3-2021

On incentive compatible, individually rational public good provision mechanisms

Takashi KUNIMOTO
Singapore Management University, tkunimoto@smu.edu.sg

Cuiling ZHANG
Dongbei University of Finance And Economics

On Incentive Compatible, Individually Rational Public Good Provision Mechanisms*

Takashi Kunimoto† Cuiling Zhang‡

First Version: June 2018
This Version: March 2021

Abstract

This paper characterizes mechanisms satisfying incentive compatibility and individual rationality in the classical public good provision problem. Many papers in the literature obtain the results in the so-called standard model of ex ante identical agents with a continuous, closed interval of types. The main contribution of this paper is the characterization of the budget-surplus maximizing mechanism satisfying incentive compatibility and individual rationality (Theorem 1 for Bayesian implementation and Theorem 3 for dominant strategy implementation) that applies to a finite discretization over the standard model. Making use of the proposed budget-surplus maximizing mechanisms, we show that some known results do not need the agents' risk neutrality, whereas some others do rely on the agents' risk neutrality in a subtle manner. Furthermore, we improve upon some known results and obtain new results which do not exist in the standard model.
JEL Classification: C72, D78, D82.

Keywords: Budget balance, decision efficiency, incentive compatibility, individual rationality, mechanisms, public goods

*We thank the editor, advisory editor, and two anonymous referees for helpful comments that have significantly improved the paper. We are especially indebted to Steven Williams for many useful comments on the first draft of the paper. We thank Tilman Börgers, Shurojit Chatterji, Yi-Chun Chen, Atsushi Kajii, Michihiro Kandori, Yukio Koriyama, Maciej Kotowski, Jiangtao Li, Simon Loertscher, Shigehiro Serizawa, Shuichi Tsugawa, Takashi Ui, Jingyi Xue, Huaxia Zeng, and audiences at 2018 Nanjing International Conference on Game Theory and 2018 International Conference on Economic Theory and Applications in Chengdu for helpful comments.

†School of Economics, Singapore Management University; tkunimoto@smu.edu.sg

‡The Institute for Advanced Economic Research, Dongbei University of Finance and Economics, clzhang@dufe.edu.cn

1 Introduction

This paper revisits the classical public good provision problem in which a group of agents have to decide whether to produce some indivisible and non-excludable public good. This has been a central application of the theory of mechanism design (See, for example, Krishna (2009, Chapter 5), Mas-Colell, Whinston, and Green (1995, Chapter 23), Börgers (2015, Chapters 3) for this). To analyze this problem, many papers in the literature consider the model of ex ante identical agents with a continuous, closed interval of types.[1] In what follows, we call such a model the *standard model*. One practical benefit of using the standard model is that we can appeal to the revenue equivalence theorem, which says that all incentive compatible mechanisms with the same decision rule are revenue equivalent up to a constant. For example, we see this power of reduction in Krishna and Perry (2000) and Williams (1999). We emphasize that our paper exhibits a contrast with many papers in the literature because the revenue equivalence property fails in our setup (Claim 1).

We believe that whether a discrete or continuous type space is employed is entirely a matter of mathematical tractability. No substantive issue should depend on this modelling choice. Therefore, this paper aims to obtain new results in the classical public good provision problem with a *discrete* type space. Our main contribution is the characterization of the budget-surplus maximizing mechanisms satisfying incentive compatibility and individual rationality (Theorem 1 for Bayesian implementation and Theorem 3 for dominant strategy implementation) in a discrete setup. In what follows, we call such a mechanism simply *optimal*. All the new results we obtain in this paper come from the applications of our optimal mechanisms. Our optimal mechanisms also enable us to conduct a "stress test" for the known results by subjecting them to a finite discretization over the standard model.

We assume that each agent's type, i.e., preferences for public good, is chosen independently from an identical distribution over finitely many values. A mechanism designer is interested in implementing a *decision rule*, which is a mapping from each possible preference profile of agents to the probability that the public good is provided. Throughout this paper, we impose incentive compatibility and individual rationality on *direct mechanisms*, which map each type profile to the probability of providing a public good (i.e., decision rule) and monetary transfers across agents. A direct mechanism satisfies *Bayesian incentive compatibility* (BIC) if all agents' announcing their true type constitutes a Bayesian Nash equilibrium of the direct mechanism. By the celebrated revelation principle, we focus on direct mechanisms without loss of generality so that we call a direct mechanism simply a mechanism. A mechanism satisfies *interim individual rationality* (IIR) if each agent guarantees

[1] The reader is referred to Chapter 3.3 of Börgers (2015) for the textbook treatment of the classical public good problem with identical agents whose type space constitutes a continuous, closed interval on the real line.

an expected utility of zero (utility of non-participation), provided that all agents announce their type truthfully. We introduce two more requirements which are sometimes imposed on the mechanisms. A mechanism satisfies *ex post budget balance* (BB) if the total payments be at least as large as the *expected* cost of the public good at any type profile. We sometimes consider a more demanding version of BB. A mechanism satisfies *strong* BB if the total payments be at least as large as the cost of the public good at any type profile. A mechanism satisfies *decision efficiency* (EFF) if the public good is provided if and only if the total surplus from the public good is at least as high as the cost of the public good.

To state our results below, we introduce the following categories. By the *trivial* cases we mean that it is always efficient to provide the public good. We call any other case a *nontrivial case*. We obtain the following Bayesian implementation results in our discrete setup.

- Theorem 1: Fix an implementable decision rule. We derive the expected transfer which, associated with the decision rule, maximizes the ex ante budget surplus among all mechanisms with the same decision rule satisfying BIC and IIR.

- Lemma 3: We propose the *tight mechanism* as a most natural candidate inducing the optimal expected transfer characterized by Theorem 1.

- Proposition 1: There exists a mechanism satisfying BIC, IIR, and BB if and only if the tight mechanism generates nonnegative ex ante budget surplus.

- Theorem 2: In any nontrivial case, as the population size gets large, the ex ante probability that the public good is provided converges to zero in any mechanism satisfying BIC, IIR, and BB.

Theorem 1 is the key result of this paper, which characterizes the ex ante budget-surplus maximizing mechanism satisfying BIC and IIR (i.e., the optimal mechanism) in terms of expected payments in our discrete setup. This is a powerful result because, together with Lemma 3, it can reduce our search for the class of mechanisms satisfying BIC, IIR, and BB to the class of the tight mechanisms which was proposed by Kos and Manea (2009) in a bilateral trade model. An important innovation introduced by this result is its proof technique. We confirm by our Claim 1 that the revenue equivalence property fails in our setup. Thus, for proving Theorem 1, we follow Vohra (2011) in employing a *linear programming* technique.

Theorem 1 allows us to obtain Proposition 1 and Theorem 2. Krishna and Perry (2000) show in their Theorem 2 that there exists a mechanism satisfying BIC, IIR, EFF, and strong BB if and only if the generalized VCG mechanism generates nonnegative ex ante budget surplus.[2] Since we do not impose EFF on mechanisms, Proposition 1 is considered a generalization of Theorem 2 of Krishna and Perry (2000) in the public good provision problem with a one-dimensional type space. Moreover, we can handle general risk preferences, whereas Krishna and Perry (2000) treat risk neutral agents.

Theorem 2 makes use of Proposition 1 to uncover an important implication in large economies. It shows that in all nontrivial cases, the ex ante probability that the public good is provided converges to zero in any mechanism satisfying BIC, IIR, and BB when the population size of the economy gets infinite. This theorem is considered an improvement over Theorem 2 of Mailath and Postlewaite (1990) because we can dispense with risk neutrality of agents, which is assumed in Mailath and Postlewaite (1990). For this result, however, we impose an additional condition, *Condition* α, which says that the probability that any agent can be pivotal is approximately zero in large economies. The basic logic for Theorem 2 goes as follows. Each agent of a higher type can lower his payment by announcing a lower type. The only incentive to not do so is that the agent is pivotal, i.e., his announcement will change the provision probability. Since Condition α implies that no one is pivotal in large economies, it is prohibitively costly to induce all agents of higher types to tell the truth.

Although Condition α seems natural in large economies, it is nonetheless a nontrivial condition. To justify Condition α, we show in our Proposition 2 that when agents are risk neutral, Condition α is satisfied by any anonymous mechanism which only depends on the average surplus from the public good. This implies that the agents' risk neutrality assumed in Theorem 2 of Mailath and Postlewaite (1990) matters only to the extent that Condition α is justified by the class of anonymous mechanisms which depend only on the average surplus from the public good.[3] We consider this as an important clarification by looking at a discrete type space.

Finally, we strengthen BIC and IIR into dominant strategy incentive compatibility (DSIC) and ex post individual rationality (EPIR), respectively.[4] One benefit of doing so is that we can completely drop any distributional assumption about types and allow for any degree of correlation. We obtain the following dominant strategy implementation results.

- Theorem 3: Fix an implementable decision rule. We derive the transfer rule

[2] The VCG mechanism is based on the contribution of Vickrey (1961), Clarke (1971), and Groves (1973). The reader is referred to Krishna and Perry (2000, Subsection 5.2) for the definition of the generalized VCG mechanism.

[3] In their Theorem 2, Mailath and Postlewaite (1990) show that if we are to find a mechanism maximizing the public good provision probability among all mechanisms satisfying BIC, IIR, and strong BB, we lose nothing to focus on mechanisms in which the probability depends only on the average virtual valuation.

[4] The reader is referred to Chapter 4.3 of Börgers (2015) for the textbook treatment of the public good problem using DSIC and EPIR. Once again, a big difference from our paper is that Börgers' type space is assumed to be a closed interval in the real line.

which, associated with the decision rule, maximizes the ex post budget surplus among all mechanisms associated with the same decision rule satisfying DSIC and EPIR. Moreover, the optimal transfer rule is identical to the one in the tight mechanism.

- Theorem 4: Under a richness condition on decision rules, there are no mechanisms satisfying DSIC, EPIR, and strong BB in all nontrivial cases.

Theorem 3 is considered a dominant strategy implementation counterpart of our Theorem 1. It shows that the tight mechanism is the only optimal one maximizing ex post budget surplus among all mechanisms satisfying DSIC and EPIR. This exhibits a contrast with the so-called *guess-and-verify* approach, often used in the literature of mechanism design. In the guess-and-verify approach, we propose a particular mechanism (a guess) and later verify that the proposed mechanism satisfies the desired properties. This approach, however, entails a fundamental difficulty of how to come up with a guess (a mechanism) in the first place. By contrast, we can bypass this difficulty by using the linear programming approach which allows us to uniquely deduce the tight mechanism as the optimal mechanism.

In Theorem 4, we introduce a richness condition imposed on decision rules, saying that if all agents except one have their highest type, then the public good is provided. In all nontrivial cases with our richness condition, we have no hope in finding mechanisms satisfying DSIC, EPIR, and strong BB simultaneously. Our richness condition is very likely to be satisfied in large economies. Thus, Theorem 4 is considered a dominant strategy counterpart of our Theorem 2, which shows that there are no mechanisms satisfying BIC, IIR, and BB in large economies in all nontrivial cases. Moreover, we also compare our Theorem 4 with Green and Laffont (1977), Serizawa (1999), and Kuzmics and Steg (2017) in a comprehensive manner.

The rest of the paper is organized as follows. In Section 2, we introduce the general concepts and notation used throughout the paper. Section 3 characterizes the optimal mechanism for Bayesian implementation, identifies the condition for the existence of mechanisms satisfying BIC, IIR, and BB, and investigates the implication of the results in large economies. In Section 4, we replace BIC and IIR with DSIC and EPIR, respectively so that we characterize the optimal mechanism for dominant strategy implementation and investigate the corresponding implications. Section 5 concludes. The Appendix contains the proofs omitted from the main body of the paper and provides a discrete approximation of the ex ante budget-surplus maximizing mechanism satisfying BIC and IIR in the standard model. This discrete approximation justifies our claim that whether a discrete or continuous type space is employed is entirely a matter of mathematical tractability.

2 Preliminaries

There are $N \geq 2$ agents and we denote by $\mathcal{N} = \{1, \ldots, N\}$ the set of agents. A group of N agents must decide whether to undertake the public project and if undertaken, how to distribute the costs of the project among the members of the group. Each agent $i \in \mathcal{N}$ has $M \geq 2$ possible types $\theta_i \in \Theta \equiv \{\theta^1, \cdots, \theta^M\}$ such that $0 \leq \theta^1 < \cdots < \theta^M$ (i.e., all types take nonnegative values).[5] We assume that each agent's type is private information. Denote by $\Theta^N = \{\theta^1, \cdots, \theta^M\}^N$ the set of possible type profiles. The types are independently drawn from an identical distribution where $P(\theta^m)$ denotes the probability that θ^m is chosen. We assume that $P(\theta^m) > 0$ for all $\theta^m \in \Theta$. Therefore, there is a common prior P^N over Θ^N such that for each $\theta = (\theta_1, \ldots, \theta_N) \in \Theta^N$,

$$P^N(\theta) \equiv P(\theta_1) \times \cdots \times P(\theta_N).$$

Preferences of each agent depend upon whether or not the public project is implemented and the amount of monetary payment which is incurred by that agent. Agents evaluate lotteries over outcomes using expected utility. If the public good is provided with probability $q \in [0, 1]$ and agent i makes a payment t_i to the planner, then agent i's preferences can be represented by

$$u_i(q, t_i, \theta_i) = v(q, \theta_i) - t_i,$$

where $v(q, \theta_i)$ is agent i's valuation for the provision decision $q \in [0, 1]$ when his type is $\theta_i \in \Theta$. This formulation assumes that each agent's preferences are quasilinear in money. We assume that $v(q, \theta_i)$ is a continuous function of q and θ_i and $v(q, \theta_i) \geq 0$ for all $q \in [0, 1]$ and $\theta_i \in \Theta$.[6] We further assume that $v(q, \theta_i)$ is nondecreasing in the provision probability $q \in [0, 1]$ for each $\theta_i \in \Theta$, and that $v(q, \theta_i)$ satisfies strictly increasing differences, that is, for each $i \in \mathcal{N}$, $\hat{q} > q$, and $\hat{\theta}_i > \theta_i$,

$$v(\hat{q}, \hat{\theta}_i) - v(q, \hat{\theta}_i) > v(\hat{q}, \theta_i) - v(q, \theta_i).$$

In other words, the marginal gain from increasing the provision probability q is larger when agent i has a higher type.

