Robust testing for explosive behavior with strongly dependent errors

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Abstract

A heteroskedasticity-autocorrelation robust (HAR) test statistic is proposed to test for the presence of explosive roots in financial or real asset prices when the equation errors are strongly dependent. Limit theory for the test statistic is developed and extended to heteroskedastic models. The new test has stable size properties unlike conventional test statistics that typically lead to size distortion and inconsistency in the presence of strongly dependent equation errors. The new procedure can be used to consistently time-stamp the origination and termination of an explosive episode under similar conditions of long memory errors. Simulations are conducted to assess the finite sample performance of the proposed test and estimators. An empirical application to the S&P 500 index highlights the usefulness of the proposed procedures in practical work.

1 Introduction

The standard no-arbitrage condition for the determination of the price P_t of a financial or real asset at time t implies that

$$P_t = \frac{1}{1+R}\mathbb{E}_t\left(P_{t+1} + D_{t+1}\right), \tag{1}$$

where R, $\mathbb{E}_t$, and D_t denote the discount rate, expectation conditional on information available at time t, and fundamentals (such as the dividend for a stock or the rental income from a house) at time t. Solving (1) by forward substitution leads to the equation $P_t = P_t^F + B_t$, where $P_t^F = \sum_{i=1}^{\infty}\left(\frac{1}{1+R}\right)^i \mathbb{E}_t\left(D_{t+i}\right)$ is the fundamental price and $B_t = \frac{1}{1+R}\mathbb{E}_t(B_{t+1})$ is the bubble component. B_t is unrelated to fundamentals and emerges as part of the general solution to (1), whereas P_t^F is the particular solution driven by fundamentals measured by the sum of the discounted expectations of future dividends if P_t is the price of a stock. Under the transversality condition $\lim_{T\to\infty}\left(1+R\right)^{-T}\mathbb{E}_t P_{t+T} = 0$, the general solution $B_t = 0$ and $P_t = P_t^F$. When the transversality condition fails, $B_t \neq 0$ and is an explosive process since $1 + R > 1$ for $R > 0$. Explosive behavior in B_t ensures that P_t is an explosive process even when fundamentals P_t^F are themselves not explosive.

In practical work, many recent empirical studies have confirmed evidence of episodic explosive behavior in the price-fundamental ratio using bubble detection techniques; e.g., Phillips et al. (2015a) (hereafter PSYa) and Pedersen and Schütte (2020). A natural approach to bubble detection is to employ a right-tailed unit root test, initially employed by Diba and Grossman (1988) and subsequently used in sequential testing methods by Phillips et al. (2011) (hereafter PWY) and Phillips and Yu (2011) (hereafter PY) that provide consistent estimates of bubble initiation and termination dates. PSYa extended that work to allow for the detection of multiple bubbles by means of sequential evolving search methods for episodic bubbles in time series. Harvey et al. (2016, 2018, 2019) provided further extensions of these methods by allowing for models with heteroskedastic errors and Pedersen and Schütte (2020) emphasized the importance of treating autocorrelated errors in the small sample procedures that are inevitably involved in sequential and evolving testing algorithms. Readers are referred to Phillips and Shi (2020) and Shi and Phillips (2021) for recent overviews of these methods, including instrumental variable methods for calculating fundamentals, bootstrap methods for controlling the multiplicity issues that affect sequential testing, and algorithms for practical implementation.

The simplest model for explosive behavior testing has the following first-order autoregressive (AR) form

$$y_t = \rho y_{t-1} + \epsilon_t, \ y_0 = O_p(1), \ \text{with } \epsilon_t \overset{iid}{\sim} (0, \sigma^2), t = 1, ..., n. \tag{2}$$

Under normal market conditions, time series y_t of asset prices typically follow random wandering behavior. Correspondingly, the common null hypothesis for such conditions is that y_t is a random walk process with autoregressive coefficient $\rho = 1$. Under the alternative hypothesis of bubble behavior originating from some point of initialization in the sample, the process y_t displays explosive behavior with a fixed coefficient $\rho > 1$ or mildly explosive behavior with locally defined coefficient $\rho = 1 + \delta/n^\alpha$, $\delta > 0$, and $\alpha \in (0, 1)$, as in Phillips and Magdalinos (2007). Against both these alternatives, right-tailed unit root tests have finite sample power and are consistent as $n \to \infty$ (PWY). This framework provides the basis for more complex versions of tests for explosive behavior and bubbles that are better suited to the data in financial and real asset markets.

Pedersen and Schütte (2020) allowed for weakly dependent errors in their application, noting that failure to do so led to considerable size distortion in bubble testing algorithms, particularly those that use recursive sample methods. The present paper is motivated by similar concerns and extends the analysis of earlier work by considering a generating mechanism such as (2) in which the errors follow a strongly dependent process. The phenomenon of strong dependence is widespread in economic and financial time series. Cheung (1993) and Baillie et al. (1996) found empirical evidence of strong dependence in exchange rates. Christensen and Nielsen (2007), Andersen et al. (2003) and Ohanissian et al. (2008) provided evidence of strong dependence in volatilities of stock returns and exchange rate returns; and empirical studies by Gil-Alana et al. (2014) and Barros et al. (2014) showed similar evidence of strong dependence in housing prices in European and US cities. More recently, Chevillon and Mavroeidis (2017) utilized statistical learning methods in long memory analysis, finding strong dependence in the US monthly CPI inflation rates.

Consider the following unit root process driven by long memory errors u_t

$$\begin{cases} y_t &= y_{t-1} + u_t, \ t = 1, ..., n \\ u_t &= \Delta_+^{-d}\epsilon_t, \ d > 0, \ \epsilon_t \overset{iid}{\sim} (0, \sigma^2), \ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty, \ \delta > 0 \end{cases}, \tag{3}$$

where the operator Δ_+^{-d} associated with the memory parameter d is defined by

$$\Delta_+^{-d}\epsilon_t = (1-L)^{-d}1(t \geq 1)\epsilon_t = (1-L)^{-d}\epsilon_t 1(t \geq 1) = \sum_{j=0}^{t-1} \frac{(d)_j}{j!}\epsilon_{t-j}, \tag{4}$$

with $(d)_j = \Gamma(d+j)/\Gamma(d)$ and initialization at time $t = 0$. The moving average coefficients $(d)_j/j! =: c_{dj}$ in (4) are positive when $d > 0$ for all j. By standard gamma function asymptotic expansion $c_{dj} = \frac{1}{\Gamma(d)j^{1-d}}\{1 + O\left(\frac{1}{j}\right)\} \overset{a}{\sim} \frac{1}{\Gamma(d)j^{1-d}}$ as $j \to \infty$. If $d = 0$ in (3), model (3) reduces to (2) with $\rho = 1$. When $d > 0$, there is strong dependence in the sequence u_t, commonly written as $u_t \sim FI(d)$, so that u_t is fractionally integrated (FI) of order d or with long memory parameter d.[1] Since the first difference of y_t is $FI(d)$, it follows that $y_t \sim FI(d_y)$ with $d_y = 1 + d$.

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[1] More precisely, u_t is a Type II FI time series with fractional order d – see Marinucci and Robinson
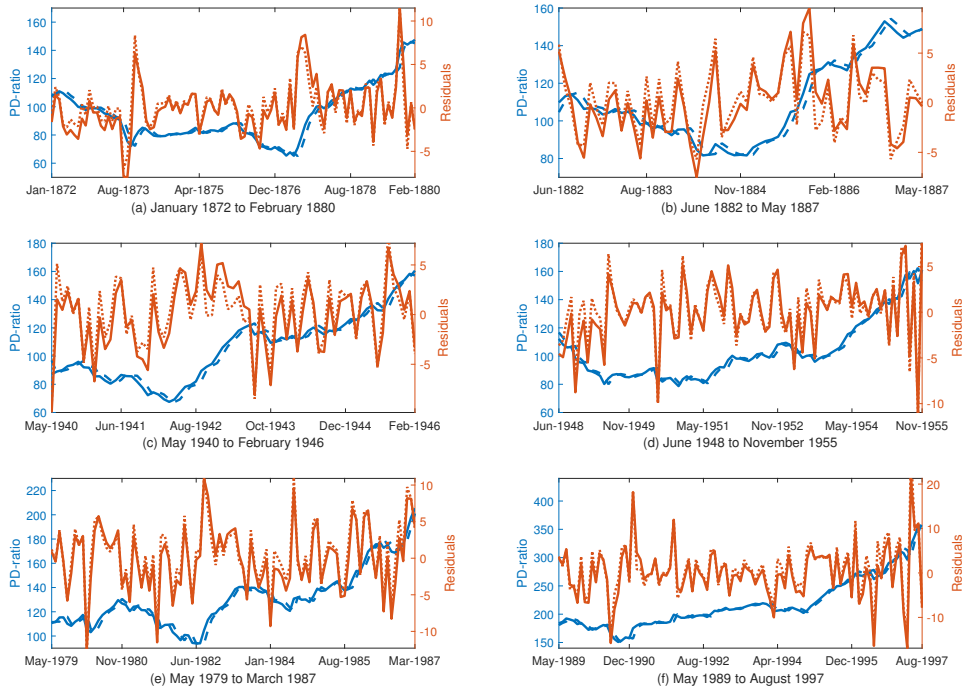
Figure 1: The unbroken and dashed blue lines are time series plots of the actual and fitted monthly PD ratios (left axis), where the fitted value is obtained by an AR(1) regression with an intercept. The unbroken and dotted red lines are time series plots of residuals from a fitted least squares autoregression and exact local Whittle (ELW) estimation of the long memory parameter (right axis). See the text for further details.

Although data y_t generated by (3) follow a unit root process driven by strongly dependent errors, it is not uncommon to observe realizations that form time paths with episodes mimicking an explosive trajectory. Solving (3) gives the partial sum representation $y_t = \sum_{i=1}^{t} u_i + y_0$. Since u_t is strongly dependent with representation (4) and moving average coefficients $c_{dj} > 0$ for all j, it is evident that any large positive shock ε_{t-j} provides a sustained positive impact on y_t due to strong dependence. Since y_t is the cumulative effect of such inputs, a succession of positive shocks produces an upward trend in the process that can mimic an explosive time series.

A standard procedure for testing explosive behavior is to fit an AR model such as (2) and employ a right-tailed unit root t-test. In this event, if data is generated according to (3), it is well known that the t-statistic diverges as $n \to \infty$ (see Sowell (1990)), so that a conventional right-tailed test will inevitably lead to rejection of a unit root null when n is large. Thus, for data from a unit root process with long memory innovations such as (3),

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(1999) and Davidson and Hashimzade (2009) for further discussion of this terminology and Type I FI time series together with definitions of corresponding fractional Brownian motion processes.

application of right-tailed tests that ignore strong dependence in the innovations when such dependence is present will lead to the mistaken conclusion of explosive behavior and spurious detection of a rational bubble in the data.

To showcase the empirical relevance of this problem, Figure 1 plots historical data for the monthly price-dividend (PD) ratio of the S&P 500 in the unbroken blue line, following PSYa.[2] The panels shown in Figure 1 cover six periods: (a) January 1872 to February 1880; (b) June 1882 to May 1887; (c) May 1940 to February 1946; (d) June 1948 to November 1955; (e) May 1979 to March 1987 and (f) May 1989 to August 1997. Each period contains a trajectory for which there is some apparent exuberance in the S&P 500 market. Under the assumption that the generating mechanism is (2) with errors that are not strongly dependent, autoregressions with an intercept are fitted for each subperiod and Dickey-Fuller t-statistics (denoted DF_n) are calculated and reported in Table 1. The results show rejection of a unit root at the 1% level for each of subperiods, indicating strong statistical evidence for a rational bubble in each case.[3]

Table 1: Right-tailed unit root tests for the S&P 500 PD ratio, exact local Whittle estimates $\hat{d}$ of d, and corresponding confidence intervals

Sampling Period	DF_n	$\hat{d}$	90% CI	95% CI
(a) Jan 1872 to Feb 1880	1.35	0.24	(0.05,0.43)	(0.02,0.46)
(b) Jun 1882 to May 1887	0.66	0.32	(0.10,0.54)	(0.06,0.58)
(c) May 1940 to Feb 1946	1.38	0.34	(0.13,0.55)	(0.09,0.59)
(d) Jun 1948 to Nov 1955	1.70	0.29	(0.10,0.48)	(0.06,0.52)
(e) May 1979 to Mar 1987	1.73	0.21	(0.02,0.40)	(-0.01,0.43)
(f) May 1989 to Apr 1998	2.78	0.24	(0.05,0.42)	(0.02,0.46)

PSYa found evidence of rational bubbles in the S&P 500 for the following periods: the long-depression period (October 1878 to April 1880), the great crash episode (November 1928 to October 1929), the postwar boom in 1954 (January 1955 to April 1956), Black Monday in October 1987 (June 1986 to September 1987), and the dot-com bubble (November 1995 to August 2001). Our sampling periods (a), (d), (e) and (f) overlap four of the PSYa estimated rational bubble periods, re-affirming the evidence for market exuberance in these periods.[4]

If the data were assumed to be fractionally integrated as in (3) the composite long memory parameter d_y could be estimated directly and the corresponding memory parameter d of the innovations could be deduced. Accordingly, the exact local Whittle (ELW)

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[2]Monthly price-dividend ratio measurements are shown on the left axis. The figure also plots the fitted monthly price-dividend ratio (the blue dashed line), obtained from least squares (LS) autoregression with an intercept on the left axis, and the residuals obtained from that regression and the LM model with memory parameter fitted by the exact local Whittle (ELW) method (shown by the red unbroken line and the red dashed line) on the right axis.

[3]The asymptotic right-tail 95% critical value for the standard t statistic for the presence of a unit root is 0.60 (Table B.6 in Hamilton, 1994).

[4]The periods where statistical significance of a positive LM parameter d is not established are not reported here.

procedure (Shimotsu and Phillips, 2005) was used to estimate d_y, deduce d, and test for short memory ($d = 0$) in the innovations against strong dependence ($d > 0$).

In Figure 1, the red solid line is the residual plot of $\{\hat{\varepsilon}_{t,LS}\}_{t=2}^{n}$ obtained from least squares (LS) autoregression and the red dotted line is the residual plot of $\{\hat{\varepsilon}_{t,ELW}\}_{t=2}^{n}$ obtained by ELW estimation[5] for each of the six sampling periods. Note that an exuberance trajectory can be generated either by an explosive AR model with an error sequence $\{\hat{\varepsilon}_{t,LS}\}_{t=2}^{n}$ or by a fractionally integrated time series (3) with $d = \hat{d}$ and the error sequence $\{\hat{\varepsilon}_{t,ELW}\}_{t=2}^{n}$. Table 1 reports the ELW estimate of d and its 90% and 95% confidence intervals for each subperiod.[6] In all cases, $\hat{d}$ is positive and the null hypothesis of short memory is rejected against the alternative of strong dependence at either the 5% or 10% level, supporting evidence of long memory in the innovations u_t in the sampling periods. These findings suggest that a plausible alternative model for the generating mechanism is a unit root model with strong dependent errors (3) instead of the explosive model indicated by the results of unit root testing. Hence, empirical rejection of a unit root null in favor of an explosive process may arise from the presence of strong dependence in the errors, raising the possibility of spurious inference concerning the presence of a rational bubbles.[7]

Motivated by these empirical findings and the potential implications for bubble detection with standard right-tail unit root tests, the present paper seeks to address the problem of spurious test outcomes from right-tail tests. We propose to modify standard test procedures by constructing a heteroskedasticity-autocorrelation robust (HAR) statistic which controls performance so that the test statistic does not diverge and has a well defined limit distribution under the null test but diverges and is consistent under the alternative of an explosive or mildly explosive root. The new HAR test has asymptotic discriminatory power in detecting explosive time series even in models driven by long memory errors. The test can be implemented in recursive algorithms to consistently timestamp origination and termination dates of episodic bubbles. The modified test statistic is constructed in the same spirit as the use of HAR statistics to perform valid testing in potentially spurious relationships (Sun, 2004; Phillips et al., 2019).

The remainder of this paper is organized as follows. Section 2 briefly reviews traditional right-tail unit root tests for explosiveness and procedures for date stamping explosive periods in data. Section 3 introduces the model with strongly dependent errors, proposes the new test, and derives asymptotic theory under the null. Section 4 examines asymptotic properties under explosive alternatives. New estimators of bubble origination and termination dates in recursive applications of the new statistic are given in Section 5. Section 6 discusses how to conduct tests in the presence of time-varying volatilities. Simulations ex-

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[5]These residuals are estimated by computing $\hat{\varepsilon}_{t,ELW} = (1 - L)_+^{-\hat{d}} \Delta y_t$.

[6]Shimotsu and Phillips (2005) show that the asymptotic distribution of the ELW estimate of d is given by $\sqrt{m}(\hat{d} - d) \xrightarrow{d} \mathcal{N}(0, 1/4)$ using bandwidth $m = n^\delta$, where n is the sample size and δ is a bandwidth parameter that controls the number of periodogram ordinates near the origin that are used in estimation. The setting $\delta = 0.65$ was used in the computations reported in Table 1.

[7]An analysis based on a larger sample size is conducted in Appendix, confirming strong dependence in the residuals using the ELW estimator and explosiveness in the data using DF_n tests.

ploring finite sample properties of the procedures are reported in Section 7. An empirical study using the S&P 500 index is conducted in Section 8 where the results are compared with earlier findings that employ standard test procedures. Section 9 concludes. Proofs of the main results in the paper and some related lemmas are given in the Appendix. An Online Supplement provides additional technical and simulation results, including the development of a sup HAR statistic, asymptotic theory and finite sample analysis of the sup HAR statistic and the proofs of the lemmas. Notation is standard with $\xrightarrow{p}, \xrightarrow{d}, \xrightarrow{as}, \Rightarrow,$ $\overset{a}{\sim}, \lfloor \cdot \rfloor, :=$ and $=:$ denoting convergence in probability, convergence in distribution, almost sure convergence, weak convergence on the relevant probability space, asymptotic equivalence, the floor function, and definitional equality.

2   A Brief Review of the Literature

This section briefly reviews right-tailed unit root tests and methods to timestamp the origination and termination of explosive episodes in time series data. Model (2) is fitted by LS regression with an intercept from the full sample giving the coefficient estimate $\hat{\rho}_n$ and associated t-statistic $DF_n = (\hat{\rho}_n - 1)/se(\hat{\rho}_n)$, where $se(\hat{\rho}_n)$ is the usual standard error of $\hat{\rho}_n$. Under the null hypothesis that $\rho = 1$, by standard methods (Phillips, 1987a) as $n \to \infty$, $DF_n \Rightarrow \int_0^1 \breve{W}(s)dW(s)/\left(\int_0^1 \breve{W}(s)^2 ds\right)^{1/2} =: DF_\infty$, where $W(s)$ is standard Brownian motion (BM) and $\breve{W}(s) = W(s) - \int_0^1 W(p)dp$ is demeaned BM. Right-tailed unit root tests are implemented by rejecting the null when DF_n exceeds its right-tailed critical value.

In practical work, potentially explosive episodes are typically investigated within sample at some point of time $\tau_e = \lfloor nr_e \rfloor$, with corresponding sample fraction $r_e \in (0,1)$. Such episodes may then end later in the sample at some time $\tau_f = \lfloor nr_f \rfloor$, with $r_e < r_f < 1$, when there is a market correction or shock that terminates exuberance. If explosive behavior emerges and collapses within sample in this way, Phillips et al. (2011) prove that $DF_n \xrightarrow{p} -\infty$, revealing that full sample right-tailed unit root tests of the type suggested in Diba and Grossman (1988) have no discriminatory power for detecting financial bubbles. Instead, PWY and PY propose a sup statistic based on recursive regressions of the form

$$y_t = \hat{\mu} + \hat{\rho}_\tau y_{t-1} + \hat{u}_t, \text{ for } t = 1, ..., \tau = \lfloor nr \rfloor, \ r > r_0 \tag{5}$$

where $\hat{\mu}_\tau$, $\hat{\rho}_\tau$, and $\hat{u}_t$ are the fitted intercept, AR coefficient, and residuals from regressions with $\tau = \lfloor nr \rfloor > \tau_0 = \lfloor nr_0 \rfloor$ and τ_0 is an initiating sample size for the recursion for which it is assumed that $\tau_0 < \tau_e$. Subsequent regressions proceed from the initiating sample of size $\tau_0 = \lfloor nr_0 \rfloor$ until the full sample size n with $r = 1$ is reached. Using the t-statistic $DF_\tau = (\hat{\rho}_\tau - 1)/s_\tau$ based on the regression with τ observations and recursive standard error $s_\tau = \left(\frac{1}{\tau}\sum_{t=1}^\tau \hat{u}_t^2 / \left[\sum_{t=1}^\tau y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^\tau y_{t-1}\right)^2\right]\right)^{1/2}$ of $\hat{\rho}_\tau$, the test statistic proposed by PWY and PY is $\sup_{\tau \in [\tau_0, n]} DF_\tau$, whose limit distribution is given by the corresponding

functional

$$SDF := \sup_{\tau \in [\tau_0, n]} DF_\tau \Rightarrow \sup_{r \in [r_0, 1]} \frac{\int_0^r \tilde{W}(s) dW(s)}{\left(\int_0^r \tilde{W}(s)^2 ds\right)^{1/2}}, \text{ as } n \to \infty.$$

The null hypothesis is rejected in favor of the presence of an explosive episode in the sample if the statistic SDF exceeds the right-tailed critical value corresponding to the specified significance level.

Once evidence of an explosive episode is detected, the origination and termination dates of the episode, represented by $\tau_e = \lfloor nr_e \rfloor$ and $\tau_f = \lfloor nr_f \rfloor$ with sample fraction forms r_e and r_f, can be estimated. Suppose the generating mechanism under the alternative of an explosive episode within the sample is given by

$$\begin{cases} y_t &= y_{t-1} 1\{t < \tau_e\} + \rho_n y_{t-1} 1\{\tau_e \leq t \leq \tau_f\} \\ &\quad + \left(\sum_{k=\tau_f+1}^{t} \epsilon_k + y^*_{\tau_f}\right) 1\{t > \tau_f\} + \epsilon_t \, 1\{t \leq \tau_f\} \\ \rho_n &= 1 + \frac{c}{n^\alpha}, \ \ c > 0, \ \ \alpha \in (0,1), \epsilon_t \overset{iid}{\sim} (0, \sigma^2), \ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty, \ \delta > 0 \end{cases}, \quad (6)$$

where $y^*_{\tau_f} = y_{\tau_e} + y^*$ with $y^* \sim O_p(1)$. Model (6) has two structural breaks. Before the first break (at $t = \tau_e$), y_t follows a unit root process. After the first break and before the second break (i.e. $\tau_e \leq t \leq \tau_f$), the process is mildly explosive with autoregressive coefficient $\rho_n = 1 + \frac{c}{n^\alpha}$ and localizing coefficient $c > 0$. At $\tau_f + 1$, the explosive period ends with a collapse in the process to $y^*_{\tau_f}$, which is assumed to be in an $O_p(1)$ neighborhood of y_{τ_e}, the value reached before the explosive episode begins. The sample fractions r_e and r_f are the true origination and termination dates of the explosive period, which may be estimated by

$$\hat{r}_e^{PWY} = \inf_{r \geq r_0} \{r : DF_\tau > cv_n\}, \quad (7)$$

$$\hat{r}_f^{PWY} = \inf_{s \geq \hat{r}_e + \frac{\gamma \ln(n)}{n}} \{s : DF_s < cv_n\}, \quad (8)$$

the latter estimate being conditional on evidence of an originating date $\hat{r}_e^{PWY}$ to the episode. In (7) and (8), the critical value cv_n increases with the sample size. If $cv_n \to \infty$ at a slower rate than $n^{1-\alpha/2}$, Phillips and Yu (2009) showed that $\hat{r}_e^{PWY} \overset{p}{\to} r_e$ and $\hat{r}_f^{PWY} \overset{p}{\to} r_f$ and the two estimates are consistent under some general regularity conditions. In empirical applications, PWY set cv_n proportional to $\ln \ln n$.

3 A New Test and Asymptotic Null Distribution

Motivated by the empirical findings in Section 1, we consider the following prototypical model

$$\begin{cases} y_t &= \rho_n y_{t-1} + u_t, \ t = 1, ..., n \\ u_t &= \Delta_+^{-d} \epsilon_t, \ d > 0, \ \epsilon_t \overset{iid}{\sim} (0, \sigma^2), \ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty, \ \delta > 0 \\ y_0 &= o_p(n^{1/2+d}) \end{cases}. \quad (9)$$

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Model (9) differs from (2) in that u_t can be strongly dependent. The model also differs from Sowell (1990), who used Type I FI $u_t = \Delta^{-d}\epsilon_t = \sum_{j=0}^{\infty} \frac{(d)_j}{j!}\epsilon_{t-j}$ with $d \in (0, 0.5)$ to model strong dependence, because the Type II formulation $u_t = \Delta_+^{-d}\epsilon_t$ allows for a wide range of stationary and nonstationary long range dependence, for which consistent estimation of d or d_y is possible using the ELW procedure with associated pivotal Gaussian inference, as noted in Shimotsu and Phillips (2005) and Hualde and Robinson (2011). We first consider the asymptotic behavior of the traditional Dickey-Fuller t-test when $\rho_n = 1$.

3.1 Asymptotic null distribution of DF_τ

Lemma 3.1 *Assume the true data generation process (DGP) is given by (9) with $\rho_n = 1$ and $d > 0$. For any $r \in (0, 1]$ and $\tau = \lfloor nr \rfloor$ as $n \to \infty$,*

$$DF_\tau = O_p\left(n^d\right). \tag{10}$$

According to Lemma 3.1, the statistic $DF_{\lfloor nr \rfloor}$ diverges with the sample size, implying rejection of the null hypothesis as $n \to \infty$ leading to spurious inference concerning explosive behavior in the data. With Type I fractional integration and $d \in (0, 0.5)$, Theorem 4 in Sowell (1990) also showed divergence $DF_n \xrightarrow{p} \infty$. Lemma 3.1 extends that result to any $d > 0$ with Type II fractional integration and the divergence rate $O_p\left(n^d\right)$ shows faster divergence for larger d holding for any $r \in (0, 1]$, so the result is relevant for subsample inference.

Remark 3.1 *To detect the presence of explosiveness, PWY and PSYa and Phillips et al. (2015b) (hereafter PSYb) proposed to use SDF and GSDF statistics defined by*

$$SDF(\tau_0) = \sup_{\tau \in [\tau_0, n]} DF_\tau \quad and \quad GSDF(\tau_0) = \sup_{\tau_2 \in [\tau_0, n], \tau_1 \in [0, \tau_2 - \tau_0]} DF_{\tau_1}^{\tau_2},$$

where $\tau_0 = \lfloor nr_0 \rfloor$ is the minimum data window and $DF_{\tau_1}^{\tau_2}$ is the t statistic based on the observations from $\tau_1 = \lfloor nr_1 \rfloor$ to $\tau_2 = \lfloor nr_2 \rfloor$. As Lemma 3.1 holds uniformly for $r \in (0, 1]$, under model (9) with $\rho_n = 1$ and $d \in (0, 0.5)$, we have

$$SDF(\tau_0) = O_p\left(n^d\right) \quad and \quad GSDF(\tau_0) = O_p\left(n^d\right).$$

Both statistics lead to the detection of spurious explosive behavior as $n \to \infty$.

Remark 3.2 *Similar to the framework in Phillips et al. (2014), model (9) can be extended to include an asymptotically negligible intercept, which can be useful in capturing the presence of a small drift in the data. In this case,*

$$y_t = \mu_n + \rho_n y_{t-1} + u_t, \tag{11}$$

where $\mu_n = O(n^{-\theta})$ with $\theta > 1/2 - d$. It can be shown that Lemma 3.1 continues to hold in this case. The result in Lemma 3.1 also continues to hold when the ADF test or the CUSUM test of Homm and Breitung (2011) is used.

