Sequential auctions with decreasing reserve prices.

Massimiliano LANDI
Singapore Management University, landim@smu.edu.sg

Domenico MENICUCCI

Massimiliano Landi[1] / Domenico Menicucci[2]

Sequential Auctions with Decreasing Reserve Prices

[1] School of Economics, Singapore Management University, 90 Stamford Road, 178903 Singapore, E-mail: landim@smu.edu.sg
[2] Dipartimento di Scienze per l'Economia e l'Impresa, Università degli Studi di Firenze, Via delle Pandette 9, I-50127 Firenze, Italia, E-mail: domenico.menicucci@unifi.it

Abstract:
We study sequential sealed bid auctions with decreasing reserve prices when there are two identical objects for sale and unit-demand bidders (existing literature has dealt with the case of weakly increasing reserve prices). Under decreasing reserve prices bidders may have an incentive not to bid in the first auction, and no equilibrium exists with a strictly increasing stage one bidding function. However, we find that an equilibrium always exists, and its shape depends on the distance between the two reserve prices. The equilibrium exhibits some pooling at the stage one auction, which disappears in the limit as the number of bidders tends to infinity. We also show revenue equivalence between first-price and second-price sequential auctions under decreasing reserve prices. Finally, our results allow us to shed some light on an optimal order problem (increasing versus decreasing exogenous reserve prices) for selling the two objects.

Keywords: sequential auctions, first-price auction, second-price auction, revenue equivalence

JEL classification: C7, D44

DOI: 10.1515/bejte-2017-0125

1 Introduction

We study a model of sequential sealed bid auctions with decreasing reserve prices. Sequential auctions are widely adopted to sell, among other things, U.S. Treasury bills, wines, and agricultural products. Nowadays, eBay is an increasingly popular venue for sequentially selling units of the same consumer product (electronic calculators, wristwatches, etcetera). It is not uncommon for the same seller to use alternate auctions – with or without reserve price – for different units of the same consumer product.

In the theoretical literature, sequential sealed bid auctions have received relatively scarce attention: Milgrom and Weber (1999) and Weber (1983), provide theoretical analyses of sequential auctions for multiple identical objects under the assumption that there are no reserve prices. Gong, Tan, and Xing (2014) (GTX henceforth) study sequential auctions for identical objects with unit-demand bidders, allowing for different reserve prices at different stages. When the reserve prices are (weakly) ascending, GTX identify an equilibrium and provide an equivalence result between sequential first-price and second-price auctions. However, when reserve prices are descending, they identify an equilibrium only if the stage one reserve price is sufficiently larger than the stage two reserve price (see the equilibrium (a) later in the introduction). Moreover, they prove the non existence of an equilibrium with a strictly increasing stage one bidding function.

Our paper completes their analysis of decreasing reserve prices. Therefore, we study sequential auctions with two identical objects and unit-demand bidders. Our main contribution shows that for any pair of decreasing reserve prices there exists a pure strategy equilibrium, which we completely characterize for both sequential first-price auctions and sequential second-price auctions. In particular, we confirm the non-existence of an equilibrium with a strictly increasing bidding function and we find that the equilibrium has a partially flat stage one bidding function.

In order to better understand our key finding, it is useful to recall a few properties of the equilibrium for the case of ascending reserve prices. To fix the ideas, let us focus on sequential second-price auctions, for which the equilibrium bidding at stage two is straightforward. Let r_1, r_2 denote the reserve prices for the first and second auction, respectively. Then, given $r_1 \le r_2$, a bidder with value $x \in [r_1, r_2]$ participates only in the first auction, and bids as in a single-unit auction. Meanwhile, a bidder with value $x > r_2$ participates in both auctions; in the stage one auction, he bids less aggressively than in a single-unit auction because, if he loses, he will have another opportunity to win.

Domenico Menicucci is the corresponding author.

With descending reserve prices, the incentive to shade bids at stage one is magnified because, all else being equal, $r_2 < r_1$ makes the stage two auction more profitable for a bidder that can participate in both stages. In particular, a bidder may choose not to compete at stage one even though his value is greater than r_1, because winning the object at price r_1 may be less profitable than competing in the second auction.[1] This force must be taken into account when constructing an equilibrium, and we prove that its shape depends on the distance between r_1 and r_2.

Specifically, the equilibrium for second-price sequential auctions is such that:

a. When r_1 is sufficiently larger than r_2, none of the bidder types participates in the stage one auction.

b. When $r_1 - r_2$ takes on intermediate values, there is a threshold $\gamma > r_1$ such that in the stage one auction, bidders with value smaller than γ do not bid, and bidders with value at least γ bid r_1.

c. If r_1 is close to r_2, the equilibrium is characterized by two thresholds, γ (whose value is different from before) and $\lambda > \gamma$, such that in the stage one auction, bidders with value smaller than γ do not bid, bidders with value between γ and λ bid r_1, and bidders with value bigger than λ adopt a strictly increasing bidding function.

Existence of a pure strategy equilibrium is obtained by allowing for a partially flat bidding function. This implies that a bidder's beliefs about the values of the other active bidders at stage two do not depend only on the stage one winning bid, but also on the bid he has submitted at stage one. These beliefs are important because they affect a bidder's total payoff as evaluated at stage one. Deriving and handling these beliefs is not entirely straightforward, but plays a key role in establishing the existence of the equilibrium.

We then remark that the flat portion of the stage one bidding function tends to disappear when the number of bidders tends to infinity. We prove that for each $r_1 > r_2$, the equilibrium in (c) emerges if the number of bidders is sufficiently large, and in that case γ and λ are both close to r_1. Therefore, (almost) every bidder with value above r_1 bids at the stage one auction – and according to a strictly increasing function.

An additional finding in our paper is that sequential first-price and second-price auctions are revenue equivalent *even* under decreasing reserve prices. This result is useful since sequential first-price auctions are somewhat more complicated to deal with than sequential second-price auctions. This is because, at stage two, there exists no dominant bid; hence, the beliefs outlined above affect also the stage two equilibrium bids.

Finally, our findings allow us to deal with a problem introduced in GTX about the optimal order in which two objects should be auctioned. The setting considered has two sellers, each of whom owns one of the two objects; it is commonly known that one seller has value $r > 0$ for her object while the other seller has value zero for her object. The objects are offered through sequential auctions such that at each stage the reserve price is equal to the seller's value for the object auctioned at that stage. An auctioneer chooses the object which is put for sale first in order to maximize the sum of the sellers' profits. GTX use the equilibrium in (a) to show that decreasing reserve prices are optimal if r is large. However, our equilibria in (b) and (c) allow us to consider any r, and we prove that increasing reserve prices are optimal when r is small. We also obtain very specific results if the values are uniformly distributed.[2]

To the best of our knowledge, a partially flat equilibrium bidding function is a novel feature in standard symmetric auctions. Che and Gale (1998) obtain it in auctions with financially constrained bidders; Niedermayer, Shneyerov, and Xu (2016) in foreclosure auctions with asymmetric bidders.

In order to see the effects of financial constraints, consider a single unit second-price auction in which each bidder cannot pay, and bid, more than a certain amount $w > 0$ which is independent of his value for the object, x.[3] The equilibrium bid of type x is the smaller number between x and w. This generates a flat portion in the equilibrium bidding function for values above w. Our result in case (c) is different since the stage one bidding function is flat only for intermediate values, and strictly increasing thereafter. The flat portion in our model is driven by the existence of two stages and decreasing reserve prices, not by budget constraints.

Niedermayer et al. (2016) examine foreclosure auctions in which the bidders are a lender and some real estate brokers. In these auctions, the lender has a credit of $v > 0$ (the balance of the mortgage) towards the house owner, and will earn the smaller amount between the revenues from the auction and v. In addition, the lender privately observes the resale price x of the house.

In equilibrium, the lender's bidding function has a flat region for a range of intermediate values of x. Specifically, when x is much smaller than v, the lender bids an amount that increases with x because his bid plays the role of a reserve price which increases his revenue. When x takes on larger values that are still below v, the lender bids v, because this is the maximum amount that he can receive from the winner of the auction. (Winning the house and reselling it in the market would generate even lower revenues.) Finally, for x larger than v, the lender bids again an amount that increases with x (but is less than x) because the lender wants to win the auction to cash the favorable resale price.

The mechanism which generates a partially flat bidding function is very different from ours, as it depends on the asymmetry between the lender and ordinary bidders, and the fact that the lender may turn out to be either a buyer or a seller.

The remainder of the paper is organized as follows: Section 2 describes the model in detail. Section 3 provides the analysis for sequential second-price auctions. Section 4 proves that sequential first-price auctions and sequential second-price auctions are revenue equivalent. Section 5 contains our analysis for the optimal order problem, while Section 6 concludes. All the proofs are either in the Appendix or the Supplementary Material.

2 The Model

Two identical objects are offered to $n \geq 3$ bidders through two sequential (sealed bid) auctions with reserve prices – r_1 and r_2, respectively – which are common knowledge. In particular, one object is offered using an auction with reserve price r_1, and the winning bid is publicly announced. In case the highest bid is submitted by $m \geq 2$ bidders, the winning bidder is selected randomly, and each highest bidder has probability $\frac{1}{m}$ to win. If no bid is submitted at stage one, then this information is revealed to each bidder before stage two, and the object remains unsold. The other object is offered using an auction with reserve price r_2.[4]

We assume that each bidder is risk neutral, has no time discount, and has unit demand so that no bidder wants to buy more than one object. All else being equal, bidders are therefore indifferent between getting the object at either stage. Moreover, unit demand implies that the winner of the first object does not join the second auction. Last, we assume that each bidder i has value X_i for the object and each X_i is independent and identically distributed (i.i.d. henceforth) on the non-negative support $[\underline{x}, \bar{x}]$, with cumulative distribution function (c.d.f. henceforth) F and density $f \equiv F' > 0$ that is continuous in $[\underline{x}, \bar{x}]$. We write x_i to denote a realization of X_i, which is privately observed by bidder i.

We are interested in analyzing sequential sealed bid first-price (F) and second-price (S) auctions when the reserve prices are descending, that is when $\underline{x} \leq r_2 < r_1 \leq \bar{x}$.[5] A bidding strategy in auction $A = F, S$ for bidder i consists of a pair of functions $(b_{A,i}^{(1)}, b_{A,i}^{(2)})$ which specify bidder i's bid in stage one, $b_{A,i}^{(1)}$, as a function of i's value x_i, and bidder i's bid in stage two, $b_{A,i}^{(2)}$ (conditional on i not winning in stage one), as a function of i's value x_i and of any other information that bidder i has obtained at stage one. Since bidders are symmetric ex ante, we restrict the analysis to strategies that do not depend on the bidders' identities. Therefore, a strategy will be indicated as a pair $(b_A^{(1)}, b_A^{(2)})$.[6]

We are interested in equilibria which satisfy the definition of Perfect Bayesian Equilibrium, that is: (i) the stage two beliefs are obtained from the stage one bidding functions using Bayes' rule whenever possible (otherwise beliefs are unrestricted); (ii) at both stage one and stage two, each bidder's strategy is sequentially rational given the bidder's beliefs; in particular, following any outcome of the stage one auction, the strategies for the stage two auction form an equilibrium, given the beliefs.

As it will appear repeatedly, we recall from (for example) Krishna (2010) a feature of a single stage first-price auction with $k \geq 2$ bidders and reserve price r_2, where each bidder's beliefs about the highest value among the other $k-1$ bidders are given by a c.d.f. G with density g.[7] The equilibrium bidding function β satisfies

$$\beta(r_2) = r_2 \qquad \text{and} \qquad \beta'(x) = (x - \beta(x))\frac{g(x)}{G(x)} \quad \text{for } x > r_2 \tag{1}$$

Under the specific assumption that values are i.i.d. random variables each with c.d.f. F and support $[\underline{x}, \bar{x}]$, it follows that $G(x) = F^{k-1}(x)$ and $\frac{g(x)}{G(x)} = \frac{(k-1)f(x)}{F(x)}$. Therefore, from (1) we obtain the following equilibrium bidding function:

$$\beta_{k,r_2}(x) = x - \int_{r_2}^{x} \frac{F^{k-1}(s)}{F^{k-1}(x)} ds = \int_{\underline{x}}^{x} \frac{\max\{r_2, s\} dF^{k-1}(s)}{F^{k-1}(x)} \quad \text{for } x \geq r_2 \tag{2}$$

In this equilibrium, the expected payoff of a bidder with type $x \geq r_2$ is

$$v_k(x) = \int_{r_2}^{x} F^{k-1}(s) ds \tag{3}$$

As it is well known,[8] $v_k(x)$ is also the expected payoff of a bidder with type x in a single stage second-price auction with the same information structure described above – assuming bidders play the unique undominated equilibrium.

In the next two sections we identify equilibria of sequential auctions. We begin from second-price auctions, as they are simpler.

3 Second-Price Auctions

We start the analysis by working backwards. Thus, we look at the equilibrium bid in the second stage auction, for all the active bidders. This is actually straightforward, since in a one-shot second-price auction each bidder has a (weakly) dominant strategy: to bid his value, regardless of the information obtained at stage one. Hence, we introduce $b_S^{(2)} = b_S^{(2)*}$ as follows:

$$b_S^{(2)*}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ x & \text{if } x \in [r_2, \bar{x}] \end{cases} \tag{4}$$

Conversely, the equilibrium bidding function at stage one is not as straightforward, and depends on the relationship between r_1 and r_2.

In order to carry out our analysis, we need some additional notation. Given a candidate equilibrium $(b_S^{(1)}, b_S^{(2)*})$, for each x and y in $[\underline{x}, \bar{x}]$ we use $u_S(x,y)$ to denote the payoff of a bidder with value x if in stage one he bids as a bidder with value y does according to $b_S^{(1)}$, given that the other bidders follow $(b_S^{(1)}, b_S^{(2)*})$. For $x \geq r_2$, we can write

$$u_S(x,y) = p(y)(x - t(y)) + (1 - p(y)) \int_{r_2}^{x} G_{b_S^{(1)}(y)}(s)ds \tag{5}$$

where $p(y)$ denotes the probability to win at stage one with the bid $b_S^{(1)}(y)$; $t(y)$ denotes the bidder's expected payment at stage one, conditional on winning at stage one with the bid $b_S^{(1)}(y)$; and $G_{b_S^{(1)}(y)}$ denotes the expected c.d.f., conditional on losing after bidding $b_S^{(1)}(y)$, about the highest value among the other bidders who lost at stage one.

In order to make sense of (5), notice that the first term indicates the expected payoff from winning the stage one auction times the probability of winning it, while the second term indicates the expected payoff from participating in the second auction times the probability of losing the stage one auction. In particular, for a bidder with value $x \in [r_2, \bar{x}]$ the payoff in stage two, conditional on losing in stage one, is given by $\int_{\underline{x}}^{x} (x - \max\{r_2, s\}) dG_{b_S^{(1)}(y)}(s)$. This reduces to $\int_{r_2}^{x} G_{b_S^{(1)}(y)}(s)ds$ after applying integration by parts.

It is straightforward to derive $G_{b_S^{(1)}(y)}$ when $b_S^{(1)}$ is strictly increasing, but not if $b_S^{(1)}$ is constant over an interval. This complicates the analysis substantially, as we will see below.

3.1 Ascending Reserve Prices

The case with (weakly) ascending reserve prices is solved by GTX. The following is an adaptation of their result which allows for $r_1 > \underline{x}$ (GTX assume that $[\underline{x}, \bar{x}] = [0,1]$ and $r_1 = 0$).

Proposition 1 (Proposition 2 in GTX)
Suppose that two objects are offered through sequential second-price auctions with (weakly) ascending reserve prices r_1, r_2 such that $\underline{x} \leq r_1 \leq r_2 \leq \bar{x}$. Then, there exists an equilibrium in which

$$b_S^{(1)}(x) = \begin{cases} \textit{no bid} & \textit{if } x \in [\underline{x}, r_1) \\ x & \textit{if } x \in [r_1, r_2) \\ \beta_{n-1,r_2}(x) & \textit{if } x \in [r_2, \bar{x}] \end{cases} \tag{6}$$

and $b_S^{(2)}(x) = b_S^{(2)*}(x)$ from (4).

The rationale behind the equilibrium bidding function (6) is as follows. A bidder whose value is between r_1 and r_2 only participates in the first auction; from his perspective he is joining a one shot second-price auction. As a result, it is still weakly dominant for him to bid his own value. A bidder with value above r_2 has two attempts at getting the object; at the second (and last) he will bid his own value. In the first stage, he bids the expected payment he would make if he were to lose the first auction and win the second. This corresponds to the equilibrium bid in a one shot first-price auction with reserve price r_2 and $n-1$ bidders.[9]Krishna (2010) illustrates this result for sequential auctions with no reserve prices, so that the main difference in (6) comes from bidders with values below r_2.

