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Diagnostic Tests for Homoskedasticity in Spatial Cross-Sectional or Panel Models

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Abstract

We propose tests for homoskedasticity in spatial econometric models, based on joint or concentrated score functions and an *Outer-Product-of-Martingale-Difference* (OPMD) estimate of the variance of the joint or concentrated score functions. Versions of these tests robust against non-normality are also given. Asymptotic properties of the proposed tests are formally examined using a cross-section model and a panel model with fixed effects. Monte Carlo results show that the proposed tests based on the concentrated score function have good finite sample properties. Finally, the generality of the proposed approach in constructing tests for homoskedasticity is further demonstrated using a spatial dynamic panel data model with short panels.

Keywords: Adjusted quasi-scores; Dynamics; Fixed effects; Heteroskedasticity; Non-normality; Martingale difference; Score tests; Short panels; Spatial effects.

JEL Classifications: C12, C18, C21, C23.

1. Introduction

The spatial dimension in panel data econometrics has attracted a lot of research recently, see the textbook chapter of Baltagi (2013) and the nice survey by Lee and Yu (2015). Many panels exhibit a spatial structure, which could be due to network issues, competition between cross-sectional units, spillover effects, *etc.* Spatial empirical illustrations in the recent literature include health care expenditures, house prices, convergence of EU economies, determinants of employment growth, car traffic, contagion problems to mention only a few.

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Generally, the existence of spatial effects could be related to spatial dependence, a specific case of cross-sectional dependence, or to spatial heterogeneity, which could be considered as a special case of cross-sectional heterogeneity. It is important to deal with this last point because cross-sectional units usually vary in size and as a result may exhibit heteroskedasticity.

In this context, the homoskedasticity assumption of the disturbances could often be restrictive in many empirical applications. Anselin (1988, pp. 119-120) places great emphasis on the link between spatial heterogeneity and heteroskedasticity underlying the consequences of the estimated spatial models for statistical validity (misleading significance levels, suboptimal forecasts,...). Moreover, for the cross-section spatial autoregressive model with spatial autoregressive errors (SARAR), Kelejian and Prucha (2010) noted that if the disturbances are heteroskedastic, the ML estimator considered in Lee (2004) is inconsistent, and the asymptotic distribution given in Kelejian and Prucha (1998) for the generalized spatial two-stage least squares (GS2SLS) estimator is not appropriate. Overall, this means that ignoring heteroskedastic disturbances can lead to misleading inference. To ensure proper statistical inference, it is now standard to use a Heteroskedasticity and Autocorrelation Consistent (HAC) procedure, based on the influential work of White (1980), and Newey and West (1987). For the cross-sectional spatial case, see Conley (1999), Kelejian and Prucha (2007), Kim and Sun (2011), and Driscoll and Kraay (1998), and Arellano (1987), Hansen (2007) and Vogelsang (2012) for panel data models, to mention a few. Furthermore, for cross-section spatial models, Kelejian and Prucha (2010), Lin and Lee (2010) and Debarsy, Lin and Lee (2015) developed generalized method of moments (GMM) estimators that are robust to heteroskedasticity. Nevertheless, when the heteroskedasticity is of an unknown form, these estimators are inefficient due to the fact that the best set of moment functions is generally not available. Moreover, the finite sample inference properties are largely unexplored, except by Kelejian and Prucha (2010). The articles mentioned above reflect that for many spatial models, especially those of panel data, a specific procedure taking into account heteroskedasticity is not necessarily available. Thus, the development of tests for heteroskedasticity is highly desirable, so that the 'source' of heteroskedasticity is identified and that the model estimation and inference proceed with heteroskedasticity being taken into account.

Mazodier and Trognon (1978) seem to be the first to deal with heteroskedasticity using panel data, see Baltagi (2013) for a textbook treatment of this subject. The homoskedasticity assumption of the disturbances could be tested in the spirit of Breusch and Pagan (1979), Verbon (1980), Randolph (1988), Holly and Gardiol (2000), Lejeune (2006), Baltagi, Bresson and Pirotte (2006), Baltagi, Song and Jung (2010) and Montes-Rojas and Sosa-Escudero (2011), to mention a few. More specifically, Baltagi, Song and Jung (2010) considered a panel data regression model with heteroskedastic as well as serially correlated disturbances, and derived joint and conditional LM tests, whereas Montes-Rojas and Sosa-Escudero (2011) derived tests for homoskedasticity in the error components model that possess robustness properties. This panel literature on heteroskedasticty testing ignores spatial correlation.[1]

This paper contributes to the spatial heteroskedasticity literature by first introducing general principles in constructing score-type tests for homoskedasticity in a general spatial econometric model where the disturbances of the model may exhibit potential heteroskedastic structure, and then giving detailed treatments on two popular spatial models for cross-section as well as panel data. The tests for homoskedasticity that we propose are based on joint or concentrated (quasi) score functions and *Outer-Product-of-Martingale-Difference* (OPMD) estimate of the variance of the joint or concentrated (quasi) score functions. Finite sample improved tests are also proposed and their performance is investigated using Monte Carlo simulations. The unobservable individual heterogeneity in panel data is captured using individual fixed effects rather than random effects. The generality of the proposed approach is further demonstrated using a spatial dynamic panel data model with small T.

The organization of the paper is as follows. In section 2, we introduce general principles in constructing score-type tests for homoskedasticity. Section 3 describes these tests as well as their improved versions to test for homoskedasticity in a spatial cross-sectional model. Section 4 describes these tests as well as their improved versions to test for homoskedasticity in a spatial panel data model with fixed effects. Section 5 focuses on the Monte Carlo experiments investigating the finite sample performance of the proposed tests. Section 6 discusses possible extensions of the proposed methods, in particular, a fixed effects spatial dynamic panel data model with small T. Section 7 concludes with suggestions for further work.

2. Tests for Homoskedasticity: General Principle

In this section, we outline the general principles leading to the development of score or Quasi-Score (QS) based tests for homoskedasticity in a general econometric model:

$$q(Y_n, X_n, W_{1n}, \ldots, W_{mn}; \beta, \lambda) = V_n, \tag{2.1}$$

where Y_n is an $n \times 1$ vector of observations on the dependent variable, X_n is an $n \times p$ matrix of observations on the p exogenous variables. W_{jn}, $j = 1, \ldots, m$, are the given $n \times n$ weighting matrices capturing the interactions among the n spatial units. (β, λ) are the common model parameters representing the covariate and spatial effects, respectively. V_n is an $n \times 1$ vector of independent disturbances which may exhibit unknown heteroskedasticity. In particular, the elements $\{v_{ni}\}$ of V_n have zero mean but heteroskedastic variances $\sigma^2 h(z'_{ni}\alpha)$ with the $k \times 1$ vectors z_{ni} and α being, respectively, the heteroskedasticity variables and the

[1] In contrast to the tests for heteroskedasticity, a huge literature exists on testing for spatial autocorrelation, see Anselin (1988) and Elhorst (2014) for a textbook treatment of this subject. See also, Baltagi, Song and Koh (2003), Baltagi, Song, Jung and Koh (2007), Baltagi, Song and Kwon (2009), Debarsy and Ertur (2010), Yang (2010), Baltagi and Yang (2013a,b). More specifically, Baltagi, Song and Kwon (2009) derived LM tests in the context of a one-way error components random effects model with heteroskedastic random individual effects, and spatial errors; Debarsy and Ertur (2010), and Baltagi and Yang (2013b) derived LM tests for spatial dependence in fixed effects panel regressions that are robust against heteroskedasticity and non-normality.

heteroskedasticity parameters. The heteroskedasticity function $h(\cdot)$ is an unknown smooth function such that $h(0) = 1$. Thus, when $\alpha = 0$, the model becomes homoskedastic. A test for homoskedasticity against heteroskedasticity becomes a test of:

$$H_0 : \alpha = 0 \text{ vs. } H_a : \alpha \neq 0. \tag{2.2}$$

See Breusch and Pagan (1979) for the original idea behind this type of tests; Koenker (1981), Godfrey et al. (2006), and Yang and Tse (2008), among others, for some further developments; and Anselin (1988, p. 121) for this type of tests in a spatial framework. The variables in z_{ni} may contain some elements of the x_{ni}, the ith value of the set of regressors. In spatial models, z_{ni} may contain variables that relate to the spatial weight matrices, e.g., the number of non-zero elements in each row of W_{1n} (number of neighbors), *etc.* This makes the test of H_0 in the context of spatial models more appealing. In certain spatial models such as models with large group interaction (Lee, 2004), the elements of W_n depend on n and hence the values z_{ni} of the heteroskedasticity variables may also depend on n. The values of the exogenous variables x_{ni} are allowed to be n-dependent as well, because the models to be discussed are allowed to contain spatial Durbin effects (Anselin, 1988, p. 40).

We will develop score-type tests. The score-type tests require only the estimation of the model under the null, which makes the tests usually simpler to derive. In the current set-up, we only need to know that the heteroskedasticity is of the general form $h(z'\alpha)$ but the function $h(\cdot)$ remains unknown. Denote $\theta = (\beta', \sigma^2, \lambda')'$ and $\psi = (\theta', \alpha')'$. Let ψ_0 be the true value of ψ. The usual expectation operator 'E' and variance operator 'Var' correspond to the true parameter values. Let $S_n(\psi)$ be the score or QS function of ψ, and $S_{n,\theta}(\psi)$ and $S_{n,\alpha}(\psi)$ be its components corresponding to θ and α, respectively.

Score Test. The central idea that we follow in developing tests of homoskedasticity is to find ways to decompose $S_n(\psi_0)$ into a sum of Martingale Differences (MD), i.e.,

$$S_n(\psi_0) = \sum_{i=1}^n \mathbf{g}_{ni}(\psi_0), \tag{2.3}$$

where $\{\mathbf{g}_{ni}(\psi_0)\}$ form a vector MD sequence with respect to $\mathcal{F}_{ni}$: the increasing σ-field generated by $\{v_{n1}, \ldots, v_{ni}\}$. Then, $\text{Var}[S_n(\psi_0)] = \sum_{i=1}^n \text{E}[\mathbf{g}_{ni}(\psi_0)\mathbf{g}'_{ni}(\psi_0)]$. It follows that the average of the estimated OPMD, i.e.,

$$\tfrac{1}{n} \sum_{i=1}^n \tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}'_{ni}, \tag{2.4}$$

gives a consistent estimate of $\text{Var}[\frac{1}{\sqrt{n}} S_n(\psi_0)]$, where $\tilde{\mathbf{g}}_{ni} \equiv \mathbf{g}_{ni}(\tilde{\psi}_n)$, and $\tilde{\psi}_n$ is the restricted Maximum Likelihood (ML) or Quasi ML (QML) estimator under some general linear or non-linear constraints on ψ, including the constraints imposed by H_0 in (2.2).

When $S_n(\psi_0)$ is indeed the score vector, i.e., when the error distribution is correctly specified, an OPMD form of score test for a linear or non-linear constraint on ψ is as follows:

$$T_n = (\textstyle\sum_{i=1}^n \tilde{\mathbf{g}}'_{ni})(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}'_{ni})^{-1}(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni}). \tag{2.5}$$

In a special case where the elements of Y_n are independent, so that $S_n(\psi_0)$ is automatically the sum of individual scores or gradients, the test reduces to the well known *Outer-Product-of-Gradients* (OPG) test (see, e.g., Wooldridge, 2010). Clearly, this is not the case for the type of models we consider here, and hence an MD representation for $S_n(\psi_0)$ is needed.

In the special case of testing for homoskedasticity in (2.1), $\tilde{\psi}_n = (\tilde{\theta}_n', 0_k')'$ and $S_{n,\theta}(\tilde{\psi}_n) = 0$, where $\tilde{\theta}_n$ is the ML or QML estimator of θ of the homoskedastic model and 0_k is a $k \times 1$ vector of zeros. Partitioning $\mathbf{g}_{ni}(\psi_0) = (\mathbf{g}_{ni,\theta}'(\psi_0), \mathbf{g}_{ni,\alpha}'(\psi_0))'$ according to θ and α, the OPMD form of the score test for homoskedasticity given in (2.5) reduces to:

$$T_{\mathtt{S}} = \big(\textstyle\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}'\big)\big[\big(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}_{ni}'\big)^{-1}\big]_{\alpha\alpha}\big(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\big), \tag{2.6}$$

where $[\,\cdot\,]_{\alpha\alpha}$ denotes the α-α block of the corresponding matrix. In this case,

$$\begin{aligned} \big[\big(\textstyle\sum_{i=1}^n \mathbf{g}_{ni}\mathbf{g}_{ni}'\big)^{-1}\big]_{\alpha\alpha} &= \big[\textstyle\sum_{i=1}^n \mathbf{g}_{ni,\alpha}\mathbf{g}_{ni,\alpha}' - K_n\big(\sum_{i=1}^n \mathbf{g}_{ni,\theta}\mathbf{g}_{ni,\alpha}'\big)\big]^{-1}, \\ &= \big[\textstyle\sum_{i=1}^n (\mathbf{g}_{ni,\alpha} - K_n\mathbf{g}_{ni,\theta})\big(\mathbf{g}_{ni,\alpha} - K_n\mathbf{g}_{ni,\theta}\big)'\big]^{-1}, \end{aligned} \tag{2.7}$$

where $K_n = \big(\sum_{i=1}^n \mathbf{g}_{ni,\alpha}\mathbf{g}_{ni,\theta}'\big)\big(\sum_{i=1}^n \mathbf{g}_{ni,\theta}\mathbf{g}_{ni,\theta}'\big)^{-1}$.[2] The validity of this full OPMD form of the score test lies with the Information Matrix Equality (IME); see, e.g., Cameron and Trivedi (2005) and Wooldridge (2010). The following discussion offers some simpler and clearer explanation for the role played by the IME, in the context of tests for homoskedasticity.

Quasi-Score Test. If the error distribution is misspecified, $S_n(\psi)$ is no longer the true score function. While the generalized IME, $\mathrm{E}[-\frac{\partial}{\partial\psi'}S_n(\psi_0)] = \mathrm{E}[S_n(\psi_0)S_n^{\diamond\prime}(\psi_0)]$, holds as long as $\mathrm{E}[S_n(\psi_0)] = 0$, it cannot be used since the true score function $S_n^{\diamond}(\psi_0)$ is unknown (see, e.g., Cameron and Trivedi, 2005). Note that the QS test of H_0 depends on the subvector $S_{n,\alpha}(\tilde{\theta}_n, 0_k)$ and its variance. To facilitate the discussion on constructing the QS test and for the proof of its asymptotic property, let $\Sigma_n \equiv \Sigma_n(\theta_0) = -E[\frac{\partial}{\partial\psi'}S_n(\psi_0)|_{H_0}]$ and $\Omega_n \equiv \Omega_n(\theta_0) = \mathrm{Var}[S_n(\psi_0)|_{H_0}]$, both partitioned according to θ and α with the distinct submatrices of Σ_n denoted by $\Sigma_{n,\theta\theta}$, $\Sigma_{n,\alpha\theta}$, and $\Sigma_{n,\alpha\alpha}$, and those of Ω_n by $\Omega_{n,\theta\theta}$, $\Omega_{n,\alpha\theta}$, and $\Omega_{n,\alpha\alpha}$. Under mild conditions, Taylor expansions lead to an asymptotic MD representation:

$$\frac{1}{\sqrt{n}}S_{n,\alpha}(\tilde{\theta}_n, 0_k) = \frac{1}{\sqrt{n}}S_{n,\alpha}(\theta_0, 0_k) - \frac{1}{\sqrt{n}}\Sigma_{n,\alpha\theta}\Sigma_{n,\theta\theta}^{-1}S_{n,\theta}(\theta_0, 0_k) + o_p(1), \tag{2.8}$$

$$= \frac{1}{\sqrt{n}}\sum_{i=1}^n(\mathbf{g}_{ni,\alpha} - \Gamma_n\mathbf{g}_{ni,\theta}) + o_p(1), \tag{2.9}$$

where $\Gamma_n = \Sigma_{n,\alpha\theta}\Sigma_{n,\theta\theta}^{-1}$ and $\{\mathbf{g}_{ni,\alpha} - \Gamma_n\mathbf{g}_{ni,\theta}\}$ form a vector MD sequence with respect to $\mathcal{F}_{n,i}$. It follows that $\mathrm{Var}[\frac{1}{\sqrt{n}}S_{n,\alpha}(\tilde{\theta}_n, 0_k)] = \frac{1}{n}\sum_{i=1}^n \mathrm{E}[(\mathbf{g}_{ni,\alpha} - \Gamma_n\mathbf{g}_{ni,\theta})(\mathbf{g}_{ni,\alpha} - \Gamma_n\mathbf{g}_{ni,\theta})'] + o(1)$, leading immediately to an OPMD estimator of $\mathrm{Var}[\frac{1}{\sqrt{n}}S_{n,\alpha}(\tilde{\theta}_n, 0_k)]$, and an OPMD form of the QS test for homoskedasticity robust against non-normality:

$$T_{\mathtt{QS}} = S_{n,\alpha}'(\tilde{\theta}_n, 0_k)\big\{\textstyle\sum_{i=1}^n(\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{\Gamma}_n\tilde{\mathbf{g}}_{ni,\theta})(\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{\Gamma}_n\tilde{\mathbf{g}}_{ni,\theta})'\big\}^{-1}S_{n,\alpha}(\tilde{\theta}_n, 0_k), \tag{2.10}$$

where $\tilde{\Gamma}_n = \tilde{\Sigma}_{n,\alpha\theta}\tilde{\Sigma}_{n,\theta\theta}^{-1}$, with $\tilde{\Sigma}_{n,\alpha\theta}$ and $\tilde{\Sigma}_{n,\theta\theta}$ being either the plug-in estimates of $\Sigma_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta}$, or simply $-\frac{\partial}{\partial\theta'}S_{n,\alpha}(\tilde{\theta}_n, 0_k)$ and $-\frac{\partial}{\partial\theta'}S_{n,\theta}(\tilde{\theta}_n, 0_k)$. When the error distribution

[2] We prefer to use the term *score test* instead of *LM test* as the more general and robust tests are developed through adjusting the concentrated scores.

is correctly specified, we have $\Sigma_{n,\alpha\theta} = \Omega_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta} = \Omega_{n,\theta\theta}$ by IME. Hence, $\tilde{\Sigma}_{n,\alpha\theta} = \sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\tilde{\mathbf{g}}'_{ni,\theta}$ and $\tilde{\Sigma}_{n,\theta\theta} = \sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\theta}\tilde{\mathbf{g}}'_{ni,\theta}$ can be used, and $T_{\mathtt{QS}}$ reduces to $T_{\mathtt{S}}$, justifying the asymptotic validity of the test in (2.6).

Adjusted Score or Quasi-Score Tests. The score test given in (2.6) and the QS test given in (2.10) may not have satisfactory finite sample properties when there are many *nuisance* parameters. However, if θ contains many linear and scale parameters, of which the constrained estimates given the 'other' parameters possess analytical expressions, then one can concentrate the joint score $S_n(\psi)$, and then *recenter* the numerator of the concentrated score to give a set of unbiased estimation functions for the 'other' parameters (see, e.g., Baltagi and Yang 2013*a*). To fix ideas, let $\theta = (\delta', \lambda')'$, where given (λ, α), the restricted ML estimator of δ has an analytical expression $\tilde{\delta}_n(\lambda, \alpha)$. Let $S_n(\psi) = (S'_{n,\delta}(\psi), S'_{n,\lambda}(\psi), S'_{n,\alpha}(\psi))'$. Then the concentrated score function of (λ, α) has components $S_{n,\lambda}(\tilde{\delta}(\lambda, \alpha), \lambda, \alpha)$ and $S_{n,\alpha}(\tilde{\delta}(\lambda, \alpha), \lambda, \alpha)$ corresponding to λ and α, respectively. Denote the unbiased estimating function obtained through recentering the numerators of the concentrated score functions by $S_n^*(\lambda, \alpha)$, and its components by $S_{n,\lambda}^*(\lambda, \alpha)$ and $S_{n,\alpha}^*(\lambda, \alpha)$. Define the adjusted estimator of λ_0 under H_0 as

$$\tilde{\lambda}_n^* = \arg\{S_{n,\lambda}^*(\lambda, 0_k) = 0\}.$$

Then, the *Adjusted Score* (AS) test or the *Adjusted Quasi-Score* (AQS) test is constructed based on $S_{n,\alpha}^*(\tilde{\lambda}_n^*, 0)$. Assume $S_n^*(\lambda_0, 0)$ possess an MD representation, i.e, $S_n^*(\lambda_0, 0) = \sum_{i=1}^n \mathbf{g}_{ni}^*$, where $\{\mathbf{g}_{ni}^*\}$ form a vector MD sequence with respect to $\mathcal{F}_{ni}$. Similar to the case of QS test, let $\Sigma_n^* \equiv \Sigma_n^*(\lambda_0) = -E[\frac{\partial}{\partial(\lambda', \alpha')} S_n^*(\lambda_0, \alpha)|_{H_0}]$, partitioned according to λ and α with the distinct submatrices of Σ_n^* denoted by $\Sigma_{n,\lambda\lambda}^*$, $\Sigma_{n,\alpha\lambda}^*$, and $\Sigma_{n,\alpha\alpha}^*$. It is easy to show that $S_{n,\alpha}^*(\tilde{\lambda}_n^*, 0)$ possesses the following asymptotic MD representation:

$$\tfrac{1}{\sqrt{n}} S_{n,\alpha}^*(\tilde{\lambda}_n^*, 0) = \tfrac{1}{\sqrt{n}} \textstyle\sum_{i=1}^n (\mathbf{g}_{ni,\alpha}^* - \Gamma_n^* \mathbf{g}_{ni,\lambda}^*) + o_p(1), \tag{2.11}$$

where $\Gamma_n^* = \Sigma_{n,\alpha\lambda}^* \Sigma_{n,\lambda\lambda}^{*-1}$ and $\mathbf{g}_{ni,\lambda}^*$ and $\mathbf{g}_{ni,\alpha}^*$ are, respectively, the λ- and α-component of $\mathbf{g}_{ni}^*$. Thus, $\frac{1}{n}\mathrm{Var}[S_{n,\alpha}^*(\tilde{\lambda}_n^*, 0)] = \frac{1}{n}\sum_{i=1}^n (\mathbf{g}_{ni,\alpha}^* - \Gamma_n^* \mathbf{g}_{ni,\lambda}^*)(\mathbf{g}_{ni,\alpha}^* - \Gamma_n^* \mathbf{g}_{ni,\lambda}^*)' + o_p(1)$. It follows that an OPMD form of the AQS test for homoskedasticity is:

$$T_{\mathtt{QS}}^* = S_{n,\alpha}^{*\prime}(\tilde{\lambda}_n^*, 0)\big[\textstyle\sum_{i=1}^n (\tilde{\mathbf{g}}_{ni,\alpha}^* - \tilde{\Gamma}_n^* \tilde{\mathbf{g}}_{ni,\lambda}^*)(\tilde{\mathbf{g}}_{ni,\alpha}^* - \tilde{\Gamma}_n^* \tilde{\mathbf{g}}_{ni,\lambda}^*)'\big]^{-1} S_{n,\alpha}^*(\tilde{\lambda}_n^*, 0), \tag{2.12}$$

where $\tilde{\mathbf{g}}_{ni,\lambda}^*$ and $\tilde{\mathbf{g}}_{ni,\alpha}^*$ are the plug-in estimates of $\mathbf{g}_{ni,\lambda}^*$ and $\mathbf{g}_{ni,\alpha}^*$ at H_0, and $\tilde{\Gamma}_n^* = \tilde{\Sigma}_{n,\alpha\lambda}^* \tilde{\Sigma}_{n,\lambda\lambda}^{*-1}$.

There are in general two choices for $\tilde{\Gamma}_n^*$: (*i*) the *plug-in* estimate, $\Sigma_{n,\alpha\lambda}^*(\tilde{\lambda}_n^*)\Sigma_{n,\lambda\lambda}^{*-1}(\tilde{\lambda}_n^*)$, and (*ii*) the *Hessian* estimate, $[\frac{\partial}{\partial\lambda'} S_{n,\alpha}^*(\tilde{\lambda}_n^*, 0_k)][\frac{\partial}{\partial\lambda'} S_{n,\lambda}^*(\tilde{\lambda}_n^*, 0_k)]^{-1}$. The tests based on these estimates are robust against misspecification of the error distribution, and are referred to as the AQS tests. If the error distribution is correctly specified, we have a third choice: $\tilde{\Gamma}_n^* = [\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}^* \tilde{\mathbf{g}}_{ni,\lambda}^{*\prime}][\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\lambda}^* \tilde{\mathbf{g}}_{ni,\lambda}^{*\prime}]^{-1}$, leading to a test that is referred to as the AS test, and is denoted by $T_{\mathtt{S}}^*$ for easy reference. This is justified by an IME with respect to the underlining 'adjusted likelihood' that generates the AS function $S_n^*(\lambda, \alpha)$.[3]

[3] Alternatively, the OPMD estimate, $\tilde{\Gamma}_n^* = [\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}^* \tilde{\mathbf{g}}'_{ni,\lambda}][\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\lambda}^* \tilde{\mathbf{g}}'_{ni,\lambda}]^{-1}$, can also be used where

The general ideas and principles outlined above are seen to be very simple, and yet quite general. They are not restricted to the spatial econometric models, and the ideas behind the score-, AS-, QS-, and AQS-test can readily be generalized to the case of a general *estimating function* (EF) in the context of M-estimation or GMM to give an OPMD-form of M-tests (Cameron and Trivedi, 2005, Wooldridge, 2010). The key is the development of an MD representation for the EF. One advantage of such an MD representation is that it avoids the analytical expression of the variance of the EF, which typically involves higher-order moments making the estimation unstable numerically, or may be difficult to obtain in certain complicated econometrics models, or may be infeasible in spatial dynamic panel data models with short panels due to the unobserved past history of the process (see Su and Yang, 2015). Furthermore, in certain models such as spatial dynamic panel data models, such a plug-in method may not be applicable. In the subsequent two sections, we use a spatial cross section model, and a spatial panel data model to demonstrate these ideas and principles in details. In Section 6, we discuss possible extensions of the methodology to, e.g., higher-order spatial models and spatial dynamic panel data models.

3. Tests for Homoskedasticity for the SAC Model

In this section, we demonstrate the general ideas and principles outlined in Section 2 using the spatial autoregressive model with spatial autoregressive errors (SARAR), also termed Spatial Autoregressive Combined model, or SAC model (see LeSage and Pace, 2009; Vega and Elhorst, 2015), for easy reference. Four tests for homoskedasticity in the SAC model are introduced, and formal asymptotic theory for the proposed tests are presented. The proofs are relegated to Appendix B. The simplest heteroskedastic SAC model takes the form:

$$Y_n = \lambda_1 W_{1n} Y_n + X_n\beta + U_n, \;\; U_n = \lambda_2 W_{2n} U_n + V_n, \tag{3.1}$$

where all the quantities are defined in Model (2.1). It is easy to see that Model (3.1) has the reduced form: $B_{2n}(\lambda_2)[B_{1n}(\lambda_1)Y_n - X_n\beta] = V_n$ where $B_{rn}(\lambda_r) = I_n - \lambda_r W_{rn}, r = 1, 2$. This is a special case of Model (2.1).[4] The spatial weights matrices W_{1n} and W_{2n} are assumed to be exogenously given with zero diagonal elements. The null hypothesis is given in (2.2).

