{\theta\Delta}\left(1-e^{\theta T}\right)}{1-e^{\theta\Delta}}\right]. \tag{6.52}$$

Case 3: $H \in (1/2, 1)$. Using the Cauchy-Schwarz inequality, for $n \to \infty$ and $\Delta \to 0$, we have

$$\mathbb{E}\left|\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{2\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dB_s^H\right|$$
$$\leq \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{2\theta i\Delta}\left(\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB_s^H\right)^2\right)^{1/2}\left(\mathbb{E}\left(\int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dB_s^H\right)^2\right)^{1/2}$$
$$= \alpha_H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{2\theta i\Delta}\left(\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta(s+r)}|s-r|^{2H-2}dsdr\right)^{1/2}$$
$$\times\left(\int_{(j-1)\Delta}^{j\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)}|s-r|^{2H-2}dsdr\right)^{1/2}$$
$$= 2\alpha_H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta}e^{-\theta T}e^{\theta(i-j)\Delta}\int_0^{\Delta}\int_0^{r} e^{-\theta(s+r)}|s-r|^{2H-2}dsdr$$

$$\sim \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{2\theta\Delta} e^{-\theta T} e^{\theta(i-j)\Delta} \Delta^{2H}$$
$$= \frac{e^{\theta\Delta} T^{2H} e^{-\theta T}}{1-e^{-\theta\Delta}} + \frac{e^{\theta\Delta}\Delta^{2H} e^{-\theta T}\left(1-e^{-\theta T}\right)}{\left(1-e^{-\theta\Delta}\right)^2} \leq CTe^{-\theta T}\Delta^{2H-2}. \tag{6.53}$$

Combining (6.51), (6.52) with (6.53), we have $A_{3n} \stackrel{a.s.}{\to} 0$ using the Markov inequality and the Borel-Cantelli lemma.

Using the above results, as $\Delta \to 0$ and $\frac{\log(\Delta)}{T} \to 0$, we have

$$e^{-2\theta\Delta} \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_{i\Delta} \left(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T \right) \stackrel{a.s.}{\to} 0\,. \tag{6.54}$$

Moreover, from (6.9), we can write

$$\frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \sum_{i=1}^{n} \epsilon_{i\Delta} = T^H \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \Xi_T. \tag{6.55}$$

Combining (6.55), Lemma 6.14 and Proposition 6.2, we can obtain

$$\frac{1}{n} e^{-\theta T} e^{-\theta\Delta} \sum_{i=1}^{n} \epsilon_{i\Delta} \sum_{i=1}^{n} X_{(i-1)\Delta} \stackrel{\mathcal{L}}{\to} 0\,. \tag{6.56}$$

Using (6.54) and (6.56), we complete the proof of $R_{1n} \stackrel{\mathcal{L}}{\to} 0$.

(ii) Next, we deal with $R_{2n} \stackrel{a.s.}{\to} 0$. Using (6.8), we can write the first term of R_{2n} as

$$e^{-2\theta\Delta} \sum_{i=2}^{n} e^{-2\theta(n-i+1)\Delta} \left(\widetilde{X}^2_{(i-1)\Delta} - \widetilde{X}^2_{(i-2)\Delta} \right)$$
$$= e^{-4\theta\Delta} \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \left(\widetilde{X}_{(i-1)\Delta} + \widetilde{X}_{(i-2)\Delta} \right) \left(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_{(i-2)\Delta} \right)$$
$$= 2\sigma^2 e^{-4\theta\Delta} \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \int_0^{(i-2)\Delta} e^{-\theta s} dB_s^H \cdot \int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta s} dB_s^H$$
$$+ \sigma^2 e^{-4\theta\Delta} \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \left(\int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta s} dB_s^H \right)^2$$
$$+ e^{-4\theta\Delta} \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \left(2X_0 + \frac{\mu}{\theta}\left(2 - e^{-\theta(i-1)\Delta} - e^{-\theta(i-2)\Delta}\right)\right) \frac{\mu}{\theta}\left(e^{-\theta(i-2)\Delta} - e^{-\theta(i-1)\Delta}\right)$$
$$+ 2\sigma e^{-4\theta\Delta} \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \left(X_0 + \frac{\mu}{\theta}\left(1 - e^{-\theta(i-1)\Delta}\right)\right) \int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta s} dB_s^H$$
$$+ 2\sigma e^{-4\theta\Delta} \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \frac{\mu}{\theta}\left(e^{-\theta(i-2)\Delta} - e^{-\theta(i-1)\Delta}\right) \int_0^{(i-2)\Delta} e^{-\theta s} dB_s^H$$

$$:= I_{1n} + I_{2n} + I_{3n} + I_{4n} + I_{5n}. \tag{6.57}$$

For I_{1n}, we consider it in three cases.

Case 1: $H \in (0, 1/2)$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\left|\sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \int_0^{(i-2)\Delta} e^{-\theta s} dB_s^H \cdot \int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta s} dB_s^H\right| \\
&\leq \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \left(\mathbb{E}\left(\int_0^{(i-2)\Delta} e^{-\theta s} dB_s^H\right)^2\right)^{1/2} \left(\mathbb{E}\left(\int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta s} dB_s^H\right)^2\right)^{1/2} \\
&\leq C \sum_{i=2}^{n} e^{-2\theta T} e^{\theta i\Delta} \Delta^H \sim C\Delta^{H-1} e^{-\theta T}.
\end{aligned} \tag{6.58}$$

Case 2: $H = 1/2$. We can see that

$$\begin{aligned}
&\mathbb{E}\left|\sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \int_0^{(i-2)\Delta} e^{-\theta s} dW_s \cdot \int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta s} dW_s\right| \\
&\leq \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \left(\int_0^{(i-2)\Delta} e^{-2\theta s} ds\right)^{1/2} \left(\int_{(i-2)\Delta}^{(i-1)\Delta} e^{-2\theta s} ds\right)^{1/2} \\
&= \frac{e^{2\theta\Delta}}{2\theta} \sum_{i=2}^{n} e^{-2\theta n\Delta} e^{\theta i\Delta} \left(1 - e^{-2\theta(i-2)\Delta}\right)^{1/2} \left(1 - e^{-2\theta\Delta}\right)^{1/2} \\
&\leq \frac{e^{2\theta\Delta}}{2\theta} \sum_{i=2}^{n} e^{-2\theta n\Delta} e^{\theta i\Delta} \left(1 - e^{-2\theta\Delta}\right)^{1/2} \\
&= \frac{e^{4\theta\Delta} \left(1 - e^{-2\theta\Delta}\right)^{1/2}}{2\theta \left(1 - e^{\theta\Delta}\right)} e^{-2\theta n\Delta} \left(1 - e^{\theta(n-1)\Delta}\right) \\
&\sim \frac{1}{\sqrt{2}\theta^{3/2}} \Delta^{-1/2} e^{-\theta T} \left(1 - e^{-\theta(n-1)\Delta}\right).
\end{aligned} \tag{6.59}$$

Case 3: $H \in (1/2, 1)$. We can write the following formula immediately:

$$\begin{aligned}
&\mathbb{E}\left|\sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \int_0^{(i-2)\Delta} e^{-\theta s} dB_s^H \cdot \int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta s} dB_s^H\right| \\
&\leq \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} \alpha_H \left(\int_0^{(i-2)\Delta} \int_0^{(i-2)\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr\right)^{1/2} \\
&\quad \times \left(\int_{(i-2)\Delta}^{(i-1)\Delta} \int_{(i-2)\Delta}^{(i-1)\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr\right)^{1/2} \\
&\leq C \sum_{i=2}^{n} e^{-2\theta T} e^{2\theta i\Delta} e^{-\theta(i-2)\Delta} \left(\int_0^{\Delta} \int_0^{\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr\right)^{1/2}
\end{aligned}$$

$$\sim C\Delta^{H-1}e^{-\theta T}\left(1-e^{-\theta T}\right). \tag{6.60}$$

Combining (6.58), (6.59) and (6.60) and using the Borel-Cantelli lemma, we can get

$$I_{1n}\overset{a.s.}{\to}0. \tag{6.61}$$

For I_{2n}, using the same arguments as (6.44), for $H\in(0,1)$, we can obtain that

$$I_{2n}\overset{a.s.}{\to}0\,. \tag{6.62}$$

For I_{3n}, it is easy to verify that, as $T\to\infty$ and $\Delta\to 0$,

$$e^{-4\theta\Delta}\sum_{i=2}^{n}e^{-2\theta T}e^{2\theta i\Delta}\left(2X_0+\frac{\mu}{\theta}\left(2-e^{-\theta(i-1)\Delta}-e^{-\theta(i-2)\Delta}\right)\right)\frac{\mu}{\theta}\left(e^{-\theta(i-2)\Delta}-e^{-\theta(i-1)\Delta}\right)\to 0.$$

For I_{4n}, using the same arguments as (6.44), for $H\in(0,1)$, as $n\to\infty$, we have

$$\begin{aligned}&\mathbb{E}\left(\sum_{i=2}^{n}e^{-2\theta T}e^{2\theta i\Delta}\left(X_0+\frac{\mu}{\theta}\left(1-e^{-\theta(i-1)\Delta}-e^{-\theta(i-2)\Delta}\right)\right)\int_{(i-2)\Delta}^{(i-1)\Delta}e^{-\theta s}dB_s^H\right)^2\\&\le\left(\mathbb{E}X_0^2+\frac{\mu^2}{\theta^2}\right)e^{-2\theta T}\mathbb{E}\left(\sum_{i=2}^{n}e^{-\theta T}e^{2\theta i\Delta}\int_{(i-2)\Delta}^{(i-1)\Delta}e^{-\theta s}dB_s^H\right)^2\le Ce^{-2\theta T}.\end{aligned} \tag{6.63}$$

Using the Borel-Cantelli lemma, we can obtain

$$I_{4n}\overset{a.s.}{\to}0. \tag{6.64}$$

For I_{5n}, using the same arguments as (6.44), for $H\in(0,1)$, as $\Delta\to 0$ and $T\to\infty$, we have

$$\begin{aligned}I_{5n}=&2\sigma e^{-4\theta\Delta}\sum_{i=2}^{n}e^{-2\theta T}e^{2\theta i\Delta}\frac{\mu}{\theta}\left(e^{-\theta(i-2)\Delta}-e^{-\theta(i-1)\Delta}\right)\int_0^{(i-2)\Delta}e^{-\theta s}dB_s^H\\\sim&2\mu e^{-2\theta\Delta}\sum_{i=2}^{n}e^{-2\theta T}e^{\theta i\Delta}\Delta\int_0^{(i-2)\Delta}e^{-\theta s}dB_s^H\overset{a.s.}{\to}0.\end{aligned} \tag{6.65}$$

Combining (6.57)-(6.65), it follows that

$$e^{-2\theta\Delta}\sum_{i=2}^{n}e^{-2\theta(n-i+1)\Delta}\left(\widetilde{X}^2_{(i-1)\Delta}-\widetilde{X}^2_{(i-2)\Delta}\right)\overset{a.s.}{\to}0 \tag{6.66}$$

It is obvious that

$$e^{-2\theta\Delta}e^{-2\theta T}\widetilde{X}_0^2\overset{a.s.}{\to}0.$$

Moreover, Lemma 6.14 tells us that

$$\frac{\left(1-e^{-2\theta\Delta}\right)e^{-2\theta T}}{n}\left(\sum_{i=1}^{n}X_{(i-1)\Delta}\right)^{2}\overset{a.s.}{\to}0. \tag{6.67}$$

Finally, from (6.66), (6.67) and (6.15), we have $R_{2n}\overset{a.s.}{\to}0$.

Proof of Proposition 6.4. (i). Firstly, we show $R_{3n}\overset{\mathcal{L}}{\to}0$. By (6.16), we have

$$\begin{aligned}
R_{3n} &\sim \sigma\Xi_T e^{-2\theta\Delta}\sum_{i=2}^{n}e^{-2\theta(T-i\Delta+\Delta)}\left(\widetilde{X}^2_{(i-1)\Delta}-\widetilde{X}^2_{(i-2)\Delta}\right)-2\sigma\mu\Delta e^{-2\theta T}\Xi_T\sum_{i=1}^{n}X_{(i-1)\Delta}\\
&\quad+2\sigma\mu T^{1-H}e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}+2\sigma\theta T^{1-H}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}\\
&=\sigma\Xi_T e^{-2\theta\Delta}\sum_{i=2}^{n}e^{-2\theta(T-i\Delta+\Delta)}\left(\widetilde{X}^2_{(i-1)\Delta}-\widetilde{X}^2_{(i-2)\Delta}\right)-S_{1n}+S_{2n}+S_{3n}\,,
\end{aligned}\tag{6.68}$$

where $S_{1n}=2\sigma\mu\Delta e^{-2\theta T}\Xi_T\sum_{i=1}^{n}X_{(i-1)\Delta}$, $S_{2n}=2\sigma\mu T^{1-H}e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}$, $S_{3n}=2\sigma\theta T^{1-H}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}$.

Firstly, by Proposition 6.2 and Proposition 6.3, we have $\sigma\Xi_T R_{2n}\overset{\mathcal{L}}{\to}0$. Moreover, by Lemma 6.2 and Lemma 6.14, we can get $S_{1n}\overset{\mathcal{L}}{\to}0$.

For S_{2n}, using the Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
&\mathbb{E}\left|T^{1-H}e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}\right|\\
&\le T^{1-H}\left(\mathbb{E}\left(e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}\right)^2\right)^{1/2}\\
&=T^{1-H}e^{-2\theta T}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathbb{E}\epsilon^2_{i\Delta}\right)^{1/2}\left(\mathbb{E}X^2_{(j-1)\Delta}\right)^{1/2}\\
&\le T^{1-H}e^{-2\theta T}T^{2H}e^{2\theta\Delta}\sum_{j=1}^{n}\left(\mathbb{E}X^2_{(j-1)\Delta}\right)^{1/2}\\
&\le n^{2-2H}T^{1+H}e^{2\theta\Delta}e^{-\theta T}\left(\mathbb{E}\widetilde{X}^2_{\infty}\right)^{1/2}.
\end{aligned}$$

Hence, the Markov inequality and the Borel-Cantelli lemma imply that $S_{2n}\overset{a.s}{\to}0$.

For S_{3n}, from Lemma 6.14, we have

$$T^{1-H}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}\overset{a.s}{\to}0. \tag{6.69}$$

Combining Proposition 6.1, Proposition 6.2 and Lemma 6.3, we have

$$e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}\overset{\mathcal{L}}{\to}\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^{H}}\nu\widetilde{X}_{\infty}, \tag{6.70}$$

Thus, by (6.69) and (6.70), $S_{3n}\overset{\mathcal{L}}{\to}0$ holds. Using the above results, we have $R_{3n}\overset{\mathcal{L}}{\to}0$.

(ii). Secondly, we prove $R_{4n}\overset{a.s.}{\to}0$. From (6.17), we can see that

$$\begin{aligned}
R_{4n}\sim&e^{-2\theta\Delta}\sum_{i=2}^{n}e^{-2\theta(T-i\Delta+\Delta)}\left(\widetilde{X}^2_{(i-1)\Delta}-\widetilde{X}^2_{(i-2)\Delta}\right)+e^{-2\theta\Delta}e^{-2\theta T}\widetilde{X}^2_0\\
&+2\theta\left(T^{1/2}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}\right)^2+2\sigma e^{-\theta\Delta}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}\\
&-2\sigma e^{-\theta\Delta}e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}\\
:=&e^{-2\theta\Delta}\sum_{i=2}^{n}e^{-2\theta(T-i\Delta+\Delta)}\left(\widetilde{X}^2_{(i-1)\Delta}-\widetilde{X}^2_{(i-2)\Delta}\right)+e^{-2\theta\Delta}e^{-2\theta T}\widetilde{X}^2_0+S_{4n}+S_{5n}-S_{6n}\,,
\end{aligned}$$

where $S_{4n}=2\theta\left(T^{1/2}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}\right)^2$, $S_{5n}=2\sigma e^{-\theta\Delta}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}$ and $S_{6n}=2\sigma e^{-\theta\Delta}e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}$. Proposition 6.3 and Lemma 6.14 tell us that the summation of first two terms converges to 0 almost surely and $S_{4n}\overset{a.s.}{\to}0$, respectively.

For S_{5n}, we consider the asymptotic behavior of $e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}$. Indeed, By (6.40)-(6.43), we have

$$\mathbb{E}\left|e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}\right|\le e^{-\theta T}\left(\mathbb{E}\left(\sum_{i=1}^{n}\epsilon_{i\Delta}\right)^2\right)^{1/2}\sim T^{H}e^{-\theta T}.$$

Together with the Borel-Cantelli lemma, it implies that

$$e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}\overset{a.s}{\to}0. \tag{6.71}$$

Combining (6.71) and the fact that $\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}\overset{a.s}{\to}0$, we have $S_{5n}\overset{a.s}{\to}0$.