3.2 Descending Reserve Prices

In the equilibrium of Proposition 1, where we have that $r_1 \leq r_2$, each bidder participates in the auction at stage j if (and only if) his value is at least r_j, for $j = 1, 2$. Conversely, when $r_1 > r_2$, not all bidders with value at least r_1 bid at stage one since it could be more profitable to try to win the second stage auction at a lower reserve price. For instance, a bidder with type $x = r_1$ does not bid in the stage one auction because he cannot make a positive payoff in that auction, but has a positive payoff from participating in the stage two auction.[10] The greater r_1 is with respect to r_2, the stronger this effect for types in $(r_1, \bar{x}]$.

This suggests that the equilibrium analysis is more complicated with descending reserve prices. In fact, no equilibrium with a strictly increasing stage one bidding function exists (see GTX or Subsubsection 3.2.2). Nevertheless, we identify an equilibrium in which the stage one bidding function is partially flat, and its features depend on the magnitude of r_1 relative to r_2. Our main result is reported in the following proposition.

Proposition 2

Suppose that the two objects are offered through sequential second-price auctions with descending reserve prices, that is $\underline{x} \leq r_2 < r_1 \leq \bar{x}$. Let

$$\bar{r}_1 \equiv \bar{x} - v_n(\bar{x})$$

There exists a unique $\tilde{r}_1 \in (r_2, \bar{r}_1)$ such that we have an equilibrium in which each bidder bids according to $b_S^{(2)*}$ in stage two and:

i. *if $r_1 \in (r_2, \tilde{r}_1)$, each bidder adopts the stage one bidding function*

$$\tilde{b}_S^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, \gamma) \\ r_1 & \text{if } x \in [\gamma, \lambda] \\ \beta_{n-1,r_2}(x) & \text{if } x \in (\lambda, \bar{x}] \end{cases} \tag{7}$$

where γ, λ are uniquely determined by suitable indifference conditions (see eqs. (16)-(17) below);

ii. *if $r_1 \in [\tilde{r}_1, \bar{r}_1)$, each bidder adopts the stage one bidding function*

$$\hat{b}_S^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, \gamma) \\ r_1 & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{8}$$

where γ (different from γ in case (i)) is uniquely determined by a suitable indifference condition (see eq. (14) below);

iii. *if $r_1 \in [\bar{r}_1, \bar{x}]$, no bidder bids at stage one.*

Proposition 2 shows that some pooling in the stage one equilibrium bidding function arises. To the best of our knowledge, this is a novel feature of an equilibrium bidding function in standard symmetric auctions. In order to understand the logic behind this result we discuss each case separately, starting from the last. It may also be convenient to refer to the black curves in Figure 1, Subsection 3.3 as they offer a graphical representation of the stage one equilibrium bidding functions described by Proposition 2(i)-(ii) for uniformly distributed values.

3.2.1 Case *(iii)*: Large r_1

If $r_1 \geq \bar{r}_1$, no bidder wants to bid at stage one because bidding at stage one is less profitable than competing at stage two, where the reserve price is considerably lower. In more detail, given that no other bidder bids in stage one, each type $x \geq r_2$ earns the payoff (i) $v_n(x)$ if he does not bid at stage one; (ii) $x - r_1$ if he bids r_1 at stage one. The first option is superior to the second as the inequality $v_n(x) \geq x - r_1$ holds for each $x \in [r_2, \bar{x}]$, when $r_1 \geq \bar{r}_1$.[11]

3.2.2 Case *(ii)*: Intermediate r_1

When $r_2 < r_1 < \bar{r}_1$, it may be natural to inquire into the existence of an equilibrium $(b_S^{(1)}, b_S^{(2)*})$ with a cutoff γ, such that only bidders with value greater than or equal to γ participate in stage one, and adopt a strictly increasing bidding function. However, such an equilibrium does not exist. First, notice that type γ must be indifferent between bidding $b_S^{(1)}(\gamma)$ and not bidding. It turns out that this indifference is determined under the circumstance that type γ has the highest value, and it boils down to $\beta_{n,r_2}(\gamma) = r_1$. Second, if $b_S^{(1)}$ is strictly increasing in $(\gamma, \bar{x}]$ then $b_S^{(1)}(x)$ is equal to type x's expected payment in the stage two auction, conditional on winning at stage two (as for ascending reserve prices). That is, $b_S^{(1)}(x) = \beta_{n-1,r_2}(x)$ for $x > \gamma$. Hence, for x slightly greater than γ we find $\beta_{n-1,r_2}(x) < \beta_{n,r_2}(\gamma)$, or $b_S^{(1)}(x) < r_1$; that is, types slightly above γ bid less than r_1 which is a contradiction.

We may obtain an intuitive explanation of this result if we compare the tradeoffs faced by a bidder of type γ and a bidder of type x marginally bigger than γ. The former needs to choose between bidding or not in stage one. As we noticed, the indifference condition is based on being the type with the largest value. Therefore, if he chooses not to bid he will face all $n-1$ other bidders in stage two. The type marginally bigger than γ, instead faces a tradeoff between bidding slightly more or slightly less. If he loses at stage one, he will face $n-2$ opponents at stage two, because someone must have won stage one. This indicates that the stage two auction is seen as more competitive by type γ than by type x slightly bigger than γ, and this explains why type x would bid less in the first auction.

The argument above suggests that with $r_1 < \bar{r}_1$ some pooling at the reserve price r_1 occurs in equilibrium; that is, types slightly greater than γ bid r_1 just like type γ. Indeed, this occurs in (7) and (8). When r_1 is smaller (but not much smaller) than $\bar{r}_1$, the stage one equilibrium bidding function is $\hat{b}_S^{(1)}$ in which pooling involves type γ and all types bigger than γ. Some features of this equilibrium are worth emphasizing:

First, any two types in $[\gamma, \bar{x}]$ have the same probability to win in stage one. Therefore, the stage two beliefs of a losing bidder about the highest value among the other bidders who did not win at stage one are not straightforward.

Second, type γ is obtained from the indifference condition between bidding r_1 and not participating in the stage one auction. Therefore, each type above γ prefers to bid r_1 rather than not to bid, and each type below γ prefers not to bid rather than to bid r_1.

Third, no type of bidder wants to bid more than r_1 at stage one, despite the fact that by doing so he wins the object for sure.

Consistent with our previous notation, we use $\hat{u}_S(x, y)$ to denote the payoff of a bidder with value x if he bids $\hat{b}_S^{(1)}(y)$ in stage one (i.e., if he bids as a bidder with value y does according to $\hat{b}_S^{(1)}$), given that every other bidder follows the strategy $(\hat{b}_S^{(1)}, b_S^{(2)*})$. Hence, $\hat{u}_S(x, \underline{x})$ is the payoff of a bidder with value x from not bidding at stage one (like type $\underline{x}$ does); and $\hat{u}_S(x, \gamma)$ is the payoff from bidding r_1 at stage one (like type γ does). Last, we write $\hat{G}$ to denote the c.d.f. representing the stage two beliefs of a losing bidder about the highest value among the other bidders who did not win at stage one.[12] We remark that $\hat{G}$ depends on the stage one winning bid and the bidder's stage one bid.

We now describe how $\hat{u}_S(x, \underline{x})$ and $\hat{u}_S(x, \gamma)$ are derived as well as how γ is determined. In order to shorten the notation, we write Γ to indicate $F(\gamma)$.

3.2.2.1 Derivation of $\hat{u}_S(x, \underline{x})$

Consider a bidder who has not bid at stage one and learns that there has been no bid by any bidder. In light of (8) he must conclude that none of the $n-1$ other bidders has a value bigger than γ. This event has probability Γ^{n-1} from his ex ante point of view. Thus, his beliefs are given by the c.d.f. $\hat{G}(\cdot|\text{no,no})$ such that

$$\hat{G}(s|\text{no}, \text{no}) = \begin{cases} \frac{F^{n-1}(s)}{\Gamma^{n-1}} & \text{if } s \in [\underline{x}, \gamma] \\ 1 & \text{if } s \in (\gamma, \bar{x}] \end{cases} \tag{9}$$

Given $s \leq \gamma$, the unconditional probability that the largest value of all the other $n-1$ bidders is less than or equal to s is $F^{n-1}(s)$. Therefore, the conditional probability is given by the line at the top in (9). The expression at the bottom naturally follows upon learning that all the other $n-1$ bidders have a type smaller than γ.

On the other hand, if a bidder has not bid at stage one but learns that the winning bid has been r_1, he must conclude that at least one of the other $n-1$ bidders has a type bigger than γ. This event has probability $1-\Gamma^{n-1}$ from his ex ante point of view. Thus, his beliefs are given by the c.d.f. $\hat{G}(\cdot|\text{no}, r_1)$ such that

$$\hat{G}(s|\text{no}, r_1) = \begin{cases} \frac{(n-1)(1-\Gamma)}{1-\Gamma^{n-1}} F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma] \\ \frac{1-\Gamma}{1-\Gamma^{n-1}} \frac{F^{n-1}(s)-\Gamma^{n-1}}{F(s)-\Gamma} & \text{if } s \in (\gamma, \bar{x}] \end{cases} \tag{10}$$

Given $s \leq \gamma$, we obtain the expression at the top of (10) when exactly one of the other $n-1$ bidders has a value bigger than γ, while the remaining have values no greater than s. The unconditional probability of this event is $(n-1)(1-\Gamma)F^{n-2}(s)$ and therefore the conditional probability is as described. To derive the expression at the bottom of (10) we need to consider all the cases in which at least two of the other bidders have values bigger than γ. We leave the details of the derivation to the proof of Proposition 2 in the Appendix.

At the time of choosing to make no bid, the bidder's expected c.d.f. for the highest value among the other losing bidders is $\hat{G}_{\text{no}}$ such that

$$\hat{G}_{\text{no}}(s) = \Gamma^{n-1}\hat{G}(s|\text{no},\text{no}) + (1-\Gamma^{n-1})\hat{G}(s|\text{no}, r_1)$$

Therefore, in view of (5) the payoff of type $x \geq r_2$ from not bidding at stage one is

$$\hat{u}_S(x, \underline{x}) = \int_{r_2}^{x} \hat{G}_{\text{no}}(s)ds \tag{11}$$

3.2.2.2 Derivation of $\hat{u}_S(x,\gamma)$ and γ.

Let $\hat{p}(\gamma)$ denote the probability to win at stage one for a bidder bidding r_1. Hence $1-\hat{p}(\gamma)$ is the probability to lose at stage one. In eq. (55) in the Appendix we show that $\hat{p}(\gamma) = (1/n)\sum_{i=0}^{n-1}\Gamma^i$. In case the bidder loses, his beliefs are given by

$$\hat{G}(s|r_1, r_1) = \begin{cases} \frac{(n-1)(1-\Gamma)}{2(1-\hat{p}(\gamma))}F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma] \\ \frac{1-\Gamma}{n}\frac{(n-1)F^n(s) - n\Gamma F^{n-1}(s) + \Gamma^n}{(1-\hat{p}(\gamma))(F(s)-\Gamma)^2} & \text{if } s \in (\gamma, \bar{x}] \end{cases} \tag{12}$$

The derivation of eq. (12) relies on a logic similar to that applied to derive eq. (10). For $s \leq \gamma$, we obtain the expression at the top of eq. (12) when one of the other $n-1$ bidders has a type bigger than γ, while each of the remaining has a type no greater than s. In this context our bidder loses with probability $1/2$, and therefore the unconditional probability is given by $\frac{1}{2}(n-1)(1-\Gamma)F^{n-2}(s)$. To derive the expression at the bottom we need to consider all the cases in which at least two other bidders have a value bigger than γ. The details of the derivation are in the proof of Proposition 2 in the Appendix. We conclude that the payoff of a type $x \geq r_2$ from bidding r_1 is

$$\hat{u}_S(x,\gamma) = \hat{p}(\gamma)(x-r_1) + (1-\hat{p}(\gamma))\int_{r_2}^{x}\hat{G}(s|r_1, r_1)ds \tag{13}$$

After defining $\hat{u}_S(x,\underline{x})$ and $\hat{u}_S(x,\gamma)$, we identify γ in $\hat{b}_S^{(1)}$ as the unique solution to the equation

$$\hat{u}_S(\gamma,\underline{x}) = \hat{u}_S(\gamma,\gamma) \tag{14}$$

that is, type γ is indifferent between bidding r_1 at stage one and not bidding. In the proof of Proposition 2 we show that there indeed exists a unique solution to eq. (14) in the interval $(r_2, \bar{x})$.

3.2.2.3 On bidding more than r_1.

As stated earlier, no type of bidder prefers to bid more than r_1. In particular, for a type bigger than γ, this means that winning the object for sure at the price of r_1 gives a lower expected utility than winning it at the price of r_1, with probability $\hat{p}(\gamma)$; and joining the second stage with probability $1-\hat{p}(\gamma)$. Formally:

$$x - r_1 < \hat{p}(\gamma)(x-r_1) + (1-\hat{p}(\gamma))\int_{r_2}^{x}\hat{G}(s|r_1, r_1)ds \tag{15}$$

To see why, remark that if a bidder with value $x \geq \gamma$ bids more than r_1, he improves his stage one payoff by $(1-\hat{p}(\gamma))(x-r_1)$ and reduces his stage two payoff by $(1-\hat{p}(\gamma))\int_{r_2}^{x}\hat{G}(s|r_1,r_1)ds$. We obtain that the payoff from winning at stage one, $x-r_1$, is smaller than the expected payoff conditional on losing, $\int_{r_2}^{x}\hat{G}(s|r_1,r_1)ds$. In other words, and interestingly enough, bidders prefer to participate in the stage one auction and lose it rather than win it. This may not be intuitive, but it is simple to see when r_1 is slightly smaller than $\bar{r}_1$. To fix the ideas, consider type $\bar{x}$; although our argument applies to each type in $(\gamma, \bar{x}]$. Since r_1 is close to $\bar{r}_1$, it follows that $\bar{x} - r_1$

is close to $\bar{x} - \bar{r}_1$. That is, when bidding more than r_1 a bidder with value $\bar{x}$ earns a payoff of about $v_n(\bar{x})$, the expected payoff of type $\bar{x}$ in a second-price auction with $n-1$ opponents. Moreover, when r_1 is close to $\bar{r}_1$, γ is close to $\bar{x}$ and this implies that $\hat{G}(s|r_1, r_1)$ is close to $F^{n-2}(s)$, for each $s \in [r_2, \bar{x}]$. As a result, type $\bar{x}$'s payoff conditional on losing at stage one is approximately equal to $\int_{r_2}^{\bar{x}} F^{n-2}(s)ds = v_{n-1}(\bar{x})$; his expected payoff in a second-price auction with $n-2$ opponents. That is, losing after bidding r_1 makes him join a less competitive auction, and this is better for him since $v_{n-1}(\bar{x}) > v_n(\bar{x})$.

Remark that this argument does not imply that type $\bar{x}$ prefers not to bid at stage one. If he does not bid, he will likely face $n-1$ opponents at stage two because γ close to $\bar{x}$ makes it unlikely that there is a winner at stage one. However, if he bids r_1 at stage one and loses then necessarily there has been a winner at stage one; therefore the stage two auction is with $n-2$ other bidders. After bidding r_1 in stage one, learning he has lost the stage one auction is good news for him. Therefore, there is a significant difference between losing from not bidding and losing after bidding r_1.

3.2.3 Case *(i)*: Small r_1

When r_1 is close to r_2, $\hat{b}_S^{(1)}$ cannot be part of an equilibrium because high types prefer to bid above r_1 in the stage one auction, as inequality (15) is reversed. Following the same logic as before, we just need to show that $x - r_1 > \int_{r_2}^{x} \hat{G}(s|r_1, r_1)ds$; that is, winning and paying r_1 is better than losing. For r_1 close to r_2, the payoff from winning at stage one is close to $x - r_2 = \int_{r_2}^{x} 1ds > \int_{r_2}^{x} \hat{G}(s|r_1, r_1)ds$. This explains why $\tilde{b}_S^{(1)}$ in (7) prescribes that some types offer r_1, while other types offer more than r_1.