Remark: A special case of the valuation function is $v(q, \theta_i) = q\theta_i$ for each $q \in [0, 1]$ and $\theta_i \in \Theta$. In other words, agents are risk neutral. In this case, $v(q, \theta_i)$ is nondecreasing in q for each θ_i and satisfies strictly increasing differences.

[5] In the special case where each agent i is risk-neutral, i.e., his valuation for the provision decision is $v(q, \theta_i) = q\theta_i$ for each $q \in [0, 1]$ and $\theta_i \in \Theta$, our nonnegative type assumption is consistent with the nonnegative valuation assumption which will be introduced later.

[6] After introducing the formal definition of IIR, we will show that if the valuation functions are nonnegative valued, the IIR constraints can be incorporated into part of the BIC constraints by introducing a dummy type. Due to this methodology we employ, we exclude negative valuations.

A *direct* mechanism is defined as a triplet $\Gamma = (\Theta^N, x, (t_i)_{i\in\mathcal{N}})$ where $\Theta = \{\theta^1, \cdots, \theta^M\}$ is the set of actions available to agent i, i.e., each agent is asked to reveal his type; $x : \Theta^N \to [0,1]$ is the *decision rule* which specifies the probability that the public good is provided; and $t_i : \Theta^N \to \mathbb{R}$ is the payment or subsidy to agent i and $t = (t_1, \ldots, t_N)$ is called the *transfer rule*. By the well known revelation principle, we lose nothing to focus on direct mechanisms. In what follows, we denote by (x,t) a direct mechanism or simply a mechanism.

For the ease of notation, for each agent $i \in \mathcal{N}$ in a mechanism (x,t), we denote by $\bar{x}_i(\hat{\theta}_i)$ the interim expected probability that the public good is provided and by $\bar{t}_i(\hat{\theta}_i)$ agent i's interim expected transfer when he announces type $\hat{\theta}_i$ and all the other agents announce their types truthfully, respectively. That is,

$$\bar{x}_i(\hat{\theta}_i) \equiv \sum_{\theta_{-i}\in\Theta^{N-1}} P^{N-1}(\theta_{-i})x(\hat{\theta}_i, \theta_{-i}),$$

and

$$\bar{t}_i(\hat{\theta}_i) \equiv \sum_{\theta_{-i}\in\Theta^{N-1}} P^{N-1}(\theta_{-i})t_i(\hat{\theta}_i, \theta_{-i}).$$

Since all agents' types are independently drawn from an identical distribution, it is without loss of generality to focus on symmetric mechanisms in the sense that if agent i and j report the same type, they face the same interim expected transfer, i.e., for any $i, j \in \mathcal{N}$, if $\theta_i = \theta_j$, then $\bar{t}_i(\theta_i) = \bar{t}_j(\theta_j)$.

By abuse of notation, we let $v(\bar{x}_i(\hat{\theta}_i), \theta_i)$ be type θ_i's interim expected valuation for the provision decision when he announces type $\hat{\theta}_i$ and all the other agents announce their types truthfully. That is,

$$v(\bar{x}_i(\hat{\theta}_i), \theta_i) \equiv \sum_{\theta_{-i}\in\Theta^{N-1}} P^{N-1}(\theta_{-i})v(x(\hat{\theta}_i, \theta_{-i}), \theta_i).$$

Definition 1. A mechanism (x,t) satisfies *Bayesian incentive compatibility* (BIC) if, for each $i \in \mathcal{N}$, $\theta_i, \hat{\theta}_i \in \Theta$,

$$v(\bar{x}_i(\theta_i), \theta_i) - \bar{t}_i(\theta_i) \geq v(\bar{x}_i(\hat{\theta}_i), \theta_i) - \bar{t}_i(\hat{\theta}_i).$$

The literature often assumes that every agent must participate in the mechanism; otherwise, he obtains a utility of zero. See, for example, Börgers (2015) for the details of this argument.[7] With this, we introduce the individual rationality constraint.

[7] Saijo and Yamato (1999) assume instead that each agent can not be excluded from the consumption of the public good even if he decides not to participate in the mechanism. Although the individual rationality of Saijo and Yamato (1999) is a lot more demanding than our IIR, we nevertheless establish a few negative results. Thus, we rather stick to our weaker individual rationality. The reader is referred to Saijo and Yamato (1999) for the discussion of their individual rationality constraints. Yenmez (2013) considers a similar constraint in a one-to-one matching environment.

Definition 2. A mechanism (x, t) satisfies the *interim individual rationality* (IIR) if, for each $i \in \mathcal{N}$ and $\theta_i \in \Theta$,

$$v(\bar{x}_i(\theta_i), \theta_i) - \bar{t}_i(\theta_i) \geq 0.$$

Since the valuation functions always take nonnegative values, we add a dummy type θ^0 satisfying the following property for any mechanism (x, t): $v(x(\theta^0, \theta_{-i}), \theta_i) = t_i(\theta^0, \theta_{-i}) = 0$ for each $i \in \mathcal{N}$, $\theta_{-i} \in \Theta^{N-1}$, and $\theta_i \in \Theta$. Then, we can incorporate the IIR constraints into part of the BIC constraints. So, from now on, Θ contains the dummy type θ^0 and in particular, we let $\theta^0 < \theta^1 < \cdots < \theta^M$.

We introduce a stronger version of incentive compatibility and individual rationality.

Definition 3. A mechanism (x, t) satisfies *dominant strategy incentive compatibility* (DSIC) if, for each $i \in \mathcal{N}$, $\theta \in \Theta^N$ and $\hat{\theta}_i \in \Theta$,

$$v(x(\theta_i, \theta_{-i}), \theta_i) - t_i(\theta_i, \theta_{-i}) \geq v(x(\hat{\theta}_i, \theta_{-i}), \theta_i) - t_i(\hat{\theta}_i, \theta_{-i}).$$

DSIC implies BIC and it does not need to make any distributional assumption about how each agent's type is realized.

Definition 4. A mechanism (x, t) satisfies *ex post individual rationality* (EPIR) if, for each $i \in \mathcal{N}$, $\theta_i \in \Theta$ and $\theta_{-i} \in \Theta^{N-1}$,

$$v(x(\theta_i, \theta_{-i}), \theta_i) - t_i(\theta_i, \theta_{-i}) \geq 0.$$

Note that EPIR implies IIR. Recall that the valuation functions are nonnegative valued and thus we can add a dummy type θ^0 satisfying the following property for any mechanism (x, t): $v(x(\theta^0, \theta_{-i}), \theta_i) = t_i(\theta^0, \theta_{-i}) = 0$ for each $i \in \mathcal{N}$, $\theta_{-i} \in \Theta^{N-1}$, and $\theta_i \in \Theta$. With such a dummy type, we also incorporate the EPIR constraints into part of the DSIC constraints.

When there are N agents in the economy, providing the public good will incur a cost equal to $c(N) > 0$ which is assumed to be an increasing function in N. This is consistent with the setup of Mailath and Postlewaite (1990).[8] For any agent i, let $v(1, \theta^1)$ be his valuation for the public good when he has the lowest type θ^1. Throughout the paper, we assume that $Nv(1, \theta^1) < c(N)$. We do not discuss the case where $Nv(1, \theta^1) \geq c(N)$ because it is considered a trivial case in the sense that the public good should be provided and its cost is shared equally even if all agents have the lowest type θ^1. In the trivial case, the public good should always be provided and the non-rivalry property of a pure public good does not hold here.

[8]Hellwig (2003) points out that this assumption is crucial for the result. Indeed, he completely overturns the result of Mailath and Postlewaite (1990) by isolating the effect of changes in the number of participants, while keeping cost technologies fixed.

Definition 5. A mechanism (x, t) satisfies *decision efficiency* (EFF) if, for each $\theta \in \Theta^N$,

$$x(\theta) = \begin{cases} 1 & \text{if } \sum_{i \in \mathcal{N}} v(1, \theta_i) \geq c(N) \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we denote by $x^*(\cdot)$ the efficient decision rule.

Remark: If each agent i's valuation of the provision decision is $v(q, \theta_i) = q\theta_i$ for each $q \in [0, 1]$ and $\theta_i \in \Theta$, then the efficient decision rule reduces to the following: for each $\theta \in \Theta^N$,

$$x^*(\theta) = \begin{cases} 1 & \text{if } \sum_{i \in \mathcal{N}} \theta_i \geq c(N) \\ 0 & \text{otherwise.} \end{cases}$$

For any stochastic decision rule $x : \Theta^N \to [0, 1]$, we introduce the following budget balance constraint.

Definition 6. A mechanism (x, t) satisfies the *ex post budget balance* (BB) if, for each $\theta \in \Theta^N$,

$$\sum_{i \in \mathcal{N}} t_i(\theta) - c(N)x(\theta) \geq 0.$$

In words, BB requires that the total payments be at least as large as the *expected* cost of the public good. Thus, the total payments may be insufficient to cover the cost of the public good in some type profile. On the other hand, Mailath and Postlewaite (1990) propose a different budget balance constraint, which we call *strong* BB.

Definition 7. A mechanism (x, t) satisfies the *stronger version of ex post budget balance* (strong BB) if, for each $\theta \in \Theta^N$,

$$x(\theta) \left(\sum_{i \in \mathcal{N}} t_i(\theta) - c(N) \right) \geq 0.$$

In words, the total payments must be at least sufficient to cover the cost of the public good in any type profile. So, this is stronger than our BB constraint.

We can justify the use of our weak version of budget balance constraint because our main focus is on establishing negative results. To appreciate this point, we discuss our Theorem 2, showing that the probability that the public good is provided is approximately zero in any mechanisms satisfying BIC, IIR, and BB in large economies. Therefore, this result remains the same if we use any stronger version of budget balance constraints, such as strong BB.

3 Existence of Mechanisms Satisfying BIC, IIR, and BB

In this section, we will fix a decision rule x and investigate the existence of a transfer rule t such that the mechanism (x, t) satisfies BIC, IIR, and BB.

This section is organized as follows. In Subsection 3.1, we introduce a set of machineries which allows us to formulate our mechanism design question as the shortest path problem in a network flow problem (See, for example, Vohra (2011, Chapter 3) for Network Flow Problem). Section 3.2 uses the linear programming approach to characterize the ex ante budget-surplus maximizing mechanisms satisfying BIC and IIR (i.e., the optimal mechanism) in terms of expected transfers (Theorem 1). We then propose the tight mechanism as the one inducing the optimal expected transfer rule (Lemma 3). Finally, we show in Proposition 1 that there exists a mechanism satisfying BIC, IIR, and BB if and only if the tight mechanism generates nonnegative ex ante budget surplus. Section 3.3 establishes an impossibility result for the public good provision problem in large economies (Theorem 2). In Section 3.4, we argue that an additional condition (Condition α) needed for Theorem 2 can be justified by the class of anonymous mechanisms which depend only on the average surplus from the public good in an economy with risk neutral agents (Proposition 2).

3.1 Preliminaries

Recall that $\bar{x}_i(\theta_i)$ is the interim expected probability of public good provision and that $\bar{t}_i(\theta_i)$ is agent i's interim expected transfer when he announces type θ_i and all the other agents announce their type truthfully, respectively. We characterize the mechanisms satisfying BIC below. We say that a decision rule x is *implementable in Bayesian Nash equilibrium* (IBN) if there exists a transfer rule $t : \Theta^N \to \mathbb{R}^N$ such that the mechanism (x, t) satisfies BIC. We first characterize the implementability in terms of monotonicity of decision rules. Since the following monotonicity result is well-known in the literature, we omit the proof.

Lemma 1. A decision rule $x : \Theta^N \to [0, 1]$ is implementable in Bayesian Nash equilibrium (IBN) if and only if $\bar{x}_i : \Theta \to [0, 1]$ is monotone, i.e., for each $i \in \mathcal{N}$ and $\theta^m, \theta^n \in \Theta$ with $\theta^m > \theta^n$, $\bar{x}_i(\theta^m) \geq \bar{x}_i(\theta^n)$.

Fix an IBN decision rule x. We shall find out the transfer rule t^* which maximizes the ex ante budget surplus among all mechanisms (x, t) satisfying BIC and IIR. This result will reduce our search for an appropriate mechanism to the mechanism (x, t^*). To establish this, we need to compute the ex ante budget surplus of any mechanism (x, t) when all agents report their types truthfully.

We can compute the ex ante budget surplus $\Pi_{ea}(x, t)$ of any mechanism (x, t)

in terms of the interim expected transfer $\bar{t}_i(\cdot)$:

$$\begin{aligned}
\Pi_{ea}(x,t) &\equiv \sum_{\theta\in\Theta^N} P^N(\theta)\left(\sum_{i\in\mathcal{N}} t_i(\theta) - c(N)x(\theta)\right) \\
&= \sum_{i\in\mathcal{N}}\sum_{\theta\in\Theta^N} P^N(\theta)t_i(\theta) - c(N)\sum_{\theta\in\Theta^N} P^N(\theta)x(\theta) \\
&= \sum_{i\in\mathcal{N}}\sum_{\theta_i\in\Theta} P(\theta_i)\left(\sum_{\theta_{-i}\in\Theta^{N-1}} P^{N-1}(\theta_{-i})t_i(\theta_i,\theta_{-i})\right) - c(N)\sum_{\theta\in\Theta^N} P^N(\theta)x(\theta) \\
&= \sum_{i\in\mathcal{N}}\sum_{m=1}^{M} P(\theta^m)\bar{t}_i(\theta^m) - c(N)\sum_{\theta\in\Theta^N} P^N(\theta)x(\theta).
\end{aligned}$$

So, the total ex ante expected transfers from the agents, $\sum_{i\in\mathcal{N}}\sum_{m=1}^{M} P(\theta^m)\bar{t}_i(\theta^m)$, must be as large as possible in order to maximize the ex ante budget surplus. Our objective here is to find their maximum values among all mechanisms satisfying BIC and IIR.