3.2 A new test statistic

The failure of the standard t statistic stems from the use of an inappropriate standard error based on the sample variance of residuals, $\frac{1}{\tau}\sum_{t=1}^{\tau}\hat{u}_t^2$, which results in the divergence of DF_τ. Instead, we use a self-normalized statistic that employs a robust standard error estimate, leading to a well defined limit distribution as $n \to \infty$ for $d \in [0, 0.5)$. Specifically, allowing for potential strong dependence in u_t, we employ the HAR estimate

$$\hat{\Omega}_{HAR} = \sum_{j=-\tau+1}^{\tau} K\left(\frac{j}{M}\right)\hat{\gamma}_j, \tag{12}$$

where $K(\cdot)$ is a kernel function with bandwidth $M = M_\tau$, and $\hat{\gamma}_j = \frac{1}{\tau}\sum_{t=j+1}^{\tau}\Delta y_t \Delta y_{t-j}$ is the jth order sample autocovariance over the subsample $t = 1, ..., \tau$. Based on $\hat{\Omega}_{HAR}$, the t statistic becomes

$$DF_{\tau,HAR} = \frac{\hat{\rho}_\tau - 1}{s_{\tau,HAR}}, \tag{13}$$

in which the robust standard error is

$$s_{\tau,HAR} = \sqrt{\frac{\hat{\Omega}_{HAR}}{\sum_{t=1}^{\tau}\bar{y}_{t-1}^2}}, \quad \text{where } \bar{y}_t = y_t - \frac{1}{\tau}\sum_{t=1}^{\tau}y_{t-1}. \tag{14}$$

For HAR estimation, the bandwidth is selected by the fixed-b setting $M_\tau = b \times \tau$ with $b \in (0, 1]$ so the bandwidth is of the same order of magnitude as the subsample τ in the regression window. This approach follows Kiefer and Vogelsang (2002b,a, 2005), Bunzel et al. (2001), Vogelsang (2003) and many subsequent works that employ the fixed-b method. In the present setting, the HAR normalization of the test statistic plays the same role as in Sun (2004) in the context of potentially spurious co-integration.

Theorem 3.1 *Suppose $M_\tau = b\tau$, and $K(x) = K_B(x)$ is the Bartlett kernel with $K_B(x) = (1-|x|)\mathbf{1}(|x| \le 1)$. Under model (9), with $\tau = \lfloor nr \rfloor$ and $r \in (0, 1]$ as $n \to \infty$, $DF_{\tau=\lfloor nr \rfloor, HAR}$ has the following fixed-b asymptotic distribution,*

$$DF_{\tau,HAR} \Rightarrow \begin{cases} \dfrac{b^{1/2}r\int_0^r \tilde{W}(s)dW(s)}{\left[2\int_0^r \tilde{W}(s)^2 ds\left(\int_0^r W(p)^2 dp - \int_0^{(1-b)r} W(p)W(p+br)dp\right)\right]^{1/2}} =: F_{r,0}^* & \text{for } d = 0 \\[4ex] \dfrac{b^{1/2}\left[\frac{r}{2}\left(W^H(r)\right)^2 - \left(\int_0^r W^H(s)ds\right)W^H(r)\right]}{\left[2\int_0^r\left(\tilde{W}^H(s)\right)^2 ds\left(\int_0^r W^H(p)^2 - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right)\right]^{1/2}} =: F_{r,d} & \text{for } d \in (0, 0.5) \end{cases}, \tag{15}$$

where $W^H(r) = \frac{1}{\Gamma(H+1/2)}\int_0^r (r-s)^{H-1/2}dW(s)$ is Type II fractional Brownian motion (fBM) with the Hurst parameter $H = 1/2 + d$ and $\tilde{W}^H(r) = W^H(r) - \frac{1}{r}\int_0^r W^H(s)ds$ is demeaned Type II fBM.[8]

In contrast to the divergence of DF_τ, Theorem 3.1 shows that $DF_{\tau,HAR}$ converges weakly to a well-defined limit distribution for any $\tau = \lfloor nr \rfloor$ whether $d = 0$ or $d > 0$.

---

[8]It can be shown that $DF_{\tau,HAR} \Rightarrow F_{r,d}$ for $d > 0.5$.

Hence, provided the DGP does not have an explosive root, $DF_{\tau,HAR}$ has well-behaved asymptotics for $d \geq 0$ and in this respect is better suited to testing. However, although the statistic is well normalized for $d \geq 0$, the limit distribution of $DF_{\tau,HAR}$ in (15) is not uniform over d. The lack of uniformity arises because the centered LS estimator $\hat{\rho}_\tau - 1$ involves a component involving $n^{-1-2d} \left(\sum_{t=1}^{\tau} u_t^2 \right)$ that is asymptotically negligible when $d > 0$ but non-negligible when $d = 0$, thereby affecting the limit theory in that case and leading to a discontinuity in the limit distribution when $d \to 0$. As is evident from the form of the two limits given in (15), the denominator of $F_{r,d}$ is equivalent to the denominator of $F_{r,0}^*$ as $d \to 0$ (and $H \to 1/2$) but this is not true of the numerators. The discrepancy produces the discontinuity in the limit theory as $d \to 0$.

Simulations (not reported here) show that critical values obtained from the limit distribution of $DF_{\tau,HAR}$ do not provide satisfactory performance and lead to size distortion in testing when d is close to zero. This distortion stems from two factors. First, when $d > 0$ but is close to zero, the component $n^{-1-2d} \left(\sum_{t=1}^{\tau} u_t^2 \right)$ converges in probability to zero very slowly and the limit distribution $F_{r,d}$ does not provide a good finite sample approximation to the true distribution. Second, when $d = 0$, use of $F_{r,\hat{d}}$ for an asymptotic approximation with a plug-in estimate $\hat{d} > 0$ can be a poor approximation to the correct distribution $F_{r,0}^*$ which should be used to provide critical values.

The size distortion in the use of $DF_{\tau,HAR}$ and the source of the discrepancy in the limit theory motivates the design of a modified statistic $\widetilde{DF}_{\tau,HAR}$ whose limit expression smooths over the discontinuity as $d \to 0$ and assists in delivering satisfactory size performance. The modified statistic has the form

$$\widetilde{DF}_{\tau,HAR} = \frac{\tilde{\rho}_\tau - 1}{s_{\tau,HAR}}, \tag{16}$$

where $\tilde{\rho}_\tau = \hat{\rho}_\tau + \frac{\frac{1}{2} \sum_{t=1}^{\tau} \Delta y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}$ and $\Delta y_t = y_t - y_{t-1}$. This correction is analogous to the weak dependence correction in semiparametric unit root tests in Phillips (1987a).

Theorem 3.2 *Under the same assumptions as Theorem 3.1, for $\tau = \lfloor nr \rfloor$ with $r \in (0,1]$ and $n \to \infty$,*

$$\widetilde{DF}_{\tau,HAR} \Rightarrow F_{r,d}, \ for \ d \geq 0. \tag{17}$$

where $F_{r,d}$ is defined in (15).

Theorem 3.2 shows that the limit theory given by $F_{r,d}$ provides a smooth transition to $F_{r,0}$ as $d \to 0$, replacing $F_{r,0}^*$ when $d = 0$. When $H = 1/2$ we have $W^H(r) = W(r)$ and $F_{r,d} \to F_{r,0}$ as $d \to 0$. The continuity in the limit theory is achieved by simple algebraic removal of the component $n^{-1-2d} \left(\sum_{t=1}^{\tau} u_t^2 \right)$ in the centered LS estimator $\hat{\rho}_\tau - 1$. The component $n^{-1-2d} \left(\sum_{t=1}^{\tau} u_t^2 \right)$ is no longer relevant in the limit theory and there is no abrupt shift in the asymptotic behavior of $\widetilde{DF}_{\tau,HAR}$ at $d = 0$.

To perform a right-tailed unit root test based on the sample $\{y_t\}_{t=1}^{\tau}$ with $\tau = \lfloor nr \rfloor$ the statistic $\widetilde{DF}_{\tau,HAR}$ can be used in conjunction with the $\beta \times 100\%$ asymptotic critical

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value $cv^\beta_{r,HAR}(d)$, for which

$$\Pr\left(F_{r,d} > cv^\beta_{r,HAR}(d)\right) = \beta, \;\; \text{for } r \in (0,1], \tag{18}$$

where $F_{r,d}$ is defined in (15). This procedure applies to the full sample statistic $\widetilde{DF}_{n,HAR}$ with limit variate $F_{1,d}$ and corresponding critical value $cv^\beta_{1,HAR}$ satisfying (18).

Remark 3.3 *The limit distributions given in Theorem 3.1 and Remark 3.4 below apply when the error term u_t follows a stationary ARFIMA(p, d, q) process with $d > 0$. Indeed, $n^{-1-2d}\left(\sum^\tau_{t=1} u^2_t\right)$ vanishes asymptotically when $d > 0$ as $n \to \infty$, thereby removing the sample variance term which is dependent on the specific form of the ARFIMA(p, d, q) process.*

Remark 3.4 *Other kernel functions ($K_2(\cdot)$) may be used in place of the Bartlett kernel $K_B(\cdot)$ and similarly lead to a fixed-b limit distribution of the corresponding statistic $\widetilde{DF}_{\tau,HAR}$ constructed with this kernel. For instance, suppose $\hat\Omega_{HAR} = \sum^\tau_{j=-\tau+1} K_2\left(\frac{j}{M}\right)\hat\gamma_j$ with some twice continuously differentiable positive and symmetric kernel function $K_2(\cdot)$. Then, for all $d \in [0, 0.5)$ and $\tau = \lfloor nr \rfloor$ with $r \in (0, 1]$ we have the limit theory as $n \to \infty$,*

$$\widetilde{DF}_{\tau,HAR} \Rightarrow \frac{\frac{br^{3/2}}{2}\left(W^H(r)\right)^2 - br^{1/2}\left(\int^r_0 W^H(s)ds\right)W^H(r)}{\left(\left(\int^r_0 \tilde{W}^H(s)^2 ds\right)\int^r_0\int^r_0 -K''_2\left(\frac{p-q}{br}\right)W^H(p)W^H(q)dpdq\right)^{1/2}} =: \tilde{F}_{r,d}, \tag{19}$$

where $K''_2(\cdot)$ is the second derivative of $K_2(\cdot)$.

Remark 3.5 *The preceding results are given for long memory time series formulated in terms of Type II FI and the corresponding limit theory involves Type II fBM. Similar results apply for innovations involving Type I FI time series with limit theory involving a Type I fBM process. Specifically, when $d \in (0, 0.5)$ we can replace $u_t = \Delta^{-d}_+\epsilon_t$ in (9) with $u_t = (1-L)^{-d}\epsilon_t = \sum^\infty_{j=0}\frac{(d)_j}{j!}\epsilon_{t-j}$ and $W^H(t)$ in (15), (18) and (19) with*

$$B^H(t) = W^H(t) + \frac{1}{\Gamma(H+1/2)}\int^0_{-\infty}[(r-s)^{H-1/2} - (-s)^{H-1/2}]dW(s).$$

For discussion of Type I and Type II formulations of fBM see Marinucci and Robinson (1999, 2000); Davidson and Hashimzade (2009).

Remark 3.6 *As in PWY, a sup statistic $\sup_{\tau\in[\tau_0,n]}\widetilde{DF}_{\tau,HAR}$ can be constructed from recursive regression for empirical testing covering subsamples of the full sample. The limit theory can be obtained by continuous mapping in the usual way (PSYa; PSYb) and employed in practical work to identify explosive behavior in a subsample. This construction is discussed in detail in the Online Supplement.*

Both d_y and d can be consistently estimated by ELW estimation (Shimotsu and Phillips, 2005) or quasi-maximum likelihood estimation (QMLE) (Hualde and Robinson, 2011). Let $\widehat{d}$ denote the estimate of d so obtained. Critical values for testing can then be found for $F_{r,\widehat{d}}$ by simulations. Alternatively, we can tabulate critical values of $F_{r,d}$ for a set of grid points for $d \in [0, 0.5)$ and interpolation can be employed to obtain the critical value of $F_{r,\widehat{d}}$. Simulations reported in section 7 show that this plug-in approach delivers good size performance in finite samples even for $n = 100$ and performs nearly as well as the infeasible method where the true value of d is used to construct critical values. For practical implementation it is convenient to impose bounds in estimation so that $d_y \in [1, 1 + \bar{d}]$ for some $0 < \bar{d} < \infty$ when performing optimization in calculating estimates of d_y. Then $\widehat{d_y} \in [1, 1 + \bar{d}]$ and $\widehat{d} \in [0, \bar{d}]$. In the simulations reported in Section 7 we set $\bar{d} = 0.49$.

The plug-in method does not take into account the randomness in $\hat{d}$. An alternative feasible inferential procedure that does so is to use the fractional differencing bootstrap algorithm of Kapetanios et al. (2019) to obtain asymptotically correct critical values or bootstrap p-values. This approach involves the following five steps.

1. Let $e_{n,t} = \Delta_+^{1+\hat{d}} y_t$ and $\tilde{e}_{n,t} = (e_{n,t} - \bar{e}_{n,t})/\hat{\sigma}_e$ where $\hat{d}$ is an estimate of d, $\bar{e}_{n,t}$ and $\hat{\sigma}_e$ are the sample mean and sample standard deviation of $e_{n,t}$.

2. Redraw i.i.d. samples $\left\{ e_{n,t}^* \right\}$ from the empirical distribution of $\tilde{e}_{n,t}$ with replacement.

3. Let
$$u_t^* = \hat{\sigma}_e \Delta_+^{-\hat{d}} e_{n,t}^*, y_t^* = y_{t-1}^* + u_t^*, \ \text{ with } y_0 = 0,$$
and calculate $\widetilde{DF}_{n,HAR}^*$ as in (16).

4. Repeat Steps 2-4 B times and calculate the bootstrap empirical cdf
$$\hat{F}^*(x) = \frac{1}{B} \sum_{j=1}^{B} 1(\widetilde{DF}_{n,HAR}^{*,j} \leq x).$$

Define the $\beta \times 100\%$ bootstrap critical value (bcv^β) as the $1 - \beta$ quantile of $\hat{F}^*$ and let the bootstrap p-value be
$$p^*(\widetilde{DF}_{n,HAR}) = 1 - \hat{F}^*(\widetilde{DF}_{n,HAR}). \tag{20}$$

5. Reject the unit root null hypothesis when $\widetilde{DF}_{n,HAR} > bcv^\beta$ or when $p^*(\widetilde{DF}_{n,HAR}) < \beta$.

The following theorem shows that this bootstrap approach delivers the correct test size asymptotically.

Theorem 3.3 *Suppose we reject the hypothesis $\rho_n = 1$ when $p^*(\widetilde{DF}_{n,HAR})$ in (20) is less than β. Under the assumptions specified in Theorem 3.1 and if, $n^\gamma(\hat{d} - d) \xrightarrow{d} \mathcal{N}(0, V)$, with $1/4 < \gamma \leq 1/2$ and $V > 0$, then as $n \to \infty$, we have*

$$
\begin{aligned}
\widetilde{DF}^*_{n,HAR} &\Rightarrow F_{1,d}, \\
p^*(\widetilde{DF}_{n,HAR}) &\Rightarrow U[0,1].
\end{aligned}
$$

4 Alternative Hypothesis and Asymptotic Theory

To study the asymptotic behavior of the proposed test statistic under an alternative hypothesis, we follow the literature and use two popular ways of modeling explosive departures from unity. The first alternative adopts the local to unit root (LUR) framework of Phillips (1987b) – see Harvey et al. (2016, 2018, 2019). The advantage of using the locally explosive model is that it facilitates the computation of local power. The second alternative is the mildly explosive model of Phillips and Magdalinos (2007); see PWY, PSYa, PSYb and PY. Under a mildly explosive alternative a consistent test is obtained.

4.1 Locally explosive model

We first consider the alternative hypothesis with the following locally explosive setting:

$$
\begin{cases}
y_t &= (y_{t-1} + u_t)\,1\{t < \tau_e\} + (\rho_n y_{t-1} + u_t)\,1\{\tau_e \leq t \leq n\}, t = 1, ..., n, \\
u_t &= \Delta_+^{-d} \epsilon_t, d \geq 0, \epsilon_t \overset{iid}{\sim} (0, \sigma^2),\ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty,\ \delta > 0, \\
\rho_n &= 1 + c/n, c > 0, 1 + c/n, c > 0, \\
y_0 &= o_p\left(n^{1/2+d}\right).
\end{cases}
\tag{21}
$$

In model (21), y_t has a unit autoregressive root generating mechanism before time τ_e and becomes mildly explosive after τ_e, producing a structural break at τ_e. During both periods the errors in the AR model have strong dependence with the same memory parameter d. We now consider the asymptotic behavior of $\widetilde{DF}_{\tau,HAR}$.

Theorem 4.1 *Under model (21), for $\tau = \lfloor nr \rfloor$ with any $r > r_e$, as $n \to \infty$,*

$$
\widetilde{DF}_{\tau,HAR} \Rightarrow \frac{\left(\frac{\left(\frac{1}{2}C_{r,d} - \frac{1}{r}A_{r,d}W^H(r)\right)r}{B_{r,d} - \frac{1}{r}A_{r,d}^2} + cr\right)\left(B_{r,d} - \frac{1}{r}A_{r,d}^2\right)^{1/2}}{\left[\frac{2}{b}\left(\int_0^r G_{r_e,c}(d,p)^2 dp - \int_0^{(1-b)r} G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp\right)\right]^{1/2}} =: F_{r,d}^c,
\tag{22}
$$

where

$$
\begin{aligned}
A_{r,d} &:= \int_0^r \left(e^{(x-r_e)c}W^H(r_e) + \int_{r_e}^x e^{(x-s)c}dW^H(s)\right)dx, \\
B_{r,d} &:= \int_0^r \left(e^{(x-r_e)c}W^H(r_e) + \int_{r_e}^x e^{(x-s)c}dW^H(s)\right)^2 dx,
\end{aligned}
$$

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$$C_{r,d} := \left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^r e^{(r-s)c} dW^H(s) \right)^2,$$

$$G_{r_e,c}(p) := W^H(p) - cA_{p,d} - \int_0^{r_e} W^H(p) dp.$$

The limit distribution in Theorem 4.1 depends on the non-centrality localizing scale parameter c. This parameter differentiates $F_{r,d}^c$ from $F_{r,d}$ and evidently $F_{r,d}^c = F_{r,d}$ for $c = 0$ from (17). Since both $F_{r,d}^c$ and $F_{r,d}$ are $O_p(1)$, they may be used to compute local power of the proposed test.

4.2 Mildly explosive model

Next consider the alternative hypothesis with the following mildly explosive setting

$$\begin{cases} y_t &= (y_{t-1} + u_t) \, 1\{t < \tau_e\} + (\rho_n y_{t-1} + u_t) \, 1\{\tau_e \le t \le n\}, \; y_0 = o_p(n^{1/2+d_1}) \\ u_t &= \begin{cases} \Delta_+^{-d_1} \epsilon_t & \text{if } t < \tau_e, \\ \Delta_+^{-d_2} \epsilon_t & \text{if } \tau_e \le t \le n, \end{cases} \epsilon_t \overset{iid}{\sim} (0, \sigma^2), d_1, d_2 \ge 0, \mathbb{E}|\epsilon_1|^{2+\delta} < \infty, \delta > 0, \end{cases} \quad (23)$$

where

$$\rho_n = 1 + \frac{c}{n^\alpha}, \; c > 0, \; \alpha \in (0,1). \quad (24)$$

In model (23) y_t has unit root behavior before τ_e and becomes mildly explosive after τ_e, implying a structural break at τ_e. For both periods the errors in the AR model have strong dependence but with memory parameter d_1 prior to the break and memory parameter d_2 after the break. Since the localizing rate parameter $0 < \alpha < 1$, the specification (24) delivers stronger explosive behavior than the explosive LUR model considered earlier. Note that under the LUR model, the memory parameters has a role to determine whether the locally explosive trajectory is relevant asymptotically. Suppose that $\alpha = 1$, and $d_1 \ne d_2$, model (23) becomes the LUR model with long memory errors with memory parameters d_1 and d_2 at $t < \tau_e$ and $t \in [\tau_e, n]$, respectively. It can be shown that the persistence of y_t is solely determined by $\max(d_1, d_2)$ as $n \to \infty$, if $d_1 > d_2$, the locally explosive part will be asymptotically dominated by the non-explosive episode.

Theorem 4.2 *Under model (23) with (24), as $n \to \infty$,*

$$DF_n = O_p(n^{1-\alpha/2}) \overset{p}{\to} \infty \text{ and } \widetilde{DF}_{n,HAR} = O_p\left(n^{\frac{1-\alpha}{2}}\right) \overset{p}{\to} \infty.$$

Theorem 4.2 shows that the statistic DF_n diverges to infinity under mildly explosive alternatives. Combining with the result in Lemma 3.1, divergence of DF_n may be due to either strongly dependent errors or mildly explosive autoregression. But the modified HAR statistic $\widetilde{DF}_{n,HAR}$ diverges only under the alternative hypothesis given by (23) and (24). For any $\beta \times 100\%$ critical value $cv_{HAR}^\beta(d)$, we have $\Pr\left(\widetilde{DF}_{n,HAR} > cv_{HAR}^\beta(d)\right) \to 1$ under model (23) with condition (24), giving a consistent test for mildly explosive alternatives. Note that DF_n diverges at the same rate $n^{1-\alpha/2}$ as that obtained in PWY and PSYa under i.i.d. errors. The divergence rate of both statistics does not depend on d.

5 Dating Origination and Termination

We now discuss estimation of the origination and termination dates of an explosive period. Following PWY and PY, we consider the following model:

$$
\begin{cases}
y_t &= (y_{t-1} + u_t)\, 1\{t < \tau_e\} + (\rho_n y_{t-1} + u_t)\, 1\{\tau_e \le t \le \tau_f\} \\
&\quad + \left(\sum_{k=\tau_f+1}^{t} u_k + y_{\tau_f}^*\right) 1\{t > \tau_f\}, \\
\rho_n &= 1 + \frac{c}{n^\alpha},\ c > 0,\ \alpha \in (0,1), \\
u_t &= \Delta_+^{-d_t} \epsilon_t,\ \epsilon_t \overset{iid}{\sim} (0, \sigma^2),\ \mathbb{E}|\epsilon_1|^{2+\delta} < \infty,\ \delta > 0, \\
d_t &= d_1 \text{ for } t \in [1, \tau_e) \cup [\tau_f + 1, n],\ d_t = d_2 \text{ for } t \in [\tau_e, \tau_f],\ \tau_e = \lfloor n r_e \rfloor,\ \tau_f = \lfloor n r_f \rfloor, \\
y_{\tau_f}^* &= y_{\tau_e} + y^*,\ \text{and } y^* = O_p(1).
\end{cases}
\tag{25}
$$

This model extends (6) by allowing for potential strong dependence in the errors. As in (6), the notations τ_e (r_e) and τ_f (r_f) are the true temporal (fractional) origination and termination dates of the explosive period.

Break point estimators of r_e and r_f are defined by employing the HAR statistic $\widetilde{DF}_{n,HAR}$ in the usual criteria

$$
\begin{aligned}
\hat{r}_e^{HAR} &= \inf_{r \ge r_0}\{r : \widetilde{DF}_{\tau,HAR} > cv_{n,HAR}\}, \\
\hat{r}_f^{HAR} &= \inf_{r > \hat{r}_e + \gamma \ln(n)/n}\{r : \widetilde{DF}_{\tau,HAR} < cv_{n,HAR}\}.
\end{aligned}
\tag{26}
$$

The following theorem shows that $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ deliver consistent estimates of r_e and r_f when $cv_{n,HAR}$ passes to infinity at a controlled rate.

Theorem 5.1 *Under model (25) with $\tau = \lfloor nr \rfloor$, $\widetilde{DF}_{\tau,HAR}$ has the following asymptotic behavior:*

$$
\begin{aligned}
\widetilde{DF}_{\tau,HAR} = O_p\left(n^{\frac{1-\alpha}{2}}\right) &\overset{p}{\to} \infty \text{ if } \tau \in [\tau_e, \tau_f], \\
\widetilde{DF}_{\tau,HAR} = O_p\left(n^{\frac{1-\alpha}{2}}\right) &\overset{p}{\to} -\infty \text{ if } \tau \in [\tau_f + 1, n].
\end{aligned}
\tag{27}
$$

If $r_e \ge r_0$ and the critical value $cv_{n,HAR}$ satisfies the following condition

$$
\frac{1}{cv_{n,HAR}} + \frac{cv_{n,HAR}}{n^{(1-\alpha)/2}} \to 0,
\tag{28}
$$

then, as $n \to \infty$,

$$
\hat{r}_e^{HAR} \overset{p}{\to} r_e \quad and \quad \hat{r}_f^{HAR} \overset{p}{\to} r_f.
$$

Remark 5.1 *Under the null hypothesis of no explosive behavior, i.e. model (9), if $cv_{n,HAR} \to \infty$ the probability of detecting an explosive episode in the data using $\widetilde{DF}_{\tau,HAR}$ goes to zero as $n \to \infty$. This is because $\widetilde{DF}_{\tau,HAR} \sim O_p(1)$ under model (9).*

Remark 5.2 *Under the alternative hypothesis, consistent estimation of the origination and termination dates of an explosive period requires that the critical value $cv_{n,HAR} \to \infty$ but at a rate slower than $n^{(1-\alpha)/2}$. This is a stronger control condition for consistency than that obtained in Phillips and Yu (2011) for the model without strongly dependent errors (where the rate is required to be slower than $n^{(2-\alpha)/2}$) and is due to the influence of long memory equation errors on the asymptotic behavior of the statistic.*

Remark 5.3 *As in* PWY, *the procedure provides real-time estimates of* r_e *and* r_f *because the date estimates* $\hat{r}_e^{HAR}$ *and* $\hat{r}_f^{HAR}$ *only use subsamples of data observed to those points.*

6   Heteroskedastic model

This section explains how to conduct right-tailed unit root tests in the presence of unconditional heteroskedasticity. Time series models with unconditionally heteroskedastic errors were studied in Cavaliere and Taylor (2005, 2007) and Xu and Phillips (2008). More recently, Harvey et al. (2016, 2018, 2019) and Astill et al. (2021) adopted an AR model with time-varying volatilities and proposed new tests for explosive behavior in such settings. The following provides a simple extension of those ideas under strongly dependent errors.

Consider the model

$$\begin{cases} y_t &= y_{t-1} + u_t, \ y_0 = o_p(n^{1/2+d}), t = 1, ..., n, \\ u_t &= \Delta_+^{-d}\epsilon_t = \Delta_+^{-d}\sigma_{t,n}\varepsilon_t = \Delta_+^{-d}g\left(t/n\right)\varepsilon_t, \ \varepsilon_t \overset{iid}{\sim} (0,1), \ d \geq 0, \\ \mathbb{E}|\varepsilon_t|^q &< K < \infty \text{ and } q \geq 4, \end{cases} \tag{29}$$

where g is a strictly positive, non-stochastic and continuously differentiable function on $[0,1]$ with $\sup_s g\left(s\right) < C < \infty$. Model (29) has strongly dependent errors (captured by the parameter d) that are also unconditionally heteroskedastic (captured by the weakly trending function $\sigma_{t,n} = g(t/n)$).

Lemma 6.1 *Under model (29), as* $n \to \infty$, *we have*

$$\frac{1}{n^{1/2+d}} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow \frac{1}{\Gamma(H+1/2)} \int_0^r g(s)(r-s)^{H-1/2} dW(s) =: W_g^H(r). \tag{30}$$

Remark 6.1 *When* $d = 0$, Cavaliere (2005) *and* Cavaliere and Taylor (2005, 2007) *showed that*

$$n^{-1/2} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow \int_0^r g(s) dW(s),$$

and Lemma 6.1 extends that result to the case where $d > 0$ *and the limit involves a weighted functional of fBM.*

Using the functional law (30) yields the corresponding limit theory for the statistic $\widetilde{DF}_{\tau,HAR}$ in the case of strong dependence and unconditional heterogeneity.

Theorem 6.1 *Under model (29), as* $n \to \infty$, *we have*

$$\widetilde{DF}_{\tau,HAR} \Rightarrow \frac{b^{1/2}\left[\frac{r}{2}\left(W_g^H(r)\right)^2 - \left(\int_0^r W_g^H(s)ds\right)W_g^H(r)\right]}{\left[2\int_0^r \left(\tilde{W}_g^H(s)\right)^2 ds \left(\int_0^r W_g^H(p)^2 - \int_0^{(1-b)r} W_g^H(p)W_g^H(p+br)dp\right)\right]^{1/2}} =: F_{r,d}^g, \tag{31}$$

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where $\tilde{W}_g^H(r) = W_g^H(r) - \frac{1}{r}\int_0^r W_g^H(s)ds$, $b = M/\tau$ where $M = M_\tau$ is the bandwidth in the kernel function used to construct the modified HAR statistic $\widetilde{DF}_{\tau,HAR}$.

The limit functional $F_{r,d}^g$ depends on the unknown quantities d and g. One approach to operationalize inference is to consistently estimate d and g and obtain critical values for the functional $F_{r,\hat{d}}^{\hat{g}}$ using these plug-in estimates. For example, we can consistently estimate $d_y = 1 + d$ directly from the given data, and hence d, under model (29) by ELW or QML estimation. Then $\{y_t\}$ can be filtered using $\hat{d}$ by calculating $\hat{u}_t = \Delta_+^{1+\hat{d}}y_t$, and the adaptive kernel method (Beare, 2004; Phillips and Xu, 2006; Xu and Phillips, 2008; Cavaliere et al., 2022; Astill et al., 2021) can be used to estimate g.