For S_{6n}, straightforward calculations lead to

$$\begin{aligned}
\mathbb{E}\left|e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}\right|\le&\left(\mathbb{E}\left(e^{-2\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}\right)^2\right)^{1/2}\\
=&e^{-2\theta T}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathbb{E}\epsilon^2_{i\Delta}\right)^{1/2}\left(\mathbb{E}X^2_{(j-1)\Delta}\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le e^{-2\theta T}T^{2H}e^{2\theta\Delta}\sum_{j=1}^{n}\left(\mathbb{E}X_{(j-1)\Delta}^2\right)^{1/2}\\
&\le n^2\Delta^{2H}e^{2\theta\Delta}e^{-\theta T}\left(\mathbb{E}\widetilde{X}_\infty^2\right)^{1/2}.
\end{aligned}
$$

By the Markov inequality and the Borel-Cantelli lemma, we have $S_{6n}\overset{a.s}{\to}0$. Consequently, we complete the proof of $R_{4n}\overset{a.s.}{\to}0$.

(3). Finally, we consider the asymptotic behavior for $T^{H-1}R_{3n}$. By (6.68), we have

$$
T^{H-1}R_{3n}\sim\sigma T^{H-1}\Xi_Te^{-2\theta\Delta}\sum_{i=2}^{n}e^{-2\theta(T-i\Delta+\Delta)}\left(\widetilde{X}_{(i-1)\Delta}^2-\widetilde{X}_{(i-2)\Delta}^2\right)-T^{H-1}S_{1n}+T^{H-1}S_{2n}+T^{H-1}S_{3n}.
$$

Using (6.40)-(6.43), we can deduce that

$$
T^{H-1}\Xi_T\overset{a.s.}{\to}0. \tag{6.72}
$$

Together with Proposition 6.3, it holds

$$
T^{H-1}\Xi_TR_{2n}\overset{a.s.}{\to}0.
$$

Recall that

$$
S_{1n}=2\sigma\mu\Delta e^{-2\theta T}\Xi_T\sum_{i=1}^{n}X_{(i-1)\Delta}.
$$

By (6.72) and Lemma 6.14, we can get

$$
T^{H-1}S_{1n}\overset{a.s.}{\to}0.
$$

Combining (6.72) and the fact that $S_{2n}\overset{a.s.}{\to}0$, we can obtain

$$
T^{H-1}S_{2n}\overset{a.s.}{\to}0.
$$

Therefore, to prove that $T^{H-1}R_{3n}\overset{a.s.}{\to}0$, it is sufficient to show

$$
T^{H-1}S_{3n}\overset{a.s.}{\to}0.
$$

Recall that

$$
S_{3n}=2\sigma\theta T^{1-H}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta}.
$$

By (6.18), we can write $T^{H-1}S_{3n}$ as

$$
T^{H-1}S_{3n}=2\sigma\theta\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}X_{(i-1)\Delta},
$$

$$= 2\theta \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} e^{-\theta\Delta} \widetilde{U}_T \widetilde{X}_T$$
$$+ 2\sigma\theta \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} e^{-\theta\Delta} \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_{i\Delta} \left(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T \right).$$

Using Lemma 6.14 and (6.54), we have

$$\frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} e^{-\theta\Delta} \sum_{i=1}^{n} e^{-\theta(n-i)\Delta} \epsilon_{i\Delta} \left(\widetilde{X}_{(i-1)\Delta} - \widetilde{X}_T \right) \stackrel{a.s.}{\to} 0.$$

Therefore, to show $T^{H-1} S_{3n} \stackrel{a.s.}{\to} 0$, we only need to prove

$$e^{-\theta\Delta} \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \widetilde{U}_T \widetilde{X}_T \stackrel{a.s.}{\to} 0. \tag{6.73}$$

In fact, since $\Delta \to 0$, $\frac{(\log \Delta)^2}{T^3} \to 0$, for any $\epsilon > 0$, as $n \to \infty$, it holds for any $\ell \geq 2$

$$\begin{aligned}
&\mathbb{P}\left(\left| e^{-\theta\Delta} \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \widetilde{U}_T \widetilde{X}_T \right| > \epsilon \right) \\
&\leq \mathbb{P}\left(\left| \sum_{i=1}^{n} X_{(i-1)\Delta} \right| > \frac{\epsilon n e^{\theta T}}{\ell^2 \log n} \right) + \mathbb{P}\left(\left| \widetilde{U}_T \right| > \ell (\log n)^{1/2} \right) + \mathbb{P}\left(\left| \widetilde{X}_T \right| > \ell (\log n)^{1/2} \right) \\
&\leq C \exp\left\{ - \frac{\epsilon^2 n^2 e^{2\theta T}}{\ell^2 (\log n)^2 \sigma^2_{\sum_{i=1}^{n} X_{(i-1)\Delta}}} \right\} + C \exp\left\{ - \frac{\ell \log n}{\sigma^2_{\widetilde{U}_T}} \right\} + C \exp\left\{ - \frac{\ell \log n}{\sigma^2_{\widetilde{X}_T}} \right\} \\
&\leq C_1 \exp\left\{ -C_2 \frac{\epsilon^2 T^2}{(\log n)^2} \right\} + C \exp\{-\ell \log n\}.
\end{aligned}$$

By the Borel-Cantelli lemma, we can get (6.73), and the proof of this proposition is completed.

6.2 Proofs of Theorems 2.2-2.4

Proof of Theorem 2.2. (i). We first prove the strong consistency of $\widehat{\theta}_\Delta$. By Proposition 6.1, we have

$$\begin{aligned}
\frac{e^{-\theta\Delta}}{\Delta} \frac{U_T}{V_T} &= \frac{e^{-\theta\Delta} \Delta^{-1} \left(1 - e^{-2\theta\Delta}\right) e^{-2\theta T} U_T}{\left(1 - e^{-2\theta\Delta}\right) e^{-2\theta T} V_T} \\
&= \frac{\Delta^{-1} \left(1 - e^{-2\theta\Delta}\right) e^{-2\theta\Delta} e^{-\theta T} \widetilde{U}_T \widetilde{X}_T + \Delta^{-1} \left(1 - e^{-2\theta\Delta}\right) e^{-\theta T} R_{1n}}{e^{-2\theta\Delta} \widetilde{X}^2_{(n-1)\Delta} - R_{2n}}.
\end{aligned}$$

Using the fact

$$\mathbb{E}\left|e^{-\theta T}\widetilde{U}_T\widetilde{X}_T\right| \le e^{-\theta T}\left(\mathbb{E}\widetilde{U}_T^2\right)^{1/2}\left(\mathbb{E}\widetilde{X}_T^2\right)^{1/2} \le Ce^{-\theta T},$$

and the Borel-Cantelli lemma, we obtain that

$$e^{-\theta T}\widetilde{U}_T\widetilde{X}_T \xrightarrow{a.s.} 0. \tag{6.74}$$

Moreover, using (6.14), we can write

$$\begin{aligned}
e^{-\theta T}R_{1n} =& e^{-\theta T}\sigma e^{-2\theta\Delta}\sum_{i=1}^{n} e^{-\theta(n-i)\Delta}\epsilon_{i\Delta}\left(\widetilde{X}_{(i-1)\Delta}-\widetilde{X}_T\right)\\
&- e^{-\theta T}\sigma\frac{e^{-\theta(n+1)\Delta}}{n}\sum_{i=1}^{n}\epsilon_{i\Delta}\sum_{i=1}^{n}X_{(i-1)\Delta}\\
=&\sigma e^{-\theta T}e^{-2\theta\Delta}\sum_{i=1}^{n} e^{-\theta(n-i)\Delta}\epsilon_{i\Delta}\left(\widetilde{X}_{(i-1)\Delta}-\widetilde{X}_T\right)\\
&- \sigma e^{-\theta\Delta}\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}\cdot e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta}.
\end{aligned} \tag{6.75}$$

By (6.54), we have

$$e^{-\theta T}e^{-2\theta\Delta}\sum_{i=1}^{n} e^{-\theta(n-i)\Delta}\epsilon_{i\Delta}\left(\widetilde{X}_{(i-1)\Delta}-\widetilde{X}_T\right) \xrightarrow{a.s.} 0. \tag{6.76}$$

Combining Lemma 6.14 and (6.71), we have

$$\frac{1}{n}e^{-\theta T}\sum_{i=1}^{n}X_{(i-1)\Delta}\cdot e^{-\theta T}\sum_{i=1}^{n}\epsilon_{i\Delta} \xrightarrow{a.s.} 0. \tag{6.77}$$

Hence, using (6.75)-(6.77), we can obtain

$$e^{-\theta T}R_{1n} \xrightarrow{a.s.} 0. \tag{6.78}$$

Using Propositions 6.2-6.3, (6.74) and (6.78), we have

$$\frac{e^{-\theta\Delta}}{\Delta}\frac{U_T}{V_T} \xrightarrow{a.s.} 0. \tag{6.79}$$

By (6.1), (6.79) and the Taylor expansion, we have

$$\widehat{\theta}_\Delta - \theta = \frac{1}{\Delta}\log\left(1+e^{-\theta\Delta}\frac{U_T}{V_T}\right) \xrightarrow{a.s.} 0.$$

(ii). Now, we turn to prove the asymptotic distribution of $\widehat{\theta}_\Delta$, that is, (2.18). By (6.1), Proposition 6.1, Proposition 6.3, and (6.79), we can get

$$\begin{aligned}\frac{\Delta e^{\theta T}}{1-e^{-2\theta\Delta}}\left(\widehat{\theta}_\Delta-\theta\right) &= \frac{e^{\theta T}}{1-e^{-2\theta\Delta}}\log\left(1+e^{-\theta\Delta}\frac{U_T}{V_T}\right)\\ &= \frac{e^{-\theta(n+1)\Delta}U_T}{(1-e^{-2\theta\Delta})e^{-2\theta T}V_T}(1+o(1))\\ &= \frac{e^{-2\theta\Delta}\widetilde{U}_T\widetilde{X}_T+R_{1n}}{e^{-2\theta\Delta}\widetilde{X}^2_{(n-1)\Delta}-R_{2n}}(1+o(1)) = \frac{\widetilde{U}_T\widetilde{X}_T}{\widetilde{X}^2_{(n-1)\Delta}}(1+o_p(1)). \end{aligned} \tag{6.80}$$

where U_T and V_T are defined by (6.3) and (6.4), respectively.

Combining Propositions 6.1-6.2, and (6.80), we can obtain that

$$\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta-\theta\right)\xrightarrow{\mathcal{L}}\frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\nu}{X_0+\frac{\mu}{\theta}+\sigma\frac{\sqrt{H\Gamma(2H)}}{\theta^H}\omega},$$

which completes the proof of this theorem.

Proof of Theorem 2.3. Note that $\widehat{\theta}\xrightarrow{a.s.}\theta$ and

$$\widehat{\mu}_\Delta=\mu+\left(\widehat{\theta}_\Delta-\theta\right)\left(\frac{\mu}{\theta}+\frac{M_T}{N_T}\right)+\theta\cdot\frac{M_T}{N_T}.$$

To verify the strong consistency of $\widehat{\mu}$, we just need to show that $\frac{M_T}{N_T}\xrightarrow{a.s.}0$.

As $\Delta\to0$ and $T\to\infty$, by using (6.13), Proposition 6.2 and Proposition 6.4, we have

$$\left(n(e^{\theta\Delta}-1)\right)^{-1}\left(1-e^{-2\theta\Delta}\right)e^{-2\theta T}N_T\xrightarrow{a.s.}\widetilde{X}^2_\infty. \tag{6.81}$$

As $\Delta\to0$ and $\frac{T^2}{(\log n)^3}\to\infty$, using (6.12), Proposition 6.2, Proposition 6.4 and (6.72), we obtain

$$\left(n(e^{\theta\Delta}-1)\right)^{-1}\left(1-e^{-2\theta\Delta}\right)e^{-2\theta T}M_T\xrightarrow{a.s.}0. \tag{6.82}$$

Combing (6.81) and (6.82), we have $\frac{M_T}{N_T}\xrightarrow{a.s.}0$, which completes the proof of this theorem.

Proof of Theorem 2.4. Using (6.80), we can write

$$\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta-\theta\right)=\frac{\widetilde{U}_T\widetilde{X}_T}{\widetilde{X}^2_{(n-1)\Delta}}+o_p(1). \tag{6.83}$$

Note that

$$T^{1-H}\left(\widehat{\mu}_\Delta-\mu\right)=\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta-\theta\right)\left(2\mu e^{-\theta T}T^{1-H}+2e^{-\theta T}\frac{T^{-H}M_T}{(\theta n\Delta)^{-1}N_T}\right)+\frac{T^{-H}M_T}{(\theta n\Delta)^{-1}N_T},$$

where M_T and N_T are defined by (6.5) and (6.6), respectively. Together with Proposition 6.1 and Proposition 6.4, we can obtain

$$T^{1-H}\left(\widehat{\mu}_\Delta - \mu\right) = \sigma \Xi_T + o_p(1). \tag{6.84}$$

From (6.83) and (6.84), we have

$$\left(\frac{e^{\theta T}}{2\theta}\left(\widehat{\theta}_\Delta - \theta\right), T^{1-H}\left(\widehat{\mu}_\Delta - \mu\right)\right) = \left(\frac{\widetilde{U}_T \widetilde{X}_T}{\widetilde{X}^2_{(n-1)\Delta}}, \sigma \Xi_T\right) + o_p(1).$$

Using Proposition 6.2, we can complete the proof of this theorem.

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Online Supplement: Proofs of Lemmas 6.1-6.14

In this online supplement, we give proofs of Lemmas 6.1-6.14.
Proof of Lemma 6.1. A standard calculation shows

$$\alpha_H \sum_{i,j=1}^{n} \sum_{j\neq i} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta j\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds$$

$$= 2\alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s+j\Delta-i\Delta|^{2H-2} dtds$$

$$\sim 2\alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} |t-s+j\Delta-i\Delta|^{2H-2} dsdt$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} [(j\Delta-i\Delta+\Delta)^{2H} - 2(j\Delta-i\Delta)^{2H} + (j\Delta-i\Delta-\Delta)^{2H}]$$

$$= \sum_{i=1}^{n} \sum_{j=i+2}^{n+1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} (j\Delta-i\Delta)^{2H} - 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} (j\Delta-i\Delta)^{2H}$$

$$+ \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{3\theta\Delta} (j\Delta-i\Delta)^{2H}$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} \left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right) (j\Delta-i\Delta)^{2H} - \sum_{i=1}^{n} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta\Delta} \Delta^{2H}$$

$$+ \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{2\theta\Delta} (n\Delta+\Delta-i\Delta)^{2H} - \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{3\theta\Delta} (n\Delta-i\Delta)^{2H}$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} \left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right) (j\Delta-i\Delta)^{2H} - \frac{e^{4\theta\Delta}\left(1-e^{-2\theta T}\right)}{e^{2\theta\Delta}-1} \Delta^{2H}$$

$$+ \sum_{i=1}^{n} e^{-\theta i\Delta} e^{3\theta\Delta} (i\Delta)^{2H} - \sum_{i=1}^{n-1} e^{-\theta i\Delta} e^{3\theta\Delta} (i\Delta)^{2H}$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{\theta\Delta} \left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right) (j\Delta-i\Delta)^{2H}$$

$$- \frac{e^{4\theta\Delta}\left(1-e^{-2\theta T}\right)}{e^{2\theta\Delta}-1} \Delta^{2H} + e^{-\theta T} e^{3\theta\Delta} T^{2H}$$

$$\sim \theta^2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} (j\Delta-i\Delta)^{2H} \Delta^2 - \frac{1-e^{-2\theta T}}{2\theta} \Delta^{2H-1} + e^{-\theta T} T^{2H}$$

$$\sim \theta^2 \int_0^T \int_s^T e^{-2\theta T} e^{\theta t} e^{\theta s} (t-s)^{2H} dtds + o(1),$$

where for the first '$\sim$', $e^{-\theta t}e^{-\theta s}$ is approximated as 1, which is because $e^{-2\theta\Delta} \leq e^{-\theta t}e^{-\theta s} \leq$

1 for $t, s \in [0, \Delta]$.

Furthermore, we can obtain

$$
\begin{aligned}
&\theta^2 \int_0^T \int_s^T e^{-2\theta T} e^{\theta t} e^{\theta s} (t-s)^{2H} dtds \\
&= \theta^2 \int_0^T \int_0^{T-s} e^{-2\theta T} e^{\theta t} e^{2\theta s} t^{2H} dtds \\
&= \theta^2 \int_0^T \int_0^{T-t} e^{-2\theta T} e^{\theta t} e^{2\theta s} t^{2H} dsdt \\
&= \frac{\theta}{2} \int_0^T e^{-2\theta T} (e^{2\theta T} e^{-2\theta t} - 1) e^{\theta t} t^{2H} dt \\
&= \frac{\theta}{2} \int_0^T e^{-\theta t} t^{2H} dt - \frac{\theta}{2} e^{-2\theta T} \int_0^T e^{\theta t} t^{2H} dt \\
&\sim \frac{\theta}{2} \int_0^T e^{-\theta t} t^{2H} dt - \frac{1}{2} e^{-\theta T} T^{2H} \\
&\to \frac{H\Gamma(2H)}{\theta^{2H}}, \text{ as } T \to \infty \text{ and } \Delta \to 0.
\end{aligned}
$$

When $j = i$, it is easy to see that

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^n e^{-2\theta T} e^{4\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&= \alpha_H \sum_{i=1}^n e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&\sim \sum_{i=1}^n e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta\Delta} \Delta^{2H} \\
&= O(\Delta^{2H-1}).
\end{aligned}
\tag{6.1}
$$

Hence, the proof of this lemma is completed.