Recall that we consider symmetric mechanisms in the sense that if agents i and j report the same type, they face the same interim expected transfer. Then, it suffices to focus on one agent, say, agent i, and the per capita ex ante expected revenue is exactly the ex ante expected revenue from agent i, which is $\sum_{m=1}^{M} P(\theta^m)\bar{t}_i(\theta^m)$. As we argue before, since all agents' valuations are nonnegative, the IIR constraints can be incorporated into part of the BIC constraints by adding a dummy type θ^0. Then, the optimization problem can be simplified as follows:

$$\begin{aligned}
&\max_{\{\bar{t}_i(\theta^m)\}_{m=1}^M} \sum_{m=1}^{M} P(\theta^m)\bar{t}_i(\theta^m) \\
\text{s.t.}\quad & v(\bar{x}_i(\theta^m),\theta^m) - \bar{t}_i(\theta^m) \geq v(\bar{x}_i(\theta^n),\theta^m) - \bar{t}_i(\theta^n) \quad \forall m,n\in\{0,\cdots,M\},
\end{aligned}$$

where $v(\bar{x}_i(\hat{\theta}_i),\theta_i)$ denotes agent i's interim expected valuation of the provision decision when his true type is θ_i and he announces type $\hat{\theta}_i$. The BIC constraints can be rewritten as follows: for all $m,n\in\{0,\cdots,M\}$,

$$\bar{t}_i(\theta^m) - \bar{t}_i(\theta^n) \leq v(\bar{x}_i(\theta^m),\theta^m) - v(\bar{x}_i(\theta^n),\theta^m).$$

It follows from Vohra (2011, Chapter 6) that the derived inequality system has the expected network interpretation.[9] Introduce one node for each type (the node

[9]Heydenreich et al. (2009), Carbajal and Müller (2015), and Edelman and Weymark (2020) all employ a graph theoretic approach to characterize dominant strategy incentive compatibility and revenue equivalence. Moreover, Heydenreich et al. (2009) state in their footnote 3 that with appropriate adjustments, their characterization of revenue equivalence extends to the case of Bayesian incentive compatibility, as we do here.

corresponding to the dummy type θ^0 will be the source) and, to each arc (θ^n, θ^m),[10] assign a length of $c_{nm} = v(\bar{x}_i(\theta^m), \theta^m) - v(\bar{x}_i(\theta^n), \theta^m)$. We denote this network by T_x^i. Let θ^M be the terminal node in the network T_x^i. We aim at finding a shortest path from θ^0 to θ^M over network T_x^i, which is called the shortest-path problem. Since the arc length can also be interpreted as arc cost, finding a shortest path from θ^0 to θ^M is equivalent to determining the minimum cost of transferring one unit of the good from θ^0 to θ^M.

To properly describe the shortest-path problem, let us introduce additional pieces of notation. Denote by y_{nm} the flow from θ^n to θ^m on arc (θ^n, θ^m). Moreover, let b denote the net demand vector such that $b_0 = -1$ at the source node θ^0, $b_M = 1$ at the terminal node θ^M, and $b_m = 0$ at all the other nodes.

We assume the conservation of flow, requiring that the total flow into any node θ^m minus the total flow out of that node must equal the net demand b_m at the node, that is,

$$\sum_{n \neq m} y_{nm} - \sum_{n \neq m} y_{mn} = b_m \quad \forall m \in \{0, \cdots, M\}.$$[11]

In addition to the conservation equations, we also assume that the flow on each arc is nonnegative, that is, $y_{nm} \geq 0$ for all $n \neq m$. Then, the shortest-path problem is to determine a feasible flow $\{y_{nm}\}_{n \neq m}$ that minimizes $\sum_{m=0}^{M} \sum_{n \neq m} c_{nm} y_{nm}$:

$$\begin{aligned}
\min_{\{y_{nm}\}_{n \neq m}} \quad & \sum_{m=0}^{M} \sum_{n \neq m} c_{nm} y_{nm} \\
\text{s.t.} \quad & \sum_{n \neq m} y_{nm} - \sum_{n \neq m} y_{mn} = b_m \quad \forall m \in \{0, \cdots, M\} \\
& y_{nm} \geq 0 \quad \forall n \neq m.
\end{aligned}$$

Summing all the conservation equations gives $\sum_{m=0}^{M} \sum_{n \neq m} y_{nm} - \sum_{m=0}^{M} \sum_{n \neq m} y_{mn} = \sum_{m=0}^{M} b_m$. It is easy to see that both sides are equal to zero. In other words, any one of the conservation equations is redundant, since it is equal to the opposite of the sum of all other equations.

The shortest-path problem has its dual:

$$\begin{aligned}
\max_{\{z_m\}_{m=0,\dots,M}} \quad & z_M - z_0 \\
\text{s.t.} \quad & z_m - z_n \leq c_{nm} \quad \forall n \neq m.
\end{aligned}$$

[10] An arc (θ^n, θ^m) can be transversed from θ^n to θ^m but not the other way around.

[11] We impose the restriction $n \neq m$ to avoid the self-loop which is an arc from a node to itself.

To see this, we start from the dual constraints:

$$\begin{aligned} z_m - z_n &\leq c_{nm} \quad \forall n \neq m \\ \Rightarrow (z_m - z_n) y_{nm} &\leq c_{nm} y_{nm} \quad \forall n \neq m \;\; (\because y_{nm} \geq 0 \; \forall n \neq m) \\ \Rightarrow \sum_{m=0}^{M} \sum_{n \neq m} (z_m - z_n) y_{nm} &\leq \sum_{m=0}^{M} \sum_{n \neq m} c_{nm} y_{nm}. \end{aligned} \tag{1}$$

Note that the left-hand side of (1) can be rewritten as follows:

$$\sum_{m=0}^{M} z_m \sum_{n \neq m} (y_{nm} - y_{mn}) = \sum_{m=0}^{M} z_m b_m, \tag{2}$$

where the equality follows because of the primal constraints. Substituting (2) into the left-hand side of (1), we obtain

$$\sum_{m=0}^{M} z_m b_m \leq \sum_{m=0}^{M} \sum_{n \neq m} c_{nm} y_{nm}.$$

Moreover, recall that $b_0 = -1, b_M = 1$, and $b_m = 0$ at all the other nodes. Then, we have

$$z_M - z_0 \leq \sum_{m=0}^{M} \sum_{n \neq m} c_{nm} y_{nm}.$$

Therefore, $z_M - z_0$ is a lower bound on the value of the objective function in the primal. The problem of finding the largest such lower bound is called the dual in the literature (See, for example, Vohra (2005, Section 4.2) for this).

Since any one of the primal constraints is redundant, then by (2), we can set any one of the dual variables to zero. Set $z_0 = 0$ and the dual becomes

$$\begin{aligned} \max_{\{z_m\}_{m=1,\ldots,M}} \quad & z_M \\ \text{s.t.} \quad & z_m - z_n \leq c_{nm} \quad \forall n \neq m \\ & z_0 = 0. \end{aligned}$$

Let $(z_1^*, \cdots, z_M^*)$ be a solution to the dual. By the duality theorem (see Dantzig (1998)), the optimal values of the objective functions in the primal and dual must be the same. Therefore, z_M^* is the length of the shortest path from the source node θ^0 to the terminal node θ^M. Indeed, given any feasible $(z_1, \cdots, z_M)$ with $z_0 = 0$ in the dual, each z_m is bounded from above by the length of the shortest path from the source θ^0 to node θ^m. This can be deduced by adding up the dual constraints that correspond to the arcs on the shortest path from θ^0 to θ^m.

Recall that our optimization problem is

$$\max_{\{\bar{t}_i(\theta^m)\}_{m=1,\dots,M}} \sum_{m=1}^{M} P(\theta^m)\bar{t}_i(\theta^m)$$
$$\text{s.t.} \quad \bar{t}_i(\theta^m) - \bar{t}_i(\theta^n) \leq v(\bar{x}_i(\theta^m),\theta^m) - v(\bar{x}_i(\theta^n),\theta^m) \quad \forall m,n \in \{0,\cdots,M\}.$$

Recall also that the arc length is $c_{nm} = v(\bar{x}_i(\theta^m),\theta^m) - v(\bar{x}_i(\theta^n),\theta^m)$ for all $n \neq m$. Then, it is easy to see that our inequality constraints coincide with the dual constraints. As a result, each $\bar{t}_i(\theta^m)$ is bounded from above by the length of the shortest path from the source θ^0 to node θ^m, and it is optimal to set each $\bar{t}_i(\theta^m)$ equal to the length of the shortest path from the source θ^0 to node θ^m.

Lemma 2. Let $x : \Theta^N \to [0,1]$ be an IBN decision rule. Then, there exists a shortest path from the source node θ^0 to the terminal node θ^M with respect to $\{c_{nm}\}_{n\neq m}$ in the network T_x^i.

Proof. By Theorem 4.6.1 of Vohra (2011), a decision rule x is IBN if and only if for each agent $i \in \mathcal{N}$, there are no negative length cycles in the network T_x^i.[12] Moreover, by Corollary 3.4.2 of Vohra (2011), there exists a shortest path from the source node to the terminal node in a network if and only if the network contains no negative length cycles. Since x is IBN, there exists a shortest path from θ^0 to θ^M in the network T_x^i.[13] ■

In the next subsection, we determine the length of the shortest path in the network T_x^i.

3.2 Characterizations of Mechanisms Satisfying BIC, IIR, and BB

Recall that a decision rule x is IBN if there exists a transfer rule t such that the mechanism (x,t) satisfies BIC. The following is the main result of this paper.

Theorem 1. Let $x : \Theta^N \to [0,1]$ be an IBN decision rule[14] and t^* be a transfer rule such that the interim expected transfer for each agent $i \in \mathcal{N}$ and each type

[12] A cycle is a path whose initial and terminal nodes are the same.

[13] Our Lemma 2 also follows from Observation 2 and Lemma 1 of Heydenreich et al. (2009), both of which are the allocation graph counterpart of Theorem 4.6.1 and Corollary 3.4.2 of Vohra (2011), respectively. Although Observation 2 is concerned with dominant strategy implementation, Heydenreich et al. (2009, footnote 3 and p.312) say that their analysis extends to Bayesian implementation and type graphs.

[14] The existence of such a decision rule is automatically guaranteed because we consider the mechanism (x,t) such that the public good is never provided and no transfers are made. Such x is an IBN decision rule.

$\theta^m \in \{\theta^1, \cdots, \theta^M\}$ is given as follows:

$$\bar{t}_i^*(\theta^m) = \sum_{l=1}^{m} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right]. \tag{3}$$

Then, the mechanism (x, t^*) maximizes the ex ante budget surplus among all mechanisms (x, t) satisfying BIC and IIR. Moreover, we obtain the maximal ex ante budget surplus as follows:

$$\Pi_{ea}(x, t^*) = N \sum_{m=1}^{M} \left[v(\bar{x}_i(\theta^m), \theta^m) - v(\bar{x}_i(\theta^{m-1}), \theta^m) \right] \sum_{l=m}^{M} P(\theta^l) - c(N) \sum_{\theta \in \Theta^N} P^N(\theta) x(\theta). \tag{4}$$

Proof. The proof is in the Appendix. ■

Thus far, we know that if we fix an IBN decision rule x, the interim expected transfer (3) maximizes the ex ante budget surplus among all mechanisms (x, t) satisfying BIC and IIR. The interim expected transfer described in (3) is determined by the decision rule x only. What remains to be resolved is the existence of transfer rule t^* which induces the interim expected transfer (3). To do so, we propose the *tight mechanism* (x, t^T) as a most natural candidate inducing the interim expected transfer (3).[15]

Definition 8. A mechanism (x, t^T) is called the *tight mechanism* if, for each $i \in \mathcal{N}$, $\theta^m \in \Theta$, and $\theta_{-i} \in \Theta^{N-1}$,

$$t_i^T(\theta^m, \theta_{-i}) = \sum_{l=1}^{m} \left[v(x(\theta^l, \theta_{-i}), \theta^l) - v(x(\theta^{l-1}, \theta_{-i}), \theta^l) \right].$$

In the tight mechanism, an agent's payment is equal to his marginal contribution to the public good. Suppose all agents other than i announce their types truthfully. Then, the interim expected transfer for agent i of type θ^m in the tight mechanism is

$$\begin{aligned} \bar{t}_i^T(\theta^m) &= \sum_{\theta_{-i} \in \Theta^{N-1}} P^{N-1}(\theta_{-i}) \sum_{l=1}^{m} \left[v(x(\theta^l, \theta_{-i}), \theta^l) - v(x(\theta^{l-1}, \theta_{-i}), \theta^l) \right] \\ &= \sum_{l=1}^{m} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right]. \end{aligned}$$

The reader is referred to Section 2 to check how $v(\bar{x}_i(\theta^l), \theta^l)$ and $v(\bar{x}_i(\theta^{l-1}), \theta^l)$ are defined. The following lemma is obvious.

[15] The tight mechanism was originally proposed by Kos and Manea (2009) in a bilateral trade environment. We adapt it to our public good environment.

Lemma 3. Let $x : \Theta^N \to [0,1]$ be an IBN decision rule. The interim expected transfer in the tight mechanism (x, t^T) is identical with the interim expected transfer (3) which maximizes the ex ante budget surplus among all mechanisms (x, t) satisfying BIC and IIR.

Then, by Theorem 1, we know that for any IBN decision rule x, the tight mechanism (x, t^T) maximizes the ex ante budget surplus among all mechanisms (x, t) satisfying BIC and IIR. Moreover, the ex ante budget surplus generated by the tight mechanism is exactly equal to $\Pi_{ea}(x, t^*)$.

Proposition 1 below reduces our search for desirable mechanisms to the class of the tight mechanisms. Its necessity part says that if there exists a mechanism satisfying BIC, IIR, and BB, then the tight mechanism must achieve a nonnegative ex ante budget surplus. Its sufficiency part says that if the tight mechanism generates a nonnegative ex ante budget surplus, we may redistribute the ex ante surplus in such a way that we can construct a mechanism that satisfies BIC, IIR, and BB. Since our definition of BB is weaker than strong BB, the sufficiency part of the result is weaker accordingly. We formally state the result below.