A second approach is to note that, under (29), we have

$$\widetilde{x}_p = \sum_{t=1}^p \Delta_+^{-d}\left(\frac{\Delta_+^{1+d}y_t}{g(t/n)}\right) = \sum_{t=1}^p \Delta_+^{-d}\varepsilon_t = \widetilde{x}_{p-1} + \Delta_+^{-d}\varepsilon_p.$$

Hence, based on $\hat{d}$ and $\hat{g}$, we can define

$$x_{\lfloor ns \rfloor} = \sum_{t=1}^{\lfloor ns \rfloor} \Delta_+^{-\hat{d}}\left(\frac{\Delta_+^{1+\hat{d}}y_t}{\hat{g}(t/n)}\right), \ x_0 = 0, \ \text{for } s \in [0,1], \tag{32}$$

where $\hat{g}^2(t/n) = \sum_{i=1}^{\tau} k_{ti}\hat{\epsilon}_i^2$ with $k_{ti} = \frac{K_\nu(t-i)}{\sum_{i=1}^{\tau} K_\nu(t-i)}$, $\hat{\epsilon}_t = \Delta_+^{1+\hat{d}}y_t$, $K_\nu(\cdot) = K(\frac{\cdot}{\nu})$ and $K(\cdot)$ is a kernel function with bandwidth ν. Following Astill et al. (2021), we assume that the kernel $K(\cdot)$ satisfies the conditions given in Theorem 6.2.

The following limit theory holds when the $\widetilde{DF}_{\tau,HAR}$ test is applied to $\{x_t\}_{t=1}^\tau$.

Theorem 6.2 *Assume $\{y_t\}_{t=1}^n$ is generated from model (29). Suppose the kernel function $K(\cdot)$ satisfies the following conditions: it is continuously differentiable over the interval $(0,1)$; $K(x) = 0$, for $x \leq 0$ and $x \geq 1$; $\int_0^1 K dx > 0$, $\int_0^1 |K(x)|dx < \infty$, $\int_0^1 |K(x)x|dx < \infty$, and the characteristic function of K is absolutely integrable. Suppose that $n^\gamma\left(\hat{d} - d\right) \overset{d}{\to} \mathcal{N}(0, V)$ with $1/4 < \gamma \leq 1/2$ and $V > 0$. Furthermore, assume the bandwidth ν satisfies $\nu \to \infty$, $\frac{\nu}{n} \to 0$ and $\frac{\nu^2}{n} \to \infty$ as $n \to \infty$. Let $\widetilde{DF}_{\tau,HAR}^x$ denote the test statistic $\widetilde{DF}_{\tau,HAR}$ applied to data $\{x_t\}_{t=1}^\tau$ constructed as in (32). As $n \to \infty$, we then have*

$$\widetilde{DF}_{\tau,HAR}^x \Rightarrow F_{r,d}. \tag{33}$$

Remark 6.2 *Theorem 6.2 states that one can use the same limit distribution as in Theorem 3.2 to obtain critical values for the test under the heteroskedastic model (29). It is therefore possible to extend the bootstrap procedures given earlier to accommodate the presence of error variance heterogeneity. Simulations, not reported here, were conducted to check size performance in these tests for different forms of error variance function $g(\cdot)$. Overall, the finite sample performance was found to be comparable to that based on the statistic $\widetilde{DF}_{\tau,HAR}$ for the homogeneous case where σ^2 is fixed - see Table 2 in the following section.*

7 Monte Carlo Studies

This section reports the results of simulation experiments designed (i) to explore the size and power performance of the proposed tests for the presence of explosive behavior in the data, and (ii) to study performance of the procedures for estimating the origination and termination dates in finite samples. The reported results relate to the model with homogeneous error variance.[9] Normalized partial sums of $u_t = \Delta_+^{-d}\epsilon_t$, with $\epsilon_t \overset{iid}{\sim} (0,1)$, were used to approximate the Type II fBM that appears in the limit theory.[10] This approximation allows us to simulate DF_∞, $F_{r,0}$ and $F_{r,d}$ to obtain the critical values. The number of replications in all experiments is 2,500.

To investigate the empirical size of the tests we use the following DGP,

$$\begin{cases} y_t & = & y_{t-1} + u_t,\ t=1,...,n \\ u_t & = & \Delta_+^{-d}\epsilon_t,\ \epsilon_t \overset{iid}{\sim} \mathcal{N}(0,1) \end{cases}, \tag{34}$$

with parameter settings: $d \in \{0, 0.05, 0.1, ..., 0.45\}$, $y_0 = 0$, and $n \in \{100, 500\}$.

Table 2: Empirical sizes of DF_n, $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$ for various d based on a nominal 5% right-tailed critical value

	$n = 100$									
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
DF_n	0.05	0.09	0.14	0.20	0.26	0.31	0.36	0.41	0.45	0.48
$DF_{n,HAR}(\hat{d})$	0.12	0.12	0.11	0.09	0.06	0.05	0.04	0.03	0.04	0.04
$\widetilde{DF}_{n,HAR}(d)$	0.05	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06
$\widetilde{DF}_{n,HAR}(\hat{d})$	0.04	0.05	0.05	0.06	0.05	0.05	0.06	0.06	0.06	0.06
$\widetilde{DF}^{*}_{n,HAR}(\hat{d})$	0.04	0.04	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.06
	$n = 500$									
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
DF_n	0.05	0.10	0.18	0.27	0.34	0.40	0.45	0.49	0.53	0.56
$DF_{n,HAR}(\hat{d})$	0.15	0.11	0.07	0.03	0.02	0.03	0.03	0.04	0.04	0.05
$\widetilde{DF}_{n,HAR}(d)$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$\widetilde{DF}_{n,HAR}(\hat{d})$	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
$\widetilde{DF}^{*}_{n,HAR}(\hat{d})$	0.05	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.05	0.05

For each parameter setting right-tailed unit tests were conducted using the statistics DF_n, $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$. For the standard right-tailed test based on DF_n, the null hypothesis is rejected when the statistic exceeds the 5% right-tail critical value of the corresponding asymptotic distribution or bootstrap distribution.[11] Critical values for $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$ were obtained via simulations. The critical values of $DF_{n,HAR}$

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[9] As indicated earlier, similar findings were obtained in the heterogeneous variance case for several variance functions but these are not reported to save space.

[10] The sums $\frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor nr\rfloor}\epsilon_t$ and $\frac{1}{n^{1/2+d}}\sum_{t=1}^{\lfloor nr\rfloor}u_t$ are used to approximate $W(r)$ and $W^H(r)$ with $n=5000$.

[11] The critical values for DF_n were obtained from Table B.6 in Hamilton (1994)

were obtained from the simulated limit distribution (15) with the true value of d being replaced by the ELW estimate $\hat{d}$ Shimotsu and Phillips (2005). There are three critical values for our test statistic $\widetilde{DF}_{n,HAR}$. First, we assume d is known and obtain the asymptotic (infeasible) critical value from $F_{r,d}$, which provides a benchmark for calibrating the empirical size of the feasible tests. Second, we obtain the feasible asymptotic critical values from $F_{r,\hat{d}}$.[12] Finally, we obtain critical values from the bootstrap approach. The fixed-b scale parameter $b = 0.05$ was used for calculating $\hat{\Omega}_{HAR}$.[13]

Table 2 reports the empirical sizes of DF_n, $DF_{n,HAR}$ and $\widetilde{DF}_{n,HAR}$ with the corresponding 5% critical values. For $\widetilde{DF}_{n,HAR}$, we report the test sizes using critical values obtained from $F_{r,d}$ (denoted $\widetilde{DF}_{n,HAR}(d)$), $F_{r,\hat{d}}$ (denoted $\widetilde{DF}_{n,HAR}(\hat{d})$), and the bootstrap method (denoted $\widetilde{DF}^{*}_{n,HAR}(\hat{d})$). Several observations can be made on the findings from Table 2. First, DF_n has satisfactory performance only when $d = 0$ and the test is oversized when $d > 0$. For instance, when $d = 0.3$ and $n = 500$, the test rejects the null about 40% of the time. These simulation results are consistent with the asymptotic theory in Sowell (1990) and the predictions from Lemma 3.1, which imply severe false detection of explosiveness as d increases. Second, use of $DF_{n,HAR}$ does not lead to a divergent empirical size. But, when the true value of d is equal to or close to zero, some size distortion in the feasible statistic $DF_{n,HAR}(\hat{d})$ is noticeable. Finally and most importantly, use of the modified test $\widetilde{DF}_{n,HAR}$ shows good size performance irrespective of the value of d and alternative ways of obtaining the critical value. The simulation evidence suggests that $\widetilde{DF}_{n,HAR}(\hat{d})$ with critical values obtained from $F_{r,\hat{d}}$ delivers overall good size performance in finite samples across all values of d.

Given its good size performance, the finite sample power properties of the $\widetilde{DF}_{n,HAR}$ test were explored next. The experiment was designed using model (23) with the following parameter settings: $n = 100$, $y_0 = 100$, $r_e = 0.5$, $d_1 = d_2 = d \in \{0, 0.01, 0.02, ..., 0.49\}$, $\rho_n = 1 + c/n^\alpha$, $c = 1$, and $\alpha \in \{0.50, 0.55, 0.56, ..., 1\}$, which corresponds to the autoregressive root ρ_n ranged from 1.1 to 1.01. [14]

Table 3 reports the empirical rejection rates (empirical power) under selected values of α and d while Figure 2 plots the power as a function of α and d. Several finding are notable. First, the smaller is the value of α, the higher is the power. This is expected since the stronger explosiveness enhances detection. A sharp contrast can be observed when $d = 0.45$, the empirical rejection rate is 1 at $\alpha = 0.50$, while the rejection rate becomes 0.12 at $\alpha = 1$. Second, when α is small, variations in memory parameter only have a small effect on the empirical rejection rates. It can be seen that for α less than or equal to 0.7, different values of d only slightly change the empirical rejection rate, while different values

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[12]We also estimate d by the QMLE method of Hualde and Robinson (2011). The empirical sizes are similar to those based on the ELW method.

[13]This value of b was chosen because extensive simulations showed that for any $b > 0.05$ the test delivered empirical size close to the nominal value. Notably, however, lower values of b were found to yield higher power, as is known from other applications of fixed-b methods.

[14]The initial condition $y_0 = 100$ is used to ensure a positive sample path for the simulated data and produce an explosive episode that has an upward trajectory. This choice matches the real data considered later.
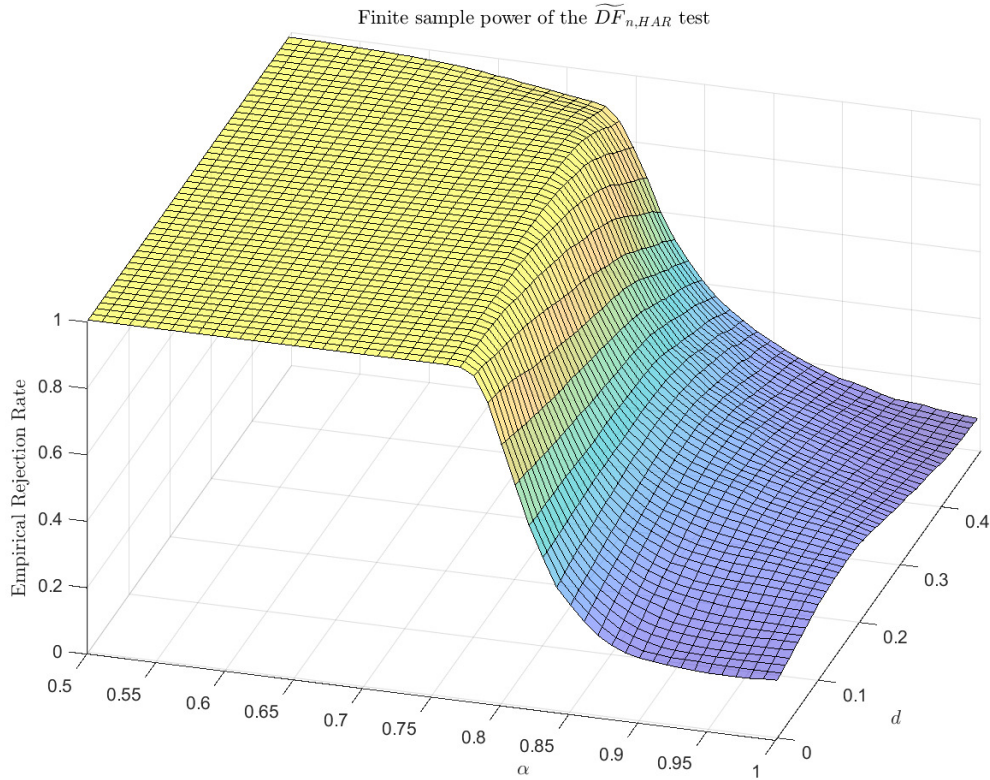
Figure 2: The empirical power of the $\widetilde{DF}_{n,HAR}$ test as a function of α and d.

of d change the empirical rejection rate materially when α moves closer to 1.

To study the accuracy of the date detectors $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ in finite samples, we used an experimental design based on model (25) with the following parameter settings: $n = 100$, $y_0 = 100$, $c = 1$, $\alpha \in \{0.5, 0.55, ..., 0.7\}$, $d_1 = d_2 = d \in \{0, 0.05, 0.1, ..., 0.45\}$, $\epsilon_t \overset{iid}{\sim} \mathcal{N}(0,1)$, $y_{\tau_f}^* = y_{\tau_e}$, $r_e = 0.5$, $r_f = 0.7$, $r_0 = 0.4$, and $\gamma \ln(n)/n = 0.1$. To obtain $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$, we first calculate $\{\widetilde{DF}_{\tau,HAR}\}_{\tau=\lfloor nr_0 \rfloor}^n$ and then obtain $\{\hat{d}_\tau\}_{\tau=\tau_0}^n$ using ELW estimation based on the data $\{y_t\}_{t=1}^\tau$. The following critical values for $cv_{n,HAR}$ are employed

$$cv_{n,HAR} = cv_{n,HAR}^{3\%}\left(\hat{d}_\tau\right) + \frac{\ln(\ln(ns))}{100}, \tag{35}$$

where ns is proportional to the sample size n and $s \in (0,1]$ is a fractional of the sample corresponding to n. These critical values are constructed using the 3% critical value of $\widetilde{DF}_{n,HAR}$ under $\hat{d}_\tau$ augmented with the slowly diverging factor $\frac{\ln(\ln(ns))}{100}$ [15] This diverging factor guarantees that $cv_{n,HAR}$ satisfies condition (28) and hence, lead to consistent $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$. However, in our finite sample setting $\frac{\ln(\ln(ns))}{100}$ takes values between 0.01 and

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[15]The 3% critical value is adopted here as extensive simulations suggest that it yields a successful detection rate (defined below) than the 5% critical value in most cases.

Table 3: The empirical rejection rates of $\widetilde{DF}_{n,HAR}$

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\alpha = 0.50$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\alpha = 0.55$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\alpha = 0.60$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99
$\alpha = 0.65$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.98
$\alpha = 0.70$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.96
$\alpha = 0.75$	1.00	1.00	1.00	1.00	1.00	0.99	0.99	0.96	0.92	0.85
$\alpha = 0.80$	0.78	0.75	0.72	0.69	0.67	0.64	0.61	0.58	0.52	0.47
$\alpha = 0.85$	0.32	0.35	0.37	0.39	0.40	0.41	0.40	0.37	0.34	0.30
$\alpha = 0.90$	0.19	0.22	0.26	0.29	0.32	0.32	0.31	0.27	0.24	0.21
$\alpha = 0.95$	0.17	0.20	0.24	0.26	0.27	0.26	0.24	0.21	0.18	0.15
$\alpha = 1.00$	0.17	0.20	0.23	0.24	0.24	0.21	0.18	0.16	0.13	0.12

0.015 and $cv_{n,HAR}^{3\%}\left(\hat{d}_\tau\right)$ has a greater magnitude than $\frac{\ln(\ln(ns))}{100}$.

Table 4 reports the successful detection rate and the means of $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ where successful detection is obtained. The numbers in parentheses below the means are the root mean square errors of the estimates. Successful detection is defined whenever $\hat{r}_e^{HAR}$ falls into the interval $[r_e, r_f]$ (i.e. $\hat{r}_e^{HAR} \in [r_e, r_f]$). Several findings emerge from this simulation. First, when α is small, the successful detection rate is only slightly affected by changes in the memory parameter: the successful detection rate in Table 4 drops only by 0.01 when $\alpha = 0.50$ and d increases from 0 to 0.45, whereas it drops by 0.19 when $\alpha = 0.70$. Further, the estimates of $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ are less accurate when both α and d are large: the root mean square errors of $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ are 0.09 and 0.03 respectively when $d = 0.45$ and $\alpha = 0.70$, in contrast to the corresponding root mean square errors of 0.01 and 0.00 when $d = 0$ and $\alpha = 0.50$.

8   Empirical Application

To highlight the usefulness of the proposed test and date-stamping strategy we conduct an empirical study using the same time series as in Table 1. We calculate the $\widetilde{DF}_{n,HAR}$ statistic and use 10%, 5% and 1% critical values when performing the right-tailed unit root test. Since these data are price-dividend ratios which take account of fundamental values, explosive behavior in the time series is indicative of a rational bubble.

Table 5 reports the HAR test statistic $\widetilde{DF}_{n,HAR}$ together with 10%, 5%, and 1% critical values computed for the six different sample periods. In Table 1 it was found that standard testing using the DF_n statistic exceeded the 5% critical value for each sample period, indicating strong evidence for the presence of bubbles. Table 5 updates the analysis by using the new HAR statistic to allow for the possible presence of strong dependence in the data. The results show that for the sample period (b) the test fails to reject a unit root null at the 10% level; for period (c) the test rejects the null at the 10% level but fails

Table 4: Finite sample performance of $\hat{r}_e^{HAR}, \hat{r}_f^{HAR}$ when $r_e = 0.5, r_f = 0.7$

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$\alpha = 0.50$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.91	0.91	0.90	0.90
$\hat{r}_e^{HAR}$	0.50	0.51	0.51	0.51	0.51	0.51	0.52	0.52	0.53	0.53
	(0.01)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.03)	(0.03)	(0.04)	(0.05)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)
$\alpha = 0.55$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.91	0.90	0.90	0.88
$\hat{r}_e^{HAR}$	0.51	0.51	0.51	0.51	0.52	0.52	0.52	0.53	0.53	0.54
	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)	(0.03)	(0.04)	(0.04)	(0.05)	(0.06)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.02)
$\alpha = 0.60$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.91	0.90	0.88	0.85
$\hat{r}_e^{HAR}$	0.51	0.51	0.51	0.52	0.52	0.53	0.53	0.54	0.54	0.55
	(0.01)	(0.02)	(0.02)	(0.02)	(0.03)	(0.04)	(0.05)	(0.05)	(0.06)	(0.06)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.71
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)
$\alpha = 0.65$										
Detect. Rate	0.90	0.90	0.91	0.91	0.91	0.91	0.90	0.88	0.85	0.82
$\hat{r}_e^{HAR}$	0.51	0.52	0.52	0.52	0.53	0.53	0.54	0.55	0.55	0.56
	(0.02)	(0.02)	(0.03)	(0.03)	(0.04)	(0.05)	(0.06)	(0.07)	(0.07)	(0.08)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.71	0.71	0.71
	(0.00)	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.01)	(0.02)	(0.02)	(0.02)
$\alpha = 0.70$										
Detect. Rate	0.90	0.90	0.91	0.90	0.90	0.90	0.88	0.83	0.77	0.71
$\hat{r}_e^{HAR}$	0.52	0.52	0.53	0.53	0.54	0.55	0.55	0.56	0.56	0.57
	(0.02)	(0.03)	(0.04)	(0.04)	(0.05)	(0.06)	(0.07)	(0.08)	(0.08)	(0.09)
$\hat{r}_f^{HAR}$	0.70	0.70	0.70	0.70	0.70	0.70	0.71	0.701	0.71	0.71
	(0.00)	(0.00)	(0.00)	(0.01)	(0.01)	(0.02)	(0.02)	(0.03)	(0.03)	(0.03)

to reject a unit root null at the 5% level; and for periods (a), (d), (e) and (f), the test rejects the null at the 5% level. Thus, using the conventional 5% level the four periods (a), (d), (e) and (f) show significant evidence of being bubble episodes in the S&P stock market. Taking into account the findings for the other periods, it is clear that allowing for the presence of strong dependence does change the outcomes, giving statistical evidence only for the existence of explosive behavior in periods (a), (d), (e) and (f). However, these results continue to support the presence of stock market bubble behavior, including the internet bubble of the late 1990s even in the presence of strong dependence.

The bubble dating methodology was used to estimate the origination and termination dates r_e and r_f in sample periods (a), (d), (e) and (f) where bubble behavior was evident

Table 5: Empirical results for the S&P 500 with $\widetilde{DF}_{n,HAR}$ and critical values

Sampling Period	$\hat{d}$	$\widetilde{DF}_{n,HAR}$	$cv_{HAR}^{10\%}\left(\hat{d}\right)$	$cv_{HAR}^{5\%}\left(\hat{d}\right)$	$cv_{HAR}^{1\%}\left(\hat{d}\right)$
(a) Jan 1872 to Feb 1880	0.24	1.25	0.70	0.92	1.30
(b) Jun 1882 to May 1887	0.32	0.62	0.76	0.97	1.36
(c) May 1940 to Feb 1946	0.34	0.89	0.77	0.98	1.38
(d) Jun 1948 to Nov 1955	0.29	1.54	0.74	0.94	1.33
(e) May 1979 to Mar 1987	0.21	1.28	0.67	0.90	1.26
(f) May 1989 to Aug 1997	0.24	1.18	0.70	0.92	1.30

in the data. For this implementation 48 monthly observations were used to initialize estimation, the minimum explosive episode duration was 4 months, and the statistic $\widetilde{DF}_{\tau,HAR}$ and critical value $cv_{n,HAR}^{3\%}\left(\hat{d}_\tau\right)$ were computed recursively, as in PSYa.

Table 6: Empirical results for bubble origination and termination $(\hat{r}_e^{HAR},\ \hat{r}_f^{HAR})$

Sampling period	$\hat{r}_e^{HAR}$	$\hat{r}_f^{HAR}$	Duration
(a) Jan 1872 to May 1880	Oct 1879	Apr 1880	6 months
(d) Jun 1948 to Feb 1957	Dec 1954	Feb 1956	14 months
(e) May 1979 to Jan 1988	Feb 1987	Sep 1987	7 months
(f) May 1989 to Jan 1998	Feb 1997	Nov 1997	9 months

Table 6 reports the estimates $\hat{r}_e^{HAR}$ and $\hat{r}_f^{HAR}$ and associated bubble duration (in months) for episodes (a), (d), (e) and (f). The following conclusions can be drawn from these results. First, in episode (a) a rational bubble is found to originate in October 1879 and collapse in April 1880, lasting six months. Second, in episode (d), the bubble lasts for fourteen months from December 1954 to February 1956. Third, in episode (e), the explosive period begins in February 1987 and ends in September 1987, lasting seven months. Finally, in episode (f), the bubble starts in February 1997 and ends in November 1997, lasting night months.

These findings coincide with those of PSYa in rational bubble identification. In particular, the explosive episodes in the Great Depression, postwar boom, before Black Monday, and the dotcom bubble period are also found using our estimation method. However, while explosive behavior is detected using our methods, the episode durations are often shorter than those obtained by PSYa. In PSYa the explosive episodes in the Great Depression, postwar boom, Black Monday, and dotcom bubble periods were estimated to last for 18 months, 15 months, 15 months and 87 months, respectively. The explosive episodes identified by our method last for 6 months, 14 months, 7 months and 9 months, respectively. The presence of strong dependence in the data therefore does affect bubble duration. Nonetheless, the most striking overall result is that the empirical findings in PSYa of several major bubble episodes in the historical S&P 500 data are sustained using methods that are more robust to data dependence, including possible long memory in the

data.

9 Conclusion

This paper introduces a new right-tailed test and new dating algorithm to detect the presence of explosive episodes in time series data. The approach is motivated by showing empirical evidence of strong dependence in the errors of the autoregressive model employed for estimation and inference. Strongly dependent errors lead to divergent unit root test statistics, thereby leading to potential spurious detection of explosive behavior in traditional right-tailed unit root test statistics. To avert problems of spurious detection, this paper proposes a robust approach to inference using an appropriately self-normalized HAR statistic that accommodates potential strong dependence in the errors. Recursive implementation of this procedure enables consistent estimation of the origination and termination dates of explosive episodes in the data. The proposed test and asymptotics are extended to models with unconditional heteroskedasticity, thereby accommodating features that are known to be relevant in practice, particularly in financial data. Simulations show reliable finite sample performances of the new method in terms of both size and power. An empirical application corroborates the robustness of earlier findings on certain bubble episodes in historical S&P 500 data but leads typically to shorter duration periods of financial exuberance.

This paper has not addressed the complex additional issue of possible multiple bubble episodes in the same time series sample. However, the procedures developed here can be extended to allow for such multiple periods and break points in the data in precisely the same way as PSYa and PSYb. This extension simply involves replacing the use of the DF_τ statistic by $\widetilde{DF}_{\tau,HAR}$ in the PSY algorithm and imposing the conditions used here for consistency in the presence of strong dependence. We expect that when modified in this way the algorithm will retain validity for multiple bubble detection using the robust statistic $\widetilde{DF}_{\tau,HAR}$. This investigation is left for future study.

10 Appendix

10.1 Proofs of the main results

We begin with the following result and two useful lemmas.

Lemma 10.1 (Theorem 3.1 in Silveira (1991)) *Suppose* $u_t = \Delta_+^{-d} \epsilon_t$, $\epsilon_t \overset{iid}{\sim} (0, \sigma^2)$
$\mathbb{E}|\epsilon_1|^{2+\delta} < \infty$, *for some* $\delta > 0$ *and* $d > -0.5$. *Then, as* $n \to \infty$,

$$\frac{1}{\sigma n^{1/2+d}} \sum_{t=1}^{\lfloor nr \rfloor} u_t \Rightarrow W^H(r), \tag{36}$$

in $\mathcal{D}[0,1]$ *with the uniform metric.*

Lemma 10.2 *Suppose the DGP is given by model (9). Let* $\tau = \lfloor nr \rfloor$ *with* $r \in (0,1]$.
Then, as $n \to \infty$,

$$\frac{1}{n^{1+2d}} \sum_{t=1}^{\tau} y_{t-1} u_t \Rightarrow \begin{cases} \frac{\sigma^2}{2} \left[W^2(r) - r \right] & \text{if } d = 0 \\ \frac{\sigma^2}{2} \left(W^H(r) \right)^2 & \text{if } d \in (0, 0.5) \end{cases}, \tag{37}$$

$$\frac{1}{n^{3/2+d}} \sum_{t=1}^{\tau} y_{t-1} \Rightarrow \sigma \int_0^r W^H(s) ds, \tag{38}$$

$$\frac{1}{n^{2+2d}} \sum_{t=1}^{\tau} y_{t-1}^2 \Rightarrow \sigma^2 \int_0^r \left(W^H(s) \right)^2 ds. \tag{39}$$

Suppose the empirical regression (5) is based on $\{y_t\}_{t=1}^{\tau}$. *For* $r \in (0,1]$, *as* $n \to \infty$, *we*
have

$$\tau(\hat{\rho}_\tau - 1) \Rightarrow \begin{cases} \frac{\frac{r}{2} \left((W(r))^2 - r \right)^2 - \left(\int_0^r W(s) ds \right) W(r)}{\int_0^r \left(\tilde{W}(s) \right)^2 ds} & \text{if } d = 0, \\ \frac{\frac{r}{2} \left(W^H(r) \right)^2 - \left(\int_0^r W^H(s) ds \right) W^H(r)}{\int_0^r \left(\tilde{W}^H(s) \right)^2 ds} & \text{if } d \in (0, 0.5) \end{cases}. \tag{40}$$

Furthermore, let $\tilde{\rho}_\tau = \hat{\rho}_\tau + \frac{\frac{1}{2} \sum_{t=1}^{\tau} \triangle y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}$. *We have*

$$\tau(\tilde{\rho}_\tau - 1) \Rightarrow \frac{\frac{r}{2} \left(W^H(r) \right)^2 - \left(\int_0^r W^H(s) ds \right) W^H(r)}{\int_0^r \left(\tilde{W}^H(s) \right)^2 ds} , \text{ for } d \in [0, 0.5). \tag{41}$$

Proofs of Theorem 3.1 and Theorem 3.2

Write

$$DF_{\tau, HAR} = \frac{\hat{\rho}_\tau - 1}{s_{\tau, HAR}} = \frac{\tau(\hat{\rho}_\tau - 1)}{\left(\tau^2 s_{\tau, HAR}^2 \right)^{1/2}}, \tag{42}$$

$$\widetilde{DF}_{\tau, HAR} = \frac{\tau(\tilde{\rho}_\tau - 1)}{\left(\tau^2 s_{\tau, HAR}^2 \right)^{1/2}}. \tag{43}$$

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and to show the limit we first study the denominator in (42) and (43). Note that $s_{\tau,HAR}^2 = \frac{\hat{\Omega}_{HAR}}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}$. For $\hat{\Omega}_{HAR}$, letting $K_{i,j} = K\left(\frac{i-j}{b\tau}\right)$ and $S_t = \sum_{i=1}^{t} \Delta y_i$, we have

$$
\begin{aligned}
\hat{\Omega}_{HAR} &= \sum_{j=-\tau+1}^{\tau} K\left(\frac{j}{b\tau}\right) \hat{\gamma}_j = \frac{1}{\tau} \sum_{i=1}^{\tau} \sum_{i=1}^{\tau} \Delta y_i K_{i,j} \Delta y_j \\
&= \frac{1}{\tau} \sum_{i=1}^{\tau-1} \frac{1}{\tau} \sum_{j=1}^{\tau-1} \tau^2 \left[(K_{i,j} - K_{i,j+1}) - (K_{i+1,j} - K_{i+1,j+1})\right] \frac{1}{\sqrt{\tau}} \hat{S}_i \frac{1}{\sqrt{\tau}} \hat{S}_j \\
&= \frac{1}{\tau} \sum_{i=1}^{\tau-1} \frac{1}{\tau} \sum_{j=1}^{\tau-1} \tau^2 D_\tau\left(\frac{i-j}{b\tau}\right) \frac{1}{\sqrt{\tau}} S_i \frac{1}{\sqrt{\tau}} S_j, \tag{44}
\end{aligned}
$$

where $D_\tau\left(\frac{i-j}{b\tau}\right) = (K_{i,j} - K_{i,j+1}) - (K_{i+1,j} - K_{i+1,j+1})$. The last equality follows from Equation (A.1) in Kiefer and Vogelsang (2002b).