Proof of Lemma 6.2. By some basic calculations, we can see that

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^n \sum_{j\neq i} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&= \alpha_H \sum_{i=1}^n \sum_{j=i+1}^n e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} |t-s+j\Delta-i\Delta|^{2H-2} dtds \\
&\quad + \alpha_H \sum_{i=1}^n \sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} |-t+s+i\Delta-j\Delta|^{2H-2} dtds \\
&:= K_{1n} + K_{2n}
\end{aligned}
\tag{6.2}
$$

where $K_{1n} = \alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} |t - s + j\Delta - i\Delta|^{2H-2} dtds$ and $K_{2n} = \alpha_H \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} |-t + s + i\Delta - j\Delta|^{2H-2} dtds$.

For K_{1n}, it follows that

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} |t - s + j\Delta - i\Delta|^{2H-2} dtds \\
&\sim \alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta (t - s + j\Delta - i\Delta)^{2H-2}\, dtds \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \Big[(j\Delta - i\Delta + \Delta)^{2H} - 2\,(j\Delta - i\Delta)^{2H} + (j\Delta - i\Delta - \Delta)^{2H} \Big] \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+2}^{n+1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{3\theta\Delta} (j\Delta - i\Delta)^{2H} - \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} (j\Delta - i\Delta)^{2H} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} (j\Delta - i\Delta)^{2H} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \Big(1 + e^{2\theta\Delta} - 2e^{\theta\Delta} \Big) (j\Delta - i\Delta)^{2H} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{2\theta\Delta} (n\Delta + \Delta - i\Delta)^{2H} - \frac{1}{2} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \Delta^{2H} \\
&\quad - \frac{1}{2} \sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta\Delta} (n\Delta - i\Delta)^{2H} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \Big(1 + e^{2\theta\Delta} - 2e^{\theta\Delta} \Big) (j\Delta - i\Delta)^{2H} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} e^{-\theta T} e^{-\theta i\Delta} e^{3\theta\Delta} (i\Delta)^{2H} - \frac{1}{2} e^{-\theta T} e^{2\theta\Delta} n\Delta^{2H} - \frac{1}{2} \sum_{i=1}^{n-1} e^{-\theta T} e^{-\theta i\Delta} e^{\theta\Delta} (i\Delta)^{2H} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \Big(1 + e^{2\theta\Delta} - 2e^{\theta\Delta} \Big) (j\Delta - i\Delta)^{2H} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} e^{-\theta T} e^{-\theta i\Delta} e^{\theta\Delta} \Big(e^{2\theta\Delta} - 1 \Big) (i\Delta)^{2H} - \frac{1}{2} e^{-\theta T} e^{2\theta\Delta} n\Delta^{2H} + \frac{1}{2} e^{-2\theta T} e^{\theta\Delta} T^{2H} \\
&\sim \frac{\theta^2}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} (j\Delta - i\Delta)^{2H} \Delta^2 + \theta \sum_{i=1}^{n} e^{-\theta T} e^{-\theta i\Delta} e^{\theta\Delta} (i\Delta)^{2H} \Delta \\
&\quad - \frac{1}{2} e^{-\theta T} e^{2\theta\Delta} n\Delta^{2H} + \frac{1}{2} e^{-2\theta T} e^{\theta\Delta} T^{2H} \\
&\sim \frac{\theta^2}{2} \int_0^T \int_s^T e^{-\theta T} e^{\theta s} e^{-\theta t} e^{\theta\Delta} (t - s)^{2H}\, dtds + \theta \int_0^T e^{-\theta T} e^{-\theta t} e^{\theta\Delta} t^{2H} dt
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}e^{-\theta T}e^{2\theta\Delta}n\Delta^{2H}+\frac{1}{2}e^{-2\theta T}e^{\theta\Delta}T^{2H}\\
&=\frac{\theta^2}{2}e^{\theta\Delta}e^{-\theta T}\int_0^T\int_s^T e^{\theta s}e^{-\theta t}\left(t-s\right)^{2H}dtds+\theta e^{\theta\Delta}e^{-\theta T}\int_0^T e^{-\theta t}t^{2H}dt\\
&-\frac{1}{2}e^{2\theta\Delta}Te^{-\theta T}\Delta^{2H-1}+\frac{1}{2}e^{\theta\Delta}e^{-2\theta T}T^{2H}.
\end{aligned}
$$

Using

$$
\begin{aligned}
&\int_0^T\int_s^T e^{\theta s}e^{-\theta t}\left(t-s\right)^{2H}dtds\\
&=\int_0^T\int_0^{T-s} e^{-\theta t}e^{\theta\Delta}t^{2H}dtds\\
&=e^{\theta\Delta}\int_0^T\left(T-t\right)e^{-\theta t}t^{2H}dt\\
&=T\int_0^T e^{-\theta t}t^{2H}dt-\int_0^T e^{-\theta t}t^{2H+1}dt, \qquad (6.3)
\end{aligned}
$$

we can approximate K_{1n} as $O(Te^{-\theta T})$, which equals to $o(1)$.

For K_{2n}, we have

$$
\begin{aligned}
&\alpha_H\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t}e^{-\theta s}|-t+s+i\Delta-j\Delta|^{2H-2}dtds\\
&\sim\alpha_H\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta\left(-t+s+i\Delta-j\Delta\right)^{2H-2}dtds\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\left[\left(i\Delta-j\Delta+\Delta\right)^{2H}-2\left(i\Delta-j\Delta\right)^{2H}+\left(i\Delta-j\Delta-\Delta\right)^{2H}\right]\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{i-2}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{\theta\Delta}\left(i\Delta-j\Delta\right)^{2H}-\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\left(i\Delta-j\Delta\right)^{2H}\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=2}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{3\theta\Delta}\left(i\Delta-j\Delta\right)^{2H}\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(i\Delta-j\Delta\right)^{2H}\\
&+\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{\theta i\Delta}e^{\theta\Delta}\left(i\Delta\right)^{2H}-\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{2\theta\Delta}\Delta^{2H}-\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\left(i\Delta-\Delta\right)^{2H}\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(i\Delta-j\Delta\right)^{2H}
\end{aligned}
$$

$$+\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{\theta i\Delta}e^{\theta\Delta}\left(i\Delta\right)^{2H}-\frac{1}{2}e^{-\theta T}e^{2\theta\Delta}n\Delta^{2H}-\frac{1}{2}\sum_{i=1}^{n-1}e^{-\theta T}e^{\theta i\Delta}e^{3\theta\Delta}\left(i\Delta\right)^{2H}$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(i\Delta-j\Delta\right)^{2H}$$

$$-\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{\theta i\Delta}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)\left(i\Delta\right)^{2H}+\frac{1}{2}e^{3\theta\Delta}T^{2H}-\frac{1}{2}e^{-\theta T}e^{2\theta\Delta}n\Delta^{2H}$$

$$\sim\frac{\theta^2}{2}e^{\theta\Delta}e^{-\theta T}\int_0^T\int_0^s e^{\theta s}e^{-\theta t}\left(s-t\right)^{2H}dtds-\theta e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H}dt$$

$$+\frac{1}{2}e^{3\theta\Delta}T^{2H}-\frac{1}{2}e^{2\theta\Delta}Te^{-\theta T}\Delta^{2H-1}. \tag{6.4}$$

Substitute

$$\int_0^T\int_0^s e^{\theta s}e^{-\theta t}\left(s-t\right)^{2H}dtds$$

$$=\int_0^T\int_0^s e^{\theta t}t^{2H}dtds$$

$$=\int_0^T(T-t)e^{\theta t}t^{2H}dt$$

$$=T\int_0^T e^{\theta t}t^{2H}dt-\int_0^T e^{\theta t}t^{2H+1}dt$$

$$=T\int_0^T e^{\theta t}t^{2H}dt-\frac{1}{\theta}\left[e^{\theta T}T^{2H+1}-(2H+1)\int_0^T e^{\theta t}t^{2H}dt\right]$$

$$=\frac{T}{\theta}\left[e^{\theta T}T^{2H}-2H\int_0^T e^{\theta t}t^{2H-1}dt\right]-\frac{1}{\theta}e^{\theta T}T^{2H+1}$$

$$+\frac{2H+1}{\theta^2}\left[e^{\theta T}T^{2H}-2H\int_0^T e^{\theta t}t^{2H-1}dt\right]$$

$$=\frac{1}{\theta}e^{\theta T}T^{2H-1}-\frac{2HT}{\theta^2}\left[e^{\theta T}T^{2H-1}-(2H-1)\int_0^T e^{\theta t}t^{2H-2}dt\right]-\frac{1}{\theta}e^{\theta T}T^{2H+1}$$

$$+\frac{2H+1}{\theta^2}\left[e^{\theta T}T^{2H}-2H\int_0^T e^{\theta t}t^{2H-1}dt\right]$$

$$\sim\frac{1}{\theta}e^{\theta T}T^{2H+1}-\frac{2H}{\theta^2}e^{\theta T}T^{2H}+\frac{2H(2H-1)}{\theta^3}T^{2H-1}-\frac{1}{\theta}e^{\theta T}T^{2H+1}$$

$$+\frac{2H+1}{\theta^2}e^{\theta T}T^{2H}-\frac{2H(2H+1)}{\theta^3}e^{\theta T}T^{2H-1}$$

$$=\frac{1}{\theta^2}e^{\theta T}T^{2H}-\frac{4H}{\theta^3}e^{\theta T}T^{2H-1} \tag{6.5}$$

into (6.4), we have

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta}\int_0^{\Delta} e^{-\theta t} e^{-\theta s} |-t+s+i\Delta-j\Delta|^{2H-2} dtds \\
&\sim \frac{\theta^2}{2} e^{\theta\Delta} e^{-\theta T}\left(\frac{1}{\theta^2} e^{\theta T} T^{2H} - \frac{4H}{\theta^3} e^{\theta T} T^{2H-1}\right) - e^{\theta\Delta}(T^{2H} - \frac{2H}{\theta} T^{2H-1}) \\
&\quad + \frac{1}{2} e^{3\theta\Delta} T^{2H} - \frac{1}{2} e^{2\theta\Delta} T e^{-\theta T}\Delta^{2H-1} \\
&= \frac{1}{2} e^{\theta\Delta} T^{2H} - e^{\theta\Delta} T^{2H} + \frac{1}{2} e^{3\theta\Delta} T^{2H} - \frac{1}{2} e^{2\theta\Delta} T e^{-\theta T}\Delta^{2H-1} \\
&\sim \theta T^{2H}\Delta + o(1).
\end{aligned} \tag{6.6}
$$

When $j = i$, it holds that

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&= \alpha_H \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \int_0^{\Delta}\int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&\sim T e^{-\theta T}\Delta^{2H-1}.
\end{aligned}
$$

Together with $K_{1n} \sim o(1)$, (6.6) and (6.2), we can obtain (6.21).

Proof of Lemma 6.3. Firstly, in the case of $j \neq i$, we can see that

$$
\begin{aligned}
&\alpha_H \sum_{i,j=1}^{n}\sum_{j\neq i} e^{\theta(i+j)\Delta} \int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta t} |t-s|^{2H-2} dtds \\
&= 2\alpha_H \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta} \int_0^{\Delta}\int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s+j\Delta-i\Delta|^{2H-2} dtds \\
&\sim 2\alpha_H \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta} \int_0^{\Delta}\int_0^{\Delta} (t-s+j\Delta-i\Delta)^{2H-2}\, dtds \\
&= \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta}\left[(j\Delta-i\Delta+\Delta)^{2H} - 2\,(j\Delta-i\Delta)^{2H} + (j\Delta-i\Delta-\Delta)^{2H}\right] \\
&= \sum_{i=1}^{n}\sum_{j=i+2}^{n+1} e^{2\theta\Delta}\,(j\Delta-i\Delta)^{2H} - 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta}\,(j\Delta-i\Delta)^{2H} \\
&\quad + \sum_{i=1}^{n}\sum_{j=i+1}^{n-1} e^{2\theta\Delta}\,(j\Delta-i\Delta)^{2H} \\
&= \sum_{i=1}^{n} e^{2\theta\Delta}\,(n\Delta+\Delta-i\Delta)^{2H} - \sum_{i=1}^{n} e^{2\theta\Delta}\Delta^{2H} - \sum_{i=1}^{n} e^{2\theta\Delta}\,(n\Delta-i\Delta)^{2H}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} e^{2\theta\Delta} (i\Delta)^{2H} - \sum_{i=1}^{n-1} e^{2\theta\Delta} (i\Delta)^{2H} - e^{2\theta\Delta} n\Delta^{2H} \\
&= e^{2\theta\Delta} T^{2H} - e^{2\theta\Delta} T^{2H} n^{1-2H} \\
&= e^{2\theta\Delta} T^{2H} + o(T^{2H}).
\end{aligned} \tag{6.7}
$$

Second, in the case of $j = i$, using the fact that

$$
\int_0^{\Delta} \int_0^{s} e^{-\theta(t+s)} |t-s|^{2H-2} dtds \sim \frac{\Delta^{2H}}{2H(2H-1)}, \tag{6.8}
$$

we have

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta(t+s)} |t-s|^{2H-2} dtds \\
&= 2\alpha_H \sum_{i=1}^{n} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{s} e^{-\theta(t+s)} |t-s|^{2H-2} dtds \\
&\sim e^{2\theta\Delta} n\Delta^{2H} \\
&= e^{2\theta\Delta} T^{2H} n^{1-2H} \\
&= o\left(T^{2H}\right).
\end{aligned} \tag{6.9}
$$

Combining (6.7) with (6.9), we can get (6.21).

Proof of Lemma 6.4. It holds that

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&= \alpha_H \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s+j\Delta-i\Delta|^{2H-2} dtds.
\end{aligned}
$$

In the case of $j = i$, using (6.8), we have

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n} e^{-\theta i\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s|^{2H-2} dtds \\
&= \Delta^{2H} e^{\theta\Delta} \frac{1-e^{-\theta T}}{1-e^{-\theta\Delta}} = O\left(\Delta^{2H-1}\right) = o(1).
\end{aligned} \tag{6.10}
$$

Now, for $j > i$, we have

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s+j\Delta-i\Delta|^{2H-2} dtds \\
&\sim \alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} |t-s+j\Delta-i\Delta|^{2H-2} dtds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\left[(j\Delta-i\Delta+\Delta)^{2H}-2\,(j\Delta-i\Delta)^{2H}+(j\Delta-i\Delta-\Delta)^{2H}\right]\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+2}^{n+1} e^{-\theta j\Delta}e^{3\theta\Delta}\,(j\Delta-i\Delta)^{2H}-\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\,(j\Delta-i\Delta)^{2H}\\
&\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n-1} e^{-\theta j\Delta}e^{\theta\Delta}\,(j\Delta-i\Delta)^{2H}\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H}+\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{2\theta\Delta}\,(n\Delta+\Delta-i\Delta)^{2H}\\
&\quad-\frac{1}{2}\sum_{i=1}^{n}e^{-\theta i\Delta}e^{2\theta\Delta}\Delta^{2H}-\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{\theta\Delta}\,(n\Delta-i\Delta)^{2H}\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H}+\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{2\theta\Delta}\,(i\Delta)^{2H}\\
&\quad-\frac{1-e^{-\theta T}}{2(1-e^{-\theta\Delta})}e^{\theta\Delta}\Delta^{2H}-\frac{1}{2}\sum_{i=1}^{n-1}e^{-\theta T}e^{\theta\Delta}\,(i\Delta)^{2H}\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H}+\frac{1}{2}\sum_{i=1}^{n}e^{-\theta T}e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H}\\
&\quad-\frac{1-e^{-\theta T}}{2(1-e^{-\theta\Delta})}e^{\theta\Delta}\Delta^{2H}+\frac{1}{2}e^{-\theta T}e^{\theta\Delta}T^{2H}\\
&\sim \frac{\theta^2}{2}\int_0^T\int_s^T e^{-\theta t}e^{\theta\Delta}\,(t-s)^{2H}\,dtds+\frac{\theta}{2}e^{-\theta T}e^{\theta\Delta}\int_0^T t^{2H}dt\\
&\quad-\frac{1-e^{-\theta T}}{2\theta}e^{2\theta\Delta}\Delta^{2H-1}+\frac{1}{2}e^{-\theta T}e^{\theta\Delta}T^{2H}\\
&= \frac{\theta^2}{2}e^{\theta\Delta}\int_0^T\int_s^T e^{-\theta t}\,(t-s)^{2H}\,dtds+o(1).
\end{aligned}
$$