3.1. ML or QML Estimation of the Cross-Sectional SAC Model

We now outline the ML or QML estimation procedure for the SAC model and the asymptotic properties of the estimates at the null as they are essential for the study of the asymptotic

$\tilde{\mathbf{g}}_{ni,\lambda}$ is the λ-component of $\tilde{\mathbf{g}}_{ni}$ defined in the score test corresponding to the original joint score function. This is because when the error distribution is correctly specified, the generalized IME can be used, i.e., $\Sigma^*_{n,\lambda\lambda} = \mathrm{E}[S^*_{n,\lambda}(\lambda_0, 0_k)S^\circ_{n,\lambda}(\theta_0)]$ and $\Sigma^*_{n,\alpha\lambda} = \mathrm{E}[S^*_{n,\alpha}(\lambda_0, 0_k)S^\circ_{n,\lambda}(\theta_0)]$. See the following sections for details.

[4]Model (3.1) can be extended by adding higher-order spatial lags in Y_n and U_n, and/or by adding a spatial Durbin term, $W_{3n}X_{1n}\gamma$, where X_{1n} is a submatrix of X_n. The former extension incurs some extra algebra in the subsequent developments, but the latter extension does not. See, e.g., Elhorst (2014) and Lee and Yu (2016) for discussions on spatial Durbin models and the associated issue of parameter identification.

properties of the tests to be introduced later. Let $\mathcal{H}_n(\alpha) = \texttt{diag}(\{h(z'_{ni}\alpha)\})$, where $\texttt{diag}(\cdot)$ forms a diagonal matrix based on the given elements or a given vector. The full Gaussian loglikelihood function for $\psi = (\beta', \sigma^2, \lambda', \alpha')'$ is given by:

$$\begin{aligned} \ell_{\texttt{SAC}}(\psi) = \;& -\tfrac{n}{2}\log(2\pi\sigma^2) + \log|B_{1n}(\lambda_1)| + \log|B_{2n}(\lambda_2)| - \tfrac{1}{2}\log|\mathcal{H}_n(\alpha)| \\ & -\tfrac{1}{2\sigma^2}V'_n(\beta,\lambda)\mathcal{H}_n^{-1}(\alpha)V_n(\beta,\lambda), \end{aligned} \tag{3.2}$$

where $V_n(\beta,\lambda) = Y_n(\lambda) - X_n(\lambda_2)\beta$, $Y_n(\lambda) = B_{2n}(\lambda_2)B_{1n}(\lambda_1)Y_n$, $X_n(\lambda_2) = B_{2n}(\lambda_2)X_n$, and $\lambda = (\lambda_1, \lambda_2)'$. Maximizing $\ell_{\texttt{SAC}}(\psi)$ gives the ML Estimate (MLE) or QML Estimate (QMLE) of ψ of the full model. Maximizing $\ell_{\texttt{SAC}}(\psi)$ at the null, $\ell_{\texttt{SAC}}(\psi)|_{H_0}$, gives the MLE or QMLE for the null model. Given λ, $\ell_{\texttt{SAC}}(\psi)|_{H_0}$, is partially maximized at:

$$\tilde\beta_n(\lambda) = [X'_n(\lambda_2)X_n(\lambda_2)]^{-1}X'_n(\lambda_2)Y_n(\lambda) \text{ and } \tilde\sigma^2_n(\lambda) = \tfrac{1}{n}Y'_n(\lambda)M_n(\lambda_2)Y_n(\lambda), \tag{3.3}$$

where $M_n(\lambda_2) = I_n - X_n(\lambda_2)[X'_n(\lambda_2)X_n(\lambda_2)]^{-1}X'_n(\lambda_2)$. The concentrated null loglikelihood function for λ is obtained by substituting $\tilde\beta_n(\lambda)$ and $\tilde\sigma^2_n(\lambda)$ into $\ell_{\texttt{SAC}}(\psi)|_{H_0}$:

$$\ell^c_{\texttt{SAC}}(\lambda)|_{H_0} = -\tfrac{n}{2}[\log(2\pi)+1] - \tfrac{n}{2}\log(\tilde\sigma^2_n(\lambda)) + \log|B_{1n}(\lambda_1)| + \log|B_{2n}(\lambda_2)|. \tag{3.4}$$

Maximizing $\ell^c_{\texttt{SAC}}(\lambda)|_{H_0}$ leads to the null MLE or QMLE $\tilde\lambda_n$ of λ.[5] Upon substitution, the null (Q)MLEs of β and σ^2 are denoted as $\tilde\beta_n \equiv \tilde\beta_n(\tilde\lambda_n)$ and $\tilde\sigma^2_n \equiv \tilde\sigma^2_n(\tilde\lambda_n)$. The (Q)MLE of $\theta_0 = (\beta'_0, \sigma^2_0, \lambda'_0)'$ for the null model is thus $\tilde\theta_n = (\tilde\beta_n, \tilde\sigma^2_n, \tilde\lambda'_n)'$.

Jin and Lee (2013) show that under some regularity conditions, $\sqrt{n}(\tilde\theta_n - \theta_0)$ is asymptotically normal with mean 0 and VC matrix $n\Sigma^{-1}_{n,\theta\theta}\Omega_{n,\theta\theta}\Sigma^{-1}_{n,\theta\theta}$, where $\Sigma_{n,\theta\theta}$ and $\Omega_{n,\theta\theta}$ are, respectively, the expected negative Hessian and the variance of the score of the null model. Note that we will use the same notation as in Section 2 (Σ_n, Ω_n and Σ^*_n, and their submatrices including $\Sigma_{n,\theta\theta}$ and $\Omega_{n,\theta\theta}$) in subsequent developments.

3.2. Score or Quasi-Score Tests

The (quasi) score function $S_{\texttt{SAC}}(\psi) = \frac{\partial}{\partial\psi}\ell_{\texttt{SAC}}(\psi)$ has components for $\beta, \sigma^2, \lambda_1, \lambda_2$, and α:

$$S_{\texttt{SAC}}(\psi) = \begin{cases} \frac{1}{\sigma^2}X'_n(\lambda_2)\mathcal{H}_n^{-1}(\alpha)V_n(\beta,\lambda), \\ \frac{1}{2\sigma^4}V'_n(\beta,\lambda)\mathcal{H}_n^{-1}(\alpha)V_n(\beta,\lambda) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2}V'_n(\beta,\lambda)\mathcal{H}_n^{-1}(\alpha)B_{2n}(\lambda_2)W_{1n}Y_n - \text{tr}[G_{1n}(\lambda_1)], \\ \frac{1}{\sigma^2}V'_n(\beta,\lambda)\mathcal{H}_n^{-1}(\alpha)G_{2n}(\lambda_2)V_n(\beta,\lambda) - \text{tr}[G_{2n}(\lambda_2)], \\ \frac{1}{2\sigma^2}\dot h(z'_{ni}\alpha)\sum_{i=1}^n\left[\left(\frac{v^2_{ni}(\beta,\lambda)}{h(z'_{ni}\alpha)} - \sigma^2\right)\frac{z_{ni}}{h(z'_{ni}\alpha)}\right], \end{cases} \tag{3.5}$$

where $G_{rn}(\lambda_r) = W_{rn}B^{-1}_{rn}(\lambda_r)$, $r = 1,2$, and $\dot h(x) = \frac{d}{dx}h(x)$.

Under $H_0 : \alpha = 0$, $h(0) = 1$ and $\dot h(0)$ becomes a constant, independent of i. The score

[5] When n is large, the computational burden in maximizing $\ell^c_{\texttt{SAC}}(\lambda)|_{H_0}$ can be alleviated by the identity: $|I_n - \lambda W_{rn}| = \prod_{i=1}^n(1-\lambda_r\omega_{ri})$, where ω_{ri} are the eigenvalues of W_{rn}, $r = 1,2$, which need not be updated in each iteration of the numerical maximization process; see Griffith (1988).

function at the null, $S^{\circ}_{\texttt{SAC}}(\theta) = S_{\texttt{SAC}}(\psi)|_{H_0}$, simplifies to

$$S^{\circ}_{\texttt{SAC}}(\theta) = \begin{cases} \frac{1}{\sigma^2} X'_n(\lambda_2) V_n(\beta, \lambda), \\ \frac{1}{2\sigma^4} V'_n(\beta, \lambda) V_n(\beta, \lambda) - \frac{n}{2\sigma^2}, \\ \frac{1}{\sigma^2} V'_n(\beta, \lambda) B_{2n}(\lambda_2) W_{1n} Y_n - \text{tr}[G_{1n}(\lambda_1)], \\ \frac{1}{\sigma^2} V'_n(\beta, \lambda) G_{2n}(\lambda_2) V_n(\beta, \lambda) - \text{tr}[G_{2n}(\lambda_2)], \\ \frac{1}{2\sigma^2} \dot{h}(0) \sum_{i=1}^n \left[\left(v^2_{ni}(\beta, \lambda) - \sigma^2 \right) z_{ni} \right]. \end{cases} \tag{3.6}$$

To derive the score or QS tests for homoskedasticity in the SAC model, we first develop an MD representation for $S^{\circ}_{\texttt{SAC}}(\theta_0)$. For ease of exposition, we drop the arguments of a quantity evaluated at the true parameter values, e.g., $V_n = V_n(\beta_0, \lambda_0)$, $B_{rn} = B_{rn}(\lambda_{r0})$, $G_{rn} = G_{rn}(\lambda_{r0})$, *etc.* We have, for the two key quantities in (3.6),

$$\begin{aligned} V'_n(\beta_0, \lambda_0) B_{2n}(\lambda_{20}) W_{1n} Y_n &= V'_n B_{2n} G_{1n} B^{-1}_{2n} V_n + V'_n B_{2n} G_{1n} B^{-1}_{2n} X_n(\lambda_{20}) \beta_0, \\ V'_n(\beta_0, \lambda_0) G_{2n}(\lambda_{20}) V_n(\beta_0, \lambda_0) &= V'_n G_{2n} V_n, \end{aligned}$$

noting that under H_0 the elements of V_n are $iid(0, \sigma^2_0)$. Using these results, the score vector $S^{\circ}_{\texttt{SAC}}(\theta_0)$ under the true parameter values is further simplified into the following general form:

$$S^{\circ}_{\texttt{SAC}}(\theta_0) = \begin{cases} \Pi'_1 V_n, \\ V'_n \Phi_1 V_n - \text{E}(V'_n \Phi_1 V_n), \\ V'_n \Phi_2 V_n - \text{E}(V'_n \Phi_2 V_n) + V'_n \Pi_2, \\ V'_n \Phi_3 V_n - \text{E}(V'_n \Phi_3 V_n), \\ \frac{1}{2\sigma^2_0} \dot{h}(0) \sum_{i=1}^n \left[\left(v^2_{ni} - \sigma^2_0 \right) z_{ni} \right], \end{cases} \tag{3.7}$$

where $\Pi_1 = \frac{1}{\sigma^2_0} X_n(\lambda_{20})$, and $\Pi_2 = \frac{1}{\sigma^2_0} B_{2n} G_{1n} B^{-1}_{2n} X_n(\lambda_{20}) \beta_0$; $\Phi_1 = \frac{1}{2\sigma^4_0} I_n$, $\Phi_2 = \frac{1}{\sigma^2_0} B_{2n} G_{1n} B^{-1}_{2n}$, and $\Phi_3 = \frac{1}{\sigma^2_0} G_{2n}$; and 'E' corresponds to the null model.

Using the Central Limit Theorem (CLT) for Linear-Quadratic (LQ) forms of Kelejian and Prucha (2001) or its simpler version under homoskedasticity stated in Lemma A.5 of this paper for easy reference, one can easily prove that $\frac{1}{\sqrt{n}} S^{\circ}_{\texttt{SAC}}(\theta_0) \xrightarrow{D} N(0, \lim_{n \to \infty} \frac{1}{n} \Omega_n)$, where $\Omega_n = \text{Var}[S^{\circ}_{\texttt{SAC}}(\theta_0)]$. See the proof of Theorem 3.1 in Appendix B. A crucial step in constructing a score test is to find a consistent estimate of Ω_n. A popular way is to find the analytical expression of Ω_n and then plug-in the null estimates. As we allow for non-normality of the errors, such an expression would involve skewness and kurtosis of the errors which need to be estimated. In a more complicated model, such as the panel SAC model to be considered in the next section, estimation of these quantities may not be trivial, not to mention the issue of the numerical stability in estimating higher-order moments. Furthermore, in certain models such as spatial dynamic panel data models, such a plug-in method may not be applicable due to the unobserved past history of the process (see Su and Yang, 2015). The proposed method, however, does not require the explicit expression of the variance of the score. Instead, it decomposes the score at the null into sums of MD sequences, so that an averaged OPMD gives a consistent estimate. This method is fairly general and the resulting estimate of Ω_n is automatically robust against non-normality. The details are as follows:

For a general n-dimensional square matrix Φ_n, denote its upper, lower, and diagonal matrices by Φ_n^u, Φ_n^l and Φ_n^d, respectively, such that $\Phi_n = \Phi_n^u + \Phi_n^l + \Phi_n^d$. This gives $V_n'\Phi_n V_n = V_n'(\Phi_n^u + \Phi_n^l + \Phi_n^d)V_n = V_n'(\Phi_n^{u\prime} + \Phi_n^l + \Phi_n^d)V_n = V_n'\xi_n + V_n'\Phi_n^d V_n$, where $\xi_n = (\Phi_n^{u\prime} + \Phi_n^l)V_n$. As $V_n'\Phi_n^u V_n$ is a scalar, we have $V_n'\Phi_n^u V_n = V_n'\Phi_n^{u\prime} V_n$. It follows that,

$$V_n'\Phi_n V_n - \mathrm{E}(V_n'\Phi_n V_n) = \textstyle\sum_{i=1}^n \left[v_{ni}\xi_{ni} + (v_{ni}^2 - \sigma_0^2)\phi_{n,ii}\right] \equiv \sum_{i=1}^n g_{ni}(\theta_0), \tag{3.8}$$

where $\{\phi_{n,ii}\}$ are the diagonal elements of Φ_n. Noting that the elements v_{ni} are *iid* under H_0, thus $\{g_{ni}(\theta_0)\}$ form an MD sequence with respect to the increasing sequence of σ-fields $\{\mathcal{F}_{in}\}$ generated by $(v_{n1}, \ldots, v_{ni})$. See, e.g., Baltagi and Yang (2013*b*) for details.

Following (3.8), define $g_{r,ni}(\theta_0), r = 1, 2, 3$, corresponding to the three quadratic forms given in (3.7) associated with $\Phi_r, r = 1, 2, 3$. Using these results, we have

$$S_{\mathtt{SAC}}^{\circ}(\theta_0) = \textstyle\sum_{i=1}^n \mathbf{g}_{ni}(\theta_0), \tag{3.9}$$

where $\mathbf{g}_{ni}(\theta_0) = (\mathbf{g}_{ni,\theta}'(\theta_0), \mathbf{g}_{ni,\alpha}'(\theta_0))'$, $\mathbf{g}_{ni,\theta}(\theta_0) = \{\Pi_{1i}'v_{ni},\ \ g_{1i},\ \ g_{2i} + \Pi_{2i}v_{ni},\ \ g_{3i}\}'$, and $\mathbf{g}_{ni,\alpha}(\theta_0) = \frac{1}{2\sigma_0^2}\dot{h}(0)(v_{ni}^2 - \sigma_0^2)z_{ni}$. Obviously, $\{\mathbf{g}_{ni}(\theta_0), \mathcal{F}_{in}\}_{i=1}^n$ form a vector MD sequence. Thus, $\mathrm{Var}[S_{\mathtt{SAC}}^{\circ}(\theta_0)] = \sum_{i=1}^n \mathrm{E}[\mathbf{g}_{ni}(\theta_0)\mathbf{g}_{ni}'(\theta_0)]$, and its sample analogue $\sum_{i=1}^n \tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}_{ni}'$ gives a consistent estimator in the sense that $\frac{1}{n}\sum_{i=1}^n \tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}_{ni}' - \frac{1}{n}\mathrm{Var}[S_{\mathtt{SAC}}^{\circ}(\theta_0)] = o_p(1)$. The proof of this result follows the Weak Law of Large Numbers (WLLN) for MD arrays in, e.g., Davidson (1994, p. 299), with details given in the proof of Theorem 3.1, Appendix B.

Thus, following (2.6) and (2.7), a score test for testing $H_0 : \alpha = 0$ for the SAC model, based on $S_{\mathtt{SAC}}^{\circ}(\theta)$ defined in (3.6) and the MD representation (3.9), has the form:

$$T_{\mathtt{SAC}} = \big(\textstyle\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}'\big)\big[\sum_{i=1}^n (\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{K}_n\tilde{\mathbf{g}}_{ni,\theta})\big(\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{K}_n\tilde{\mathbf{g}}_{ni,\theta}\big)'\big]^{-1}\big(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\big), \tag{3.10}$$

where $\tilde{K}_n = \big(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\tilde{\mathbf{g}}_{ni,\theta}'\big)\big(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\theta}\tilde{\mathbf{g}}_{ni,\theta}'\big)^{-1}$. As in (2.5), the test in (3.10) can simply be written as $T_{\mathtt{SAC}} = S_{\mathtt{SAC}}^{\circ\prime}(\tilde{\theta}_n)\big(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}_{ni}'\big)^{-1}S_{\mathtt{SAC}}^{\circ}(\tilde{\theta}_n)$.[6] Obviously, $T_{\mathtt{SAC}}$ is invariant to the unknown $\dot{h}(0)$ appearing in $\mathbf{g}_{ni,\alpha}(\theta_0)$, and hence it can be removed or simply set to 1.

When the normality of V_n is in doubt, one can simply replace $\tilde{K}_n$ in (3.10) by $\tilde{\Gamma}_n = \tilde{\Sigma}_{n,\alpha\theta}\tilde{\Sigma}_{n,\theta\theta}^{-1}$, where $\tilde{\Sigma}_{n,\alpha\theta} = -\frac{\partial}{\partial\theta'}S_{\mathtt{SAC},\alpha}^{\circ}(\tilde{\theta}_n)$ and $\tilde{\Sigma}_{n,\theta\theta} = -\frac{\partial}{\partial\theta'}S_{\mathtt{SAC},\theta}^{\circ}(\tilde{\theta}_n)$, noting that $\tilde{v}_{ni} = v_{ni}(\tilde{\beta}_n, \tilde{\lambda}_n)$ and $(S_{\mathtt{SAC},\theta}^{\circ\prime}(\tilde{\theta}_n), S_{\mathtt{SAC},\alpha}^{\circ\prime}(\tilde{\theta}_n))' = S_{\mathtt{SAC}}^{\circ\prime}(\tilde{\theta}_n)$, to give a QS-test as in (2.10), robust against non-normality:

$$T_{\mathtt{SAC}}^{\mathtt{r}} = \big(\textstyle\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}'\big)\big[\sum_{i=1}^n (\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{\Gamma}_n\tilde{\mathbf{g}}_{ni,\theta})\big(\tilde{\mathbf{g}}_{ni,\alpha} - \tilde{\Gamma}_n\tilde{\mathbf{g}}_{ni,\theta}\big)'\big]^{-1}\big(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\big). \tag{3.11}$$

The expressions for $\frac{\partial}{\partial\theta'}S_{\mathtt{SAC},\alpha}^{\circ}(\theta)$ and $\frac{\partial}{\partial\theta'}S_{\mathtt{SAC},\theta}^{\circ}(\theta)$ can easily be obtained from (3.5), and are given in Appendix B following the proof of Theorem 3.1. The asymptotic null distributions

[6] Note that $\tilde{\mathbf{g}}_{ni} = \mathbf{g}_{ni}(\tilde{\theta}_n)$, implicitly indicating that it is obtained from $\mathbf{g}_{ni}(\theta_0)$ by replacing θ_0 by $\tilde{\theta}_n$ and $V_n = V_n(\beta_0, \lambda_0)$ by $\tilde{V}_n = V_n(\tilde{\beta}_n, \tilde{\lambda}_n)$. Setting $\lambda_1 = 0$ and using expected information matrix instead of OPMD, the test (3.10) reduces to that of Anselin (1988, p. 121).

of the tests are established under the following standard regularity conditions:

Assumption 3.1. *The disturbances* $\{v_{ni}, i = 1, \ldots, n\}$ *are independent with means 0, variances* $\sigma^2 h(z_i'\alpha)$*, and* $E|v_{ni}|^{4+\epsilon} < \infty$ *for some* $\epsilon > 0$*.*

Assumption 3.2. *The elements of* X_n *are nonstochastic and are uniformly bounded, and* $\lim_{n\to\infty} \frac{1}{n} X_n' X_n$ *exists and is nonsingular.*

Assumption 3.3. W_{1n} *and* W_{2n} *are uniformly bounded in both row and column sums in absolute value, and their diagonal elements are zero.*[7]

Assumption 3.4. B_{1n}^{-1} *and* B_{2n}^{-1} *are uniformly bounded in both row and column sums in absolute value, uniformly in* λ *in a neighborhood of its true value.*

Theorem 3.1. *Under Assumptions 3.1-3.4, if* $\tilde{\theta}_n$ *is* $\sqrt{n}$*-consistent for* θ_0 *under* H_0*, and* $\frac{1}{n}\Sigma_{n,\theta\theta}$ *and* $\frac{1}{n}\Omega_n$ *are positive definite for large enough* n*, then,* $T_{\mathtt{SAC}}^{\mathtt{r}}|_{H_0} \xrightarrow{D} \chi_k^2$ *when the errors are either normal or non-normal;* $T_{\mathtt{SAC}}|_{H_0} \xrightarrow{D} \chi_k^2$ *when the errors are normal.*

3.3. Adjusted Score or Quasi-Score Tests

The finite sample properties of the score tests $T_{\mathtt{SAC}}$ and $T_{\mathtt{SAC}}^{\mathtt{r}}$ may be improved by working with the concentrated score functions for λ and α, concentrating out β and σ^2. The intuition behind this is that the concentrated scores capture the variability inherent from the estimation of β and σ^2. Furthermore, the process of deriving the test statistic involves the 'standardization' (centering and rescaling) of the key quantities in the concentrated score functions. This standardization makes the quantiles of the resulting statistic closer to the corresponding asymptotic values, compared with the test statistics $T_{\mathtt{SAC}}$ and $T_{\mathtt{SAC}}^{\mathtt{r}}$. Hence, it can be expected that such 'standardized' tests based on the concentrated scores would have better finite sample performance; see, e.g., Baltagi and Yang (2013*a*, *b*).

Substituting $\tilde{\beta}_n(\lambda)$ and $\tilde{\sigma}_n^2(\lambda)$ defined in (3.3) into the last three components of (3.5), we obtain the concentrated scores at the null: $\tilde{\sigma}_n^{-2}(\lambda)Y_n'(\lambda)M_n(\lambda_2)B_{2n}(\lambda_2)G_{1n}(\lambda_1)B_{2n}^{-1}(\lambda_2)Y_n(\lambda) - \text{tr}[G_{1n}(\lambda_1)]$, $\tilde{\sigma}_n^{-2}(\lambda)Y_n'(\lambda)M_n(\lambda_2)G_{2n}(\lambda_2)M_n(\lambda_2)Y_n(\lambda) - \text{tr}[G_{2n}(\lambda_2)]$, and $\tilde{\sigma}_n^{-2}(\lambda)\dot{h}(0)Z_n'\tilde{\zeta}_n(\lambda)$, respectively for λ_1, λ_2 and α, which are rewritten in the form

$$S_{\mathtt{SAC}}^{c}(\lambda,\alpha)|_{H_0} = \begin{cases} \tilde{\sigma}_n^{-2}(\lambda)\big\{Y_n'(\lambda)M_n(\lambda_2)\big[B_{2n}(\lambda_2)G_{1n}(\lambda_1)B_{2n}^{-1}(\lambda_2) - \bar{G}_{1n}(\lambda_1)I_n\big]Y_n(\lambda)\big\}, \\ \tilde{\sigma}_n^{-2}(\lambda)\big\{Y_n'(\lambda)M_n(\lambda_2)\big[G_{2n}(\lambda_2) - \bar{G}_{2n}(\lambda_2)I_n\big]M_n(\lambda_2)Y_n(\lambda)\big\}, \\ \tilde{\sigma}_n^{-2}(\lambda)\big\{\dot{h}(0)Z_n'\tilde{\zeta}_n(\lambda)\big\}, \end{cases} \tag{3.12}$$

where $\tilde{\zeta}_n(\lambda) = \frac{1}{2}\{v_{ni}^2(\tilde{\beta}_n(\lambda),\lambda) - \tilde{\sigma}_n^2(\lambda)\}$, and $\bar{G}_{rn}(\lambda_r) = \frac{1}{n}\text{tr}(G_{rn}(\lambda_r)),\ r = 1, 2$.

Under mild conditions, the constrained QMLE $\tilde{\lambda}_n$ defined in Section 3.1 is equivalent to the solution of the following estimating equations: $Y_n'(\lambda)M_n(\lambda_2)\big[B_{2n}(\lambda_2)G_{1n}(\lambda_1)B_{2n}^{-1}(\lambda_2) -$

[7]The elements of W_{rn} may be of uniform order h_n^{-1}, where h_n is a rate sequence such that $\lim_{n\to\infty}(h_n/n) = 0$, to reflect the fact that the degree of spatial dependence may grow with the sample size n (Lee, 2004). While the main results are stated without explicitly accounting for h_n to avoid unnecessary complications in applications, their proofs reflect explicitly the role played by h_n, in particular the lemmas in Appendix A.