Therefore, two cutoffs appear in $\tilde{b}_S^{(1)}$: γ and λ, such that γ is different from the one that appears in $\hat{b}_S^{(1)}$ in (8), and λ is greater than γ. For each x and y in $[\underline{x}, \bar{x}]$ we use $\tilde{u}_S(x, y)$ to denote the payoff of a bidder with value x if he bids $\tilde{b}_S^{(1)}(y)$ in stage one, given that all the other bidders follow the strategy $(\tilde{b}_S^{(1)}, b_S^{(2)*})$. (The tilde symbol – rather than the hat symbol – indicates the case of small r_1.)

The values of γ and λ are given by the unique solution of the following equations:

$$\tilde{u}_S(\gamma, \underline{x}) = \tilde{u}_S(\gamma, \gamma) \tag{16}$$

$$\tilde{u}_S(\lambda, \gamma) = \lim_{y \downarrow \lambda} \tilde{u}_S(\lambda, y) \tag{17}$$

Equation (16) is analogous to eq. (14) as it states that type γ is indifferent between not bidding, and bidding r_1. Equation (17) states that type λ is indifferent between bidding r_1, and bidding just above $\beta_{n-1,r_2}(\lambda)$, which is greater than r_1. Hence $\tilde{b}_S^{(1)}$ is discontinuous at $x = \lambda$ (see Figure 1a).[13]

Remark If we study the effect of increasing the number of bidders, we can see that for a large n, $\tilde{r}_1$ approaches $\bar{x}$ and, therefore, the equilibrium described by Proposition 2(i) arises (unless r_1 is very close to $\bar{x}$). Moreover, γ and λ approach r_1 so that almost every type of bidder with value above r_1 bids more than r_1 in the stage one auction. The logic for this result is simple: given $r_1 > r_2$, a bidder with value greater than r_1 may not want to bid at stage one because he prefers to compete under the more favorable terms of stage two. However, when n is large, the intensity of the competition in both stages is mainly determined by the number of bidders rather than by reserve prices. Therefore, it is unprofitable for a bidder not to bid at stage one unless his value is very close to r_1.

3.3 Example with Uniformly Distributed Values

In this subsection we examine a specific example in which values are uniformly distributed on $[0, 1]$ and $r_2 = 0$; in this context we compare the equilibrium bids under increasing and decreasing reserve prices. Given our assumptions, for small r_1 the eqs. (16)-(17) reduce to

$$\frac{1}{n} \cdot \frac{\lambda^n - \gamma^n}{\lambda - \gamma}(\gamma - r_1) - \frac{1}{2}\gamma^{n-1}\lambda + \frac{n-2}{2n}\gamma^n \quad = 0 \tag{18}$$

$$-\frac{n-2}{2}\gamma^{n-1} - \frac{(n-1)\lambda^n - n\lambda^{n-1}\gamma + \gamma^n}{(\lambda - \gamma)^2}(\lambda - r_1) + \lambda\frac{\lambda^{n-1} - \gamma^{n-1}}{\lambda - \gamma} \quad = 0 \tag{19}$$

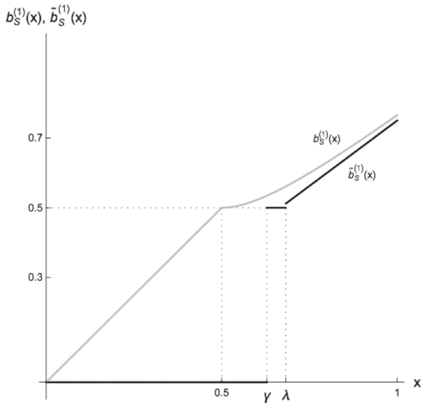
For $n = 3$, this system of equations can be solved analytically: the solution is $\gamma = (1 + \frac{\sqrt{3}}{3})r_1$, $\lambda = (1 + \frac{2\sqrt{3}}{3})r_1$. For larger n, analytical solutions are difficult or impossible to obtain, but inspection of eqs. (18)-(19) reveals that the solution is homogeneous of degree one in r_1; that is, $\gamma = c_\gamma r_1$ and $\lambda = c_\lambda r_1$ for suitable coefficients c_γ and c_λ with $c_\lambda > c_\gamma > 1$. Therefore $\lambda < 1$ if (and only if) $r_1 < \frac{1}{c_\lambda}$, hence $\tilde{r}_1 = \frac{1}{c_\lambda}$. Finally, we have that $\bar{r}_1 = \frac{n-1}{n}$.

Table 1 reports numerical estimates of c_γ, c_λ, $\tilde{r}_1$, and $\bar{r}_1$ for several values of n. In this way, we can tell for which range of values of r_1 each of the cases in Proposition 2 applies, and which types of bidder bid r_1. Moreover, Table 1 allows us to visualize the convergence results we have mentioned at the end of the previous subsection.

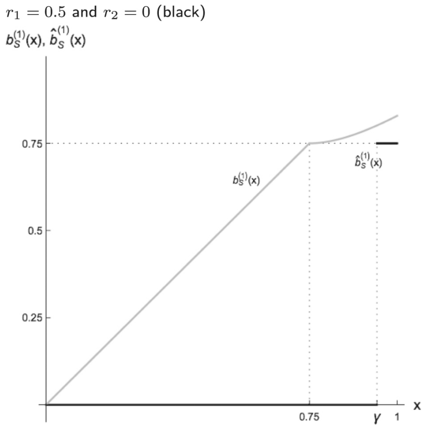
Table 1: Numerical estimates of the cutoff values when bidders' values are uniformly distributed in the unit interval, $r_2 = 0$ and $r_1 \in (0, 1]$

n	3	5	10	15	20	30	50	75
c_γ	1.577350	1.257289	1.111677	1.071574	1.052689	1.034499	1.020411	1.013514
c_λ	2.154700	1.365022	1.131076	1.079438	1.056924	1.036304	1.021040	1.013790
$\tilde{r}_1$	0.464102	0.732589	0.884114	0.926408	0.946142	0.964968	0.979393	0.986398
$\bar{r}_1$	0.666667	0.800000	0.900000	0.933333	0.950000	0.966667	0.980000	0.986667

For intermediate values of r_1, Proposition 2(ii) applies and γ is obtained by solving (18) given $\lambda = 1$. Figure 1 reports the equilibrium bidding functions for the first stage under ascending ($r_1 = 0$ and $r_2 = r$) and descending ($r_1 = r$ and $r_2 = 0$) reserve prices with five bidders, and $r \in \{0.5, 0.75\}$. The equilibrium bidding function for the case of ascending reserve prices is given by $b_S^{(1)}$ in (6) and is plotted in grey. When $r_1 = 0.5$, the equilibrium bidding function for the case of descending reserve prices is given by $\tilde{b}_S^{(1)}$ in (7) with $\gamma = \frac{1.257289}{2}$ and $\lambda = \frac{1.365022}{2}$, and is plotted in black in Figure 1a. When $r_1 = 0.75$, the equilibrium bidding function for descending reserve prices is given by $\hat{b}_S^{(1)}$ in eq. (8) with $\gamma = 0.942$, and is plotted in black in Figure 1b.



(a) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.5$ (grey) or when $r_1 = 0.5$ and $r_2 = 0$ (black)



(b) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.75$ (grey) or when $r_1 = 0.75$ and $r_2 = 0$ (black)

Figure 1: Equilibrium bidding functions in the first of two sequential second-price auctions when reserve prices are 0 and 0.5 (top) or 0 and 0.75 (bottom). There are five bidders with values independently drawn from a uniform distribution in the unit interval.

4 First-Price Auctions

This section establishes a revenue equivalence result – given decreasing reserve prices – between sequential first-price auctions and sequential second-price auctions. Precisely, we prove that if the objects are offered through two sequential first-price auctions with $r_1 > r_2$, there then exists an equilibrium which generates the same outcome (in terms of allocation of the objects and of bidders' expected payments) as the equilibrium described by Proposition 2 for sequential second-price auctions. GTX prove a similar equivalence result for the case of $r_1 \le r_2$. Combined, these results establish revenue equivalence between sequential first-price and second-price auctions for all $r_1 \gtreqless r_2$.

We focus on symmetric strategy profiles and use $(b_F^{(1)}, b_F^{(2)})$ to denote each bidder's bidding functions at stage one and two. A significant difference with respect to sequential second-price auctions is that with sequential first-price auctions, no dominant bid exists at stage two. A bidder's equilibrium behavior at stage two depends on his beliefs about the values of the other losing bidders at stage one, which in turn depend on his stage one bid $\mathfrak{b}$, and the stage one winning bid $\mathfrak{b}_w$. Therefore, the analysis of sequential first-price auctions requires extra care. This is mainly true for the case of decreasing reserve prices, because when $r_1 \le r_2$ there exists an equilibrium in which $b_F^{(1)}$ is strictly increasing (for $x \ge r_1$), and this generates beliefs which are relatively simple to manage: see GTX. We use the notation $b_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w)$, and for example, $b_F^{(2)}(x|\text{no}, r_1)$ is the bid of type x at stage two given that he has not bid at stage one ($\mathfrak{b} = no$), and given that the winning bid at stage one has been r_1 ($\mathfrak{b}_w = r_1$). With $\mathfrak{b}_w$ =*no* we represent the case in which no bid has been submitted at stage one.

4.1 Descending Reserve Prices

In this subsection (and in the Appendix), for each of the three cases considered in Proposition 2, we identify an equilibrium for sequential first-price auctions which is equivalent to the equilibrium described by Proposition 2 for sequential second-price auctions.

4.1.1 Intermediate r_1

Consider $r_1 > r_2$ such that r_1 is in the interval $[\tilde{r}_1, \bar{r}_1)$, for which Proposition 2(ii) identifies $(\hat{b}_S^{(1)}, b_S^{(2)*})$ where $\hat{b}_S^{(1)}$ is from (8) and $b_S^{(2)*}$ is from (4). For sequential first-price auctions, we find an equilibrium given by the functions $(\hat{b}_F^{(1)}, \hat{b}_F^{(2)})$ below; in which γ is the unique solution to eq. (14) as in Proposition 2(ii), and $\hat{G}(\cdot|\text{no}, r_1)$ and $\hat{G}(\cdot|r_1, r_1)$ are given by (10), (12), respectively:

$$\hat{b}_F^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, \gamma) \\ r_1 & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{20}$$

$$\hat{b}_F^{(2)}(x|\text{no}, \text{no}) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \beta_{n,r_2}(\gamma) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{21}$$

$$\hat{b}_F^{(2)}(x|\text{no}, r_1) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \hat{b}_F^{(2)}(\hat{y}(x)|r_1, r_1) \text{ such that } \hat{y}(x) \text{ is in} & \\ \arg\max_{y \in [\gamma, x]} (x - \hat{b}_F^{(2)}(y|r_1, r_1))\hat{G}(y|\text{no}, r_1) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{22}$$

$$\hat{b}_F^{(2)}(x|r_1, r_1) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \frac{\beta_{n-1,r_2}(\gamma)\hat{G}(\gamma|r_1,r_1) + \int_\gamma^x s\hat{g}(s|r_1,r_1)ds}{\hat{G}(x|r_1,r_1)} & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{23}$$

$$\hat{b}_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, r_2) \\ \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \bar{x}] \end{cases} \quad \text{for each } \mathfrak{b}_w > r_1,\ \mathfrak{b}_w \ge \mathfrak{b} \tag{24}$$

It is important to notice that (i) $\hat{b}_F^{(1)}$ coincides with $\hat{b}_S^{(1)}$; (ii) $\hat{b}_F^{(2)}(\cdot|\text{no},\text{no})$ and $\hat{b}_F^{(2)}(\cdot|\text{no},r_1)$ are both strictly increasing in the interval $[r_2,\gamma)$; (iii) $\hat{b}_F^{(2)}(\cdot|r_1,r_1)$ is strictly increasing in $[\gamma,\bar{x}]$. Therefore, eqs. (20)–(24) generate the same allocation of the two objects as the equilibrium described by Proposition 2(ii) for sequential second-price auctions. Since in both cases each bidder with type $\underline{x}$ has payoff equal to zero, the Revenue Equivalence Theorem implies that each type of bidder and the seller have the same payoff in both cases.

In the rest of this subsection (and in the Appendix) we explain why $(\hat{b}_F^{(1)},\hat{b}_F^{(2)})$ plus suitable beliefs constitute an equilibrium. The first step is to notice that the beliefs at stage two of each losing bidder are the same as for sequential second-price auctions; given by (9), (10), and (12) since $\hat{b}_F^{(1)}$ coincides with $\hat{b}_S^{(1)}$. Given these beliefs, we show that (21)–(24) are sequentially rational.[14] Notice that (21)–(24) cover all the possible stage one outcomes.

4.1.1.1 Stage two: The case of $\mathfrak{b}_w$ = *no*.

Consider $\hat{b}_F^{(2)}(\cdot|\text{no,no})$ in (21). Each bidder's beliefs are given by $\hat{G}(\cdot|\text{no,no})$ in (9), and $\frac{\hat{g}(s|\text{no,no})}{\hat{G}(s|\text{no,no})}=(n-1)f(s)/F(s)$ for $s\in(r_2,\gamma)$. Therefore, (1) reveals that the equilibrium bid for type $x\in[r_2,\gamma)$ is equal to $\beta_{n,r_2}(x)$, as specified by $\hat{b}_F^{(2)}(\cdot|\text{no,no})$. Moreover, $\hat{b}_F^{(2)}(\cdot|\text{no,no})$ also needs to specify a bid at stage two for each type $x\in[\gamma,\bar{x}]$, given $\mathfrak{b}_w=no$.[15] In this case type x expects that all the other bidders have a value in $[\underline{x},\gamma)$, and then it is optimal for him to bid $\beta_{n,r_2}(\gamma)$; the minimum bid that guarantees a win. This is what (21) prescribes for $x\in[\gamma,\bar{x}]$.

4.1.1.2 Stage two: The case of $\mathfrak{b}_w=r_1$.

This case is more involved, since a losing bidder's beliefs and bidding at stage two depend on his bid at stage one, which could have been *no*, or r_1. In particular, a type who has not won at stage one expects an opponent of type x to bid $\hat{b}_F^{(2)}(x|\text{no},r_1)$ if $x\in[r_2,\gamma)$, and to bid $\hat{b}_F^{(2)}(x|r_1,r_1)$ if $x\in[\gamma,\bar{x}]$. Then assume momentarily that the function

$$\hat{b}_{F,r_1}^{(2)}(x)=\begin{cases}\hat{b}_F^{(2)}(x|\text{no},r_1) & \text{if } x\in[r_2,\gamma)\\ \hat{b}_F^{(2)}(x|r_1,r_1) & \text{if } x\in[\gamma,\bar{x}]\end{cases}\tag{25}$$

is strictly increasing and notice that the beliefs of a losing bidder are: $\hat{G}(\cdot|\text{no},r_1)$ in eq. (10) if the bidder has not bid at stage one (according to $\hat{b}_F^{(1)}$, these are the types in $[r_2,\gamma)$, neglecting the types in $[\underline{x},r_2)$); and $\hat{G}(\cdot|r_1,r_1)$ in eq. (12) if the bidder has bid r_1 at stage one (according to $\hat{b}_F^{(1)}$, these are the types in $[\gamma,\bar{x}]$). Then solving

$$b(r_2)=r_2 \qquad \text{and} \qquad b'(x)=(x-b(x))\frac{\hat{g}(x|\text{no},r_1)}{\hat{G}(x|\text{no},r_1)} \qquad \text{for } x\in(r_2,\gamma)$$

yields $\hat{b}_F^{(2)}(x|\text{no},r_1)=\beta_{n-1,r_2}(x)$ for $x\in[r_2,\gamma)$, consistently with (22). Likewise, solving

$$b(\gamma)=\beta_{n-1,r_2}(\gamma) \qquad \text{and} \qquad b'(x)=(x-b(x))\frac{\hat{g}(x|r_1,r_1)}{\hat{G}(x|r_1,r_1)} \qquad \text{for } x\in(\gamma,\bar{x}]$$

yields $\hat{b}_F^{(2)}(x|r_1,r_1)=\frac{\beta_{n-1,r_2}(\gamma)\hat{G}(\gamma|r_1,r_1)+\int_\gamma^x s\hat{g}(s|r_1,r_1)ds}{\hat{G}(x|r_1,r_1)}$ for $x\in[\gamma,\bar{x}]$, as in (23). Hence the resulting function $\hat{b}_{F,r_1}^{(2)}$ in (25) is indeed strictly increasing.[16]

4.1.1.3 Stage two: The case in which $\mathfrak{b}_w>r_1$.

The equilibrium strategies also include (24), which covers the off-the-equilibrium case in which the stage one winning bid is greater than r_1. Then we suppose that the beliefs of each losing bidder are equal to the initial beliefs. Therefore, the stage two auction is an ordinary first-price auction with $n-1$ bidders and reserve price r_2, for which (24) is the equilibrium bidding function.