With the fact that

$$
\begin{aligned}
&\frac{\theta^2}{2}\int_0^T\int_s^T e^{-\theta t}\,(t-s)^{2H}\,dtds\\
&=\frac{\theta^2}{2}\int_0^T\int_0^{T-s} e^{-\theta t}e^{-\theta s}t^{2H}dtds\\
&=\frac{\theta}{2}\int_0^T\left(1-e^{-\theta T}e^{\theta t}\right)e^{-\theta t}t^{2H}dt\\
&=\frac{\theta}{2}\int_0^T e^{-\theta t}t^{2H}dt-\frac{1-e^{-\theta T}}{2\theta}e^{\theta\Delta}\Delta^{2H-1}+\frac{1}{2}e^{-\theta T}T^{2H}\\
&=O(1),
\end{aligned}
$$

we have

$$\alpha_H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} |t-s+j\Delta-i\Delta|^{2H-2} dtds \sim O(1). \qquad (6.11)$$

Similarly, if $j<i$, we can obtain

$$\begin{aligned}
&\alpha_H \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} |-t+s+i\Delta-j\Delta|^{2H-2} dtds \\
&\sim \alpha_H \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} (-t+s+i\Delta-j\Delta)^{2H-2}\, dtds \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j\Delta} e^{2\theta\Delta} \left[(i\Delta-j\Delta+\Delta)^{2H} - 2(i\Delta-j\Delta)^{2H} + (i\Delta-j\Delta-\Delta)^{2H} \right] \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=0}^{i-2} e^{-\theta j\Delta} e^{\theta\Delta} (i\Delta-j\Delta)^{2H} - \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j\Delta} e^{2\theta\Delta} (i\Delta-j\Delta)^{2H} \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=2}^{i-1} e^{-\theta j\Delta} e^{3\theta\Delta} (i\Delta-j\Delta)^{2H} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j\Delta} e^{\theta\Delta} (1+e^{2\theta\Delta}-2e^{\theta\Delta}) (i\Delta-j\Delta)^{2H} + \frac{1}{2} \sum_{i=1}^{n} e^{\theta\Delta} (i\Delta)^{2H} \\
&\quad - \frac{1}{2} \sum_{i=1}^{n} e^{-\theta i\Delta} e^{2\theta\Delta} \Delta^{2H} - \frac{1}{2} \sum_{i=1}^{n} e^{2\theta\Delta} (i\Delta-\Delta)^{2H} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j\Delta} e^{\theta\Delta} (1+e^{2\theta\Delta}-2e^{\theta\Delta}) (i\Delta-j\Delta)^{2H} - \frac{1}{2} \sum_{i=1}^{n} e^{\theta\Delta} \left(e^{\theta\Delta} - 1 \right) (i\Delta)^{2H} \\
&\quad - \frac{1-e^{-\theta T}}{2(1-e^{-\theta\Delta})} e^{\theta\Delta} \Delta^{2H} + \frac{1}{2} e^{2\theta\Delta} T^{2H} \\
&\sim \frac{\theta^2}{2} e^{\theta\Delta} \int_0^T \int_0^s e^{-\theta t} (s-t)^{2H} dtds - \frac{\theta}{2} e^{\theta\Delta} \int_0^T t^{2H} dt \\
&\quad - \frac{1-e^{-\theta T}}{2(1-e^{-\theta\Delta})} e^{\theta\Delta} \Delta^{2H} + \frac{1}{2} e^{2\theta\Delta} T^{2H} \\
&= \frac{\theta^2}{2} e^{\theta\Delta} \int_0^T \int_0^s e^{-\theta s} e^{\theta t} t^{2H} dtds - \frac{\theta}{2(2H+1)} e^{\theta\Delta} T^{2H+1} + \frac{1}{2} e^{2\theta\Delta} T^{2H} + o(1) \\
&= \frac{\theta}{2} e^{\theta\Delta} \int_0^T \left(e^{-\theta t} - e^{-\theta T} \right) e^{\theta t} t^{2H} dt - \frac{\theta}{2(2H+1)} e^{\theta\Delta} T^{2H+1} + \frac{1}{2} e^{2\theta\Delta} T^{2H} + o(1) \\
&= \frac{1}{2} e^{2\theta\Delta} T^{2H} - \frac{\theta}{2} e^{\theta\Delta} e^{-\theta T} \int_0^T e^{\theta t} t^{2H} dt + o(1)
\end{aligned}$$

$$= \frac{1}{2}e^{2\theta\Delta}T^{2H} - \frac{1}{2}e^{\theta\Delta}e^{-\theta T}\left(e^{\theta T}T^{2H} - 2H\int_0^T e^{\theta t}t^{2H-1}dt\right) + o(1)$$
$$\sim \frac{1}{2}e^{\theta\Delta}(e^{\theta\Delta}-1)T^{2H} + \theta H e^{\theta\Delta}T^{2H-1} + o(1)$$
$$\sim \frac{\theta}{2}e^{\theta\Delta}T^{2H}\Delta + \theta H T^{2H-1} + o(1). \tag{6.12}$$

Combining (6.10)-(6.12), we can complete the proof of this lemma.

Proof of Lemma 6.5. It holds that

$$\alpha_H \sum_{i=1}^n \sum_{j=1}^n e^{-\theta T}e^{\theta(i+2j)\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta t}e^{-\theta s}|t-s|^{2H-2}dtds$$
$$= \alpha_H \sum_{i=1}^n \sum_{j=1}^n e^{-\theta T}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}|t-s+j\Delta-i\Delta|^{2H-2}dtds.$$

When $j=i$, we have

$$\alpha_H \sum_{i=1}^n e^{-\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}|t-s|^{2H-2}dtds$$
$$\sim e^{-\theta T}\Delta^{2H}\sum_{i=1}^n e^{\theta i\Delta} = O\left(\Delta^{2H-1}\right). \tag{6.13}$$

When $j>i$, similar to the proof of Lemma 6.1, for some constant $\Theta\in(0,1)$, we have

$$\alpha_H \sum_{i=1}^n \sum_{j=i+1}^n e^{-\theta T}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}|t-s+j\Delta-i\Delta|^{2H-2}dtds$$
$$\sim \alpha_H \sum_{i=1}^n \sum_{j=i+1}^n e^{-\theta T}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} (t-s+j\Delta-i\Delta)^{2H-2}dtds$$
$$= \frac{1}{2}\sum_{i=1}^n \sum_{j=i+1}^n e^{-\theta T}e^{\theta j\Delta}e^{2\theta\Delta}\left[(j\Delta-i\Delta+\Delta)^{2H} - 2(j\Delta-i\Delta)^{2H} + (j\Delta-i\Delta-\Delta)^{2H}\right]$$
$$= \frac{1}{2}\sum_{i=1}^n \sum_{j=i+2}^{n+1} e^{-\theta T}e^{\theta j\Delta}e^{\theta\Delta}(j\Delta-i\Delta)^{2H} - \sum_{i=1}^n \sum_{j=i+1}^n e^{-\theta T}e^{\theta j\Delta}e^{2\theta\Delta}(j\Delta-i\Delta)^{2H}$$
$$+ \frac{1}{2}\sum_{i=1}^n \sum_{j=i+1}^{n-1} e^{-\theta T}e^{\theta j\Delta}e^{3\theta\Delta}(j\Delta-i\Delta)^{2H}$$
$$= \frac{1}{2}\sum_{i=1}^n \sum_{j=i+1}^n e^{-\theta T}e^{\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H} + \frac{1}{2}\sum_{i=1}^n e^{2\theta\Delta}(n\Delta+\Delta-i\Delta)^{2H}$$
$$- \frac{1}{2}\sum_{i=1}^n e^{-\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\Delta^{2H} - \frac{1}{2}\sum_{i=1}^n e^{3\theta\Delta}(n\Delta-i\Delta)^{2H}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H} + \frac{1}{2}\sum_{i=1}^{n} e^{2\theta\Delta}(i\Delta)^{2H} \\
&\quad - \frac{1-e^{-\theta T}}{2(e^{\theta\Delta}-1)} e^{3\theta\Delta}\Delta^{2H} - \frac{1}{2}\sum_{i=1}^{n-1} e^{3\theta\Delta}(i\Delta)^{2H} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H} - \frac{1}{2}\sum_{i=1}^{n} e^{2\theta\Delta}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H} \\
&\quad - \frac{1-e^{-\theta T}}{2(e^{\theta\Delta}-1)} e^{3\theta\Delta}\Delta^{2H} + \frac{1}{2} e^{3\theta\Delta} T^{2H} \\
&\sim \frac{\theta^2}{2}\int_0^T\int_s^T e^{-\theta T} e^{\theta t} e^{\theta\Delta}(t-s)^{2H} dtds - \frac{\theta}{2} e^{2\theta\Delta}\int_0^T t^{2H} dt + o(1) + \frac{1}{2} e^{3\theta\Delta} T^{2H} \\
&= \frac{\theta^2}{2} e^{\theta\Delta} e^{-\theta T}\int_0^T\int_0^{T-s} e^{\theta t} e^{\theta s} t^{2H} dtds - \frac{\theta}{2(2H+1)} e^{2\theta\Delta} T^{2H+1} + o(1) + \frac{1}{2} e^{3\theta\Delta} T^{2H} \\
&= \frac{\theta}{2} e^{\theta\Delta} e^{-\theta T}\int_0^T (e^{\theta T} e^{-\theta t}-1) e^{\theta t} t^{2H} dt - \frac{\theta}{2(2H+1)} e^{2\theta\Delta} T^{2H+1} + o(1) + \frac{1}{2} e^{3\theta\Delta} T^{2H} \\
&= \frac{1}{2} e^{3\theta\Delta} T^{2H} - \frac{\theta}{2} e^{\theta\Delta} e^{-\theta T}\int_0^T e^{\theta t} t^{2H} dt + o(1) \\
&\sim \frac{1}{2} e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right) T^{2H} + o(1) \\
&\sim \theta e^{\theta\Delta} T^{2H}\Delta + o(1). \qquad (6.14)
\end{aligned}
$$

When $j < i$, it follows that

$$
\begin{aligned}
&\alpha_H \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t} e^{-\theta s}|t-s+j\Delta-i\Delta|^{2H-2} dtds \\
&\sim \alpha_H \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta (-t+s+i\Delta-j\Delta)^{2H-2} dtds \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta}\left[(i\Delta-j\Delta+\Delta)^{2H} - 2(i\Delta-j\Delta)^{2H} + (i\Delta-j\Delta-\Delta)^{2H}\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{i-2} e^{-\theta T} e^{\theta j\Delta} e^{3\theta\Delta}(i\Delta-j\Delta)^{2H} - \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta}(i\Delta-j\Delta)^{2H} \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=2}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta}(i\Delta-j\Delta)^{2H} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta}(1+e^{2\theta\Delta}-2e^{\theta\Delta})(i\Delta-j\Delta)^{2H} + \frac{1}{2}\sum_{i=1}^{n} e^{-\theta T} e^{3\theta\Delta}(i\Delta)^{2H}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{2\theta\Delta}\Delta^{2H} - \frac{1}{2}\sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta}\left(i\Delta-\Delta\right)^{2H} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta}(1+e^{2\theta\Delta}-2e^{\theta\Delta})(i\Delta-j\Delta)^{2H} + \frac{1}{2}\sum_{i=1}^{n} e^{-\theta T} e^{3\theta\Delta}\left(i\Delta\right)^{2H} \\
&\quad - \frac{1-e^{-\theta T}}{2(e^{\theta\Delta}-1)} e^{3\theta\Delta}\Delta^{2H} - \frac{1}{2}\sum_{i=1}^{n-1} e^{-\theta T} e^{2\theta\Delta}\left(i\Delta\right)^{2H} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta}(1+e^{2\theta\Delta}-2e^{\theta\Delta})(i\Delta-j\Delta)^{2H} + \frac{1}{2}\sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta}\left(e^{\theta\Delta}-1\right)\left(i\Delta\right)^{2H} \\
&\quad - \frac{1-e^{-\theta T}}{2(e^{\theta\Delta}-1)} e^{3\theta\Delta}\Delta^{2H} + \frac{1}{2} e^{-\theta T} e^{2\theta\Delta} T^{2H} \\
&\sim \frac{\theta^2}{2}\int_0^T\int_0^s e^{-\theta T} e^{\theta t} e^{\theta\Delta}(s-t)^{2H} dtds + \frac{\theta}{2} e^{-\theta T} e^{2\theta\Delta}\int_0^T t^{2H} dt + o(1) \\
&= \frac{\theta^2}{2} e^{\theta\Delta} e^{-\theta T}\int_0^T\int_0^s e^{\theta s} e^{-\theta t} t^{2H} dtds + \frac{\theta}{2\left(2H+1\right)} e^{-\theta T} e^{2\theta\Delta} T^{2H+1} + o(1) \\
&= \frac{\theta}{2} e^{\theta\Delta} e^{-\theta T}\int_0^T \left(e^{\theta T}-e^{\theta t}\right) e^{-\theta t} t^{2H} dt + o(1) \\
&= \frac{\theta}{2} e^{\theta\Delta}\int_0^T e^{-\theta t} t^{2H} dt - \frac{\theta}{2\left(2H+1\right)} e^{\theta\Delta} e^{-\theta T} T^{2H+1} + o(1) \\
&= O(1).
\end{aligned}
\tag{6.15}
$$

Combining (6.13)-(6.15), we can complete the proof of this lemma.

Proof of Lemma 6.6. Firstly, we can see that

$$
\begin{aligned}
&\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s}\frac{\partial R(t,s)}{\partial t} dtds \\
&+ \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(i-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right) dt \\
&= \theta\int_0^{\Delta}\int_0^{\Delta} e^{-\theta(t+(i-1)\Delta)} e^{-\theta(s+(j-1)\Delta)}\frac{\partial R(t+(i-1)\Delta, s+(j-1)\Delta)}{\partial t} dtds \\
&\quad + \int_0^{\Delta} e^{-\theta(t+(i-1)\Delta)} e^{-\theta j\Delta}\frac{\partial R(t+(i-1)\Delta, j\Delta)}{\partial t} dt \\
&\quad - \int_0^{\Delta} e^{-\theta(t+(i-1)\Delta)} e^{-\theta(j-1)\Delta}\frac{\partial R(t+(i-1)\Delta, (j-1)\Delta)}{\partial t} dt \\
&= \theta e^{-\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t} e^{-\theta s}\frac{\partial\frac{1}{2}(|t+(i-1)\Delta|^{2H} - |t-s+i\Delta-j\Delta|^{2H})}{\partial t} dtds \\
&\quad + e^{-\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}(|t+(i-1)\Delta|^{2H} - |t+i\Delta-j\Delta-\Delta|^{2H})}{\partial t} dt
\end{aligned}
$$

$$
\begin{aligned}
&- e^{-\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}(|t+(i-1)\Delta|^{2H}-|t+i\Delta-j\Delta|^{2H})}{\partial t}dt\\
&= e^{-\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&- e^{-\theta i\Delta}e^{-\theta j\Delta}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta-\Delta\right|^{2H}}{\partial t}dt\\
&- \theta e^{-\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds. \qquad (6.16)
\end{aligned}
$$

Using (6.16), we can obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{2\theta(i+j)\Delta}\left(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\right.\\
&\left.+\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t}-e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\right)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&\quad-\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta-\Delta\right|^{2H}}{\partial t}dt\\
&\quad-\theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&\quad-\sum_{i=1}^{n}\sum_{j=2}^{n+1} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&\quad-\theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&\quad-\sum_{i=1}^{n} e^{-\theta T}e^{\theta i\Delta}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-n\Delta-\Delta\right|^{2H}}{\partial t}dt\\
&\quad+\sum_{i=1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-\Delta\right|^{2H}}{\partial t}dt
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds\\
&=-\sum_{i=1}^{n}e^{-\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|-t+i\Delta\right|^{2H}}{\partial t}dt\\
&\quad+\sum_{i=0}^{n-1}e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta\right|^{2H}}{\partial t}dt\\
&\quad+\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&\quad-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds\\
&:=J_{1n}+J_{2n}+J_{3n},
\end{aligned}
\tag{6.17}
$$

where

$$
\begin{aligned}
J_{1n}&=-\sum_{i=1}^{n}e^{-\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|-t+i\Delta\right|^{2H}}{\partial t}dt+\sum_{i=0}^{n-1}e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta\right|^{2H}}{\partial t}dt,\\
J_{2n}&=\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt,\\
J_{3n}&=-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds.
\end{aligned}
$$