Straightforward calculations show that

$$
D_\tau\left(\frac{i-j}{b\tau}\right) = \begin{cases} \frac{2}{b\tau} & \text{if } |i-j| = 0 \\ -\frac{1}{b\tau} & \text{if } |i-j| = \lfloor b\tau \rfloor, \\ 0 & \text{otherwise} \end{cases}
$$

which implies

$$
\begin{aligned}
\hat{\Omega}_{HAR} &= \sum_{i=1}^{\tau-1} \sum_{j=1}^{\tau-1} D_\tau\left(\frac{i-j}{b\tau}\right) \frac{1}{\sqrt{\tau}} S_i \frac{1}{\sqrt{\tau}} S_j \\
&= \frac{2}{b\tau} \sum_{i=1}^{\tau-1} \left(\frac{1}{\sqrt{\tau}} S_i\right)^2 - \frac{2}{b\tau} \sum_{i=1}^{\tau-\lfloor b\tau \rfloor-1} \left(\frac{1}{\sqrt{\tau}} S_i\right) \left(\frac{1}{\sqrt{\tau}} S_{i+\lfloor b\tau \rfloor}\right) \\
&= \frac{2}{b} \frac{n}{\lfloor nr \rfloor} \frac{1}{n} \sum_{i=1}^{\tau-1} \left(\frac{1}{\sqrt{\tau}} S_i\right)^2 - \frac{2}{b} \frac{n}{\lfloor nr \rfloor} \frac{1}{n} \sum_{i=1}^{\tau-\lfloor b\tau \rfloor-1} \left(\frac{1}{\sqrt{\tau}} S_i\right) \left(\frac{1}{\sqrt{\tau}} S_{i+\lfloor b\tau \rfloor}\right). \tag{45}
\end{aligned}
$$

Thus, with $i = \lfloor np \rfloor$ and under the assumption $\rho_n = 1$, we have $S_i = \sum_{j=1}^{i} \Delta y_j = \sum_{j=1}^{\lfloor np \rfloor} u_j$. This implies that

$$
\frac{1}{n^d} \frac{1}{\sqrt{\tau}} S_{\lfloor np \rfloor} = \left(\frac{n}{\tau}\right)^{1/2} \frac{1}{n^{1/2+d}} \sum_{t=1}^{\lfloor np \rfloor} u_i \Rightarrow \frac{\sigma}{r^{1/2}} W^H(p). \tag{46}
$$

Therefore

$$
\begin{aligned}
\frac{1}{n^{2d}} \hat{\Omega}_{HAC} &= \frac{2n}{b\tau} \frac{1}{n} \sum_{i=1}^{\tau-1} \left(\frac{1}{n^d} \frac{1}{\sqrt{\tau}} S_{\lfloor np \rfloor}\right)^2 - \frac{2n}{b\tau} \frac{1}{n} \sum_{i=1}^{\tau-\lfloor b\tau \rfloor-1} \left(\frac{1}{n^d} \frac{1}{\sqrt{\tau}} S_i\right) \left(\frac{1}{n^d} \frac{1}{\sqrt{\tau}} S_{i+\lfloor b\tau \rfloor}\right) \\
&\Rightarrow \frac{2}{br} \int_0^r \left(\frac{\sigma}{r^{1/2}} W^H(p)\right)^2 dp - \frac{2}{br} \int_0^{(1-b)r} \frac{\sigma^2}{r} W^H(p) W^H(p+br) dp \tag{47}
\end{aligned}
$$

$$= \frac{2\sigma^2}{br^2} \left(\int_0^r \left(W^H(p) \right)^2 dp - \int_0^{(1-b)r} W^H(p) W^H(p+br) dp \right), \tag{48}$$

where we have applied (46) and continuous mapping to obtain the limit (47).

Combining (38) and (47), upon normalization we have

$$\begin{aligned}
\tau^2 s_{\tau,HAR}^2 &= \left(\frac{\tau}{n} \right)^2 \frac{\frac{1}{n^{2d}} \hat{\Omega}_{HAR}}{\frac{1}{n^{2+2d}} \left(\sum_{t=1}^\tau y_{t-1}^2 - \tau^{-1} \left(\sum_{t=1}^\tau y_{t-1} \right)^2 \right)} \\
&\Rightarrow \frac{2 \left(\int_0^r W^H(p)^2 dp - \int_0^{(1-b)r} W^H(p) W^H(p+br) dp \right)}{b \int_0^r \left(\tilde{W}^H(s) \right)^2 ds}.
\end{aligned} \tag{49}$$

We now proceed to obtain the limit of $DF_{\tau,HAR}$. When $d=0$, we have

$$DF_{\tau,HAR} = \frac{\tau \left(\hat{\rho}_\tau - 1 \right)}{\left(\tau^2 s_{\tau,HAR}^2 \right)^{1/2}} = \frac{\lfloor nr \rfloor}{nr} r \frac{n \left(\hat{\rho}_\tau - 1 \right)}{\left(\tau^2 s_{\tau,HAR}^2 \right)^{1/2}}$$

$$\Rightarrow \frac{r \int_0^r \tilde{W}(s) dW(s)}{\int_0^r \left(\tilde{W}(s) \right)^2 ds} \left(\frac{b \int_0^r \tilde{W}(s)^2 ds}{2 \left(\int_0^r \left(W(p) \right)^2 dp - \int_0^{(1-b)r} W(p) W(p+br) dp \right)} \right)^{1/2}$$

$$= \frac{b^{1/2} r \int_0^r \tilde{W}(s) dW(s)}{\left[2 \int_0^r \left(\tilde{W}(s) \right)^2 ds \left(\int_0^r \left(W(p) \right)^2 dp - \int_0^{(1-b)r} W(p) W^H(p+br) dp \right) \right]^{1/2}},$$

where the standard result $n \left(\hat{\rho}_\tau - 1 \right) \Rightarrow \int_0^r \tilde{W}(s) dW(s) / \int_0^r \left(\tilde{W}(s) \right)^2 ds$ and (49) are used with $H = 1/2$.

For $d \in (0, 0.5)$, similarly write

$$DF_{\tau,HAR} = \frac{\tau \left(\hat{\rho}_\tau - 1 \right)}{\left(\tau^2 s_{\tau,HAR}^2 \right)^{1/2}}$$

$$\Rightarrow \frac{\frac{r}{2} \left(W^H(r) \right)^2 - \left(\int_0^r W^H(s) ds \right) W^H(r)}{\int_0^r \left(\tilde{W}^H(s) \right)^2 ds} \left(\frac{b \int_0^r \tilde{W}^H(s)^2 ds}{2 \left(\int_0^r \left(W^H(p) \right)^2 dp - \int_0^{(1-b)r} W^H(p) W^H(p+br) dp \right)} \right)^{1/2}$$

$$= \frac{\frac{rb^{1/2}}{2} \left(W^H(r) \right)^2 - b^{1/2} \left(\int_0^r W^H(s) ds \right) W^H(r)}{\left[2 \int_0^r \left(\tilde{W}^H(s) \right)^2 ds \left(\int_0^r \left(W^H(p) \right)^2 dp - \int_0^{(1-b)r} W^H(p) W^H(p+br) dp \right) \right]^{1/2}}, \tag{50}$$

where the limit is obtained using (40) and (49).

For $\widetilde{DF}_{\tau,HAR}$, using (41) and (49), we have

$$\widetilde{DF}_{\tau,HAR} = \frac{\tau \left(\tilde{\rho}_\tau - 1 \right)}{\left(\tau^2 s_{\tau,HAR}^2 \right)^{1/2}}$$

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$$\Rightarrow \quad \frac{\frac{rb^{1/2}}{2}\left(W^H(r)\right)^2 - b^{1/2}\left(\int_0^r W^H(s)ds\right)W^H(r)}{\left[2\int_0^r\left(\tilde{W}^H(s)\right)^2 ds\left(\int_0^r W^H(p)^2 dp - \int_0^{(1-b)r} W^H(p)W^H(p+br)dp\right)\right]^{1/2}},$$

which completes the proof of Theorems 3.1 and 3.2. ∎

Proof of Theorem 3.3 The next lemmas are useful in what follows.

Lemma 10.3 *Under the assumptions of Theorem 3.3, as $n \to \infty$ with $m = n^\gamma$, we have*

$$\Delta_+^{1+\hat{d}}y_t = \Delta_+^{1+d}y_t + O_p(m^{-1}). \tag{51}$$

Lemma 10.4 *[Lemma 12 in Mikusheva (2007)] Let $\{\epsilon_{n,j}^* : j = 1,...,n\}$ be a triangular array of random variables, such that for every n, $\{\epsilon_{n,j}^*\}_{j=1}^n$ are i.i.d. with a cumulative distribution function (CDF) $F_n \in \mathcal{L}_r(K,M,\theta)$. Then we can construct a process $\eta_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor nr \rfloor}\epsilon_{n,j}^*$ and a Brownian motion w_n on a common probability space such that for every $\varepsilon > 0$*

$$\lim_{n\to\infty}\sup_{F_n\in\mathcal{L}_r(K,M,\theta)}\Pr\left\{\sup_{0\le t\le 1}|\eta_n(t) - w_n(t)| > \varepsilon n^{-\delta}\right\} = 0, \tag{52}$$

for some $\delta > 0$, where $\mathcal{L}_r(K,M,\theta)$ is a class which satisfies the following three conditions:

1. $\mu_1(F_n) = 0$;

2. $\mu_2(F_n) = \sigma_n^2$, *where $|\sigma_n^2 - 1| \le Mn^{-\theta}$*;

3. $\sup_n |\mu|_r(F_n) < K$.

   where $\mu_j(F)$ and $|\mu|_j(F)$ are the jth central and absolute moments of F, respectively, and M and K are positive constants.

The results in Lemma 10.4 use an expanded common probability space in which a weakly convergent sequence can be represented by a sequence that converges almost surely via Skorohod representation (see, e.g. Pollard (1984)). Throughout the proof of Theorem 3.3, random sequences are assumed to belong to this common probability space.

We can now prove Theorem 3.3. We first show that the bootstrap residuals fall into the class of $\mathcal{L}_r(K,M,\theta)$ in Lemma 10.4 and verify the three conditions in the lemma. The first condition is satisfied because the residuals are centered. For the third condition, note that

$$e_{n,t} = \Delta_+^{1+\hat{d}}y_t = \Delta_+^{1+d}y_t + R_n = \varepsilon_t + R_n,$$

where $R_n = O_p(m^{-1})$, and the second equality is established from Lemma 10.3. Further,

$$\frac{1}{n}\sum_{t=1}^n |e_{n,t}|^r = \frac{1}{n}\sum_{t=1}^n |\varepsilon_t + R_n|^r \le C_r\frac{1}{n}\sum_{t=1}^n |\varepsilon_t|^r + C_r\frac{1}{n}\sum_{t=1}^n |R_n|^r. \tag{53}$$

Note that the first term $\frac{1}{n}\sum_{t=1}^n |\varepsilon_t|^r$ is bounded almost surely by virtue of the strong law of large numbers, and with Lemma 10.3, the second term converges almost surely to zero via the Skorohod representation theorem. This verifies the third condition in Lemma 10.4.

For the second condition, since $\bar{e}_{n,t} \equiv \frac{1}{n} \sum_{t=1}^{n} e_{n,t} \overset{as}{\to} 0$ and $\hat{\sigma}_e^2 \equiv \frac{1}{n} \sum_{t=1}^{n} (\varepsilon_t + R_n)^2 \overset{as}{\to} \sigma^2$, therefore

$$
\begin{aligned}
\frac{1}{n} \sum_{t=1}^{n} e_{n,t}^{*}{}^2 - 1 &= \frac{1}{n} \sum_{t=1}^{n} \left(\frac{e_{n,t} - \bar{e}_{n,t}}{\hat{\sigma}_e} \right)^2 - 1 = \frac{1}{\hat{\sigma}_e^2} \left(\frac{1}{n} \sum_{t=1}^{n} \left(e_{n,t}^2 - 2e_{n,t}\bar{e}_{n,t} + \bar{e}_{n,t}^2 \right) \right) - 1 \\
&= \frac{1}{\hat{\sigma}_e^2} \left(\frac{1}{n} \sum_{t=1}^{n} e_{n,t}^2 - 2\bar{e}_{n,t} \frac{1}{n} \sum_{t=1}^{n} e_{n,t} + \bar{e}_{n,t}^2 \right) - 1 \overset{as}{\to} 1 - 1 = 0.
\end{aligned}
$$

This verifies the second condition and we can apply the approximation in (52).

Note that $y_t^* = y_{t-1}^* + u_t^*$ with $u_t^* = \Delta_+^{-\hat{d}} e_{n,t}^*$ and so $y_t^* = \Delta_+^{-(1+\hat{d})} e_{n,t}^*$. Let $\pi_j^d = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$ and applying a similar argument to (51) we have

$$
\begin{aligned}
y_t^* &= \hat{\sigma}_e \Delta_+^{-(1+\hat{d})} e_{n,t}^* = \hat{\sigma}_e \Delta_+^{-(1+d)} e_{n,t}^* + O_p(m^{-1}) \\
&= \hat{\sigma}_e \Delta_+^{-d_y} e_{n,t}^* + O_p(m^{-1}) = \hat{\sigma}_e \sum_{j=1}^{t} \pi_{t-j}^{d_y} e_{n,j}^* + O_p(m^{-1}).
\end{aligned}
$$

Set $\phi = \lfloor nr \rfloor$, $S_j^* = \sum_{t=1}^{j} e_{n,t}^*$, $Y_{\lfloor nr \rfloor}^* = n^{1/2 - d_y} y_{\lfloor nr \rfloor}^*$ and write

$$
Y_{\lfloor nr \rfloor}^* = n^{1/2 - d_y} \hat{\sigma}_e \sum_{j=1}^{\lfloor nr \rfloor} \pi_{t-j}^{d_y} e_{n,j}^* + o_p(1) = n^{1/2 - d_y} \hat{\sigma}_e \sum_{j=1}^{\phi} \pi_{t-j}^{d_y} \left(S_j^* - S_{j-1}^* \right) + o_p(1).
$$

Following Silveira (1991), letting $V_j = \sum_{i=1}^{j} z_i$ and $z_t \overset{iid}{\sim} \mathcal{N}(0,1)$, we have

$$
Y_n^*(r) = Q_{1n}(r) + Q_{2n}(r) + Q_{3n}(r) + Q_{4n}(r) + o_p(1),
$$

where

$$
\begin{aligned}
Q_{1n}(r) &= \hat{\sigma}_e \left(n^{1/2 - d_y} \sum_{j=1}^{\phi-1} \frac{(\phi - j)^{d_y - 1}}{\Gamma(d_y)} (V_j - V_{j-1}) \right), \\
Q_{2n}(r) &= \hat{\sigma}_e n^{1/2 - d_y} \sum_{j=1}^{\phi-1} \pi_{\phi-j}^{d_y} \left[\left(S_j^* - S_{j-1}^* \right) - (V_j - V_{j-1}) \right], \\
Q_{3n}(r) &= \hat{\sigma}_e n^{1/2 - d_y} \sum_{j=1}^{\phi-1} \left(\pi_{\phi-j}^{d_y} - \frac{(\phi - j)^{d_y - 1}}{\Gamma(d_y)} \right) (V_j - V_{j-1}), \\
Q_{4n}(r) &= \hat{\sigma}_e n^{1/2 - d_y} \left(S_\phi^* - S_{\phi-1}^* \right).
\end{aligned}
$$

Silveira (1991) shows that

$$
n^{1/2 - d_y} \sum_{j=1}^{\phi-1} \frac{(\phi - j)^{d_y - 1}}{\Gamma(d_y)} (V_j - V_{j-1}) \Rightarrow W^H(r),
$$

$$n^{1/2-d_y} \sum_{j=1}^{\phi-1} \left(\pi_{\phi-j}^{d_y} - \frac{(\phi-j)^{d_y-1}}{\Gamma(d_y)} \right) (V_j - V_{j-1}) \xrightarrow{p} 0.$$

We can also show $Q_{4n}(r) = o_p(1)$ by applying Donsker's theorem for martingale difference arrays (MDAs) as in Theorem 27.14 of Davidson (1994). Coupled with $\hat{\sigma}_e \xrightarrow{p} \sigma$, we find that $Q_{1n}(r) \Rightarrow \sigma W^H(r)$, $Q_{3n}(r) = o_p(1)$, and $Q_{4n}(r) = o_p(1)$.

To show $Q_{2n}(r) = o_p(1)$, note that

$$\sup_r |Q_{2n}(r)| \leq \hat{\sigma}_e \sup_r \sum_{j=1}^{\phi-1} |\pi_{\phi-j}^{d_y-1}| \sup_{j \leq n} n^{1/2-d_y} |S_j - V_j|$$

$$= \hat{\sigma}_e \frac{1}{n^d} \sup_r \sum_{j=1}^{\phi-1} |\pi_{\phi-j}^{d_y-1}| \sup_{j \leq n} \left| \frac{S_j - V_j}{n^{1/2}} \right| \leq C\hat{\sigma}_e \frac{1}{n^d} \sum_{j=1}^{n-1} (n-j)^{d_y-2} \sup_{j \leq n} \left| \frac{S_j - V_j}{n^{1/2}} \right|$$

$$= C\hat{\sigma}_e \frac{1}{n^d} \sum_{j=1}^{n-1} j^{d_y-2} \times O_p(n^{-\delta}), \tag{54}$$

where C is a constant and $S_j = \sum_{t=1}^{j} e_t$ with $e_t \sim_{i.i.d.} \mathcal{N}(0,1)$. Note that $\sup_r \sum_{j=1}^{\phi-1} |\pi_{\phi-j}^{d_y-1}| \leq \sum_{j=1}^{n-1} (n-j)^{d_y-2}$ is obtained by applying Lemma 3-A-2 in Silveira (1991) and the last equality is due to Lemma 10.4.

If $d = 0$, $\sum_{j=1}^{n-1} j^{d_y-2} = \sum_{j=1}^{n-1} \frac{1}{j}$ diverges at a $\log n$ rate and is dominated by $O_p(n^{-\delta})$. If $d > 0$, $\sum_{j=1}^{n-1} j^{d_y-2}$ diverges at the n^d rate and this divergence is neutralized by the factor $\frac{1}{n^d}$, so that the whole term in (54) is of order $O_p(n^{-\delta})$ in this case. We deduce that $Q_{2n}(r) = o_p(1)$ and $\frac{1}{n^{1/2+d}} y_{\lfloor nr \rfloor}^* \Rightarrow \sigma W^H(r)$. Then by repeated application of the continuous mapping theorem (CMT) and analysis analogous to Lemma 10.2 and Theorem 3.2, we obtain $\widetilde{DF}_{\tau,HAR} \Rightarrow F_{r,d}$.

This result implies that the CDF of $\widetilde{DF}_{n,HAR}$ converges to the CDF of $F_{1,d}$ uniformly in probability. Therefore, $p^*(\widetilde{DF}_{n,HAR}) \Rightarrow U[0,1]$ under the null hypothesis and the proof of Theorem 3.3 is completed. ∎

Proof of Remark 3.4

As in (44), write

$$\hat{\Omega}_{HAR} = \frac{1}{\tau} \sum_{i=1}^{\tau-1} \frac{1}{\tau} \sum_{j=1}^{\tau-1} \tau^2 D_\tau \left(\frac{i-j}{b\tau} \right) \frac{1}{\sqrt{\tau}} S_i \frac{1}{\sqrt{\tau}} S_j,$$

and following the steps in proving Theorem 4 in Sun (2004), we can show $\lim_{n \to \infty} \tau^2 D_\tau \left(\frac{i-j}{b\tau} \right) = -\frac{1}{b^2 r^2} K'' \left(\frac{p-q}{br} \right)$, given $(i/n, j/n) \to (p, q)$. Combining (44), (46) and applying the CMT, we have

$$\frac{1}{n^{2d}} \hat{\Omega}_{HAR} = \frac{1}{\tau} \sum_{i=1}^{\tau-1} \frac{1}{\tau} \sum_{j=1}^{\tau-1} D_\tau \left(\frac{i-j}{b\tau} \right) \frac{1}{n^d} \frac{1}{\sqrt{\tau}} \hat{S}_i \frac{1}{n^d} \frac{1}{\sqrt{\tau}} \hat{S}_j$$

$$\Rightarrow \quad -\frac{\sigma^2}{b^2 r^3} \int_0^r \int_0^r K'' \left(\frac{p-q}{br}\right) W^H(p) W^H(q) dp dq. \qquad (55)$$

Since $s^2_{\tau,HAR} = \frac{\hat{\Omega}_{HAR}}{\sum_{t=1}^{\tau} y_{t-1}^2 - \tau^{-1} \left(\sum_{t=1}^{\tau} y_{t-1}\right)^2}$, Lemma 10.2 and (55) give

$$\tau^2 s^2_{\tau,adj} \Rightarrow \frac{\int_0^r \int_0^r -K'' \left(\frac{p-q}{br}\right) W^H(p) W^H(q) dp dq}{b^2 r \int_0^r \left(\tilde{W}^H(s)\right)^2 ds}. \qquad (56)$$

Finally, since $\widetilde{DF}_{\tau,HAR} = \frac{\tau(\tilde{\rho}_\tau - 1)}{\left(\tau^2 s^2_{\tau,HAR}\right)^{1/2}}$, and using (41) and (56), standard calculation yields

$$\widetilde{DF}_{\tau,HAR} \Rightarrow \frac{\frac{br^{3/2}}{2} \left(W^H(r)\right)^2 - br^{1/2} \left(\int_0^r W^H(s) ds\right) W^H(r)}{\left(\left(\int_0^r \left(\tilde{W}^H(s)\right)^2 ds\right) \int_0^r \int_0^r -K'' \left(\frac{p-q}{br}\right) W^H(p) W^H(q) dp dq\right)^{1/2}}.$$

This completes the proof of Remark 3.4. ∎

In proving Theorem 4.1 the following lemma is useful.

Lemma 10.5 *Let $\tau = \lfloor nr \rfloor$ with $r \in (r_e, 1]$. Then, under the local alternative model (21), as $n \to \infty$.*

1. $\frac{1}{n^{1/2+d}} y_\tau \Rightarrow \sigma \left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^r e^{(r-s)c} dW^H(s)\right)$;

2. $\frac{1}{n^{3/2+d}} \sum_{t=1}^{\tau} y_{t-1} \Rightarrow \sigma A_{r,d}$;

3. $\frac{1}{n^{2+2d}} \sum_{t=1}^{\tau} y_{t-1}^2 \Rightarrow \sigma^2 B_{r,d}$;

4. $\frac{1}{n^{1+2d}} \left(\sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2} \sum_{t=1}^{\tau} \Delta y_t^2\right) \Rightarrow \frac{\sigma^2}{2} C_{r,d}$;

5. $n(\tilde{\rho}_\tau - \rho_c) \Rightarrow X_c(r, d)$;

6. $n(\tilde{\rho}_\tau - 1) \Rightarrow X_c(r, d) + c$,

where

$$A_{r,d} = \int_0^r \left(e^{(x-r_e)c} W^H(r_e) + \int_{r_e}^x e^{(x-s)c} dW^H(s)\right) dx,$$

$$B_{r,d} = \int_0^r \left(e^{(x-r_e)c} W^H(r_e) + \int_{r_e}^x e^{(x-s)c} dW^H(s)\right)^2 dx,$$

$$C_{r,d} = \left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^r e^{(r-s)c} dW^H(s)\right)^2 - W^H(r_e)^2,$$

$$X_c(r,d) = \frac{\frac{1}{2} C_{r,d} - \frac{1}{r} A_{r,d} W^H(r)}{B_{r,d} - \frac{1}{r} A_{r,d}^2},$$

$$Y_c(r,d) = \frac{B_{r,d} W^H(r) - \frac{1}{2} C_{r,d} A_{r,d}}{r \left(B_{r,d} - A_{r,d}^2\right)}.$$

Proof of Theorem 4.1

From (45)

$$\hat{\Omega}_{HAR} = \frac{2}{b} \frac{n}{\lfloor nr \rfloor} \frac{1}{n} \sum_{i=1}^{\tau-1} \left(\frac{1}{\sqrt{\tau}} S_i \right)^2 - \frac{2}{b} \frac{n}{\lfloor nr \rfloor} \frac{1}{n} \sum_{i=1}^{\tau-\lfloor b\tau \rfloor -1} \left(\frac{1}{\sqrt{\tau}} S_i \right) \left(\frac{1}{\sqrt{\tau}} S_{i+\lfloor b\tau \rfloor} \right), \tag{57}$$

where $S_{\lfloor np \rfloor} = \sum_{i=1}^{\lfloor np \rfloor} \Delta y_i$. Write the partial sum $S_{\lfloor np \rfloor} = \sum_{i=1}^{\lfloor np \rfloor} \Delta y_i$ as

$$
\begin{aligned}
S_{\lfloor np \rfloor} &= \sum_{i=1}^{\tau_e-1} \Delta y_i + \sum_{i=\tau_e}^{\lfloor np \rfloor} \Delta y_i = \sum_{i=1}^{\tau_e-1} u_i + \frac{c}{n} \sum_{i=\tau_e}^{\lfloor np \rfloor} y_{i-1} + \sum_{i=\tau_e}^{\lfloor np \rfloor} u_i \\
&= \sum_{i=1}^{\lfloor np \rfloor} u_i + \frac{c}{n} \sum_{i=1}^{\lfloor np \rfloor} y_{i-1} - \frac{c}{n} \sum_{i=1}^{\tau_e-1} y_{i-1}.
\end{aligned}
$$

Upon normalization, we have

$$
\begin{aligned}
&\frac{1}{n^{1/2+d}} \hat{S}_{\lfloor np \rfloor} \\
&= \frac{1}{n^{1/2+d}} \sum_{i=1}^{\lfloor np \rfloor} u_i + \frac{c}{n^{3/2+d}} \sum_{i=1}^{\lfloor np \rfloor} y_{i-1} - \frac{c}{n^{3/2+d}} \sum_{i=1}^{\tau_e-1} y_{i-1} \\
&\Rightarrow \sigma \left(W^H(p) + cA_{p,d} - \int_0^{r_e} W^H(p)dp \right) := \sigma G_{r_e,c}(p). \tag{58}
\end{aligned}
$$

Thus, combining (57) and (58), as $n \to \infty$,

$$\frac{1}{n^{2d}} \hat{\Omega}_{HAR} \Rightarrow \frac{2\sigma^2}{br^2} \left(\int_0^r G_{r_e,c}(d,p)^2 dp - \int_0^{(1-b)r} G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp \right).$$

With Lemma 10.5,

$$
\begin{aligned}
\tau^2 s_{\tau,HAR}^2 &= \left(\frac{\tau}{n} \right)^2 \frac{\frac{1}{n^{2d}} \hat{\Omega}_{HAR}}{\frac{1}{n^{2+2d}} \left(\sum_{t=1}^{\tau} y_{t-1}^2 - \tau^{-1} \left(\sum_{t=1}^{\tau} y_{t-1} \right)^2 \right)} \\
&\Rightarrow \frac{\frac{2}{b} \left(\int_0^r G_{r_e,c}(d,p)^2 dp - \int_0^{(1-b)r} G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp \right)}{B_{r,d} - \frac{1}{r} A_{r,d}^2}. \tag{59}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widetilde{DF}_{\tau,HAR} &= \frac{\lfloor nr \rfloor}{nr} \frac{nr (\tilde{\rho}_\tau - 1)}{\left(\tau^2 s_{\tau,HAR}^2 \right)^{1/2}} \\
&\Rightarrow \frac{\frac{\left(\frac{1}{2} C_{r,d} - \frac{1}{r} A_{r,d} W^H(r) \right) r}{B_{r,d} - \frac{1}{r} A_{r,d}^2} + cr}{\sqrt{\frac{\frac{2}{b} \left(\int_0^r G_{r_e,c}(d,p)^2 dp - \int_0^{(1-b)r} G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp \right)}{B_{r,d} - \frac{1}{r} A_{r,d}^2}}}
\end{aligned}
$$

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$$= \frac{\left(\frac{\left(\frac{1}{2}C_{r,d} - \frac{1}{r}A_{r,d}W^H(r) \right)r}{B_{r,d} - \frac{1}{r}A_{r,d}^2} + cr \right) \left(B_{r,d} - \frac{1}{r}A_{r,d}^2 \right)^{1/2}}{\left[\frac{2}{b} \left(\int_0^r G_{r_e,c}(d,p)^2 dp - \int_0^{(1-b)r} G_{r_e,c}(d,p)G_{r_e,c}(d,p+br)dp \right) \right]^{1/2}},$$

where the limit holds by applying Lemma 10.5.6 and (59) since $\frac{\lfloor nr \rfloor}{nr} \to 1$. This completes the proof of Theorem 4.1. ∎

Proofs of Theorems 4.2 and 5.1

These proofs are similar and are combined. Since the error u_t involves two memory parameters in non-explosive periods and the explosive period (viz., d_1 and d_2), we write u_t as $u_{t,d}$ when u_t is an $FI(d)$ process. The lemmas below are useful in the following analysis.