Proposition 1. Let $x : \Theta^N \to [0,1]$ be an IBN decision rule. Then, there exists a transfer rule $t : \Theta^N \to \mathbb{R}^N$ such that the mechanism (x, t) satisfies BIC, IIR, and BB if and only if the tight mechanism (x, t^T) generates nonnegative ex ante budget surplus, i.e., $\Pi_{ea}(x, t^*) \geq 0$.

Proof. The proof is in the Appendix. ■

We conclude that ex ante budget surplus is the key to obtaining possibility results in our paper. Similar insights can be found in Mailath and Postlewaite (1990), Schweizer (2006), and Segal and Whinston (2011) in different setups. For example, Mailath and Postlewaite (1990) show in their Theorem 1 that in a continuous, one-dimensional type space, there exists a mechanism satisfying BIC, IIR, and strong BB in the public good provision problem if and only if the *expected virtual surplus* is nonnegative.

Before proceeding to the next subsection, we introduce the formal definition of revenue equivalence for Bayesian implementation and show that revenue equivalence fails in our setup. Here we adapt the analysis of Heydenreich et al. (2009) from dominant strategy to Bayesian implementation.

Definition 9. An IBN decision rule $x : \Theta^N \to [0,1]$ satisfies the *revenue equivalence property* if, for any two Bayesian incentive compatible mechanisms (x, t) and (x, t') and any agent $i \in \mathcal{N}$, there exists a constant $C_i \in \mathbb{R}$ such that the interim expected transfers $\bar{t}_i(\cdot)$ and $\bar{t}'_i(\cdot)$ satisfy the following equation: for each $\theta_i \in \Theta$,

$$\bar{t}_i(\theta_i) = \bar{t}'_i(\theta_i) + C_i.$$

The reader is referred to Section 2 for the definitions of $\bar{t}_i(\theta_i)$ and $\bar{t}'_i(\theta_i)$. Corollary 1 of Heydenreich et al. (2009) characterizes the necessary and sufficient condition for revenue equivalence under dominant strategy implementation. With appropriate adjustments, it is easy to show that their characterization result extends to Bayesian implementation. To simplify the notation, we denote by $\text{dist}_{T_x^i}(\theta^m, \theta^n)$ the length of the shortest path from node θ^m to θ^n in a type graph T_x^i.

Lemma 4. Let $x : \Theta^N \to [0,1]$ be an IBN decision rule. Then, x satisfies the revenue equivalence property if and only if $\text{dist}_{T_x^i}(\theta^m, \theta^n) = -\text{dist}_{T_x^i}(\theta^n, \theta^m)$ for all $\theta^m, \theta^n \in \Theta$ and all $i \in \mathcal{N}$.

We show below that the revenue equivalence property imposes a stringent condition on decision rules.

Claim 1. If an IBN decision rule $x : \Theta^N \to [0,1]$ satisfies the revenue equivalence property, then, no agent is pivotal "on average" in the provision decision, i.e., for any agent $i \in \mathcal{N}$ and all $\theta^m, \theta^n \in \Theta$,

$$\bar{x}_i(\theta^m) = \bar{x}_i(\theta^n).$$

Proof. The proof is in the Appendix. ■

We next show by means of an example that the derived condition in the above claim can easily be violated. Hence, the revenue equivalence property fails in our setup. For example, there are only two agents and each agent has only two types, i.e., $\mathcal{N} = \{1,2\}$ and $\Theta = \{\theta^1, \theta^2\}$. Suppose that each agent i is risk-neutral, i.e., his valuation for the provision decision is $v(q, \theta_i) = q\theta_i$ for each $q \in [0,1]$ and $\theta_i \in \Theta$. We further assume that the cost of providing the public good equals $\theta^1 + \theta^2$. Consider the efficient decision rule x^*:

$$x^*(\theta) = \begin{cases} 1 & \text{if } \sum_{j\in\mathcal{N}} \theta_j \geq \theta^1 + \theta^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then, we compute the interim expected provision probability for each agent i as follows:

$$\begin{aligned} \bar{x}_i^*(\theta^1) &= \sum_{\theta_{-i}\in\Theta} P(\theta_{-i}) x^*(\theta^1, \theta_{-i}) = P(\theta^2) < 1; \\ \bar{x}_i^*(\theta^2) &= \sum_{\theta_{-i}\in\Theta} P(\theta_{-i}) x^*(\theta^2, \theta_{-i}) = P(\theta^1) + P(\theta^2) = 1. \end{aligned}$$

Obviously, $\bar{x}_i^*(\theta^1) \neq \bar{x}_i^*(\theta^2)$, violating the revenue equivalence property.

3.3 Mechanisms Satisfying BIC, IIR, and BB in Large Economies

Now, let us investigate the implication of mechanisms satisfying BIC, IIR, and BB in large economies. Let $x[N]$ and $t^T[N]$ denote an IBN decision rule and the transfer rule of the tight mechanism in an economy with N agents, respectively. We have $\lim_{N\to\infty} c(N)/N > v(1,\theta^1)$ because we only consider nontrivial cases. Recall that we call it a trivial case if it is efficient to provide the public good even if all agents have the lowest type.

Recall that $\bar{x}_i[N](\theta_i)$ denotes the interim expected probability that the public good is provided when agent i announces type θ_i and all the other agents announce their type truthfully. We first introduce the following condition:

Definition 10. A sequence of decision rules $\{x[N]\}_{N\in\mathbb{N}}$ satisfies *Condition* α, if for any $\theta^m, \theta^n \in \Theta$ and $i \in \mathcal{N}$,

$$\lim_{N\to\infty} \bar{x}_i[N](\theta^m) = \lim_{N\to\infty} \bar{x}_i[N](\theta^n).$$

Condition α says that the probability that any agent can be pivotal is approximately zero in large economies.

In the theorem below, we shall show that, under Condition α, the probability that the public good is provided converges to zero as the population size goes to infinity in any mechanism satisfying BIC, IIR, and BB.

Theorem 2. Let $\{x[N]\}_{N\in\mathbb{N}}$ be a sequence of decision rules satisfying Condition α such that for each population size N, there exists a transfer rule $t[N]$ for which the mechanism $(x[N], t[N])$ satisfies BIC, IIR, and BB in the N-agent economy.[16] Then, in any nontrivial case, $\lim_{N\to\infty} \sum_{\theta\in\Theta^N} P^N(\theta)x[N](\theta) = 0$, i.e., the ex ante probability that the public good is provided converges to zero as the economy gets infinite ($N \to \infty$).

Proof. The proof is in the Appendix. ■

Remark: Theorem 2 is considered a discrete type space counterpart of Theorem 2 of Mailath and Postlewaite (1990). Although we need Condition α for this result, we argue later in Proposition 2 that this condition is a mild requirement. So, as the economy gets large, all mechanisms satisfying BIC, IIR, and BB share the same feature: the ex ante probability that the public good is provided converges to zero. This is quite a negative result.

[16]The existence of such a sequence is automatically guaranteed because we consider the mechanism (x, t) such that the public good never be provided and no transfers be made. Such a mechanism trivially satisfies BIC, IIR, and BB and it works for any number of agents. Moreover, since the public good is never provided, the interim expected probability that the public good is provided is always zero for any agent of any type. Hence, Condition α is trivially satisfied in this case.

This leads us to the need of weakening IIR if we seek for more positive results in large economies. For example, Grüner and Koriyama (2012) consider a connected, one-dimensional type space and assume that the set of possible provision decisions is binary, i.e., $\{0,1\}$. To define a weaker individual rationality requirement, they replace the outside option utility of zero with the outside option utility induced by a majority voting game with equal-cost sharing, which dictates that, if the unanimous agreement for the provision of the public good is not made, the public good is provided and its cost is shared equally if and only if more than half of the agents vote for the provision. Then, they establish a possibility result for the existence of mechanisms satisfying BIC, EFF, strong BB (See Section 2 for the definition of strong BB), and their IIR. Schweizer (2006) and Segal and Whinston (2011) also enhance the possibility results in different setups by including outside options as part of the design. It would be interesting to investigate how to incorporate the design of outside option into our discrete framework. However, we leave it for future work.

3.4 Justifying Condition α

In this subsection, we provide a justification for Condition α, which is needed for Theorem 2. To do so, we assume that each agent i is risk-neutral, i.e., each agent i's valuation of the provision decision is $v(q,\theta_i) = q\theta_i$ for any $q \in [0,1]$ and $\theta_i \in \Theta$. Then, $\sum_{j\in\mathcal{N}} \theta_j/N$ can be interpreted as the average surplus from the provision of the public good. Note that $[\theta^1,\theta^M]$ spans the space of all possible values the average surplus takes.

We shall consider the finest partition of the interval $[\theta^1,\theta^M]$ with the property that conditional upon the type announcement of the other agents, any change in agent i's type announcement leading to a new average surplus must fall in a different sub-interval. Let $h^{min} = \min_{m\in\{2,\cdots,M\}}(\theta^m - \theta^{m-1})$, i.e., the minimum difference between any two consecutive types of each agent. Then, the finest partition of the interval $[\theta^1,\theta^M]$ is defined as

$$\left\{[\theta^1,\theta^1 + h^{min}/N), [\theta^1 + h^{min}/N, \theta^1 + 2h^{min}/N), \cdots, \left[\theta^1 + K(N)h^{min}/N, \theta^M\right]\right\},$$

where $K(N) = \lfloor N(\theta^M - \theta^1)/h^{min}\rfloor$ is the greatest integer less than or equal to $N(\theta^M - \theta^1)/h^{min}$ such that $\theta^1 + K(N)h^{min}/N \leq \theta^M$ and $\theta^1 + (K(N)+1)\,h^{min}/N > \theta^M$. To simplify the notation, we let $\mathcal{A}_{k_N}[N] \equiv [\theta^1 + k_N h^{min}/N, \theta^1 + (k_N+1)h^{min}/N)$ for each $k_N \in \{0,1,\cdots,K(N)\}$. Then, we denote the finest partition of the interval $[\theta^1,\theta^M]$ by $\{\mathcal{A}_{k_N}[N]\}_{k_N=0}^{K(N)}$.

We impose the following assumption on decision rules.

Assumption 1. A decision rule $x : \Theta^N \to [0,1]$ satisfies Assumption 1 if, for any integer $N \geq 2$ and any two type profiles $\theta, \hat{\theta} \in \Theta^N$, if there exists $k_N \in \{0,\ldots,K(N)\}$ such that $(\sum_{j\in\mathcal{N}} \theta_j/N), (\sum_{j\in\mathcal{N}} \hat{\theta}_j/N) \in \mathcal{A}_{k_N}[N]$, then

$$x[N](\theta) = x[N](\hat{\theta}).$$

Remark: A similar mechanism can be found in the proof of Theorem 2 of Mailath and Postlewaite (1990, p.357). In their mechanism, the provision probability depends only on the average virtual valuation and such a mechanism maximizes the provision probability among all mechanisms satisfying BIC, IIR, and strong BB in a continuous type space.

Assumption 1 says that the public good provision decision depends only on the average surplus from the public good. In particular, it does not matter whether a certain amount of surplus is contributed by agent i or agent j. This implies a version of anonymity and this anonymity strikes us as being natural in large economies. Moreover, Assumption 1 is satisfied under the efficient decision rule x^*: for each type profile $\theta \in \Theta^N$,

$$x^*(\theta) = \begin{cases} 1 & \text{if } \sum_{j \in \mathcal{N}} \theta_j \geq c(N), \\ 0 & \text{otherwise.} \end{cases}$$

The reason is as follows. Under the efficient decision rule x^*, we can divide all the possible values of the average surplus into the following two sub-intervals: $[\theta^1, c(N)/N)$ and $[c(N)/N, \theta^M]$. Then, whenever two possible values of the average surplus fall within the same sub-interval, the efficient provision decisions are the same.

By Assumption 1, we can rewrite the decision rule $x[N] : \Theta^N \to [0,1]$ as $x[N] : \{\mathcal{A}_{k_N}[N]\}_{k_N=0}^{K(N)} \to [0,1]$, which is constant over any atom of the finest partition $\{\mathcal{A}_{k_N}[N]\}_{k_N=0}^{K(N)}$. We will show that Condition α is satisfied under Assumption 1.

Proposition 2. Let $\{x[N]\}_{N \in \mathbb{N}}$ be a sequence of decision rules satisfying Assumption 1. Then, Condition α is satisfied, i.e., for any $i \in \mathcal{N}$ and $\theta^m, \theta^n \in \Theta$,

$$\lim_{N \to \infty} \bar{x}_i[N](\theta^m) = \lim_{N \to \infty} \bar{x}_i[N](\theta^n).$$

Proof. The proof is in the Appendix. ■

If we only consider the class of anonymous mechanisms which only depend on the average surplus from the public good, which is embodied by Assumption 1, Condition α is automatically satisfied so that Theorem 2 is re-established without Condition α. However, risk neutrality is important for Theorem 2 to the extent that our average surplus interpretation of mechanisms in Assumption 1 makes sense.

Before concluding this section, we find it worthwhile to highlight two main differences between Theorem 2 of Mailath and Postlewaite (1990) and Theorem 2 of our paper. First, the implications are different. Theorem 2 of Mailath and Postlewaite (1990) shows that the following decision rule $\rho^N(\theta; \alpha^*(N))$, which depends only on the average virtual surplus, maximizes the provision probability among all mechanisms satisfying BIC, IIR, and strong BB in a continuous, one-dimensional type space. That is, for some $\alpha^*(N) > 0$,

$$\rho^N(\theta; \alpha^*(N)) = \begin{cases} 1 & \text{if } N^{-1} \sum_i \beta_i^N(\theta_i) + \alpha^*(N) \geq c(N)/N \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta_i^N(\theta_i) = \theta_i - \left(1 - F_i^N(\theta_i)\right) / f_i^N(\theta_i)$ is agent i's virtual valuation, and $F_i^N(\theta_i)$ and $f_i^N(\theta_i)$ are the cumulative distribution function and probability density function of agent i's type, respectively. On the contrary, our paper restricts attention to the decision rules which depend only on the average surplus to justify our Condition α which is needed for our Theorem 2.