First, for J_{1n}, we have

$$
\begin{aligned}
&-\sum_{i=1}^{n}e^{-\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|-t+i\Delta\right|^{2H}}{\partial t}dt+\sum_{i=0}^{n-1}e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta\right|^{2H}}{\partial t}dt\\
&=H\sum_{i=1}^{n}e^{-\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}\left(-t+i\Delta\right)^{2H-1}dt+H\sum_{i=0}^{n-1}e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\int_0^{\Delta}\left(t+i\Delta\right)^{2H-1}dt\\
&\sim\frac{1}{2}\sum_{i=1}^{n}e^{-\theta i\Delta}e^{2\theta\Delta}\left[(i\Delta)^{2H}-(i\Delta-\Delta)^{2H}\right]+\frac{1}{2}\sum_{i=0}^{n-1}e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\left[(i\Delta+\Delta)^{2H}-(i\Delta)^{2H}\right]\\
&=\frac{1}{2}\sum_{i=1}^{n}e^{-\theta i\Delta}e^{2\theta\Delta}\left(i\Delta\right)^{2H}-\frac{1}{2}\sum_{i=1}^{n}e^{-\theta i\Delta}e^{\theta\Delta}\left(i\Delta\right)^{2H}\\
&\quad-\frac{1}{2}\sum_{i=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{\theta\Delta}\left(i\Delta\right)^{2H}+\frac{1}{2}\sum_{i=1}^{n}e^{-2\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\left(i\Delta\right)^{2H}\\
&=\frac{1}{2}\sum_{i=1}^{n}e^{-\theta i\Delta}e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\left(i\Delta\right)^{2H}+\frac{1}{2}e^{-\theta T}e^{\theta\Delta}T^{2H}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H}+\frac{1}{2}e^{-\theta T}e^{2\theta\Delta}T^{2H}\\
&=\frac{1}{2}\sum_{i=1}^{n} e^{-\theta i\Delta} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H}+\frac{1}{2}e^{-\theta T}e^{\theta\Delta}T^{2H}\\
&\quad-\frac{1}{2}\sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H}+\frac{1}{2}e^{-\theta T}e^{2\theta\Delta}T^{2H}\\
&=\frac{1}{2}e^{-\theta T}e^{\theta\Delta}\left(e^{\theta\Delta}+1\right)T^{2H}+\frac{\theta}{2}\sum_{i=1}^{n} e^{-\theta i\Delta} e^{\theta\Delta}(i\Delta)^{2H}\Delta-\frac{\theta}{2}\sum_{i=1}^{n} e^{-2\theta T}e^{\theta i\Delta} e^{\theta\Delta}(i\Delta)^{2H}\Delta\\
&\sim\frac{\theta}{2}\int_0^T e^{-\theta t}t^{2H}dt-\frac{\theta}{2}e^{-2\theta T}\int_0^T e^{\theta t}t^{2H}dt+o(1)\\
&=\frac{\theta}{2}\int_0^T e^{-\theta t}t^{2H}dt+o(1)\\
&\to\frac{H\Gamma(2H)}{\theta^{2H}}. \qquad (6.18)
\end{aligned}
$$

Second, for J_{2n}, we can see that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^\Delta e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&=H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^\Delta e^{-\theta t}\left(t+i\Delta-j\Delta\right)^{2H-1}dt\\
&\quad+H\sum_{i=1}^{n} e^{-2\theta T} e^{2\theta i\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^\Delta e^{-\theta t}t^{2H-1}dt\\
&\quad-H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^\Delta e^{-\theta t}\left(-t+j\Delta-i\Delta\right)^{2H-1}dt\\
&\sim H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^\Delta \left(t+i\Delta-j\Delta\right)^{2H-1}dt\\
&\quad+H\sum_{i=1}^{n} e^{-2\theta T} e^{2\theta i\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^\Delta t^{2H-1}dt\\
&\quad-H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^\Delta \left(-t+j\Delta-i\Delta\right)^{2H-1}dt\\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\left[\left(i\Delta-j\Delta+\Delta\right)^{2H}-\left(i\Delta-j\Delta\right)^{2H}\right]\\
&\quad+\frac{1}{2}e^{2\theta\Delta}\left(1-e^{-2\theta T}\right)\Delta^{2H}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\left[(j\Delta-i\Delta)^{2H}-(j\Delta-i\Delta-\Delta)^{2H}\right]\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\Big[\,(j\Delta-i\Delta+\Delta)^{2H}-2\,(j\Delta-i\Delta)^{2H}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (j\Delta-i\Delta-\Delta)^{2H}\,\Big]\\
&+ \frac{1}{2}e^{2\theta\Delta}\left(1-e^{-2\theta T}\right)\Delta^{2H}\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+2}^{n+1} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{-\theta\Delta}\left(e^{2\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H}\\
&- \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H}\\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n-1} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H}\\
&+ \frac{1}{2}e^{2\theta\Delta}\left(1-e^{-2\theta T}\right)\Delta^{2H}\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{-\theta\Delta}\left(e^{2\theta\Delta}-1\right)\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H}\\
&+ \frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{\theta i\Delta}\left(e^{2\theta\Delta}-1\right)(n\Delta+\Delta-i\Delta)^{2H}\\
&- \frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{\theta i\Delta}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)(n\Delta-i\Delta)^{2H}\\
&- \frac{1}{2}\sum_{i=1}^{n} e^{-2\theta T}e^{2\theta i\Delta}\left(e^{2\theta\Delta}-1\right)\Delta^{2H} + \frac{1}{2}e^{2\theta\Delta}\left(1-e^{-2\theta T}\right)\Delta^{2H}\\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{-\theta\Delta}\left(e^{2\theta\Delta}-1\right)\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)(j\Delta-i\Delta)^{2H}\\
&+ \frac{1}{2}\sum_{i=1}^{n} e^{-\theta i\Delta}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)(i\Delta)^{2H} - \frac{1}{2}\sum_{i=1}^{n-1} e^{-\theta i\Delta}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)(i\Delta)^{2H}\\
&\sim \theta^{3}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-2\theta T}e^{\theta i\Delta}e^{\theta j\Delta}e^{-\theta\Delta}(j\Delta-i\Delta)^{2H}\Delta^{3} + \frac{1}{2}e^{-\theta T}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)T^{2H}\\
&\sim \theta^{3}\int_{0}^{T}\int_{s}^{T} e^{-2\theta T}e^{\theta t}e^{\theta s}(t-s)^{2H}dtds\Delta + o(1)\\
&= \frac{\theta^{2}}{2}\int_{0}^{T} e^{-\theta t}t^{2H}dt\Delta - \frac{\theta^{2}}{2}e^{-2\theta T}\int_{0}^{T} e^{\theta t}t^{2H}dt\Delta + o(1)
\end{aligned}
$$

$$= o(1). \tag{6.19}$$

Third, for J_{3n}, it follows that

$$
\begin{aligned}
& -\theta \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2} |t-s+i\Delta-j\Delta|^{2H}}{\partial t} dtds \\
&= -\theta H \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} (t-s+i\Delta-j\Delta)^{2H-1} dtds \\
&\quad + \theta H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} (-t+s+j\Delta-i\Delta)^{2H-1} dtds \\
&\quad - \theta \sum_{i=1}^{n} e^{-2\theta T} e^{2\theta i\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2} |t-s|^{2H}}{\partial t} dtds \\
&\sim -\theta H \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} (t-s+i\Delta-j\Delta)^{2H-1} dtds \\
&\quad + \theta H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} (-t+s+j\Delta-i\Delta)^{2H-1} dtds \\
&= -\frac{\theta}{2} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \Big[(i\Delta - j\Delta + \Delta)^{2H+1} - 2(i\Delta - j\Delta)^{2H+1} \\
&\qquad\qquad + (i\Delta - j\Delta - \Delta)^{2H+1} \Big] \\
&\quad + \frac{\theta}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta j\Delta} e^{2\theta\Delta} \Big[(j\Delta - i\Delta + \Delta)^{2H+1} - 2(j\Delta - i\Delta)^{2H+1} \\
&\qquad\qquad + (j\Delta - i\Delta - \Delta)^{2H+1} \Big] \\
&= 0.
\end{aligned} \tag{6.20}
$$

Combining (6.17)-(6.20), we can obtain (6.24).

Proof of Lemma 6.7. Using (6.16), we can obtain

$$
\begin{aligned}
& \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \Big(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dtds \\
&\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta (j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Big) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2} |t+i\Delta-j\Delta|^{2H}}{\partial t} dt \\
&\quad - \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \int_0^{\Delta} e^{-\theta t} \frac{\partial \frac{1}{2} |t+i\Delta-j\Delta-\Delta|^{2H}}{\partial t} dt
\end{aligned}
$$

$$
\begin{aligned}
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t} dt \\
&- \sum_{i=1}^{n}\sum_{j=2}^{n+1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&= \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-\Delta\right|^{2H}}{\partial t} dt \\
&- \sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-n\Delta-\Delta\right|^{2H}}{\partial t} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&= H \sum_{i=0}^{n-1} e^{-\theta T} e^{\theta i\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \left(t+i\Delta\right)^{2H-1} dt \\
&+ \sum_{i=1}^{n} e^{-\theta T} e^{-\theta i\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \left(-t+i\Delta\right)^{2H-1} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&:= A_{1n} + A_{2n}, \qquad (6.21)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{1n} = & H \sum_{i=0}^{n-1} e^{-\theta T} e^{\theta i\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \left(t+i\Delta\right)^{2H-1} dt \\
& + \sum_{i=1}^{n} e^{-\theta T} e^{-\theta i\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \left(-t+i\Delta\right)^{2H-1} dt
\end{aligned}
$$

and

$$
A_{2n} = -\theta \; sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds.
$$

For A_{1n}, it holds that

$$
H \sum_{i=0}^{n-1} e^{-\theta T} e^{\theta i\Delta} e^{2\theta\Delta} \int_0^\Delta \left(t+i\Delta\right)^{2H-1} dt + \sum_{i=1}^{n} e^{-\theta T} e^{-\theta i\Delta} e^{2\theta\Delta} \int_0^\Delta \left(-t+i\Delta\right)^{2H-1} dt
$$

$$= \frac{1}{2}\sum_{i=0}^{n-1} e^{-\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\left[(i\Delta+\Delta)^{2H}-(i\Delta)^{2H}\right]$$

$$+\frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{-\theta i\Delta}e^{2\theta\Delta}\left[(i\Delta)^{2H}-(i\Delta-\Delta)^{2H}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{\theta i\Delta}e^{\theta\Delta}(i\Delta)^{2H}-\frac{1}{2}\sum_{i=1}^{n-1} e^{-\theta T}e^{\theta i\Delta}e^{2\theta\Delta}(i\Delta)^{2H}$$

$$+\frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{-\theta i\Delta}e^{2\theta\Delta}(i\Delta)^{2H}-\frac{1}{2}\sum_{i=1}^{n-1} e^{-\theta T}e^{-\theta i\Delta}e^{\theta\Delta}(i\Delta)^{2H}$$

$$= -\frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{\theta i\Delta}e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H}+\frac{1}{2}e^{2\theta\Delta}T^{2H}$$

$$+\frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{-\theta i\Delta}e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H}+\frac{1}{2}e^{-2\theta T}e^{\theta\Delta}T^{2H}$$

$$\sim -\frac{\theta}{2}\int_0^T e^{-\theta T}e^{\theta t}e^{\theta\Delta}t^{2H}dt+\frac{1}{2}e^{2\theta\Delta}T^{2H}+\frac{\theta}{2}\int_0^T e^{-\theta T}e^{-\theta t}e^{\theta\Delta}t^{2H}dt+o(1)$$

$$= -\frac{\theta}{2}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H}dt+\frac{1}{2}e^{2\theta\Delta}T^{2H}+o(1)$$

$$\sim \frac{1}{2}e^{2\theta\Delta}T^{2H}-\frac{1}{2}e^{\theta\Delta}T^{2H}+\frac{H}{2}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H-1}dt+o(1)$$

$$\sim \frac{\theta}{2}T^{2H}\Delta+o(1). \tag{6.22}$$

For A_{2n}, a standard calculation shows

$$-\theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds$$

$$= -\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t}e^{-\theta s}\left(t-s+i\Delta-j\Delta\right)^{2H-1}dtds$$

$$-\theta\sum_{i=1}^{n} e^{-\theta T}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s\right|^{2H}}{\partial t}dtds$$

$$+\theta H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t}e^{-\theta s}\left(-t+s+j\Delta-i\Delta\right)^{2H-1}dtds$$

When $j < i$, it holds that

$$-\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t}e^{-\theta s}\left(t-s+i\Delta-j\Delta\right)^{2H-1}dtds$$

$$
\begin{aligned}
&\sim -\theta H \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta (t - s + i\Delta - j\Delta)^{2H-1}\, dtds \\
&= -\frac{\theta}{2(2H+1)} \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \Big[(i\Delta - j\Delta + \Delta)^{2H+1} - 2\,(i\Delta - j\Delta)^{2H+1} \\
&\qquad\qquad + (i\Delta - j\Delta - \Delta)^{2H+1} \Big] \\
&= -\frac{\theta}{2(2H+1)} \sum_{i=1}^{n}\sum_{j=0}^{i-2} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta}\, (i\Delta - j\Delta)^{2H+1} \\
&\quad + \frac{\theta}{2H+1} \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta}\, (i\Delta - j\Delta)^{2H+1} \\
&\quad - \frac{\theta}{2(2H+1)} \sum_{i=1}^{n}\sum_{j=2}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{3\theta\Delta}\, (i\Delta - j\Delta)^{2H+1} \\
&= -\frac{\theta}{2(2H+1)} \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) (i\Delta - j\Delta)^{2H+1} \\
&\quad - \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{\theta\Delta}\, (i\Delta)^{2H+1} + \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \Delta^{2H+1} \\
&\quad + \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{2\theta\Delta}\, (i\Delta - \Delta)^{2H+1} \\
&= -\frac{\theta}{2(2H+1)} \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) (i\Delta - j\Delta)^{2H+1} \\
&\quad - \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{\theta\Delta}\, (i\Delta)^{2H+1} + \frac{\theta}{2(2H+1)} T e^{-\theta T} e^{2\theta\Delta} \Delta^{2H} \\
&\quad + \frac{\theta}{2(2H+1)} \sum_{i=1}^{n-1} e^{-\theta T} e^{\theta i\Delta} e^{3\theta\Delta}\, (i\Delta)^{2H+1} \\
&= -\frac{\theta}{2(2H+1)} \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) (i\Delta - j\Delta)^{2H+1} \\
&\quad + \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{\theta\Delta} \left(e^{2\theta\Delta} - 1\right) (i\Delta)^{2H+1} - \frac{\theta}{2(2H+1)} e^{3\theta\Delta} T^{2H+1} \\
&\quad + \frac{\theta}{2(2H+1)} T e^{-\theta T} e^{2\theta\Delta} \Delta^{2H} \\
&\sim -\frac{\theta^3}{2(2H+1)} \int_0^T \int_0^s e^{-\theta T} e^{\theta s} e^{-\theta t} e^{\theta\Delta} (s-t)^{2H+1} dtds
\end{aligned}
$$

$$+\frac{\theta^2}{2H+1}\int_0^T e^{-\theta T}e^{\theta t}e^{\theta\Delta}t^{2H+1}dt-\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}+o(1)$$
$$=-\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\int_0^T\int_0^s e^{\theta t}t^{2H+1}dtds+\frac{\theta^2}{2H+1}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt$$
$$-\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}+o(1)$$
$$=-\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\int_0^T (T-t)\,e^{\theta t}t^{2H+1}dt+\frac{\theta^2}{2H+1}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt$$
$$-\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}+o(1)$$
$$=-\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}Te^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt+\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+2}dt$$
$$+\frac{\theta^2}{2H+1}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt-\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}+o(1).$$