Lemma 10.6 *Let $B = [\tau_e, \tau_f]$ be the bubble period, $N_0 \in [1, \tau_e)$ and $N_1 = [\tau_f + 1, n]$ are the normal market periods before and after the bubble period. Under the DGP (25), with $t = \lfloor nr \rfloor$, we have the following asymptotic approximations:*

1. *For $t \in N_0$, $y_t \overset{a}{\sim} n^{1/2+d_1} \sigma W^{H_1}(r_e)$.*

2. *For $t \in B$, $y_t \overset{a}{\sim} \rho_n^{(t-\tau_e)} n^{1/2+d_1} \sigma W^{H_1}(r_e)$.*

3. *For $t \in N_1$, $y_{\lfloor nr \rfloor} \overset{a}{\sim} n^{1/2+d_1} \left[\sigma \left(W^{H_1}(r) - W^{H_1}(r_f) \right) + \sigma W^{H_1}(r_e) \right],$*

   where $W^H(r)$ is a Type II fBM with the Hurst parameter $H = 1/2 + d$.

Lemma 10.7 *For the sample average,*

1. *For $\tau \in B$, $\frac{1}{\tau} \sum_{j=1}^{\tau} y_j \overset{a}{\sim} n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e)$.*

2. *For $\tau \in N_1$, $\frac{1}{\tau} \sum_{j=1}^{\tau} y_j \overset{a}{\sim} n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e)$.*

Lemma 10.8 *Let $\bar{y}_t = y_t - \frac{1}{\tau} \sum_{j=1}^{\tau} y_{j-1}$.*

1. *For $\tau \in B$, if $t \in N_0$,*

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e); \tag{60}$$

   if $t \in B$,

$$\bar{y}_t \overset{a}{\sim} \left(\rho_n^{(t-\tau_e)} - \frac{n^\alpha}{nrc} \rho_n^{\tau-\tau_e} \right) n^{1/2+d_1} \sigma W^{H_1}(r_e). \tag{61}$$

2. *For $\tau \in N_1$, if $t \in N_0$,*

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e), \tag{62}$$

   if $t \in B$,

$$\bar{y}_t \overset{a}{\sim} \left(\rho_n^{(t-\tau_e)} - \frac{n^\alpha}{nrc} \rho_n^{\tau-\tau_e} \right) n^{1/2+d_1} \sigma W^{H_1}(r_e), \tag{63}$$

   if $t \in N_1$,

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e). \tag{64}$$

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Lemma 10.9 *The sample variance terms involving $\bar{y}_t$ behave as follows.*

1. *If $\tau \in B$,*

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+2d_1+\alpha} \frac{\rho_n^{2(\tau-\tau_e)}}{2c} \sigma^2 W^{H_1}(r_e)^2. \tag{65}$$

2. *If $\tau \in N_1$,*

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+\alpha+2d_1} \frac{\rho_n^{2(\tau_f-\tau_e)}}{2c} \sigma^2 W^{H_1}(r_e)^2. \tag{66}$$

Lemma 10.10 *The sample variances of $\bar{y}_t$ and u_t behave as follows:*

1. *For $\tau \in B$,*

$$\sum_{j=1}^{\tau} \bar{y}_{j-1} u_j = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{ \frac{1+\alpha+f(d_1)}{2}+d_1, \alpha+2d_1 \right\}} \right). \tag{67}$$

2. *For $\tau \in N_1$,*

$$\sum_{j=1}^{\tau} \bar{y}_{j-1} u_j = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{ \frac{1+\alpha+f(d_2)}{2}+d_1, \alpha+2d_1 \right\}} \right), \tag{68}$$

where

$$f(d_2) = \begin{cases} 1 & \text{if } d_2 \in [0, 0.5) \\ 1+\epsilon & \text{if } d_2 = 0.5 \\ 2d_2 & \text{if } d_2 > 0.5 \end{cases}, \epsilon > 0.$$

Lemma 10.11 *The sample covariances of $\bar{y}_{j-1}$ and $y_j - \rho_n y_{j-1}$ behave as follows:*

1. *For $\tau \in B$,*

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{ \frac{1+\alpha+f(d_2)}{2}+d_1, 2d_1+1 \right\}} \right). \tag{69}$$

2. *For $\tau \in N_1$,*

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) \overset{a}{\sim} -\rho_n^{2(\tau_f-\tau_e)} n^{1+2d_1} \sigma^2 \left(W^{H_1}(r_e) \right)^2.$$

Lemma 10.12 *For $\sum_{t=1}^{\tau} \Delta y_t^2$, the following asymptotics apply:*

1. *When $\tau \in B$,*

$$\sum_{t=1}^{\tau} \Delta y_t^2 = O_p(n^{1+2d_1-\alpha} \rho_n^{2(\tau_f-\tau_e)}). \tag{70}$$

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2. When $\tau \in N_1$,

$$\sum_{t=1}^{\tau} \Delta y_t^2 = O_p(n^{1+2d_1} \rho_n^{2(\tau_f - \tau_e)}). \tag{71}$$

Lemma 10.13 *For the LS estimator $\hat{\rho}_\tau$, the following asymptotics hold:*

1. *When $\tau \in B$, $n(\tilde{\rho}_\tau - 1) = n^{1-\alpha}c + o_p(1) \xrightarrow{p} \infty$.*

2. *When $\tau \in N_1$, $n(\tilde{\rho}_\tau - 1) = -n^{1-\alpha}c + o_p(1) \xrightarrow{p} -\infty$.*

Lemma 10.14 *Under model (25), we have the following asymptotics*

$$\hat{\Omega}_{HAR} = O_p\left(n^{2d_1} \rho_n^{2(\tau - \tau_e)}\right),$$

$$s_\tau^2 = \frac{\frac{1}{\tau}\sum_{t=1}^{\tau} \hat{u}_t^2}{\frac{1}{\tau}\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} = O_p(n^{-2-\alpha}), \ for \ \tau \in B.$$

We are now in the position to prove Theorems 4.2 and 5.1. Recall $DF_\tau = \frac{\hat{\rho}_\tau - 1.}{s_\tau}$ and suppose that $\tau \in B$. Applying Lemma 10.9.1, 10.13.1 and 10.14, we obtain

$$\frac{\hat{\rho}_\tau - 1}{s_\tau} = O_p(n^{1+\alpha/2})\frac{c}{n^\alpha} = O_p(n^{1-\alpha/2}). \tag{72}$$

This proves the first claim in Theorem 4.2.

Note that $\widetilde{DF}_{\tau,HAR} = \left(\frac{\sum_{i=1}^{\tau} \bar{y}_{i-1}^2}{\hat{\Omega}_{HAR}}\right)^{1/2} (\tilde{\rho}_\tau - 1)$. Suppose that $\tau \in B$. As in showing (72), we find that

$$\left(\frac{\sum_{i=1}^{\tau} \bar{y}_{i-1}^2}{\hat{\Omega}_{HAR}}\right)^{1/2} (\tilde{\rho}_\tau - 1) = O_p\left(\frac{n^{1+\alpha+2d_1} \rho_n^{2(\tau - \tau_e)}}{n^{2d_1} \rho_n^{2(\tau - \tau_e)}}\right)^{1/2} \frac{c}{n^\alpha}$$

$$= O_p\left(n^{\frac{1-\alpha}{2}}\right) \to \infty,$$

which gives the second claim of Theorem 4.2.

Suppose that $\tau \in N_1$. Applying the results in Lemma 10.9.1, 10.13.2 and 10.14, we have

$$\left(\frac{\sum_{i=1}^{\tau} \bar{y}_{i-1}^2}{\hat{\Omega}_{HAR}}\right)^{1/2} (\tilde{\rho}_\tau - 1) = O_p\left(\frac{n^{1+\alpha+2d_1} \rho_n^{2(\tau_f - \tau_e)}}{n^{2d_1} \rho_n^{2(\tau - \tau_e)}}\right)^{1/2} \left(-\frac{c}{n^\alpha}\right)$$

$$= -O_p\left(n^{\frac{1-\alpha}{2}}\right) \to -\infty.$$

To show $\hat{r}_e^{HAR} \xrightarrow{p} r_e$ and $\hat{r}_f^{HAR} \xrightarrow{p} r_f$, note that if $\tau \in N_0$,

$$\lim_{n\to\infty} \Pr(\widetilde{DF}_{\tau,HAR} > cv_{n,HAR}) = \Pr(F_{r,d} > \infty) = 0.$$

If $\tau \in B$, $\lim_{n\to\infty} \Pr(\widetilde{DF}_{\tau,HAR} > cv_{n,HAR}) = 1$, given that $\frac{cv_{n,HAR}}{n^{(1-\alpha)/2}} \to 0$. If $\tau \in N_1$, $\lim_{n\to\infty} \Pr(\widetilde{DF}_{\tau,HAR} > cv_{n,HAR}) = 0$, as $\widetilde{DF}_{\tau,HAR} = -O_p\left(n^{\frac{1-\alpha}{2}}\right)$. It follows that, for any $\eta, \vartheta > 0$, we have

$$\Pr(\hat{r}_e^{HAR} > r_e + \eta) \to 0, \text{ and } \Pr(\hat{r}_f^{HAR} < r_f + \vartheta) \to 0,$$

due to the fact that $\Pr(\widetilde{DF}_{(\tau_e + \alpha_\eta/n),HAR} > r_e + \eta) \to 1$ for all $0 < \alpha_\eta < \eta$ and $\Pr(\widetilde{DF}_{(\tau_f - \alpha_\vartheta/n),HAR} > cv_{n,HAR}) \to 1$ for all $0 < \alpha_\vartheta < \vartheta$. As η and ϑ are arbitrary and $\Pr(\hat{r}_e^{HAR} < r_e) \to 0$ and $\Pr(\hat{r}_f^{HAR} > r_f) \to 0$, we deduce that $\Pr(|\hat{r}_e^{HAR} - r_e| > \eta) \to 0$ and $\Pr(|\hat{r}_f^{HAR} - r_f| > \vartheta) \to 0$ as $n \to \infty$, provided that

$$\frac{1}{cv_{n,HAR}} + \frac{cv_{n,HAR}}{n^{(1-\alpha)/2}} \to 0.$$

This completes the proof of Theorem 5.1. ∎

Proof of Theorem 6.2

We shall only prove that under the assumptions in Theorem 6.2, we have

$$\frac{1}{n^{1/2+d}} x_{\lfloor ns \rfloor} \Rightarrow W^H(s), \tag{73}$$

as when (73) holds we can use the steps in proving Theorem 3.1 to establish the claim in Theorem 6.2. We first show the following two results which will be useful in establishing (73).

Letting $m = n^\gamma$ we have

$$\Delta_+^{1+\hat{d}} y_t = \Delta_+^{1+d} y_t + O_p(m^{-1}). \tag{74}$$

and

$$\max_{1 \leq t \leq n} \left|\hat{g}^2(t/n) - g^2(t/n)\right| = o_p(1). \tag{75}$$

Set $\xi_n = \hat{d} - d$, and $z_t = \Delta_+^{1+d} y_t = g(t/n)\varepsilon_t$. To show (74), note that

$$\Delta_+^{1+\hat{d}} y_t = \Delta_+^{\hat{d}-d}\left(\Delta_+^{1+d} y_t\right) = \Delta_+^{\xi_n} z_t, \text{ and}$$

$$\Delta_+^{\xi_n} z_t = \sum_{k=0}^{t-1} \binom{\xi_n}{k} (-L)^k z_t = z_t - \xi_n \left(\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}\right) + O_p(\xi_n^2). \tag{76}$$

Since $\xi_n = O_p(m^{-1})$, $Var(\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}) = \sum_{k=1}^{t-1} \frac{g^2((t-k)/n)}{k^2} \leq \sup_t g^2(t/n) \sum_{k=0}^{t-1} \frac{1}{k^2}$ which is asymptotically bounded (let $t = \lfloor ns \rfloor$, $s \in [0,1]$ and $n \to \infty$), we have $\xi_n \left(\sum_{k=1}^{t-1} \frac{z_{t-k}}{k}\right) = O_p(m^{-1})$, giving (74).

To show (75), note that

$$\hat{g}^2\left(\frac{t}{n}\right) = \sum_{j=1}^{\tau} k_{tj}\left(\Delta_+^{1+\hat{d}} y_j\right)^2 = \sum_{j=1}^{\tau} k_{tj}\left[g(j/n)\varepsilon_j + R_n\right]^2$$

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$$= \sum_{j=1}^{\tau} k_{tj} g^2(j/n)\varepsilon_j^2 + 2R_n \sum_{j=1}^{\tau} k_{tj} g(j/n)\varepsilon_j + R_n^2, \qquad (77)$$

where R_n is $O_p(m^{-1})$, as indicated by (74).

We now show that the second term in (77) is $o_p(1)$. Note that

$$\sum_{j=1}^{\tau} k_{tj} g(j/n)\varepsilon_j = \frac{\sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j}{\sum_{i=1}^{\tau} K\left(\frac{t-i}{\nu}\right)}. \qquad (78)$$

First consider the numerator of (78). As in Theorem 2.8 of Pagan and Ullah (2006), write

$$\begin{aligned}
\sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j &= \frac{1}{2\pi} \sum_{j=1}^{\tau} \int \exp\left(-iv\left(\frac{t-j}{\nu}\right)\right) g(j/n)\varepsilon_j \phi(v) dv \\
&= \frac{1}{2\pi} \int \sum_{j=1}^{\tau} \exp\left(\frac{ivj}{\nu}\right) g(j/n)\varepsilon_j \phi(v) \exp\left(\frac{-ivt}{\nu}\right) dv \\
&= \frac{\nu}{2\pi} \int \sum_{j=1}^{\tau} \exp(ixj) g(j/n)\varepsilon_j \phi(\nu x) \exp(-ixt) dx,
\end{aligned}$$

where $\phi(\cdot)$ is the characteristic function of K and we let $v = \nu x$ to obtain the third equality. Thus,

$$\begin{aligned}
\max_{t<\tau} \left| \sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j \right| &= \max_{t<\tau} \left| \frac{\nu}{2\pi} \int \sum_{j=1}^{\tau} \exp(ixj) g(j/n)\varepsilon_j \phi(\nu x) \exp(-ixt) dx \right| \\
&\leq \frac{\nu}{2\pi} \int \left| \sum_{j=1}^{\tau} \exp(ixj) g(j/n)\varepsilon_j \right| \left(\max_{t<\tau} |\exp(-ixt)| \right) |\phi(\nu x)| dx \\
&\leq \frac{\nu}{2\pi} \int \left| \sum_{j=1}^{\tau} \exp(ixj) g(j/n)\varepsilon_j \right| |\phi(\nu x)| dx.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{1}{\sqrt{n}} \sum_{j=1}^{\tau} \exp(ixj) g(j/n)\varepsilon_j &= \frac{1}{\sqrt{n}} \sum_{j=1}^{\tau} \cos(xj) g(j/n)\varepsilon_j + i\frac{1}{\sqrt{n}} \sum_{j=1}^{\tau} \sin(xj) g(j/n)\varepsilon_j \\
&= O_p(1),
\end{aligned}$$

which implies $\sum_{j=1}^{\tau} \exp(ixj) g(j/n)\varepsilon_j = O_p(\sqrt{n})$. Therefore,

$$\max_{t<\tau} \left| \sum_{j=1}^{\tau} K\left(\frac{t-j}{\nu}\right) g(j/n)\varepsilon_j \right| \leq \frac{\nu}{2\pi} O_p(\sqrt{n}) \int |\phi(\nu x)| dx = O_p(\sqrt{n}).$$

Note that the denominator in (78) is $O(\nu)$. Hence, the second term in (77) is

$$R_n \sum_{j=1}^{t} k_{tj} g(j/n)\varepsilon_j = O_p(m^{-1}) \frac{O_p(\sqrt{n})}{O(\nu)} = O_p\left(\frac{n^{1/2-\gamma}}{\nu}\right) = O_p\left(\left(\frac{n}{\nu^2} \frac{1}{n^{2\gamma}}\right)^{1/2}\right) = o_p(1).$$

For the first term in (77), given the rate condition of ν and the kernel function of $K(\cdot)$, Lemma 1 in Astill et al. (2021) shows $\max_t \left| \sum_{i=1}^{\tau} k_{ti} g^2(i/n)\varepsilon_i^2 - g^2(t/n) \right| = o_p(1)$. This implies that

$$\max_{1 \le t \le n} \left| \widehat{g}^2(t/n) - g^2(t/n) \right| = o_p(1). \tag{79}$$

Since $\Delta_+^{1+\hat{d}} y_t = g(t/n)\varepsilon_t + O_p(m^{-1})$, letting $R_n = O_p(m^{-1})$ we have

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \left(\frac{\Delta_+^{1+\hat{d}} y_s}{\widehat{g}(s/n)} \right)$$

$$= \frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \left[\frac{g(s/n)\varepsilon_s}{\widehat{g}(s/n)} + \frac{R_n}{\widehat{g}(s/n)} \right]$$

$$= \frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \frac{g(s/n)}{\widehat{g}(s/n)} \varepsilon_s + \frac{1}{n^{1/2+d}} \sum_{s=1}^{t} \Delta_+^{-\hat{d}} \frac{R_n}{\widehat{g}(s/n)}. \tag{80}$$

Equation (79) implies that second term of (80) is $O_p(n^{-1/2-\gamma})$. By Stirling's approximation, for large enough S there is a constant C such that, as $n \to \infty$,

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} R_n \le \frac{1}{n^{1/2+d}} C + R_n \frac{C}{n^{1/2+d}} \sum_{s=S}^{\lfloor nr \rfloor} s^{d-1} \le R_n \frac{C}{n^{1/2+d}} \int_0^n s^{d-1} ds = \frac{R_n C}{n^{1/2}} \to 0.$$

Further, (79) implies that $\widehat{g}^2(t/n) = O_p(1)$ for all t. Therefore, the second term of (80) is $O_p(n^{-1/2-\gamma})$.

For the first term in (80), let $t = \lfloor nr \rfloor$ and note that

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \frac{g(s/n)}{\widehat{g}(s/n)} \varepsilon_s$$

$$= \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left[\left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)} \right) + 1 \right] \varepsilon_s$$

$$= \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \varepsilon_s + \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)} \right) \varepsilon_s. \tag{81}$$

We shall prove

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \varepsilon_s \Rightarrow W^H(r) \text{ and} \tag{82}$$

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)} \right) \varepsilon_s = o_p(1). \tag{83}$$

Since

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \varepsilon_s = \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}+d} \left(\Delta_+^{-d} \varepsilon_s \right),$$

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using the same technique as in (76), we have

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}+d} \left(\Delta_+^{-d} \varepsilon_s \right)$$

$$= \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \left(\Delta_+^{-d} \varepsilon_s + \xi_n \left(\sum_{j=1}^{s-1} \frac{\Delta_+^{-d} \varepsilon_{s-j}}{j} \right) \right) + \frac{\lfloor nr \rfloor}{n^{1/2+d}} O_p(\xi_n^2)$$

$$= \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-d} \varepsilon_s + \frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} \frac{\Delta_+^{-d} \varepsilon_{s-j}}{j} + o_p(1), \tag{84}$$

where $\frac{\lfloor nr \rfloor}{n^{1/2+d}} O_p(\xi_n^2) = O_p(n^{-(1/2+d-1+2\gamma)}) = o_p(1)$ because $\gamma > 1/4$.

For the first term in (84), by Lemma 10.1,

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-d} \varepsilon_s \Rightarrow W^H(r).$$

For the second term in (84), since ε_s is i.i.d., we have

$$\frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} \frac{\Delta_+^{-d} \varepsilon_{s-j}}{j} \stackrel{d}{=} \frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} \frac{\Delta_+^{-d} \varepsilon_s}{j}.$$

For $d > 0$,

$$\frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} \frac{\Delta_+^{-d} \varepsilon_s}{j} \leq \frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} \frac{\Delta_+^{-d} |\varepsilon_s|}{j}$$

$$\leq \frac{C}{\Gamma(d)} \frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} j^{d-2} |\varepsilon_s|$$

$$\leq \sup_{t \in [1,n]} |\varepsilon_t| \frac{C}{\Gamma(d)} \frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} j^{d-2}.$$

Note that $\sup_{t \in [1,n]} |\varepsilon_t|$ is $O_p(1)$ and $\frac{C}{\Gamma(d)} \frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} j^{d-2} \leq C' \frac{\xi_n}{n^{1/2+d}} n^d \to 0$ where C and C' are some constants. Therefore, $\frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} \frac{\Delta_+^{-d} \varepsilon_{s-j}}{j} = o_p(1)$.

Let $X_s = \sum_{j=1}^{s-1} \frac{\varepsilon_k}{j}$. When $d = 0$, $\frac{\xi_n}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=1}^{s-1} \frac{\Delta_+^{-d} \varepsilon_s}{j} = \frac{\xi_n}{n^{1/2}} \sum_{s=1}^{\lfloor nr \rfloor} X_k$ and

$$Var \left(\frac{\xi_n}{n^{1/2}} \sum_{s=1}^{\lfloor nr \rfloor} X_s \right) = \frac{\xi_n^2}{n} \left[\sum_{s=1}^{\lfloor nr \rfloor} Var(X_s) + 2 \sum_{k<j}^{\lfloor nr \rfloor} Cov(X_j, X_k) \right] \leq C \frac{\xi_n^2}{n} \ln(n) \to 0.$$

Therefore, $\frac{\xi_n}{n^{1/2}} \sum_{s=1}^{\lfloor nr \rfloor} \sum_{j=0}^{s-1} \frac{\varepsilon_{s-j}}{j} = o_p(1)$ and (82) is established.

40

To show (83), note that

$$\left| \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)} \right) \varepsilon_s \right| \le \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left| \left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)} \right) \right| |\varepsilon_s|$$

$$\le \max_s \left| \frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)} \right| \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} |\varepsilon_s| = o_p(1) \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} |\varepsilon_s|$$

$$= o_p(1) \left[\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} (|\varepsilon_s| - \mathbb{E}|\varepsilon_s|) + \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \mathbb{E}|\varepsilon_s| \right].$$

By Lemma 10.1,

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} (|\varepsilon_s| - \mathbb{E}|\varepsilon_s|) = \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-d} (|\varepsilon_s| - \mathbb{E}|\varepsilon_s|) + o_p(1) = O_p(1),$$

and

$$\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \mathbb{E}|\varepsilon_s| = \frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-d} \mathbb{E}|\varepsilon_s| + o_p(1).$$

There exists a bound for which $\frac{C}{n^{1/2+d}} \frac{1}{\Gamma(d)} \sum_{s=1}^{\lfloor nr \rfloor} s^{d-1} < \frac{C}{n^{1/2+d}} \frac{1}{\Gamma(d)} n^d \to 0$. This implies $\frac{1}{n^{1/2+d}} \sum_{s=1}^{\lfloor nr \rfloor} \Delta_+^{-\hat{d}} \left(\frac{g(s/n) - \widehat{g}(s/n)}{\widehat{g}(s/n)} \right) \varepsilon_s = o_p(1)$.

Combining (81), (82) and (83), we have $\frac{1}{n^{1/2+d}} x_{\lfloor ns \rfloor} \Rightarrow W^H(s)$. The limit theorem in (33) follows in a straightforward way.

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11 Online Supplement (Not for Publication)

11.1 Additional empirical results

Table 7: Right-tailed unit root tests for the S&P 500 PD ratio, exact local Whittle (ELW) estimates $\hat{d}$ of d, and corresponding confidence intervals

Sampling Period	DF_n	$\hat{d}$	90% CI	95% CI
(a') Jan 1871 to Feb 1880	1.32	0.24	(0.06,0.42)	(0.03,0.45)
(b') Jan 1882 to May 1887	0.71	0.31	(0.10,0.52)	(0.06,0.56)
(c') Nov 1936 to Jun 1946	0.61	0.28	(0.08,0.48)	(0.04,0.52)
(d') Aug 1947 to Nov 1955	1.42	0.23	(0.04,0.42)	(0.01,0.45)
(e') Jun 1977 to Mar 1987	1.93	0.21	(0.03,0.39)	(0.00,0.42)
(f') May 1988 to Apr 1998	3.76	0.24	(0.06,0.42)	(0.03,0.45)

Table 7 supports the empirical findings in Table 1 using the same time series but over a longer observation period. As in Table 1, DF_n, $\hat{d}$, and the confidence intervals for d are reported, and the results support the conclusions that (i) the time series has a strongly dependent error in view of the ELW estimates $\hat{d}$ and the associated confidence intervals for d, and (ii) there is evidence of explosive behavior in the data using right sided DF_n tests.

11.2 Sup statistic

As mentioned in Remark 3.6, a version of right-tailed sup statistics can be employed. The sup statistic $\widetilde{SDF}_{HAR}(\tau_0)$ is given by

$$\widetilde{SDF}_{HAR}(\tau_0) = \sup_{\tau \in [\tau_0, n]} \frac{\tilde{\rho}_\tau - 1}{s_{\tau, HAR}},$$

and its limit theory under null and alternative hypotheses are given in the following result.

Theorem 11.1 *Let $M = \lfloor b\tau \rfloor$ and $K_B(x)$ be the Bartlett kernel function. Under model (9), as $n \to \infty$,*

$$\widetilde{SDF}_{HAR}(\tau_0) \tag{85}$$
$$\Rightarrow \sup_{r \in [r_0, 1]} \frac{b^{1/2} \left[\frac{r}{2}(W^H(r)^2) - \left(\int_0^r W^H(s)ds W^H(r) \right) \right]}{\left[2r \left(\int_0^r \tilde{W}^H(s)^2 ds \right) \left(\int_0^1 W^H(p)^2 dp - \int_0^{1-b} W^H(p)W^H(p + br)dp \right) \right]^{1/2}}.$$

Under model (21), as $n \to \infty$,

$$\widetilde{SDF}_{HAR}(\tau_0)$$
$$\Rightarrow \sup_{r \in [r_0, 1]} \frac{b^{1/2} \left(\frac{r}{2}C_{r,d} - A_{r,d}W^H(r) + B_{r,d}cr - cA_{r,d}^2 \right)}{\left[2r \left(B_{r,d} - \frac{1}{r}A_{r,d}^2 \right) \left(\int_0^1 G_{r_e,c}(p)^2 dp - \int_0^{(1-b)} G_{r_e,c}(p)G_{r_e,c}(p + br)dp \right) \right]^{1/2}}.$$

Under model (23)-(24) if $\tau_0 < \tau_f$, $\widetilde{SDF}_{HAR}(\tau_0) \overset{p}{\to} \infty$, as $n \to \infty$.