4.1.1.4 Stage one.

After examining stage two, we move to stage one and evaluate the total expected payoff (over two stages) for each bidder. Let $\hat{u}_F(x,y)$ denote the payoff of a bidder with value x if he bids $\hat{b}_F^{(1)}(y)$ in stage one, as type y is supposed to do. Then consider a type $x\in[r_2,\gamma)$ who expects that all the other bidders follow (20). We prove that it is optimal for him not to bid at stage one, consistent with (20). His payoff from not bidding at stage one is

$$\hat{u}_F(x,\underline{x})=\Gamma^{n-1}\left(x-\hat{b}_F^{(2)}(x|\text{no,no})\right)\hat{G}(x|\text{no,no})+(1-\Gamma^{n-1})\left(x-\hat{b}_F^{(2)}(x|\text{no},r_1)\right)\hat{G}(x|\text{no},r_1)$$

which coincides with $\hat{u}_S(x,\underline{x}) = v_n(x) + (n-1)(1-\Gamma)v_{n-1}(x)$ as it is obtained from (11). Moreover, his payoff from bidding r_1 is

$$\hat{u}_F(x,\gamma) = \hat{p}(\gamma)(x-r_1) + (1-\hat{p}(\gamma))(x - \hat{b}_F^{(2)}(x|r_1,r_1))\hat{G}(x|r_1,r_1)$$

in which $\hat{p}(\gamma)$ is the probability to win at stage one with a bid r_1, introduced in Section 3 just before (12). Using (13) we see that $\hat{u}_F(x,\gamma) = \hat{u}_S(x,\gamma) = \hat{p}(\gamma)(x-r_1) + \frac{n-1}{2}(1-\Gamma)v_{n-1}(x)$. We know from Proposition 2(ii) that $\hat{u}_S(x,\underline{x}) \geq \max\{\hat{u}_S(x,\gamma), x-r_1\}$ for each $x \in [r_2,\gamma)$, hence $\hat{u}_F(x,\underline{x}) \geq \max\{\hat{u}_F(x,\gamma), x-r_1\}$ for each $x \in [r_2,\gamma)$. That is, not bidding at stage one is a best reply for each type in $[r_2,\gamma)$. The argument regarding types in $[\gamma,\bar{x}]$ is in the proof of Proposition 3(ii).

4.1.2 Small r_1, or Large r_1

When $r_1 \in (r_2,\tilde{r}_1)$ we identify an equilibrium which is equivalent to the equilibrium of Proposition 2(i) for sequential second-price auctions. The analysis is more complicated than for the case of $r_1 \in [\tilde{r}_1,\bar{r}_1)$, as the equilibrium bidding function at stage one is partially flat and then strictly increasing for x close to $\bar{x}$. Therefore, we have reported this part in the proof of Proposition 3(i).

When $r_1 \in [\bar{r}_1,\bar{x}]$ we identify the following equilibrium, which is equivalent to the equilibrium of Proposition 2(iii) for sequential second-price auctions:

$$\bar{b}_F^{(1)}(x) = \text{no bid for each } x \in [\underline{x},\bar{x}] \tag{26}$$

$$\bar{b}_F^{(2)}(x|\text{no},\text{no}) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x},r_2) \\ \beta_{n,r_2}(x) & \text{if } x \in [r_2,\bar{x}] \end{cases} \tag{27}$$

$$\bar{b}_F^{(2)}(x|\mathfrak{b},\mathfrak{b}_w) = \hat{b}_F^{(2)}(x|\mathfrak{b},\mathfrak{b}_w) \qquad \text{for each } \mathfrak{b}_w \geq r_1,\ \mathfrak{b}_w \geq \mathfrak{b} \tag{28}$$

Given (26), when $\mathfrak{b}_w = no$ the beliefs of each bidder at stage two coincide with the initial beliefs, and therefore an ordinary first-price auction with n bidders and reserve price r_2 is held, for which (27) is the equilibrium bidding function. In case some bids have been submitted at stage one (an off-the-equilibrium event), we can argue as for $\hat{b}_F^{(2)}$ in (24).

Moving to stage one, for each bidder it is a best reply not to bid if he expects the other bidders to follow (26)–(27). If a type x bids at stage one, his payoff is not larger than $x - r_1$, which is smaller than the payoff $v_n(x)$ he obtains from not bidding at stage one, since $r_1 \geq \bar{r}_1$.

Proposition 3

Suppose that the two objects are offered through sequential first-price auctions with descending reserve prices, that is $\underline{x} \leq r_2 < r_1 \leq \bar{x}$. Let $\tilde{r}_1, \bar{r}_1$ be defined as in Proposition 2. Then

i. *if $r_1 \in (r_2,\tilde{r}_1)$, there exists an equilibrium which generates the same outcome as the equilibrium described by Proposition 2(i) for sequential second-price auctions;*

ii. *if $r_1 \in [\tilde{r}_1,\bar{r}_1)$, there exists an equilibrium in which each bidder follows the strategy $\hat{b}_F^{(1)}, \hat{b}_F^{(2)}$ in* (20)–(24);

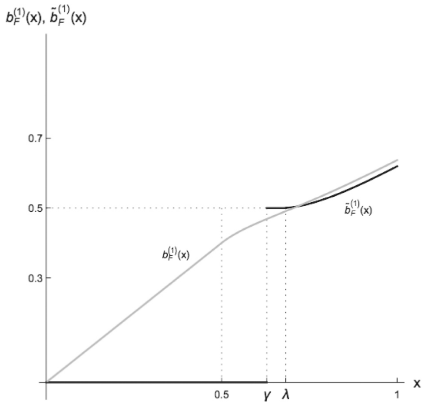
iii. *if $r_1 \in [\bar{r}_1,\bar{x}]$, there exists an equilibrium in which each bidder follows the strategy $\bar{b}_F^{(1)}, \bar{b}_F^{(2)}$ in* (26)–(28).

4.2 Example with Uniformly Distributed Values – Continued

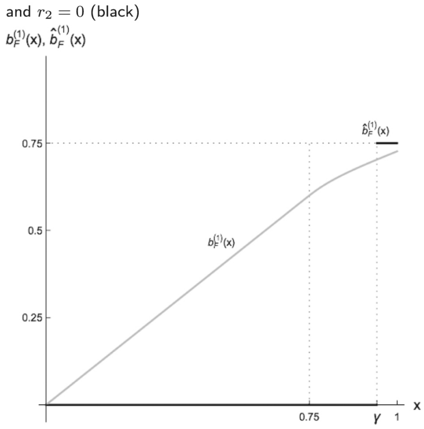
Figure 2 reports the plots of the equilibrium bidding functions in the first stage with sequential first-price auctions when values are uniformly distributed, there are five bidders, one item has no reserve price and the other has a reserve price $r \in \{0.5, 0.75\}$. The stage one equilibrium bidding function for the sequential first-price auctions with ascending reserve prices (where $r_1 = 0$ and $r_2 = r > 0$) is obtained by GTX, and is given by

$$b_F^{(1)}(x) = \begin{cases} \frac{4}{5}x & \text{if } x \leqslant r \\ \frac{3}{5}x + \frac{r^4}{x^3} - \frac{4}{5}\frac{r^5}{x^4} & \text{if } x \in (r,1] \end{cases}$$

It is plotted in grey in Figure 2. When $r_1 = 0.5$, $r_2 = 0$, the stage one equilibrium bidding function is given by $\tilde{b}_F^{(1)}$ in (57) and is plotted in black in Figure 2a with $\lambda = \frac{1.257289}{2}$ and $\gamma = \frac{1.365022}{2}$. When $r_1 = 0.75$, $r_2 = 0$ the stage one equilibrium bidding function is given by $\hat{b}_F^{(1)}$ in eq. (20) with $\gamma = 0.942$ and is plotted in black in Figure 2b. The cutoff values λ and γ are the same as for sequential second-price auctions (see Figures 1(a) and 1(b)) in view of Proposition 3.



(a) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.5$ (grey) or $r_1 = 0.5$ and $r_2 = 0$ (black)



(b) Equilibrium bidding functions when $r_1 = 0$ and $r_2 = 0.75$ (grey) or $r_1 = 0.75$ and $r_2 = 0$ (black)

Figure 2: Equilibrium bidding function in the first of two sequential first-price auctions when reserve prices are 0 and 0.5 (top) or 0 and 0.75 (bottom). There are five bidders with values independently drawn from a uniform distribution in the unit interval.

5 The Optimal Order Problem

In this section we use Proposition 2 to relook at an optimal order problem introduced by GTX. Specifically, we assume that values are randomly drawn from the unit interval and there are two different sellers, each of whom owns one of the two objects that are auctioned. One seller has value zero for her object, and the other seller has a commonly known value $r \in (0, 1)$ for her object. We assume that the objects are offered through sequential second-price auctions, such that at each stage the reserve price coincides with the seller's value for the object auctioned at that stage.[17] Our results are unaffected if sequential first-price auctions are adopted, given the revenue equivalence established in Section 4.

An auctioneer chooses the object to put on sale first in order to maximize the sum of each seller's expected profit, $\pi = \pi^0 + \pi^r$. Here, π^0 is just the expected revenue from the sale of the object that has no reserve price, while π^r is the difference between the expected revenue from the object with reserve price r and the reserve price times the probability of making the sale.[18]

Essentially, the auctioneer chooses between $r_1 = 0$, $r_2 = r$, and $r_1 = r$, $r_2 = 0$. When $r_1 = 0$, $r_2 = r$, the object with zero reserve price is auctioned first – that is reserve prices are increasing – and we use IRP to denote this case. We use DRP to denote the case of decreasing reserve prices, such that $r_1 = r$, $r_2 = 0$. Therefore, $\pi_{\text{IRP}} = \pi^0_{\text{IRP}} + \pi^r_{\text{IRP}}$ will indicate the sellers' total profits given IRP, and $\pi_{\text{DRP}} = \pi^r_{\text{DRP}} + \pi^0_{\text{DRP}}$ will indicate the sellers' total profits given DRP.[19]

With respect to this optimal order problem, GTX only study the case of $r > \bar{r}_1$ because they do not identify equilibria under DRP when $r < \bar{r}_1$. Proposition 4(i) below slightly generalizes their results. In addition, we exploit Propositions 1 and 2 to obtain some results about the optimal order when $r < \bar{r}_1$.

Proposition 4

Suppose $\underline{x} = 0$, $\bar{x} = 1$ and for only one unit we have a positive reserve price $r \in (0, 1)$. The following holds:

i. *If r is close to 1, then $\pi_{IRP} < \pi_{DRP}$. Moreover, if $\pi_{IRP} \leq \pi_{DRP}$ holds when $r = \bar{r}_1$, then $\pi_{IRP} < \pi_{DRP}$ holds for eachr $\in (\bar{r}_1, 1)$.*

ii. *If r is close to 0, then$\pi_{\text{IRP}} > \pi_{\text{DRP}}$.*

Proposition 4 allows us to conclude that $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ when r is close to 1, whereas $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ when r is close to 0. At the end of this section we show a sharper comparison, made under the assumption of uniformly distributed values.

In order to better understand Proposition 4, we start from the following result:

$$\pi^r_{\text{IRP}} > \pi^r_{\text{DRP}} \qquad \text{for each } r \in (0, 1) \tag{29}$$

that is, for each r the profit from the object with reserve price r is higher under IRP. This property holds because DRP discourage bidders' participation at stage one more than IRP do at stage two. Under DRP, a bidder knows that if he loses at stage one he will compete at stage two in a more favorable auction with $r_2 = 0 < r_1 = r$; under IRP, at stage two each bidder has his last opportunity to win an object. This generates higher bidding from a larger set of bidders under IRP.

On the other hand, we find that the comparison between π^0_{IRP} and π^0_{DRP} depends on the value of r: when r is close to 1, $\pi^0_{\text{DRP}} > \pi^0_{\text{IRP}}$ by a magnitude that outweighs eq. (29), therefore $\pi_{\text{DRP}} > \pi_{\text{IRP}}$ if r is close to 1 (Proposition 4(i)). When, instead, r is close to 0, $\pi^0_{\text{IRP}} > \pi^0_{\text{DRP}}$ and this reinforces the effect from eq. (29): therefore $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ if r is close to 0 (Proposition 4(ii)).

In order to see why the sign of $\pi^0_{\text{IRP}} - \pi^0_{\text{DRP}}$ depends on r, consider first r close to 1. Then π^0_{DRP} is equal to the expectation of the second highest value because with DRP, Proposition 2(iii) applies and each bidder (does not bid at stage one and) bids his own value at stage two. With IRP, π^0_{IRP} would be equal to the expectation of the second highest value if each bidder was bidding his own value at stage one (this occurs if $r = 1$). However, we know from Proposition 1 that each bidder with value greater than $r_2 = r$ bids less than his true value. Therefore, we have that $\pi^0_{\text{DRP}} > \pi^0_{\text{IRP}}$ when r is close to 1, and it turns out that this effect dominates the effect described by eq. (29). For the case of r close to zero, observe that $\pi^0_{\text{IRP}} = \pi^0_{\text{DRP}}$ if $r = 0$, and a small $r > 0$ increases both π^0_{IRP} and π^0_{DRP}. Precisely, it increases π^0_{IRP} because under IRP it makes less attractive for bidders to bid in stage two, thus promoting more aggressive bidding in stage one. A small $r > 0$ also increases π^0_{DRP}, but under DRP the revenue in stage two increases only for a small set of types, and by a small amount. As a consequence we obtain $\pi^0_{\text{IRP}} > \pi^0_{\text{DRP}}$ for a small r.

Example Under the assumption of uniform distribution of values, we can obtain more specific results. Specifically, if $n = 3$, $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ for $r < 0.641$, and $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ for $r > 0.641$. If $n \geq 4$, then $\pi_{\text{IRP}} > \pi_{\text{DRP}}$ for $r \leq \min\{\frac{5n-12}{5n-5}, \tilde{r}_1\}$, and $\pi_{\text{IRP}} < \pi_{\text{DRP}}$ for $r \geq \bar{r}_1$. In particular, as n tends to infinity we have that $\min\{\frac{5n-12}{5n-5}, \tilde{r}_1\}$ approaches 1 and therefore, for large n, IRP provide bigger profits at all r except those very close to 1.

6 Conclusions

In this paper, we prove the existence of (and we completely characterize) an equilibrium for two sequential first-price or second-price auctions, given any pair of decreasing reserve prices. Decreasing reserve prices induce a few types of bidders to skip the stage one auction because it is more profitable to compete in stage two. Although this implies that the stage one bidding function cannot be strictly increasing for the types that are active at stage one, we show that allowing for a (partially) flat bidding function restores existence. This comes at the cost of complicating the beliefs for bidders who do not win the first auction. Finally, as the number of bidders increases without bound, the flat portion of the bidding function tends to disappear.

It is natural to ask whether – and how – our equilibrium generalizes to sequential auctions with more than two objects. While we think that several of our insights hold beyond the two-good setting, things get complicated as soon as the number of objects for sale increases beyond two. As an instance of the first point, consider the case of three goods and reserve prices such that $r_1 > r_2 > r_3$, with r_1 large relative to r_2, r_3; no type of bidder participates in the stage one auction, and in stages two and three bidders bid as described by Proposition 2 or Proposition 3. However, and this addresses the second point, if r_1 is not too large some types of bidder will participate in the stage one auction, possibly according to a partially flat bidding function. This generates beliefs in stage two which may be different among different bidders (like $\hat{G}(\cdot|\text{no}, r_1)$ and $\hat{G}(\cdot|r_1, r_1)$ in (10) and (12) for instance), which make it complicated to apply the logic of Proposition 2 to the remaining two stages. Tackling this issue is part of our research agenda.

Appendix

A Proof of Proposition 2

Remember that for each x and y in $[\underline{x}, \bar{x}]$ we use $\tilde{u}_S(x, y)$ to denote the payoff of a bidder with value x if he bids $\tilde{b}_S^{(1)}(y)$ in stage one, given that all the other bidders follow the strategy $(\tilde{b}_S^{(1)}, b_S^{(2)*})$. For instance, $\tilde{u}_S(x, \underline{x})$ is the payoff of type x from not bidding in stage one; $\tilde{u}_S(x, \gamma)$, or $\tilde{u}_S(x, \lambda)$, is his payoff from bidding r_1 in stage one.