Since

$$\int_0^T e^{\theta t}t^{2H}dt\sim\frac{1}{\theta}e^{\theta T}T^{2H}-\frac{2H}{\theta^2}e^{\theta T}T^{2H-1},$$
$$\int_0^T e^{\theta t}t^{2H+1}dt\sim\frac{1}{\theta}e^{\theta T}T^{2H+1}-\frac{2H+1}{\theta^2}e^{\theta T}T^{2H}+\frac{2H\,(2H+1)}{\theta^3}e^{\theta T}T^{2H-1},$$

and

$$\int_0^T e^{\theta t}t^{2H+2}dt\sim\frac{1}{\theta}e^{\theta T}T^{2H+2}-\frac{2H+2}{\theta^2}e^{\theta T}T^{2H+1}+\frac{(2H+2)\,(2H+1)}{\theta^3}e^{\theta T}T^{2H}$$
$$-\frac{2H\,(2H+1)\,(2H+2)}{\theta^4}e^{\theta T}T^{2H-1},$$

we have

$$-\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}Te^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt+\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+2}dt$$
$$+\frac{\theta^2}{2H+1}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt-\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}+o(1)$$
$$\sim-\frac{1}{2(2H+1)}e^{\theta\Delta}\left[\theta^2T^{2H+2}-\theta\,(2H+1)\,T^{2H+1}+2H\,(2H+1)\,T^{2H}\right]$$
$$+\frac{1}{2(2H+1)}e^{\theta\Delta}\Big[\theta^2T^{2H+2}-\theta\,(2H+2)\,T^{2H+1}+(2H+2)\,(2H+1)\,T^{2H}$$
$$-\frac{2H\,(2H+1)\,(2H+2)}{\theta}T^{2H-1}\Big]$$
$$+\frac{1}{2H+1}e^{\theta\Delta}\left[\theta T^{2H+1}-(2H+1)\,T^{2H}+\frac{2H\,(2H+1)}{\theta}T^{2H-1}\right]$$

$$
\begin{aligned}
&- \frac{\theta}{2(2H+1)} e^{3\theta\Delta} T^{2H+1} + o(1) \\
&= \frac{\theta}{2} e^{\theta\Delta} T^{2H+1} - H e^{\theta\Delta} T^{2H} - \frac{\theta\,(2H+2)}{2(2H+1)} e^{\theta\Delta} T^{2H+1} + (H+1) e^{\theta\Delta} T^{2H} \\
&\quad + \frac{\theta}{2H+1} e^{\theta\Delta} T^{2H+1} - e^{\theta\Delta} T^{2H} - \frac{\theta}{2(2H+1)} e^{3\theta\Delta} T^{2H+1} + o(1) \\
&\sim -\frac{\theta^2}{2H+1} T^{2H+1}\Delta + o(1).
\end{aligned}
\tag{6.23}
$$

When $j = i$, it is easy to verify that

$$
\theta H \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \left(-t+s\right)^{2H-1} dtds = o(1). \tag{6.24}
$$

When $j > i$, we have

$$
\begin{aligned}
&\theta H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \left(-t+s+j\Delta-i\Delta\right)^{2H-1} dtds \\
&\sim \theta H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta \left(-t+s+j\Delta-i\Delta\right)^{2H-1} dtds \\
&= \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \Big[\left(j\Delta - i\Delta + \Delta\right)^{2H+1} - 2\left(j\Delta - i\Delta\right)^{2H+1} \\
&\qquad + \left(j\Delta - i\Delta - \Delta\right)^{2H+1} \Big] \\
&= \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} \sum_{j=i+2}^{n+1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{3\theta\Delta} \left(j\Delta - i\Delta\right)^{2H+1} \\
&\quad - \frac{\theta}{2H+1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{2\theta\Delta} \left(j\Delta - i\Delta\right)^{2H+1} \\
&\quad + \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \left(j\Delta - i\Delta\right)^{2H+1} \\
&= \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) \left(j\Delta - i\Delta\right)^{2H+1} \\
&\quad + \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{2\theta\Delta} \left(n\Delta + \Delta - i\Delta\right)^{2H+1} \\
&\quad - \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \Delta^{2H+1} - \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} e^{-2\theta T} e^{\theta i\Delta} e^{\theta\Delta} \left(n\Delta - i\Delta\right)^{2H+1} \\
&= \frac{\theta}{2(2H+1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta i\Delta} e^{-\theta j\Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) \left(j\Delta - i\Delta\right)^{2H+1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta}{2(2H+1)}\sum_{i=1}^{n}e^{-\theta T}e^{-\theta i\Delta}e^{3\theta\Delta}\left(i\Delta\right)^{2H+1}-\frac{\theta}{2(2H+1)}Te^{-\theta T}e^{2\theta\Delta}\Delta^{2H}\\
&-\frac{\theta}{2(2H+1)}\sum_{i=1}^{n-1}e^{-\theta T}e^{-\theta i\Delta}e^{\theta\Delta}\left(i\Delta\right)^{2H+1}\\
=&\frac{\theta}{2(2H+1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-\theta T}e^{\theta i\Delta}e^{-\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(j\Delta-i\Delta\right)^{2H+1}\\
&+\frac{\theta}{2(2H+1)}\sum_{i=1}^{n}e^{-\theta T}e^{-\theta i\Delta}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)\left(i\Delta\right)^{2H+1}-\frac{\theta}{2(2H+1)}Te^{-\theta T}e^{2\theta\Delta}\Delta^{2H}\\
&+\frac{\theta}{2(2H+1)}e^{-2\theta T}e^{\theta\Delta}T^{2H+1}\\
\sim&\frac{\theta^3}{2(2H+1)}\int_0^T\int_s^T e^{-\theta T}e^{\theta s}e^{-\theta t}e^{\theta\Delta}\left(t-s\right)^{2H+1}dtds\\
&+\frac{\theta^2}{2H+1}\int_0^T e^{-\theta T}e^{-\theta t}e^{\theta\Delta}t^{2H+1}dt+o(1)\\
=&\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\int_0^T\int_0^{T-s}e^{-\theta t}t^{2H+1}dtds\\
&+\frac{\theta^2}{2H+1}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{-\theta t}t^{2H+1}dt+o(1)\\
=&\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}T\int_0^T e^{-\theta t}t^{2H+1}dt-\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{-\theta t}t^{2H+2}dt\\
&+\frac{\theta^2}{2H+1}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{-\theta t}t^{2H+1}dt+o(1)\\
=&\,o(1).
\end{aligned}
\tag{6.25}
$$

Combining (6.23)-(6.25), we have $A_{2n}\sim-\frac{\theta^2}{2H+1}T^{2H+1}\Delta+o(1)$.

Together with (6.22), we complete the proof of this lemma.

Proof of Lemma 6.8. Using (6.16), we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\theta(i+j)\Delta}\Big(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\\
&+\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t}-e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Big)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}e^{2\theta\Delta}\int_0^\Delta e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&\quad-\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\theta\Delta}\int_0^\Delta e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta-\Delta\right|^{2H}}{\partial t}dt
\end{aligned}
$$

$$
\begin{aligned}
&-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds\\
=&\sum_{i=1}^{n}\sum_{j=1}^{n}e^{2\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&-\sum_{i=1}^{n}\sum_{j=2}^{n+1}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds\\
=&\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&-\sum_{i=1}^{n}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-n\Delta-\Delta\right|^{2H}}{\partial t}dt+\sum_{i=1}^{n}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-\Delta\right|^{2H}}{\partial t}dt\\
&-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds\\
=&-\sum_{i=1}^{n}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|-t+i\Delta\right|^{2H}}{\partial t}dt+\sum_{i=0}^{n-1}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta\right|^{2H}}{\partial t}dt\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt\\
&-\theta\sum_{i=1}^{n}\sum_{j=1}^{n}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds \qquad (6.26)
\end{aligned}
$$

First,

$$
\begin{aligned}
&-\sum_{i=1}^{n}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|-t+i\Delta\right|^{2H}}{\partial t}dt+\sum_{i=0}^{n-1}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\frac{\partial\frac{1}{2}\left|t+i\Delta\right|^{2H}}{\partial t}dt\\
&=H\sum_{i=1}^{n}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\left(-t+i\Delta\right)^{2H-1}dt+H\sum_{i=0}^{n-1}e^{\theta\Delta}\int_0^{\Delta}e^{-\theta t}\left(-t+i\Delta\right)^{2H-1}dt\\
&\sim H\sum_{i=1}^{n}e^{\theta\Delta}\int_0^{\Delta}\left(-t+i\Delta\right)^{2H-1}dt+H\sum_{i=0}^{n-1}e^{\theta\Delta}\int_0^{\Delta}\left(-t+i\Delta\right)^{2H-1}dt\\
&=\frac{1}{2}e^{\theta\Delta}\sum_{i=1}^{n}\left[(i\Delta)^{2H}-(i\Delta-\Delta)^{2H}\right]+\frac{1}{2}e^{\theta\Delta}\sum_{i=0}^{n-1}\left[(i\Delta+\Delta)^{2H}-(i\Delta)^{2H}\right]\\
&=e^{\theta\Delta}\sum_{i=1}^{n}(i\Delta)^{2H}-e^{\theta\Delta}\sum_{i=0}^{n-1}(i\Delta)^{2H}
\end{aligned}
$$

$$= e^{\theta\Delta}T^{2H}. \tag{6.27}$$

Second, for the third term in (6.27), we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt$$
$$+ \sum_{i=1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}t^{2H}}{\partial t}dt$$
$$+ \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|-t+j\Delta-i\Delta\right|^{2H}}{\partial t}dt$$
$$= H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\left(t+i\Delta-j\Delta\right)^{2H-1}dt$$
$$+ H\sum_{i=1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}t^{2H-1}dt$$
$$- H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\left(-t+j\Delta-i\Delta\right)^{2H-1}dt$$
$$\sim H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} \left(t+i\Delta-j\Delta\right)^{2H-1}dt$$
$$+ H\sum_{i=1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} t^{2H-1}dt$$
$$- H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\int_0^{\Delta} \left(-t+j\Delta-i\Delta\right)^{2H-1}dt$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\left[\left(i\Delta-j\Delta+\Delta\right)^{2H}-\left(i\Delta-j\Delta\right)^{2H}\right]+\frac{1}{2}e^{\theta\Delta}n\left(e^{\theta\Delta}-1\right)\Delta^{2H}$$
$$- \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\left[\left(j\Delta-i\Delta\right)^{2H}-\left(j\Delta-i\Delta-\Delta\right)^{2H}\right]$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)\left[\left(j\Delta-i\Delta+\Delta\right)^{2H}-2\left(j\Delta-i\Delta\right)^{2H}+\left(j\Delta-i\Delta-\Delta\right)^{2H}\right]$$
$$+ \frac{1}{2}e^{\theta\Delta}n\left(e^{\theta\Delta}-1\right)\Delta^{2H}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+2}^{n+1} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H} - \sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H} \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n-1} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H} + \frac{1}{2}e^{\theta\Delta}n\left(e^{\theta\Delta}-1\right)\Delta^{2H} \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(e^{\theta\Delta}-1\right)(n\Delta+\Delta-i\Delta)^{2H} - \frac{1}{2}\sum_{i=1}^{n} e^{\theta\Delta}\left(e^{\theta\Delta}-1\right)(n\Delta-i\Delta)^{2H} \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H} - \frac{1}{2}\sum_{i=1}^{n-1}\left(e^{\theta\Delta}-1\right)(i\Delta)^{2H} \\
&= \frac{1}{2}\left(e^{\theta\Delta}-1\right)T^{2H} \\
&\sim o(T^{2H}). \qquad (6.28)
\end{aligned}
$$

Third, for the last term in (6.27), we can see that

$$
\begin{aligned}
&-\theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds \\
&= -\theta\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds \\
&\quad -\theta\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds \\
&= -\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}(t-s+i\Delta-j\Delta)^{2H-1}dtds \\
&\quad +\theta H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}(-t+s+j\Delta-i\Delta)^{2H-1}dtds \\
&\sim -\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}(t-s+i\Delta-j\Delta)^{2H-1}dtds \\
&\quad +\theta H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}(-t+s+j\Delta-i\Delta)^{2H-1}dtds \\
&= -\frac{\theta e^{2\theta\Delta}}{2(2H+1)}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\left[(i\Delta-j\Delta+\Delta)^{2H+1}-2(i\Delta-j\Delta)^{2H+1}+(i\Delta-j\Delta-\Delta)^{2H+1}\right] \\
&\quad +\frac{\theta e^{2\theta\Delta}}{2(2H+1)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left[(j\Delta-i\Delta+\Delta)^{2H+1}-2(j\Delta-i\Delta)^{2H+1}+(j\Delta-i\Delta-\Delta)^{2H+1}\right] \\
&= 0. \qquad (6.29)
\end{aligned}
$$

Combining (6.26)-(6.29), we can get (6.26).

Proof of Lemma 6.9. Using (6.16), straightforward calculations lead to

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\theta i\Delta}\Big(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds \\
&\qquad\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Big) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta j\Delta}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta-\Delta\right|^{2H}}{\partial t}dt \\
&\quad - \theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt \\
&\quad - \sum_{i=1}^{n}\sum_{j=2}^{n+1} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt \\
&\quad - \theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds \\
&= \sum_{i=1}^{n} e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-\Delta\right|^{2H}}{\partial t}dt \\
&\quad - \sum_{i=1}^{n} e^{-\theta T}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-n\Delta-\Delta\right|^{2H}}{\partial t}dt \\
&\quad - \theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds \\
&= H\sum_{i=0}^{n-1} e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\left(t+i\Delta\right)^{2H-1}dt + H\sum_{i=1}^{n} e^{-\theta T}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\left(-t+i\Delta\right)^{2H-1}dt \\
&\quad - \theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds. \qquad (6.30)
\end{aligned}
$$

First, it is easy to see that

$$
H\sum_{i=0}^{n-1} e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\left(t+i\Delta\right)^{2H-1}dt + H\sum_{i=1}^{n} e^{-\theta T}e^{\theta\Delta}\int_0^{\Delta} e^{-\theta t}\left(-t+i\Delta\right)^{2H-1}dt
$$

$$
\begin{aligned}
&\sim H\sum_{i=0}^{n-1} e^{\theta\Delta}\int_0^{\Delta}(t+i\Delta)^{2H-1}\,dt + H\sum_{i=1}^{n} e^{-\theta T}e^{\theta\Delta}\int_0^{\Delta}(-t+i\Delta)^{2H-1}\,dt\\
&=\frac{1}{2}\sum_{i=0}^{n-1} e^{\theta\Delta}\left[(i\Delta+\Delta)^{2H}-(i\Delta)^{2H}\right]+\frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{\theta\Delta}\left[(i\Delta)^{2H}-(i\Delta-\Delta)^{2H}\right]\\
&=\frac{1}{2}e^{\theta\Delta}T^{2H}+\frac{1}{2}e^{-\theta T}e^{\theta\Delta}T^{2H}\\
&=\frac{1}{2}e^{\theta\Delta}T^{2H}+o(1).
\end{aligned}
\tag{6.31}
$$

Second, the last term in (6.30) can be decomposed into

$$
\begin{aligned}
&-\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\left(t-s+i\Delta-j\Delta\right)^{2H-1}dtds\\
&+\theta H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta} e^{-\theta t}e^{-\theta s}\left(-t+s+j\Delta-i\Delta\right)^{2H-1}dtds\\
&\sim-\theta H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}\left(t-s+i\Delta-j\Delta\right)^{2H-1}dtds\\
&\quad+\theta H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\int_0^{\Delta}\int_0^{\Delta}\left(-t+s+j\Delta-i\Delta\right)^{2H-1}dtds\\
&=-\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta j\Delta}e^{2\theta\Delta}\Big[\left(i\Delta-j\Delta+\Delta\right)^{2H+1}-2\left(i\Delta-j\Delta\right)^{2H+1}\\
&\qquad\qquad+\left(i\Delta-j\Delta-\Delta\right)^{2H+1}\Big]\\
&\quad+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta j\Delta}e^{2\theta\Delta}\Big[\left(j\Delta-i\Delta+\Delta\right)^{2H+1}-2\left(j\Delta-i\Delta\right)^{2H+1}\\
&\qquad\qquad+\left(j\Delta-i\Delta-\Delta\right)^{2H+1}\Big]\\
&=-\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=0}^{i-2} e^{-\theta j\Delta}e^{\theta\Delta}\left(i\Delta-j\Delta\right)^{2H+1}\\
&\quad+\frac{\theta}{2H+1}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta j\Delta}e^{2\theta\Delta}\left(i\Delta-j\Delta\right)^{2H+1}\\
&\quad-\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=2}^{i-1} e^{-\theta j\Delta}e^{3\theta\Delta}\left(i\Delta-j\Delta\right)^{2H+1}\\
&\quad+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=i+2}^{n+1} e^{-\theta j\Delta}e^{3\theta\Delta}\left(j\Delta-i\Delta\right)^{2H+1}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\theta}{2H+1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta j \Delta} e^{2\theta\Delta} \left(j\Delta - i\Delta\right)^{2H+1} \\
&+ \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} e^{-\theta j \Delta} e^{\theta\Delta} \left(j\Delta - i\Delta\right)^{2H+1} \\
= &- \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j \Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) \left(i\Delta - j\Delta\right)^{2H+1} \\
&- \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{\theta\Delta} \left(i\Delta\right)^{2H+1} + \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{-\theta i\Delta} e^{2\theta\Delta} \Delta^{2H+1} \\
&+ \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{2\theta\Delta} \left(i\Delta - \Delta\right)^{2H+1} \\
&+ \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta j \Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) \left(j\Delta - i\Delta\right)^{2H+1} \\
&+ \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{-\theta T} e^{2\theta\Delta} \left(n\Delta + \Delta - i\Delta\right)^{2H+1} - \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{-\theta i\Delta} e^{2\theta\Delta} \Delta^{2H+1} \\
&- \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{-\theta T} e^{\theta\Delta} \left(n\Delta - i\Delta\right)^{2H+1} \\
= &- \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta j \Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) \left(i\Delta - j\Delta\right)^{2H+1} \\
&+ \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{\theta\Delta} \left(e^{\theta\Delta} - 1\right) \left(i\Delta\right)^{2H+1} - \frac{\theta}{2\left(2H+1\right)} e^{2\theta\Delta} T^{2H+1} \\
&+ \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta j \Delta} e^{\theta\Delta} \left(1 + e^{2\theta\Delta} - 2e^{\theta\Delta}\right) \left(j\Delta - i\Delta\right)^{2H+1} \\
&- \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{2\theta\Delta} e^{-\theta i\Delta} \Delta^{2H+1} + \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} e^{-\theta T} e^{\theta\Delta} \left(e^{\theta\Delta} - 1\right) \left(i\Delta\right)^{2H+1} \\
&+ \frac{\theta}{2\left(2H+1\right)} e^{-\theta T} e^{\theta\Delta} T^{2H+1} \\
\sim &- \frac{\theta^3}{2\left(2H+1\right)} \int_0^T \int_0^s e^{-\theta t} e^{\theta\Delta} \left(s - t\right)^{2H+1} dt ds \\
&+ \frac{\theta^2}{2\left(2H+1\right)} \int_0^T e^{\theta\Delta} t^{2H+1} dt - \frac{\theta}{2\left(2H+1\right)} e^{2\theta\Delta} T^{2H+1} \\
&+ \frac{\theta^3}{2\left(2H+1\right)} \int_0^T \int_s^T e^{-\theta t} e^{\theta\Delta} \left(t - s\right)^{2H+1} dt ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta^2}{2\,(2H+1)}e^{-\theta T}e^{\theta\Delta}\int_0^T t^{2H+1}dt+o(1)\\
&=-\frac{\theta^3}{2\,(2H+1)}e^{\theta\Delta}\int_0^T\int_0^s e^{\theta t}e^{-\theta s}t^{2H+1}dtds+\frac{\theta^2}{2\,(2H+1)\,(2H+2)}e^{\theta\Delta}T^{2H+2}\\
&-\frac{\theta}{2\,(2H+1)}e^{2\theta\Delta}T^{2H+1}+\frac{\theta^3}{2\,(2H+1)}e^{\theta\Delta}\int_0^T\int_0^{T-s}e^{-\theta t}e^{-\theta s}t^{2H+1}dtds+o(1)
\end{aligned}
\tag{6.32}
$$