Theorem 11.1 establishes the asymptotic behavior of $\widetilde{SDF}_{HAR}(\tau_0)$ under the null hypothesis, local alternative, and mildly explosive alternative. Under the null the sup statistic has a well-defined limit. Under the local alternative, the limit of the test statistic can be used to obtain the local power function. Under the mildly explosive alternative, the divergent behavior of the test statistic implies consistency. Further, the limit distribution (85) and consistent estimation of d allow us to obtain the $\beta \times 100\%$ critical value, denoted by $scv_{HAR}^{\beta}\left(\hat{d}\right)$, for practical implementation of the test.

To investigate the empirical size of $\widetilde{SDF}_{HAR}(\tau_0)$, we perform a Monte Carlo study based on the DGP (34). Let $d \in \{0, 0.05, ..., 0.45\}$. To calculate $\widetilde{SDF}_{HAR}(\tau_0)$, as in Section 7, we let $b = 0.05$. For the minimum window, based on extensive simulations, we find that the following rule of thumb gives satisfactory size and power performance in finite samples: $r_0 = 0.01 + 4.9/\sqrt{n}$. So $r_0 \approx 0.5$ if $n = 100$ and $r_0 \approx 0.23$ if $n = 500$. For comparison we report both the empirical size of $SDF(\tau_0)$ and $\widetilde{SDF}_{HAR}(\tau_0)$ based on the 5% critical value in Table 8. The findings echo those in Table 2. Mostly importantly, $SDF(\tau_0)$ suffers severe oversizing when d is large, whereas $\widetilde{SDF}_{HAR}(\tau_0)$ has empirical size close to the nominal level.

Table 8: Empirical size of $SDF(\tau_0)$ and $\widetilde{SDF}_{HAR}(\tau_0)$

	$n = 100, r_0 = 0.50$									
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$SDF(\tau_0)$	0.04	0.08	0.15	0.23	0.32	0.41	0.50	0.56	0.63	0.68
$\widetilde{SDF}_{HAR}(\tau_0)$	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.06	0.06	0.06
	$n = 500, r_0 = 0.23$									
d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
$SDF(\tau_0)$	0.04	0.15	0.32	0.49	0.64	0.75	0.83	0.87	0.90	0.93
$\widetilde{SDF}_{HAR}(\tau_0)$	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05

To explore power of the sup test the simulations are based on model (25) with the following parameter settings: $y_0 = 100$, $n = 100$, $c = 1$, $\alpha = 0.6$, $r_e = 0.6$, $r_f \in \{0.7, 0.75, 0.85\}$ and $d \in \{0, 0.05, ..., 0.45\}$. Similar to Table 8, we report the power of $\widetilde{SDF}_{HAR}(\tau_0)$ based on the 5% critical value in Table 9. The results show that $\widetilde{SDF}_{HAR}(\tau_0)$ has good power performance in detecting explosive behavior.

Table 9: Power of $\widetilde{SDF}_{HAR}(\tau_0)$ when $n = 100, r_0 = 0.5$

d	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45
	$r_e = 0.6, r_f = 0.7$									
$\widetilde{SDF}_{HAR}(\tau_0)$	1.00	1.00	1.00	0.99	0.97	0.94	0.90	0.85	0.80	0.75
	$r_e = 0.6, r_f = 0.75$									
$\widetilde{SDF}_{HAR}(\tau_0)$	1.00	1.00	1.00	1.00	0.99	0.98	0.95	0.91	0.87	0.82
	$r_e = 0.6, r_f = 0.8$									
$\widetilde{SDF}_{HAR}(\tau_0)$	1.00	1.00	1.00	1.00	1.00	0.98	0.98	0.95	0.92	0.87

11.3 Proofs of the Lemmas

Proof of Lemma 3.1

Recall that by definition $s_\tau^2 = \dfrac{\frac{1}{\tau}\sum_{t=1}^{\tau}\hat{u}_t^2}{\sum_{t=1}^{\tau}y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau}y_{t-1}\right)^2}$. As $\hat{u}_t$ $(= u_t + (1-\hat{\rho}_\tau)y_{t-1} - \hat{\mu})$ involves $\hat{\mu}$, we first study the properties of $\hat{\mu}$. Write

$$\hat{\mu} = \frac{\sum_{t=1}^{\tau}y_{t-1}^2\sum_{t=1}^{\tau}u_t - \sum_{t=1}^{\tau}y_{t-1}u_t\sum_{t=1}^{\tau}y_{t-1}}{\frac{\tau}{n}\sum_{t=1}^{\tau}y_{t-1}^2 - \frac{1}{n}\left(\sum_{t=1}^{\tau}y_{t-1}\right)^2}.$$

and upon normalization we have

$$
\begin{aligned}
n^{1/2-d}\hat{\mu} &= \frac{\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}y_{t-1}^2\frac{1}{n^{1/2+d}}\sum_{t=1}^{\tau}u_t - \frac{1}{n^{1+2d}}\sum_{t=1}^{\tau}y_{t-1}u_t\frac{1}{n^{3/2+d}}\sum_{t=1}^{\tau}y_{t-1}}{\frac{\tau}{n}\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}y_{t-1}^2 - \left(\frac{1}{n^{3/2+d}}\sum_{t=1}^{\tau}y_{t-1}\right)^2} \\
&\Rightarrow \frac{\sigma^3\int_0^r\left(W^H(s)\right)^2 ds\, W^H(r) - \frac{\sigma^3}{2}\left(W^H(r)\right)^2\int_0^r W^H(s)ds}{r\sigma^2\int_0^r\left(W^H(s)\right)^2 ds - \sigma^2\left(\int_0^r W^H(s)ds\right)^2}, \quad (86)
\end{aligned}
$$

which implies $\hat{\mu} = O_p(n^{-1/2+d}) = o_p(1)$.

For the mean squared residuals $\frac{1}{\tau}\sum_{t=1}^{\tau}\hat{u}_t^2$, write

$$
\begin{aligned}
\frac{1}{\tau}\sum_{t=1}^{\tau}\hat{u}_t^2 &= \frac{1}{\tau}\sum_{t=1}^{\tau}\left(u_t + (1-\hat{\rho}_\tau)y_{t-1} - \hat{\mu}\right)^2 \\
&= \frac{1}{\tau}\sum_{t=1}^{\tau}u_t^2 + \frac{2(1-\hat{\rho}_\tau)}{\tau}\sum_{t=1}^{\tau}y_{t-1}u_t + \frac{(1-\hat{\rho}_\tau)^2}{\tau}\sum_{t=1}^{\tau}y_{t-1}^2 \\
&\quad - 2\hat{\mu}\frac{1}{\tau}\sum_{t=1}^{\tau}u_t - 2\frac{(1-\hat{\rho}_\tau)\hat{\mu}}{\tau}\sum_{t=1}^{\tau}y_{t-1} + \frac{\hat{\mu}^2}{\tau}\sum_{t=1}^{\tau}1. \\
&= \frac{1}{\tau}\sum_{t=1}^{\tau}u_t^2 + \frac{2\tau(1-\hat{\rho}_\tau)}{\tau^2}\sum_{t=1}^{\tau}y_{t-1}u_t + \frac{(\tau(1-\hat{\rho}_\tau))^2}{\tau^3}\sum_{t=1}^{\tau}y_{t-1}^2 - 2\tau(1-\hat{\rho}_\tau)\hat{\mu}\left(\frac{1}{\tau^2}\sum_{t=1}^{\tau}y_{t-1}\right) + \hat{\mu}^2.
\end{aligned}
$$

From Lemma 10.2, $\tau(1-\hat{\rho}_\tau) = O_p(1)$, $\tau^{-2}\sum_{t=1}^{\tau}y_{t-1}u_t = O_p\left(n^{2d-1}\right) = o_p(1)$, $\tau^{-3}\sum_{t=1}^{\tau}y_{t-1}^2 = O_p(n^{2d-1}) = o_p(1)$, $\tau^{-2}\sum_{t=1}^{\tau}y_{t-1} = O_p\left(n^{d-1/2}\right)$. From (86), $\hat{\mu}^2 = O_p\left(n^{-1+2d}\right) = o_p(1)$, and so $\frac{1}{\tau}\sum_{t=1}^{\tau}\hat{u}_t^2 = \frac{1}{\tau}\sum_{t=1}^{\tau}u_t^2 + o_p(1) = O_p(1)$.

Applying Lemma 10.2,

$$
\begin{aligned}
n^{-(2+2d)}\left[\sum_{t=1}^{\tau}y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau}y_{t-1}\right)^2\right] &= n^{-(2+2d)}\sum_{t=1}^{\tau}y_{t-1}^2 - \frac{\left(n^{-3/2-d}\sum_{t=1}^{\tau}y_{t-1}\right)^2}{\frac{\tau}{n}} \quad (87) \\
&\Rightarrow \sigma^2\left(\int_0^r\left(W^H(s)\right)^2 ds - \frac{1}{r}\left(\int_0^r W^H(s)ds\right)^2\right),
\end{aligned}
$$

and it is straightforward to show

$$n^{-d}DF_\tau = \frac{n(\hat{\rho}_\tau - 1)}{(n^{2+2d}s_\tau^2)^{1/2}} = \frac{\frac{n}{\tau}\tau(\hat{\rho}_\tau - 1)}{(n^{2+2d}s_\tau^2)^{1/2}} = O_p(1).$$

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This implies that $DF_\tau \sim O_p(n^d)$ and the proof of Lemma 3.1 is complete. ∎

Proof of Lemma 6.1

We sketch the proof as it is similar to Silveira (1991). Let $d_y = 1 + d$, $\pi_j^d = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$, and write $y_t = y_{t-1} + \Delta_+^{-d}\epsilon_t = \Delta_+^{-d_y}\epsilon_t + y_0 = \sum_{j=1}^t \pi_{t-j}^{d_y}\epsilon_j + y_0$. Let $Y_n(r) = n^{1/2-d_y}\, y_{\lfloor nr \rfloor}$, and then

$$
\begin{aligned}
Y_n(r) &= n^{1/2-d_y}\sum_{j=1}^{\lfloor nr \rfloor}\pi_{\lfloor nr \rfloor - j}^{d_y}\sigma_j\varepsilon_j + o_p(1)\\
&= n^{1/2-d_y}\sum_{j=1}^{\phi}\pi_{\phi-j}^{d_y}\sigma_j\left(S_j - S_{j-1}\right) + o_p(1),
\end{aligned}
$$

where $\phi = \lfloor nr \rfloor$ and $S_j = \sum_{i=1}^j \varepsilon_i$.

Let $V_j = \sum_{i=1}^j z_i$ and $z_t \overset{iid}{\sim} \mathcal{N}(0,1)$, we can write

$$
Y_n(r) = Q_{1n}(r) + Q_{2n}(r) + Q_{3n}(r) + Q_{4n}(r) + o_p(1),
$$

where

$$
\begin{aligned}
Q_{1n}(r) &= n^{1/2-d_y}\sum_{j=1}^{\phi-1}\frac{(\phi-j)^{d_y-1}}{\Gamma(d_y)}\sigma_j\left(V_j - V_{j-1}\right),\\
Q_{2n}(r) &= n^{1/2-d_y}\sum_{j=1}^{\phi-1}\pi_{\phi-j}^{d_y}\sigma_j\left[\left(S_j - S_{j-1}\right) - \left(V_j - V_{j-1}\right)\right],\\
Q_{3n}(r) &= n^{1/2-d_y}\sum_{j=1}^{\phi-1}\left(\pi_{\phi-j}^{d_y} - \frac{(\phi-j)^{d_y-1}}{\Gamma(d_y)}\right)\sigma_j\left(V_j - V_{j-1}\right),\\
Q_{4n}(r) &= n^{1/2-d_y}\sigma_\phi\left(S_\phi - S_{\phi-1}\right).
\end{aligned}
$$

The ideas is to show $Q_{1n}(r) \Rightarrow W_g^H(r)$ and $Q_{2n}(r)$, $Q_{3n}(r)$ and $Q_{4n}(r)$ are all $o_p(1)$. Given the finiteness of $\sup_{s\in[0,1]} g(s)$, it is straightforward to show $Q_{2n}(r)$, $Q_{3n}(r)$ and $Q_{4n}(r)$ are $o_p(1)$ following the approach in Silveira (1991). For the weak convergence of $Q_{1n}(r)$, Silveira (1991) showed that we need to verify the following conditions:

1. $\lim_{n\to\infty} \mathbb{E}\left[Q_{1n}(r)\right] = \mathbb{E}\left[W_g^H(r)\right]$.

2. $\lim_{n\to\infty} \mathbb{E}\left[Q_{1n}(r)Q_{1n}(s)\right] = \mathbb{E}\left[W_g^H(r)W_g^H(s)\right]$.

3. $\mathbb{E}\left[Q_{1n}(r) - Q_{1n}(q)\right]^2 \mathbb{E}\left[Q_{1n}(s) - Q_{1n}(r)\right]^2 \leq D|s-q|^\gamma$, for all $n \geq 1$ and $0 \leq q < r < s \leq 1$ and D and γ are some positive constants.

4. $\mathbb{E}\left[W_g^H(s) - W_g^H(r)\right]^2 \leq D|s-r|^\gamma$, for $0 \leq s < r \leq 1$ and D and γ are some positive constants.

It is trivial to verify the first condition since both $Q_{1n}(r)$ and $W_g^H(r)$ are Gaussian random variables with a zero mean. For the second condition, note that

$$
\mathbb{E}\left[Q_{1n}(r)Q_{1n}(s)\right] = \mathbb{E}\left[n^{1/2-d_y}\sum_{j=1}^{\lfloor nr \rfloor-1}\frac{(\lfloor nr \rfloor - j)^{d_y-1}}{\Gamma(d_y)}\sigma_j z_j,\, n^{1/2-d_y}\sum_{j=1}^{\lfloor ns \rfloor-1}\frac{(\lfloor ns \rfloor - j)^{d_y-1}}{\Gamma(d_y)}\sigma_j z_j\right]
$$

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$$= \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=1}^{\lfloor nr \rfloor - 1} \sigma_j^2 \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y - 1} \left(\frac{\lfloor ns \rfloor - j}{n} \right)^{d_y - 1}. \tag{88}$$

Clearly, (88) converges to $\frac{1}{\Gamma(d_y)^2} \int_0^r g(x)^2 (r-x)^{d_y-1} (s-x)^{d_y-1} ds$ by the dominated convergence theorem. And the second condition is satisfied.

For the third condition, note that

$$Q_{1n}(r) - Q_{1n}(q)$$

$$= \frac{n^{1/2-d_y}}{\Gamma(d_y)} \left[\sum_{j=1}^{\lfloor nr \rfloor - 1} (\lfloor nr \rfloor - j)^{d_y-1} \sigma_j z_j - n^{1/2-d_y} \sum_{j=1}^{\lfloor nq \rfloor - 1} (\lfloor nq \rfloor - j)^{d_y-1} \sigma_j z_j \right]$$

$$= \frac{n^{1/2-d_y}}{\Gamma(d_y)} \left[\sum_{j=1}^{\lfloor nq \rfloor - 1} \left((\lfloor nr \rfloor - j)^{d_y-1} - (\lfloor nq \rfloor - j)^{d_y-1} \right) \sigma_j z_j + \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor - 1} (\lfloor nr \rfloor - j)^{d_y-1} \sigma_j z_j \right]$$

$$= \frac{1}{\Gamma(d_y)} \frac{1}{n^{1/2}} \sum_{j=1}^{\lfloor nq \rfloor - 1} \left(\left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} - \left(\frac{\lfloor nq \rfloor - j}{n} \right)^{d_y-1} \right) \sigma_j z_j$$

$$+ \frac{1}{\Gamma(d_y)} \frac{1}{n^{1/2}} \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor - 1} \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} \sigma_j z_j.$$

Therefore, we have

$$\mathbb{E} \left[Q_{1n}(r) - Q_{1n}(q) \right]^2 = \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=1}^{\lfloor nq \rfloor - 1} \left(\left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} - \left(\frac{\lfloor nq \rfloor - j}{n} \right)^{d_y-1} \right)^2 \sigma_j^2$$

$$+ \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor - 1} \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{2(d_y-1)} \sigma_j^2$$

$$\leq \tilde{\sigma}^2 (\tilde{Q}_{a,n}(r) + \tilde{Q}_{b,n}(r)),$$

where

$$\tilde{Q}_{a,n}(r) = \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=1}^{\lfloor nq \rfloor - 1} \left(\left(\frac{\lfloor nr \rfloor - j}{n} \right)^{d_y-1} - \left(\frac{\lfloor nq \rfloor - j}{n} \right)^{d_y-1} \right)^2,$$

$$\tilde{Q}_{b,n}(r) = \frac{1}{\Gamma(d_y)^2} \frac{1}{n} \sum_{j=\lfloor nq \rfloor}^{\lfloor nr \rfloor - 1} \left(\frac{\lfloor nr \rfloor - j}{n} \right)^{2(d_y-1)}$$

and $\tilde{\sigma}^2 = \sup_{j \in [1,n]} \sigma_j^2$.

Lemma 3-A-3 in Silveira (1991) shows $\tilde{Q}_{a,n}(r) + \tilde{Q}_{b,n}(r) \leq c(r-q)^\gamma$ for some constants c and γ. This implies the third condition is satisfied.

Finally, for the fourth condition, $\mathbb{E} \left[W_g^H(s) - W_g^H(r) \right]^2$, given $\sup_s g(s)^2 < \infty$, using the same steps to show the third condition, it is immediate to see the fourth condition holds.

With the weak convergence of $Q_{1n}(r)$, since $Q_{2n}(r)$, $Q_{3n}(r)$ and $Q_{4n}(r)$ are $o_p(1)$, the proof of Lemma 6.1 is completed. ∎

Proof of Lemma 10.2.

When $d = 0$, the error term is an i.i.d. process and the results in (37), (38), (39) and (40) are well known in the literature. Only claims for $d > 0$ need proving. For the first claim, since $\sum_{t=1}^{\tau} y_{t-1} u_t = \frac{1}{2} \left(y_{\lfloor nr \rfloor}^2 - y_0^2 - \sum_{t=1}^{\tau} u_t^2 \right)$, we have

$$n^{-1-2d} \sum_{t=1}^{\tau} y_{t-1} u_t = \frac{1}{2} \left[\left(n^{-1/2-d} y_{\lfloor nr \rfloor} \right)^2 - \frac{1}{n^{2d}} \left(\frac{1}{n} \sum_{t=1}^{\tau} u_t^2 \right) \right] + o_p(1) \Rightarrow \frac{\sigma^2}{2} \left(W^H(r) \right)^2,$$

where the last step is due to $n^{-1/2-d} y_{\lfloor nr \rfloor} \Rightarrow \sigma W^H(r)$ (from Lemma 10.1) and $\frac{1}{n^{2d}} \left(\frac{1}{n} \sum_{t=1}^{\tau} u_t^2 \right) \xrightarrow{p} 0$. To see why convergence in probability applies, note that for $d \in (0, 0.5)$ we have $\sum_{t=1}^{\tau} u_t^2 = O_p(n)$; when $d = 0.5$, $\sum_{t=1}^{\tau} u_t^2 = O_p(n (\ln(n))^2)$ (see Duffy and Kasparis (2018, 2021)); and when $d > 0.5$ Lemma 10.1 implies

$$\frac{u_t}{n^{1/2+d-1}} \Rightarrow \sigma W^H(r), \text{ for } t = \lfloor nr \rfloor,$$

so that $\frac{u_t}{n^{1/2+d-1}} = O_p(1)$. By continuous mapping $\frac{1}{n} \sum_{t=1}^{\tau} \left(\frac{u_t^2}{n^{1/2+d-1}} \right) = O_p(1)$, and $\sum_{t=1}^{\tau} u_t^2 = O_p(n^{2d})$. Hence, $\frac{1}{n^{2d}} \left(\frac{1}{n} \sum_{t=1}^{\tau} u_t^2 \right) \xrightarrow{p} 0$ for any $d > 0$, and we obtain (37).

For the second claim, since $\sum_{t=1}^{\tau} y_{t-1} = \sum_{t=1}^{\tau} \left(\sum_{i=1}^{t-1} u_i + y_0 \right)$, applying Lemma 10.1 and continuous mapping gives

$$\frac{1}{n^{3/2+d}} \sum_{t=1}^{\tau} y_{t-1} = \frac{1}{n} \sum_{t=1}^{\tau} \left(\frac{1}{n^{1/2+d}} \sum_{i=1}^{t-1} u_i \right) + o_p(1) \Rightarrow \sigma \int_0^r W^H(s) ds.$$

Applying similar arguments, the third claim follows as

$$\frac{1}{n^{2+2d}} \sum_{t=1}^{\tau} y_{t-1}^2 = \frac{1}{n} \sum_{t=1}^{\tau} \left(\frac{1}{n^{1/2+d}} \sum_{i=1}^{\tau-1} u_i \right)^2 + o_p(1) \frac{1}{n^{2+2d}} \sum_{t=1}^{\tau} y_{t-1}^2 \Rightarrow \sigma^2 \int_0^r \left(W^H(s) \right)^2 ds.$$

For the normalized centered LS estimator $\tau(\hat{\rho}_\tau - 1)$, we have

$$
\begin{aligned}
\tau(\hat{\rho}_\tau - 1) &= \frac{\tau}{n} \frac{n^{-1-2d}}{n^{-2-2d}} \left[\frac{\sum_{t=1}^{\tau} y_{t-1} u_t - \frac{1}{\tau} \sum_{t=1}^{\tau} y_{t-1} \sum_{t=1}^{\tau} u_t}{\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{\tau} \left(\sum_{t=1}^{\tau} y_{t-1} \right)^2} \right] \\
&= \frac{\tau}{n} \frac{n^{-1-2d} \sum_{t=1}^{\tau} y_{t-1} u_t - \frac{n}{\tau} n^{-3/2-d} \sum_{t=1}^{\tau} y_{t-1} n^{-1/2-d} \sum_{t=1}^{\tau} u_t}{n^{-2-2d} \sum_{t=1}^{\tau} y_{t-1}^2 - \frac{n}{\tau} \left(n^{-3/2-d} \sum_{t=1}^{\tau} y_{t-1} \right)^2} \\
&\Rightarrow \frac{\frac{r}{2} \left(W^H(r) \right)^2 - \int_0^r W^H(s) ds W^H(r)}{\int_0^r \left(W^H(s) \right)^2 ds - \frac{1}{r} \left(\int_0^r W^H(s) ds \right)^2},
\end{aligned}
\tag{89}
$$

where the last result is obtained by applying (36), (38) and (39). As (89) is equivalent to (40) when $d \in (0, 0.5)$, we have proved (40).

To show (41), note that

$$\tilde{\rho}_\tau - 1 = \hat{\rho}_\tau - 1 + \frac{\frac{1}{2} \sum_{t=1}^{\tau} \triangle y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}$$

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$$= \frac{\sum_{t=1}^{\tau} y_{t-1} u_t - \frac{1}{\tau} \sum_{t=1}^{\tau} y_{t-1} \sum_{t=1}^{\tau} u_t}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} + \frac{\frac{1}{2} \sum_{t=1}^{\tau} \triangle y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}$$

$$= \frac{D_\tau}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} - \frac{\frac{1}{\tau} \sum_{t=1}^{\tau} y_{t-1} \sum_{t=1}^{\tau} u_t}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2},$$

where $D_\tau = \sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2} \sum_{t=1}^{\tau} \triangle y_t^2$. Again $\sum_{t=1}^{\tau} y_{t-1} u_t = \frac{1}{2} \left(y_\tau^2 - y_0^2 \right) - \frac{1}{2} \sum_{t=1}^{\tau} u_t^2$. So when $\rho_n = 1$, $\triangle y_t = u_t$ and $\sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2} \sum_{t=1}^{\tau} \triangle y_t^2 = \frac{1}{2} \left(y_\tau^2 - y_0^2 \right)$. Then

$$n^{-1-2d} D_\tau \Rightarrow \frac{1}{2} W^H(r)^2. \tag{90}$$

Hence,

$$
\begin{aligned}
\tau(\tilde{\rho}_\tau - 1) &= \frac{\tau}{n} \frac{n^{-1-2d} D_\tau - \frac{n}{\tau} n^{-3/2-d} \sum_{t=1}^{\tau} y_{t-1} n^{-1/2-d} \sum_{t=1}^{\tau} u_t}{n^{-2-2d} \sum_{t=1}^{\tau} y_{t-1}^2 - \frac{n}{\tau} \left(n^{-3/2-d} \sum_{t=1}^{\tau} y_{t-1} \right)^2} \\
&\Rightarrow \frac{\frac{r}{2} \left[\left(W^H(r) \right)^2 \right] - \int_0^r W^H(s) ds W^H(r)}{\int_0^r \left(W^H(s) \right)^2 ds - \frac{1}{r} \left(\int_0^r W^H(s) ds \right)^2}, \quad \text{for } d \in [0, 0.5).
\end{aligned}
$$

∎

Proof of Lemma 10.3

Let $\xi_n = \hat{d} - d$ and note that

$$\Delta_+^{1+\hat{d}} y_t = \Delta_+^{\xi_n} \left(\Delta_+^{1+d} y_t \right) = \Delta_+^{\xi_n} \epsilon_t.$$

$$\Delta_+^{\xi_n} \epsilon_t = \sum_{k=0}^{t-1} \binom{\xi_n}{k} (-L)^k \epsilon_t = \epsilon_t - \xi_n \sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k} + O_p(\xi_n^2).$$

Since $\xi_n = O_p(m^{-1})$, $Var(\sum_{k=1}^{t-1} \frac{\epsilon_{t-k}}{k}) = \sigma^2 \sum_{k=1}^{t-1} \frac{1}{k^2}$ is asymptotically bounded as $n \to \infty$, establishing (51). ∎

Proof of Lemma 10.5

To show the first claim, backward substitution gives

$$y_{\lfloor nr \rfloor} = \rho_n^{\lfloor nr \rfloor - (\lfloor nr_e \rfloor - 1)} y_{\lfloor nr_e \rfloor - 1} + \sum_{j=\lfloor r_e \rfloor}^{\lfloor nr \rfloor} \left(1 + \frac{c}{n} \right)^{\lfloor nr \rfloor - j} u_j,$$

and $\rho_n^{\lfloor nr \rfloor - (\lfloor nr_e \rfloor - 1)} = (1 + c/n)^{\lfloor nr \rfloor - (\lfloor nr_e \rfloor - 1)} = \exp((r - r_e)c) + o(1)$. Therefore,

$$
\begin{aligned}
\frac{1}{n^{1/2+d}} y_{\lfloor nr \rfloor} &= \exp((r - r_e)c) \frac{1}{n^{1/2+d}} y_{\lfloor nr_e \rfloor - 1} + \frac{1}{n^{1/2+d}} \sum_{j=\lfloor r_e \rfloor}^{\lfloor nr \rfloor} \left(1 + \frac{c}{n} \right)^{\lfloor nr \rfloor - j} u_j \\
&\Rightarrow \sigma \left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^r e^{(r-s)c} dW^H(s) \right), \tag{91}
\end{aligned}
$$

where the limit follows by Lemma 10.1 and the CMT as in Lui et al. (2020).

For the second and the third claims, write $\frac{1}{n^{2/3+d}}\sum_{t=1}^{\tau}y_{t-1} = \frac{1}{n}\sum_{t=1}^{\tau}\left(\frac{1}{n^{1/2+d}}y_{t-1}\right)$ and $\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}y_{t-1}^2 = \frac{1}{n}\sum_{t=1}^{\tau}\left(\frac{1}{n^{1/2+d}}y_{t-1}\right)^2$. Application of (91) and the CMT then yields the two results.