$\bar{G}_{1n}(\lambda_1)I_n\big]Y_n(\lambda) = 0$ and $Y_n'(\lambda)M_n(\lambda_2)\big[G_{2n}(\lambda_2) - \bar{G}_{2n}(\lambda_2)I_n\big]M_n(\lambda_2)Y_n(\lambda) = 0$, obtained from the first two components of (3.12). However, neither estimation functions have an expectation zero, which constitutes a major source of finite sample bias of $\tilde{\lambda}_n$ (Yang, 2015; Liu and Yang, 2015), and a major source of size distortion for the tests constructed, for e.g., in Section 3.1 (Baltagi and Yang, 2013*b*). Noting that $\tilde{\sigma}_n^2(\lambda_0) \xrightarrow{p} \sigma_0^2$, we construct a test that potentially has better finite sample properties. This is done by working on the numerators of (3.12) or the quantities in the curling brackets, i.e., $\tilde{\sigma}_n^2(\lambda)S^c_{\mathtt{SAC}}(\lambda,\alpha)|_{H_0}$. Under H_0 and λ_0,

$$\begin{aligned}Y_n'(\lambda_0)M_n(B_{2n}G_{1n}B_{2n}^{-1} - \bar{G}_{1n}I_n)Y_n(\lambda_0) &= V_n'M_n(B_{2n}G_{1n}B_{2n}^{-1} - \bar{G}_{1n}I_n)V_n \\ &\quad + V_n'M_n(B_{2n}G_{1n}B_{2n}^{-1} - \bar{G}_{1n}I_n)X_n(\lambda_2)\beta_0 \equiv V_n'\Phi_1V_n + \Pi'V_n, \\ Y_n'(\lambda_0)M_n(G_{2n} - \bar{G}_{2n}I_n)M_nY_n(\lambda_0) &= V_n'M_n(G_{2n} - \bar{G}_{2n}I_n)M_nV_n \equiv V_n'\Phi_2V_n,\end{aligned}$$

where $\Pi = M_n(B_{2n}G_{1n}B_{2n}^{-1} - \bar{G}_{1n}I_n)X_n(\lambda_{20})\beta_0$, $\Phi_1 = M_n(B_{2n}G_{1n}B_{2n}^{-1} - \bar{G}_{1n}I_n)$ and $\Phi_2 = M_n(G_{2n} - \bar{G}_{2n}I_n)M_n$. These show that the expectations of the first two components of the numerator of (3.12) are, respectively, $\sigma_0^2\mathrm{tr}(\Phi_r), r = 1, 2$. Also, for the numerator of the last component of (3.12), we have $V_n(\tilde{\beta}_n(\lambda_0), \lambda_0) = M_n(\lambda_{20})Y_n(\lambda_0) = M_nV_n$. It follows that $\mathrm{E}[v_{ni}^2(\tilde{\beta}_n(\lambda_0), \lambda_0)] = \mathrm{E}[(M_{ni}V_n)^2] = \sigma_0^2\sum_{j=1}^n M_{n,ij}^2 \equiv \sigma_0^2 m_i$, where M_{ni} denotes the ith row of M_n and $M_{n,ij}$ the ijth element of M_n. Define

$$\tilde{\zeta}_n^*(\lambda) = \tfrac{1}{2}\{\tfrac{1}{m_i(\lambda_2)}v_{ni}^2(\tilde{\beta}_n(\lambda), \lambda) - \tfrac{n}{n-p}\tilde{\sigma}_n^2(\lambda)\}_{n\times 1}. \tag{3.13}$$

The set of adjusted concentrated quasi-scores (ACQS) at H_0 thus have the simple form:

$$S^*_{\mathtt{SAC}}(\lambda) = \begin{cases} Y_n'(\lambda)\Phi_1(\lambda)Y_n(\lambda) - \frac{n}{n-p}\tilde{\sigma}_n^2(\lambda)\mathrm{tr}[\Phi_1(\lambda)], \\ Y_n'(\lambda)\Phi_2(\lambda)Y_n(\lambda) - \frac{n}{n-p}\tilde{\sigma}_n^2(\lambda)\mathrm{tr}[\Phi_2(\lambda)], \\ Z_n'\tilde{\zeta}_n^*(\lambda). \end{cases} \tag{3.14}$$

It is easy to see that $\mathrm{E}[S^*_{\mathtt{SAC}}(\lambda_0)|_{H_0}] = 0$, and hence $S^*_{\mathtt{SAC}}(\lambda_0)$ may lead to a potentially improved score-type test. To find its variance estimator, noting that $\tilde{\sigma}_n^2(\lambda_0) = \frac{1}{n}V_n'M_nV_n$, we have at H_0: $Y_n'(\lambda_0)\Phi_1Y_n(\lambda_0) - \frac{n}{n-p}\tilde{\sigma}_n^2(\lambda_0)\mathrm{tr}(\Phi_1) = V_n'\Phi_1^*V_n + V_n'\Pi_n$, and $Y_n'(\lambda_0)\Phi_2Y_n(\lambda_0) - \frac{n}{n-p}\tilde{\sigma}_n^2(\lambda_0)\mathrm{tr}(\Phi_2) = V_n'\Phi_2^*V_n$, where $\Phi_r^* = \Phi_r - \frac{1}{n-p}\mathrm{tr}(\Phi_r)M_n, r = 1, 2$. Similar to the developments in Section 3.1, we can write $V_n'\Phi_r^*V_n = \sum_{i=1}^n g_{ri}(\theta_0), r = 1, 2$, where $\{g_{ri}(\theta_0), \mathcal{F}_{in}\}$ form an MD sequence. The elements of $\tilde{\xi}_n^*(\lambda)$ are asymptotically independent. Define,

$$\mathbf{g}^*_{ni}(\theta_0) = \{g_{1i} + \Pi_i v_{ni},\ g_{2i},\ z_{ni}'\tilde{\xi}^*_{ni}(\lambda_0)\}'. \tag{3.15}$$

Then, $S^*_{\mathtt{SAC}}(\lambda_0) = \sum_{i=1}^n \mathbf{g}^*_{ni}(\theta_0)$, and it can be shown that

$$\tfrac{1}{n}\mathrm{Var}[S^*_{\mathtt{SAC}}(\lambda_0)] = \tfrac{1}{n}\sum_{i=1}^n \mathrm{E}[\mathbf{g}^*_{ni}(\theta_0)\mathbf{g}^{*\prime}_{ni}(\theta_0)] + o(1).$$

A score-type test statistic, or the AQS test, for testing $H_0: \alpha = 0$ takes the following form:

$$T^{\mathtt{r}*}_{\mathtt{SAC}} = \big(\textstyle\sum_{i=1}^n \tilde{\mathbf{g}}^{*\prime}_{ni,\alpha}\big)\big[\sum_{i=1}^n(\tilde{\mathbf{g}}^*_{ni,\alpha} - \tilde{\Gamma}^*_n\tilde{\mathbf{g}}^*_{ni,\lambda})\big(\tilde{\mathbf{g}}^*_{ni,\alpha} - \tilde{\Gamma}^*_n\tilde{\mathbf{g}}^*_{ni,\lambda}\big)'\big]^{-1}\big(\sum_{i=1}^n \tilde{\mathbf{g}}^*_{ni,\alpha}\big), \tag{3.16}$$

where $\tilde{\Gamma}^*_n = \tilde{\Sigma}^*_{n,\alpha\lambda}\tilde{\Sigma}^{*-1}_{n,\lambda\lambda}$, $\tilde{\Sigma}^*_{n,\alpha\lambda} = -\frac{\partial}{\partial\lambda}S^*_{\mathtt{SAC},\alpha}(\tilde{\lambda}_n)$, and $\tilde{\Sigma}^*_{n,\lambda\lambda} = -\frac{\partial}{\partial\lambda}S^*_{\mathtt{SAC},\lambda}(\tilde{\lambda}_n)$. These derivatives can be obtained from (3.14) after some tedious algebra, and their detailed expressions are given in Appendix B following the proof of Theorem 3.2.[8] When the errors are normally distributed, one could simply use $\tilde{\Sigma}^*_{n,\alpha\lambda} = \sum_{i=1}^n \tilde{\mathbf{g}}^*_{ni,\alpha}\tilde{\mathbf{g}}^{*\prime}_{ni,\lambda}$ and $\tilde{\Sigma}^*_{n,\lambda\lambda} = \sum_{i=1}^n \tilde{\mathbf{g}}^*_{ni,\lambda}\tilde{\mathbf{g}}^{*\prime}_{ni,\lambda}$, leading to an adjusted score (AS) test, denoted by $T^*_{\mathtt{SAC}}$ for easy reference.[9]

The process of deriving $T^*_{\mathtt{SAC}}$ or $T^{\mathtt{r}*}_{\mathtt{SAC}}$ starts from the concentrated score where the variability from the estimation of β and σ^2 is captured, then recenters the numerator of the concentrated scores, and then rescales the 'recentered' score. Thus, these tests are expected to perform better in finite samples than $T_{\mathtt{SAC}}$ or $T^{\mathtt{r}}_{\mathtt{SAC}}$. Note that unlike the case with joint scores, $S^*_{\mathtt{SAC},\lambda}(\tilde{\lambda}_n)$ is not identically zero, as $\tilde{\lambda}_n$ is not the solution of the estimating equation $S^*_{\mathtt{SAC},\lambda}(\lambda) = 0$. In this case, an adjusted estimator that solves the ACPS equations, i.e.,

$$\tilde{\lambda}^*_n = \arg\{S^*_{\mathtt{SAC},\lambda}(\lambda) = 0\}, \tag{3.17}$$

should be used to ensure a good finite sample performance of the AQS test. This is confirmed by the Monte Carlo results presented in Section 5. The asymptotic null behavior of $T^*_{\mathtt{SAC}}$ or $T^{\mathtt{r}*}_{\mathtt{SAC}}$ is summarized in the following theorem:

Theorem 3.2. *Under the assumptions of Theorem 3.1, $T^{\mathtt{r}*}_{\mathtt{SAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the errors are either normal or non-normal; and $T^*_{\mathtt{SAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the errors are normal.*

4. Tests for Homoskedasticity for the FESAC Model

In this section, we consider the panel SARAR model with fixed effects, which is also called Fixed Effects (FE) spatial autoregressive combined panel data, or FESAC, model:

$$Y_{nt} = \lambda_1 W_{1n} Y_{nt} + X_{nt}\beta + \mu_n + U_{nt}, \ \ U_{nt} = \lambda_2 W_{2n} U_{nt} + V_{nt}, \ \ t = 1, 2, \ldots, T, \tag{4.1}$$

where Y_{nt} denotes the $n \times 1$ vector of observations on the dependent variable in period t. X_{nt} denotes the $n \times p$ matrix of observations on the exogenous regressors in period t (which may contain the constant term). The parameters β, λ_1 and λ_2 are defined in the same way as in model (3.1). μ_n represents the vector of unit-specific effects that are allowed to be correlated with some of the regressors. The elements $\{v_{it}\}$ of V_{nt} are independent across i and t with means 0 and variances $\sigma^2 h(z'_{ni}\alpha)$. Again, a test for homoskedasticity across the cross-section dimension corresponds to the test of the null hypothesis $H_0 : \alpha = 0$.[10]

Six tests for homoskedasticity in the FESAC model are introduced, and formal asymptotic theory for the proposed tests is presented with the proofs relegated to Appendix C. At

[8] Numerical derivatives can be used in place of analytical ones: $\frac{\partial}{\partial \lambda_1} S^*_{\mathtt{SAC}}(\lambda) = [S^*_{\mathtt{SAC}}(\lambda + (\epsilon, 0)') - S^*_{\mathtt{SAC}}(\lambda)]/\epsilon$ and $\frac{\partial}{\partial \lambda_2} S^*_{\mathtt{SAC}}(\lambda) = [S^*_{\mathtt{SAC}}(\lambda + (0, \epsilon)') - S^*_{\mathtt{SAC}}(\lambda)]/\epsilon$, where ϵ is a small positive number, e.g., 0.00001.

[9] This is justified by an IME with respect to the underlining distribution (adjusted likelihood) that generates the ACQS (3.14). Alternatively, the generalized IME can be applied to give $\tilde{\Sigma}^*_{n,\alpha\lambda} = \sum_{i=1}^n \tilde{\mathbf{g}}^*_{ni,\alpha}\tilde{\mathbf{g}}'_{ni,\lambda}$ and $\tilde{\Sigma}^*_{n,\lambda\lambda} = \sum_{i=1}^n \tilde{\mathbf{g}}^*_{ni,\lambda}\tilde{\mathbf{g}}'_{ni,\lambda}$, where $\tilde{\mathbf{g}}_{ni,\lambda}$ is the restricted estimate of the λ-element of the full $\mathbf{g}_{ni}$ in (3.9). However, the numerical results show that the former performs better in finite samples.

[10] Similar to the case of the SAC model, the FESAC model can also be extended by adding the spatial Durbin terms, higher-order spatial lags of the response, and higher-order lags of the disturbances.

the end of the section, two important extensions are discussed: (*i*) allowing for time-wise heteroskedasticity, and (*ii*) allowing for time-specific fixed effects.

4.1. ML or QML Estimation of the Panel FESAC Model

The ML or QML estimation of the FESAC model under $H_0 : \alpha = 0$ proceeds with the transformation approach followed by Lee and Yu (2010) and Yang et al. (2016). To eliminate the individual effects, define $J_T = (I_T - \frac{1}{T} l_T l_T')$ and let $[F_{T,T-1}, \frac{1}{\sqrt{T}} l_T]$ be the orthonormal eigenvector matrix of J_T, where $F_{T,T-1}$ is the $T \times (T-1)$ submatrix corresponding to the eigenvalues of one, I_T is a $T \times T$ identity matrix and l_T is a $T \times 1$ vector of ones. For any $n \times T$ matrix $[A_{n1}, \ldots, A_{nT}]$, define the $n \times (T-1)$ transformed matrix as

$$[A_{n1}^*, \ldots, A_{n,T-1}^*] = [A_{n1}, \ldots, A_{nT}] F_{T,T-1}. \tag{4.2}$$

This leads to the transformed vectors: Y_{nt}^*, U_{nt}^*, V_{nt}^*, and $X_{nt,j}^*$ for the jth regressor, for $t = 1, \ldots, T-1$. Let $X_{nt}^* = [X_{nt,1}^*, X_{nt,2}^*, \ldots, X_{nt,k}^*]$.

The transformed model takes the form:

$$Y_{nt}^* = \lambda_1 W_{1n} Y_{nt}^* + X_{nt}^* \beta + U_{nt}^*, \quad U_{nt}^* = \lambda_2 W_{2n} U_{nt}^* + V_{nt}^*, \quad t = 1, \ldots, T-1. \tag{4.3}$$

After the transformation, the effective sample size becomes $N = n(T-1)$. Stacking the vectors and matrices, i.e., letting $\mathbf{Y}_N = (Y_{n1}^{*\prime}, \ldots, Y_{n,T-1}^{*\prime})'$, $\mathbf{U}_N = (U_{n1}^{*\prime}, \ldots, U_{n,T-1}^{*\prime})'$, $\mathbf{V}_N = (V_{n1}^{*\prime}, \ldots, V_{n,T-1}^{*\prime})'$, $\mathbf{X}_N = (X_{n1}^{*\prime}, \ldots, X_{n,T-1}^{*\prime})'$, and denoting $\mathbf{W}_{rN} = I_{T-1} \otimes W_{rn}, r = 1, 2$, we have the following compact expression for the transformed model:

$$\mathbf{Y}_N = \lambda_1 \mathbf{W}_{1N} \mathbf{Y}_N + \mathbf{X}_N \beta + \mathbf{U}_N, \;\; \mathbf{U}_N = \lambda_2 \mathbf{W}_{2N} \mathbf{U}_N + \mathbf{V}_N, \tag{4.4}$$

which is identical in form to the cross-sectional SAC model, showing that the QML estimation of the FESAC model is similar to that of the cross-sectional SAC model. The key difference is that the elements $\{v_{it}^*\}$ of the transformed error vector $\mathbf{V}_N$ may not be totally independent unless the original errors are independent and normal. When the original errors are independent but non-normal, $\{v_{it}^*\}$ are independent across i by definition but only uncorrelated across t, as seen using the identity $(V_{n1}^{*\prime}, \ldots, V_{n,T-1}^{*\prime})' = (F_{T,T-1}' \otimes I_n)(V_{n1}', \ldots, V_{nT}')'$,

$$\begin{aligned} & E(V_{n1}^{*\prime}, \ldots, V_{n,T-1}^{*\prime})'(V_{n1}^{*\prime}, \ldots, V_{n,T-1}^{*\prime}) \\ = \;& \sigma^2 (F_{T,T-1}' \otimes I_n)(I_T \otimes \mathcal{H}_n(\alpha))(F_{T,T-1} \otimes I_n) \\ = \;& \sigma^2 (I_{T-1} \otimes \mathcal{H}_n(\alpha)) \equiv \sigma^2 \mathbf{H}_N(\alpha), \end{aligned} \tag{4.5}$$

where $\mathcal{H}_n(\alpha)$ is defined in Section 3. It follows that the full quasi Gaussian log likelihood function for $\psi = (\beta', \sigma^2, \lambda', \alpha')'$ (required for the derivation of the score-type tests later) is,

$$\begin{aligned} \ell_{\texttt{FESAC}}(\psi) = \;& -\tfrac{N}{2} \log(2\pi\sigma^2) + \log|\mathbf{B}_{1N}(\lambda_1)| + \log|\mathbf{B}_{2N}(\lambda_2)| \\ & -\tfrac{1}{2} \log|\mathbf{H}_N(\alpha)| - \tfrac{1}{2\sigma^2} \mathbf{V}_N'(\beta, \lambda) \mathbf{H}_N^{-1}(\alpha) \mathbf{V}_N(\beta, \lambda), \end{aligned} \tag{4.6}$$

where $\mathbf{V}_N(\beta, \lambda) = \mathbf{Y}_N(\lambda) - \mathbf{X}_N(\lambda_2)\beta$, $\mathbf{Y}_N(\lambda) = \mathbf{B}_{2N}(\lambda_2)\mathbf{B}_{1N}(\lambda_1)\mathbf{Y}_N$, $\mathbf{X}_N(\lambda_2) = \mathbf{B}_{2N}(\lambda_2)\mathbf{X}_N$, $\mathbf{B}_{1N}(\lambda_1) = I_N - \lambda_1 \mathbf{W}_{1N}$, and $\mathbf{B}_{2N}(\lambda_2) = I_N - \lambda_2 \mathbf{W}_{2N}$.

Maximizing $\ell_{\texttt{FESAC}}(\psi)$ gives the (Q)MLE of ψ for the full model and maximizing $\ell_{\texttt{FESAC}}(\psi)|_{H_0}$ gives the (Q)MLEs of the parameters in the null model. Now, similar to the SAC model, for a given λ, $\ell_{\texttt{FESAC}}(\psi)|_{H_0}$ is partially maximized at:

$$\tilde{\beta}_N(\lambda) = [\mathbf{X}'_N(\lambda_2)\mathbf{X}_N(\lambda_2)]^{-1}\mathbf{X}'_N(\lambda_2)\mathbf{Y}_N(\lambda) \;\text{ and }\; \tilde{\sigma}^2_N(\lambda) = \tfrac{1}{N}\mathbf{Y}'_N(\lambda)\mathbf{M}_N(\lambda_2)\mathbf{Y}_N(\lambda),$$

where $\mathbf{M}_N(\lambda_2) = I_N - \mathbf{X}_N(\lambda_2)[\mathbf{X}'_N(\lambda_2)\mathbf{X}_N(\lambda_2)]^{-1}\mathbf{X}'_N(\lambda_2)$. Substituting $\tilde{\beta}_N(\lambda)$ and $\tilde{\sigma}^2_N(\lambda)$ back into (4.6) gives the null concentrated log likelihood function of λ:

$$\ell^c_{\texttt{FESAC}}(\lambda)|_{H_0} = -\tfrac{N}{2}(\log(2\pi)+1) + \log|\mathbf{B}_{1N}(\lambda_1)| + \log|\mathbf{B}_{2N}(\lambda_2)| - \tfrac{N}{2}\ln\tilde{\sigma}^2_N(\lambda). \tag{4.7}$$

Maximizing $\ell^c_{\texttt{FESAC}}(\lambda)|_{H_0}$ gives the null QMLE $\tilde{\lambda}_N$ of λ, which upon substitutions gives the null QMLEs of β and σ^2 as $\tilde{\beta}_N \equiv \tilde{\beta}_N(\tilde{\lambda}_N)$ and $\tilde{\sigma}^2_N \equiv \tilde{\sigma}^2_N(\tilde{\lambda}_N)$.[11] The QMLE of $\theta = (\beta', \sigma^2, \lambda')'$, the parameter vector of the null model, is thus $\tilde{\theta}_N = (\tilde{\beta}_N, \tilde{\sigma}^2_N, \tilde{\lambda}'_N)'$.

Lee and Yu (2010) show that, as n goes large (where T can go large or stays fixed), $\sqrt{N}(\tilde{\theta}_N - \theta)$ is asymptotically normal with mean 0 and VC matrix $N\mathbf{\Sigma}^{-1}_{N,\theta\theta}\mathbf{\Omega}_{N,\theta\theta}\mathbf{\Sigma}^{-1}_{N,\theta\theta}$, where $\mathbf{\Sigma}_{N,\theta\theta}$ and $\mathbf{\Omega}_{N,\theta\theta}$ are, respectively, the expected negative Hessian matrix and the variance of the score of the null model.[12] Note that a similar set of notation, $\mathbf{\Sigma}_N$, $\mathbf{\Omega}_N$ and $\mathbf{\Sigma}^*_N$ for the full model at H_0 with $\mathbf{\Sigma}_{N,\theta\theta}$ and $\mathbf{\Omega}_{N,\theta\theta}$, *etc.*, being their submatrices, will be followed in the subsequent developments. The use of bold face to reflect that the underlining model is panel.

4.2. Score or Quasi-Score Tests

The same idea as in the earlier subsection can be followed to give a score or QS test of homoskedasticity in the FESAC model. However, it should be noted that when the original errors are non-normal, the transformed errors are independent along the cross-sectional dimension only, not along the time dimension although they are still uncorrelated. While this makes the proof of the theorems more difficult, it emphasizes the advantage of the proposed OPMD method. This is because under the transformed QML approach, the explicit VC matrix of the score vector involves the unknown 3rd and 4th moments of the original errors v_{it}, but only the estimated residuals on the transformed scale are available.

The score function $S_{\texttt{FESAC}}(\psi) = \frac{\partial}{\partial\psi}\ell_{\texttt{FESAC}}(\psi)$ has the form:

$$S_{\texttt{FESAC}}(\psi) = \begin{cases} \frac{1}{\sigma^2}\mathbf{X}'_N(\lambda_2)\mathbf{H}^{-1}_N(\alpha)\mathbf{V}_N(\beta,\lambda), \\ \frac{1}{2\sigma^4}\mathbf{V}'_N(\beta,\lambda)\mathbf{H}^{-1}_N(\alpha)\mathbf{V}_N(\beta,\lambda) - \frac{N}{2\sigma^2}, \\ \frac{1}{\sigma^2}\mathbf{V}'_N(\beta,\lambda)\mathbf{H}^{-1}_N(\alpha)\mathbf{B}_{2N}(\lambda_2)\mathbf{W}_{1N}\mathbf{Y}_N - \mathrm{tr}[\mathbf{G}_{1N}(\lambda_1)], \\ \frac{1}{\sigma^2}\mathbf{V}'_N(\beta,\lambda)\mathbf{H}^{-1}_N(\alpha)\mathbf{G}_{2N}(\lambda_2)\mathbf{V}_N(\beta,\lambda) - \mathrm{tr}[\mathbf{G}_{2N}(\lambda_2)], \\ -\frac{1}{2\sigma^2}\dot{h}(z'_{ni}\alpha)\sum_{t=1}^{T-1}\sum_{i=1}^{n}\left[\frac{v^{*2}_{it}(\beta,\lambda)}{h(z'_{ni}\alpha)} - \sigma^2\right]\frac{z_{ni}}{h(z'_{ni}\alpha)}, \end{cases} \tag{4.8}$$

[11]Similarly, numerical maximization of $\ell^c_{\texttt{FESAC}}(\lambda)|_{H_0}$ can be made easier by $|\mathbf{B}_{rN}(\lambda_r)| = \big(|I_n - \lambda W_{rn}|\big)^{T-1} = \big(\prod_{i=1}^n(1-\lambda_r\omega_{ri})\big)^{T-1}$, where ω_{ri} are the eigenvalues of W_{rn}; see Lee and Yu (2010) and Griffith (1988).

[12]In general, the tests to be introduced in the subsequent sections would require only the $\sqrt{N}$-consistency of $\tilde{\theta}_N$ (the $\sqrt{n}$-consistency of $\tilde{\theta}_n$ for the SAC model), which is implied by the asymptotic normality result.

where $\mathbf{G}_{rN}(\lambda_r) = I_{T-1} \otimes G_{rn}(\lambda_r),\ r = 1, 2$. Under H_0, $h(0) = 1$ and $\dot{h}(0)$ becomes a constant free of i and t. Hence, the score function at the null, $S^{\circ}_{\texttt{FESAC}}(\theta) = S_{\texttt{FESAC}}(\psi)|_{H_0}$, simplifies to

$$S^{\circ}_{\texttt{FESAC}}(\theta) = \begin{cases} \frac{1}{\sigma^2}\mathbf{X}'_N(\lambda_2)\mathbf{V}_N(\beta, \lambda), \\ \frac{1}{2\sigma^4}\mathbf{V}'_N(\beta, \lambda)\mathbf{V}_N(\beta, \lambda) - \frac{N}{2\sigma^2}, \\ \frac{1}{\sigma^2}\mathbf{V}'_N(\beta, \lambda)\mathbf{B}_{2N}(\lambda_2)\mathbf{W}_{1N}\mathbf{Y}_N - \text{tr}[\mathbf{G}_{1N}(\lambda_1)], \\ \frac{1}{\sigma^2}\mathbf{V}'_N(\beta, \lambda)\mathbf{G}_{2N}(\lambda_2)\mathbf{V}_N(\beta, \lambda) - \text{tr}[\mathbf{G}_{2N}(\lambda_2)], \\ \frac{1}{2\sigma^2}\dot{h}(0)\sum_{t=1}^{T-1}\sum_{i=1}^{n}\left[v^{*2}_{it}(\beta, \lambda) - \sigma^2\right]z_{ni}. \end{cases} \tag{4.9}$$

At the null and the true parameter values $\theta_0 = (\beta'_0, \sigma^2_0, \lambda'_0)'$, $\mathbf{H}_N^{-1}(0) = I_N$, $\mathbf{V}_N(\beta_0, \lambda_0) = \mathbf{V}_N$, $\mathbf{B}_{2N}(\lambda_{20}) = \mathbf{B}_{2N}$, and $\mathbf{G}_{rN}(\lambda_{r0}) = \mathbf{G}_{rN}$. To derive a variance estimator, we again express $S^{\circ}_{\texttt{FESAC}}(\theta_0)$ in terms of $\mathbf{V}_N$ and θ_0, in a form identical to (3.7):

$$S^{\circ}_{\texttt{FESAC}}(\theta_0) = \begin{cases} \Pi'_1\mathbf{V}_N, \\ \mathbf{V}'_N\mathbf{\Phi}_1\mathbf{V}_N - \text{E}(\mathbf{V}'_N\mathbf{\Phi}_1\mathbf{V}_N), \\ \mathbf{V}'_N\mathbf{\Phi}_2\mathbf{V}_N - \text{E}(\mathbf{V}'_N\mathbf{\Phi}_2\mathbf{V}_N) + \mathbf{V}'_N\Pi_2, \\ \mathbf{V}'_N\mathbf{\Phi}_3\mathbf{V}_N - \text{E}(\mathbf{V}'_N\mathbf{\Phi}_3\mathbf{V}_N), \\ \frac{1}{2\sigma_0^2}\dot{h}(0)\sum_{t=1}^{T-1}\sum_{i=1}^{n}(v^{*2}_{it} - \sigma^2_0)z_{ni}, \end{cases} \tag{4.10}$$

where $\Pi_1 = \frac{1}{\sigma_0^2}\mathbf{X}_N(\lambda_2)$, $\Pi_2 = \frac{1}{\sigma_0^2}\mathbf{B}_{2N}\mathbf{G}_{1N}\mathbf{B}_{2N}^{-1}\mathbf{X}_N(\lambda_2)\beta$, $\mathbf{\Phi}_1 = \frac{1}{2\sigma_0^4}I_N$, $\mathbf{\Phi}_2 = \frac{1}{\sigma_0^2}\mathbf{B}_{2N}\mathbf{G}_{1N}\mathbf{B}_{2N}^{-1}$, $\mathbf{\Phi}_3 = \frac{1}{\sigma_0^2}\mathbf{G}_{2N}$, and the expectation 'E' corresponds to the null model. In an identical way leading to (3.9), we can write $S_{\texttt{FESAC}}(\psi_0)|_{H_0} = \sum_{j=1}^{N}\mathbf{g}_{Nj}(\theta_0)$, where j $(= 1, \ldots, N)$ is the combined index for (i, t) with $i = 1, \ldots, n$ for each $t = 1, \ldots, T-1$, and the detailed expression of $\mathbf{g}_{Nj}(\theta_0)$ is given in (C.1) of Appendix C.