Proof of Proposition 2(i)

We prove that there exists a unique $\tilde{r}_1 \in (r_2, \bar{r}_1)$ such that if $r_1 \in (r_2, \tilde{r}_1)$, then there exists a unique solution to eqs. (16)-(17) and there exists an equilibrium in which each bidder bids according to the strategy $(\tilde{b}_S^{(1)}, b_S^{(2)*})$.

Although some details of the proof are cumbersome, its logic is simple. We employ several steps to obtain this result. First, we determine the payoff for each bidder type from not bidding, bidding r_1, or bidding $\beta_{n-1,r_2}(y)$ for some $y \in [\lambda, \bar{x}]$. To this purpose, we need to use (5). Hence, we need to derive a bidder's beliefs in case he loses, as a function of his stage one bid. Second, we show that the system of (16)-(17) has a unique solution. Last, we show (in the Supplementary Material) that the bidding function we obtain is strictly increasing in the interval $[\lambda, \bar{x}]$, and that no profitable deviation exists for any type of bidder.

Step 1: Derivation of $\tilde{u}_S(x, \underline{x})$, $\tilde{u}_S(x, \gamma)$ and $\tilde{u}_S(x, y)$

We start by illustrating how $\tilde{u}_S(x, \underline{x})$, $\tilde{u}_S(x, \gamma)$ and $\tilde{u}_S(x, y)$ are derived. To this end, we need to determine the updated beliefs for a bidder who lost at stage one – because he did not participate in the auction, bid r_1, or he bid $\beta_{n-1,r_2}(y)$. These beliefs are conditional on the information the bidder learns at stage one: his bid at stage one (which we denote with $\mathfrak{b}$) and the winning bid at stage one (which we denote with $\mathfrak{b}_w$). In order to shorten the notation, we set $\Gamma \equiv F(\gamma)$ and $\Lambda \equiv F(\lambda)$.

Step 1.1: Updated Beliefs for a Bidder who has not Bid at Stage One, and $\tilde{u}_S(x, \underline{x})$.

Consider a bidder with type x who has made no bid at stage one. Here we describe his beliefs upon learning $\mathfrak{b}_w$, and his expected payoff $\tilde{u}_S(x, \underline{x})$ from (5).

– In case there has been no bid by any bidder, an event with probability Γ^{n-1} from the bidder's ex ante point of view, his beliefs are given by the c.d.f. $\tilde{G}(\cdot|\text{no,no})$ such that

$$\tilde{G}(s|\text{no, no}) = \begin{cases} \frac{F^{n-1}(s)}{\Gamma^{n-1}} & \text{if } s \in [\underline{x}, \gamma) \\ 1 & \text{if } s \in [\gamma, \bar{x}] \end{cases} \tag{30}$$

– In case $\mathfrak{b}_w = r_1$, an event with probability $\Lambda^{n-1} - \Gamma^{n-1}$ from the bidder's ex ante point of view, his beliefs are given by the c.d.f. $\tilde{G}(\cdot|\text{no}, r_1)$ such that

$$\tilde{G}(s|\text{no}, r_1) = \begin{cases} \frac{(n-1)(\Lambda-\Gamma)}{\Lambda^{n-1}-\Gamma^{n-1}} F^{n-2}(s) & \text{if } s \in [\underline{x}, \gamma) \\ \frac{\Lambda-\Gamma}{\Lambda^{n-1}-\Gamma^{n-1}} \frac{F^{n-1}(s)-\Gamma^{n-1}}{F(s)-\Gamma} & \text{if } s \in [\gamma, \lambda] \\ 1 & \text{if } s \in (\lambda, \bar{x}] \end{cases} \tag{31}$$

About the derivation of $\tilde{G}(s|\text{no}, r_1)$, consider the point of view of, say, bidder 1; the following probabilities refer to the $n - 1$ bidders different from 1. For $s \in [\underline{x}, \gamma)$, $\tilde{G}(s|\text{no}, r_1)$ is obtained from the probability that one of the other bidders has value in $[\gamma, \lambda]$ and each other bidder has value smaller than s. This probability is equal to $(n-1)(\Lambda - \Gamma)F^{n-2}(s)$.

For $s \in [\gamma,\lambda]$, $\tilde{G}(s|\text{no},r_1)$ is obtained from the probability that at least one of the other bidders has value in $[\gamma,\lambda]$, none of them has value bigger than λ, and each of the non winning bidders with value $x \in [\gamma,\lambda]$ is such that $x \le s$. Such a probability is given by[20]

$$(n-1)(\Lambda-\Gamma)\sum_{j=0}^{n-2}\frac{C_{n-2,j}}{j+1}\Gamma^{n-2-j}(F(s)-\Gamma)^j \tag{32}$$

Specifically, $\Lambda-\Gamma$ is the probability that a bidder (the winner) has value in $[\gamma,\lambda]$ and we have $n-1$ possible ways of picking a winner. If there are j other bidders (from the remaining $n-2$) whose value is greater than γ, we need each of them to have value less than s, and $\frac{1}{j+1}$ is the probability that our initially selected bidder wins. Remark that

$$\frac{C_{n-2,j}}{j+1}\Gamma^{n-2-j}(F(s)-\Gamma)^j = \frac{C_{n-1,j+1}}{(n-1)(F(s)-\Gamma)}\Gamma^{n-2-j}(F(s)-\Gamma)^{j+1} \tag{33}$$

for $j=0,1,...,n-2$. The right hand side of (33) is equal to $\frac{C_{n-1,h}}{(n-1)(F(s)-\Gamma)}\Gamma^{n-1-h}(F(s)-\Gamma)^h$, for $h=1,2,...,n-1$ (with $h=j+1$). Hence (32) is equal to

$$(n-1)(\Lambda-\Gamma)\sum_{h=1}^{n-1}\frac{C_{n-1,h}}{(n-1)(F(s)-\Gamma)}\Gamma^{n-1-h}(F(s)-\Gamma)^h = \frac{\Lambda-\Gamma}{F(s)-\Gamma}\left(F^{n-1}(s)-\Gamma^{n-1}\right)$$

– In case $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda,\bar{x}]$, an event with probability $1-\Lambda^{n-1}$ from the bidder's ex ante point of view, his beliefs are given by the c.d.f. with value $F^{n-2}(s)/F^{n-2}(z)$ if $s \in [\underline{x},z)$, with value 1 if $s \in [z,\bar{x}]$. This c.d.f. applies for each stage one bid $\mathfrak{b} \le \tilde{b}_S^{(1)}(z)$ as long as the winning bid has been $\tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda,\bar{x}]$; hence we define $\tilde{G}(s|\mathfrak{b},\tilde{b}_S^{(1)}(z))$ such that

$$\tilde{G}(s|\mathfrak{b},\tilde{b}_S^{(1)}(z)) = \begin{cases}\frac{F^{n-2}(s)}{F^{n-2}(z)} & \text{if } s\in[\underline{x},z)\\ 1 & \text{if } s \in [z,\bar{x}]\end{cases} \text{ for each } \mathfrak{b} \le \tilde{b}_S^{(1)}(z) \tag{34}$$

When he decides to make no bid, the bidder's expected beliefs are represented by the c.d.f. $\tilde{G}_{\text{no}}$ such that

$$\begin{aligned}\tilde{G}_{\text{no}}(s) &= \Gamma^{n-1}\tilde{G}(s|\text{no},\text{no}) + (\Lambda^{n-1}-\Gamma^{n-1})\tilde{G}(s|\text{no},r_1) + \int_\lambda^{\bar{x}}\tilde{G}(s|\text{no},\tilde{b}_S^{(1)}(z))dF^{n-1}(z)\\ &= \begin{cases}F^{n-1}(s)+(n-1)(1-\Gamma)F^{n-2}(s) & \text{if } s\in[\underline{x},\gamma)\\ \Gamma^{n-1}+\frac{(\Lambda-\Gamma)(F^{n-1}(s)-\Gamma^{n-1})}{F(s)-\Gamma}+(n-1)(1-\Lambda)F^{n-2}(s) & \text{if } s\in[\gamma,\lambda]\\ (n-1)F^{n-2}(s)-(n-2)F^{n-1}(s) & \text{if } s \in (\lambda,\bar{x}]\end{cases}\end{aligned}$$

using eqs. (30), (31) and (34). Hence the payoff of a type x from not bidding at stage one is

$$\tilde{u}_S(x,\underline{x}) = \int_{r_2}^{x}\tilde{G}_{\text{no}}(s)ds \tag{35}$$

Step 1.2: Updated Beliefs for a Bidder who has Bid r_1 at Stage one but has not Won at Stage one, and $\tilde{u}_S(x,\gamma)$.

For future convenience, we introduce the following function M, defined for $a \in [0,1]$ and $b \in [0,1]$:

$$M(a,b) = \begin{cases}\frac{(n-1)a^n - na^{n-1}b + b^n}{(a-b)^2} & \text{if } a \ne b\\ \frac{n(n-1)}{2}a^{n-2} & \text{if } a=b\end{cases} \tag{36}$$

Multiplying $(a-b)^2$ by $(n-1)a^{n-2}+(n-2)a^{n-3}b+...+2ab^{n-3}+b^{n-2}$ reveals that

$$M(a,b) = (n-1)a^{n-2}+(n-2)a^{n-3}b+...+2ab^{n-3}+b^{n-2} \tag{37}$$

and therefore M is strictly increasing both with respect to a and with respect to b.

For a bidder bidding r_1, the probability to win at stage one is

$$\begin{aligned}\tilde{p}(\gamma) &= \sum_{j=0}^{n-1} \frac{C_{n-1,j}}{j+1}\Gamma^{n-1-j}(\Lambda-\Gamma)^j = \sum_{j=0}^{n-1} \frac{C_{n,j+1}}{n(\Lambda-\Gamma)}\Gamma^{n-1-j}(\Lambda-\Gamma)^{j+1} \\ &= \sum_{h=1}^{n} \frac{C_{n,h}}{n(\Lambda-\Gamma)}\Gamma^{n-h}(\Lambda-\Gamma)^h = \frac{\Lambda^n-\Gamma^n}{n(\Lambda-\Gamma)} \end{aligned} \tag{38}$$

Let $\tilde{p}_\ell$ denote the probability that another bidder wins at stage one with a bid of r_1. The probability that another bidder wins at stage one with a bid $\tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$ is $1-\Lambda^{n-1}$. Since $\tilde{p}(\gamma)+\tilde{p}_\ell+1-\Lambda^{n-1}=1$, it follows that $\tilde{p}_\ell = \Lambda^{n-1}-\tilde{p}(\gamma)$, that is

$$\tilde{p}_\ell = \frac{\Lambda-\Gamma}{n}M(\Lambda,\Gamma) = (n-1)(\Lambda-\Gamma)\sum_{j=0}^{n-2}\frac{C_{n-2,j}}{j+2}\Gamma^{n-2-j}(\Lambda-\Gamma)^j \tag{39}$$

Now consider a bidder who has bid r_1 at stage one but has not won. Then either $\mathfrak{b}_w = r_1$, or $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$.

– In case $\mathfrak{b}_w = r_1$ and another bidder has won, an event with probability $\tilde{p}_\ell$ from the bidder's ex ante point of view, his beliefs are given by $\tilde{G}(\cdot|r_1, r_1)$ such that

$$\tilde{G}(s|r_1,r_1) = \begin{cases} \frac{(n-1)(\Lambda-\Gamma)}{2\tilde{p}_\ell}F^{n-2}(s) & \text{if } s \in [\underline{x},\gamma) \\ \frac{\Lambda-\Gamma}{n\tilde{p}_\ell}M(F(s),\Gamma) & \text{if } s \in [\gamma,\lambda] \\ 1 & \text{if } s \in (\lambda,\bar{x}] \end{cases} \tag{40}$$

Considering the point of view of bidder 1, the derivation of $\tilde{G}(s|r_1,r_1)$ for $s \in [\underline{x},\gamma)$ is similar to the derivation of $\tilde{G}(s|\text{no},r_1)$ for $s \in [\underline{x},\gamma)$, taking into account that bidder 1 has bid r_1 rather than abstaining from bidding. For $s \in [\gamma,\lambda]$, $\tilde{G}(s|r_1,r_1)$ is obtained from the probability that none of the other bidders has value greater than λ, at least one of them has value in $[\gamma,\lambda]$ and wins, and each losing bidder with value $x \in [\gamma,\lambda]$ is such that $x \le s$. This probability is equal to

$$(n-1)(\Lambda-\Gamma)\sum_{j=0}^{n-2}\frac{C_{n-2,j}}{j+2}\Gamma^{n-2-j}(F(s)-\Gamma)^j \tag{41}$$

From (39) we see that $\sum_{j=0}^{n-2}\frac{C_{n-2,j}}{j+2}\Gamma^{n-2-j}(\Lambda-\Gamma)^j = \frac{M(\Lambda,\Gamma)}{n(n-1)}$. Hence (41) is equal to

$$(n-1)(\Lambda-\Gamma)\sum_{j=0}^{n-2}\frac{C_{n-2,j}}{j+2}\Gamma^{n-2-j}(F(s)-\Gamma)^j = \frac{\Lambda-\Gamma}{n}M(F(s),\Gamma)$$

– In case $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \in (\lambda,\bar{x}]$, an event with probability $1-\Lambda^{n-1}$ from the bidder's ex ante point of view, his beliefs are given by $\tilde{G}(\cdot|r_1,\tilde{b}_S^{(1)}(z))$ in (34).

When he decides to bid r_1 at stage one, the bidder expects to lose with probability $1-\tilde{p}(\gamma) = \tilde{p}_\ell + 1 - \Lambda^{n-1}$, hence his expected beliefs are represented by the c.d.f. $\tilde{G}_{r_1}$ such that

$$\begin{aligned}\tilde{G}_{r_1}(s) &= \frac{\tilde{p}_\ell\tilde{G}(s|r_1,r_1) + \int_\lambda^{\bar{x}}\tilde{G}(s|r_1,\tilde{b}_S^{(1)}(z))dF^{n-1}(z)}{1-\tilde{p}(\gamma)}n \\ &= \frac{1}{1-\tilde{p}(\gamma)}\begin{cases} \frac{(n-1)(2-\Gamma-\Lambda)}{2}F^{n-2}(s) & \text{if } s \in [\underline{x},\gamma) \\ \frac{\Lambda-\Gamma}{n}M(F(s),\Gamma) + (n-1)(1-\Lambda)F^{n-2}(s) & \text{if } s \in [\gamma,\lambda]n \\ (n-1)F^{n-2}(s) - (n-2)F^{n-1}(s) - \tilde{p}(\gamma) & \text{if } s \in (\lambda,\bar{x}] \end{cases}\end{aligned}$$

using (40) and (34). Hence the payoff of a type x from bidding r_1 at stage one is

$$\tilde{u}_S(x,\gamma) = \tilde{p}(\gamma)(x - r_1) + (1 - \tilde{p}(\gamma)) \int_{r_2}^{x} \tilde{G}_{r_1}(s)ds \quad (42)$$

Step 1.3: Updated Beliefs for a Bidder who has Bid $\tilde{b}_S^{(1)}(y)$, with $y \in (\lambda, \bar{x}]$, at Stage one but has not Won at Stage one, and $\tilde{u}_S(x,y)$.