For the fourth claim, first consider the second component $\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2$. Since

$$\Delta y_t = \begin{cases} u_t & \text{if } t < \tau_e, \\ \frac{c}{n}y_{t-1} + u_t & \text{otherwise} \end{cases},$$

we write

$$\sum_{t=1}^{\tau}\Delta y_t^2 = \sum_{t=1}^{\tau_e-1}u_t^2 + \sum_{t=\tau_e}^{\tau}\left(\frac{c}{n}y_{t-1} + u_t\right)^2$$

$$= \sum_{t=1}^{\tau}u_t^2 + \frac{c^2}{n^2}\sum_{t=\tau_e}^{\tau}y_{t-1}^2 + \frac{2c}{n}\sum_{t=\tau_e}^{\tau}y_{t-1}$$

$$= \sum_{t=1}^{\tau}u_t^2 + \frac{c^2}{n^2}n^{2+2d}\left(\frac{1}{n^{2+2d}}\sum_{t=\tau_e}^{\tau}y_{t-1}^2\right) + \frac{2c}{n}n^{3/2+d}\left(\frac{1}{n^{3/2+d}}\sum_{t=\tau_e}^{\tau}y_{t-1}\right). \quad (92)$$

For the term $\frac{1}{n^{2+2d}}\sum_{t=\tau_e}^{\tau}y_{t-1}^2$, as $n \to \infty$,

$$\frac{1}{n^{2+2d}}\sum_{t=\tau_e}^{\tau}y_{t-1}^2 = \frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}y_{t-1}^2 - \frac{1}{n^{2+2d}}\sum_{t=1}^{\tau_e-1}y_{t-1}^2 \Rightarrow \sigma^2 B_{r,d} - \sigma^2\int_0^{r_e}\left(W^H(s)\right)^2 ds, \quad (93)$$

where Lemma 10.5.3 and (39) are used to obtain the limit.

For the term $\frac{1}{n^{3/2+d}}\sum_{t=\tau_e}^{\tau}y_{t-1}$, similarly, using Lemma 10.5.2 and (38), we have

$$\frac{1}{n^{3/2+d}}\sum_{t=\tau_e}^{\tau}y_{t-1} \Rightarrow \sigma A_{r,d} - \sigma\int_0^{r_e}W^H(s)ds. \quad (94)$$

Combining (92), (93) and (94),

$$\sum_{t=1}^{\tau}\Delta y_t^2 = \sum_{t=1}^{\tau}u_t^2 + R_{1,n}, \quad R_{1,n} = O_p(n^{1/2+d}).$$

Upon normalization we now have

$$\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau}y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2\right) \quad (95)$$

$$= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau}y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\tau}u_t^2 + R_{1,n}\right)$$

$$= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\lfloor nr_e\rfloor-1}y_{t-1}u_t + \frac{1}{2}\sum_{t=1}^{\lfloor nr_e\rfloor-1}u_t^2\right) + \frac{1}{n^{1+2d}}\left(\sum_{t=\lfloor nr_e\rfloor}^{\tau}y_{t-1}u_t + \frac{1}{2}\sum_{t=\lfloor nr_e\rfloor}^{\tau}u_t^2\right) + \frac{R_{1,n}}{n^{1+2d}}.$$

For the first component on the right-hand side of (95), applying (90) gives

$$\frac{1}{n^{1+2d}} \left(\sum_{t=1}^{\lfloor nr_e \rfloor - 1} y_{t-1} u_t + \frac{1}{2} \sum_{t=1}^{\lfloor nr_e \rfloor - 1} u_t^2 \right) \Rightarrow \frac{\sigma^2}{2} (W^H(r_e))^2.$$

For the second component, standard calculation yields

$$\sum_{t=\lfloor nr_e \rfloor}^{\tau} y_{t-1} u_t = \frac{1}{2\rho_n} \sum_{t=\lfloor nr_e \rfloor}^{\tau} y_t^2 - \frac{\rho_n}{2} \sum_{t=\lfloor nr_e \rfloor}^{\tau} y_{t-1}^2 - \frac{1}{2\rho_n} \sum_{t=\lfloor nr_e \rfloor}^{\tau} u_t^2.$$

Hence,

$$
\begin{aligned}
\sum_{t=\lfloor nr_e \rfloor}^{\tau} y_{t-1} u_t + \frac{1}{2} \sum_{t=\lfloor nr_e \rfloor}^{\tau} u_t^2 &= \frac{1}{2\rho_n} \sum_{t=\lfloor nr_e \rfloor}^{\tau} y_t^2 - \frac{\rho_n}{2} \sum_{t=\lfloor nr_e \rfloor}^{\tau} y_{t-1}^2 - \frac{1}{2\rho_n} \sum_{t=\lfloor nr_e \rfloor}^{\tau} u_t^2 + \frac{1}{2} \sum_{t=\lfloor nr_e \rfloor}^{\tau} u_t^2 \\
&= \frac{1}{2\rho_n} \sum_{t=\lfloor nr_e \rfloor}^{\tau} y_t^2 - \frac{\rho_n}{2} \sum_{t=\lfloor nr_e \rfloor}^{\tau} y_{t-1}^2 + \frac{1}{2} \left(1 - \frac{1}{\rho_n} \right) \sum_{t=\lfloor nr_e \rfloor}^{\tau} u_t^2.
\end{aligned}
$$

As $\rho_n = 1 + o(1)$, we have

$$\frac{1}{n^{1+2d}} \sum_{t=\lfloor nr_e \rfloor}^{\tau} y_{t-1} u_t = \frac{1}{n^{1+2d}} \frac{1}{2} \left[y_\tau^2 - y_{\lfloor nr_e \rfloor - 1}^2 \right] - \frac{1}{n^{1+2d}} \frac{1}{2} \sum_{t=\lfloor nr_e \rfloor}^{\tau} u_t^2 + o_p(1).$$

Thus,

$$
\begin{aligned}
&\frac{1}{n^{1+2d}} \left(\sum_{t=\lfloor nr_e \rfloor}^{\tau} y_{t-1} u_t + \frac{1}{2} \sum_{t=\lfloor nr_e \rfloor}^{\tau} u_t^2 \right) \\
&= \frac{1}{n^{1+2d}} \frac{1}{2} \left[y_\tau^2 - y_{\lfloor nr_e \rfloor - 1}^2 \right] + o_p(1) \\
&\Rightarrow \frac{\sigma^2}{2} \left[\left(e^{(r-r_e)c} W^H(r_e) + \int_{r_e}^{r} e^{(r-s)c} dW^H(s) \right)^2 - (W^H(r_e))^2 \right],
\end{aligned}
\qquad (96)
$$

where we obtain the limit by applying Lemma 10.5.1 and Lemma 10.1. The last term $R_{1,n}/n^{1+2d}$ in (95) vanishes because $R_{1,n} = O_p(n^{1/2+d})$ as $n \to \infty$. This confirms the fourth claim.

To show the fifth claim, note that

$$\tilde{\rho}_\tau - \rho_n = (\hat{\rho}_\tau - \rho_n) + \frac{\frac{1}{2} \sum_{t=1}^{\tau} \Delta y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2} = \frac{\sum_{t=1}^{\tau} \bar{y}_{t-1} (y_t - \rho_n y_{t-1}) + \frac{1}{2} \sum_{t=1}^{\tau} \Delta y_t^2}{\sum_{t=1}^{\tau} \bar{y}_{t-1}^2}. \qquad (97)$$

The numerator of (97) is

$$
\begin{aligned}
&\sum_{t=1}^{\tau} \bar{y}_{t-1} (y_t - \rho_n y_{t-1}) + \frac{1}{2} \sum_{t=1}^{\tau} \Delta y_t^2 \\
&= \sum_{t=1}^{\tau_e - 1} \bar{y}_{t-1} (y_t - \rho_n y_{t-1}) + \sum_{t=\tau_e}^{\tau} \bar{y}_{t-1} (y_t - \rho_n y_{t-1}) + \frac{1}{2} \sum_{t=1}^{\tau} u_t^2 + R_{1,n}
\end{aligned}
$$

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$$= \sum_{t=1}^{\tau_e-1} \bar{y}_{t-1}\left(y_{t-1} + u_t - \rho_c y_{t-1}\right) + \sum_{t=\tau_e}^{\tau} \bar{y}_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2 + R_{1,n}$$

$$= \left(\sum_{t=1}^{\tau} \bar{y}_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2\right) - \frac{c}{n}\sum_{t=1}^{\tau_e-1} \bar{y}_{t-1} + R_{1,n}. \tag{98}$$

Upon normalization, the first component in (98) is

$$\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} \bar{y}_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2\right)$$

$$= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2 - \frac{1}{\tau}\sum_{t=1}^{\tau} y_{t-1}\sum_{t=1}^{\tau} u_t\right)$$

$$= \frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau} y_{t-1} u_t + \frac{1}{2}\sum_{t=1}^{\tau} u_t^2\right) - \frac{n}{\tau}\frac{1}{n^{3/2+d}}\sum_{t=1}^{\tau} y_{t-1}\frac{1}{n^{1/2+d}}\sum_{t=1}^{\tau} u_t$$

$$\Rightarrow \frac{1}{2}\sigma^2 C_{r,d} - \frac{1}{r}\sigma^2 A_{r,d} W^H(r), \tag{99}$$

by virtue of (96), Lemma 10.5 and Lemma 10.1 to obtain the limit. For the second component, note that

$$\frac{c}{n}\sum_{t=1}^{\tau_e-1} \bar{y}_{t-1} = \frac{c}{n}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{c}{n}\frac{1}{\tau}\sum_{t=1}^{\tau_e-1}\sum_{j=1}^{\tau} y_{j-1} = \frac{c}{n}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{c}{n}\frac{\tau_e-1}{\tau}\sum_{j=1}^{\tau} y_{j-1}.$$

After normalization, we have

$$\frac{1}{n^{1/2+d}}\left(\frac{c}{n}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{c}{n}\frac{\tau_e-1}{\tau}\sum_{j=1}^{\tau} y_{j-1}\right) = \frac{c}{n^{3/2+d}}\sum_{t=1}^{\tau_e-1} y_{t-1} - \frac{\tau_e-1}{\tau}\frac{c}{n^{3/2+d}}\sum_{j=1}^{\tau} y_{j-1}$$

$$\Rightarrow c\sigma\int_0^{r_e} W^H(s)ds - c r_e \sigma A_{r,d},$$

where the limit follows by using Lemma 10.2 and 10.5.2.

Therefore, the first component in (98) is $O_p(n^{1+2d})$ which dominates $\frac{c}{n}\sum_{t=1}^{\tau_e-1}\bar{y}_{t-1} = O_p(n^{1/2+d})$ and the third component $R_{1,n} = O_p(n^{1/2+d})$. Consequently,

$$\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau}\bar{y}_{t-1}\left(y_t - \rho_n y_{t-1}\right) + \frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2\right) \Rightarrow \frac{1}{2}\sigma^2 C_{r,d} - \frac{1}{r}\sigma^2 A_{r,d}W^H(r).$$

For the denominator in (97), applying Lemma 10.5.2 and 10.5.3, we have

$$\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}\bar{y}_{t-1}^2 = \frac{1}{n^{2+2d}}\left[\sum_{t=1}^{\tau} y_{t-1}^2 - \frac{1}{\tau}\left(\sum_{t=1}^{\tau} y_{t-1}\right)^2\right] \Rightarrow \sigma^2\left(B_{r,d} - \frac{1}{r}A_{r,d}^2\right), \tag{100}$$

and then

$$n(\tilde{\rho}_\tau - \rho_n) = \frac{\frac{1}{n^{1+2d}}\left(\sum_{t=1}^{\tau}\bar{y}_{t-1}\left(y_t - \rho_n y_{t-1}\right) + \frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2\right)}{\frac{1}{n^{2+2d}}\sum_{t=1}^{\tau}\bar{y}_{t-1}^2}$$

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$$\Rightarrow \frac{\frac{1}{2}C_{r,d} - \frac{1}{r}A_{r,d}W^H(r)}{B_{r,d} - \frac{1}{r}A_{r,d}^2} := X_c(r,d) \ .$$

For the sixth claim, since $n(\tilde{\rho}_\tau - 1) = n(\tilde{\rho}_\tau - \rho_n) + n(\rho_n - 1) = n(\tilde{\rho}_\tau - \rho_n) + c$, we have $n(\hat{\rho}_\tau - 1) \Rightarrow X_c(r,d) + c$, completing the proof of Lemma 10.5. ∎

Proof of Lemma 10.6

1. From Lemma 10.1, we have $\frac{1}{n^{1/2+d_1}} y_{\lfloor nr \rfloor} \Rightarrow \sigma W^{H_1}(r_e)$.

2. For $t \in B$, we have

$$y_t = \rho_n^{t-\tau_e+1} y_{\tau_e-1} + \sum_{j=0}^{t-\tau_e} \rho_n^j u_{t-j} = \rho_n^{t-\tau_e+1} y_{\tau_e-1} + \sum_{j=0}^{t-\tau_e} \rho_n^j u_{t-j,d_2}.$$

Pre-multiplying both terms by $\rho_n^{-(t-\tau_e)}$

$$\rho_n^{-(t-\tau_e)} y_t = \rho_n y_{\tau_e-1} + \rho_n^{-(t-\tau_e)} \sum_{j=0}^{t-\tau_e} \rho_n^{-j} u_{t-j,d_2}, \tag{101}$$

and applying Cauchy–Schwarz we have

$$\sum_{j=0}^{t-\tau_e} \rho_n^{-j} u_{t-j,d_2} \leq \left(\sum_{j=0}^{t-\tau_e} \rho_n^{-2j} \right)^{1/2} \left(\sum_{j=0}^{t-\tau_e} u_{t-j,d_2}^2 \right)^{1/2} \tag{102}$$

$$= \left(\sum_{j=0}^{t-\tau_e} \rho_n^{-2j} \right)^{1/2} \left(\sum_{i=\tau_e}^{t} u_{i,d_2}^2 \right)^{1/2} = \left(\frac{\rho_n^2}{\rho_n^2} \frac{\rho_n^{-2(t-\tau_e)-2} - 1}{\rho_n^{-2} - 1} \right)^{1/2} \left(\sum_{i=\tau_e}^{t} u_{i,d_2}^2 \right)^{1/2}$$

$$= \left(\frac{\rho_n^{-2(t-\tau_e)} - \rho_n^2}{1 - \rho_n^2} \right)^{1/2} \left(\sum_{i=\tau_e}^{t} u_{i,d_2}^2 \right)^{1/2} = O_p(n^{\alpha/2}) \times O_p(n_{d_2}^{1/2}), \tag{103}$$

where

$$n_d = \begin{cases} n & \text{if } d \in [0, 0.5) \\ n \ln (n)^2 & \text{if } d = 0.5 \\ n^{2d} & \text{if } d > 0.5 \end{cases}, \tag{104}$$

where the above orders in (104) follow from Duffy and Kasparis (2018, 2021). Since $\rho_n^{-(t-\tau_e)} = \rho_n^{-(\lfloor nr \rfloor - \lfloor nr_e \rfloor)} = \rho_n^{-n(r-r_e)+o(1)}$ and $\rho_n^{-n(r-r_e)} = \exp\left(-Kn^{1-\alpha}\right) + o(1)$, where K is a positive constant, we have $\rho_n^{-(t-\tau_e)} \sum_{j=0}^{t-\tau_e} \rho_n^{-j} u_{t-j,d_2} = o_p(1)$ for any $d_2 \geq 0$, and so

$$\rho_n^{-(t-\tau_e)} \frac{1}{n^{1/2+d_1}} y_t \overset{a}{\sim} \frac{\rho_n}{n^{1/2+d_1}} y_{\tau_e-1} \overset{a}{\sim} \sigma W^{H_1}(r_e).$$

3. For $t \in N_1$, we have

$$y_{\lfloor nr \rfloor} = \sum_{k=\tau_f+1}^{\lfloor nr \rfloor} u_{k,d_1} + y_{\tau_f}^* = \sum_{k=\tau_f+1}^{\lfloor nr \rfloor} u_{k,d_1} + y_{\tau_e} + y^*$$

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$$= \sum_{k=1}^{\lfloor nr \rfloor} u_{k,d_1} - \sum_{k=1}^{\tau_f} u_{k,d_1} + y_{\tau_e} + y^*.$$

Note that $y_{\tau_e} \overset{a}{\sim} n^{1/2+d_1}\sigma W^{H_1}(r_e)$. Applying Lemma 10.1 gives

$$y_{\lfloor nr \rfloor} \overset{a}{\sim} n^{1/2+d_1} \left[\sigma \left(W^{H_1}(r) - W^{H_1}(r_f) \right) + \sigma W^{H_1}(r_e) \right].$$

This completes the proof of Lemma 10.6. ∎

Proof of Lemma 10.7 For $\tau \in B$, we have

$$\frac{1}{\tau} \sum_{j=1}^{\tau} y_j = \frac{1}{\tau} \sum_{j=1}^{\tau_e-1} y_j + \frac{1}{\tau} \sum_{j=\tau_e}^{\tau} y_j.$$

The first term is

$$\frac{1}{\tau} \sum_{j=1}^{\tau_e-1} y_j = n^{1/2+d_1} \frac{\tau_e}{\tau} \left(\frac{1}{\tau_e} \sum_{j=1}^{\tau_e-1} \frac{1}{n^{1/2+d_1}} y_j \right) \overset{a}{\sim} n^{1/2+d_1} \frac{r_e}{r} \sigma \int_0^{r_e} W^{H_1}(s) ds, \tag{105}$$

where Lemma 10.6.1 and the CMT are used to obtain (105). For the second term,

$$\frac{1}{\tau} \sum_{j=\tau_e}^{\tau} y_j \overset{a}{\sim} \frac{1}{\tau} \sum_{j=\tau_e}^{\tau} \rho_n^{(j-\tau_e)} n^{1/2+d_1} \sigma W^{H_1}(r_e) = n^{1/2+d_1} \sigma W^{H_1}(r_e) \frac{1}{\tau} \sum_{j=\tau_e}^{\tau} \rho_n^{j-\tau_e}$$

$$= \frac{n^{1/2+d_1} \sigma W^{H_1}(r_e)}{\tau} \frac{\rho_n^{\tau-\tau_e+1} - 1}{\rho_n - 1} = \frac{n^{1/2+d_1} \sigma W^{H_1}(r_e) \left[(\rho_n^{\tau-\tau_e} \rho_n) n^\alpha - n^\alpha \right]}{\lfloor nr \rfloor c}$$

$$\overset{a}{\sim} n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e), \tag{106}$$

where the last asymptotic equivalence follows because $\rho_n^{\tau-\tau_e} n^\alpha$ dominates n^α. Comparing (106) with (105) and since $\rho_n^{\tau-\tau_e}$ gives an exponential divergence rate, we have the results in Lemma 10.7.1.

For $\tau \in N_1$,

$$\frac{1}{\tau} \sum_{j=1}^{\tau} y_j = \frac{1}{\tau} \sum_{j=1}^{\tau_e-1} y_j + \frac{1}{\tau} \sum_{j=\tau_e}^{\tau_f} y_j + \frac{1}{\tau} \sum_{j=\tau_f+1}^{\tau} y_j. \tag{107}$$

For the first term, similar to (105) we have

$$\frac{1}{\tau} \sum_{j=1}^{\tau_e-1} y_j \overset{a}{\sim} n^{1/2+d_1} \frac{r_e}{r} \sigma \int_0^{r_e} W^{H_1}(s) ds.$$

For the second term, similar to (106) we have

$$\frac{1}{\tau} \sum_{j=\tau_e}^{\tau_f} y_j \overset{a}{\sim} n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e).$$

For the last term, using Lemma 10.6.3 we have

$$\frac{1}{\tau} \sum_{j=\tau_f+1}^{\tau} y_j = \frac{\tau - \tau_f}{\tau} n^{1/2+d_1} O_p(1) \overset{a}{\sim} O_p(n^{1/2+d_1}).$$

As in the proof of Lemma 10.7, the second term in (107) has the highest order. So the result in Lemma 10.7.2 follows. ∎

Proof of Lemma 10.8

1. Suppose $\tau \in B$. If $t \in N_0$, from Lemma 10.6.1, $y_t = O_p(n^{1/2+d_1})$. Following Lemma 10.7.1, we obtain $\frac{1}{\tau} \sum_{j=1}^{\tau} y_{j-1} \overset{a}{\sim} n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e)$. Hence, the second term has higher order and

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e).$$

If $t \in B$, from Lemma 10.6.2,

$$\bar{y}_t \overset{a}{\sim} \rho_n^{(t-\tau_e)} n^{1/2+d_1} \sigma W^{H_1}(r_e) - n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e).$$

2. Suppose $\tau \in N_1$. If $t \in N_0$, then similar to the proof in Lemma 10.8.1 as y_t is asymptotically dominated by the latter term, we have

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e).$$

If $t \in B$,

$$\bar{y}_t \overset{a}{\sim} \left(\rho_n^{(t-\tau_e)} - \rho_n^{\tau_f-\tau_e} \frac{n^{\alpha}}{nrc} \right) n^{1/2+d_1} \sigma W^{H_1}(r_e).$$

If $t \in N_1$, components in y_t are dominated by the components in $\frac{1}{\tau} \sum_{j=1}^{\tau} y_{j-1}$. Following the proof of Lemma 10.8.1, we have

$$\bar{y}_t \overset{a}{\sim} -n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e).$$

This completes the proof of Lemma 10.8. ∎

Proof of Lemma 10.9

1. For $\tau \in B$,

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}^2 = \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}^2 + \sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}^2. \tag{108}$$

For the first term in (108), applying (60) gives

$$\sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}^2 \overset{a}{\sim} \sum_{j=1}^{\tau_e-1} \left(-n^{\alpha+d_1-1/2} \rho_n^{\tau-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e) \right)^2$$

$$= \frac{(\tau_e - 1)}{n} n^{2(\alpha+d_1)} \rho_n^{2(\tau-\tau_e)} \frac{1}{r^2 c^2} \sigma^2 W^{H_1}(r_e)^2 \overset{a}{\sim} \frac{r_e}{r^2 c^2} n^{2(\alpha+d_1)} \rho_n^{2(\tau-\tau_e)} \sigma^2 W^{H_1}(r_e)^2.$$

$$(109)$$

For the second term in (108), applying (61) gives

$$\sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}^2 \overset{a}{\sim} \sum_{j=\tau_e}^{\tau} \left[\left(\rho_n^{(j-\tau_e)} - \frac{n^\alpha}{nrc} \rho_n^{\tau-\tau_e} \right) n^{1/2+d_1} \sigma W^{H_1}(r_e) \right]^2$$

$$= n^{1+2d_1} \sigma^2 \left(W^{H_1}(r_e) \right)^2 \sum_{j=\tau_e}^{\tau} \left(\rho_n^{(j-\tau_e)} - \frac{n^\alpha}{nrc} \rho_n^{\tau-\tau_e} \right)^2$$

$$= n^{1+2d_1} \sigma^2 \left(W^{H_1}(r_e) \right)^2 \sum_{j=\tau_e}^{\tau} \left(\rho_n^{2(j-\tau_e)} - 2\rho_n^{(j-\tau_e)} \frac{n^\alpha}{nrc} \rho_n^{\tau-\tau_e} + \frac{n^{2\alpha}}{n^2 r^2 c^2} \rho_n^{2(\tau-\tau_e)} \right)$$

$$= n^{1+2d_1} \sigma^2 \left(W^{H_1}(r_e) \right)^2 \left[\frac{n^\alpha \rho_n^{2(\tau-\tau_e)}}{2c} - 2 \frac{n^{2\alpha-1} \rho_n^{2(\tau-\tau_e)}}{nrc} + \frac{r - r_e + 1/n}{r^2 c^2} n^{2\alpha-1} \rho_n^{2(\tau-\tau_e)} \right]$$

$$\overset{a}{\sim} n^{1+2d_1+\alpha} \sigma^2 \left(W^{H_1}(r_e) \right)^2 \frac{\rho_n^{2(\tau-\tau_e)}}{2c} \text{ , as } \alpha > 2\alpha - 1.$$

Since $1 + 2d_1 + \alpha > 2(\alpha + d_1)$, $\sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}^2$ dominates $\sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}^2$ asymptotically and we have

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+2d_1+\alpha} \sigma^2 \left(W^{H_1}(r_e) \right)^2 \frac{\rho_n^{2(\tau-\tau_e)}}{2c}. \tag{110}$$

2. For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \bar{y}_{j-1}^2 = \sum_{j=\tau_1}^{\tau_e-1} \bar{y}_{j-1}^2 + \sum_{j=\tau_e}^{\tau_f} \bar{y}_{j-1}^2 + \sum_{j=\tau_f+1}^{\tau} \bar{y}_{j-1}^2. \tag{111}$$

For the first term in (111), similar to (109), we have

$$\sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}^2 \overset{a}{\sim} \frac{r_e}{r^2 c^2} n^{2(\alpha+d_1)} \rho_n^{2(\tau_f-\tau_e)} \sigma^2 \left(W^{H_1}(r_e) \right)^2.$$

For the second term, similar to (110) we have

$$\sum_{j=\tau_e}^{\tau_f} \bar{y}_{j-1}^2 \overset{a}{\sim} n^{1+\alpha+2d_1} \sigma^2 \left(W^{H_1}(r_e) \right)^2 \frac{\rho_n^{2(\tau_f-\tau_e)}}{2c}.$$

For the third term, applying (64) gives

$$\sum_{j=\tau_f+1}^{\tau} \bar{y}_{j-1}^2 \overset{a}{\sim} \sum_{j=\tau_f+1}^{\tau} \left(-n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e) \right)^2$$

$$= \frac{\tau - \tau_f}{n} n^{2(\alpha+d_1)} \rho_n^{2(\tau_f-\tau_e)} \frac{1}{r^2 c^2} \sigma^2 \left(W^{H_1}(r_e) \right)^2 \overset{a}{\sim} \frac{(r - r_f)}{r^2 c^2} n^{2(\alpha+d_1)} \rho_n^{2(\tau_f-\tau_e)} \sigma^2 \left(W^{H_1}(r_e) \right)^2.$$

As $1+\alpha+2d_1 > 2(\alpha + d_1)$, the middle term in the right hand side of (111) dominates and we have

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}^2 \stackrel{a}{\sim} n^{1+\alpha+2d_1}\sigma^2 \left(W^{H_1}(r_e)\right)^2 \frac{\rho_n^{2\left(\tau_f-\tau_e\right)}}{2c}.$$

This completes the proof of Lemma 10.9. ∎

Proof of Lemma 10.10

1. For $\tau \in B$,

$$\sum_{j=1}^{\tau_2} \bar{y}_{j-1}u_j = \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}u_{j,d_1} + \sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}u_{j,d_2}. \tag{112}$$

From (60), the first term in (112) can be written as

$$\sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}u_{j,d_1} \stackrel{a}{\sim} \sum_{j=1}^{\tau_e-1}\left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)u_{j,d_1}$$

$$= \left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)\sum_{j=1}^{\tau_e-1} u_{j,d_1}$$

$$= \frac{-n^{\alpha+d_1-1/2+d_1+1/2}}{rc}\rho_n^{\tau-\tau_e}\sigma W^{H_1}(r_e)\frac{1}{n^{1/2+d_1}}\sum_{j=1}^{\tau_e-1} u_{j,d_1}$$

$$\stackrel{a}{\sim} -\frac{n^{\alpha+2d_1}}{rc}\rho_n^{\tau-\tau_e}\sigma^2\left(W^{H_1}(r_e)\right)^2. \tag{113}$$

For the second term in (112),

$$\sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}u_{j,d_2}$$

$$\stackrel{a}{\sim} \sum_{j=\tau_e}^{\tau}\left[\left(\rho_n^{(j-\tau_e)} - \rho_n^{\tau-\tau_e}\frac{n^\alpha}{nrc}\right)n^{1/2+d_1}\sigma W^{H_1}(r_e)\right]u_{j,d_2}$$

$$= n^{1/2+d_1}\sigma W^{H_1}(r_e)\sum_{j=\tau_e}^{\tau}\left(\rho_n^{(j-\tau_e)}u_{j,d_2} - \rho_n^{\tau-\tau_e}\frac{n^\alpha}{nrc}u_{j,d_2}\right)$$

$$= n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau-\tau_e}\left[\sum_{j=\tau_e}^{\tau}\rho_n^{-(\tau-j)}u_{j,d_2} - \frac{n^\alpha}{nrc}\sum_{j=\tau_e}^{\tau} u_{j,d_2}\right]$$

$$= n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau-\tau_e}\left[O_p(n^{\alpha/2})O_p(n_{d_2}^{1/2}) - \frac{1}{rc}O_p(n^{d_2+\alpha-1/2})\right] \tag{114}$$

$$= n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau-\tau_e}O_p(n^{\alpha/2})O_p(n_{d_2}^{1/2}), \tag{115}$$

where n_{d_2} is defined in (103), (114) is obtained using the approach in (103), and the last equality follows by verifying that for any $d \geq 0$, $n^{\alpha/2}n_{d_2}^{1/2}$ diverges faster than $n^{d_2+\alpha-1/2}$.