If the original errors $\{v_{it}\}$ are *iid* normal, then the transformed errors $\{v^*_{it}\}$ or $\{v^*_j\}$ are *iid* normal, and based on the same reasoning as for the cross-sectional SAC model, $\{\mathbf{g}_{Nj}(\theta)\}$ form an MD sequence with respect to the increasing σ-fields $\{\mathcal{F}_{jN}\}$ generated by $(v^*_1, \ldots, v^*_j)$. Thus, a consistent estimator for $\mathbf{\Omega}_N = \frac{1}{N}\text{Var}[S^{\circ}_{\texttt{FESAC}}(\theta_0)]$ is

$$\widetilde{\mathbf{\Omega}}_N = \frac{1}{N}\sum_{j=1}^{N}\tilde{\mathbf{g}}_{Nj}\tilde{\mathbf{g}}'_{Nj}, \tag{4.11}$$

where $\tilde{\mathbf{g}}_{Nj} = \mathbf{g}_{Nj}(\tilde{\theta}_N)$. In a similar manner as for the cross-sectional SAC model, asymptotic normality of $S^{\circ}_{\texttt{FESAC}}(\theta_0)$ can be established using the CLT for LQ forms given in Lemma A.5, and the consistency of the variance estimator can be established using the WLLN for martingale difference arrays in Davidson (1994, p. 229). A score statistic for testing H_0 : $\alpha = 0$ has an identical form as (3.10):

$$T_{\texttt{FESAC}} = \big(\sum_{j=1}^{N}\tilde{\mathbf{g}}'_{Nj,\alpha}\big)\big[\sum_{j=1}^{N}(\tilde{\mathbf{g}}_{Nj,\alpha} - \tilde{K}_N\tilde{\mathbf{g}}_{Nj,\theta})(\tilde{\mathbf{g}}_{Nj,\alpha} - \tilde{K}_N\tilde{\mathbf{g}}_{Nj,\theta})'\big]^{-1}\big(\sum_{j=1}^{n}\tilde{\mathbf{g}}_{Nj,\alpha}\big), \tag{4.12}$$

where $\tilde{K}_N = \big(\sum_{J=1}^{n}\tilde{\mathbf{g}}_{Nj,\alpha}\tilde{\mathbf{g}}'_{Nj,\theta}\big)\big(\sum_{j=1}^{n}\tilde{\mathbf{g}}_{Nj,\theta}\tilde{\mathbf{g}}'_{Nj,\theta}\big)^{-1}$; or an identical form as the one below

(3.10): $T_{\texttt{FESAC}} = S^{\circ\prime}_{\texttt{FESAC}}(\tilde{\theta}_N)\big[\sum_{j=1}^N \tilde{\mathbf{g}}_{Nj}\tilde{\mathbf{g}}'_{Nj}\big]^{-1} S^{\circ}_{\texttt{FESAC}}(\tilde{\theta}_N)$.[13] Again, the unknown constant $\dot{h}(0)$ appearing in the score element for α cancels out, and hence it can simply be set to 1.

If $\{v_{it}\}$ are *iid* but not normal, however, $\{v^*_{it}\}$ or $\{v^*_j\}$ are not guaranteed to be totally independent in the sense that there may exist higher-order dependence among $\{v^*_{it}\}$. If this higher-order dependence does not affect the asymptotic properties of the OPMD estimate given in (4.11), then, similar to the QS test given in (3.11), a QS test for homoskedasticity in the FESAC model, allowing the errors to be non-normally distributed, can be obtained by replacing $\tilde{K}_N$ in (4.12) by $\tilde{\mathbf{\Gamma}}_N = \tilde{\mathbf{\Sigma}}_{N,\alpha\theta}\tilde{\mathbf{\Sigma}}^{-1}_{N,\theta\theta}$, where $\tilde{\mathbf{\Sigma}}_{N,\alpha\theta} = -\frac{\partial}{\partial\theta'}S^{\circ}_{\texttt{FESAC},\alpha}(\tilde{\theta}_N)$ and $\tilde{\mathbf{\Sigma}}_{N,\theta\theta} = -\frac{\partial}{\partial\theta'}S^{\circ}_{\texttt{FESAC},\theta}(\tilde{\theta}_N)$. The resulting test is denoted by $T^{\mathtt{r}}_{\texttt{FESAC}}$ for easy reference. The analytical expressions for $\frac{\partial}{\partial\theta'}S^{\circ}_{\texttt{FESAC},\alpha}(\tilde{\theta}_N)$ and $\frac{\partial}{\partial\theta'}S^{\circ}_{\texttt{FESAC},\theta}(\tilde{\theta}_N)$ can easily be obtained from (4.9), which are given in Appendix C.

However, we show in Appendix C that the correlation between v^*_{it} and v^{*2}_{is} and the correlation between v^{*2}_{it} and v^{*2}_{is}, in particular the latter, induce correlation between $\mathbf{g}_{N,it}(\theta_0)$ and $\mathbf{g}_{N,is}(\theta_0)$, $t \neq s$, which may not be ignored when the skewness and excess kurtosis of v_{it} are not zero. An extended OPMD estimate of $\mathrm{Var}[S^{\circ}_{\texttt{FESAC}}(\theta_0)]$, taking into account the possible correlation between $\mathbf{g}_{N,it}(\theta_0)$ and $\mathbf{g}_{N,is}(\theta_0)$, $t \neq s$, is given as follows:

$$\widetilde{\mathbf{\Omega}}^{\mathrm{r}}_N = \textstyle\sum_{j=1}^N \big(\tilde{\mathbf{g}}_{Nj}\tilde{\mathbf{g}}'_{Nj} + \tilde{\mathbf{d}}_{Nj}\tilde{\mathbf{d}}^{\circ\prime}_{Nj}\big), \tag{4.13}$$

where $\tilde{\mathbf{g}}_{Nj}$ is given in (4.11), and $\tilde{\mathbf{d}}_{Nj}$ and $\tilde{\mathbf{d}}^{\circ}_{Nj}$ are the null estimates of $\mathbf{d}_{Nj}$ and $\mathbf{d}^{\circ}_{Nj}$, with $\mathbf{d}_{N,it} = \big\{\Pi'_{1,it}v^*_{it}, (v^{*2}_{it} - \sigma_0^2)\phi_{1,it}, (v^{*2}_{it} - \sigma_0^2)\phi_{2,it} + \Pi_{2,it}v^*_{it}, (v^{*2}_{it} - \sigma_0^2)\phi_{3,it}, \frac{1}{2\sigma_0^2}z'_{ni}(v^{*2}_{it} - \sigma_0^2)\big\}'$, and $\mathbf{d}^{\circ}_{N,it} = \sum_{s(\neq t)=1}^{T-1}\mathbf{d}_{N,is}$. The coefficients $\phi_{r,it}$ represent the diagonal elements of $\mathbf{\Phi}_r$, $r = 1, 2, 3$. Now, the asymptotic representation of the form (2.8) leads to $\mathrm{Var}[S^{\circ}_{\texttt{FESAC},\alpha}(\tilde{\theta}_N)] = \mathbf{\Omega}_{N,\alpha\alpha} - 2\mathbf{\Omega}_{N,\alpha\theta}\mathbf{\Gamma}'_N + \mathbf{\Gamma}_N\mathbf{\Omega}_{N,\theta\theta}\mathbf{\Gamma}'_N + o(N)$. A test statistic fully robust against non-normality thus takes the form:

$$T^{\mathtt{rr}}_{\texttt{FESAC}} = \big(\textstyle\sum_{j=1}^N \tilde{\mathbf{g}}'_{Nj,\alpha}\big)\big(\widetilde{\mathbf{\Omega}}^{\mathtt{r}}_{N,\alpha\alpha} - 2\widetilde{\mathbf{\Omega}}^{\mathtt{r}}_{N,\alpha\theta}\widetilde{\mathbf{\Gamma}}'_N + \widetilde{\mathbf{\Gamma}}_N\widetilde{\mathbf{\Omega}}^{\mathtt{r}}_{N,\theta\theta}\widetilde{\mathbf{\Gamma}}'_N\big)^{-1}\big(\textstyle\sum_{j=1}^N \tilde{\mathbf{g}}_{Nj,\alpha}\big), \tag{4.14}$$

where $\widetilde{\mathbf{\Omega}}^{\mathtt{r}}_{N,\alpha\alpha}$, $\widetilde{\mathbf{\Omega}}^{\mathtt{r}}_{N,\alpha\theta}$, and $\widetilde{\mathbf{\Omega}}^{\mathtt{r}}_{N,\theta\theta}$ are the submatrices of $\widetilde{\mathbf{\Omega}}^{\mathtt{r}}_N$.

Theorem 4.1. *Extending Assumption 3.1 to $\{v_{it}\}$ and Assumption 3.2 to $\mathbf{X}_N$, and keeping Assumptions 3.3 and 3.4, if $\tilde{\psi}_N$ is $\sqrt{N}$-consistent, and $\frac{1}{N}\mathbf{\Sigma}_{N,\theta\theta}$ and $\frac{1}{N}\mathrm{Var}[S^{\circ}_{\texttt{FESAC}}(\theta_0)]$ are positive definite for large enough N, then as n goes large (where T can be large or fixed),*

(i) $T_{\texttt{FESAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the errors are normal;

(ii) $T^{\mathtt{rr}}_{\texttt{FESAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the errors are either normal or non-normal.

4.3. Adjusted Score or Quasi-Score Tests

Following the same idea of Section 3.3, treating the elements of $\mathbf{V}_N$ completely independent (recall: they are independent across i, but in general are only uncorrelated across t unless

[13] This latter form is simpler and in fact more general as it can be applied to test other linear or nonlinear constraints on parameters under normality, e.g., testing jointly homoskedasticity and lack of spatial effects.

the original errors are normal), one can derive a potentially improved test of homoskedasticity for the FESAC model. Referring to Sections 4.1 and 4.2 for the notation, define

$$\tilde{\boldsymbol{\zeta}}^*_N(\lambda) = \tfrac{1}{2}\big\{\tfrac{1}{\mathbf{m}_j(\lambda_2)} v^2_{Nj}(\tilde{\beta}_N(\lambda),\lambda) - \tfrac{N}{N-p}\tilde{\sigma}^2_N(\lambda),\ j=1,\cdots,N\big\}_{N\times 1}, \tag{4.15}$$

where $\mathbf{m}_j(\lambda_2) = \sum_{\ell=1}^N \mathbf{M}^2_{N,j\ell}(\lambda_2)$. Define the set of adjusted concentrated quasi-scores at H_0:

$$S^*_{\texttt{FESAC}}(\lambda) = \begin{cases} \mathbf{Y}'_N(\lambda)\boldsymbol{\Phi}_1(\lambda)\mathbf{Y}_N(\lambda) - \frac{N}{N-p}\tilde{\sigma}^2_N(\lambda)\mathrm{tr}[\boldsymbol{\Phi}_1(\lambda)], \\ \mathbf{Y}'_N(\lambda)\boldsymbol{\Phi}_2(\lambda_2)\mathbf{Y}_N(\lambda) - \frac{N}{N-p}\tilde{\sigma}^2_N(\lambda)\mathrm{tr}[\boldsymbol{\Phi}_2(\lambda_2)], \\ \mathbf{Z}'_N\tilde{\boldsymbol{\zeta}}^*_N(\lambda), \end{cases} \tag{4.16}$$

where $\boldsymbol{\Phi}_1(\lambda) = \mathbf{M}_N(\lambda_2)[\mathbf{B}_{2N}(\lambda_2)\mathbf{G}_{1N}(\lambda_1)\mathbf{B}^{-1}_{2N}(\lambda_2) - \bar{\mathbf{G}}_{1N}(\lambda_1)I_N]$, $\boldsymbol{\Phi}_2(\lambda_2) = \mathbf{M}_N(\lambda_2)[\mathbf{G}_{2N}(\lambda_2) - \bar{\mathbf{G}}_{2N}(\lambda_2)I_N]\mathbf{M}_N(\lambda_2)$, and $\bar{\mathbf{G}}_{rN}(\lambda_r) = \frac{1}{N}\mathrm{tr}[\mathbf{G}_{rN}(\lambda_r)]$, $r=1,2$. Then, one can easily see that $\mathrm{E}[S^{\circ c}_{\texttt{FESAC}}(\lambda_0)|_{H_0}] = 0$. Similarly, at the true parameter values, we can write the first two components of $S^*_{\texttt{FESAC}}(\lambda_0)$ as $\mathbf{V}'_N\boldsymbol{\Phi}^*_1\mathbf{V}_N + \mathbf{V}'_N\Pi$ and $\mathbf{V}'_N\boldsymbol{\Phi}^*_2\mathbf{V}_N$, where $\Pi = \mathbf{M}_N[\mathbf{B}_{2N}\mathbf{G}_{1N}\mathbf{B}^{-1}_{2N} - \bar{\mathbf{G}}_{1N}I_N]\mathbf{X}_N(\lambda_{20})\beta_0$, and $\boldsymbol{\Phi}^*_r = \boldsymbol{\Phi}_r - \frac{1}{N-p}\mathrm{tr}(\boldsymbol{\Phi}_r)\mathbf{M}_N$, $r=1,2$. Define $\mathbf{g}^*_{Nj}(\theta)$, $j=1,\ldots,N$, in the same way as $\mathbf{g}^*_{ni}(\theta)$ in (3.15), we have an AQS test for H_0 for the FESAC model:

$$T^{\mathrm{r}*}_{\texttt{FESAC}} = \big(\textstyle\sum_{j=1}^N \tilde{\mathbf{g}}^{*\prime}_{Nj,\alpha}\big)\big[\sum_{j=1}^N(\tilde{\mathbf{g}}^*_{Nj,\alpha} - \tilde{\Gamma}^*_N\tilde{\mathbf{g}}^*_{Nj,\lambda})(\tilde{\mathbf{g}}^*_{Nj,\alpha} - \tilde{\Gamma}^*_N\tilde{\mathbf{g}}^*_{Nj,\lambda})'\big]^{-1}\big(\sum_{j=1}^N \tilde{\mathbf{g}}^*_{Nj,\alpha}\big), \tag{4.17}$$

where $\tilde{\Gamma}^*_N = \tilde{\boldsymbol{\Sigma}}^*_{N,\alpha\lambda}\tilde{\boldsymbol{\Sigma}}^{*-1}_{N,\lambda\lambda}$, $\tilde{\boldsymbol{\Sigma}}^*_{N,\alpha\lambda} = -\frac{\partial}{\partial\lambda}S^*_{\texttt{FESAC},\alpha}(\tilde{\lambda}_N)$, and $\tilde{\boldsymbol{\Sigma}}^*_{N,\lambda\lambda} = -\frac{\partial}{\partial\lambda}S^*_{\texttt{FESAC},\lambda}(\tilde{\lambda}_N)$. These derivatives can be easily obtained from (4.16), and are given in Appendix C. Numerical derivatives may provide much simpler and yet quite accurate alternatives, as indicated in Footnote 8 for the SAC model. When the errors are normally distributed, one may simply use $\tilde{\boldsymbol{\Sigma}}^*_{N,\alpha\lambda} = \sum_{j=1}^N \tilde{\mathbf{g}}^*_{Nj,\alpha}\tilde{\mathbf{g}}^{*\prime}_{Nj,\lambda}$ and $\tilde{\boldsymbol{\Sigma}}^*_{N,\lambda\lambda} = \sum_{J=1}^N \tilde{\mathbf{g}}^*_{Nj,\lambda}\tilde{\mathbf{g}}^{*\prime}_{Nj,\lambda}$ based on an IME corresponding to an 'adjusted likelihood', leading to an adjusted score test, denoted by $T^*_{\texttt{FESAC}}$ for easy reference.[14] Again, to ensure good finite sample properties of the tests based on the AS or QS functions, the adjusted estimator:

$$\tilde{\lambda}^*_N = \arg\{S^*_{\texttt{FESAC},\lambda}(\lambda) = 0\} \tag{4.18}$$

may be used in place of the regular estimator $\tilde{\lambda}_N$ as it is typical that $S^*_{\texttt{FESAC},\lambda}(\tilde{\lambda}_N) \neq 0$.

Again, the statistic $T^{\mathrm{r}*}_{\texttt{FESAC}}$ may not be fully robust against non-normality. Similar to the developments leading to $T^{\mathrm{rr}}_{\texttt{FESAC}}$, a robust estimator of $\boldsymbol{\Omega}^*_N = \mathrm{Var}[S^*_{\texttt{FESAC}}(\lambda_0)]$ is

$$\widetilde{\boldsymbol{\Omega}}^{\mathrm{r}*}_N = \textstyle\sum_{j=1}^N(\tilde{\mathbf{g}}^*_{Nj}\tilde{\mathbf{g}}^{*,\prime}_{Nj} + \tilde{\mathbf{d}}^*_{Nj}\tilde{\mathbf{d}}^{*\circ\prime}_{Nj}), \tag{4.19}$$

where $\mathbf{d}^*_{N,it} = \big\{\big(v^{*2}_{N,it} - \sigma^2_0\big)\phi^*_{1,it} + \Pi_{it}v^*_{it},\ \big(v^{*2}_{N,it} - \sigma^2_0\big)\phi^*_{2,it},\ z_{ni}\tilde{\zeta}_{N,it})\big\}$, and $\mathbf{d}^{*\circ}_{N,it} = \sum_{s(\neq t)=1}^{T-1}\mathbf{d}^*_{N,it}$. The coefficients $\phi^*_{r,it}$ represent the diagonal elements of $\boldsymbol{\Phi}^*_r$, $r=1,2$. A test statistic fully

[14] Alternatively, the generalized IME can be applied to give $\tilde{\boldsymbol{\Sigma}}^*_{N,\alpha\lambda} = \sum_{j=1}^N \tilde{\mathbf{g}}^*_{Nj,\alpha}\tilde{\mathbf{g}}'_{Nj,\lambda}$ and $\tilde{\boldsymbol{\Sigma}}^*_{N,\lambda\lambda} = \sum_{J=1}^N \tilde{\mathbf{g}}^*_{Nj,\lambda}\tilde{\mathbf{g}}'_{Nj,\lambda}$, where $\tilde{\mathbf{g}}_{Nj,\lambda}$ is the λ-component of $\tilde{\mathbf{g}}_{Nj}$ defined in (4.11), but Monte Carlo results show that the early version works better in finite samples.

robust against non-normality thus takes a similar form as $T^{\text{rr}}_{\texttt{FESAC}}$:

$$T^{\text{rr}*}_{\texttt{FESAC}} = \big(\textstyle\sum_{j=1}^{N} \tilde{\mathbf{g}}^{*\prime}_{Nj,\alpha}\big)\big(\widetilde{\mathbf{\Omega}}^{\text{r}*}_{N,\alpha\alpha} - 2\widetilde{\mathbf{\Omega}}^{\text{r}*}_{N,\alpha\lambda}\widetilde{\mathbf{\Gamma}}^{*\prime}_{N} + \widetilde{\mathbf{\Gamma}}^{*}_{N}\widetilde{\mathbf{\Omega}}^{\text{r}*}_{N,\lambda\lambda}\widetilde{\mathbf{\Gamma}}^{*\prime}_{N}\big)^{-1}\big(\textstyle\sum_{j=1}^{N} \tilde{\mathbf{g}}^{*}_{Nj,\alpha}\big), \tag{4.20}$$

where $\widetilde{\mathbf{\Omega}}^{\text{r}*}_{N,\alpha\alpha}$, $\widetilde{\mathbf{\Omega}}^{\text{r}*}_{N,\alpha\lambda}$, and $\widetilde{\mathbf{\Omega}}^{\text{r}*}_{N,\lambda\lambda}$ are the submatrices of $\widetilde{\mathbf{\Omega}}^{\text{r}*}_{N}$.

Theorem 4.2. *Under the assumptions of Theorem 4.1, $T^{rr*}_{\texttt{FESAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the errors are normal or non-normal; $T^{*}_{\texttt{FESAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the errors are normal.*

4.4. Time-wise heteroskedasticity and heterogeneity

As a panel data model may allow for a much richer structure than a cross-section model, it is necessary to extend the above theory and method to a richer FESAC model to allow for at least the following two additional features: (i) time-wise heteroskedasticity and (ii) time-specific fixed effects τ in addition to the individual-specific fixed effects μ.

The time-wise heteroskedasticity can be introduced by simply allowing $\{v_{it}\}$ to be independent $(0, \sigma^2 h(z'_{n,it}\alpha))$ with the values of the heteroskedasticity variable $z_{n,it}$ being allowed to change with both i and t. In this case, (4.5) becomes,

$$E(V^{*\prime}_{n1}, \ldots, V^{*\prime}_{n,T-1})'(V^{*\prime}_{n1}, \ldots, V^{*\prime}_{n,T-1}) = \sigma^2(F'_{T,T-1} \otimes I_n)\mathcal{H}_{nT}(\alpha)(F_{T,T-1} \otimes I_n) \equiv \sigma^2\mathbf{H}_N(\alpha),$$

where $\mathcal{H}_{nT}(\alpha) = \{h(z'_{n,it}\alpha)\}$. Thus, introducing time-wise heteroskedasticity induces time-wise non-zero correlation among $\{v^*_{it}\}$ although the cross-sectional independence is kept. Changes will occur on the expressions for the α-components of the score functions. However, there will be no additional technical complications as under the null, $\mathbf{H}_N(\alpha)|_{H_0} = I_N$ and $\{v_{it}\}$ become independent across both i and uncorrelated across t.

When the individual-specific FE μ and time-specific FE τ appear in the model additively, and when the spatial weight matrices are row-normalized, another layer of orthonormal transformation can be applied to wipe out the τ. Let $F_{n,n-1}$ be the orthonormal eigenvector matrix of $J_n = I_n - \frac{1}{n} l_n l'_n$ corresponding to the eigenvalues of one. For $n \times 1$ vectors $A_{nt}, t = 1, \ldots, T$, where A_{nt} can be Y_{nt}, V_{nt}, and a column of X_{nt}, define

$$[A^*_{n-1,1}, \ldots, A^*_{n-1,T-1}] = F'_{n,n-1}[A_{n,1}, \ldots, A_{n,T}]F_{T,T-1},$$

and $W^*_{rn} = F'_{n,n-1} W_{rn} F_{n,n-1}$. Let $N = (n-1)(T-1)$ and define $\mathbf{Y}_N$, $\mathbf{X}_N$, $\mathbf{U}_N$ and $\mathbf{V}_N$ accordingly. Then, the transformed model takes an identical form as (4.4). We have, when heteroskedasticity exists along both cross-section and time dimensions,

$$E(\mathbf{V}_N\mathbf{V}'_N) = \sigma^2(F'_{T,T-1} \otimes F'_{n,n-1})\mathcal{H}_{nT}(\alpha)(F_{T,T-1} \otimes F_{n,n-1}) \equiv \sigma^2\mathbf{H}_N(\alpha).$$

Clearly, at the null we again have $\mathbf{H}_N(\alpha)|_{H_0} = I_N$. Model estimation and the construction of the tests proceed as above. Additional complications will occur in the derivation of the non-normality robust tests, due to the lack of independence among the elements of $\mathbf{V}_N$ in both cross-section and time dimensions when the original errors are non-normal. For the same reason, proofs of the asymptotic properties of these tests will be more complicated as well. To save space, formal studies for the extended FESAC model are not pursued here.

5. Monte Carlo Study

Extensive Monte Carlo experiments are performed for assessing the finite sample performance of the four tests proposed in Section 3 for the SAC model and the six tests proposed in Section 4 for the FESAC model with individual-specific fixed effects. An important purpose is to solicit accurate and reliable tests based on the Monte Carlo results, and to make recommendations to practitioners.

5.1. General Settings

Cross-Sectional Case (CSC). We use the SAC model (3.1) which includes a spatial autoregressive structure for the disturbance vector U_n. The matrix X_n contains a constant (ι_n) and one regressor (x_n). Throughout the experiment the parameters are set at $\beta_0 = 5$, $\beta_1 = 1$, λ_1, $\lambda_2 = 0.2$, 0.8, and $n = 50, 100$, 200 and 500. For the spatial matrices, we assume that $W_{1n} = W_{2n} = W_n$. We have taken the spatial matrix W_n proposed by Kelejian and Prucha (1999), which is labelled "J ahead and J behind" with the non-zero elements being $1/2J$. Clearly, as J increases, the number of non-zero elements in the spatial weight matrix increases, which is in turn increases the 'degree' of the spatial dependence. The reported results correspond to $J = 5$. Moreover, following Baltagi and Yang (2013*b*), we have also considered three other schemes for generating the spatial weights matrices: (*i*) Rook contiguity, (*ii*) Queen contiguity and (*iii*) Group interactions. In the last one, the degree of spatial dependence grows with the sample size, which is achieved by relating the number of groups k to the sample size n, e.g., $k = n^{0.5}$, see Lee (2004). Two Data Generating Processes (DGP) are considered to generate the elements $\{x_i\}$ of the regressors x_n. The first one (DGP1) assumes that $\{x_i\}$ are *iid* $N(0, 1)$, whereas the second one (DGP2) considers that there might be systematic differences in $\{x_i\}$ across the different 'sets' of spatial units, see Baltagi and Yang (2013*b*) and Lee (2004). In this case, the ith value in the jth group, $\{x_{ij}\}$ of x_n are generated according to $\{x_{ij}\} = (z_j + \epsilon_{ij})/\sqrt{2}$ where $\{z_j\} \sim iid\ N(0, 1)$, $\{\epsilon_{ij}\}$ are *iid* $N(0, 1)$, and z_j and ϵ_{ij} are independent. This second scheme gives non-*iid* $\{x_i\}$ values in contrast to the first one, or different group means in terms of group interaction, see Lee (2004). The heteroskedasticity is generated according to $\sigma^2_{v_{ni}} = \sigma^2 \exp(\alpha z_{ni})$, where z_{ni} is taken to be x_{ni}, σ is set to 1, and $\alpha = 0, 1, 2$. If $\alpha = 0$, the disturbances are homoskedastic. For the DGP of disturbances, we assume that $v_{ni} = \sigma^2_{v_{ni}} e_i$, where $\{e_i\}$ are generated from either $N(0, 1)$, or a chi-square distribution with 3 degrees of freedom.

Panel Data Case (PDC). The FESAC model (4.1) is retained. It includes a SAR structure for the disturbance vector U_{nt} and a regressor X_{nt}. The fixed effects are generated by setting $\mu_n = \frac{1}{T}\sum_{t=1}^{T} X_{nt} + \omega_n$ where $\omega_n \sim N(0, I_n)$. Two DGPs are also considered for generating the regressors' values. In DGP1, we have $x_{it} = z_{it} + 0.1t$, where $\{z_{it}\}$ are *iid* $N(0,1)$. Thus the regressor includes a time trend $0.1t$. In DGP2, we first generate X_{nt} for each t according to the DGP2 for the SAC model and then add a time trend $0.1t$ on each X_{nt}, $t = 1, \ldots, T$. Four individual dimensions are considered $n = 50$, 100, 200 and 500 combined with the time dimension $T = 5$. Throughout the experiment the parameters are set at $\beta = 1$, λ_1, $\lambda_2 = 0.2, 0.8$. The spatial matrices are those that have been defined for the SAC model. The heteroskedasticity is generated according to $\sigma^2_{v_{ni}} = \sigma^2 \exp(\alpha z_{ni})$, where $z_{ni} = \frac{1}{T}\sum_{t=1}^{T} x^*_{it}$, $\sigma = 1$, and $\alpha = 0, 1, 2$. If $\alpha = 0$, the disturbances are homoskedastic. For the DGP of the disturbances, we assume that $v_{n,it} = \sigma^2_{v_{ni}} e_{it}$, where $\{e_{it}\}$ are generated from either $N(0,1)$ or a chi-square distribution with 3 degrees of freedom.