Second, the proof techniques are different. Theorem 2 of Mailath and Postlewaite (1990) first derives the decision rule which maximizes the provision probability and then shows that the maximal provision probability converges to zero as the economy gets infinite. On the other hand, we first show that for any implementable decision rule, the maximal ex ante budget surplus must be nonnegative and then by exploiting this result, we show that the provision probability converges to zero as the economy gets infinite.

4 Dominant Strategy Incentive Compatibility and Ex Post Individual Rationality

Now, let us replace the Bayesian incentive compatibility (BIC) and interim individual rationality (IIR) constraints with dominant strategy incentive compatibility (DSIC) and ex post individual rationality (EPIR) constraints, respectively. We then investigate the existence of mechanisms satisfying DSIC, EPIR, and strong BB. In this case, we could make any distributional assumption about the type space and in particular, we could allow for any correlation among types.

This section is organized as follows. As in Subsection 3.1, Subsection 4.1 extends the shortest path problem in a network flow problem from Bayesian implementation to dominant strategy implementation. Using the machineries introduced in the previous subsection, Subsection 4.2 identifies the tight mechanism as the unique ex post budget-surplus maximizing mechanism satisfying DSIC and EPIR (Theorem 3). We also obtain Theorem 4 as a substantial implication of this section. It shows that under a richness condition imposed on decision rules, there are no mechanisms satisfying DSIC, EPIR, and strong BB in all nontrivial cases.

4.1 Preliminaries

We characterize the mechanisms satisfying DSIC below. We say that a decision rule $x : \Theta^N \to [0, 1]$ is *implementable in dominant strategies* (IDS) if there exists a transfer rule $t : \Theta^N \to \mathbb{R}^N$ such that the mechanism (x, t) satisfies DSIC (See Section 2 for the definition of DSIC). We first characterize the implementability in terms of monotonicity of a decision rule. The following monotonicity result is very well-known in the literature and so, we omit the proof.

Lemma 5. A decision rule $x : \Theta^N \to [0, 1]$ is implementable in dominant strategies (IDS) if and only if x is monotone, i.e., for each $i \in \mathcal{N}$ and $\theta_{-i} \in \Theta^{N-1}$, $\theta^m > \theta^n$ implies $x(\theta^m, \theta_{-i}) \geq x(\theta^n, \theta_{-i})$.

Fix an IDS decision rule x. We shall aim at finding a transfer rule t^{**} such that the mechanism (x, t^{**}) maximizes the ex post budget surplus among all mechanisms (x, t) satisfying DSIC and EPIR (see Section 2 for the definition of EPIR). Obviously, the ex post payment of each agent $i \in \mathcal{N}$ in each type profile $\theta \in \Theta^N$ must be as large as possible in order to maximize the ex post budget surplus. Hence, our objective here is to find their maximum values among all mechanisms (x, t) satisfying DSIC and EPIR.

Recall that since the valuation functions take nonnegative values only, the EPIR constraints can be incorporated into part of the DSIC constraints by adding a dummy type θ^0. Then, the optimization problem can be simplified as follows: fix an agent $i \in \mathcal{N}$ and the type profile of the other agents $\theta_{-i} \in \Theta^{N-1}$,

$$\max_{\{t_i(\theta_i,\theta_{-i})\}_{\theta_i \in \Theta}} \sum_{\theta_i \in \Theta} t_i(\theta_i, \theta_{-i})$$

$$\text{s.t.} \quad v(x(\theta_i, \theta_{-i}), \theta_i) - t_i(\theta_i, \theta_{-i}) \geq v(x(\hat{\theta}_i, \theta_{-i}), \theta_i) - t_i(\hat{\theta}_i, \theta_{-i}) \quad \forall \theta_i, \hat{\theta}_i \in \Theta.$$

In particular, the DSIC constraints can be rewritten as follows: for all $\theta_i \neq \hat{\theta}_i$ and $\theta_{-i} \in \Theta^{N-1}$,

$$t_i(\theta_i, \theta_{-i}) - t_i(\hat{\theta}_i, \theta_{-i}) \leq v(x(\theta_i, \theta_{-i}), \theta_i) - v(x(\hat{\theta}_i, \theta_{-i}), \theta_i).$$

It follows from Vohra (2011, Chapter 4) that the inequality system has the network interpretation. We associate with each $i \in \mathcal{N}$ and $\theta_{-i} \in \Theta^{N-1}$ a network with one node for each type $\theta_i \in \Theta$ (the nodes corresponding to the dummy type θ^0 and the highest type θ^M will be the source and terminal node, respectively) and, to each arc $(\hat{\theta}_i, \theta_i)$, assign a length of $v(x(\theta_i, \theta_{-i}), \theta_i) - v(x(\hat{\theta}_i, \theta_{-i}), \theta_i)$. We denote this network by $T^i_x(\theta_{-i})$.

Note that the network formulation is constructed verbatim in Subsection 3.1, except that we fix the other agents' types θ_{-i} instead of taking an expectation over them. Hence, we conclude that the DSIC constraints coincide with the dual constraints of the shortest-path problem and that each $t_i(\theta_i, \theta_{-i})$ is bounded from above by the length of the shortest path from the source node θ^0 to node θ_i. Therefore, it is optimal to set $t_i(\theta_i, \theta_{-i})$ equal to the length of the shortest path, and the optimization problem reduces to determining the shortest-path tree (the union of all shortest paths from the source to all nodes) in the network.

Lemma 6. Let $x : \Theta^N \to [0,1]$ be an IDS decision rule. Then, for any $\theta_{-i} \in \Theta^{N-1}$, there exists a shortest path from the source node θ^0 to the terminal node θ^M in the network $T_x^i(\theta_{-i})$.

Proof. It follows from Theorem 4.2.1 in Vohra (2011) that a decision rule x is IDS if and only if for each $i \in \mathcal{N}$ and $\theta_{-i} \in \Theta^{N-1}$, the network $T_x^i(\theta_{-i})$ does not have a finite cycle of negative length. Furthermore, it follows from Corollary 3.4.2 in Vohra (2011) that there exists a shortest path from the source node θ^0 to the terminal node θ^M if and only if the network contains no finite cycles of negative length. Since x is IDS, we conclude that there exists a shortest path from the source node θ^0 to the terminal node θ^M in the network $T_x^i(\theta_{-i})$.[17] ■

In the next subsection, we determine the length of the shortest path in the network $T_x^i(\theta_{-i})$.

4.2 Characterization of Mechanisms Satisfying DSIC, EPIR, and Strong BB

Recall that a decision rule $x : \Theta^N \to [0,1]$ is IDS if there exists a transfer rule $t : \Theta^N \to \mathbb{R}^N$ such that the mechanism (x, t) satisfies DSIC. The proof of the following theorem is completed verbatim in the proof of Theorem 1, except that we fix the other agents' types θ_{-i} instead of taking an expectation over them. Hence, we omit the proof.

Theorem 3. Let $x : \Theta^N \to [0,1]$ be an IDS decision rule[18] and t^{**} be a transfer rule such that for each $i \in \mathcal{N}$, $\theta^m \in \Theta$, and $\theta_{-i} \in \Theta^{N-1}$,

$$t_i^{**}(\theta^m, \theta_{-i}) = \sum_{l=1}^{m} \left[v(x(\theta^l, \theta_{-i}), \theta^l) - v(x(\theta^{l-1}, \theta_{-i}), \theta^l) \right]. \tag{5}$$

Then, the mechanism (x, t^{**}) maximizes the ex post budget surplus among all mechanisms (x, t) satisfying DSIC and EPIR.

Remark: The transfer rule described in (5) is determined by the decision rule x only. Moreover, it is identical to the transfer rule t_i^T in the tight mechanism. Therefore, our Theorem 3 uniquely pins down the tight mechanism as the optimal one. By Theorem 3, we show that for any IDS decision rule $x : \Theta^N \to [0,1]$, the tight mechanism (x, t^T)

[17] Our Lemma 6 also follows from Observation 2 and Lemma 1 of Heydenreich et al. (2009), both of which are the allocation graph counterpart of Theorem 4.2.1 and Corollary 3.4.2 of Vohra (2011), respectively. See Heydenreich et al. (2009, p.312) where they say "One can check that all previous arguments still apply when using type graphs."

[18] The existence of such a decision rule is automatically guaranteed because we consider the mechanism (x, t) such that the public good is never provided and no transfers are made. Such a mechanism trivially satisfies DSIC.

maximizes the ex post budget surplus among all mechanisms (x, t) satisfying DSIC and EPIR. This reduces our search for desirable mechanisms to the class of the tight mechanisms.

As we show in our Claim 1 that the revenue equivalence property fails in our Bayesian implementation setup, we can extend this result to our dominant strategy implementation setup.

Claim 2. If an IDS decision rule $x : \Theta^N \to [0, 1]$ satisfies the revenue equivalence property, then no agent is pivotal in the provision decision given any profile of other agents, i.e., for any agent $i \in \mathcal{N}$, all types $\theta^m, \theta^n \in \Theta$, and the other agents' types $\theta_{-i} \in \Theta^{N-1}$,

$$x(\theta^m, \theta_{-i}) = x(\theta^n, \theta_{-i}).$$

Proof. We omit this proof because we can easily adapt the proof of Claim 1 to the dominant strategy implementation counterpart. ■

As the derived condition in Claim 2 is obviously restrictive, the revenue equivalence property for dominant strategy implementation also fails in our setup. So, in our discrete setup, we cannot rely on the revenue equivalence property in the analysis of dominant strategy implementation. This makes our Theorem 3 particularly useful. In the theorem below, we shall invoke a richness condition imposed on decision rules in which no mechanisms satisfy DSIC, EPIR, and strong BB in all nontrivial cases.

Theorem 4. Let $x : \Theta^N \to [0, 1]$ be an IDS decision rule satisfying the following richness condition: $x(\theta) = 1$ if there exists an agent $i \in \mathcal{N}$ such that $\theta_j = \theta^M$ for all $j \neq i$. Then, there exists no transfer rule $t : \Theta^N \to \mathbb{R}^N$ such that the mechanism (x, t) satisfies DSIC, EPIR, and strong BB.

Remark: Our richness condition requires that the public good be provided if all agents except one have their highest type. Thus, it is a very mild condition in large economies. When the number of agents is very small, on the contrary, it becomes a stringent condition. For example, if there are only two agents and each agent has only two types, our richness condition implies $x(\theta^1, \theta^2) = x(\theta^2, \theta^1) = x(\theta^2, \theta^2) = 1$. Then, the only free variable is $x(\theta^1, \theta^1)$.

Proof. The proof is in the Appendix. ■

Hence, in all nontrivial cases satisfying our richness condition, we have no hope in finding mechanisms satisfying DSIC, EPIR, and strong BB simultaneously. Moreover, our richness condition becomes a very mild requirement when we consider large economies. This is considered a dominant strategy counterpart of our Theorem 2, which shows that no mechanisms satisfy BIC, IIR, and BB in large economies in all nontrivial cases.

4.3 The Relation with Green and Laffont (1977), Serizawa (1999), and Kuzmics and Steg (2017)

In this subsection, we will discuss the relation between our results in Subsection 4.2 and Green and Laffont (1977), Serizawa (1999), and Kuzmics and Steg (2017). Recall that all these three papers assume a continuous type space.

Theorem 7 of Green and Laffont (1977) shows the nonexistence of desirable mechanisms in a rich environment which accommodates all possible agents' preferences including non-quasilinear ones. They impose EFF and assume that there is one fixed-size unique public project, implying that the set of possible public good provision decisions is $\{0, 1\}$. They first show that any mechanism satisfying DSIC and EFF is a Groves mechanism. Then, they exploit the richness of their environment to conclude that no mechanisms satisfy DSIC and EFF.

Theorem 7 of Green and Laffont (1977) differs from our Theorem 4 in the following two aspects. First, we assume that agents have quasilinear preferences throughout and we do not impose decision efficiency on mechanisms. On the contrary, they allow for non-quasilinear preferences and impose EFF. Second, their richness condition and ours are different. Our richness condition dictates that if all agents other than one have the highest type, the public good should be provided. In contrast, Green and Laffont (1977) introduce the following richness condition: the space of permissible utility functions includes all constant functions of the transfer rules and the step functions for transfers above and below some level. Thus, our richness condition is very different in nature from that used by Green and Laffont (1977).

Serizawa (1999) assumes that the set of possible production levels of the public good is continuous, i.e., $[0, \bar{y}]$ where $\bar{y} \in (0, \infty)$ is its maximum capacity, and considers the set of continuous, strictly quasiconcave, and strictly monotone preferences over the level of the public good and that of transfers. He considers a different budget balance constraint, saying that the total payments of the agents must be exactly equal to the cost of providing the public good in any state. This is stronger than strong BB and henceforth, we call it strong* BB. His Theorem 3 shows that if a mechanism satisfies DSIC, EPIR, strong* BB, and *symmetry*, saying that two agents with the same preference makes the same payment, then the mechanism reduces to the one in which all agents share the cost of the public good equally and the level of the public good is determined by their minimal demand based on the fact that all agents share the cost equally. Suppose that the probability of providing the public good $[0, 1]$ can be interpreted as the set of continuous production levels, $[0, \bar{y}]$. Then, under quasilinear preferences and our richness condition in Theorem 4, we can show that his proposed mechanism satisfies all the properties only in the trivial case in which the public good should be always provided. The logic is as follows. Under our richness condition in Theorem 4, if all the other agents $j \neq i$ have the highest type θ^M, the public good is provided with probability one even when agent i has the lowest type θ^1, implying that each agent has a demand for the public good. Then, in particular, agent i's benefit from the public good must be higher than the cost share he bears, i.e., $v(1, \theta^1) > c(N)/N$, or equivalently, $Nv(1, \theta^1) > c(N)$, which is a trivial case in our paper. This implies that Serizawa (1999) essentially assumes that it never be the case that the public good is provided at its maximal capacity, making our richness condition rendered moot.