If a bidder has bid $\tilde{b}_S^{(1)}(y)$ at stage one and has not won, then $\mathfrak{b}_w = \tilde{b}_S^{(1)}(z)$ for some $z \geq y$, and his beliefs are given by $\tilde{G}(\cdot|\tilde{b}_S^{(1)}(y), \tilde{b}_S^{(1)}(z))$ in (34). Hence, the payoff of a type x from bidding $\tilde{b}_S^{(1)}(y)$ at stage one, for $y \in (\lambda, \bar{x}]$, is

$$\tilde{u}_S(x,y) = \int_{\underline{x}}^{y} \left(x - \max\{r_1, \tilde{b}_S^{(1)}(z)\}\right) dF^{n-1}(z) + \int_{y}^{\bar{x}} \int_{r_2}^{x} \tilde{G}(s|\tilde{b}_S^{(1)}(y), \tilde{b}_S^{(1)}(z)) ds dF^{n-1}(z) \quad (43)$$

and notice that the second term in the right hand side in (43) is equal to

$$\begin{cases} (n-1)(1-F(x))v_{n-1}(x) + \int_y^x \left(\frac{v_{n-1}(z)}{F^{n-2}(z)} + x - z\right) dF^{n-1}(z) & \text{if } y < x \\ (n-1)(1-F(y))v_{n-1}(x) & \text{if } y \geq x \end{cases}$$

Step 2: Derivation of γ and λ: Existence of a Unique Solution for eqs. (16)-(17), and Definition of $\bar{r}_1$

Using (35), (42), and (43) we find

$$\begin{aligned}
\tilde{u}_S(\gamma, \underline{x}) &= v_n(\gamma) + (n-1)(1-\Gamma)v_{n-1}(\gamma) \\
\tilde{u}_S(\gamma,\gamma) &= \tilde{p}(\gamma)(\gamma - r_1) + \frac{n-1}{2}(2 - \Gamma - \Lambda)v_{n-1}(\gamma) \\
\tilde{u}_S(\lambda,\gamma) &= \tilde{p}(\gamma)(\lambda - r_1) + \frac{(n-1)(\Lambda - \Gamma)}{2}v_{n-1}(\gamma) + \\
&\quad + \int_{\gamma}^{\lambda} \frac{\Lambda - \Gamma}{n} M(F(s),\Gamma)ds + (n-1)(1-\Lambda)v_{n-1}(\lambda) \\
\lim_{y \downarrow \lambda} \tilde{u}_S(x,y) &= \Lambda^{n-1}(x - r_1) + (n-1)(1-\Lambda)v_{n-1}(x) \equiv \tilde{u}_S(x,\lambda^+) \text{ for } x \leq \lambda
\end{aligned}$$

Hence eqs. (16) and (17) reduce, after some rearranging, respectively to

$$A(\gamma,\lambda) = 0, \qquad B(\gamma,\lambda) = 0 \quad (44)$$

with

$$A(\gamma,\lambda) = \tilde{p}(\gamma)(\gamma - r_1) - \tfrac{(n-1)(\Lambda-\Gamma)}{2}v_{n-1}(\gamma) - v_n(\gamma) \quad (45)$$

$$B(\gamma,\lambda) = \tfrac{n(n-1)}{2}v_{n-1}(\gamma) - M(\Lambda,\Gamma)(\lambda - r_1) + \int_{\gamma}^{\lambda} M(F(s),\Gamma)ds \quad (46)$$

Step 2.1: Definition of λ^*.

Define $\tau(\lambda) = F^{n-1}(\lambda)\,(\lambda - r_1) - v_n(\lambda)$, a strictly increasing function such that $\tau(r_1) < 0$ and $\tau(\bar{x}) = \bar{r}_1 - r_1 > 0$. Hence there exists λ in the interval $(r_1, \bar{x})$, which we denote λ^*, such that $\tau(\lambda) < 0$ for $\lambda \in (r_1, \lambda^*)$, $\tau(\lambda^*) = 0$, $\tau(\lambda) > 0$ for $\lambda \in (\lambda^*, \bar{x}]$.

Step 2.2: If $\lambda \in (r_1, \lambda^*)$, then There Exists no $\gamma \in (r_1, \lambda]$ such that $A(\gamma,\lambda) = 0$; if $\lambda \in [\lambda^*, \bar{x}]$, then There Exists a Unique $\gamma \in (r_1, \lambda]$ such that $A(\gamma,\lambda) = 0$, Denoted $\gamma_A(\lambda)$.

Given a function h of two variables, here and in the remainder of the Appendix we write h_i to denote the partial derivative of h with respect to its i-th variable, $i = 1, 2$.

First we prove that the function A is strictly increasing with respect to γ:

$$A_1(\gamma,\lambda) = \frac{\partial \tilde{p}(\gamma)}{\partial \gamma}(\gamma - r_1) + \frac{n-1}{2}f(\gamma)v_{n-1}(\gamma) + \tilde{p}(\gamma) - \frac{(n-1)(\Lambda - \Gamma)}{2}\Gamma^{n-2} - \Gamma^{n-1}$$

From the definition of $\tilde{p}(\gamma)$ in (38), we see that

$$\tilde{p}(\gamma) - \frac{(n-1)(\Lambda-\Gamma)}{2}\Gamma^{n-2} - \Gamma^{n-1} = \sum_{j=2}^{n-1} \frac{C_{n-1,j}}{j+1}\Gamma^{n-j-1}(\Lambda-\Gamma)^j > 0$$

and, moreover, $\frac{\partial \tilde{p}(\gamma)}{\partial \gamma}(\gamma - r_1) > 0$ and $\frac{n-1}{2}f(\gamma)v_{n-1}(\gamma) > 0$.

Now we examine the sign of $A(r_1,\lambda)$ and of $A(\lambda,\lambda)$. We have that

$$A(r_1,\lambda) = -\frac{(n-1)(\Lambda - F(r_1))}{2}v_{n-1}(r_1) - v_n(r_1) < 0$$

and $A(\lambda,\lambda) = \tau(\lambda)$. Therefore, if $\lambda \in (r_1,\lambda^*)$ then $A(\lambda,\lambda) < 0$ and there is no solution to $A(\gamma,\lambda) = 0$ in the interval $(r_1,\lambda]$; if $\lambda \in [\lambda^*,\bar{x}]$, then there exists ($A$ is continuous in λ) a unique solution to $A(\gamma,\lambda) = 0$ in the interval $(r_1,\lambda]$, which we denote $\gamma_A(\lambda)$.

Step 2.3: There Exists $\tilde{r}_1 \in (r_2,\bar{r}_1)$ such that the Equation $B(\gamma_A(\lambda),\lambda) = 0$ has a Unique Solution in $(\lambda^*,\bar{x})$ if $r_1 \in (r_2,\tilde{r}_1)$; the Equation $B(\gamma_A(\lambda),\lambda) = 0$ has no Solution in $(\lambda^*,\bar{x})$ if $r_1 \geq \tilde{r}_1$.

First we prove that $B(\gamma_A(\lambda),\lambda)$ is strictly decreasing in λ. Notice that Γ below is actually equal to $F(\gamma_A(\lambda))$. We have that

$$\frac{dB(\gamma_A(\lambda),\lambda)}{d\lambda} = f(\gamma_A(\lambda))\left(\int_{\gamma_A}^{\lambda}(M_2(F(s),\Gamma)ds - M_2(\Lambda,\Gamma)(\lambda-r_1)\right)\gamma_A'(\lambda) - M_1(\Lambda,\Gamma)(\lambda - r_1)f(\lambda)$$

and we prove that $\frac{dB(\gamma_A(\lambda),\lambda)}{d\lambda} < 0$. From the previous step we have that

$$\gamma_A'(\lambda) = -\frac{A_2(\gamma,\lambda)}{A_1(\gamma,\lambda)} = -\frac{\frac{\partial\tilde{p}}{\partial\lambda}(\gamma_A(\lambda)-r_1) - \frac{n-1}{2}v_{n-1}(\gamma_A(\lambda))f(\lambda)}{\frac{\partial\tilde{p}}{\partial\gamma}(\gamma_A(\lambda)-r_1) + \frac{n-1}{2}v_{n-1}(\gamma_A(\lambda))f(\gamma_A(\lambda)) + \tilde{p} - \frac{n-1}{2}(\Lambda-\Gamma)\Gamma^{n-2} - \Gamma^{n-1}} \tag{47}$$

From the proof of Step 2.2 we know that the denominator in the right hand side of (47) is positive. Therefore $\frac{dB}{d\lambda}$ has the same sign as

$$\begin{aligned}
-f(\gamma_A(\lambda))\left(\frac{\partial\tilde{p}}{\partial\lambda}(\gamma_A - r_1) - \frac{n-1}{2}v_{n-1}(\gamma_A(\lambda))f(\lambda)\right)\times \\
\left(\int_{\gamma_A(\lambda)}^{\lambda}(M_2(F(s),\Gamma)ds - M_2(\Lambda,\Gamma)(\lambda-r_1)\right)+ \\
-M_1(\Lambda,\Gamma)(\lambda-r_1)f(\lambda)\left(\frac{\partial\tilde{p}}{\partial\gamma}(\gamma_A(\lambda)-r_1) + \frac{n-1}{2}v_{n-1}(\gamma_A)f(\gamma_A(\lambda)) + K\right)
\end{aligned} \tag{48}$$

with $K = \tilde{p} - \frac{n-1}{2}(\Lambda-\Gamma)\Gamma^{n-2} - \Gamma^{n-1} > 0$. Moreover,

$$\frac{\partial\tilde{p}}{\partial\gamma} = \frac{M(\Gamma,\Lambda)}{n}f(\gamma_A(\lambda)), \qquad \frac{\partial\tilde{p}}{\partial\lambda} = \frac{M(\Lambda,\Gamma)}{n}f(\lambda)$$

hence (48) is smaller than

$$\begin{aligned}
-f(\gamma_A(\lambda))f(\lambda)\left(\frac{M(\Lambda,\Gamma)}{n}(\gamma_A - r_1) - \frac{n-1}{2}v_{n-1}(\gamma_A(\lambda))\right)\times \\
\left(\int_{\gamma_A}^{\lambda}(M_2(F(s),\Gamma)ds - M_2(\Lambda,\Gamma)(\lambda-r_1)\right)+ \\
-M_1(\Lambda,\Gamma)(\lambda-r_1)f(\lambda)f(\gamma_A(\lambda))\left(\frac{M(\Gamma,\Lambda)}{n}(\gamma_A(\lambda)-r_1) + \frac{n-1}{2}v_{n-1}(\gamma_A)\right)
\end{aligned}$$

which is equal to $-f(\gamma_A(\lambda))f(\lambda)$ times

$$\begin{aligned}
\Bigg[\frac{n-1}{2}v_{n-1}(\gamma_A(\lambda))\left((\lambda-r_1)M_1(\Lambda,\Gamma) + (\lambda-r_1)M_2(\Lambda,\Gamma) - \int_{\gamma_A}^{\lambda}(M_2(F(s),\Gamma)ds\right)+ \\
\frac{\gamma_A(\lambda)-r_1}{n}\left(M(\Lambda,\Gamma)\int_{\gamma_A}^{\lambda}M_2(F(s),\Gamma)ds + (\lambda-r_1)\left[M_1(\Lambda,\Gamma)M(\Gamma,\Lambda) - M(\Lambda,\Gamma)M_2(\Lambda,\Gamma)\right]\right)\Bigg]
\end{aligned} \tag{49}$$

We now prove that (49) is negative by showing that the terms inside the square brackets are positive.

- The inequality $\gamma_A(\lambda) > r_1$ implies $(\lambda - r_1)M_1(\Lambda,\Gamma) + (\lambda - r_1)M_2(\Lambda,\Gamma) - \int_{\gamma_A(\lambda)}^{\lambda}(M_2(F(s),\Gamma)ds > (\lambda - r_1)M_1(\Lambda,\Gamma) + \int_{\gamma_A(\lambda)}^{\lambda}(M_2(\Lambda,\Gamma) - M_2(F(s),\Gamma))\,ds$, and the right hand side is positive since $M_1 > 0$ and $M_2(a,b) = \sum_{j=0}^{n-3}(n-2-j)(j+1)a^{n-3-j}b^j$ is strictly increasing in a.
- The term $M(\Lambda,\Gamma)\int_{\gamma_A(\lambda)}^{\lambda} M_2(F(s),\Gamma)ds$ is positive since $M_2 > 0$.
- The term $M_1(\Lambda,\Gamma)M(\Gamma,\Lambda) - M(\Lambda,\Gamma)M_2(\Lambda,\Gamma)$ is positive. In fact, from (36) we have $M_1(a,b) = \frac{(n-1)(n-2)a^n - 2b^n + n(n-1)a^{n-2}b^2 - 2(n-2)na^{n-1}b}{(a-b)^3}$, and $M_2(a,b) = \frac{(n-2)(a^n-b^n)+nab(b^{n-2}-a^{n-2})}{(a-b)^3}$. Therefore,

$$M_1(\Lambda,\Gamma)M(\Gamma,\Lambda) - M(\Lambda,\Gamma)M_2(\Lambda,\Gamma) = \frac{n\Lambda^{2n-2}\Gamma}{(\Lambda-\Gamma)^4}\left(1-(n-1)^2k^{n-2} + 2n(n-2)k^{n-1} - (n-1)^2k^n + k^{2n-2}\right)$$

with $k = \frac{\Gamma}{\Lambda} \in (0,1)$. We now define

$$\mu(k) = k^{2n-2} - (n-1)^2 k^n + 2n(n-2)k^{n-1} - (n-1)^2 k^{n-2} + 1 \tag{50}$$

and show it is positive for each $k \in (0,1)$. Remark that $\mu(1) = 0$. We now prove that $\mu(k) > 0$ for each $k \in (0,1)$. We find that $\mu'(k) = k^{n-3}\nu(k)$, with $\nu(k) = 2(n-1)k^n - n(n-1)^2 k^2 + 2n(n-1)(n-2)k - (n-1)^2(n-2)$ and $\nu(1) = 0$. In addition, we have that $\nu'(k) = 2n(n-1)\left(k^{n-1} - (n-1)k + n - 2\right)$, with $\nu'(1) = 0$, and $\nu''(k) = -2n(n-1)^2(1-k^{n-2}) < 0$ for each $k \in (0,1)$. Hence, ν' is strictly decreasing and since $\nu'(1) = 0$ we can conclude that $\nu'(k) > 0$ for each $k \in (0,1)$. This, in turn, implies that ν is strictly increasing, and since $\nu(1) = 0$, we obtain that $\nu(k) < 0$ for each $k \in (0,1)$. Therefore $\mu'(k) < 0$ for each $k \in (0,1)$, and since $\mu(1) = 0$ we can conclude that $\mu(k) > 0$ for each $k \in (0,1)$.

Now we prove that $B(\gamma_A(\lambda^*),\lambda^*) > 0$ and then examine the sign of $B(\gamma_A(\bar{x}),\bar{x})$. Given that $B(\gamma_A(\lambda),\lambda)$ is a continuous function of λ, if $B(\gamma_A(\bar{x}),\bar{x}) < 0$ then there exists a unique $\tilde{\lambda} \in (\lambda^*,\bar{x})$ such that $B(\gamma_A(\tilde{\lambda}),\tilde{\lambda}) = 0$. Since $A(\gamma_A(\tilde{\lambda}),\tilde{\lambda}) = 0$, it follows that $\gamma_A(\tilde{\lambda}),\tilde{\lambda}$ is a solution to (44), i.e. to eqs. (16)-(17). We prove that there exists $\tilde{r}_1 \in (r_2,\bar{r}_1)$ such that $B(\gamma_A(\bar{x}),\bar{x}) < 0$ if and only if $r_1 \in (r_2,\tilde{r}_1)$.

Regarding $B(\gamma_A(\lambda^*),\lambda^*)$, since $A(\gamma_A(\lambda^*),\lambda^*) = 0 = \tau(\lambda^*) = A(\lambda^*,\lambda^*)$, we have that $\gamma_A(\lambda^*) = \lambda^*$; hence (37) and (46) imply that

$$\begin{aligned} B(\gamma_A(\lambda^*),\lambda^*) &= \tfrac{n(n-1)}{2}\left(v_{n-1}(\lambda^*) - F^{n-2}(\lambda^*)(\lambda^* - r_1)\right) \\ &= \tfrac{n(n-1)}{2F(\lambda^*)}\int_{r_2}^{\lambda^*} F^{n-2}(s)\left(F(\lambda^*) - F(s)\right)ds > 0 \end{aligned}$$

where the last equality follows from the definition of λ^*.

Regarding $B(\gamma_A(\bar{x}),\bar{x})$, we have that

$$B(\gamma_A(\bar{x}),\bar{x}) = \frac{n(n-1)}{2}v_{n-1}(\gamma_A(\bar{x})) - M(1,F(\gamma_A(\bar{x})))(\bar{x} - r_1) + \int_{\gamma_A(\bar{x})}^{\bar{x}} M(F(s),F(\gamma_A(\bar{x})))ds \tag{51}$$

We now take into account that $\gamma_A(\bar{x})$ is an increasing function of r_1 ($\frac{d\gamma_A(\bar{x})}{dr_1} > 0$ since $\frac{\partial A}{\partial \gamma} > 0$ and $\frac{\partial A}{\partial r_1} < 0$), and we view $B(\gamma_A(\bar{x}),\bar{x})$ as a function $\ell(r_1)$ of r_1 that is defined for $r_1 \in (r_2,\bar{r}_1)$. As $r_1 \uparrow \bar{r}_1$, we have that $\gamma_A(\bar{x}) \to \bar{x}$ and $F(\gamma_A(\bar{x})) \to 1$, hence

$$\lim_{r\uparrow\bar{r}_1}\ell(r_1) = \frac{n(n-1)}{2}v_{n-1}(\bar{x}) - \frac{n(n-1)}{2}(\bar{x} - \bar{r}_1) = \frac{n(n-1)}{2}(v_{n-1}(\bar{x}) - v_n(\bar{x})) > 0$$

As $r_1 \downarrow r_2$, we have that $\gamma_A(\bar{x}) \to r_2$, hence

$$\lim_{r_1\downarrow r_2}\ell(r_1) = \int_{r_2}^{\bar{x}}\left(M(F(s),F(r_2)) - M(1,F(r_2))\right)ds < 0$$

since $F(s) < 1$ for $s \in (r_2,\bar{x})$. The continuity of ℓ implies that there exists $\tilde{r}_1 \in (r_2,\bar{r}_1)$ such that $\ell(\tilde{r}_1) = 0$, and $\ell(r_1) < 0$ for $r_1 \in (r_2,\tilde{r}_1)$. The proof that a unique $\tilde{r}_1$ exists such that $\ell(\tilde{r}_1) = 0$ is long and is reported in Section C in the Supplementary material.