The regressors are treated as fixed in all the experiments. Each set of results, corresponding to a combination of the value of n, the values of λ_1 and λ_2, a DGP, a set of spatial weight matrices and an error distribution. Results are based on $5,000$ Monte Carlo replications. Three nominal sizes of the tests are considered: 10%, 5% and 1%.

5.2. Monte Carlo Results

Cross-Sectional Case (CSC). Tables 1-4 summarize the empirical sizes of the four tests: $T_{\texttt{SAC}}$, $T^{\mathrm{r}}_{\texttt{SAC}}$, $T^{*}_{\texttt{SAC}}$ and $T^{\mathrm{r}*}_{\texttt{SAC}}$, introduced in Section 3 for the SAC model, with Tables 1 and 2 corresponding to DGP1 and Tables 3 and 4 corresponding to DGP2. From the results, the following general observations are in order:

- (*i*) Among the four tests, the AQS test $T^{\mathrm{r}*}_{\texttt{SAC}}$ performs the best in the sense that its empirical size is in general quite close to its nominal level. The score test $T_{\texttt{SAC}}$ performs the worst, much worse than the other three in terms of size;
- (*ii*) Non-normality can have a big impact on the finite sample performance of the tests – size distortion can be much bigger when the errors are non-normal than when they are normal, except for the AQS test $T^{\mathrm{r}*}_{\texttt{SAC}}$ where the size distortions are at an 'acceptable' level even when $n = 50$;
- (*iii*) When errors are normal, the size of all tests converges to its nominal level as the sample size n increases. When the errors are non-normal, the two robust tests converge as expected. For the two non-robust tests, the score test $T_{\texttt{SAC}}$ still has a large size distortion even when sample size is 500, but the AS test $T^{*}_{\texttt{SAC}}$ has size quite close to its nominal level when n is large enough, showing that it is fairly robust against non-normality;
- (*iv*) Neither the values of spatial parameters nor the spatial weight matrices have a significant effect on the finite sample performance of the tests. One exception, is under normality, when sample size is not large, the last three tests can be slightly under-sized.
- (*v*) The way the regressor was generated (DGP1 vs DGP2) does not seem to have a significant impact on the finite sample performance of the tests.

Comparing the quasi score test $T^{r}_{\mathtt{SAC}}$ with the score test $T_{\mathtt{SAC}}$ (see Section 3.2), we see that the simple changes on $T_{\mathtt{SAC}}$ not only offer robustness against non-normality but also lead to huge improvements in its finite sample performance. Comparing the adjusted score test $T^{*}_{\mathtt{SAC}}$ with the score test $T_{\mathtt{SAC}}$, we see that some simple adjustments on the concentrated scores can lead to huge improvements in the finite sample performance of the test. Thus, a combination of the idea leading to the AS test and the idea leading to the QS test, we obtain an AQS test that not only is robust against non-normality but also has the best finite sample properties.

We have also studied the power properties of the four tests. The results (not reported for brevity) show that the size-adjusted power of the two non-robust tests is comparable, and that of the two robust tests are also comparable. Other Monte Carlo results include the empirical mean and standard deviation of the test statistics under the null. In light of the overall performance of the four tests, the AQS test $T^{\mathrm{r}*}_{\mathtt{SAC}}$ is recommended for practical applications. In case where n is fairly large, the QS test and the AS test can also be used.

[**Insert Tables 1-4**]

Panel Data Case (PDC). Tables 5-8 show the empirical size of the six tests introduced in Section 4 for the FESAC model: $T_{\mathtt{FESAC}}$, $T^{\mathrm{r}}_{\mathtt{FESAC}}$, $T^{\mathrm{rr}}_{\mathtt{FESAC}}$, $T^{*}_{\mathtt{FESAC}}$, $T^{\mathrm{r}*}_{\mathtt{FESAC}}$ and $T^{\mathrm{rr}*}_{\mathtt{FESAC}}$, with Tables 5 and 6 relating to DGP1 and Tables 7 and 8 relating to DGP2. The case of $(\lambda_1, \lambda_2) = (0.2, 0.8)$ and the power of the tests are not reported to save space. Monte Carlo experiments are also carried out under different values of T, and the results (not reported due to space constraints) reveal similar patterns.

Similar patterns are observed for the FESAC model as for the SAC model. In particular, the score test $T_{\mathtt{FESAC}}$ can have a large size distortion when n is small and the errors are non-normal, irrespective of the values of the spatial parameters, the spatial weight matrix structures, and the way the regressor was generated. Similar patterns are observed for the tests $T^{\mathrm{r}}_{\mathtt{FESAC}}$, $T^{*}_{\mathtt{FESAC}}$, and $T^{\mathrm{r}*}_{\mathtt{FESAC}}$, though the size-distortions are on a smaller scale when compared with the score test. The size of these four tests do not seem to converge to the nominal levels as the large size distortions remain even when $n = 500$ with $T = 5$.

In contrast, the two fully robust tests $T^{\mathrm{rr}}_{\mathtt{FESAC}}$ and $T^{\mathrm{rr}*}_{\mathtt{FESAC}}$ in general offer a great reduction in size distortion. The empirical size of these two tests converge to their nominal levels as n goes large where T can go large with n or stay fixed. Hence the two fully robust tests are both recommended for practical applications.

[**Insert Tables 5-8**]

A final discussion is given to the power of the tests. The proposed tests can be further compared once the 'size-adjusted power' of the similar tests is computed (i.e., score vs AS, and QS vs AQS). A small set of Monte Carlo results show that, once the tests are size-adjusted, their power performance is similar. This is expected as the tests are derived from the same set of 'score' functions. In this regard, the results on power are not reported to conserve space, but are available upon request from the authors.

6. Extensions

The methods considered in this paper have wide applicability. As discussed earlier (see, e.g., Footnotes 4 and 10), the methods developed for the cross-sectional SAC model and panel FESAC model can be extended to include spatial Durbin terms, higher-order spatial lags and higher-order spatial errors. Also, the SAR error in the model can be replaced by the spatial moving average error with simple changes in the relevant expressions. The methods can also be extended to spatial dynamic panel data (SDPD) models. The former extensions are straightforward, but the latter is not, due to the dynamic nature of the SDPD model. In this section, we will focus on the fixed effects SDPD model with short panels studied recently by Yang (2018a). We demonstrate that the OPMD method provides a consistent estimator of the variance but the traditional methods do not. This is due to the unobserved past history of the process (see Su and Yang, 2015). This point has been stressed throughout the paper for the usefulness of the OPMD method. The SDPD model (or the dynamic FESAC model) takes the following form:

$$\begin{aligned} Y_{nt} &= \rho Y_{n,t-1} + \lambda_1 W_{1n} Y_{nt} + \lambda_2 W_{2n} Y_{n,t-1} + X_{nt}\beta + \mu_n + U_{nt}, \\ & U_{nt} = \lambda_3 W_{3n} U_{nt} + V_{nt}, \quad t = 1, 2, \ldots, T. \end{aligned} \tag{6.1}$$

which extends Model (4.1) by adding the dynamic term $\rho Y_{n,t-1}$, and the space-time lag term $\lambda_2 W_{2n} Y_{n,t-1}$. It extends the model considered in Yang (2018*a*) by allowing for cross-sectional heteroskedasticity, i.e., $V_{nt} \sim (0, \sigma^2 \mathcal{H}_n(\alpha)), t = 1, \ldots, T$. Hence, a test for cross-sectional homoskedasticity corresponds to the test of null hypothesis $H_0 : \alpha = 0$.

The model specification implies that the data is available from $t = 0$. First-differencing (6.1) to eliminate the fixed effects μ_n, we have

$$\begin{aligned} \Delta Y_{nt} &= \rho \Delta Y_{n,t-1} + \lambda_1 W_{1n} \Delta Y_{nt} + \lambda_2 W_{2n} \Delta Y_{n,t-1} + \Delta X_{nt}\beta + \Delta u_t, \\ \Delta U_{nt} &= \lambda_3 W_{3n} \Delta U_{nt} + \Delta V_{nt}, \quad t = 2, \ldots, T. \end{aligned} \tag{6.2}$$

When T is fixed, the model (full or null) cannot be estimated consistently based on the conditional likelihood, conditional on the initial difference ΔY_{n1}. Yang (2018*a*) proposes an M-estimator for the null model that is consistent and asymptotically unbiased whether T is fixed or grows with n. The M-estimator is obtained by solving a set of unbiased estimating equations obtained by modifying the conditional score functions. This method can readily be extended to the SDPD model with cross-sectional heteroskedasticity $\mathcal{H}_n(\alpha)$.

Stacking the vectors and matrices in (6.2) for $t = 2, \ldots, T$, i.e., $\Delta \mathbf{Y}_N = \{\Delta Y'_{n2}, \ldots, \Delta Y'_{nT}\}'$, $\Delta \mathbf{Y}_{N,-1} = \{\Delta Y'_{n1}, \ldots, \Delta Y'_{n,T-1}\}'$, and similarly for $\Delta \mathbf{X}_N$ and $\Delta \mathbf{V}_N$. Let $\mathbf{W}_{rN} = I_{T-1} \otimes W_{rn}, r = 1, 2, 3$. Define $B_{rn}(\lambda_r) = I_n - \lambda_r W_{rn}, r = 1, 3$, and $B_{2n}(\rho, \lambda_2) = \rho I_n + \lambda_2 W_{2n}$. Let $\mathbf{B}_{rN}(\lambda_r) = I_{T-1} \otimes B_{rn}(\lambda_r), r = 1, 3$, and $\mathbf{B}_{2N}(\rho, \lambda_2) = I_{T-1} \otimes B_{2n}(\rho, \lambda_2)$. Let $\mathcal{Z}_{nj}$ be the diagonal matrix formed by the jth column Z_{nj} of Z_n, where Z_n is the $n \times k$ matrix of the k heteroskedasticity variables. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)'$ and $\theta = (\beta', \sigma^2, \rho, \lambda')'$. Let $\mathbf{C}_N = C_{T-1} \otimes I_n$, where C_{T-1} is a $(T-1)\times(T-1)$ constant matrix defined as

$$C_{T-1}=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

If V_{nt} are independent $N(0,\sigma^2\mathcal{H}_n(\alpha))$, then $\Delta\mathbf{V}_N\sim N[0,\sigma^2 C_{T-1}\otimes\mathcal{H}_n(\alpha)]$. From this, one can easily obtain the conditional (quasi) Gaussian likelihood for (θ,α), given ΔY_{n1}, and the conditional (quasi) Gaussian score function. Extending the results of Yang (2018a), we obtain a set of AQS functions for the full model.[15] To construct a test of $H_0:\alpha=0$, we have the AQS functions at H_0,

$$S^{\circ}_{\texttt{DFESAC}}(\theta)=\begin{cases}\frac{1}{\sigma^2}\Delta\mathbf{X}'_B\mathbf{B}'_{3N}(\lambda_3)\mathbf{C}_N^{-1}\Delta\mathbf{V}_N(\beta,\delta),\\ \frac{1}{2\sigma^4}\Delta\mathbf{V}'_N(\beta,\delta)\mathbf{C}_N^{-1}\Delta\mathbf{V}_N(\beta,\delta)-\frac{N}{2\sigma^2},\\ \frac{1}{\sigma^2}\Delta\mathbf{V}'_N(\beta,\delta)\mathbf{C}_N^{-1}\mathbf{B}_{3N}(\lambda_3)\Delta\mathbf{Y}_{N,-1}+\mathrm{tr}(\mathbf{C}_N^{-1}\mathbf{D}_{N,-1}),\\ \frac{1}{\sigma^2}\Delta\mathbf{V}'_N(\beta,\delta)\mathbf{C}_N^{-1}\mathbf{B}_{3N}(\lambda_3)\mathbf{W}_{1N}\Delta\mathbf{Y}_N+\mathrm{tr}(\mathbf{C}_N^{-1}\mathbf{D}_N\mathbf{W}_{1N}),\\ \frac{1}{\sigma^2}\Delta\mathbf{V}'_N(\beta,\delta)\mathbf{C}_N^{-1}\mathbf{B}_{3N}(\lambda_3)\mathbf{W}_{2N}\Delta\mathbf{Y}_{N,-1}+\mathrm{tr}(\mathbf{C}_N^{-1}\mathbf{D}_{N,-1}\mathbf{W}_{2N}),\\ \frac{1}{2\sigma^2}\Delta\mathbf{V}'_N(\beta,\delta)(C_{T-1}^{-1}\otimes G^{+}_{3n}(\lambda_3))\Delta\mathbf{V}_N(\beta,\delta)-(T-1)\mathrm{tr}(G_{3n}(\lambda_3)),\\ \frac{1}{2\sigma^2}\Delta\mathbf{V}'_N(\beta,\delta)(C_{T-1}^{-1}\otimes\mathcal{Z}_{nj})\Delta\mathbf{V}_N(\beta,\delta)-(T-1)Z'_{nj}1_n,\ j=1,\ldots,k,\end{cases}\tag{6.3}$$

where $\Delta\mathbf{V}_N(\beta,\delta)=\mathbf{B}_{3N}(\lambda_3)[\mathbf{B}_{1N}(\lambda_1)\Delta\mathbf{Y}_N-\mathbf{B}_{2N}(\rho,\lambda_2)\Delta\mathbf{Y}_{N,-1}-\Delta\mathbf{X}_N\beta]$, $\delta=(\delta,\lambda')'$, the unknown constant $\dot{h}(0)$ in the α-components of $S^{\circ}_{\texttt{DFESAC}}(\theta)$ is dropped, $G_{3n}(\lambda_3)=W_{3n}B_{3n}^{-1}(\lambda_3)$,

$$\mathbf{D}_{N,-1}=\begin{pmatrix} I_n, & 0, & \ldots & 0, & 0\\ \mathcal{B}_n-2I_n, & I_n, & \ldots & 0, & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathcal{B}_n^{T-4}(I_n-\mathcal{B}_n)^2, & \mathcal{B}_n^{T-5}(I_n-\mathcal{B}_n)^2, & \ldots & \mathcal{B}_n-2I_n, & I_n\end{pmatrix}\mathbf{B}_{1N}^{-1}(\lambda_1),$$

$$\text{and } \mathbf{D}_N=\begin{pmatrix} \mathcal{B}_n-2I_n, & I_n, & \ldots & 0\\ (I_n-\mathcal{B}_n)^2, & \mathcal{B}_n-2I_n, & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{B}_n^{T-3}(I_n-\mathcal{B}_n)^2, & \mathcal{B}_n^{T-4}(I_n-\mathcal{B}_n)^2, & \ldots & \mathcal{B}_n-2I_n\end{pmatrix}\mathbf{B}_{1N}^{-1}(\lambda_1),$$

and $\mathcal{B}_n\equiv\mathcal{B}_n(\rho,\lambda_1,\lambda_2)=B_{1n}^{-1}(\lambda_1)B_{2n}(\rho,\lambda_2)$. Note that $\mathcal{B}_n$ and hence $\mathbf{D}_{N,-1}$ and $\mathbf{D}_N$ depend on $(\rho,\lambda_1,\lambda_2)$. The same notation will be used when they are evaluated at the true parameter values $(\rho_0,\lambda_{10},\lambda_{20})$. For other parametric quantities, e.g., $\mathbf{B}_{1N}(\lambda_1)$, shorthand notation will

[15] Solving the resulting AQS equations leads to consistent and asymptotically unbiased estimators of the full model. However, the tests to be developed depend only on the estimation of the null model, and thus the the estimation of the full model will not be pursued in detail.

be used when evaluated at the true parameter values, e.g., $\mathbf{B}_{1N}$ for $\mathbf{B}_{1N}(\lambda_{10})$.

Now, from Lemma 3.2 of Yang (2018*a*) we have

$$\begin{aligned} \Delta\mathbf{Y}_N &= \mathbb{R}\ \Delta\mathbf{Y}_{n1} + \boldsymbol{\eta} + \mathbb{S}\Delta\mathbf{V}_N, \qquad (6.4)\\ \Delta\mathbf{Y}_{N,-1} &= \mathbb{R}_{-1}\Delta\mathbf{Y}_{n1} + \boldsymbol{\eta}_{-1} + \mathbb{S}_{-1}\Delta\mathbf{V}_N, \qquad (6.5)\end{aligned}$$

where $\Delta\mathbf{Y}_{n1} = 1_{T-1}\otimes\Delta Y_{n1}$, $\mathbb{R} = \text{blkdiag}(\mathcal{B}_n, \mathcal{B}_n^2, \ldots, \mathcal{B}_n^{T-1})$, $\mathbb{R}_{-1} = \text{blkdiag}(I_n, \mathcal{B}_n, \ldots, \mathcal{B}_n^{T-2})$, $\boldsymbol{\eta} = \mathbb{B}\mathbf{B}_1^{-1}\Delta\mathbf{X}\beta_0$, $\boldsymbol{\eta}_{-1} = \mathbb{B}_{-1}\mathbf{B}_1^{-1}\Delta\mathbf{X}\beta_0$, $\mathbb{S} = \mathbb{B}\mathbf{B}_1^{-1}\mathbf{B}_3^{-1}$, $\mathbb{S}_{-1} = \mathbb{B}_{-1}\mathbf{B}_1^{-1}\mathbf{B}_3^{-1}$,

$$\mathbb{B} = \begin{pmatrix} I_n & 0 & \ldots & 0 & 0 \\ \mathcal{B}_n & I_n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{B}_n^{T-2} & \mathcal{B}_n^{T-3} & \ldots & \mathcal{B}_n & I_n \end{pmatrix}, \quad \textit{and} \quad \mathbb{B}_{-1} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ I_n & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{B}_n^{T-3} & \mathcal{B}_n^{T-4} & \ldots & I_n & 0 \end{pmatrix}.$$

With (6.4) and (6.5), one immediately obtains

$$S^{\circ}_{\texttt{DFESAC}}(\theta_0) = \begin{cases} \Pi_1'\Delta\mathbf{V}_N, \\ \Delta\mathbf{V}_N'\mathbf{\Phi}_1\Delta\mathbf{V}_N - \frac{N}{2\sigma^2}, \\ \Delta\mathbf{V}_N'\mathbf{\Psi}_1\Delta\mathbf{Y}_{n1} + \Pi_2'\Delta\mathbf{V}_N + \Delta\mathbf{V}_N'\mathbf{\Phi}_2\Delta\mathbf{V}_N + \text{tr}(\mathbf{C}_N^{-1}\mathbf{D}_{N,-1}), \\ \Delta\mathbf{V}_N'\mathbf{\Psi}_2\Delta\mathbf{Y}_{n1} + \Pi_3'\Delta\mathbf{V}_N + \Delta\mathbf{V}_N'\mathbf{\Phi}_3\Delta\mathbf{V}_N + \text{tr}(\mathbf{C}_N^{-1}\mathbf{D}_N\mathbf{W}_{1N}), \\ \Delta\mathbf{V}_N'\mathbf{\Psi}_3\Delta\mathbf{Y}_{n1} + \Pi_4'\Delta\mathbf{V}_N + \Delta\mathbf{V}_N'\mathbf{\Phi}_4\Delta\mathbf{V}_N + \text{tr}(\mathbf{C}_N^{-1}\mathbf{D}_{N,-1}\mathbf{W}_{2N}), \\ \Delta\mathbf{V}_N'\mathbf{\Phi}_5\Delta\mathbf{V}_N - (T-1)\text{tr}(G_{3n}), \\ \Delta\mathbf{V}_N'\mathbf{\Phi}_{5+j}\Delta\mathbf{V}_N - (T-1)Z_{nj}'1_n, \ j = 1, \ldots, k, \end{cases} \qquad (6.6)$$

where

$$\begin{aligned} &\Pi_1 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\Delta\mathbf{X}_B, && \mathbf{\Phi}_1 = \tfrac{1}{2\sigma_0^4}\mathbf{C}_N^{-1}, \\ &\Pi_2 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\boldsymbol{\eta}_{-1}, && \mathbf{\Phi}_2 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbb{S}_{-1}, \\ &\Pi_3 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{1N}\boldsymbol{\eta}, && \mathbf{\Phi}_3 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{1N}\mathbb{S}, \\ &\Pi_4 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{2N}\boldsymbol{\eta}_{-1}, && \mathbf{\Phi}_4 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{2N}\mathbb{S}_{-1} \\ &\mathbf{\Psi}_1 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbb{R}_{-1}, && \mathbf{\Phi}_5 = \tfrac{1}{2\sigma_0^2}(C_{T-1}^{-1}\otimes G_{3n}^{+}), \\ &\mathbf{\Psi}_2 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{1N}\mathbb{R}, && \mathbf{\Phi}_{5+j} = \tfrac{1}{2\sigma_0^2}(C_{T-1}^{-1}\otimes\mathcal{Z}_{nj}), j = 1, \ldots, k \\ &\mathbf{\Psi}_3 = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{2N}\mathbb{R}_{-1}, && \end{aligned}$$

The expression for $S^{\circ}_{\texttt{DFESAC}}(\theta_0)$ given in (6.6) shows clearly that the usual plug-in method for estimating $\mathbf{\Sigma}_N = \text{Var}[S^{\circ}_{\texttt{DFESAC}}(\theta_0)]$ does not work as the analytical expression of $\mathbf{\Sigma}_N$ involves the unobservables contained in ΔY_{n1}, $\boldsymbol{\eta}_{-1}$ and $\boldsymbol{\eta}$. We show that an OPMD estimate of $\mathbf{\Sigma}_N$ can be derived when T is fixed, following the methods of Yang (2018*a*).

Now, for the general matrices Π, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ appearing in (6.6), denote by Π_t, Φ_{ts} and Ψ_{ts} their submatrices partitioned according to $t, s = 2, \ldots, T$. Define $\Psi_{t+} = \sum_{s=2}^{T}\Psi_{ts}$, $t = 2, \ldots, T$, $\Theta = \Psi_{2+}(B_{30}B_{10})^{-1}$, $\Delta Y_{n1}^{\circ} = B_{30}B_{10}\Delta Y_{n1}$, and $\Delta Y_{n1t}^{*} = \Psi_{t+}\Delta Y_{n1}$. Let $\{\mathcal{G}_{n,i}\}$ be the increasing sequence of σ-fields generated by $(v_{j1}, \ldots, v_{jT}, j = 1, \ldots, i), i = 1, \ldots, n$,

$n \geq 1$. Let $\mathcal{F}_{n,0}$ be the σ-field generated by $(v_0, \Delta y_0)$, and define $\mathcal{F}_{n,i} = \mathcal{F}_{n,0} \otimes \mathcal{G}_{n,i}$. Clearly, $\mathcal{F}_{n,i-1} \subseteq \mathcal{F}_{n,i}$, i.e., $\{\mathcal{F}_{n,i}\}_{i=1}^n$ is an increasing sequence of σ-fields, for each $n \geq 1$. Using Lemma 3.3 of Yang (2018a), the linear, quadratic and bilinear terms appearing in (6.6) can all be written as, $\Pi' \Delta \mathbf{V}_N = \sum_{i=1}^n g_{1i}$, $\Delta \mathbf{V}_N' \mathbf{\Phi} \Delta \mathbf{V}_N = \sum_{i=1}^n g_{2i}$, and $\Delta \mathbf{V}_N' \Psi \Delta \mathbf{Y}_{n1} - \mathrm{E}(\Delta \mathbf{V}_N' \Psi \Delta \mathbf{Y}_{n1}) = \sum_{i=1}^n g_{3i}$, so that $\{(g_{1i}', \ g_{2i}, \ g_{3i})', \mathcal{F}_{n,i}\}_{i=1}^n$ form a vector MD sequence, where

$$
\begin{aligned}
g_{1i} &= \textstyle\sum_{t=2}^T \Pi_{it}' \Delta v_{it}, & (6.7)\\
g_{2i} &= \textstyle\sum_{t=2}^T (\Delta v_{it} \Delta \xi_{it} + \Delta v_{it} \Delta v_{it}^* - \sigma_{v0}^2 d_{it}), & (6.8)\\
g_{3i} &= \Delta v_{2i} \Delta \zeta_i + \Theta_{ii} (\Delta v_{2i} \Delta y_{1i}^{\circ} + \sigma_{v0}^2) + \textstyle\sum_{t=3}^T \Delta v_{it} \Delta y_{1it}^*, & (6.9)
\end{aligned}
$$

$\{\Delta \xi_{it}\} = \Delta \xi_t = \sum_{s=2}^T (\Phi_{st}^{u\prime} + \Phi_{ts}^l) \Delta V_{ns}$, $\Delta V_{nt}^* = \sum_{s=2}^T \Phi_{ts}^d \Delta V_{ns}$, $\{d_{it}\}$ = diagonal elements of $\mathbf{C}_{T-1} \Phi$, $\{\Delta \zeta_i\} = \Delta \zeta = (\Theta^u + \Theta^l) \Delta Y_{n1}^{\circ}$, $\Delta Y_{n1}^{\circ} = B_{3n} B_{1n} \Delta Y_{n1}$, and $\mathrm{diag}\{\Theta_{ii}\} = \Theta^d$.

Finally, for each $\Pi_r, r = 1, 2, 3, 4$, appearing in (6.6), define g_{1ri} according to (6.7); for each $\Phi_r, r = 1, \ldots, 5+k$, define g_{2ri} according to (6.8); and for each $\Psi_r, r = 1, 2, 3$, define g_{3ri} according to (6.9). Let

$$
\mathbf{g}_{ni} = (g_{11i}', \ g_{21i}, \ g_{31i} + g_{12i} + g_{22i}, \ g_{32i} + g_{13i} + g_{23i}, \ g_{33i} + g_{14i} + g_{24i}, \ g_{25i}, \ldots, g_{2(5+k)i})'.
$$

Then, $S_{\mathtt{DFESAC}}^{\circ}(\theta_0) = \sum_{i=1}^n \mathbf{g}_{ni}$, and $\{\mathbf{g}_{ni}, \mathcal{F}_{n,i}\}$ form a vector MD sequence. It follows that $\mathbf{\Sigma}_N = \mathrm{Var}[S_{\mathtt{DFESAC}}^{\circ}(\theta_0)] = \sum_{i=1}^n \mathrm{E}(\mathbf{g}_{ni} \mathbf{g}_{ni}')$. The 'average' of the outer products of the estimated $\mathbf{g}_{ni}'s$ at H_0, i.e., $\frac{1}{N} \sum_{i=1}^n \tilde{\mathbf{g}}_{ni} \tilde{\mathbf{g}}_{ni}'$, gives a consistent estimate of $\frac{1}{N} \mathbf{\Sigma}_N$. For the same reasoning, we obtain a robust test statistic

$$
T_{\mathtt{DFESAC}}^r = \big(\textstyle\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}' \big) \big[\textstyle\sum_{i=1}^n (\tilde{\mathbf{g}}_{ni,\alpha} - \widetilde{\mathbf{\Gamma}}_N \tilde{\mathbf{g}}_{ni,\theta}) (\tilde{\mathbf{g}}_{ni,\alpha} - \widetilde{\mathbf{\Gamma}}_N \tilde{\mathbf{g}}_{ni,\theta})' \big]^{-1} \big(\textstyle\sum_{j=1}^n \tilde{\mathbf{g}}_{ni,\alpha} \big), \quad (6.10)
$$

where $\widetilde{\mathbf{\Gamma}}_N = \widetilde{\mathbf{\Sigma}}_{N,\alpha\theta} \widetilde{\mathbf{\Sigma}}_{N,\theta\theta}^{-1}$, $\widetilde{\mathbf{\Sigma}}_{N,\alpha\theta} = -\frac{\partial}{\partial \theta} S_{\mathtt{DFESAC},\alpha}(\tilde{\theta}_N)$, and $\widetilde{\mathbf{\Sigma}}_{N,\theta\theta} = -\frac{\partial}{\partial \theta} S_{\mathtt{DFESAC},\theta}(\tilde{\theta}_N)$. These derivatives can be easily obtained from (6.3). The $\tilde{\theta}_N$ is the M-estimator of Yang (2018a) for the null model, which solves $S_{\mathtt{DFESAC},\theta}(\theta) = 0$. Under regularity conditions of Yang (2018a) and additional conditions on Z_n given earlier, one can show that under H_0, $T_{\mathtt{DFESAC}}^r \xrightarrow{D} \chi_k^2$, when the errors are normal or non-normal. We note that even when the errors are normal, the test does not have a simplified version as for SAC or FESAC model. In this case the AQS function is not the true score function so that the information matrix equality (IME) does not hold, and the generalized IME cannot be applied as the true score function is unknown.