Furthermore, the summation of (6.31) and (6.32) can be written as

$$
\begin{aligned}
&\frac{1}{2}e^{\theta\Delta}T^{2H}-\frac{\theta^2e^{\theta\Delta}}{2\,(2H+1)}\int_0^T\left(e^{-\theta t}-e^{-\theta T}\right)e^{\theta t}t^{2H+1}dt\\
&+\frac{\theta^2}{2\,(2H+1)\,(2H+2)}e^{\theta\Delta}T^{2H+2}-\frac{\theta}{2\,(2H+1)}e^{2\theta\Delta}T^{2H+1}\\
&+\frac{\theta^2e^{\theta\Delta}}{2\,(2H+1)}\int_0^T\left(1-e^{-\theta T}e^{\theta t}\right)e^{-\theta t}t^{2H+1}dt+o(1)\\
&=\frac{1}{2}e^{\theta\Delta}T^{2H}-\frac{\theta^2e^{\theta\Delta}}{2\,(2H+1)\,(2H+2)}T^{2H+2}+\frac{\theta^2e^{\theta\Delta}}{2\,(2H+1)}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt\\
&\quad+\frac{\theta^2}{2\,(2H+1)\,(2H+2)}e^{\theta\Delta}T^{2H+2}-\frac{\theta}{2\,(2H+1)}e^{2\theta\Delta}T^{2H+1}\\
&\quad+\frac{\theta^2e^{\theta\Delta}}{2\,(2H+1)}\int_0^T e^{-\theta t}t^{2H+1}dt-\frac{\theta^2e^{\theta\Delta}}{2\,(2H+1)\,(2H+2)}e^{-\theta T}T^{2H+2}+o(1)\\
&=\frac{1}{2}e^{\theta\Delta}T^{2H}+\frac{\theta^2e^{\theta\Delta}}{2\,(2H+1)}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt-\frac{\theta}{2\,(2H+1)}e^{2\theta\Delta}T^{2H+1}+O(1)\\
&=\frac{1}{2}e^{\theta\Delta}T^{2H}+\frac{\theta e^{\theta\Delta}}{2\,(2H+1)}e^{-\theta T}\left[e^{\theta T}T^{2H+1}-(2H+1)\int_0^T e^{\theta t}t^{2H}dt\right]\\
&\quad-\frac{\theta}{2\,(2H+1)}e^{2\theta\Delta}T^{2H+1}+O(1)\\
&\sim\frac{1}{2}e^{\theta\Delta}T^{2H}+\frac{\theta e^{\theta\Delta}}{2\,(2H+1)}T^{2H+1}-\frac{e^{\theta\Delta}}{2}T^{2H}-\frac{\theta}{2\,(2H+1)}e^{2\theta\Delta}T^{2H+1}+O(1)\\
&\sim-\frac{\theta}{2\,(2H+1)}T^{2H+1}\Delta+O(1).
\end{aligned}
$$

Proof of Lemma 6.10. By (6.16), we can see that

$$
\begin{aligned}
&\sum_{i=1}^n\sum_{j=1}^n e^{-\theta T}e^{\theta(i+2j)\Delta}\Bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\\
&+\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t}-e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t} dt \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta-\Delta\right|^{2H}}{\partial t} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t} dt \\
&- \sum_{i=1}^{n}\sum_{j=2}^{n+1} e^{-\theta T} e^{\theta j\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} \left(e^{2\theta\Delta}-1\right) \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t} dt \\
&+ \sum_{i=1}^{n} e^{-\theta T} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-\Delta\right|^{2H}}{\partial t} dt \\
&- \sum_{i=1}^{n} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-n\Delta-\Delta\right|^{2H}}{\partial t} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&= H \sum_{i=0}^{n-1} e^{-\theta T} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} (t+i\Delta)^{2H-1} dt + H \sum_{i=1}^{n} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} (-t+i\Delta)^{2H-1} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} e^{2\theta\Delta} \int_0^\Delta \int_0^\Delta e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta} \left(e^{2\theta\Delta}-1\right) \int_0^\Delta e^{-\theta t} \frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t} dt \\
&= H \sum_{i=0}^{n-1} e^{-\theta T} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} (t+i\Delta)^{2H-1} dt + H \sum_{i=1}^{n} e^{\theta\Delta} \int_0^\Delta e^{-\theta t} (-t+i\Delta)^{2H-1} dt \\
&- \theta \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-j\theta\Delta} e^{3\theta\Delta} \int_0^\Delta \int_0^\Delta e^{\theta t} e^{\theta s} \frac{\partial \frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t} dtds
\end{aligned}
$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt. \tag{6.33}$$

First, straight calculations lead to

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt \\
&=\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt \\
&\quad+\sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t} t^{2H-1}dt \\
&\quad+\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\frac{\partial \frac{1}{2}\left|t+i\Delta-j\Delta\right|^{2H}}{\partial t}dt \\
&=H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\left(t+i\Delta-j\Delta\right)^{2H-1}dt \\
&\quad+\sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} t^{2H-1}dt \\
&\quad-H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} e^{-\theta t}\left(-t+j\Delta-i\Delta\right)^{2H-1}dt \\
&\sim H\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} \left(t+i\Delta-j\Delta\right)^{2H-1}dt \\
&\quad+H\sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta}\left(e^{2\theta\Delta}-1\right)\Delta^{2H} \\
&\quad-H\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\int_0^{\Delta} \left(-t+j\Delta-i\Delta\right)^{2H-1}dt \\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\left[\left(i\Delta-j\Delta+\Delta\right)^{2H}-\left(i\Delta-j\Delta\right)^{2H}\right] \\
&\quad+H\sum_{i=1}^{n} e^{-\theta T} e^{\theta i\Delta}\left(e^{2\theta\Delta}-1\right)\Delta^{2H} \\
&\quad-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\left[\left(j\Delta-i\Delta\right)^{2H}-\left(j\Delta-i\Delta-\Delta\right)^{2H}\right] \\
&=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=0}^{i-1} e^{-\theta T} e^{\theta j\Delta} e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)\left(i\Delta-j\Delta\right)^{2H}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)(i\Delta-j\Delta)^{2H} \\
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H} \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n-1} e^{-\theta T}e^{\theta j\Delta}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H} \\
=&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i-1} e^{-\theta T}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\left(e^{\theta\Delta}-1\right)(i\Delta-j\Delta)^{2H} \\
&+\frac{1}{2}\sum_{i=1}^{n} e^{-\theta T}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)(i\Delta)^{2H} \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{-\theta T}e^{\theta j\Delta}\left(e^{2\theta\Delta}-1\right)\left(e^{\theta\Delta}-1\right)(j\Delta-i\Delta)^{2H} \\
&-\frac{1}{2}\sum_{i=1}^{n} e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)(i\Delta)^{2H}+\frac{1}{2}e^{\theta\Delta}\left(e^{2\theta\Delta}-1\right)T^{2H} \\
\sim&\,\theta^2 e^{-\theta T}\int_0^T\int_0^s e^{\theta t}(s-t)^{2H}\,dtds+\theta e^{-\theta T}e^{\theta\Delta}\int_0^T t^{2H}dt \\
&+\theta^2\int_0^T\int_s^T e^{-\theta T}e^{\theta t}(t-s)^{2H}\,dtds-\theta e^{\theta\Delta}\int_0^T t^{2H}dt+\frac{\theta}{2}e^{\theta\Delta}T^{2H}\Delta \\
=&\,\theta^2 e^{-\theta T}\int_0^T\int_0^s e^{\theta s}e^{-\theta t}t^{2H}dtds+\frac{\theta e^{-\theta T}}{2H+1}T^{2H+1} \\
&+\theta^2\int_0^T\int_0^{T-s} e^{-\theta T}e^{\theta s}e^{\theta t}t^{2H}dtds-\frac{\theta}{2H+1}e^{\theta\Delta}T^{2H+1}+\frac{\theta}{2}e^{\theta\Delta}T^{2H}\Delta \\
=&\,\theta e^{-\theta T}\int_0^T\left(e^{\theta T}-e^{\theta t}\right)e^{-\theta t}t^{2H}dt+\frac{\theta e^{-\theta T}}{2H+1}T^{2H+1} \\
&+\theta\int_0^T\left(e^{\theta T}e^{-\theta t}-1\right)e^{-\theta T}e^{\theta t}t^{2H}dt-\frac{\theta}{2H+1}e^{\theta\Delta}T^{2H+1}+\frac{\theta}{2}e^{\theta\Delta}T^{2H}\Delta \\
\sim&\,\theta\int_0^T e^{-\theta t}t^{2H}dt+\frac{\theta}{2H+1}T^{2H+1}-T^{2H}-\frac{\theta}{2H+1}e^{\theta\Delta}T^{2H+1} \\
&+\frac{\theta}{2}e^{\theta\Delta}T^{2H}\Delta+o(1) \\
=&\,O(1)-\frac{\theta^2}{2H+1}T^{2H+1}\Delta-T^{2H}+\frac{\theta}{2}e^{\theta\Delta}T^{2H}\Delta+o(1). \qquad (6.34)
\end{aligned}
$$

Second, we have

$$
-\theta\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{\theta j\Delta}e^{2\theta\Delta}\int_0^\Delta\int_0^\Delta e^{-\theta t}e^{-\theta s}\frac{\partial\frac{1}{2}\left|t-s+i\Delta-j\Delta\right|^{2H}}{\partial t}dtds
$$

$$
\begin{aligned}
&= -\theta \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2} \left| t - s + i\Delta - j\Delta \right|^{2H}}{\partial t} dtds \\
&\quad - \theta \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial \frac{1}{2} \left| t - s + i\Delta - j\Delta \right|^{2H}}{\partial t} dtds \\
&= -\theta H \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \left(t - s + i\Delta - j\Delta \right)^{2H-1} dtds \\
&\quad + \theta H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} e^{-\theta t} e^{-\theta s} \left(-t + s + j\Delta - i\Delta \right)^{2H-1} dtds \\
&\sim -\theta H \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} \left(t - s + i\Delta - j\Delta \right)^{2H-1} dtds \\
&\quad + \theta H \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \int_0^{\Delta} \int_0^{\Delta} \left(-t + s + j\Delta - i\Delta \right)^{2H-1} dtds \\
&= -\frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \Big[\left(i\Delta - j\Delta + \Delta \right)^{2H+1} - 2 \left(i\Delta - j\Delta \right)^{2H+1} \\
&\qquad\qquad + \left(i\Delta - j\Delta - \Delta \right)^{2H+1} \Big] \\
&\quad + \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \Big[\left(j\Delta - i\Delta + \Delta \right)^{2H+1} - 2 \left(j\Delta - i\Delta \right)^{2H+1} \\
&\qquad\qquad + \left(j\Delta - i\Delta - \Delta \right)^{2H+1} \Big] \\
&= -\frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=0}^{i-2} e^{-\theta T} e^{\theta j \Delta} e^{3\theta\Delta} \left(i\Delta - j\Delta \right)^{2H+1} \\
&\quad + \frac{\theta}{2H+1} \sum_{i=1}^{n} \sum_{j=1}^{i-1} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \left(i\Delta - j\Delta \right)^{2H+1} \\
&\quad - \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=2}^{i-1} e^{-\theta T} e^{\theta j \Delta} e^{\theta\Delta} \left(i\Delta - j\Delta \right)^{2H+1} \\
&\quad + \frac{\theta}{2\left(2H+1\right)} \sum_{i=1}^{n} \sum_{j=i+2}^{n+1} e^{-\theta T} e^{\theta j \Delta} e^{\theta\Delta} \left(j\Delta - i\Delta \right)^{2H+1} \\
&\quad - \frac{\theta}{2H+1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} e^{-\theta T} e^{\theta j \Delta} e^{2\theta\Delta} \left(j\Delta - i\Delta \right)^{2H+1} \\
&\quad + \frac{\theta}{2H+1} \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} e^{-\theta T} e^{\theta j \Delta} e^{3\theta\Delta} \left(j\Delta - i\Delta \right)^{2H+1}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(i\Delta-j\Delta\right)^{2H+1}\\
&\quad-\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}e^{-\theta T}e^{3\theta\Delta}\left(i\Delta\right)^{2H+1}\\
&\quad+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}e^{-\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\Delta^{2H+1}+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}e^{-\theta T}e^{2\theta\Delta}\left(i\Delta-\Delta\right)^{2H+1}\\
&\quad+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-\theta T}e^{\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(j\Delta-i\Delta\right)^{2H+1}\\
&\quad+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}e^{2\theta\Delta}\left(n\Delta+\Delta-i\Delta\right)^{2H+1}\\
&\quad-\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}e^{-\theta T}e^{\theta i\Delta}e^{2\theta\Delta}\Delta^{2H+1}\\
&\quad-\frac{\theta}{2(2H+1)}\sum_{i=1}^{n}e^{3\theta\Delta}\left(n\Delta-i\Delta\right)^{2H+1}\\
&= -\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=1}^{i-1}e^{-\theta T}e^{\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(i\Delta-j\Delta\right)^{2H+1}\\
&\quad-\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}e^{-\theta T}e^{2\theta\Delta}\left(e^{\theta\Delta}-1\right)\left(i\Delta\right)^{2H+1}-\frac{\theta}{2\left(2H+1\right)}e^{-\theta T}e^{2\theta\Delta}T^{2H+1}\\
&\quad+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}\sum_{j=i+1}^{n}e^{-\theta T}e^{\theta j\Delta}e^{\theta\Delta}\left(1+e^{2\theta\Delta}-2e^{\theta\Delta}\right)\left(j\Delta-i\Delta\right)^{2H+1}\\
&\quad+\frac{\theta}{2\left(2H+1\right)}\sum_{i=1}^{n}e^{2\theta\Delta}\left(i\Delta\right)^{2H+1}-\frac{\theta}{2(2H+1)}\sum_{i=1}^{n-1}e^{3\theta\Delta}\left(i\Delta\right)^{2H+1}\\
&\sim -\frac{\theta^3}{2\left(2H+1\right)}e^{-\theta T}e^{\theta\Delta}\int_0^T\int_0^s e^{\theta t}\left(s-t\right)^{2H+1}dtds\\
&\quad-\frac{\theta^2}{2\left(2H+1\right)}e^{-\theta T}e^{2\theta\Delta}\int_0^T t^{2H+1}dt+o(1)\\
&\quad+\frac{\theta^3}{2\left(2H+1\right)}e^{\theta\Delta}\int_0^T\int_s^T e^{-\theta T}e^{\theta t}\left(t-s\right)^{2H+1}dtds\\
&\quad-\frac{\theta^2}{2\left(2H+1\right)}e^{2\theta\Delta}\int_0^T t^{2H+1}dt+\frac{\theta}{2\left(2H+1\right)}e^{3\theta\Delta}T^{2H+1}\\
&= -\frac{\theta^3}{2\left(2H+1\right)}e^{-\theta T}e^{\theta\Delta}\int_0^T\int_0^s e^{\theta s}e^{-\theta t}t^{2H+1}dtds\\
&\quad-\frac{\theta^2}{2\left(2H+1\right)\left(2H+2\right)}e^{-\theta T}e^{2\theta\Delta}T^{2H+2}+o(1)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\theta^3}{2(2H+1)}e^{\theta\Delta}\int_0^T\int_0^{T-s}e^{-\theta T}e^{\theta s}e^{\theta t}t^{2H+1}dtds\\
&-\frac{\theta^2}{2(2H+1)(2H+2)}e^{2\theta\Delta}T^{2H+2}+\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}\\
=&-\frac{\theta^2}{2(2H+1)}e^{-\theta T}e^{\theta\Delta}\int_0^T\left(e^{\theta T}-e^{\theta t}\right)e^{-\theta t}t^{2H+1}dt+o(1)\\
&+\frac{\theta^2}{2(2H+1)}e^{\theta\Delta}\int_0^T\left(e^{\theta T}e^{-\theta t}-1\right)e^{-\theta T}e^{\theta t}t^{2H+1}dt\\
&-\frac{\theta^2}{2(2H+1)(2H+2)}e^{2\theta\Delta}T^{2H+2}+\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}\\
=&-\frac{\theta^2}{2(2H+1)}e^{\theta\Delta}\int_0^T e^{-\theta t}t^{2H+1}dt+o(1)\\
&-\frac{\theta^2}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\int_0^T e^{\theta t}t^{2H+1}dt+\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}\\
\sim& O(1)-\frac{\theta}{2(2H+1)}e^{\theta\Delta}e^{-\theta T}\left[e^{\theta T}T^{2H+1}-(2H+1)\int_0^T e^{\theta t}t^{2H}dt\right]+\frac{\theta}{2(2H+1)}e^{3\theta\Delta}T^{2H+1}\\
\sim&\frac{\theta^2}{2H+1}e^{\theta\Delta}T^{2H+1}\Delta+\frac{1}{2}e^{\theta\Delta}T^{2H}+O(1). \qquad (6.35)
\end{aligned}
$$

Combining (6.33)-(6.35) and (6.31), we can obtain (6.28).