Kuzmics and Steg (2017) assume that the set of possible production levels of the public good is binary, i.e., $\{0, 1\}$ and restrict attention to quasilinear preferences. Their type space is a closed interval on $\mathbb{R}$, and each agent is risk neutral. They consider strong* BB, which is the same as Serizawa (1999). Their Proposition 1 shows that any mechanism satisfying DSIC, EPIR, and strong* BB with an additional property that the lowest types obtain zero ex post utility from participating in the mechanism is the *threshold* mechanism, dictating that the public good is provided if and only if all agents have types that are at least their respective thresholds and that each agent pays an amount equal to his threshold when the public is provided and pays nothing otherwise.

Proposition 1 of Kuzmics and Steg (2017) differs from our Theorem 4 in the following three aspects. First, Kuzmics and Steg (2017) require that the sum of all agents' threshold values be exactly equal to the cost in the class of the threshold mechanisms. But this property is unlikely to be satisfied in our discrete setup. Second, we argue that if the valuation functions are nonnegative valued, the EPIR constraints can be incorporated into part of the DSIC constraints by adding a dummy type. Due to this methodology employed, we exclude negative valuations. On the contrary, Kuzmics and Steg (2017) can handle negative valuations. Third, we impose a richness condition in our Theorem 4, while Kuzmics and Steg (2017) do not have its counterpart. Restricting attention to nonnegative valuations only, we can show that a threshold mechanism exists only in the trivial case where the public good should be always provided. The logic is as follows. From our richness condition in Theorem 4, we know that the threshold for each agent is θ^1 and thus the sum of all agents' threshold values is $N\theta^1$, which is considered a trivial case in the sense that the public good should be provided even when all agents have the lowest type. As the sum of all agents' threshold values must be exactly equal to the cost in a threshold mechanism, we must satisfy $N\theta^1 = c(N)$, which is generically violated in our discrete setup.

5 Concluding Remark

This paper characterizes mechanisms satisfying BIC, IIR, and BB for public good production and cost decision in a finite-type environment with quasilinear preferences and fixed-size projects. The main contribution of this paper lies in both our characterization of the budget-surplus maximizing mechanisms satisfying incentive compatibility and individual rationality (i.e., the optimal mechanisms) in a discrete setup (Theorem 1 for Bayesian implementation and Theorem 3 for dominant strategy implementation) and our comprehensive comparison between our results and the known results in the literature which deal with a continuous type space. In our discrete framework, we not only establish new results in the classical public good provision problem but also restore some known results in the literature, which we call a stress test. To the extent that we obtain new results using our discrete version of the optimal mechanisms, the analysis of our paper brings new insights to the classical public good provision problem.

We believe that whether a discrete or continuous type space is employed is entirely a matter of mathematical tractability. No substantive issue should depend on this modelling choice. The best we can do is to take a discrete approximation of the continuous type space (See Abreu and Matsushima (1992, Section 5)). Indeed, we achieve this in the Appendix: we formally show that our optimal mechanism is considered a discrete approximation of the optimal mechanism in the model with a continuous type space.

6 Appendix

In the Appendix, we provide the proofs omitted from the main body of the paper. We also provide a discrete approximation of the ex ante budget-surplus maximizing mechanism satisfying BIC and IIR over the standard model of a continuous type space.

6.1 Proof of Theorem 1

Proof. The proof is completed by the following four steps. In Step 1, we derive the length of the shortest path from the source θ^0 to every other node in the network T_x^i. In Step 2, we define a transfer rule t^* such that the interim expected transfer of each agent of each type is equal to the length of the corresponding shortest path, that is, the interim expected transfer satisfies (3). We then verify that the mechanism (x, t^*) satisfies all the adjacent BIC constraints. In Step 3, we show that the mechanism (x, t^*) satisfies BIC and IIR. In Step 4, we compute the maximal ex ante budget surplus.

Step 1: We derive the length of the shortest path from the source θ^0 to every other node in the network T_x^i.

We introduce the following lemma:

Lemma 7. If v satisfies strictly increasing differences, then the length of arc (θ^m, θ^{m+2}) must be at least as large as the length of (θ^m, θ^{m+1}) plus the length of $(\theta^{m+1}, \theta^{m+2})$ for each $m \in \{0, \cdots, M-2\}$.

Proof. Theorem 6.2.2 of Vohra (2011) establishes a similar result in an auction environment. We adopt it to our public good environment and provide the proof below.

Suppose on the contrary that the length of the arc (θ^m, θ^{m+2}) is strictly smaller than the length of (θ^m, θ^{m+1}) plus the length of $(\theta^{m+1}, \theta^{m+2})$. Reflecting the formula of arc length $v(\bar{x}_i(\theta_i), \theta_i) - v(\bar{x}_i(\hat{\theta}_i), \theta_i)$ for each arc $(\hat{\theta}_i, \theta_i)$, we have

$$\begin{aligned} & v(\bar{x}_i(\theta^{m+2}), \theta^{m+2}) - v(\bar{x}_i(\theta^m), \theta^{m+2}) \\ < \; & v(\bar{x}_i(\theta^{m+1}), \theta^{m+1}) - v(\bar{x}_i(\theta^m), \theta^{m+1}) + v(\bar{x}_i(\theta^{m+2}), \theta^{m+2}) - v(\bar{x}_i(\theta^{m+1}), \theta^{m+2}). \end{aligned} \tag{6}$$

Note that the left-hand side of (6) can be rewritten as

$$v(\bar{x}_i(\theta^{m+2}), \theta^{m+2}) - v(\bar{x}_i(\theta^{m+1}), \theta^{m+2}) + v(\bar{x}_i(\theta^{m+1}), \theta^{m+2}) - v(\bar{x}_i(\theta^m), \theta^{m+2}).$$

Substituting it into the left-hand side of (6), we obtain

$$\begin{aligned} & v(\bar{x}_i(\theta^{m+2}), \theta^{m+2}) - v(\bar{x}_i(\theta^{m+1}), \theta^{m+2}) + v(\bar{x}_i(\theta^{m+1}), \theta^{m+2}) - v(\bar{x}_i(\theta^m), \theta^{m+2}) \\ < \; & v(\bar{x}_i(\theta^{m+1}), \theta^{m+1}) - v(\bar{x}_i(\theta^m), \theta^{m+1}) + v(\bar{x}_i(\theta^{m+2}), \theta^{m+2}) - v(\bar{x}_i(\theta^{m+1}), \theta^{m+2}). \end{aligned}$$

After rearrangement, we obtain

$$v(\bar{x}_i(\theta^{m+1}), \theta^{m+2}) - v(\bar{x}_i(\theta^m), \theta^{m+2}) < v(\bar{x}_i(\theta^{m+1}), \theta^{m+1}) - v(\bar{x}_i(\theta^m), \theta^{m+1}),$$

which violates the assumption that v satisfies strictly increasing differences, as $\bar{x}_i(\theta^{m+1}) \geq \bar{x}_i(\theta^m)$ and $\theta^{m+2} > \theta^{m+1}$. ■

In view of the above, the network associated with the BIC constraints is described in Figure 1 below.

Figure 1: Network of BIC constraints

We conclude that the shortest-path tree rooted at the dummy type θ^0 must be $\theta^0 \to \theta^1 \to \theta^2 \to \cdots \to \theta^M$. Algebraically, the length of the shortest path from the source θ^0 to every other node is given as follows: for each $\theta^m \in \{\theta^1, \cdots, \theta^M\}$,

$$\sum_{l=1}^{m} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right].$$

This completes the proof of Step 1. ■

Step 2: If there exists a transfer rule t^* such that for each agent i of every type θ_i, its interim expected transfer $\bar{t}_i^*(\theta_i)$ is equal to the length of the corresponding shortest path, that is, satisfies (3), then the mechanism (x, t^*) satisfies all the adjacent BIC constraints.

We introduce the following lemma:

Lemma 8. Let t^* be a transfer rule such that the interim expected transfer for each agent $i \in \mathcal{N}$ and each type $\theta^m \in \{\theta^1, \cdots, \theta^M\}$ is given as follows:

$$\bar{t}_i^*(\theta^m) = \sum_{l=1}^{m} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right],$$

where $v(\bar{x}_i(\theta^0), \theta^1) = 0$ and $\bar{t}_i^*(\theta^0) = 0$ for the dummy type θ^0. Then, the mechanism (x, t^*) satisfies the adjacent BIC constraints:

$$\begin{aligned} v(\bar{x}_i(\theta^m), \theta^m) - \bar{t}_i(\theta^m) &\geq v(\bar{x}_i(\theta^{m-1}), \theta^m) - \bar{t}_i(\theta^{m-1}) \;\; \forall m \in \{1, \cdots, M\}; \\ v(\bar{x}_i(\theta^m), \theta^m) - \bar{t}_i(\theta^m) &\geq v(\bar{x}_i(\theta^{m+1}), \theta^m) - \bar{t}_i(\theta^{m+1}) \;\; \forall m \in \{0, \cdots, M-1\}. \end{aligned}$$

To be more precise, the downward adjacent BIC constraints are binding and the upward adjacent BIC constraints are satisfied. Similar results can be found in Lovejoy (2006, Lemma 5) and Vohra (2011, Theorem 6.2.3) in an auction environment. In particular, Lemma 8 can be proved by adapting Theorem 6.2.3 of Vohra (2011) to our public good environment and hence we omit the proof.

Step 3: We show that the mechanism (x, t^*) described in Step 2 satisfies BIC and IIR.

To show this, we introduce the following lemma, saying that the BIC constraints are implied by the adjacent BIC constraints. Similar results can be found in Lovejoy (2006, Lemma 2) and Vohra (2011, Theorem 6.2.2) in an auction environment. In particular, Lemma 9 can be proved by adapting Theorem 6.2.2 of Vohra (2011) to our public good environment. Hence we omit the proof.

Lemma 9. Suppose v satisfies strictly increasing differences. If a mechanism (x, t) satisfies the following adjacent BIC constraints, then it satisfies all the BIC constraints:

$$\begin{aligned} v(\bar{x}_i(\theta^m), \theta^m) - \bar{t}_i(\theta^m) &\geq v(\bar{x}_i(\theta^{m-1}), \theta^m) - \bar{t}_i(\theta^{m-1}) \;\; \forall m \in \{1, \cdots, M\}; \\ v(\bar{x}_i(\theta^m), \theta^m) - \bar{t}_i(\theta^m) &\geq v(\bar{x}_i(\theta^{m+1}), \theta^m) - \bar{t}_i(\theta^{m+1}) \;\; \forall m \in \{0, \cdots, M-1\}. \end{aligned}$$

Since the mechanism (x, t^*) satisfies all the adjacent BIC constraints, we obtain that the mechanism (x, t^*) satisfies BIC. Furthermore, since the IIR constraints are incorporated into the BIC constraint, we conclude that the mechanism (x, t^*) satisfies BIC and IIR.

Step 4: We compute the maximal ex ante budget surplus.

Throughout Steps 1, 2, and 3, we know that the mechanism (x, t^*) maximizes the ex ante budget surplus among all mechanisms (x, t) satisfying BIC and IIR. Thus, it only remains to compute the maximal ex ante budget surplus $\Pi_{ea}(x, t^*)$:

$$\begin{aligned}
\Pi_{ea}(x, t^*) &= \sum_{i \in \mathcal{N}} \sum_{m=1}^{M} P(\theta^m) \bar{t}_i^*(\theta^m) - c(N) \sum_{\theta \in \Theta^N} P^N(\theta) x(\theta) \\
&= N \sum_{m=1}^{M} P(\theta^m) \sum_{l=1}^{m} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right] - c(N) \sum_{\theta \in \Theta^N} P^N(\theta) x(\theta) \\
&\quad \text{(recall the formula of } \bar{t}_i^*(\theta^m)) \\
&= N \sum_{m=1}^{M} \left[v(\bar{x}_i(\theta^m), \theta^m) - v(\bar{x}_i(\theta^{m-1}), \theta^m) \right] \sum_{l=m}^{M} P(\theta^l) - c(N) \sum_{\theta \in \Theta^N} P^N(\theta) x(\theta).
\end{aligned}$$

We thus obtain the desired expression for $\Pi_{ea}(x, t^*)$ as in (4). This completes the proof of Theorem 1. ■

6.2 Proof of Proposition 1

Proof. We first prove the necessity of $\Pi_{ea}(x, t^*) \geq 0$. Suppose that (x, t) satisfies BIC, IIR, and BB. Then (x, t) has nonnegative ex ante budget surplus. By Theorem 1, we obtain $\Pi_{ea}(x, t^*) \geq 0$.

We now prove the sufficiency. Consider the mechanism (x, t) where for each $\theta \in \Theta^N$,

$$\begin{aligned}
t_1(\theta) &= \left(t_1^T(\theta) - \Pi_{ea}(x, t^*)\right) + \left(c(N) x(\theta) - \sum_{i \in \mathcal{N}} t_i^T(\theta) + \Pi_{ea}(x, t^*) \right) \\
&\quad - \left(c(N) \bar{x}_1(\theta_1) - \sum_{i \in \mathcal{N}} \bar{t}_i^T(\theta_1) + \Pi_{ea}(x, t^*) \right); \\
t_2(\theta) &= t_2^T(\theta) + \left(c(N) \bar{x}_1(\theta_1) - \sum_{i \in \mathcal{N}} \bar{t}_i^T(\theta_1) + \Pi_{ea}(x, t^*) \right); \\
t_i(\theta) &= t_i^T(\theta) \text{ for any } i \in \mathcal{N} \backslash \{1, 2\}.
\end{aligned}$$

Then, the ex post budget balance (BB) is satisfied because for all $\theta \in \Theta^N$,

$$\sum_{i \in \mathcal{N}} t_i(\theta) = \sum_{i \in \mathcal{N}} t_i^T(\theta) - \Pi_{ea}(x, t^*) + \left(c(N) x(\theta) - \sum_{i \in \mathcal{N}} t_i^T(\theta) + \Pi_{ea}(x, t^*) \right) = c(N) x(\theta).$$

Besides, the interim expected transfer of each agent $i \in \mathcal{N}$ is obtained as follows.