Tests for homogeneity can be developed in the same manner for the several interesting submodels, i.e., models obtained by dropping some λ terms. For details on the M-estimation of these submodels, see Yang (2018b).

Further Adjusted Test. To improve the finite sample performance, the test given above can be further adjusted by working with the concentrated AQS function with β and σ^2 being concentrated out. The constrained M-estimators of β and σ^2 given $\delta = (\rho, \lambda')'$ are $\tilde{\beta}_N(\delta) = [\Delta \mathbf{X}_N'(\lambda_3) \mathbf{C}_N^{-1} \Delta \mathbf{X}_N(\lambda_3)]^{-1} \Delta \mathbf{X}_N'(\lambda_3) \mathbf{C}_N^{-1} \mathbf{B}_{3N}(\lambda_3) [\mathbf{B}_{1N}(\lambda_3) \mathbf{Y}_N - \mathbf{B}_{2N}(\rho, \lambda_2) \mathbf{Y}_{N,-1}]$, and $\tilde{\sigma}_N^2(\delta) = \frac{1}{N} \Delta \widetilde{\mathbf{V}}_N'(\delta) \mathbf{C}_N^{-1} \Delta \widetilde{\mathbf{V}}_N(\delta)$, where $\Delta \widetilde{\mathbf{V}}_N(\delta) = \Delta \mathbf{V}_N(\tilde{\beta}_N(\delta), \delta)$. Substituting $\tilde{\beta}_N(\delta)$

and $\hat{\sigma}_N^2(\delta)$ back into the last five components of the AQS function in (6.3) gives the concentrated AQS functions of δ and α evaluated at H_0:

$$S^{\circ c}_{\texttt{DFESAC}}(\delta) = \begin{cases} \frac{1}{\tilde{\sigma}_N^2(\delta)} \Delta \widetilde{\mathbf{V}}'_N(\delta) \mathbf{C}_N^{-1} \mathbf{B}_{3N}(\lambda_3) \Delta \mathbf{Y}_{N,-1} + \text{tr}(\mathbf{C}_N^{-1} \mathbf{D}_{N,-1}), \\ \frac{1}{\tilde{\sigma}_N^2(\delta)} \Delta \widetilde{\mathbf{V}}'_N(\delta) \mathbf{C}_N^{-1} \mathbf{B}_{3N}(\lambda_3) \mathbf{W}_{1N} \Delta \mathbf{Y}_N + \text{tr}(\mathbf{C}_N^{-1} \mathbf{D}_N \mathbf{W}_{1N}), \\ \frac{1}{\tilde{\sigma}_N^2(\delta)} \Delta \widetilde{\mathbf{V}}'_N(\delta) \mathbf{C}_N^{-1} \mathbf{B}_{3N}(\lambda_3) \mathbf{W}_{2N} \Delta \mathbf{Y}_{N,-1} + \text{tr}(\mathbf{C}_N^{-1} \mathbf{D}_{N,-1} \mathbf{W}_{2N}), \\ \frac{1}{2\tilde{\sigma}_N^2(\delta)} \Delta \widetilde{\mathbf{V}}'_N(\delta) (C_{T-1}^{-1} \otimes G_{3n}^{+}(\lambda_3)) \Delta \widetilde{\mathbf{V}}_N(\delta) - (T-1) \text{tr}(G_{3n}(\lambda_3)), \\ \frac{1}{2\tilde{\sigma}_N^2(\delta)} \Delta \widetilde{\mathbf{V}}'_N(\delta) (C_{T-1}^{-1} \otimes \mathcal{Z}_{nj}) \Delta \widetilde{\mathbf{V}}_N(\delta) - (T-1) Z'_{nj} 1_n, j = 1, \ldots, k, \end{cases} \tag{6.11}$$

It is easy to see that $\mathbf{C}_N^{-1/2} \Delta \widetilde{\mathbf{V}}'_N(\delta_0) = \mathbf{M}_N \mathbf{C}_N^{-1/2} \mathbf{V}_N$ and $\tilde{\sigma}_N^2(\delta_0) = \frac{1}{N} \mathbf{V}'_N \mathbf{C}_N^{-1} \mathbf{V}_N$, where $\mathbf{C}_N^{1/2}$ is a asquare-root matrix of $\mathbf{C}_N$ and $\mathbf{M}_N = I_N - \Delta \mathbb{X}_N (\Delta \mathbb{X}'_N \Delta \mathbb{X}_N)^{-1} \Delta \mathbb{X}'_N$ with $\Delta \mathbb{X}_N = \mathbf{C}_N^{-1/2} \mathbf{B}_{3N} \Delta \mathbf{X}_n$. At the true δ_0, we have

$$\tilde{\sigma}_N^2(\delta_0) S^{\circ c}_{\texttt{DFESAC}}(\delta_0) = \begin{cases} \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} \mathbf{B}_{3N} \Delta \mathbf{Y}_{N,-1} + \mu_\rho \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} \mathbf{V}_N, \\ \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} \mathbf{B}_{3N} \mathbf{W}_{1N} \Delta \mathbf{Y}_N + \mu_{\lambda_1} \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} \mathbf{V}_N, \\ \Delta \mathbf{V}_N \mathbf{M}_N^{\circ} \mathbf{B}_{3N} \mathbf{W}_{2N} \Delta \mathbf{Y}_{N,-1} + \mu_{\lambda_2} \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} \mathbf{V}_N, \\ \frac{1}{2} \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} (C_{T-1} \otimes G_{3n}^{+}) \mathbf{M}_N^{\circ} \Delta \mathbf{V}_N - \mu_{\lambda_3} \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} \mathbf{V}_N, \\ \frac{1}{2} \Delta \mathbf{V}_N \mathbf{M}_N^{\circ} (C_{T-1} \otimes \mathcal{Z}_{nj}) \mathbf{M}_N^{\circ} \Delta \mathbf{V}_N - \mu_{\alpha_j} \Delta \mathbf{V}'_N \mathbf{M}_N^{\circ} \mathbf{V}_N, j = 1, \ldots, k, \end{cases}$$

where $\mathbf{M}_N^{\circ} = \mathbf{C}_N^{-1/2} \mathbf{M}_N \mathbf{C}_N^{-1/2}$, $\mu_\rho = \frac{1}{N} \text{tr}(\mathbf{C}_N^{-1} \mathbf{D}_{N,-1})$, $\mu_{\lambda_1} = \frac{1}{N} \text{tr}(\mathbf{C}_N^{-1} \mathbf{D}_N \mathbf{W}_{1N})$, $\mu_{\lambda_2} = \frac{1}{N} \text{tr}(\mathbf{C}_N^{-1} \mathbf{D}_{N,-1} \mathbf{W}_{2N})$, $\mu_{\lambda_3} = \frac{T-1}{N} \text{tr}(G_{3n}(\lambda_3))$, and $\mu_{\alpha_j} = \frac{T-1}{N} Z'_{nj} 1_n, j = 1, \ldots, k$.

Using the results of Lemma 3.1 of Yang (2018*a*): $\text{E}(\Delta \mathbf{Y}_{N,-1} \Delta V_N) = -\sigma_0^2 \mathbf{D}_{N,-1} \mathbf{B}_{3N}^{-1}$ and $\text{E}(\Delta \mathbf{Y}_N \Delta V_N) = -\sigma_0^2 \mathbf{D}_N \mathbf{B}_{3N}^{-1}$, one can easily find $\mu_N^* = \text{E}[\tilde{\sigma}_N^2(\delta_0) S^{\circ c}_{\texttt{DFESAC}}(\delta_0)]$ with elements,

$$\begin{aligned} \mu_\rho^* &= \sigma_0^2 \text{tr}(\mathbf{M}_N^{\circ} \mathbf{B}_{3N} \mathbf{D}_{N,-1} \mathbf{B}_{3N}^{-1} - \mu_\rho \mathbf{C}_N \mathbf{\Phi}_1), \\ \mu_{\lambda_1}^* &= \sigma_0^2 \text{tr}(\mathbf{M}_N^{\circ} \mathbf{B}_{3N} \mathbf{W}_{1N} \mathbf{D}_N \mathbf{B}_{3N}^{-1} - \mu_{\lambda_1} \mathbf{C}_N \mathbf{\Phi}_2), \\ \mu_{\lambda_2}^* &= \sigma_0^2 \text{tr}(\mathbf{M}_N^{\circ} \mathbf{B}_{3N} \mathbf{W}_{2N} \mathbf{D}_{N,-1} \mathbf{B}_{3N}^{-1} - \mu_{\lambda_2} \mathbf{C}_N \mathbf{\Phi}_3), \\ \mu_{\lambda_3}^* &= \sigma_0^2 \mu_{\lambda_3} \text{tr}(\mathbf{C}_N \mathbf{\Phi}_4), \\ \mu_{\alpha_j}^* &= \sigma_0^2 \mu_{\alpha_j} \text{tr}(\mathbf{C}_N \mathbf{\Phi}_{4+j}),\ j = 1, \ldots, k, \end{aligned}$$

leading to unbiased AQS functions at δ_0:

$$S^*_{\texttt{DFESAC}}(\delta_0) = \begin{cases} \Delta \mathbf{V}'_N \mathbf{\Psi}_1^* \Delta \mathbf{Y}_{n1} + \Pi_1^{*\prime} \Delta \mathbf{V}_N + \Delta \mathbf{V}'_N \mathbf{\Phi}_1^* \Delta \mathbf{V}_N - \mu_\rho^*, \\ \Delta \mathbf{V}'_N \mathbf{\Psi}_2^* \Delta \mathbf{Y}_{n1} + \Pi_2^{*\prime} \Delta \mathbf{V}_N + \Delta \mathbf{V}'_N \mathbf{\Phi}_2^* \Delta \mathbf{V}_N - \mu_{\lambda_1}^*, \\ \Delta \mathbf{V}'_N \mathbf{\Psi}_3^* \Delta \mathbf{Y}_{n1} + \Pi_3^{*\prime} \Delta \mathbf{V}_N + \Delta \mathbf{V}'_N \mathbf{\Phi}_3^* \Delta \mathbf{V}_N - \mu_{\lambda_2}^*, \\ \Delta \mathbf{V}'_N \mathbf{\Phi}_4^* \Delta \mathbf{V}_N - \mu_{\lambda_3}^*, \\ \Delta \mathbf{V}'_N \mathbf{\Phi}_{4+j}^* \Delta \mathbf{V}_N - \mu_{\alpha_j}^*,\ j = 1, \ldots, k, \end{cases} \tag{6.12}$$

where

$$\Pi_1^* = \mathbf{M}_N^\circ \mathbf{B}_{3N}\boldsymbol{\eta}_{-1}, \qquad \boldsymbol{\Psi}_1^* = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbb{R}_{-1}, \qquad \boldsymbol{\Phi}_1^* = \mathbf{M}_N^\circ \mathbf{B}_{3N}\mathbb{S}_{-1} + \mu_\rho \mathbf{M}_N^\circ,$$
$$\Pi_2^* = \mathbf{M}_N^\circ \mathbf{B}_{3N}\mathbf{W}_{1N}\boldsymbol{\eta}, \quad \boldsymbol{\Psi}_2^* = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{1N}\mathbb{R}, \quad \boldsymbol{\Phi}_2^* = \mathbf{M}_N^\circ \mathbf{B}_{3N}\mathbf{W}_{1N}\mathbb{S} + \mu_{\lambda_1} \mathbf{M}_N^\circ,$$
$$\Pi_3^* = \mathbf{M}_N^\circ \mathbf{B}_{3N}\mathbf{W}_{2N}\boldsymbol{\eta}_{-1}, \; \boldsymbol{\Psi}_3^* = \tfrac{1}{\sigma_0^2}\mathbf{C}_N^{-1}\mathbf{B}_{3N}\mathbf{W}_{2N}\mathbb{R}_{-1}, \; \boldsymbol{\Phi}_3 = \mathbf{M}_N^\circ \mathbf{B}_{3N}\mathbf{W}_{2N}\mathbb{S}_{-1} + \mu_{\lambda_2} \mathbf{M}_N^\circ,$$
$$\boldsymbol{\Phi}_4^* = \tfrac{1}{1}\mathbf{M}_N^\circ (C_{T-1} \otimes G_{3n}^+)\mathbf{M}_N^\circ - \mu_{\lambda_3}\mathbf{M}_N^\circ,$$
$$\boldsymbol{\Phi}_{4+j}^* = \tfrac{1}{2}\mathbf{M}_N^\circ (C_{T-1} \otimes \mathcal{Z}_{nj})\mathbf{M}_N^\circ - \mu_{\alpha_j}\mathbf{M}_N^\circ, j = 1, \ldots, k.$$

A potentially improved test statistic can be constructed based on $S^\circ_{\texttt{DFESAC}}(\delta_0)$ in an identical manner as for $T^r_{\texttt{DFESAC}}$:

$$T^{r*}_{\texttt{DFESAC}} = \big(\textstyle\sum_{i=1}^n \tilde{\mathbf{g}}^{*\prime}_{ni,\alpha}\big)\big[\textstyle\sum_{i=1}^n (\tilde{\mathbf{g}}^*_{ni,\alpha} - \widetilde{\boldsymbol{\Gamma}}^*_N \tilde{\mathbf{g}}^*_{ni,\delta})(\tilde{\mathbf{g}}^*_{ni,\alpha} - \tilde{\boldsymbol{\Gamma}}^*_N \tilde{\mathbf{g}}_{ni,\delta})^{*\prime}\big]^{-1}\big(\textstyle\sum_{j=1}^n \tilde{\mathbf{g}}^*_{ni,\alpha}\big), \quad (6.13)$$

where $\widetilde{\boldsymbol{\Gamma}}^*_N = \widetilde{\boldsymbol{\Sigma}}^*_{N,\alpha\delta}\widetilde{\boldsymbol{\Sigma}}^{*-1}_{N,\delta\delta}$, $\widetilde{\boldsymbol{\Sigma}}^*_{N,\alpha\delta} = -\frac{\partial}{\partial\delta}S^\circ_{\texttt{DFESAC},\alpha}(\tilde{\delta}_N)$, and $\widetilde{\boldsymbol{\Sigma}}^*_{N,\delta\delta} = -\frac{\partial}{\partial\delta}S^*_{\texttt{DFESAC},\delta}(\tilde{\delta}_N)$. These derivatives can be easily obtained from (6.11). It can be shown that under H_0, $T^{r*}_{\texttt{DFESAC}} \xrightarrow{D} \chi^2_k$, whether the errors are normal or non-normal. Similarly, improved tests can be developed for various submodels discussed in Yang (2018*b*).

7. Conclusion

In this paper, we have developed new diagnostic tests for homoskedasticity in cross-sectional and panel data spatial econometric models. We have also suggested general methodologies to robustify these tests against non-normality and finite sample dimensions. Theoretical asymptotic properties of the testing procedures are formally examined whereas their finite samples are investigated through Monte Carlo experiments. We show that our procedures can easily take into account the time-wise heteroskedasticity and a more complex structure of heterogeneity, i.e. time and individual fixed effects. In addition, these homoskedastic testing procedures can also be extended to include spatial Durbin terms, higher-order spatial lags, higher-order spatial errors and SDPD models. The former extensions are straightforward, but the latter is not. This is due to the dynamic structure of the SDPD model. We have demonstrated that the OPMD method provides a consistent estimator of the variance. Furthermore, our Monte Carlo results show that our testing procedures perform well in the context of finite samples and non-normality of the disturbances, especially for the robust versions of the tests.

Moreover, the tests can be repeatedly run with different choices of the heteroskedasticity variables. In this sense, our tests provide tools for identifying the 'source' of heteroskedasticity: the heteroskedasticity variables with which the test is rejected. In this case, one may proceed with a heteroskedastic model by 'specifying' a form for the unknown function $h(\cdot)$, e.g., the popular exponential form, or non-parametrically estimating it. This is an important point to overpass the fact that specific procedures taking into account heteroskedasticity are not necessarily available. Last, an interesting extension of these testing procedures could be to apply them to nested and non-nested multi-dimensional panels. This is part of our ongoing research agenda.

Appendix A: Some Basic Lemmas

The proofs of the main results depend on the following lemmas. The results state explicitly that the *degree of spatial dependence* may grow with the sample size, i.e., elements of W_{rn}, $r = 1, 2$, are of uniform order $O(h_n^{-1})$ where h_n is such that $\lim_{n\to\infty}(h_n/n) = 0$. See Lee (2004).

Lemma A.1. *(Kelejian and Prucha, 1999; Lee, 2002): Let $\{A_n\}$ and $\{B_n\}$ be two sequences of $n \times n$ matrices that are uniformly bounded in both row and column sums. Let C_n be a sequence of conformable matrices whose elements are uniformly $O(h_n^{-1})$, where $\{h_n\}$ is a sequence of constants bounded or divergent with n. Then,*

(i) the sequence $\{A_nB_n\}$ are uniformly bounded in both row and column sums,
(ii) the elements of A_n are uniformly bounded and $tr(A_n) = O(n)$, and
(iii) the elements of A_nC_n and C_nA_n are uniformly $O(h_n^{-1})$.

Lemma A.2. *(Lee, 2004, Appendix A): For W_{rn} and $B_{rn}(\lambda_r)$, $r = 1, 2$, defined for the SAC model, if $\|W_{rn}\|$ and $\|B_{rn}^{-1}\|$ at true λ_{r0} are uniformly bounded, where $\|\cdot\|$ is a matrix norm, then $\|B_{rn}^{-1}(l_r)\|$ is uniformly bounded for l_r in a neighborhood of λ_{r0}.*

Lemma A.3. *(Lee, 2004, Appendix A): Let X_n be an $n\times p$ matrix. If the elements X_n are uniformly bounded and $\lim_{n\to\infty} \frac{1}{n}X_n'X_n$ exists and is nonsingular, then $P_n = X_n(X_n'X_n)^{-1}X_n'$ and $M_n = I_n - P_n$ are uniformly bounded in both row and column sums.*

Lemma A.4. *(Lemma B.4, Yang, 2015, extended): Let $\{\Phi_n\}$ be a sequence of $n \times n$ matrices that are uniformly bounded in either row or column sums. Suppose that the elements $\phi_{n,ij}$ of Φ_n are $O(h_n^{-1})$ uniformly in all i and j. Let V_n be a random n-vector of iid elements with mean zero, variance σ^2 and finite 4th moment, and b_n a constant n-vector of elements of uniform order $O(h_n^{-1/2})$, where h_n is such that $\lim_{n\to\infty}(h_n/n) = 0$. Then,*

(i) $\mathrm{E}(V_n'\Phi_nV_n) = O(\frac{n}{h_n})$,	*(ii)* $\mathrm{Var}(V_n'\Phi_nV_n) = O(\frac{n}{h_n})$,
(iii) $\mathrm{Var}(V_n'\Phi_nV_n + b_n'V_n) = O(\frac{n}{h_n})$,	*(iv)* $V_n'\Phi_nV_n = O_p(\frac{n}{h_n})$,
(v) $V_n'\Phi_nV_n - \mathrm{E}(V_n'\Phi_nV_n) = O_p((\frac{n}{h_n})^{\frac{1}{2}})$,	*(vi)* $V_n'\Phi_nb_n = O_p((\frac{n}{h_n})^{\frac{1}{2}})$,

Lemma A.5. *(Lee, 2004, Appendix A): Consider the linear-quadratic form of $V_n = (v_{n1}, v_{n2}, \ldots, v_{nn})'$: $Q_n = b_n'V_n + V_n'\Phi_nV_n$, where $\{v_{ni}\}$ are iid with mean zero and variance σ^2, $\{\Phi_n\}$ is a sequence of symmetric matrices with row and column sums being uniformly bounded in absolute value,*[16] *and $\{b_n\}$ is a sequence of constant vectors with its elements being uniformly bounded. Let $\sigma^2_{Q_n}$ be the variance of Q_n. Assume that $\sigma^2_{Q_n}$ is $O(n/h_n)$ with $(h_n/n)\sigma^2_{Q_n}$ being bounded away from zero, the elements of Φ_n are of uniform order $O(h_n^{-1})$, the elements of b_n are of uniform order $O(h_n^{-1/2})$, and the moment $\mathrm{E}(|v_{ni}|^{4+2\delta})$ exists for some $\delta > 0$. If $\lim_{n\to\infty}(h_n^{1+2/\delta}/n) = 0$, then $(Q_n - \sigma^2\mathrm{tr}(\Phi_n))/\sigma_{Q_n} \xrightarrow{D} N(0, 1)$.*

[16] If Φ_n is not symmetric, it can be replaced by $\frac{1}{2}(\Phi_n + \Phi_n')$, as $V_n'\Phi_nV_n = \frac{1}{2}(V_n'\Phi_nV_n + (V_n'\Phi_nV_n)')$.

Lemma A.6. *Let* $\mathbf{Q}_n = (Q_{rn}, r = 1, \ldots, m)'$, *where* $Q_{rn} = b'_{rn}V_n + V'_n\Phi_{rn}V_n$ *with* V_n, b_{rn} *and* Φ_{rn} *satisfying the conditions of Lemma A.5. Write* $\Phi_{rn} = \Phi^u_{rn} + \Phi^l_{rn} + \Phi^d_{rn}$, *the sum of the upper triangular, lower triangular, and diagonal matrices of* Φ_{rn}. *Define*

$$g_{rn,i} = v_{ni}\xi_{rn,i} + b_{rn,i}v_{ni} + (v^2_{ni} - \sigma^2)\phi_{rn,ii},\ r = 1, \ldots, m,$$

where $\{\xi_{rn,i}\} = \xi_{rn} = (\Phi^{u\prime}_{rn} + \Phi^l_{rn})V_n$. *Let* $\mathbf{g}_{ni} = (g_{rn,i}, r = 1, \ldots, m)'$. *Then,* $\{\mathbf{g}_{ni}, \mathcal{F}_{ni}\}$ *form a vector martingale difference sequence with respect to the increasing* σ*-fields* $\mathcal{F}_{ni}$ *generated by* $\{v_{n1}, \cdots, v_{ni}\}$, *such that (i)* $\mathbf{Q}_n - \mathrm{E}(\mathbf{Q}_n) = \sum_{i=1}^n \mathbf{g}_{ni}$, *(ii)* $\mathrm{Var}(\mathbf{Q}_n) = \sum_{i=1}^n \mathrm{E}(\mathbf{g}_{ni}\mathbf{g}'_{ni})$, *and (iii)* $\frac{h_n}{n}[\sum_{i=1}^n \mathbf{g}_{ni}\mathbf{g}'_{ni} - \mathrm{Var}(\mathbf{Q}_n)] = o_p(1)$.

Proof of Lemma A.6: We have for each $Q_{jn}, j = 1, \cdots, m$,

$$\begin{aligned} Q_{rn} - \mathrm{E}(Q_{rn}) &= b'_{rn}V_n + V'_n\Phi_{rn}V_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\ &= b'_{rn}V_n + V'_n(\Phi^u_{rn} + \Phi^l_{rn} + \Phi^d_{rn})V_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\ &= V'_n(\Phi^{u\prime}_{rn} + \Phi^l_{rn})V_n + b'_{rn}V_n + V'_n\Phi^d_{rn}V_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\ &= V'_n\xi_n + b'_{rn}V_n + V'_n\Phi^d_{rn}V_n - \sigma^2\mathrm{tr}(\Phi_{rn}) \\ &= \textstyle\sum_{i=1}^n[v_{ni}\xi_{rn,i} + b_{rn,i}v_{ni} + (v^2_{ni} - \sigma^2)\phi_{rn,ii}] = \sum_{i=1}^n g_{rn,i}. \end{aligned}$$

As $\xi_{rn,i}$ is $\mathcal{F}_{n,i-1}$ measurable, $\mathrm{E}(g_{jn,i}|\mathcal{F}_{n,i-1}) = 0$ for $j = 1, \cdots, m$. It follows that $\{\mathbf{g}_{ni}, \mathcal{F}_{ni}\}$ form a vector MD sequence with respect to $\mathcal{F}_{ni}$, and that $\mathrm{Var}(\mathbf{Q}_n) = \sum_{i=1}^n \mathrm{E}(\mathbf{g}_{ni}\mathbf{g}'_{ni})$, as $\{\mathbf{g}_{ni}\}$ are uncorrelated. It left to show *(iii)*. It is easy to show that, for $r, s = 1, \cdots, m$,

$$\begin{aligned} \mathrm{Cov}(Q_{rn}, Q_{sn}) &= 2\sigma^4 \textstyle\sum_{i=1}^n \sum_{j=1}^n \phi_{rn,ij}\phi_{sn,ij} + \sigma^2 \sum_{i=1}^n b_{rn,i}b_{sn,i} \\ &\quad + (\mu^{(4)} - 3) \textstyle\sum_{i=1}^n \phi_{rn,ii}\phi_{sn,ii} + \mu^{(3)} \sum_{i=1}^n (b_{rn,i}\phi_{sn,ii} + b_{sn,i}\phi_{rn,ii}), \end{aligned}$$

where $\mu^{(3)} = \mathrm{E}(v^3_{ni})$ and $\mu^{(4)} = \mathrm{E}(v^4_{ni})$. This gives, for $r, s = 1, \cdots, m$,

$$\begin{aligned} &\textstyle\sum_{i=1}^n g_{rn,i}g'_{sn,i} - \mathrm{Cov}(Q_{rn}, Q_{sn}) \\ &= \textstyle\sum_{i=1}^n(v^2_{ni}\xi_{rn,i}\xi_{sn,i} - 2\sigma^2 \sum_{j=1, j\neq i}^n \phi_{rn,ij}\phi_{sn,ij}) + \sum_{i=1}^n[v^2_{ni}(\xi_{rn,i}b_{sn,i} + \xi_{sn,i}b_{rn,i})] \\ &\quad + \textstyle\sum_{i=1}^n[(v^3_{ni} - \sigma^2 v_{ni})(\xi_{rn,i}\phi_{sn,ii} + \xi_{sn,i}\phi_{rn,ii})] + \sum_{i=1}^n(v^2_{ni} - \sigma^2)b_{rn,i}b_{sn,i} \\ &\quad + \textstyle\sum_{i=1}^n(v^3_{ni} - \mu^{(3)})(b_{rn,i}\phi_{sn,i} + b_{sn,i}\phi_{rn,i}) \\ &\quad + \textstyle\sum_{i=1}^n[(v^3_{ni} - \mu^{(3)} - 2\sigma^2(v^2_{ni} - \sigma^2))\phi_{rn,ii}\phi_{sn,ii}, \end{aligned}$$

where each of the six terms can be shown to be the sum of one or several MD sequences. Under Assumptions 3.1-3.4 and using Lemmas A.1-A.5, the conditions for the weak law of large numbers (WLLN) for martingale difference arrays in Davidson (1994, p. 299) can be verified, leading to $\frac{h_n}{n}[\sum_{i=1}^n g_{rn,i}g'_{sn,i} - \mathrm{Cov}(Q_{rn}, Q_{sn})] = o_p(1)$, for $r, s = 1, \cdots, m$. It follows that $\frac{h_n}{n}[\sum_{i=1}^n \mathbf{g}_{ni}\mathbf{g}'_{ni} - \mathrm{Var}(\mathbf{Q}_n)] = o_p(1)$.[17] ■

[17] Details are lengthy and are made available from the authors upon request. Under an additional condition that the smallest eigenvalue of $\mathrm{Var}(\mathbf{Q}_n)$ is strictly positive, the joint asymptotic normality of the LQ vector, $\mathbf{Q}_n$, can be established using Lemma A.5 and the Cramer-Wold devise.