The asymptotic orders of $\sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}u_{j,d_1}$ and $\sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}u_{j,d_2}$ depend on the magnitude of α, d_1 and d_2; and we have

$$\sum_{j=1}^{\tau_2} \bar{y}_{j-1}u_j = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1, \alpha+2d_1\right\}}\right). \tag{116}$$

2. For $\tau_1 \in N_0$ and $\tau_2 \in N_1$,

$$\sum_{j=\tau_1}^{\tau_2} \bar{y}_{j-1}u_j = \sum_{j=\tau_1}^{\tau_e-1} \bar{y}_{j-1}u_j + \sum_{j=\tau_d}^{\tau_f} \bar{y}_{j-1}u_j + \sum_{j=\tau_f+1}^{\tau_2} \bar{y}_{j-1}u_j.$$

As in (113), the first term is

$$\sum_{j=\tau_1}^{\tau_e-1} \bar{y}_{j-1}u_j \overset{a}{\sim} -\frac{n^{\alpha+2d_1}}{rc}\rho_n^{\tau-\tau_e}\sigma^2 \left(W^{H_1}(r_e)\right)^2.$$

As in (115), the second term is

$$\sum_{j=\tau_e}^{\tau_f} \bar{y}_{j-1}u_j \overset{a}{\sim} n^{1/2+d_1}\sigma W^{H_1}(r_e)\rho_n^{\tau_f-\tau_e} \times O_p(n^{\alpha/2}) \times O_p(n_{d_2}^{1/2}).$$

The third term is

$$\sum_{j=\tau_f+1}^{\tau} \bar{y}_{j-1}u_j \overset{a}{\sim} \sum_{j=\tau_f+1}^{\tau} \left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right) u_{j,d_1}$$

$$= -n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e) \sum_{j=\tau_f+1}^{\tau} u_{j,d_1}$$

$$= -n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)n^{1/2+d_1}\left(W^{H_1}(r) - W^{H_1}(r_f)\right)$$

$$\overset{a}{\sim} -n^{\alpha+2d_1}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\left(W^{H_1}(r) - W^{H_1}(r_f)\right).$$

Then, similar to (116), we can write

$$\sum_{j=\tau_1}^{\tau_2} \bar{y}_{j-1}u_j = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1, \alpha+2d_1\right\}}\right).$$

and the proof of Lemma 10.10 is complete. ∎

Proof of Lemma 10.11

1. Separate $\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1})$ into two parts

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) = \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) + \sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1})$$

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$$= \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}\left[(1-\rho_n)\,y_{j-1} + u_t\right] + \sum_{j=\tau_e}^{\tau} \bar{y}_{j-1}u_t$$

$$= \sum_{j=1}^{\tau} \bar{y}_{j-1}u_t - \frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}y_{j-1}.$$

From (67),

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}u_t = O_p\left(\rho_n^{\tau-\tau_e}n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,\alpha+2d_1\right\}}\right). \tag{117}$$

For the second term, applying (60) leads to

$$\frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}y_{j-1} \overset{a}{\sim} \frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \left(-n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)y_{j-1}$$

$$= \left(-n^{d_1-1/2}\rho_n^{\tau-\tau_e}\frac{1}{r}\sigma W^{H_1}(r_e)\right)\tau_e\left(\frac{1}{\tau_e}\sum_{j=1}^{\tau_e-1}y_{j-1}\right)$$

$$\overset{a}{\sim} \left(-n^{d_1-1/2}\frac{1}{r}\rho_n^{\tau-\tau_e}\sigma W^{H_1}(r_e)\right)\tau_e\left(n^{1/2+d_1}\sigma\int_0^{r_e}W^{H_1}(s)ds\right)$$

$$= -n^{2d_1+1}\rho_n^{\tau-\tau_e}\frac{r_e}{r}\sigma^2 W^{H_1}(r_e)\int_0^{r_e}W^{H_1}(s)ds. \tag{118}$$

Comparing the order in (117) and (118), as $\alpha < 1$, we can write

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) = O_p\left(\rho_n^{\tau-\tau_e}n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,2d_1+1\right\}}\right).$$

2. For $\tau \in N_1$, we can express

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1})$$

$$= \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) + \sum_{j=\tau_e}^{\tau_f} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) + \bar{y}_{\tau_f}(y_{\tau_f} - \rho_n y_{\tau_f})$$

$$+ \sum_{j=\tau_f+2}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1})$$

$$= \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}\left[-\frac{c}{n^\alpha}y_{j-1} + u_t\right] + \sum_{j=\tau_e}^{\tau_f} \bar{y}_{j-1}u_t + \bar{y}_{\tau_f}(y_{\tau_f+1} - \rho_n y_{\tau_f})$$

$$+ \sum_{j=\tau_f+2}^{\tau} \bar{y}_{j-1}\left[-\frac{c}{n^\alpha}y_{j-1} + u_t\right]$$

$$= \sum_{j=1}^{\tau} \bar{y}_{j-1}u_t - \frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1}y_{j-1} - \frac{c}{n^\alpha} \sum_{j=\tau_f+2}^{\tau} \bar{y}_{j-1}y_{j-1} - \rho_n\bar{y}_{\tau_f}y_{\tau_f}.$$

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For the first term, similar to (67), we have

$$\sum_{j=1}^{\tau} \bar{y}_{j-1} u_t = O_p\left(\rho_n^{\tau-\tau_e} n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1, \alpha+2d_1\right\}}\right).$$

For the second term, following the steps in obtaining (118), we have

$$\frac{c}{n^\alpha} \sum_{j=1}^{\tau_e-1} \bar{y}_{j-1} y_{j-1} \overset{a}{\sim} -n^{2d_1+1} \rho_n^{\tau_f-\tau_e} \frac{r_e}{r} \sigma^2 W^{H_1}\int_0^{r_e} W^{H_1}(s)ds.$$

For the third term,

$$\frac{c}{n^\alpha} \sum_{j=\tau_f+2}^{\tau} \bar{y}_{j-1} y_{j-1} \overset{a}{\sim} \frac{c}{n^\alpha} \sum_{j=\tau_f+2}^{\tau} \left(-n^{\alpha+d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e)\right) y_{j-1}$$

$$= -n^{d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e) \sum_{j=\tau_f+2}^{\tau} y_{j-1}$$

$$= -n^{d_1-1/2} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma W^{H_1}(r_e) (\tau-\tau_f-1) \frac{n^{1/2+d_1}}{\tau-\tau_f-1} \sum_{j=\tau_f+2}^{\tau} \frac{1}{n^{1/2+d_1}} y_{j-1}$$

$$\overset{a}{\sim} -n^{2d_1+1} \rho_n^{\tau_f-\tau_e} \frac{1}{rc} \sigma^2 \left(W^{H_1}(r_e)\right)^2 (r-r_f) \int_{r_f}^r W^{H_1}(s)ds.$$

For the last term, from Lemma 10.6.2 and (63),

$$\rho_n \bar{y}_{\tau_f} y_{\tau_f} \quad \overset{a}{\sim} \quad \bar{y}_{\tau_f} y_{\tau_f} \ (\text{as } \rho_n \to 1)$$

$$\overset{a}{\sim} \quad \left(\rho_n^{\tau_f-\tau_e} - \rho_n^{\tau_f-\tau_e}\frac{n^\alpha}{nrc}\right) n^{1/2+d_1} \sigma W^{H_1}(r_e) \left(\rho_n^{\tau_f-\tau_e} n^{1/2+d_1} \sigma_{d_1} W^{H_1}(r_e)\right)$$

$$= \quad \rho_n^{2(\tau_f-\tau_e)} n^{1+2d_1} \sigma^2 \left(W^{H_1}(r_e)\right)^2.$$

Note that the last component involves $\rho_n^{2(\tau_f-\tau_e)}$ and thus dominates the previous terms. Finally, we have

$$\sum_{j=1}^{\tau} \bar{y}_{j-1}(y_j - \rho_n y_{j-1}) \overset{a}{\sim} -\rho_n^{2(\tau_f-\tau_e)} n^{1+2d_1} \sigma^2 \left(W^{H_1}(r_e)\right)^2.$$

This completes the proof of Lemma 10.11. ∎

Proof of Lemma 10.12

For Δy_t, note that it has different expressions at different periods, viz.,

$$\Delta y_t = \begin{cases} u_{t,d_1} & if \ \ t < \tau_e, \\ (\rho_n - 1)y_{t-1} + u_{t,d_2} & if \ \ \tau_e \le t \le \tau_f, \\ y_{\tau_e} + y^* + u_{\tau_f+1,d_1} - y_{\tau_f} & if \ \ t = \tau_f + 1, \\ u_{t,d_1} & if \ \ t > \tau_f + 1. \end{cases} \tag{119}$$

Note that
$$\sum_{t=1}^{\tau} \Delta y_t^2 = \sum_{t=1}^{\tau_e-1} \Delta y_t^2 + \sum_{t=\tau_e}^{\tau_f} \Delta y_t^2 + \Delta y_{\tau_f+1}^2 + \sum_{t=\tau_f+2}^{\tau} \Delta y_t^2, \tag{120}$$

and using (119), we can write

$$\sum_{t=1}^{\tau} \Delta y_t^2 \tag{121}$$

$$= \sum_{t=1}^{\tau_e-1} u_{t,d_1}^2 + \sum_{t=\tau_e}^{\tau_f} \left((\rho_n - 1)y_{t-1} + u_{t,d_2}\right)^2 + \left(y_{\tau_e} + y^* + u_{\tau_f+1,d_1} - y_{\tau_f}\right)^2 + \sum_{t=\tau_f+2}^{\tau} u_{t,d_1}^2$$

$$= \sum_{t=1}^{\tau_e-1} u_{t,d_1}^2 + \frac{c^2}{n^{2\alpha}} \sum_{t=\tau_e}^{\tau_f} y_{t-1}^2 + \frac{2c}{n^\alpha} \sum_{t=\tau_e}^{\tau_f} y_{t-1} u_{t,d_2} + \sum_{t=\tau_e}^{\tau_f} u_{t,d_2}^2 + \left(y_{\tau_e} + y^* - y_{\tau_f}\right)^2$$

$$+ 2\left(y_{\tau_e} + y^* - y_{\tau_f}\right) u_{\tau_f+1,d_1} + u_{\tau_f+1,d_1}^2 + \sum_{t=\tau_f+2}^{\tau} u_{t,d_1}^2 \tag{122}$$

$$= \left[\sum_{t=1}^{\tau_e-1} u_{t,d_1}^2 + \sum_{t=\tau_f+1}^{\tau} u_{t,d_1}^2 + \sum_{t=\tau_e}^{\tau_f} u_{t,d_2}^2\right] + \frac{c^2}{n^{2\alpha}} \sum_{t=\tau_e}^{\tau_f} y_{t-1}^2 + \frac{2c}{n^\alpha} \sum_{t=\tau_e}^{\tau_f} y_{t-1} u_t$$

$$+ \left(y_{\tau_e} + y^* - y_{\tau_f}\right)^2 + 2\left(y_{\tau_e} + y^* - y_{\tau_f}\right) u_{\tau_f+1}. \tag{123}$$

Now compare the stochastic orders of the components in (121). First, $\sum_{i=1}^{t} u_{i,d}^2 = O_p(n_d)$, where n_d is defined in (104). For the second term, applying Lemma 10.7, we have

$$\frac{c^2}{n^{2\alpha}} \sum_{t=\tau_e}^{\tau_f} y_{t-1}^2 \overset{a}{\sim} \frac{c^2}{n^{2\alpha}} \left(n^{1/2+d_1} \sigma W^{H_1}(r_e)\right)^2 \sum_{t=\tau_e}^{\tau_f} \rho_n^{2(t-\tau_e)}$$

$$= \frac{c^2}{n^{2\alpha}} \left(n^{1/2+d_1} \sigma W^{H_1}(r_e)\right)^2 \frac{n^\alpha \rho_n^{2(\tau_f-\tau_e)}}{2c} = \frac{c}{2} n^{1+2d_1-\alpha} \rho_n^{2(\tau_f-\tau_e)} \sigma^2 W^{H_1}(r_e)^2$$

$$= O_p(n^{1+2d_1-\alpha} \rho_n^{2(\tau_f-\tau_e)}). \tag{124}$$

Suppose $\tau \in B$. We do not have the term in (120), and (124) yields (70). For the third term, note that

$$y_{\tau_e} + y^* - y_{\tau_f} \overset{a}{\sim} n^{1/2+d_1} \sigma W^{H_1}(r_e) + O_p(1) - \rho_n^{(\tau_f-\tau_e)} n^{1/2+d_1} \sigma W^{H_1}(r_e)$$

$$= O_p\left(\rho_n^{(\tau_f-\tau_e)} n^{1/2+d_1}\right),$$

which implies $\left(y_{\tau_e} + y^* - y_{\tau_f}\right)^2 = O_p\left(\rho_n^{2(\tau_f-\tau_e)} n^{1+2d_1}\right)$; and

$$2\left(y_{\tau_e} + y^* - y_{\tau_f}\right) u_{\tau_f+1} = O_p\left(\rho_n^{(\tau_f-\tau_e)} n^{1/2+d_1}\right) \times O_p(1)$$

$$= O_p\left(\rho_n^{(\tau_f-\tau_e)} n^{1/2+d_1}\right).$$

Thus, the third term which involves $\rho_n^{2(\tau_f-\tau_e)}$ dominates the other terms as $n \to \infty$, and we have (71). This completes the proof. ∎

Proof of Lemma 10.13

By definition, $\tilde{\rho}_\tau = \hat{\rho}_\tau + \frac{\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2}$. From Lemma 10.9 and Lemma 10.12, it is clear that $\frac{1}{2}\sum_{j=1}^{\tau}\Delta y_t^2$ is at most $O_p\left(n^{1+2d_1}\rho_n^{2(\tau_f-\tau_e)}\right)$ and $\sum_{j=1}^{\tau}\bar{y}_{j-1}^2 = O_p\left(n^{1+2d_1+\alpha}\rho_n^{2(\tau_f-\tau_e)}\right)$ for $\tau \in B \cup N_1$. Hence, $\frac{\frac{1}{2}\sum_{t=1}^{\tau}\Delta y_t^2}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2} = o_p(1)$. It means we only need to study the asymptotic properties of $\hat{\rho}_\tau$. We first focus on the centered statistics $\hat{\rho}_\tau - \rho_n = \frac{\sum_{j=1}^{\tau}\bar{y}_{j-1}(y_j - \rho_n y_{j-1})}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2}$.

1. When $\tau \in B$, applying (69) and (65) gives

$$\frac{\sum_{j=1}^{\tau}\bar{y}_{j-1}(y_j - \rho_n y_{j-1})}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2} = \frac{O_p\left(\rho_n^{\tau-\tau_e}n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,2d_1+1\right\}}\right)}{O_p\left(n^{1+2d_1+\alpha}\rho_n^{2(\tau-\tau_e)}\right)}$$

$$= \frac{O_p\left(n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,2d_1+1\right\}-1-2d_1-\alpha}\right)}{O_p\left(\rho_n^{\tau-\tau_e}\right)}. \quad (125)$$

Note that (125) also implies $\tilde{\rho}_\tau - \rho_n = O_p\left(n^{\max\left\{\frac{1+\alpha+f(d_2)}{2}+d_1,2d_1+1\right\}-2d_1-\alpha}\rho_n^{-(\tau-\tau_e)}\right) = o_p(1)$ as $\rho_n^{\tau-\tau_e}$ diverges exponentially. As $n(\tilde{\rho}_\tau - 1) = n(\rho_n - 1) + n(\tilde{\rho}_\tau - \rho_n)$, we have

$$n(\rho_n - 1) + n(\tilde{\rho}_\tau - \rho_n) = n^{1-\alpha}c + o_p(1) \to \infty. \quad (126)$$

2. When $\tau \in N_1$,

$$\frac{\sum_{j=1}^{\tau}\bar{y}_{j-1}(y_j - \rho_n y_{j-1})}{\sum_{j=1}^{\tau}\bar{y}_{j-1}^2} \overset{a}{\sim} \frac{-\rho_n^{2(\tau_f-\tau_e)}n^{1+2d_1}\sigma^2\left(W^{H_1}(r_e)\right)^2}{n^{1+\alpha+2d_1}\frac{\rho_n^{2(\tau_f-\tau_e)}}{2c}\sigma^2 W^{H_1}(r_e)^2}$$

$$= -n^{-\alpha}2c. \quad (127)$$

Similarly, (127) also gives the order of $\tilde{\rho}_\tau - \rho_n$.

As $n(\tilde{\rho}_\tau - 1) = n(\rho_n - 1) + n(\tilde{\rho}_\tau - \rho_n)$, we have

$$n(\rho_n - 1) + n(\tilde{\rho}_\tau - \rho_n) = n^{1-\alpha}c - n\left(n^{-\alpha}2c\right) + o_p(1) = -n^{1-\alpha}c + o_p(1) \to -\infty. \quad (128)$$

This completes the proof of Lemma 10.13. ∎

Proof of Lemma 10.14 Recall from (44) that

$$\hat{\Omega}_{HAR} = \frac{1}{\tau}\sum_{i=1}^{\tau-1}\frac{1}{\tau}\sum_{j=1}^{\tau-1}\tau^2 D_\tau\left(\frac{i-j}{\tau}\right)\frac{1}{\sqrt{\tau}}\hat{S}_i\frac{1}{\sqrt{\tau}}\hat{S}_j,$$

and to find its order we need only study the limit of $\frac{1}{\sqrt{\tau}}\hat{S}_i$. Suppose $\tau \in B$, we have

$$\frac{1}{\sqrt{n}}\hat{S}_\tau = \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1}) = \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{\tau_e-1}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1}) + \sum_{i=\tau_e}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})\right]$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_e-1} (u_{i,d_1} - (\hat{\rho}_\tau - 1)\bar{y}_{i-1}) + \frac{1}{\sqrt{n}} \sum_{i=\tau_e}^{\tau} (u_{i,d_2} - (\hat{\rho}_\tau - \rho_n)\bar{y}_{i-1})$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_e-1} u_{i,d_1} + \frac{1}{\sqrt{n}} \sum_{i=\tau_e}^{\tau} u_{i,d_2} - (\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_e-1} \bar{y}_{i-1} - (\hat{\rho}_\tau - \rho_n)\frac{1}{\sqrt{n}} \sum_{i=\tau_e}^{\tau} \bar{y}_{j-1}. \quad (129)$$

Now compare the order of the three terms in (129). It is clear that $\frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_e-1} u_{i,d_1} + \frac{1}{\sqrt{n}} \sum_{i=\tau_e}^{\tau} u_{i,d_2} = O_p\left(n^{\max\{d_1,d_2\}}\right)$. For the second term, note that $(\hat{\rho}_\tau - 1) \overset{a}{\sim} \frac{c}{n^\alpha}$ and

$$\sqrt{n}\frac{1}{n} \sum_{i=1}^{\tau_e-1} \bar{y}_{i-1} = \sqrt{n}\frac{\lfloor nr \rfloor}{n}\frac{1}{\tau} \sum_{i=1}^{\tau_e-1} \bar{y}_{i-1} = \sqrt{n}\frac{\lfloor nr \rfloor}{n}\left(\frac{1}{\tau} \sum_{i=1}^{\tau_e-1} y_{i-1} - \frac{\tau_e-1}{\tau}\frac{1}{\tau} \sum_{i=1}^{\tau} y_i\right)$$

$$= O(\sqrt{n}) \times \left(O_p\left(n^{1/2+d_1}\right) - O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right)\right) = O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right),$$

where we obtain the third equality by continuous mapping, Lemma 10.6, and 10.7.

This makes $(\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_e-1} \bar{y}_{i-1} = \frac{c}{n^\alpha}O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right) = O_p\left(n^{d_1}\rho_n^{\tau-\tau_e}\right)$. For the last term in (129), note that from (126), we have $\hat{\rho}_\tau - \rho_n = O_p\left(\frac{1}{n^\alpha\rho_n^{\tau-\tau_e}}\right)$ and

$$\frac{1}{\sqrt{n}} \sum_{i=\tau_e}^{\tau} \bar{y}_{j-1} = r\sqrt{n}\frac{\lfloor nr \rfloor}{nr}\frac{1}{\tau} \sum_{i=\tau_e}^{\tau} \bar{y}_{j-1} = r\sqrt{n}\frac{\lfloor nr \rfloor}{nr}\frac{1}{\tau} \sum_{i=\tau_e}^{\tau}\left(y_{j-1} - \frac{1}{\tau} \sum_{i=1}^{\tau} y_j\right)$$

$$= r\sqrt{n}\frac{\lfloor nr \rfloor}{nr}\left(\frac{1}{\tau} \sum_{i=\tau_e}^{\tau} y_{j-1} - \frac{\tau-\tau_e+1}{\tau}\frac{1}{\tau} \sum_{i=1}^{\tau} y_j\right).$$

From Lemma 10.7.1 and (106), we have

$$\frac{1}{\tau} \sum_{j=\tau_e}^{\tau} y_{j-1} = O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right),$$

and

$$\frac{1}{\tau} \sum_{j=1}^{\tau} y_j = O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right).$$

Thus,

$$\frac{1}{\sqrt{n}} \sum_{i=\tau_e}^{\tau} \bar{y}_{j-1} = O\left(n^{1/2}\right)O_p\left(n^{\alpha+d_1-1/2}\rho_n^{\tau-\tau_e}\right) = O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right),$$

and this implies

$$(\hat{\rho}_\tau - \rho_n)\frac{1}{\sqrt{n}} \sum_{i=\tau_e}^{\tau} \bar{y}_{j-1} = O_p\left(\frac{1}{n^\alpha\rho_n^{\tau-\tau_e}}\right)O_p\left(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}\right) = O_p\left(n^{d_1}\right). \quad (130)$$

Comparing the order of the three terms in (129), we obtain

$$\frac{1}{\sqrt{n}}\hat{S}_\tau \overset{a}{\sim} -(\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_e-1} \bar{y}_{i-1} = O_p\left(n^{d_1}\rho_n^{\tau-\tau_e}\right). \quad (131)$$

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Then (44) and (45) imply $\hat{\Omega}_{HAR} = O_p(n^{2d_1}\rho_n^{2(\tau-\tau_e)})$.

Suppose $\tau \in N_1$. In this case

$$\frac{1}{\sqrt{n}}\hat{S}_\tau = \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})$$

$$= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{\tau_e-1}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1}) + \sum_{i=\tau_e}^{\tau_f}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1}) + \sum_{i=\tau_f+1}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})\right]$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}(u_{i,d_1} - (\hat{\rho}_\tau - 1)\bar{y}_{i-1}) + \frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau_f}(u_{i,d_2} - (\hat{\rho}_\tau - \rho_n)\bar{y}_{i-1}) + \frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}(u_{i,d_1} - (\hat{\rho}_\tau - 1)\bar{y}_{i-1})$$

$$= \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{\tau_e-1}u_{i,d_1} + \sum_{i=\tau_e}^{\tau_f}u_{i,d_2} + \sum_{i=\tau_f+1}^{\tau}u_{i,d_1}\right) + (\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1}$$

$$- (\hat{\rho}_\tau - \rho_n)\frac{1}{\sqrt{n}}\sum_{i=\tau_e}^{\tau}\bar{y}_{j-1} + (\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}\bar{y}_{i-1}. \tag{132}$$

The orders of the first three terms in (132) are $O_p(n^{\max\{d_1,d_2\}})$, $O_p\left(n^{d_1}\rho_n^{\tau-\tau_e}\right)$ (as in (131)), and $O_p\left(n^{d_1}\right)$ (as in (130)), respectively. For the last term, we have

$$(\hat{\rho}_\tau - 1)\frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}\bar{y}_{i-1} \overset{a}{\sim} \frac{c}{n^\alpha}\frac{1}{\sqrt{n}}\sum_{i=\tau_f+1}^{\tau}\left(-n^{\alpha+d_1-1/2}\rho_n^{\tau_f-\tau_e}\frac{1}{rc}\sigma W^{H_1}(r_e)\right)$$

$$= O_p(n^{d_1}\rho_n^{\tau_f-\tau_e}),$$

where we have applied (64) and (128) to obtain the asymptotic equivalence. As $\tau \in N_1$, $\tau_f < \tau$, eventually we have the same expression as in (131) and this implies that $\Omega_{HAR} = O_p(n^{2d_1}\rho_n^{2(\tau-\tau_e)})$.

For $s_\tau^2 = \frac{\frac{1}{\tau}\sum_{i=1}^{\tau}\hat{u}_t^2}{\frac{1}{\tau}\sum_{i=1}^{\tau}\bar{y}_{i-1}^2}$, note that when $\tau \in B$ we can write

$$\frac{1}{\tau}\sum_{i=1}^{\tau}\hat{u}_t^2 = \frac{1}{\tau}\sum_{i=1}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})^2 = \frac{1}{\tau}\sum_{i=1}^{\tau_e-1}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})^2 + \frac{1}{\tau}\sum_{i=\tau_e}^{\tau}(\bar{y}_i - \hat{\rho}_\tau\bar{y}_{i-1})^2$$

$$= \frac{1}{\tau}\sum_{i=1}^{\tau_e-1}(u_{i,d_1} - (\hat{\rho}_\tau - 1)\bar{y}_{i-1})^2 + \frac{1}{\tau}\sum_{i=\tau_e}^{\tau}(u_{i,d_2} - (\hat{\rho}_\tau - \rho_n)\bar{y}_{i-1})^2$$

$$= \frac{1}{\tau}\left(\sum_{i=1}^{\tau_e-1}u_{i,d_1}^2 + \sum_{i=\tau_e}^{\tau}u_{i,d_2}^2\right) - 2(\hat{\rho}_\tau - 1)\frac{1}{\tau}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1}u_{i,d_1} - 2(\hat{\rho}_\tau - \rho_n)\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}\bar{y}_{i-1}u_{i,d_2}$$

$$+ (\hat{\rho}_\tau - 1)^2\frac{1}{\tau}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1}^2 + (\hat{\rho}_\tau - \rho_n)^2\frac{1}{\tau}\sum_{i=\tau_e}^{\tau}\bar{y}_{i-1}^2. \tag{133}$$

As $n \to \infty$, note that $\frac{1}{\tau}\left(\sum_{i=1}^{\tau_e-1}u_{i,d_1}^2 + \sum_{i=\tau_e}^{\tau}u_{i,d_2}^2\right) = (O_p(n_{d_1}) + O_p(n_{d_2}))/n$, where n_d is defined in (104), and

$$(\hat{\rho}_\tau - 1)\frac{1}{\tau}\sum_{i=1}^{\tau_e-1}\bar{y}_{i-1}u_i = O_p(n^{-\alpha}) \times O_p(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}) = O_p(n^{\alpha+d_1}\rho_n^{\tau-\tau_e}),$$

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where we obtain the first equality from (113) and (126). For the fourth term in (133),

$$(\hat{\rho}_\tau - 1)^2 \frac{1}{\tau} \sum_{i=1}^{\tau} \bar{y}_{i-1}^2 = O_p(n^{-2\alpha}) \times O_p(n^{-1+2(\alpha+d_1)} \rho_n^{2(\tau-\tau_e)}) = O_p(n^{2d_1-1} \rho_n^{2(\tau-\tau_e)}),$$

where the first equality follows from (109) and (126). For the last term in (133)

$$(\hat{\rho}_\tau - \rho_n) \frac{1}{\tau} \sum_{i=\tau_e}^{\tau} \bar{y}_{i-1} u_i = O_p(n^{-\alpha} \rho_n^{-(\tau-\tau_e)}) \times O_p(\rho_n^{\tau-\tau_e} n^{\frac{1}{2}(1+\alpha)+d_1+d_1\alpha-1}) = O_p(n^{-\frac{1}{2}(1+\alpha)+d_1+d_1\alpha}),$$

where the first equality follows from (115) and (126).

Note that the fourth term in (133) asymptotically dominates the other 4 terms. Therefore, we have $\frac{1}{\tau} \sum_{i=1}^{\tau} \hat{u}_t^2 = O_p(n^{2d_1-1} \rho_n^{2(\tau-\tau_e)})$. Combining the result from Lemma 10.9.1, where $\sum_{j=1}^{\tau} \bar{y}_{j-1}^2 = O_p(n^{1+2d_1+\alpha} \rho_n^{2(\tau-\tau_e)})$, we have

$$s_\tau^2 = \frac{O_p(n^{2d_1-1} \rho_n^{2(\tau-\tau_e)})}{O_p(n^{1+2d_1+\alpha} \rho_n^{2(\tau-\tau_e)})} = O_p(n^{-2-\alpha}),$$

which completes the proof of Lemma 10.14. ∎