1. For $i = 1$, $\bar{t}_1(\theta_1) = \bar{t}_1^T(\theta_1) - \Pi_{ea}(x, t^*) \leq \bar{t}_1^T(\theta_1)$ because $\Pi_{ea}(x, t^*) \geq 0$;

2. For $i = 2$,

$$
\begin{aligned}
\bar{t}_2(\theta_2) &= \bar{t}_2^T(\theta_2) + \sum_{\theta_{-2}\in\Theta^{N-1}} P^{N-1}(\theta_{-2}) \left(c(N)\bar{x}_1(\theta_1) - \sum_{i\in\mathcal{N}} \bar{t}_i^T(\theta_1) + \Pi_{ea}(x,t^*) \right) \\
&= \bar{t}_2^T(\theta_2) + \sum_{\theta_1\in\Theta} P(\theta_1) \left(c(N)\bar{x}_1(\theta_1) - \sum_{i\in\mathcal{N}} \bar{t}_i^T(\theta_1) \right) + \Pi_{ea}(x,t^*) \\
&\quad \left(\because\ c(N)\bar{x}_1(\theta_1) - \sum_{i\in\mathcal{N}} \bar{t}_i^T(\theta_1) + \Pi_{ea}(x,t^*) \text{ only depends on } \theta_1 \right) \\
&= \bar{t}_2^T(\theta_2) + \sum_{\theta\in\Theta^N} P^N(\theta) \left(c(N)x(\theta) - \sum_{i\in\mathcal{N}} t_i^T(\theta) \right) + \Pi_{ea}(x,t^*) \\
&\quad (\because \text{ types are independently distributed}) \\
&= \bar{t}_2^T(\theta_2) - \Pi_{ea}(x,t^*) + \Pi_{ea}(x,t^*) = \bar{t}_2^T(\theta_2);
\end{aligned}
$$

3. For $i \in \mathcal{N}\backslash\{1,2\}$, $\bar{t}_i(\theta_i) = \bar{t}_i^T(\theta_i)$.

Hence, the interim expected transfers of all agents in the mechanism (x,t) are the same as those in the tight mechanism (x,t^T), except agent 1. In particular, agent 1's interim expected transfer in mechanism (x,t) differs from that in (x,t^T) by a negative constant $-\Pi_{ea}(x,t^*) \le 0$. Therefore, (x,t) also satisfies BIC and IIR. This completes the proof. ■

6.3 Proof of Claim 1

Proof. Without loss of generality, we assume $\theta^m < \theta^n$. It follows from Lemma 7 that the length of arc (θ^m, θ^{m+2}) must be at least as large as the length of (θ^m, θ^{m+1}) plus the length of $(\theta^{m+1}, \theta^{m+2})$ for each $m \in \{0, \cdots, M-2\}$. Then, we have that the shortest path from θ^m to θ^n must be $\theta^m \to \theta^{m+1} \to \theta^{m+2} \to \cdots \to \theta^n$. Since we assign a length of $v(\bar{x}_i(\theta_i),\theta_i) - v(\bar{x}_i(\hat{\theta}_i),\theta_i)$ to each arc $(\hat{\theta}_i,\theta_i)$, then the length of the shortest path from θ^m to θ^n is

$$
\mathrm{dist}_{T_x^i}(\theta^m,\theta^n) = \sum_{l=m+1}^{n} \left[v(\bar{x}_i(\theta^l),\theta^l) - v(\bar{x}_i(\theta^{l-1}),\theta^l) \right].
$$

We need to find the length of the shortest path from θ^n to θ^m. The proof of the following lemma is constructed verbatim in the proof of Lemma 7. Hence, we omit the proof.

Lemma 10. If v satisfies strictly increasing differences, then the length of arc (θ^{m+2}, θ^m) must be at least as large as the length of $(\theta^{m+2}, \theta^{m+1})$ plus the length of (θ^{m+1}, θ^m).

Recall we have assumed $\theta^m < \theta^n$. Then, the shortest path from θ^n to θ^m is $\theta^n \to \theta^{n-1} \to \cdots \to \theta^m$. Since we assign a length of $v(\bar{x}_i(\theta_i), \theta_i) - v(\bar{x}_i(\hat{\theta}_i), \theta_i)$ to each arc $(\hat{\theta}_i, \theta_i)$, we obtain the length of the shortest path from θ^n to θ^m:

$$\text{dist}_{T_x^i}(\theta^n, \theta^m) = \sum_{l=m}^{n-1} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l+1}), \theta^l) \right].$$

Finally, we compute the summation of the length of the shortest paths:

$$\begin{aligned} & \text{dist}_{T_x^i}(\theta^m, \theta^n) + \text{dist}_{T_x^i}(\theta^n, \theta^m) \\ = & \sum_{l=m+1}^{n} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right] + \sum_{l=m}^{n-1} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l+1}), \theta^l) \right] \\ = & \sum_{l=m+1}^{n} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right] - \sum_{l=m}^{n-1} \left[v(\bar{x}_i(\theta^{l+1}), \theta^l) - v(\bar{x}_i(\theta^l), \theta^l) \right]. \end{aligned}$$

It can be further rewritten as follows:

$$\begin{aligned} & \text{dist}_{T_x^i}(\theta^m, \theta^n) + \text{dist}_{T_x^i}(\theta^n, \theta^m) \\ = & \sum_{l=m+1}^{n} \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right] - \sum_{l=m+1}^{n} \left[v(\bar{x}_i(\theta^l), \theta^{l-1}) - v(\bar{x}_i(\theta^{l-1}), \theta^{l-1}) \right] \\ = & \sum_{l=m+1}^{n} \left\{ \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right] - \left[v(\bar{x}_i(\theta^l), \theta^{l-1}) - v(\bar{x}_i(\theta^{l-1}), \theta^{l-1}) \right] \right\}. \end{aligned}$$

Recall that x being IBN implies that $\bar{x}_i(\theta^l) \geq \bar{x}_i(\theta^{l-1})$ for each $l \in \{2, \cdots, M\}$ and each $i \in \mathcal{N}$. We claim that $\bar{x}_i(\theta^l) = \bar{x}_i(\theta^{l-1})$ for each $l \in \{2, \cdots, M\}$ and each $i \in \mathcal{N}$. Suppose, by way of contradiction, that there exist some $l \in \{2, \cdots, M\}$ and some $i \in \mathcal{N}$ such that $\bar{x}_i(\theta^l) > \bar{x}_i(\theta^{l-1})$. Then

$$\begin{aligned} & \text{dist}_{T_x^i}(\theta^{l-1}, \theta^l) + \text{dist}_{T_x^i}(\theta^l, \theta^{l-1}) \\ = & \left[v(\bar{x}_i(\theta^l), \theta^l) - v(\bar{x}_i(\theta^{l-1}), \theta^l) \right] - \left[v(\bar{x}_i(\theta^l), \theta^{l-1}) - v(\bar{x}_i(\theta^{l-1}), \theta^{l-1}) \right] \\ > & 0, \end{aligned}$$

where the strict inequality follows because the valuation function $v(\cdot)$ satisfies strictly increasing differences. As a result, the necessary and sufficient condition in Lemma 4 is violated so that the revenue equivalence property fails in this case. This is the desired contradiction.

■

6.4 Proof of Theorem 2

Proof. Since the mechanism $(x[N], t[N])$ satisfies BIC, IIR, and BB, we obtain by Proposition 1 that the tight mechanism $(x[N], t^T[N])$ must generate a nonnegative ex ante budget surplus, i.e., $\Pi_{ea}(x[N], t^*[N]) \geq 0$. We take the expression for $\Pi_{ea}(x[N], t^*[N])$ from Theorem 1:

$$\begin{aligned}
&\Pi_{ea}(x[N], t^*[N]) \\
= \ & N \sum_{m=1}^{M} \left[v(\bar{x}_i[N](\theta^m), \theta^m) - v(\bar{x}_i[N](\theta^{m-1}), \theta^m) \right] \sum_{l=m}^{M} P(\theta^l) - c(N) \sum_{\theta \in \Theta^N} P^N(\theta) x[N](\theta).
\end{aligned}$$

The reader is referred to Section 2 to check how $v(\bar{x}_i[N](\theta^m), \theta^m)$ and $v(\bar{x}_i[N](\theta^{m-1}), \theta^m)$ are defined. Dividing both sides of the equation by the number of agents N, we obtain

$$\begin{aligned}
&\frac{\Pi_{ea}(x[N], t^*[N])}{N} \\
= \ & \sum_{m=1}^{M} \left[v(\bar{x}_i[N](\theta^m), \theta^m) - v(\bar{x}_i[N](\theta^{m-1}), \theta^m) \right] \sum_{l=m}^{M} P(\theta^l) - \frac{c(N)}{N} \sum_{\theta \in \Theta^N} P^N(\theta) x[N](\theta).
\end{aligned}$$

By Condition α, we have $\lim_{N\to\infty} \bar{x}_i[N](\theta^m) = \lim_{N\to\infty} \bar{x}_i[N](\theta^{m-1})$ for any $\theta^m, \theta^{m-1} \in \Theta$ and $i \in \mathcal{N}$. By continuity of the valuation function $v(\cdot)$, we further obtain

$$\lim_{N\to\infty} \left[v(\bar{x}_i[N](\theta^m), \theta^m) - v(\bar{x}_i[N](\theta^{m-1}), \theta^m) \right] = 0$$

for any $\theta^m, \theta^{m-1} \in \Theta$ and $i \in \mathcal{N}$. Therefore,

$$\lim_{N\to\infty} \frac{\Pi_{ea}(x[N], t^*[N])}{N} \ = \ -\lim_{N\to\infty} \frac{c(N)}{N} \sum_{\theta \in \Theta^N} P^N(\theta) x[N](\theta).$$

Recall that the tight mechanism must generate nonnegative ex ante budget surplus, i.e., $\lim_{N\to\infty} \Pi_{ea}(x[N], t^*[N])/N \geq 0$. Since $\lim_{N\to\infty} c(N)/N > v(1, \theta^1) \geq 0$, we conclude that $\lim_{N\to\infty} \sum_{\theta\in\Theta^N} P^N(\theta) x[N](\theta) = 0$ must be satisfied. This completes the proof. ■

6.5 Proof of Proposition 2

Proof. Fix $N \in \mathbb{N}$, $i \in \mathcal{N}$ and $\theta^m, \theta^n \in \Theta$ arbitrarily. Recall the following definition:

$$\bar{x}_i[N](\theta^m) \ = \ \sum_{\theta_{-i} \in \Theta^{N-1}} P^{N-1}(\theta_{-i}) x[N](\theta^m, \theta_{-i}).$$

Translating the type profiles into the partition $\{\mathcal{A}_{k_N}[N]\}_{k_N=0}^{K(N)}$, we obtain

$$\bar{x}_i[N](\theta^m) \ = \ \sum_{k_N=0}^{K(N)} \sum_{\theta_{-i}:\theta^m/N + \sum_{j\neq i}\theta_j/N \in \mathcal{A}_{k_N}[N]} P^{N-1}(\theta_{-i}) x[N](\mathcal{A}_{k_N}[N]),$$

where $x[N] : \{\mathcal{A}_{k_N}[N]\}_{k_N=0}^{K(N)} \to [0, 1]$. Similarly, we have

$$\bar{x}_i[N](\theta^n) = \sum_{\theta_{-i} \in \Theta^{N-1}} P^{N-1}(\theta_{-i}) x[N](\theta^n, \theta_{-i}) = \sum_{k_N=0}^{K(N)} \sum_{\theta_{-i}:\theta^n/N + \sum_{j\neq i}\theta_j/N \in \mathcal{A}_{k_N}[N]} P^{N-1}(\theta_{-i}) x[N](\mathcal{A}_{k_N}[N]).$$

For all $N \geq 2$, $k_N \in \{0, \cdots, K(N)\}$, and $\theta^m \in \Theta$, we define

$$\tilde{\mathcal{A}}_{k_N}[N] \equiv \left[\theta^1 + \frac{k_N h^{min}}{N} - \frac{\theta^m}{N}, \theta^1 + \frac{(k_N+1)h^{min}}{N} - \frac{\theta^m}{N}\right).$$

So, we also have that for all $N \geq 2$, $k_N \in \{0, \cdots, K(N)\}$, and $\theta^m \in \Theta$,

$$\lim_{N\to\infty} \tilde{\mathcal{A}}_{k_N}[N] = \left[\theta^1 + \lim_{N\to\infty} \frac{k_N h^{min}}{N}, \theta^1 + \lim_{N\to\infty} \frac{(k_N+1)h^{min}}{N}\right) = \lim_{N\to\infty} \mathcal{A}_{k_N}[N].$$

By construction of $\tilde{A}_{k_N}[N]$, we have that for each $\theta^m \in \Theta$,

$$\begin{aligned} & \sum_{k_N=0}^{K(N)} \sum_{\theta_{-i}:\theta^m/N+\sum_{j\neq i}\theta_j/N \in \mathcal{A}_{k_N}[N]} P^{N-1}(\theta_{-i})x[N](\mathcal{A}_{k_N}[N]) \\ = & \sum_{k=0}^{K(N)} \sum_{\theta_{-i}:\sum_{j\neq i}\theta_j/N \in \tilde{\mathcal{A}}_{k_N}[N]} P^{N-1}(\theta_{-i})x[N](\mathcal{A}_{k_N}[N]). \end{aligned}$$

When N is chosen large enough, we further obtain the following approximation, which is denoted by $\approx$ below:

$$\sum_{k_N=0}^{K(N)} \sum_{\theta_{-i}:\sum_{j\neq i}\theta_j/N \in \tilde{\mathcal{A}}_{k_N}[N]} P^{N-1}(\theta_{-i})x[N](\mathcal{A}_{k_N}[N]) \approx \sum_{k_N=0}^{K(N)} \sum_{\theta_{-i}:\sum_{j\neq i}\theta_j/N \in \mathcal{A}_{k_N}[N]} P^{N-1}(\theta_{-i})x[N](\mathcal{A}_{k_N}[N]),$$

because $\lim_{N\to\infty} \tilde{\mathcal{A}}_{k_N}[N] = \lim_{N\to\infty} \mathcal{A}_{k_N}[N]$. Note that the right-hand side of the above expression does not depend on θ^m. This implies $\lim_{N\to\infty} \bar{x}_i[N](\theta^m) = \lim_{N\to\infty} \bar{x}_i[N](\theta^n)$ for any $\theta^m, \theta^n \in \Theta$. This completes the proof. ■

6.6 Proof of Theorem 4

Proof. The proof is completed by the following two steps. In Step 1, we show that the tight mechanism generates ex post budget deficit. In Step 2, we invoke Theorem 3 to show that the same must be true for any mechanism satisfying DSIC and EPIR.