Appendix B: Proofs for Cross-Sectional SAC Model

Proof of Theorem 3.1: To show $T^{\mathtt{r}}_{\mathtt{SAC}}|_{H_0} \xrightarrow{D} \chi^2_k$, it suffices to show

(a) $\frac{1}{\sqrt{n}} S^{\circ}_{\mathtt{SAC},\alpha}(\tilde\theta_n) \xrightarrow{D} N(0_k, \lim_{n\to\infty} \frac{1}{n}\Upsilon_n)$, where $\Upsilon_n = \mathrm{Var}[S^{\circ}_{\mathtt{SAC},\alpha}(\tilde\theta_n)]$.

(b) $\frac{1}{n}\sum_{i=1}^n \mathbf{g}_{ni}(\theta_0)\mathbf{g}'_{ni}(\theta_0) - \frac{1}{n}\mathrm{Var}[S^{\circ}_{\mathtt{SAC}}(\theta_0)] = o_p(1)$;

(c) $\frac{1}{n}\sum_{i=1}^n [\tilde{\mathbf{g}}_{ni}\tilde{\mathbf{g}}'_{ni} - \mathbf{g}_{ni}(\theta_0)\mathbf{g}'_{ni}(\theta_0)] = o_p(1)$;

(d) $\frac{1}{n}(\tilde\Sigma_{n,\alpha\theta} - \Sigma_{n,\alpha\theta}) = o_p(1)$ and $\frac{1}{n}(\tilde\Sigma_{n,\theta\theta} - \Sigma_{n,\theta\theta}) = o_p(1)$.

To show (a), consider the score function at the null $S^{\circ}_{\mathtt{SAC}}(\theta)$ given in (3.6) and simplified at the true θ_0 to (3.7). Under Assumptions 3.1-3.4, it is easy to show by Lemma A.1 that $\Phi_r, r = 1, 2, 3$, defined below (3.7) are uniformly bounded in both row and column sums in absolute value. Thus, the CLT for LQ form given in Lemma A.5 is applicable to give asymptotic normality for the middle three LQ forms in (3.7). The asymptotic normality of the first and last components of (3.7) can be proved by verifying the conditions of Linderberg-Feller CLT. Finally, Cramer-Wold device leads to the asymptotic normality of $S^{\circ}_{\mathtt{SAC}}(\theta_0)$.

Now, consider the α-component of $S^{\circ}_{\mathtt{SAC}}(\theta_0)$, $S^{\circ}_{\mathtt{SAC},\alpha}(\tilde\theta_n) = \sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}$ evaluated at $\tilde\theta_n$. The joint asymptotic normality of $S^{\circ}_{\mathtt{SAC}}(\theta_0)$ and the asymptotic representation of the form (2.8) applied to $S^{\circ}_{\mathtt{SAC},\alpha}(\tilde\theta_n)$ show that $\Upsilon_n = \Omega_{n,\alpha\alpha} - 2\Omega_{n,\alpha\theta}\Gamma'_n + \Gamma_n\Omega_{n,\theta\theta}\Gamma'_n + o(n)$, and that

$$\frac{1}{\sqrt{n}} S^{\circ}_{\mathtt{SAC},\alpha}(\tilde\theta_n) \sim N(0_k, \lim_{n\to\infty} \tfrac{1}{n}\Upsilon_n), \tag{B.1}$$

where $\Gamma_n = \Sigma_{n,\alpha\theta}\Sigma^{-1}_{n,\theta\theta}$, and $\Omega_{n,\alpha\alpha}$, $\Omega_{n,\alpha\theta}$ and $\Omega_{n,\theta\theta}$ are the submatrices of $\Omega_n = \mathrm{Var}[S^{\circ}_{\mathtt{SAC}}(\theta_0)]$.

The result in (b) follows from Lemma A.6. To show (c), it is easy to see that $\frac{\partial}{\partial\theta'}\mathbf{g}_{ni}(\theta_0) = O_p(1)$ for all i. A Taylor series expansion of $\tilde{\mathbf{g}}_{ni} = \mathbf{g}_{ni}(\tilde\theta_n)$ at θ_0, and the $\sqrt{n}$-consistency of $\tilde\theta_n$ lead to the result (c). The results in (d) are proved in a similar manner with $\Sigma_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta}$ corresponding to either the expected information matrix or the negative Hessian matrix, and $\tilde\Sigma_{n,\alpha\theta}$ and $\tilde\Sigma_{n,\theta\theta}$ being their plug-in estimates. Finally, define the estimator of Υ_n as,

$$\begin{aligned}\tilde\Upsilon_n &= \textstyle\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\tilde{\mathbf{g}}'_{ni,\alpha} - 2(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\tilde{\mathbf{g}}'_{ni,\theta})\tilde\Gamma'_n + \tilde\Gamma_n(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\theta}\tilde{\mathbf{g}}'_{ni,\theta})\tilde\Gamma'_n \\ &= \textstyle\sum_{i=1}^n (\tilde{\mathbf{g}}_{ni,\alpha} - \tilde\Gamma_n\tilde{\mathbf{g}}_{ni,\theta})(\tilde{\mathbf{g}}_{ni,\alpha} - \tilde\Gamma_n\tilde{\mathbf{g}}_{ni,\theta})'.\end{aligned} \tag{B.2}$$

With the results (b)-(c), it is easy to show that $\frac{1}{n}(\tilde\Upsilon_n - \Upsilon_n) = o_p(1)$. Positive definiteness of $\frac{1}{n}\tilde\Upsilon_n$ (for large enough n) follows from the positive definiteness of $\Sigma_{n,\theta\theta}$ and Ω_n stated in the theorem, completing the proof of the result for the robust test.

If V_n is normally distributed, $\Sigma_{n,,\alpha\theta} = \Omega_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta} = \Omega_{n,\theta\theta}$. Hence, Γ_n can be consistently estimated by $(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\alpha}\tilde{\mathbf{g}}'_{ni,\theta})(\sum_{i=1}^n \tilde{\mathbf{g}}_{ni,\theta}\tilde{\mathbf{g}}'_{ni,\theta})^{-1}$, leading to the test $T_{\mathtt{SAC}}$ and the second part of the results in Theorem 3.1. ■

Estimation of $\Sigma_{n,\alpha\theta}$ and $\Sigma_{n,\theta\theta}$. The negative Hessian matrix, $\mathbb{H}^{\circ}_{n,\alpha\theta} = -\frac{\partial}{\partial\theta'}S^{\circ}_{\mathtt{SAC},\alpha}(\theta)$, for estimating $\Sigma_{n,\alpha\theta}$ has elements: $\frac{1}{\sigma^2}[V_n(\beta,\lambda)\odot Z_n]'X_n(\lambda_2)$, $\frac{1}{2\sigma^4}Z'_n\mathtt{diag}(V_n(\beta,\lambda)V'_n(\beta,\lambda))$, $\frac{1}{\sigma^2}Z'_n[(B_{2n}(\lambda_2)W_{1n}Y_n)\odot V_n(\beta,\lambda)]$, and $\frac{1}{\sigma^2}Z'_n[(W_{1n}B_{1n}(\lambda_1)Y_n - W_{1n}X_n\beta)\odot V_n(\beta,\lambda)]$; and the negative Hessian matrix, $\mathbb{H}^{\circ}_{n,\theta\theta} = -\frac{\partial}{\partial\theta'} S^{\circ}_{\texttt{SAC},\theta}(\theta)$, required for estimating $\Sigma_{n,\theta\theta}$ equals:

$$\begin{pmatrix} \mathbb{H}_{n,\beta\beta}, & \frac{1}{\sigma^4}X_n'(\lambda_2)V_n(\beta,\lambda), & \frac{1}{\sigma^2}X_n'(\lambda_2)B_{2n}(\lambda_2)W_{1n}Y_n, & \frac{1}{\sigma^2}X_n'A_n(\lambda_2)U_n(\beta,\lambda_1) \\ \sim, & \frac{1}{\sigma^6}\|V_n(\beta,\lambda)\|^2 - \frac{n}{2\sigma^4}, & \frac{1}{\sigma^4}V_n'(\beta,\lambda)B_{2n}(\lambda_2)W_{1n}Y_n, & \frac{1}{\sigma^4}V_n'(\beta,\lambda)G_{2n}(\lambda_2)V_n(\beta,\lambda) \\ \sim, & \sim, & \frac{1}{\sigma^2}\|B_{2n}(\lambda_2)W_{1n}Y_n\|^2 + \text{tr}[G_{1n}^2(\lambda_1)], & \frac{1}{\sigma^2}Y_n'W_{1n}'A_n(\lambda_2)U_n(\beta,\lambda_1) \\ \sim, & \sim, & \sim, & \frac{1}{\sigma^2}\|G_{2n}(\lambda_2)V_n(\beta,\lambda)\|^2 \end{pmatrix}$$

where $\odot$ denotes the Hadamard product, $\|\cdot\|$ the Euclidean norm, and $\texttt{diag}(\cdot)$ forms a vector by the diagonal elements of a square matrix; $\mathbb{H}_{n,\beta\beta} = \frac{1}{\sigma^2}X_n'(\lambda_2)X_n(\lambda_2)$, $A_n(\lambda_2) = W_{2n}'B_{2n}(\lambda_2) + B_{2n}'(\lambda_2)W_{2n}$ and $U_n(\beta,\lambda_1) = B_{1n}(\lambda_1)Y_n - X_n\beta$.

Proof of Theorem 3.2: Similar to the proof of Theorem 3.1. ■

Estimation of $\Sigma^*_{n,\alpha\lambda}$ and $\Sigma^*_{n,\lambda\lambda}$. To facilitate the derivations of the Hessian matrices required for estimating $\Sigma^*_{n,\alpha\lambda}$ and $\Sigma^*_{n,\lambda\lambda}$, write the first two components of (3.14) as

$$\begin{aligned} Y_n'(\lambda)\Phi_1(\lambda)Y_n(\lambda) - \tfrac{n}{n-p}\tilde{\sigma}_n^2(\lambda)\text{tr}[\Phi_1(\lambda)] &= Y_n'B_{1n}'(\lambda_1)M_n^*(\lambda_2)W_{1n}Y_n \\ &\quad - \tfrac{1}{n-p}\text{tr}[G_{1n}(\lambda_1) - X_nD_n^{-1}(\lambda_2)X_n'C_n(\lambda_2)G_{1n}(\lambda_1)]Y_n'B_{1n}'(\lambda_1)M_n^*(\lambda_2)B_{1n}(\lambda_1)Y_n, \\ Y_n'(\lambda)\Phi_2(\lambda)Y_n(\lambda) - \tfrac{n}{n-p}\tilde{\sigma}_n^2(\lambda)\text{tr}[\Phi_2(\lambda)] &= Y_n'B_{1n}'(\lambda_1)M_n^{**}(\lambda_2)B_{1n}(\lambda_1)Y_n \\ &\quad - \tfrac{1}{n-p}\text{tr}[G_{2n}(\lambda_2) - B_{2n}'(\lambda_2)W_{2n}X_nD_n^{-1}(\lambda_2)X_n']Y_n'B_{1n}'(\lambda_1)M_n^*(\lambda_2)B_{1n}(\lambda_1)Y_n, \end{aligned}$$

where $C_n(\lambda_2) = B_{2n}'(\lambda_2)B_{2n}(\lambda_2)$, $D_n = X_n'C_n(\lambda_2)X_n$, $M_n^*(\lambda_2) = B_{2n}'(\lambda_2)M_n(\lambda_2)B_{2n}(\lambda_2)$, and $M_n^{**}(\lambda_2) = B_{2n}'(\lambda_2)M_n(\lambda_2)G_{2n}(\lambda_2)M_n(\lambda_2)B_{2n}(\lambda_2)$.

To simplify the presentation, we write $B_{rn} \equiv B_n(\lambda_r), r = 1,2$, $G_{rn} \equiv B_n(\lambda_r), r = 1,2$, $C_n \equiv C_n(\lambda_2)$, $D_n \equiv D_n(\lambda_2)$, $M_n = M_n(\lambda_2)$, $M_n^* = M_n^*(\lambda_2)$, and $M_n^* = M_n^*(\lambda_2)$. Let $\dot{C}_n$, $\dot{M}_n^*$ and $\dot{M}_n^{**}$ be, respectively, the derivatives of C_n, M_n^* and M_n^{**}, and $\check{D}_n$ the derivative of D_n^{-1}, with respect to λ_2.[18]. The negative Hessian matrix, $\mathbb{H}^*_{n,\alpha\lambda} = -\frac{\partial}{\partial\lambda'}S^*_{\texttt{SAC},\alpha}(\lambda)$, takes the form

$$\mathbb{H}^*_{n,\alpha\lambda} = \begin{cases} \frac{1}{2}Z_n'\left[\frac{q^*_{1i}}{m_i(\lambda_2)} + \frac{1}{n-p}\left(Y_n'(B_{1n}M_n^*W_{1n}' + W_{1n}M_n^*B_{1n}')Y_n\right)\right]_{(n\times 1)}, \\ \frac{1}{2}Z_n'\left[q^*_{2i} - \frac{1}{n-p}\left(Y_n'B_{1n}'\dot{M}_n^*B_{1n}Y_n\right)\right]_{(n\times 1)}, \end{cases}$$

where $q_1^* = -2\widetilde{V}_n(\lambda)\odot(B_{2n}W_{1n}Y_n)$, $q_2^* = \left[-\frac{2}{m_i(\lambda_2)^2}w_{1i}\tilde{v}_{ni}^2(\lambda) + \frac{1}{m_i(\lambda_2)}w_{2i}\tilde{v}_{ni}(\lambda)\right]_{(n\times 1)}$, $w_1 = (M_n\odot\varphi)\iota_n$, $\varphi = W_{2n}X_nD_n^{-1}X_n'B_{2n}' + B_{2n}X_nD_n^{-1}X_n'W_{2n} - B_{2n}X_n\check{D}_nX_n'B_{2n}'$, $w_2 = -2\widetilde{V}_n(\lambda)\odot(W_{2n}B_{1n}Y_n - W_{2n}X_n\beta)$, and $\{\tilde{v}_{ni}(\lambda)\} = \widetilde{V}_n(\lambda) = V_n(\tilde{\beta}_n(\lambda),\lambda)$.

The negative Hessian matrix, $\mathbb{H}^*_{n\lambda\lambda} = -\frac{\partial}{\partial\lambda'}S^*_{\texttt{SAC},\lambda}(\lambda)$ has the elements:

$$\begin{aligned} \mathbb{H}^*_{n\lambda_1\lambda_1} &= Y_n'W_{1n}'M_n^*W_{1n}Y_n + \tfrac{1}{n-p}\text{tr}(G_{1n}^2 - X_nD_n^{-1}X_n'C_nG_{1n}^2)Y_n'B_{1n}'M_n^*B_{1n}Y_n \\ &\quad - \tfrac{1}{n-p}\text{tr}(G_{1n} - X_nD_n^{-1}X_n'C_nG_{1n})Y_n'(B_{1n}'M_n^*W_{1n} + W_{1n}'M_n^*B_{1n})Y_n, \\ \mathbb{H}^*_{n\lambda_1\lambda_2} &= -Y_n'B_{1n}'\dot{M}_n^*W_{1n}Y_n + \tfrac{1}{n-p}\text{tr}(G_{1n} - X_nD_n^{-1}X_n'C_nG_{1n})Y_n'B_{1n}'\dot{M}_n^*B_{1n}Y_n \\ &\quad - \tfrac{1}{n-p}\text{tr}(X_n\check{D}_nX_n'C_nG_{1n} + X_nD_n^{-1}X_n'\dot{C}_nG_{1n})Y_n'B_{1n}'M_n^*B_{1n}Y_n, \end{aligned}$$

[18] We have $\dot{C}_n = -(B_{2n}'W_{2n} + W_{2n}'B_{2n})$, $\check{D}_n = -D_n^{-1}X_n\dot{C}_nX_n'D_n^{-1}$, $\dot{M}_n^* = \dot{C}_n - \dot{C}_nX_nD_n^{-1}X_n'C_n + C_nX_nD_n^{-1}X_n'\dot{C}_n + C_nX_n\check{D}_nX_n'C_n$, and $\dot{M}_n^{**}$ can easily be expressed in terms of $\dot{C}_n$, $\check{D}_n$, and $\dot{M}_n^*$.

$$\mathbb{H}^*_{n\lambda_2\lambda_1} = -Y_n'(B_{1n}'M_n^{**}W_{1n} + W_{1n}'M_n^{**}B_{1n})Y_n$$
$$+\tfrac{1}{n-p}[\mathrm{tr}(G_{2n} - B_{2n}'W_{2n}X_nD_n^{-1}X_n')Y_n'(B_{1n}'M_n^*W_{1n} + W_{1n}'M_n^*B_{1n})Y_n,$$
$$\mathbb{H}^*_{n\lambda_2\lambda_2} = Y_n'B_{1n}'\dot{M}_n^{**}B_{1n}Y_n - \tfrac{1}{n-p}\mathrm{tr}(G_{2n} - W_{2n}'B_{2n}X_nD_n^{-1}X_n')Y_n'B_{1n}'\dot{M}_n^*B_{1n}Y_n$$
$$-\tfrac{1}{n-p}\mathrm{tr}(G_{2n}^2 + W_{2n}'W_{2n}X_nD_n^{-1}X_n' - W_{2n}'B_{2n}X_n\check{D}_nX_n')Y_n'B_{1n}'M_n^*B_{1n}Y_n.$$

Appendix C: Proofs for Panel FESAC Model

Proof of Theorem 4.1: To show $T_{\texttt{FESAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ when the original errors $\{v_{it}\}$ are iid normal, with the help of Lemmas A.1-A.6, using the fact that the elements $\{v_j^*\}$ of $\mathbf{V}_N$ are totally independent (iid normal), and referring to the increasing σ-fields $\mathcal{F}_{jN}$ generated by $(v_1^*, \cdots, v_j^*)$, one can easily show, in the same way as the proof of Theorem 3.1, the following:

(a) $\frac{1}{\sqrt{N}}S^{\circ}_{\texttt{FESAC},\alpha}(\tilde{\theta}_N) \xrightarrow{D} N(0_k, \lim_{N\to\infty}\frac{1}{N}\Upsilon_N)$, where $\Upsilon_N = \mathrm{Var}[S^{\circ}_{\texttt{FESAC},\alpha}(\tilde{\theta}_N)]$.

(b) $\frac{1}{N}\sum_{j=1}^N \mathbf{g}_{Nj}(\theta_0)\mathbf{g}'_{Nj}(\theta_0) - \frac{1}{N}\mathrm{Var}[S^{\circ}_{\texttt{FESAC}}(\theta_0)] = o_p(1)$;

(c) $\frac{1}{N}\sum_{j=1}^N[\tilde{\mathbf{g}}_{Nj}\tilde{\mathbf{g}}'_{Nj} - \mathbf{g}_{Nj}(\theta_0)\mathbf{g}'_{Nj}(\theta_0)] = o_p(1)$.

The result, $T_{\texttt{FESAC}}|_{H_0} \xrightarrow{D} \chi^2_k$, thus follows when $\{v_{it}\}$ are iid normal.

The proof of $T^{\mathrm{r}}_{\texttt{FESAC}}|_{H_0} \xrightarrow{D} \chi^2_k$ is much trickier when the original errors $\{v_{it}\}$ are allowed to be nonnormal (though still iid), since in this case it is not guaranteed that $\{v_j^*\}$ will be again totally independent. It amounts to show that (a)-(c) still hold when $\{v_{it}\}$ are iid nonnormal, and that (d) $\frac{1}{N}(\tilde{\mathbf{\Sigma}}_{N,\alpha\theta} - \mathbf{\Sigma}_{N,\alpha\theta}) = o_p(1)$ and $\frac{1}{N}(\tilde{\mathbf{\Sigma}}_{N,\theta\theta} - \mathbf{\Sigma}_{N,\theta\theta}) = o_p(1)$.

To show (a), noting that $\mathbf{V}_N = (F'_{T,T-1} \otimes I_n)\mathbb{V}_{nT}$, the components of the score function $S^{\circ}_{\texttt{FESAC}}(\theta_0)$ given in (4.10) can all be written as linear, or quadratic, or linear-quadratic forms of $\mathbb{V}_{nT}$, a vector of iid elements. Lemma A.5 and Cramer-Wold device lead to the asymptotic normality of $\frac{1}{\sqrt{N}}S^{\circ}_{\texttt{FESAC}}(\theta_0)$, and hence the asymptotic normality of $\frac{1}{\sqrt{N}}S^{\circ}_{\texttt{FESAC},\alpha}(\tilde{\theta}_N)$.

To prove (b), note that $S^{\circ}_{\texttt{FESAC}}(\theta_0) = \sum_{j=1}^N \mathbf{g}_{Nj}(\theta_0) \equiv \sum_{j=1}^N \mathbf{g}_{Nj}$, where

$$\mathbf{g}_{Nj} = \begin{cases} \Pi_{1j}v_j^*, \\ v_j^*\xi_{1j} + (v_j^{*2} - \sigma_0^2)\phi_{1j}, \\ v_j^*\xi_{2j} + (v_j^{*2} - \sigma_0^2)\phi_{2j} + \Pi_{2j}v_j^*, \\ v_j^*\xi_{3j} + (v_j^{*2} - \sigma_0^2)\phi_{3j}, \\ \frac{1}{2\sigma_0^2}z_j(v_j^{*2} - \sigma_0^2), \end{cases} \tag{C.1}$$

where $\{\xi_{rj}\} = \xi_r = (\mathbf{\Phi}_r^{u\prime} + \mathbf{\Phi}_r^l)\mathbf{V}_N$, and ϕ_{rj} are the diagonal elements of $\mathbf{\Phi}_r, r = 1, 2, 3$. All quantities are defined in (4.10), and $\dot{h}(0)$ in the last element of $\mathbf{g}_{Nj}$ is dropped as it is canceled out in the final expression of the test statistic. We have,

$$\mathrm{Var}[S^{\circ}_{\texttt{FESAC}}(\theta_0)] = \sum_{j=1}^N \mathrm{Var}(\mathbf{g}_{Nj}) + \sum_{j=1}^N\sum_{\ell\neq j}^N \mathrm{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell}). \tag{C.2}$$

Let $\odot$ denote the Hadamard product. A vector raised to rth power is operated elementwise. Let $\mathbf{f}_j$ be the jth column of $F_{T,T-1} \otimes I_n$ and $\mathbf{q}_{rj}$ be the jth column of $(F_{T,T-1} \otimes I_n)(\mathbf{\Phi}_r^u + \mathbf{\Phi}_r^{l\prime})$, for $j = 1, \ldots, N$. We have $v_j^* = \mathbf{f}_j' \mathbb{V}_{nT}$ and $\xi_{rj} = \mathbf{q}_{rj}' \mathbb{V}_{nT}$; $v_j^* \xi_{rj} = \mathbb{V}_{nT}' (\mathbf{f}_j \mathbf{q}_{rj}') \mathbb{V}_{nT}$; and $v_j^{*2} = \mathbb{V}_{nT}' (\mathbf{f}_j \mathbf{f}_j') \mathbb{V}_{nT}$. Using the following easily proved results:

$$\begin{aligned}&\text{Cov}(c_N' \mathbb{V}_{nT}, \mathbb{V}_{nT}' A \mathbb{V}_{nT}) = \mu_0^{(3)} c_N' a_N, \quad \text{and} \\ &\text{Cov}(\mathbb{V}_{nT}' A_N \mathbb{V}_{nT}, \mathbb{V}_{nT}' B_N \mathbb{V}_{nT}) = (\mu_0^{(4)} - 3\sigma_0^4) a_N' b_N + \sigma_0^4 \text{tr}[A_N(B_N + B_N')],\end{aligned}$$

for conformable matrices A_N and B_N and vector c_N, with a_N and b_N being the vectors formed by the diagonal elements of A_N and B_N, respectively, and $\mu_0^{(3)}$ and $\mu_0^{(4)}$ being, respectively, the 3rd and 4th moments of v_{it}, we have the key elements in $\text{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell})$:

$$\begin{aligned}
\text{Cov}(v_j^*, v_\ell^* \xi_{r\ell}) &= \mu_0^{(3)} \mathbf{f}_j' (\mathbf{f}_\ell \odot \mathbf{q}_{r\ell}), \\
\text{Cov}(v_j^*, v_\ell^{*2}) &= \mu_0^{(3)} \mathbf{f}_j' (\mathbf{f}_\ell \odot \mathbf{f}_\ell), \\
\text{Cov}(v_j^* \xi_{rj}, v_\ell^* \xi_{r\ell}) &= (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{q}_{rj})' (\mathbf{f}_\ell \odot \mathbf{q}_{r\ell}) + \sigma_0^4 \text{tr}[(\mathbf{f}_j \mathbf{q}_{rj}')(\mathbf{f}_\ell \mathbf{q}_{r\ell}' + \mathbf{q}_{r\ell} \mathbf{f}_\ell')], \\
\text{Cov}(v_j^{*2}, v_\ell^* \xi_{r\ell}) &= (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{f}_j)' (\mathbf{f}_\ell \odot \mathbf{q}_{r\ell}) + \sigma_0^4 \text{tr}[(\mathbf{f}_j \mathbf{f}_j')(\mathbf{f}_\ell \mathbf{q}_{r\ell}' + \mathbf{q}_{r\ell} \mathbf{f}_\ell')], \\
\text{Cov}(v_j^{*2}, v_\ell^{*2}) &= (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{f}_j)' (\mathbf{f}_\ell \odot \mathbf{f}_\ell) + \sigma_0^4 \text{tr}[(\mathbf{f}_j \mathbf{f}_j')(\mathbf{f}_\ell \mathbf{f}_\ell' + \mathbf{f}_\ell \mathbf{f}_\ell')],
\end{aligned}$$

$r = 1, 2, 3$. It is easy to see that (i) $\mathbf{f}_j' \mathbf{f}_\ell = 0$ for all $j \neq \ell$, (ii) $\mathbf{f}_j' \mathbf{q}_{r\ell} = 0$ for $\ell \leq j$, and (iii) $\mathbf{f}_j \odot \mathbf{q}_{rj} = 0$.[19] Thus, all terms vanish except $\mathbf{f}_j'(\mathbf{f}_\ell \odot \mathbf{f}_\ell)$ and $(\mathbf{f}_j \odot \mathbf{f}_j)'(\mathbf{f}_\ell \odot \mathbf{f}_\ell)$, and subsequently all covariances vanish except,

$$\text{Cov}(v_j^*, v_\ell^{*2}) = \mu_0^{(3)} \mathbf{f}_j' (\mathbf{f}_\ell \odot \mathbf{f}_\ell) \ \text{ and } \ \text{Cov}(v_j^{*2}, v_\ell^{*2}) = (\mu_0^{(4)} - 3\sigma_0^4)(\mathbf{f}_j \odot \mathbf{f}_j)' (\mathbf{f}_\ell \odot \mathbf{f}_\ell). \quad \text{(C.3)}$$

Note that (i) the vector $\mathbf{f}_j$ has only $(T-1)$ nonzero elements, and (ii) for integers $k \geq 1$ and $m \geq 1$, $\mathbf{f}_j^k \odot \mathbf{f}_\ell^m \neq 0_n$ only when the indices $j = (i, t)$ and $\ell = (i, s)$, $t \neq s$. These show that,

$$\sum_{j=1}^N \sum_{\ell \neq j}^N \text{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell}) = \sum_{i=1}^n \sum_{t=1}^{T-1} \Big(\sum_{s(\neq t)=1}^{T-1} \text{E}(\mathbf{d}_{N,it} \mathbf{d}_{N,is}') \Big) = \sum_{i=1}^n \sum_{t=1}^{T-1} \text{E}(\mathbf{d}_{N,it} \mathbf{d}_{N,it}^{\circ\prime}), \quad \text{(C.4)}$$

where $\mathbf{d}_{N,it} = \big\{ \Pi_{1,it}' v_{it}^*, (v_{it}^{*2} - \sigma_0^2)\phi_{1,it}, (v_{it}^{*2} - \sigma_0^2)\phi_{2,it} + \Pi_{2,it} v_{it}^*, (v_{it}^{*2} - \sigma_0^2)\phi_{3,it}, \frac{1}{2\sigma_0^2} z_{ni}'(v_{it}^{*2} - \sigma_0^2) \big\}'$, and $\mathbf{d}_{N,it}^\circ = \sum_{s(\neq t)=1}^{T-1} \mathbf{d}_{N,is}$. Letting $\tilde{\mathbf{d}}_{N,it}$ and $\tilde{\mathbf{d}}_{N,it}^\circ$ be the estimates of $\mathbf{d}_{N,it}$ and $\mathbf{d}_{N,it}^\circ$ at the null, one can show (details are available from the authors upon request) that

$$\frac{1}{N} \sum_{j=1}^N \sum_{\ell \neq j}^N \text{Cov}(\mathbf{g}_{Nj},\ \mathbf{g}_{N\ell}) - \frac{1}{N} \sum_{i=1}^n \sum_{t=1}^{T-1} (\tilde{\mathbf{d}}_{N,it} \tilde{\mathbf{d}}_{N,it}^{\circ\prime}) = o_p(1). \quad \text{(C.5)}$$

It left to prove $\frac{1}{N} \big\{ \sum_{j=1}^N \mathbf{g}_{Nj} \mathbf{g}_{Nj}' - \sum_{j=1}^N \text{E}(\mathbf{g}_{Nj} \mathbf{g}_{Nj}') \big\} = o_p(1)$, which can be done by referring to the proof of Lemma A.6.