Step 3: $\tilde{b}_S^{(1)}$ is Strictly Increasing in the Interval $[\lambda, \bar{x}]$

It is immediate to see that $\tilde{b}_S^{(1)}$ is strictly increasing in $(\lambda, \bar{x}]$, and we prove that $\lim_{x \downarrow \lambda} \tilde{b}_S^{(1)}(x) > r_1$ in Section D in the Supplementary material.

Step 4: Proof that no Profitable Deviation Exists

Now that $\tilde{b}_S^{(1)}$ is well defined, we prove that if a bidder expects that each other bidder follows the strategy $(\tilde{b}_S^{(1)}, b_S^{(2)*})$, then no profitable deviation exists for him. Precisely, we prove that the following inequalities hold:[21]

$$\text{for each } x \in [\gamma, \lambda],\ \tilde{u}_S(x, \gamma) \geqslant \max\{\tilde{u}_S(x, \underline{x}),\ \tilde{u}_S(x, y)\}, \text{ for each } y \in (\lambda, \bar{x}] \tag{52}$$

$$\text{for each } x \in [\gamma, \lambda], \tilde{u}_S(x, \gamma) \geqslant \max\{\tilde{u}_S(x, \underline{x}), \tilde{u}_S(x, y)\}, \text{for each } y \in (\lambda, \bar{x}] \tag{53}$$

$$\text{for each } x \in (\lambda, \bar{x}] \text{ and } y \in (\lambda, \bar{x}], \quad \tilde{u}_S(x, x) \geqslant \max\{\ \tilde{u}_S(x, x\ ), \tilde{u}_S(x, \gamma),\ \tilde{u}_S(x, y)\} \tag{54}$$

These inequalities are proved in Section D in the Supplementary material.

Proof of Proposition 2(ii)

This proof is largely given by the proof of Proposition 2(i), after setting $\lambda = \bar{x}$, consistently with the remark in footnote 13. Precisely, $\widehat{G}(s|\text{no}, r_1)$ in (10) and $\widehat{G}(s|r_1, r_1)$ in (12) can be seen as special cases of eqs. (31) and (40) with $\lambda = \bar{x}$ and $\Lambda = 1$. Remark that the probability of winning when bidding r_1 is now given by (see (38) with $\Lambda = 1$)

$$\hat{p}(\gamma) = \sum_{j=0}^{n-1} \frac{C_{n-1,j}}{j+1} \Gamma^{n-1-j}(1-\Gamma)^j = \frac{1-\Gamma^n}{n(1-\Gamma)} \tag{55}$$

and in (12), $1 - \hat{p}(\gamma)$ replaces $\tilde{p}_\ell$ at the denominator because $1 - \hat{p}(\gamma)$ is a bidder's probability of losing after a bid of r_1 given $\hat{b}_S^{(1)}$, the analog of $\tilde{p}_\ell$.

The proofs that a unique solution to eq. (14) exists and that $\hat{u}_S(x, \underline{x}) \geq \hat{u}_S(x, \gamma)$ for each $x \in [r_2, \gamma)$ and $\hat{u}_S(x, \underline{x}) \leq \hat{u}_S(x, \gamma)$ for each $x \in (\gamma, \bar{x}]$ are special cases of Step 2.2 above, and Steps 4.1 and 4.2 in Section D in the Supplementary material.

Finally, we need to explore the profitability of bidding slightly more than r_1 and to prove that $\max\{\hat{u}_S(x, \underline{x}), \hat{u}_S(x, \gamma)\} \geq x - r_1$ holds for each $x \in [r_2, \bar{x}]$. Since $\hat{u}_{S1}(x, \underline{x}) < 1$ for $x \in [r_2, \gamma)$ and $\hat{u}_{S1}(x, \gamma) < 1$ for $x \in [\gamma, \bar{x}]$, it suffices to prove that $\hat{u}_S(\bar{x}, \gamma) \geq \bar{x} - r_1$. Using (13) and rearranging the inequality we obtain

$$\frac{n(n-1)}{2} v_{n-1}(\gamma) + \int_\gamma^{\bar{x}} M(F(s), \Gamma) ds - M(1, \Gamma)(\bar{x} - r_1) \geq 0 \tag{56}$$

In examining this inequality, we need to take into account that γ is the unique solution to eq. (14) given r_1. Since (14) is equivalent to eq. (16) (i.e., to $A(\gamma, \lambda) = 0$ in eq. (44) when $\lambda = \bar{x}$), it follows that the left hand side of (56) is a function of r_1 which coincides with $\ell(r_1)$ introduced in Step 2.3 in the proof of Proposition 2(i). From Step 2.3 we know that $\ell(r_1) \geq 0$ if and only if $r_1 \in [\tilde{r}_1, \bar{r}_1]$.

Proof of Proposition 2(iii)

The proof of this part (from GTX) has been already presented in subsection 3.2.1.

B Proof for Proposition 3

Proof of Proposition 3(i)

The proof proceeds along these steps. First, we completely describe a strategy profile which we claim constitutes an equilibrium when $r_1 \in (r_2, \tilde{r}_1)$. That requires to specify the bidding behavior of each type at stage two, given any possible outcome at stage one. Then, in Step 1 we show that for each possible stage one outcome the bidding behavior we have specified is an equilibrium at stage two. This leads us to consider various cases. In Step 2 (in the Supplementary Material), we show that no stage one deviation is profitable.

Consider $r_1 \in (r_2, \tilde{r}_1)$, and let γ, λ be the unique solution to eqs. (16)-(17). Here we prove that the following bidding functions constitute an equilibrium:[22]

$$\tilde{b}_F^{(1)}(x) = \begin{cases} \text{no bid} & \text{if } x \in [\underline{x}, \gamma) \\ r_1 & \text{if } x \in [\gamma, \lambda] \\ \int_{\underline{x}}^{x} \frac{\max\{r_1, \tilde{b}_S^{(1)}(s)\} dF^{n-1}(s)}{F^{n-1}(x)} & \text{if } x \in (\lambda, \bar{x}] \end{cases} \tag{57}$$

$$\tilde{b}_F^{(2)}(x|\text{no}, \text{no}) = \begin{cases} \beta_{n,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \beta_{n,r_2}(\gamma) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{58}$$

$$\tilde{b}_F^{(2)}(x|\text{no}, r_1) = \begin{cases} \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \tilde{b}_F^{(2)}(\tilde{y}(x)|r_1, r_1) \text{ such that } \tilde{y}(x) \text{ is in} \\ \arg\max_{y \in [\gamma,\lambda]} (x - \tilde{b}_F^{(2)}(y|r_1, r_1)) \tilde{G}(y|\text{no}, r_1) & \text{if } x \in [\gamma, \bar{x}] \end{cases} \tag{59}$$

$$\tilde{b}_F^{(2)}(x|r_1, r_1) = \begin{cases} \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, \gamma) \\ \frac{\beta_{n-1,r_2}(\gamma)\tilde{G}(\gamma|r_1,r_1) + \int_\gamma^x s\tilde{g}(s|r_1,r_1)ds}{\tilde{G}(x|r_1,r_1)} & \text{if } x \in [\gamma, \lambda] \\ \frac{\beta_{n-1,r_2}(\gamma)\tilde{G}(\gamma|r_1,r_1) + \int_\gamma^\lambda s\tilde{g}(s|r_1,r_1)ds}{\tilde{G}(\lambda|r_1,r_1)} & \text{if } x \in (\lambda, \bar{x}] \end{cases} \tag{60}$$

$$\tilde{b}_F^{(2)}(x|\mathfrak{b}, \tilde{b}_F^{(1)}(z)) = \begin{cases} \beta_{n-1,r_2}(x) & \text{if } x \in [r_2, z) \\ \beta_{n-1,r_2}(z) & \text{if } x \in [z, \bar{x}] \end{cases} \text{for each } z \in (\lambda, \bar{x}], \mathfrak{b} \le \tilde{b}_F^{(1)}(z) \tag{61}$$

$$\tilde{b}_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w) = \beta_{n-1,r_2}(x) \text{ if } x \in [r_2, \bar{x}] \qquad \text{for each } \mathfrak{b}_w > \tilde{b}_F^{(1)}(\bar{x}), \mathfrak{b}_w \ge \mathfrak{b} \tag{62}$$

Remark that, in light of $\tilde{b}_F^{(1)}(x)$, $\tilde{b}_F^{(2)}(x|\text{no,no})$ for $x \in [\gamma, \bar{x}]$, $\tilde{b}_F^{(2)}(x|\text{no}, r_1)$ for $x \in [\gamma, \bar{x}]$, $\tilde{b}_F^{(2)}(x|r_1, r_1)$ for $x \notin [\gamma, \lambda]$, $\tilde{b}_F^{(2)}(x|\mathfrak{b}, \tilde{b}_F^{(1)}(z))$ for $x \in [z, \bar{x}]$, and $\tilde{b}_F^{(2)}(x|\mathfrak{b}, \mathfrak{b}_w)$ for $x \in [\underline{x}, \bar{x}]$ relate to off-the-equilibrium play. Remark also that $\tilde{b}_F^{(2)}(x|r_1, r_1)$ is constant for $x \in (\lambda, \bar{x}]$.

Step 1: Proof for Stage Two

In this first step we prove that for each possible outcome at stage one, the bidding specified by eqs. (58)-(62) constitutes an equilibrium at stage two. We start by noticing that $\tilde{b}_F^{(1)}$ generates the same stage two beliefs for losing bidders as $\tilde{b}_S^{(1)}$. Precisely, by comparing (57) with (7), we see that this property is true if $\mathfrak{b}_w = no$, or if $\mathfrak{b}_w = r_1$; in these cases the updated beliefs are given by (30), (31), and (40). But the property is true also if $\mathfrak{b}_w = \tilde{b}_F^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$, as $\tilde{b}_S^{(1)}$ is strictly increasing in the interval $(\lambda, \bar{x}]$: in this case the updated beliefs are given by the c.d.f.

$$\tilde{G}(s|\mathfrak{b}, \tilde{b}_F^{(1)}(z)) = \begin{cases} \frac{F^{n-2}(s)}{F^{n-2}(z)} & \text{if } s \in [\underline{x}, z) \\ 1 & \text{if } s \in [z, \bar{x}] \end{cases} \text{for each } \mathfrak{b} \le \tilde{b}_F^{(1)}(z) \tag{63}$$

which is essentially equivalent to (34) for each $z \in (\lambda, \bar{x}]$, $\mathfrak{b} \le \tilde{b}_F^{(1)}(z)$.

Regarding $\tilde{b}_F^{(2)}(\cdot|\text{no,no})$ in (58), we can argue as for (21), and regarding $\tilde{b}_F^{(2)}(\cdot|\mathfrak{b}, \mathfrak{b}_w)$ in (62) we can argue as for (24).

In order to consider the case in which $\mathfrak{b}_w = r_1$ (the bidding functions (59) and (60)) we first prove a stochastic dominance relation between $\tilde{G}(\cdot|r_1, r_1)$ and $\tilde{G}(\cdot|\text{no}, r_1)$.

Step 1.1: $\tilde{G}(\cdot|r_1, r_1)$ Dominates $\tilde{G}(\cdot|\text{no}, r_1)$ in Terms of the Reverse Hazard Rate.

$$\frac{\tilde{g}(s|\text{no}, r_1)}{\tilde{G}(s|\text{no}, r_1)} = \frac{\tilde{g}(s|r_1, r_1)}{\tilde{G}(s|r_1, r_1)} \quad \text{for } s \in [\underline{x}, \gamma), \qquad \frac{\tilde{g}(s|\text{no}, r_1)}{\tilde{G}(s|\text{no}, r_1)} < \frac{\tilde{g}(s|r_1, r_1)}{\tilde{G}(s|r_1, r_1)} \quad \text{for } s \in (\gamma, \lambda]$$

It is immediate to verify that $\frac{\tilde{g}(s|\text{no},r_1)}{\tilde{G}(s|\text{no},r_1)} = \frac{\tilde{g}(s|r_1,r_1)}{\tilde{G}(s|r_1,r_1)}$ for each $s \in [\underline{x}, \gamma)$. Now consider $s \in (\gamma, \lambda]$, and in order to prove that $\frac{\tilde{g}(s|\text{no},r_1)}{\tilde{G}(s|\text{no},r_1)} < \frac{\tilde{g}(s|r_1,r_1)}{\tilde{G}(s|r_1,r_1)}$, notice that

$$\frac{\tilde{g}(s|\text{no},r_1)}{\tilde{G}(s|\text{no},r_1)} = \frac{f(s)}{F(s)-\Gamma}\frac{(n-1)F^{n-2}(s)(F(s)-\Gamma)-(F^{n-1}(s)-\Gamma^{n-1})}{F^{n-1}(s)-\Gamma^{n-1}},$$
$$\frac{\tilde{g}(s|r_1,r_1)}{\tilde{G}(s|r_1,r_1)} = \frac{f(s)}{F(s)-\Gamma}\frac{n(n-1)F^{n-2}(s)(F(s)-\Gamma)^2-2[(n-1)F^n(s)-nF^{n-1}(s)\Gamma+\Gamma^n]}{(n-1)F^n(s)-nF^{n-1}(s)\Gamma+\Gamma^n}$$

After defining $k \equiv \frac{\Gamma}{F(s)} \in (0,1)$, we can write $\frac{\tilde{g}(s|r_1,r_1)}{\tilde{G}(s|r_1,r_1)} - \frac{\tilde{g}(s|\text{no},r_1)}{\tilde{G}(s|\text{no},r_1)}$ as

$$\frac{f(s)}{F(s)-\Gamma}\left(\frac{n(n-1)(1-k)^2-2(n-1-nk+k^n)}{n-1-nk+k^n} - \frac{(n-1)(1-k)-1+k^{n-1}}{1-k^{n-1}}\right)$$

and rearranging the last expression, we see that it has the same sign as

$$k^{2n-2}-(n-1)^2k^n+2n(n-2)k^{n-1}-(n-1)^2k^{n-2}+1$$

Remark that this is equal to $\mu(k)$ in (50) that we know is positive for each $k \in (0,1)$.

Step 1.2: The Bidding Function $\tilde{b}_F^{(2)}(\cdot|\text{no},r_1)$.