Step 1: The tight mechanism (x, t^T) generates ex post budget deficit.

Proof. Fix a type profile θ such that $\theta_j = \theta^M$ for all $j \in \mathcal{N}$. We compute each agent

j's payment in the tight mechanism (x, t^T):

$$\begin{aligned} t_j^T(\theta^M, \cdots, \theta^M) &= \sum_{l=1}^{M} \left[v(x(\theta^l, \theta^M, \cdots, \theta^M), \theta^l) - v(x(\theta^{l-1}, \theta^M, \cdots, \theta^M), \theta^l) \right] \\ &= \left[v(x(\theta^1, \theta^M, \cdots, \theta^M), \theta^1) - v(x(\theta^0, \theta^M, \cdots, \theta^M), \theta^1) \right] + \sum_{l=2}^{M} \left[v(1, \theta^l) - v(1, \theta^l) \right] \\ &\quad \left(\because \text{By our richness condition, } x(\theta^l, \theta^M, \cdots, \theta^M) = 1 \text{ for all } l \in \{1, \cdots, M\} \right) \\ &= \left[v(x(\theta^1, \theta^M, \cdots, \theta^M), \theta^1) - v(x(\theta^0, \theta^M, \cdots, \theta^M), \theta^1) \right] \\ &= v(1, \theta^1), \end{aligned}$$

where the last equality follows because $x(\theta^1, \theta^M, \cdots, \theta^M) = 1$ by the richness condition and $v(x(\theta^0, \theta^M, \cdots, \theta^M), \theta^1) = 0$ by the very definition of dummy type θ^0. Then, the total payments at profile $(\theta^M, \cdots, \theta^M)$ are

$$\sum_{j \in \mathcal{N}} t_j^T(\theta^M, \cdots, \theta^M) = N v(1, \theta^1),$$

where the right-hand side stands for the total surplus from the public good when all agents have the lowest type θ^1. Since we assume $Nv(1, \theta^1) < c(N)$ to focus only on nontrivial cases, we obtain $\sum_{j \in \mathcal{N}} t_j^T(\theta^M, \cdots, \theta^M) < c(N)$, implying that the tight mechanism generates ex post budget deficit at $(\theta^M, \ldots, \theta^M)$.[19] ■

Step 2: Any mechanism (x, t) satisfying DSIC and EPIR also generates ex post budget deficit.

Proof. Recall that the tight mechanism (x, t^T) maximizes the ex post budget surplus among all mechanisms (x, t) satisfying DSIC and EPIR. Therefore, any mechanism (x, t) satisfying DSIC and EPIR also generates ex post budget deficit. ■

By Steps 1 and 2, we can conclude that any mechanism (x, t) satisfying DSIC and EPIR necessarily violates strong BB. That is, there are no mechanisms satisfying DSIC, EPIR, and strong BB in all nontrivial cases. This completes the proof of the theorem. ■

6.7 Discrete Approximation of the Optimal Mechanism in a Continuous Type Space

Vohra (2011, Section 6.2.7) provides a discrete approximation of the revenue maximizing mechanism satisfying BIC and IIR in a continuous type space in an auction setup. Here we adapt it to our public good environment.

[19] The tight mechanism also generates ex post budget deficit at $(\theta^M, \ldots, \theta^M)$ even if we weaken strong BB into our BB constraint. This is because our BB is equivalent to strong BB at the type profile $(\theta^M, \cdots, \theta^M)$, due to our richness condition.

Proposition 3. Suppose that each agent i is risk-neutral, i.e., his valuation for the provision decision is $v(q,\theta_i) = q\theta_i$ for each provision probability $q \in [0,1]$ and type $\theta_i \in \Theta$. Then, the ex ante budget-surplus maximizing mechanism satisfying BIC and IIR in a continuous type space $[1, M]$ where $M \in \mathbb{N}$ can be approximated by the one in a discrete type space $\{1, 2 \cdots, M\}$.

Proof. Let $x : \Theta^N \to [0,1]$ be an IBN decision rule. Denote by $\bar{x}_i(m)$ the interim expected probability that the public good is provided and by $\bar{t}_i(m)$ agent i's interim expected transfer when he announces type m and all the other agents announce their type truthfully, respectively. In a discrete type space, let t^* be a transfer rule such that the interim expected transfer for each agent $i \in \mathcal{N}$ and each type $m \in \{1, \cdots, M\}$ is

$$\bar{t}_i^*(m) = \sum_{l=1}^{m} l\left(\bar{x}_i(l) - \bar{x}_i(l-1)\right),$$

where type 0 is a dummy type such that $l \cdot \bar{x}_i(0) = 0$ for all $l \in \{1, \cdots, M\}$ and $i \in \mathcal{N}$. According to Theorem 1 of our paper, the mechanism (x, t^*) maximizes the ex ante budget surplus among all mechanisms satisfying BIC and IIR. Note that the interim expected transfer $\bar{t}_i^*(m)$ can be rewritten as follows:

$$\bar{t}_i^*(m) \;=\; m\bar{x}_i(m) - \sum_{l=1}^{m-1} \bar{x}_i(l). \tag{7}$$

On the other hand, if the type space is continuous, i.e., $\Theta = [1, M]$, then it is well-known that for any mechanism (x, t) satisfying BIC, the interim expected transfer for each agent $i \in \mathcal{N}$ and each type $m \in [1, M]$ must satisfy

$$\bar{t}_i(m) = m\bar{x}_i(m) - \int_1^m \bar{x}_i(l)dl - U_i(1), \tag{8}$$

where $U_i(1)$ is the interim expected utility for agent i of the lowest type 1 when all agents announce their type truthfully (See, for example, Vohra (2011, Section 6.2.7) and Mas-Colell, Whinston, and Green (1995, Proposition 23.D.2) for this.). After rearrangement, we can further transform (8) into the following:

$$U_i(m) = m\bar{x}_i(m) - \bar{t}_i(m) \;=\; \int_1^m \bar{x}_i(l)dl + U_i(1)$$

where $U_i(m)$ denotes the interim expected utility for agent i of type m when all agents announce their type truthfully. Recall that the outside option utility is always zero. We introduce the following lemma:

Lemma 11. Let $x : \Theta^N \to [0,1]$ be an IBN decision rule. Suppose that $t^c : \Theta^N \to \mathbb{R}^N$ is a transfer rule such that the mechanism (x, t^c) maximizes the ex ante budget surplus among all mechanisms satisfying BIC and IIR in the continuous type space

$[1, M]$. Then, the IIR constraint for each agent i with the lowest type $\theta_i = 1$ must be binding in the mechanism (x, t^c), i.e., for each agent $i \in \mathcal{N}$,

$$U_i^c(1) = 0,$$

where $U_i^c(1)$ denotes the expected utility of agent i with type 1 in the mechanism (x, t^c).

Proof. Since the interim expected transfer for agent i with type m, denoted by $\bar{t}_i(m)$, can be rewritten as $m\bar{x}_i(m) - U_i(m)$, then our task is reduced to solving the following optimization problem:

$$\begin{aligned} &\max_{\{U_i(m)\}_{i \in \mathcal{N}, m \in [1,M]}} \sum_{i=1}^{N} \int_1^M \left(m\bar{x}_i(m) - U_i(m)\right) f_i(m) dm \\ \text{s.t.} \quad &\text{(i)} \quad U_i(m) = \int_1^m \bar{x}_i(l) dl + U_i(1) \qquad \forall i \in \mathcal{N}, \forall m \in [1, M], \\ &\text{(ii)} \quad U_i(m) \geq 0 \qquad \forall i \in \mathcal{N}, \forall m \in [1, M], \end{aligned}$$

where $f_i(\cdot)$ is the probability density function of agent i's type. Constraints (i) and (ii) correspond to the BIC and IIR constraints, respectively.

Note first that if constraint (i) is satisfied, then constraint (ii) will be satisfied if and only if $U_i(1) \geq 0$ for each agent $i \in \mathcal{N}$. The logic is as follows. If constraint (ii) is satisfied, then in particular, $U_i(1) \geq 0$ must be satisfied for each agent $i \in \mathcal{N}$. On the other hand, if $U_i(1) \geq 0$ for each agent $i \in \mathcal{N}$, then it follows from constraint (i) that $U_i(m) \geq 0$ for each agent $i \in \mathcal{N}$ and each type $m \in [1, M]$, implying that constraint (ii) is satisfied.

As a result, we can replace constraint (ii) with

$$\text{(ii')} \quad U_i(1) \geq 0 \qquad \forall i \in \mathcal{N}.$$

Next, plugging $U_i(m)$ expressed in terms of constraint (i) into the objective function, we can rewrite the optimization problem as the one choosing the values $\{U_i(1)\}_{i \in \mathcal{N}}$ to maximize

$$\begin{aligned} &\sum_{i=1}^{N} \int_1^M \left(m\bar{x}_i(m) - \int_1^m \bar{x}_i(l) dl - U_i(1) \right) f_i(m) dm \\ = \; &\sum_{i=1}^{N} \int_1^M \left(m\bar{x}_i(m) - \int_1^m \bar{x}_i(l) dl \right) f_i(m) dm - \sum_{i=1}^{N} U_i(1), \end{aligned}$$

subject to constraint (ii'). It is evident that the solution must have $U_i(1) = 0$ for each agent $i \in \mathcal{N}$. ■

Substituting $U_i^c(1) = 0$ for (8), we obtain that for each agent $i \in \mathcal{N}$ and each type $m \in [1, M]$,

$$\bar{t}_i^c(m) = m\bar{x}_i(m) - \int_1^m \bar{x}_i(l)dl.$$

Clearly, this interim expected transfer can be approximated by that in (7) in the sense that the integral of expected allocations over a continuous, closed interval of types is approximated by the corresponding summation over a finite discretization of the continuous type space. ■

References

[1] Abreu, D. and H. Matsushima, "Virtual Implementation in Iteratively Undominated Strategies: Complete Information," *Econometrica*, vol. 60, (1992), 993-1008.

[2] d'Aspremont, C. and L.A. Gérard-Varet, "Incentives and Incomplete Information," *Journal of Public Economics*, vol. 11, (1979), 25-45.

[3] Börgers, T., *An Introduction to the Theory of Mechanism Design*, Oxford University Press, 2015.

[4] Carbajal, J.C. and R. Müller, "Implementability under monotonic transformations in differences," *Journal of Economic Theory*, vol. 160, (2015), 114-131.

[5] Clarke, E., "Multipart Pricing of Public Goods," *Public Choice*, vol. 2, (1971), 19-33.

[6] Dantzig, G.B., *Linear programming and extensions*, Princeton university press, 1998.

[7] Edelman, P.H. and J.A. Weymark, "Dominant strategy implementability and zero length cycles." *Economic Theory*, (2020), 1-30.

[8] Green, J. and J.J. Laffont, "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," *Econometrica*, vol. 45, (1977), 427-438.

[9] Groves, T., "Incentives in Teams," *Econometrica*, vol. 41, (1973), 617-631.

[10] Grüner, H.P. and Y. Koriyama,"Public goods, participation constraints, and democracy: A possibility theorem," *Games and Economic Behavior*, vol. 75, (2012), 152-167.

[11] Hellwig, M., "Public-Good Provision with Many Participants," *The Review of Economic Studies*, vol. 70, (2003), 589-614.

[12] Heydenreich, B., R. Müller, M. Uetz, and R.V. Vohra, “Characterization of revenue equivalence,” *Econometrica*, vol. 77(1), (2009), 307-316.

[13] Kos, N. and M. Manea, “Efficient Trade Mechanism with Discrete Values,” *working paper*, 2009.

[14] Krishna, V., *Auction theory, the second edition*, Academic press, 2009.

[15] Krishna, V. and M. Perry, “Efficient Mechanism Design,” *mimeo*, 2000.

[16] Kuzmics, C. and J-H Steg, “On Public Good Provision Mechanisms with Dominant strategies and Balanced Budget,” *Journal of Economic Theory*, vol. 170, (2017), 56-69.

[17] Lovejoy, W.S., “Optimal mechanisms with finite agent types,” *Management Science*, vol. 52, (2006), 788-803.

[18] Mailath, G.J. and A. Postlewaite, “Asymmetric Information Bargaining Problems with Many Agents,” *The Review of Economic Studies*, vol. 57, (1990), 351-367.

[19] Mas-Colell, A., M. Whinston, and J. Green, *Microeconomic Theory*, Oxford University Press, 1995.

[20] Saijo, T. and T. Yamato, “A voluntary participation game with a non-excludable public good,” *Journal of Economic Theory*, vol. 84, (1999), 227-242.

[21] Schweizer, U., “Universal possibility and impossibility results,” *Games and Economic Behavior*, vol. 57, (2006), 73-85.

[22] Segal, I. and M.D. Whinston, “A simple status quo that ensures participation (with application to efficient bargaining),” *Theoretical Economics*, vol. 6, (2011), 109-125.

[23] Serizawa, S., “Strategy-Proof and Symmetric Social Choice Functions for Public Good Economies,” *Econometrica*, vol. 67, (1999), 121-145.

[24] Vickrey, W., “Counterspeculation, Auctions, and Competitive Sealed Tenders,” *Journal of Finance*, vol. 16, (1961), 8-37.

[25] Vohra, R.V., *Advanced mathematical economics*, Routledge, 2005.

[26] Vohra, R.V., *Mechanism design: a linear programming approach*, Cambridge University Press, 2011.

[27] Williams, S., “A Characterization of Efficient, Bayesian Incentive Compatible Mechanisms,” *Economic Theory*, vol. 14, (1999), 155-180.

[28] Yenmez, M.B., "Incentive-compatible matching mechanisms: consistency with various stability notions," *American Economic Journal: Microeconomics*, vol. 5, (2013), 120-141.