6.3 Proofs of Lemma 6.11-Lemma 6.14

Proof of Lemma 6.11. Case 1: $H\in(0,1/2)$. By Lemma 6.7 and $T^{2H+1}\Delta\to0$, we can get

$$
\begin{aligned}
&\mathbb{E}\left(\widetilde{U}_T\widetilde{X}_T\right)\\
&=\sigma^2\sum_{i=1}^n\sum_{j=1}^n e^{-\theta T}e^{2\theta i\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta}e^{-\theta s}dB_s^H\int_{(j-1)\Delta}^{j\Delta}e^{-\theta s}dB_s^H\right)\\
&=\sigma^2\sum_{i=1}^n\sum_{j=1}^n e^{-\theta T}e^{2\theta i\Delta}\Bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\\
&\qquad+\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t}-e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)\\
&\to0. \qquad (6.36)
\end{aligned}
$$

Case 2: $H=1/2$. By (6.7) and (6.8), we have

$$
\mathbb{E}\left(\widetilde{U}_T\widetilde{X}_T\right)=\mathbb{E}\left(\sigma^2\sum_{i=1}^n e^{-\theta T}e^{2\theta i\Delta}\left(\int_{(i-1)\Delta}^{i\Delta}e^{-\theta s}dB_s\right)^2\right)
$$

$$
\begin{aligned}
&= \sigma^2 \sum_{i=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s} ds \\
&= \frac{\sigma^2}{2\theta} \sum_{i=1}^{n} e^{-\theta T} \left(e^{2\theta\Delta} - 1 \right) \\
&\sim \sigma^2 T e^{-\theta T} \\
&\to 0\,,
\end{aligned} \tag{6.37}
$$

as $T \to \infty$ and $\Delta \to 0$.

Case 3: $H \in (1/2, 1)$. From (6.7), (6.8), Lemma 6.2 and $T^{2H}\Delta \to 0$, we can write the following result immediately,

$$
\begin{aligned}
&\mathbb{E}\left(\widetilde{U}_T \widetilde{X}_T \right) \\
&= \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right) \\
&= \sigma^2 \alpha_H \sum_{i=1}^{n} \sum_{j=1}^{n} e^{-\theta T} e^{2\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} \int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)} |s-r|^{2H-2} ds dr \\
&\to 0\,.
\end{aligned} \tag{6.38}
$$

Using (6.36)-(6.38), we obtain (6.29).

Proof of Lemma 6.12. Case 1: $H \in (0, 1/2)$. By Lemma 6.9 and $T^{H+1}\Delta \to 0$, it holds that

$$
\begin{aligned}
&\mathbb{E}\left(\Xi_T \widetilde{X}_T \right) \\
&= \sigma T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}\left(e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s} dB_s^H \right) \\
&= \sigma T^{-H} \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta i\Delta} \Bigg(\theta \int_{(j-1)\Delta}^{j\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} e^{-\theta s} \frac{\partial R(t,s)}{\partial t} dt ds \\
&\qquad + \int_{(i-1)\Delta}^{i\Delta} e^{-\theta t} \left(e^{-\theta j\Delta} \frac{\partial R(t, j\Delta)}{\partial t} - e^{-\theta(j-1)\Delta} \frac{\partial R(t,(j-1)\Delta)}{\partial t} \right) dt \Bigg) \\
&\to 0\,,
\end{aligned} \tag{6.39}
$$

Case 2: $H = 1/2$. Straightforward calculations lead to

$$
\begin{aligned}
\mathbb{E}\left(\Xi_T \widetilde{X}_T \right) &= \sigma T^{-1/2} \sum_{i=1}^{n} e^{\theta i\Delta} \mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s} dB_s \right)^2 \\
&= \sigma T^{-1/2} \sum_{i=1}^{n} e^{\theta i\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{-2\theta s} ds
\end{aligned}
$$

$$=\sigma\frac{T^{-1/2}}{2\theta}\frac{\left(e^{2\theta\Delta}-1\right)\left(1-e^{-\theta T}\right)}{e^{\theta\Delta}-1}\to 0\,, \tag{6.40}$$

as $T\to\infty$ and $\Delta\to 0$.

Case 3: $H\in(1/2,1)$. By Lemma 6.4 and $T^H\Delta\to 0$, it holds

$$\begin{aligned}
&\mathbb{E}\left(\Xi_T\widetilde{X}_T\right)\\
&=\sigma T^{-H}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left(e^{\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta}e^{-\theta s}dB_s^H\int_{(j-1)\Delta}^{j\Delta}e^{-\theta s}dB_s^H\right)\\
&=\sigma\alpha_H T^{-H}\sum_{i=1}^{n}\sum_{j=1}^{n}e^{\theta i\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta}e^{-\theta(s+r)}|s-r|^{2H-2}dsdr\\
&\to 0\,,
\end{aligned}$$

which together with (6.39) and (6.40) completes the proof of this lemma.

Proof of Lemma 6.13. Case 1: $H\in(0,1/2)$. By Lemma 6.10 and $T^{H+1}\Delta\to 0$, it holds that

$$\begin{aligned}
&\mathbb{E}\left(\Xi_T\sum_{j=1}^{n}e^{-\theta(n-j)\Delta}\epsilon_{j\Delta}\right)\\
&=T^{-H}\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-\theta T}e^{\theta(i+2j)\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta}e^{-\theta s}dB_s^H\int_{(j-1)\Delta}^{j\Delta}e^{-\theta s}dB_s^H\right)\\
&=T^{-H}\sum_{i=1}^{n}\sum_{j=1}^{n}e^{-\theta T}e^{\theta(i+2j)\Delta}\Bigg(\theta\int_{(j-1)\Delta}^{j\Delta}\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}e^{-\theta s}\frac{\partial R(t,s)}{\partial t}dtds\\
&\qquad\qquad+\int_{(i-1)\Delta}^{i\Delta}e^{-\theta t}\left(e^{-\theta j\Delta}\frac{\partial R(t,j\Delta)}{\partial t}-e^{-\theta(j-1)\Delta}\frac{\partial R(t,(j-1)\Delta)}{\partial t}\right)dt\Bigg)\\
&\to 0.
\end{aligned} \tag{6.41}$$

Case 2: $H=1/2$. It follows that

$$\begin{aligned}
\mathbb{E}\left(\Xi_T\widetilde{U}_T\right)=&\sigma\mathbb{E}\left(T^{-1/2}\sum_{i=1}^{n}\epsilon_{i\Delta}\sum_{j=1}^{n}e^{-\theta(n-j)\Delta}\epsilon_{j\Delta}\right)\\
=&\sigma T^{-1/2}\sum_{i=1}^{n}e^{-\theta T}e^{3\theta i\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta}e^{-\theta s}dW_s\right)^2\\
=&\sigma\frac{e^{\theta\Delta}}{2\theta}T^{-1/2}\frac{e^{2\theta\Delta}-1}{e^{\theta\Delta}-1}\left(1-e^{-\theta T}\right)\\
=&O(T^{-1/2})\,.
\end{aligned} \tag{6.42}$$

Case 3: $H \in (1/2, 1)$. By Lemma 6.5 and $T^{H+1}\Delta \to 0$, we have

$$
\begin{aligned}
&\mathbb{E}\left(\Xi_T \sum_{j=1}^{n} e^{-\theta(n-j)\Delta}\epsilon_{j\Delta}\right) \\
&= T^{-H}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{\theta(i+2j)\Delta}\mathbb{E}\left(\int_{(i-1)\Delta}^{i\Delta} e^{-\theta s}dB_s^H \int_{(j-1)\Delta}^{j\Delta} e^{-\theta s}dB_s^H\right) \\
&= \alpha_H T^{-H}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-\theta T}e^{\theta(i+2j)\Delta}\int_{(i-1)\Delta}^{i\Delta}\int_{(j-1)\Delta}^{j\Delta} e^{-\theta(s+r)}|s-r|^{2H-2}dsdr \\
&\to 0. \qquad (6.43)
\end{aligned}
$$

Together with (6.41), (6.42) and (6.43), we complete the proof of this lemma.

Proof of Lemma 6.14. First, using (2.6), we can write

$$
X_{(i-1)\Delta} = X_0 e^{\theta(i-1)\Delta} + \frac{\mu}{\theta}\left(e^{\theta(i-1)\Delta} - 1\right) + \sigma e^{\theta(i-1)\Delta}\int_0^{(i-1)\Delta} e^{-\theta s}dB_s^H. \qquad (6.44)
$$

It is obvious that

$$
\sum_{i=1}^{n} X_{(i-1)\Delta} - \left[\left(X_0 + \frac{\mu}{\theta}\right)\frac{1-e^{\theta T}}{1-e^{\theta\Delta}} - \frac{\mu}{\theta}n\right] \sim \mathcal{N}\left(0, \sigma^2_{\sum_{i=1}^{n} X_{(i-1)\Delta}}\right), \qquad (6.45)
$$

where $\sigma^2_{\sum_{i=1}^{n} X_{(i-1)\Delta}} = \mathbb{E}\left(\sum_{i=1}^{n}\sigma e^{\theta(i-1)\Delta}\int_0^{(i-1)\Delta} e^{-\theta s}dB_s^H\right)^2$.

Case 1: $H = 1/2$. We can deduce that, as $n \to \infty$

$$
\begin{aligned}
&\sigma^2_{\sum_{i=1}^{n} X_{(i-1)\Delta}} \\
&= \sigma^2\sum_{i=1}^{n} e^{2\theta(i-1)\Delta}\int_0^{(i-1)\Delta} e^{-2\theta s}ds + 2\sigma^2\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta(i-1)\Delta}e^{\theta(j-1)\Delta}\int_0^{(i-1)\Delta} e^{-2\theta s}ds \\
&= \frac{\sigma^2}{2\theta}\sum_{i=1}^{n} e^{2\theta(i-1)\Delta}\left(1-e^{-2\theta(i-1)\Delta}\right) + \frac{\sigma^2}{\theta}\sum_{i=1}^{n}\sum_{j=i+1}^{n} e^{\theta(i-1)\Delta}e^{\theta(j-1)\Delta}\left(1-e^{-2\theta(i-1)\Delta}\right) \\
&= \frac{\sigma^2}{2\theta}\left(\frac{1-e^{2\theta T}}{1-e^{2\theta\Delta}} - n\right) + \frac{\sigma^2}{\theta\left(e^{\theta\Delta}-1\right)}\left[e^{\theta T}\left(\frac{1-e^{\theta T}}{1-e^{\theta\Delta}} - \frac{1-e^{-\theta T}}{1-e^{-\theta\Delta}}\right)\right. \\
&\qquad\qquad \left. - \frac{\left(1-e^{2\theta T}\right)e^{\theta\Delta}}{1-e^{2\theta\Delta}} + ne^{\theta\Delta}\right] \\
&\le Ce^{2\theta T}\Delta^{-2}. \qquad (6.46)
\end{aligned}
$$

Case 2: $H \in (0, 1/2) \cup (1/2, 1)$. Using the Cauchy-Schwarz inequality, we have

$\sigma^2_{\sum_{i=1}^{n} X_{(i-1)\Delta}}$

$$
\begin{aligned}
&= \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta(i-1)\Delta} e^{\theta(j-1)\Delta} \mathbb{E} \left(\int_0^{(i-1)\Delta} e^{-\theta s} dB_s^H \int_0^{(j-1)\Delta} e^{-\theta s} dB_s^H \right) \\
&\leq \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta(i-1)\Delta} e^{\theta(j-1)\Delta} \mathbb{E} \left| \int_0^{(i-1)\Delta} e^{-\theta s} dB_s^H \int_0^{(j-1)\Delta} e^{-\theta s} dB_s^H \right| \\
&\leq \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta(i-1)\Delta} e^{\theta(j-1)\Delta} \left(\mathbb{E} \left(\int_0^{(i-1)\Delta} e^{-\theta s} dB_s^H \right)^2 \right)^{1/2} \left(\mathbb{E} \left(\int_0^{(j-1)\Delta} e^{-\theta s} dB_s^H \right)^2 \right)^{1/2} \\
&\leq \sigma^2 \sum_{i=1}^{n} \sum_{j=1}^{n} e^{\theta(i-1)\Delta} e^{\theta(j-1)\Delta} \left(\mathbb{E} \left(\int_0^{\infty} e^{-\theta s} dB_s^H \right)^2 \right) \\
&= \sigma^2 \frac{H\Gamma(2H)}{\theta^{2H}} \left(\frac{1-e^{\theta T}}{1-e^{\theta \Delta}} \right)^2 \leq C e^{2\theta T} \Delta^{-2}. \qquad (6.47)
\end{aligned}
$$

Using (6.46) and (6.47), for $H \in (0,1)$, we have

$$
\sigma^2_{\sum_{i=1}^n X_{(i-1)\Delta}} \leq C e^{2\theta T} \Delta^{-2}. \qquad (6.48)
$$

Note that

$$
\begin{aligned}
&T^m \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \\
&= T^m \frac{1}{n} e^{-\theta T} \sigma_{\sum_{i=1}^n X_{(i-1)\Delta}} \frac{1}{\sigma_{\sum_{i=1}^n X_{(i-1)\Delta}}} \left\{ \sum_{i=1}^{n} X_{(i-1)\Delta} - \left[\left(X_0 + \frac{\mu}{\theta} \right) \frac{1-e^{\theta T}}{1-e^{\theta \Delta}} - \frac{\mu}{\theta} n \right] \right\} \\
&\quad + T^m \frac{1}{n} e^{-\theta T} \left[\left(X_0 + \frac{\mu}{\theta} \right) \frac{1-e^{\theta T}}{1-e^{\theta \Delta}} - \frac{\mu}{\theta} n \right] \qquad (6.49)
\end{aligned}
$$

and by (6.48), we can get

$$
\begin{aligned}
T^m \frac{1}{n} e^{-\theta T} \sigma_{\sum_{i=1}^n X_{(i-1)\Delta}} &\leq C T^{m-1}, \\
T^m \frac{1}{n} e^{-\theta T} \left[\left(X_0 + \frac{\mu}{\theta} \right) \frac{1-e^{\theta T}}{1-e^{\theta \Delta}} - \frac{\mu}{\theta} n \right] &\leq C T^{m-1} - C T^m e^{-\theta T}.
\end{aligned}
$$

Therefore, we have

$$
\mathbb{P} \left(\left| T^m \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \right| \geq \epsilon \right) \leq 2 \exp \left\{ -C \epsilon^2 T^{2-2m} \right\},
$$

which implies that

$$
\sum_{n=1}^{\infty} \mathbb{P} \left(\left| T^m \frac{1}{n} e^{-\theta T} \sum_{i=1}^{n} X_{(i-1)\Delta} \right| \geq \epsilon \right) < \infty.
$$

By the Borel-Cantelli lemma, we complete the proof of this lemma.