Consider a bidder of type $x \geq r_2$ who has submitted no bid at stage one, and has learned that $\mathfrak{b}_w = r_1$. Then his beliefs on the highest value among the other losing bidders are given by $\tilde{G}(s|\text{no},r_1)$ in (31), and we prove that it is optimal for him to bid $\tilde{b}_F^{(2)}(x|\text{no},r_1)$ as specified in (59) if he expects each other losing bidder with value in $[r_2,\gamma)$ to bid according to $\tilde{b}_F^{(2)}(\cdot|\text{no},r_1)$, and each other losing bidder with value in $[\gamma,\lambda]$ to bid according to $\tilde{b}_F^{(2)}(\cdot|r_1,r_1)$ in (60).[23]

In detail, we formulate his bidding problem as the problem of selecting optimally $y \in [r_2,\lambda]$, with the interpretation that choosing $y \in [r_2,\gamma)$ is equivalent to bidding $\tilde{b}_F^{(2)}(y|\text{no},r_1)$, and choosing $y \in [\gamma,\lambda]$ is equivalent to bidding $\tilde{b}_F^{(2)}(y|r_1,r_1)$. Therefore, for this type of bidder the stage two payoff is

$$\tilde{u}_F^{(2)}(x,y|\text{no},r_1) = \begin{cases}(x-\tilde{b}_F^{(2)}(y|\text{no},r_1))\tilde{G}(y|\text{no},r_1) & \text{if } y \in [r_2,\gamma)\\ (x-\tilde{b}_F^{(2)}(y|r_1,r_1))\tilde{G}(y|\text{no},r_1) & \text{if } y \in [\gamma,\lambda]\end{cases}$$

and

$$\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y} = \begin{cases}\tilde{G}(y|\text{no},r_1)\left(-\frac{\partial \tilde{b}_F^{(2)}(y|\text{no},r_1)}{\partial y} + (x-\tilde{b}_F^{(2)}(y|no,r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}\right) & \text{if } y \in [r_2,\gamma)\\ \tilde{G}(y|\text{no},r_1)\left(-\frac{\partial \tilde{b}_F^{(2)}(y|r_1,r_1)}{\partial y} + (x-\tilde{b}_F^{(2)}(y|r_1,r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}\right) & \text{if } y \in (\gamma,\lambda]\end{cases}$$

Then notice that $\tilde{b}_F^{(2)}(\cdot|\text{no},r_1)$ and $\tilde{b}_F^{(2)}(\cdot|r_1,r_1)$ satisfy the following differential equations in $[r_2,\gamma)$ and in $(\gamma,\lambda]$, respectively:

$$\frac{\partial \tilde{b}_F^{(2)}(y|\text{no},r_1)}{\partial y} = (y-\tilde{b}_F^{(2)}(y|\text{no},r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)} \quad \text{for } y \in [r_2,\gamma) \tag{64}$$

$$\frac{\partial \tilde{b}_F^{(2)}(y|r_1,r_1)}{\partial y} = (y-\tilde{b}_F^{(2)}(y|r_1,r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)} \quad \text{for } y \in (\gamma,\lambda] \tag{65}$$

and find that

$$\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y} = \begin{cases}(x-y)\tilde{g}(y|\text{no},r_1) & \text{if } y \in [r_2,\gamma)\\ \tilde{G}(y|\text{no},r_1)\left(-(y-\tilde{b}_F^{(2)}(y|r_1,r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)} + (x-\tilde{b}_F^{(2)}(y|r_1,r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}\right) & \text{if } y \in (\gamma,\lambda]\end{cases}$$

Consider a type $x \in [r_2,\gamma)$. Then $\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y}$ is positive for $y \in [r_2,x)$, negative for $y \in (x,\gamma)$, and negative also for $y \in (\gamma,\lambda]$ because $\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)} > \frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)}$ for $y \in (\gamma,\lambda]$ implies $-(y-\tilde{b}_F^{(2)}(y|r_1,r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)} + (x-\tilde{b}_F^{(2)}(y|r_1,r_1))\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)} < (x-y)\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)} < 0$ given $x < \gamma < y$. Hence the optimal y is equal to x, i.e. the optimal bid is $\tilde{b}_F^{(2)}(x|\text{no},r_1)$.

Now consider a type $x \in [\gamma, \bar{x}]$. Then $\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y} > 0$ for $y \in [r_2, \gamma)$, hence the optimal y is in $[\gamma, \lambda]$, as specified by (59). Moreover, we have seen above that $\frac{\partial \tilde{u}_F^{(2)}(x,y|\text{no},r_1)}{\partial y} \le (x-y)\tilde{g}(y|\text{no}, r_1)$ for $y \in (\gamma, \lambda]$, hence for $x = \gamma$ the optimal y is equal to γ.

Step 1.3: The Bidding Function $\tilde{b}_F^{(2)}(\cdot|r_1, r_1)$.

Consider a bidder of type $x \ge r_2$ who has bid r_1 at stage one, and has learned that another bidder has won at stage one with a bid r_1. Then his beliefs on the highest value among the other losing bidders at stage one are given by $\tilde{G}(s|r_1, r_1)$ in (40) and we prove that it is optimal for him to bid $\tilde{b}_F^{(2)}(x|r_1, r_1)$ as specified in (60) if he expects each other losing bidder with value in $[r_2, \gamma)$ to bid according to $\tilde{b}_F^{(2)}(\cdot|\text{no}, r_1)$ in (59), and each other losing bidder with value in $[\gamma, \lambda]$ to bid according to $\tilde{b}_F^{(2)}(\cdot|r_1, r_1)$.[24]

Arguing as in the proof of Step 1.2, we can write the bidder's payoff at stage two as a function of y as follows:

$$\tilde{u}_F^{(2)}(x, y|r_1, r_1) = \begin{cases} (x - \tilde{b}_F^{(2)}(y|\text{no}, r_1))\tilde{G}(y|r_1, r_1) & \text{if } y \in [r_2, \gamma) \\ (x - \tilde{b}_F^{(2)}(y|r_1, r_1))\tilde{G}(y|r_1, r_1) & \text{if } y \in [\gamma, \lambda] \end{cases}$$

and

$$\frac{\partial \tilde{u}_F^{(2)}(x, y|r_1, r_1)}{\partial y} = \begin{cases} \tilde{G}(y|r_1, r_1)\left(-\frac{\partial \tilde{b}_F^{(2)}(y|\text{no},r_1)}{\partial y} + (x - \tilde{b}_F^{(2)}(y|\text{no}, r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)}\right) & \text{if } y \in [r_2, \gamma) \\ \tilde{G}(y|r_1, r_1)\left(-\frac{\partial \tilde{b}_F^{(2)}(y|r_1,r_1)}{\partial y} + (x - \tilde{b}_F^{(2)}(y|r_1, r_1))\frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)}\right) & \text{if } y \in (\gamma, \lambda] \end{cases}$$

Then we use (64)-(65) plus $\frac{\tilde{g}(y|\text{no},r_1)}{\tilde{G}(y|\text{no},r_1)} = \frac{\tilde{g}(y|r_1,r_1)}{\tilde{G}(y|r_1,r_1)}$ for $y \in [r_2, \gamma)$ to find

$$\frac{\partial \tilde{u}_F^{(2)}(x, y|r_1, r_1)}{\partial y} = \begin{cases} (x-y)\tilde{g}(y|r_1, r_1) & \text{if } y \in [r_2, \gamma) \\ (x-y)\tilde{g}(y|r_1, r_1) & \text{if } y \in (\gamma, \lambda] \end{cases}$$

This reveals that the optimal y is equal to x for each $x \in [r_2, \lambda]$; and it is equal to λ, for each $x \in (\lambda, \bar{x}]$. Hence, in either case, the optimal bid is $\tilde{b}_F^{(2)}(x|r_1, r_1)$.

Step 1.4: The Bidding Function $\tilde{b}_F^{(2)}(\cdot|\mathfrak{b}, \tilde{b}_F^{(1)}(z))$.

If $\mathfrak{b}_w = \tilde{b}_F^{(1)}(z)$ for some $z \in (\lambda, \bar{x}]$, then the beliefs of each losing bidder are given by the c.d.f. $\tilde{G}(\cdot|\mathfrak{b}, \tilde{b}_F^{(1)}(z))$ in (63). Then essentially the argument relative to $\hat{b}_F^{(2)}(x|\text{no,no})$ in (21) applies in this case. We find that

$$\frac{\tilde{g}(s|\mathfrak{b}, \tilde{b}_F^{(1)}(z))}{\tilde{G}(s|\mathfrak{b}, \tilde{b}_F^{(1)}(z))} = \frac{(n-2)f(s)}{F(s)} \text{ for } s \in (r_2, z)$$

hence (1) reveals that the equilibrium bidding function for $x \in [r_2, z)$ is $\beta_{n-1,r_2}(x)$, as specified by $\tilde{b}_F^{(2)}(\cdot|\mathfrak{b}, \tilde{b}_F^{(1)}(z))$. Finally, given $\mathfrak{b}_w = \tilde{b}_F^{(1)}(z)$, a type $x \in [z, \bar{x}]$ expects each other bidder to have value smaller than z, and $\beta_{n-1,r_2}(z)$ is his payoff maximizing bid, as prescribed by (61).

Step 2: Proof for Stage One

We need to consider the point of view of a bidder at stage one, given (58)-(61), and prove that it is profitable for him to bid as specified in $\tilde{b}_F^{(1)}$ in (57), if he expects the other bidders to do so. The proof is in Section G in the Supplementary material.

Proof of Proposition 3(ii)

The proof is largely given by the proof of Proposition 3(i), after setting $\lambda = \bar{x}$. Precisely, regarding stage two, in the main text we have taken care of (21)–(24), with the exception of $\hat{b}_F^{(2)}(x|\text{no}, r_1)$ and $\hat{b}_F^{(2)}(x|r_1, r_1)$ given off-the-equilibrium play at stage one. In particular, $\hat{b}_F^{(2)}(x|\text{no}, r_1)$ for $x \in [\gamma, \bar{x}]$ is the payoff maximizing bid for a type $x \in [\gamma, \bar{x}]$ who has not bid at stage one, given the beliefs $\hat{G}(\cdot|\text{no}, r_1)$ and given that the opponents bid according to (25): we find that for such a type it is sub-optimal to bid less than $\hat{b}_F^{(2)}(\gamma|r_1, r_1)$. Likewise, $\hat{b}_F^{(2)}(x|r_1, r_1)$ is the payoff maximizing bid for a type $x \in [r_2, \gamma)$ who has bid r_1 in stage one. We find that $\hat{b}_F^{(2)}(x|r_1, r_1) = \hat{b}_F^{(2)}(x|\text{no}, r_1)$ (equal to $\beta_{n-1,r_2}(x)$) in the interval $[r_2, \gamma)$ because the equality $\frac{\hat{g}(s|\text{no},r_1)}{\hat{G}(s|\text{no},r_1)} = \frac{\hat{g}(s|r_1,r_1)}{\hat{G}(s|r_1,r_1)}$ (equal to $\frac{(n-2)f(s)}{F(s)}$) holds for $s \in [r_2, \gamma)$. Regarding stage one, we have proved in the main text that $\hat{u}_F(x, \underline{x}) \ge \max\{\hat{u}_F(x, \gamma), x - r_1\}$ for each $x \in [r_2, \gamma)$. We can argue as in Steps 2.2 and 2.4 in the proof of Proposition 3(i) to conclude that for each $x \in [\gamma, \bar{x}]$, (i) $\hat{u}_F(x, \gamma) = \hat{u}_S(x, \gamma)$, hence $\hat{u}_F(x, \gamma) \ge x - r_1$; (ii) $\hat{u}_F(x, \gamma) \ge \hat{u}_F(x, \underline{x})$.

Notes

1 A similar phenomenon occurs in McAfee and Vincent (1997), in which the seller of a single object auctions the object multiple times, until it is sold. At each stage, he chooses a reserve price given his beliefs on the bidders values, as determined by the information that the object has not been sold in the past. At each given stage, a bidder with value larger than the current reserve price may choose not to bid but to wait for a successive auction with a lower reserve price.
2 Benoit and Krishna (2001) and Elmaghraby (2003) study related optimal order problems, but they consider heterogeneous objects and no reserve prices. In addition, Benoit and Krishna (2001) assume common values and financially constrained bidders.
3 Che and Gale (1998) analyze a less restrictive context but the forces at work are the same as in our simplified example.
4 We adopt the same informational structure as GTX. Notice, however, that the results we obtain for sequential second-price auctions would be the same if each bidder at stage two knew only whether or not he has won the stage one auction.
5 Notice that the stage two reserve price has no effect if $r_2 < \underline{x}$, just like $r_2 = \underline{x}$. The stage one reserve price prevents each bidder from participating in the stage one auction if $r_1 > \bar{x}$, just like $r_1 = \bar{x}$.
6 Notice that F is used to indicate the c.d.f. for each bidder's value, and as a subscript to indicate, in Section 4, that $b_F^{(1)}, b_F^{(2)}$ are bidding functions for first-price auctions. This choice allows for lighter notation and, we think, does not generate risks of confusion, as it will be clear from the context what F stands for.
7 Here we use k rather than n because later in the paper we employ the current analysis for both the cases of $k = n$ and $k = n - 1$.
8 In order to simplify the notation we use $v_k(x)$ rather than $v_{k,r_2}(x)$: no ambiguity is possible as $v_k(x)$ refers to the stage two auction, whose reserve price is r_2. The same principle applies to a few other cases below in which there is no risk of ambiguity.
9 We notice that this result would not hold if bidders were not risk neutral or if there was some time discounting. Risk averse bidders would then shade their bids less in the first stage to reduce the risk of losing the object when the winning bid is still below their value. Impatient bidders will also bid more aggressively in the first stage because to them, *de facto*, an object is more valuable today than tomorrow.
10 Furthermore, given $r_1 > r_2$, we clearly have a situation where bidders with value smaller than r_2 never bid, since they cannot make a positive payoff in either stage. For this reason, we will consider only $x \geq r_2$.
11 The equilibrium for $r_1 \geq \bar{r}_1$ is also found by GTX.
12 For brevity, from now on we will refer to them just as *beliefs*.
13 Notice also that $\hat{b}_S^{(1)}$ is a special case of $\tilde{b}_S^{(1)}$, obtained when $\lambda = \bar{x}$, and in such a case we find that eq. (16) is equivalent to eq. (14).
14 This is the main additional complication of first-price auctions with respect to second-price auctions, for which the sequential rationality of eq. (4) is immediate.
15 This occurs if the bidder did not follow $\hat{b}_F^{(1)}$ at stage one, perhaps because he made a mistake or because he chose to deviate from $\hat{b}_F^{(1)}$.
16 In the proof of Proposition 3(ii) we describe how $\hat{b}_F^{(2)}(x|\text{no}, r_1)$ is determined for $x \in [\gamma, \bar{x}]$, and how $\hat{b}_F^{(2)}(x|r_1, r_1)$ is determined for $x \in [r_2, \gamma)$; these are bids which follow off-the-equilibrium bids at stage one.
17 For this assumption, GTX offer justifications which rely on both regulations and on competition among auctioneers, who are eager to attract many bidders.
18 If the auctioneer can choose freely the reserve prices, GTX use the mechanism design approach to show that profits are maximized by selling the object with lowest value to the seller first, and by properly setting ascending reserve prices.
19 We assume that under IRP (under DRP) the bidders play as described by Proposition 1 (Proposition 2), but in principle other equilibria may exist.
20 For any pair of non negative integers $k \geq h$ we write $C_{k,h}$ to denote $\frac{k!}{h!(k-h)!}$.
21 We can neglect bids between r_1 and $\lim_{x \downarrow \lambda} \tilde{b}_S^{(1)}(x)$, since each bid between r_1 and $\lim_{x \downarrow \lambda} \tilde{b}_S^{(1)}(x)$ has the same effect as bidding $\lim_{x \downarrow \lambda} \tilde{b}_S^{(1)}(x)$. We can also neglect bids strictly greater than $\tilde{b}_S^{(1)}(\overline{x})$ as they cannot increase the probability of winning while potentially increasing the price to be paid.
22 For the sake of brevity, in each bidding function relative to stage two we consider only $x \geq r_2$, since each type with value smaller than r_2 does not bid at stage two, regardless of the outcome of stage one.
23 In view of $\tilde{G}(\cdot|\text{no}, r_1)$, he expects that no losing bidder has value greater than λ.
24 In view of $\tilde{G}(\cdot|r_1, r_1)$, he expects that no losing bidder has value greater than λ.

References

Benoit, J., and V. Krishna. 2001. "Multiple-Object Auctions with Budget Constrained Bidders." *Review of Economic Studies* 68(1):155–179.
Che, Y., and I. Gale. 1998. "Auctions with Financially Constrained Bidders." *The Review of Economic Studies* 65 (1):1–21.
Elmaghraby, W. 2003. "The Importance of Ordering in Sequential Auctions." *Management Science* 49 (5):673–682.
Gong, Q., X. Tan, and Y. Xing. 2014. "Ordering Sellers in Sequential Auctions." *Review of Economic Design* 18(1):11–35.
Krishna, V. 2010. *Auction Theory*. San Diego, CA: Academic Press.
McAfee, P., and D. Vincent. 1997. "Sequentially Optimal Auctions." *Games and Economic Behavior* 18(2):246–276.
Milgrom, P., and R. Weber. 1999. "A Theory of Auctions and Competitive Bidding, II." In *The Economic Theory of Auctions*, edited by P. Klemperer. Cheltenham, UK: Edward Edgar Publishing.
Niedermayer, A., A. Shneyerov, and P. Xu. 2016. "Foreclosure Auctions." Available at http://andras.niedermayer.ch/wp-content/uploads/2016/11/foreclosure_auctions.pdf.
Weber, R. 1983. Multiple-Object Auctions. In *Auctions, Bidding and Contracting: Uses and Theory*, edited by R. Engelbrecht-Wiggans, M. Shubik, and R. Stark. New York, NY: New York University Press.