The proofs of (c) and (d) can be carried out by referencing to the proofs of (c) and (d) of Theorem 3.1, with details being available from the authors upon request. ∎

[19] The result (ii) is due to the fact that v_j^* is uncorrelated with ξ_ℓ for $\ell \leq j$, and (iii) follows from $(F_{T,T-1} \otimes I_n)(\mathbf{\Phi}_r^u + \mathbf{\Phi}_r^{l\prime}) = F_{T,T-1} \otimes (\Phi_r^u + \Phi_r^{l\prime})$ and hence $(F_{T,T-1} \otimes I_n) \odot [(F_{T,T-1} \otimes (\Phi_r^u + \Phi_r^{l\prime})] = 0$, where $\mathbf{\Phi}_r = I_{T-1} \otimes \Phi_r$.

Estimation of $\mathbf{\Sigma}_{N,\alpha\theta}$ and $\mathbf{\Sigma}_{N,\theta\theta}$. The corresponding negative Hessian matrices take identical forms as these for the SAC model mode given in Appendix B except that n is replaced by N and the relevant quantities are replaced by the corresponding bold-faced quantities for the panel SAC model, and hence are not repeated here.

Proof of Theorem 4.2: Similar to the proof of Theorem 4.1. ∎

Estimation of $\mathbf{\Sigma}^*_{N,\alpha\lambda}$ and $\mathbf{\Sigma}^*_{N,\lambda\lambda}$. The corresponding negative Hessian matrices take identical forms as these for the SAC model mode given in Appendix B except that n is replaced by N and the relevant quantities are replaced by the corresponding bold-faced quantities for the panel SAC model, and hence are not repeated here.

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Table 1 – Empirical Sizes of the Tests for SAC Model, **DGP1 – Normal disturbances**

			Circular world			Rook contiguity			Queen contiguity			Group interaction		
(λ_1, λ_2)	n	Tests	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(0.2,0.2)	50	T_{SAC}	23.68	15.02	5.04	21.16	13.18	4.06	21.36	12.96	4.30	22.72	14.86	4.62
	50	T_{SAC}^{r}	8.88	4.60	0.92	8.12	3.98	0.52	7.10	3.60	0.76	11.74	5.90	1.20
	50	T_{SAC}^{*}	14.24	7.42	1.66	12.62	5.88	0.66	12.82	5.92	0.72	13.38	7.70	1.58
	50	T_{SAC}^{r*}	11.56	5.78	1.06	10.44	4.34	0.36	10.48	4.48	0.30	11.06	5.52	1.00
	100	T_{SAC}	16.70	9.56	2.90	17.68	10.14	2.66	17.72	10.62	3.20	17.30	9.98	2.88
	100	T_{SAC}^{r}	8.76	4.10	0.92	8.02	3.76	0.86	8.94	4.20	0.86	9.12	4.54	0.84
	100	T_{SAC}^{*}	11.74	5.38	0.94	11.80	5.64	0.82	12.00	5.90	1.00	11.64	5.82	1.08
	100	T_{SAC}^{r*}	10.60	4.56	0.74	10.28	4.60	0.56	10.68	4.98	0.66	9.90	4.98	0.76
	200	T_{SAC}	13.86	8.06	2.14	13.86	7.44	1.86	13.30	7.18	1.82	13.86	7.82	2.26
	200	T_{SAC}^{r}	9.28	4.76	0.96	9.64	4.56	0.98	8.52	4.36	0.98	10.74	5.46	1.20
	200	T_{SAC}^{*}	10.82	5.42	0.98	10.76	5.30	0.86	10.26	5.00	0.88	11.00	5.34	1.06
	200	T_{SAC}^{r*}	10.40	5.00	0.82	10.16	4.72	0.76	9.72	4.52	0.74	10.22	4.98	0.90
	500	T_{SAC}	12.56	6.78	1.64	11.12	5.78	1.10	11.98	6.08	1.62	12.26	6.54	1.40
	500	T_{SAC}^{r}	10.66	5.54	1.24	9.46	4.36	0.84	10.08	4.82	1.08	10.08	5.12	0.98
	500	T_{SAC}^{*}	11.02	5.46	1.18	9.76	4.60	0.88	10.42	4.82	1.02	10.78	5.36	0.98
	500	T_{SAC}^{r*}	10.80	5.42	1.12	9.52	4.48	0.84	10.26	4.68	1.00	10.42	5.18	0.98
(0.2,0.8)	50	T_{SAC}	21.80	13.86	4.98	22.24	13.52	3.98	21.88	14.00	4.48	22.82	15.04	5.16
	50	T_{SAC}^{r}	10.30	5.66	1.16	12.30	6.48	1.26	12.60	6.94	1.26	14.04	8.00	1.68
	50	T_{SAC}^{*}	12.58	5.94	0.74	13.04	6.14	0.60	13.50	6.18	0.74	14.58	7.32	1.10
	50	T_{SAC}^{r*}	10.40	4.04	0.42	10.34	4.28	0.38	10.40	4.24	0.26	9.32	4.00	0.28
	100	T_{SAC}	16.42	9.96	2.74	13.18	7.78	1.92	16.44	10.02	2.98	16.64	9.54	2.84
	100	T_{SAC}^{r}	11.86	6.28	1.38	5.28	2.16	0.44	11.28	6.10	1.42	12.48	6.60	1.58
	100	T_{SAC}^{*}	10.08	4.96	0.60	12.40	6.02	1.04	11.52	5.98	0.92	10.04	4.76	0.72
	100	T_{SAC}^{r*}	9.04	4.10	0.44	7.76	3.28	0.52	10.34	4.82	0.56	7.38	3.10	0.32
	200	T_{SAC}	14.32	7.80	1.84	14.08	7.96	2.14	13.72	7.88	2.04	14.04	7.64	2.16
	200	T_{SAC}^{r}	11.48	5.74	1.10	10.38	5.48	1.14	10.74	5.68	1.04	11.84	6.00	1.38
	200	T_{SAC}^{*}	11.14	5.28	0.82	10.92	5.08	0.82	10.56	5.28	0.76	11.12	5.44	0.86
	200	T_{SAC}^{r*}	10.54	4.90	0.70	10.22	4.68	0.66	9.76	5.10	0.58	9.88	4.62	0.68
	500	T_{SAC}	11.70	5.74	1.36	12.10	6.62	1.60	11.72	6.24	1.42	11.32	6.18	1.24
	500	T_{SAC}^{r}	10.62	5.06	1.20	10.06	5.32	1.12	10.22	5.12	1.12	10.38	5.44	1.04
	500	T_{SAC}^{*}	10.36	4.76	1.02	11.10	5.52	1.04	10.36	5.16	1.00	10.14	4.96	0.78
	500	T_{SAC}^{r*}	10.00	4.56	0.98	10.80	5.18	0.94	10.00	5.02	0.96	9.38	4.28	0.64
(0.8,0.2)	50	T_{SAC}	22.28	13.06	3.86	21.04	12.94	3.84	21.12	13.12	4.18	22.26	13.88	4.04
	50	T_{SAC}^{r}	8.50	4.52	0.80	10.22	5.22	0.98	9.90	5.58	1.02	11.28	5.58	1.08
	50	T_{SAC}^{*}	11.60	5.24	0.46	12.22	5.02	0.74	12.02	5.78	0.80	12.48	5.52	0.52
	50	T_{SAC}^{r*}	9.50	3.54	0.30	9.32	3.40	0.36	9.58	4.44	0.40	9.16	3.78	0.26
	100	T_{SAC}	16.08	9.72	2.64	26.28	17.64	8.00	16.82	10.02	2.72	16.62	9.62	2.74
	100	T_{SAC}^{r}	9.70	4.82	0.88	10.54	5.92	1.64	8.52	4.18	0.58	9.62	4.46	1.18
	100	T_{SAC}^{*}	11.38	5.30	0.62	12.34	5.96	0.76	11.24	5.26	0.82	11.46	4.88	0.80
	100	T_{SAC}^{r*}	10.02	4.56	0.50	6.44	2.82	0.36	10.00	4.44	0.50	9.80	4.08	0.60
	200	T_{SAC}	12.94	7.10	2.20	13.34	7.10	1.74	14.48	8.16	2.06	13.44	7.30	1.54
	200	T_{SAC}^{r}	10.20	5.06	1.38	8.06	3.84	0.82	9.76	5.24	0.98	10.14	4.86	1.04
	200	T_{SAC}^{*}	10.68	5.04	0.98	10.00	4.52	0.90	11.44	5.54	0.92	10.58	4.78	0.72
	200	T_{SAC}^{r*}	10.00	4.64	0.82	9.52	4.04	0.74	10.82	5.10	0.76	9.76	4.08	0.62
	500	T_{SAC}	11.90	6.06	1.26	11.82	6.22	1.44	11.44	5.82	1.48	11.34	5.80	1.48
	500	T_{SAC}^{r}	9.68	4.56	0.82	7.60	3.72	0.72	9.32	4.62	1.20	9.74	4.76	1.18
	500	T_{SAC}^{*}	10.66	5.00	0.72	10.50	5.14	0.94	10.16	5.02	1.00	9.90	4.72	1.10
	500	T_{SAC}^{r*}	10.26	4.82	0.64	10.26	4.98	0.90	9.86	4.90	0.96	9.76	4.54	0.96
(0.8,0.8)	50	T_{SAC}	21.32	13.36	3.86	21.60	13.62	4.18	22.32	13.80	4.06	22.58	14.58	4.74
	50	T_{SAC}^{r}	13.14	6.96	1.24	13.12	6.66	1.48	13.74	7.48	1.56	13.98	7.76	1.84
	50	T_{SAC}^{*}	12.16	5.98	1.12	11.82	5.48	0.84	13.30	5.76	0.66	14.06	7.60	1.28
	50	T_{SAC}^{r*}	9.02	4.00	0.72	9.20	3.78	0.40	9.68	4.08	0.38	9.26	4.32	0.50
	100	T_{SAC}	17.50	10.08	3.00	16.76	9.88	2.76	16.60	9.64	2.44	16.86	9.74	3.18
	100	T_{SAC}^{r}	12.58	6.56	1.52	12.52	6.52	1.58	11.86	6.24	0.96	12.80	6.78	1.78
	100	T_{SAC}^{*}	11.52	5.96	0.88	10.62	4.94	0.70	11.48	5.06	0.68	10.32	4.92	0.86
	100	T_{SAC}^{r*}	9.86	4.76	0.52	9.08	4.02	0.58	10.00	4.12	0.36	7.64	3.28	0.54
	200	T_{SAC}	13.66	7.54	2.22	14.20	7.80	2.44	14.18	7.58	1.58	14.14	8.28	2.02
	200	T_{SAC}^{r}	11.22	5.90	1.42	11.72	5.96	1.58	11.50	5.82	1.04	11.62	6.48	1.32
	200	T_{SAC}^{*}	10.74	5.16	0.88	11.16	5.28	0.94	10.80	4.96	0.64	11.38	5.70	0.78
	200	T_{SAC}^{r*}	10.12	4.70	0.78	10.32	4.84	0.72	9.96	4.52	0.54	10.38	4.74	0.64
	500	T_{SAC}	11.44	6.10	1.18	12.02	6.26	1.58	11.96	6.40	1.60	11.36	6.00	1.42
	500	T_{SAC}^{r}	10.62	5.40	0.88	10.74	5.74	1.34	10.56	5.80	1.34	10.32	5.42	1.10
	500	T_{SAC}^{*}	10.36	4.94	0.60	10.56	5.28	1.02	10.38	5.50	1.16	10.12	5.10	0.92
	500	T_{SAC}^{r*}	10.06	4.84	0.56	10.26	5.02	0.92	10.10	5.34	1.10	9.66	4.70	0.78

Table 2 – Empirical Sizes of the Tests for SAC Model, **DGP1 – Non-normal Disturbances**

			Circular world			Rook contiguity			Queen contiguity			Group interaction		
(λ_1, λ_2)	n	Tests	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(0.2,0.2)	50	T_{SAC}	41.90	32.34	17.84	41.08	31.84	17.14	40.78	31.26	17.00	42.08	32.48	17.94
	50	T_{SAC}^{r}	17.02	10.04	3.72	15.48	9.32	3.16	15.08	9.22	3.06	18.04	11.22	3.98
	50	T_{SAC}^{*}	20.06	11.82	3.00	19.02	9.72	1.90	19.44	10.36	1.66	20.44	11.74	3.18
	50	T_{SAC}^{r*}	15.28	8.12	1.00	14.30	6.64	1.02	13.98	6.68	0.82	14.58	7.58	1.44
	100	T_{SAC}	38.90	29.64	17.00	38.30	29.60	16.50	37.86	29.00	15.42	38.12	29.52	15.84
	100	T_{SAC}^{r}	14.94	8.98	2.98	13.94	8.20	2.74	13.76	7.62	2.36	14.92	9.28	3.30
	100	T_{SAC}^{*}	16.64	8.84	1.96	16.18	8.64	1.68	15.86	7.56	1.62	16.38	9.36	2.18
	100	T_{SAC}^{r*}	12.92	6.82	1.26	12.54	6.56	1.18	12.28	5.56	1.16	12.68	7.06	1.20
	200	T_{SAC}	34.20	25.42	13.94	34.00	25.66	14.02	33.80	25.52	14.20	33.30	24.78	12.60
	200	T_{SAC}^{r}	13.68	7.76	2.26	13.38	7.48	2.08	13.28	7.62	2.26	13.76	7.42	2.18
	200	T_{SAC}^{*}	13.82	7.30	1.36	14.04	7.04	1.30	13.54	7.00	1.18	13.02	6.72	1.26
	200	T_{SAC}^{r*}	12.02	5.64	1.02	11.22	5.74	0.94	11.48	5.58	1.02	10.80	4.90	0.74
	500	T_{SAC}	29.86	21.06	9.94	29.68	20.94	9.76	29.42	20.70	9.34	30.54	21.38	10.12
	500	T_{SAC}^{r}	12.32	6.26	1.70	11.90	6.08	1.58	11.82	6.18	1.32	12.20	6.66	1.48
	500	T_{SAC}^{*}	11.52	5.84	1.16	11.66	5.42	0.98	11.24	5.74	1.02	12.06	6.02	1.06
	500	T_{SAC}^{r*}	10.64	5.14	0.98	10.66	4.90	0.74	10.28	5.22	0.88	11.08	5.34	0.80
(0.2,0.8)	50	T_{SAC}	42.12	33.00	18.62	41.30	33.04	18.28	41.80	32.76	18.46	41.00	32.58	17.94
	50	T_{SAC}^{r}	18.36	12.26	4.70	20.96	13.64	5.04	21.10	14.18	4.86	19.62	12.92	4.48
	50	T_{SAC}^{*}	18.58	9.26	1.48	18.86	9.50	1.46	19.90	9.84	1.48	20.68	11.22	2.28
	50	T_{SAC}^{r*}	13.20	5.86	0.72	13.36	5.74	0.60	13.68	6.04	0.68	12.02	5.38	0.60
	100	T_{SAC}	38.14	29.06	16.44	21.74	14.24	5.08	37.921	29.10	15.66	38.02	28.42	16.18
	100	T_{SAC}^{r}	18.14	10.96	3.62	6.14	2.86	0.48	7.50	10.64	3.90	18.70	11.70	3.92
	100	T_{SAC}^{*}	14.32	7.26	1.90	16.10	8.16	1.72	15.60	7.74	1.68	14.72	8.02	1.80
	100	T_{SAC}^{r*}	11.06	5.40	1.26	8.58	4.06	0.74	12.06	5.20	0.96	10.74	4.78	0.82
	200	T_{SAC}	33.02	24.40	12.26	32.98	24.28	12.32	34.10	25.42	13.30	32.72	23.98	12.20
	200	T_{SAC}^{r}	14.26	7.90	2.21	13.96	7.92	2.34	15.20	8.38	2.36	14.04	8.32	2.40
	200	T_{SAC}^{*}	12.36	5.78	1.12	12.84	6.50	1.24	13.26	6.72	1.30	13.34	6.92	1.32
	200	T_{SAC}^{r*}	10.18	4.34	0.78	10.22	4.84	0.76	10.98	5.30	0.94	10.24	4.86	0.78
	500	T_{SAC}	30.88	21.80	10.42	27.90	19.42	8.56	27.84	19.28	8.66	28.54	20.00	9.22
	500	T_{SAC}^{r}	13.20	7.54	1.64	11.44	5.84	1.56	11.56	6.02	1.18	11.68	6.20	1.24
	500	T_{SAC}^{*}	12.12	6.02	0.94	11.46	5.66	1.18	11.44	5.62	0.88	11.48	5.72	0.82
	500	T_{SAC}^{r*}	11.00	5.26	0.68	9.94	4.56	0.78	9.92	4.76	0.64	9.76	4.50	0.48
(0.8,0.2)	50	T_{SAC}	40.48	30.98	16.98	40.82	31.30	16.52	42.32	32.20	18.32	40.72	31.36	17.12
	50	T_{SAC}^{r}	16.26	10.68	3.66	18.26	11.54	3.98	19.58	12.28	4.00	18.38	11.58	3.92
	50	T_{SAC}^{*}	17.46	8.78	1.54	17.32	8.90	1.32	18.40	10.60	2.54	17.80	9.32	1.74
	50	T_{SAC}^{r*}	12.26	5.44	0.70	12.26	5.44	0.54	13.66	7.22	1.22	12.02	4.94	0.62
	100	T_{SAC}	38.36	29.22	16.18	28.16	20.34	8.94	38.00	29.46	16.32	38.30	29.32	15.92
	100	T_{SAC}^{r}	16.04	9.34	3.06	10.86	6.20	1.76	15.38	9.08	2.94	15.80	9.36	3.02
	100	T_{SAC}^{*}	16.44	8.38	1.42	16.46	8.66	1.60	15.56	7.66	1.46	15.90	8.32	1.50
	100	T_{SAC}^{r*}	12.88	6.12	0.86	7.12	3.48	0.58	12.10	5.56	0.96	12.26	5.76	0.70
	200	T_{SAC}	35.02	26.58	13.66	34.00	25.18	13.04	32.72	24.00	12.36	35.80	26.58	13.78
	200	T_{SAC}^{r}	14.76	8.56	2.54	12.52	6.90	1.96	12.08	6.74	1.72	15.02	8.08	2.28
	200	T_{SAC}^{*}	13.54	6.68	1.20	12.74	6.40	1.06	12.48	6.20	1.22	14.06	6.82	1.18
	200	T_{SAC}^{r*}	11.42	5.26	0.76	10.54	5.04	0.86	10.64	5.38	0.88	11.00	5.46	0.88
	500	T_{SAC}	30.54	21.60	10.14	28.66	20.90	9.40	29.08	21.00	10.50	29.86	20.72	9.62
	500	T_{SAC}^{r}	11.82	6.40	1.46	9.92	5.22	1.16	11.74	6.24	1.34	11.48	6.16	1.82
	500	T_{SAC}^{*}	11.88	5.66	0.98	11.22	5.18	0.94	11.92	5.86	1.00	11.54	5.56	1.32
	500	T_{SAC}^{r*}	10.90	4.90	0.72	10.12	4.44	0.88	10.84	5.04	0.86	10.26	4.96	1.06
(0.8,0.8)	50	T_{SAC}	43.24	34.40	19.38	40.64	31.22	16.56	41.90	32.54	18.24	42.30	33.34	19.20
	50	T_{SAC}^{r}	21.56	13.70	5.18	21.16	13.30	4.40	21.76	14.08	4.80	22.36	14.56	6.20
	50	T_{SAC}^{*}	20.36	11.48	2.96	17.40	8.46	1.40	18.12	9.06	1.86	21.18	12.22	3.60
	50	T_{SAC}^{r*}	13.94	7.16	1.44	11.24	4.94	0.48	11.66	4.98	0.92	14.34	7.74	1.66
	100	T_{SAC}	38.06	28.96	16.42	28.36	19.80	8.92	37.02	28.30	15.46	40.16	31.40	17.14
	100	T_{SAC}^{r}	18.44	11.34	3.64	19.94	12.40	4.28	17.36	10.52	3.24	19.16	11.74	3.88
	100	T_{SAC}^{*}	14.14	7.90	2.20	16.14	8.10	1.52	15.74	8.20	1.32	15.54	8.54	2.40
	100	T_{SAC}^{r*}	10.90	5.74	1.36	9.78	4.08	0.78	11.74	5.66	0.62	10.62	5.20	1.40
	200	T_{SAC}	33.28	24.44	12.52	33.90	25.68	12.92	34.66	25.46	13.00	34.94	25.48	13.28
	200	T_{SAC}^{r}	14.86	8.40	2.48	15.62	8.98	2.34	14.82	8.80	2.62	15.62	8.98	2.54
	200	T_{SAC}^{*}	13.28	6.54	0.98	13.72	7.34	1.42	13.38	6.40	1.40	15.12	7.86	1.64
	200	T_{SAC}^{r*}	11.04	4.78	0.78	10.92	5.28	0.92	10.86	5.06	0.90	11.68	5.58	0.78
	500	T_{SAC}	29.76	20.80	9.84	28.76	20.14	9.52	28.22	19.72	9.28	29.28	21.90	10.62
	500	T_{SAC}^{r}	13.14	6.70	1.88	12.94	7.02	1.82	11.84	6.14	1.36	12.70	6.72	1.86
	500	T_{SAC}^{*}	11.98	6.00	1.04	12.90	6.32	1.34	11.48	5.60	0.84	12.08	5.96	1.04
	500	T_{SAC}^{r*}	10.86	5.12	0.82	10.76	5.10	0.90	10.02	4.40	0.70	12.48	4.86	0.70

Table 3 – Empirical Sizes of the Tests for SAC Model, **DGP2 – Normal Disturbances**

(λ_1,λ_2)	n	Tests	Circular world 10%	Circular world 5%	Circular world 1%	Rook contiguity 10%	Rook contiguity 5%	Rook contiguity 1%	Queen contiguity 10%	Queen contiguity 5%	Queen contiguity 1%	Group interaction 10%	Group interaction 5%	Group interaction 1%
(0.2,0.2)	50	T_{SAC}	22.42	14.06	4.74	22.02	14.40	4.66	22.54	14.38	4.82	23.26	14.68	5.26
	50	T_{SAC}^{r}	10.12	5.48	1.20	9.84	5.24	1.10	9.62	5.18	0.88	7.80	4.06	0.82
	50	T_{SAC}^{*}	13.70	6.52	0.72	13.66	7.16	1.38	13.50	7.60	1.56	14.34	7.28	1.02
	50	T_{SAC}^{r*}	11.52	5.18	0.46	11.34	5.74	0.90	11.66	5.80	1.02	12.00	5.60	0.50
	100	T_{SAC}	16.68	9.96	2.88	17.12	10.32	3.16	15.80	9.84	3.20	16.16	9.48	2.72
	100	T_{SAC}^{r}	8.22	4.16	1.06	10.22	5.04	1.04	7.98	4.38	1.02	7.66	3.84	0.68
	100	T_{SAC}^{*}	11.76	5.32	0.92	9.90	5.00	1.20	9.44	5.34	1.46	11.12	5.32	0.84
	100	T_{SAC}^{r*}	10.50	4.54	0.74	9.08	4.44	1.04	8.62	4.60	1.28	10.16	4.70	0.64
	200	T_{SAC}	13.72	7.44	2.18	13.64	7.58	1.94	14.40	8.12	2.16	13.36	6.96	1.62
	200	T_{SAC}^{r}	7.94	3.84	0.82	9.02	4.52	0.66	10.04	5.34	1.18	9.12	4.18	0.98
	200	T_{SAC}^{*}	10.66	5.32	0.86	10.78	5.34	0.74	11.34	5.78	1.10	10.34	4.54	0.84
	200	T_{SAC}^{r*}	10.20	4.98	0.76	10.14	4.88	0.74	10.72	5.24	0.92	9.56	4.34	0.68
	500	T_{SAC}	11.52	5.76	1.72	11.14	5.74	1.24	11.88	6.42	1.60	11.46	5.94	1.46
	500	T_{SAC}^{r}	7.04	3.56	0.76	9.24	4.38	0.90	9.84	5.30	1.14	9.82	5.18	1.04
	500	T_{SAC}^{*}	10.20	5.02	1.22	9.64	4.46	0.92	10.40	5.22	1.08	9.88	4.84	0.98
	500	T_{SAC}^{r*}	9.98	4.94	1.20	9.42	4.24	0.90	10.26	5.10	1.06	9.76	4.72	0.96
(0.2,0.8)	50	T_{SAC}	23.04	14.50	4.94	23.04	14.04	4.26	22.36	14.18	4.22	20.74	12.78	4.42
	50	T_{SAC}^{r}	9.72	5.20	0.90	9.68	4.96	0.90	10.64	5.12	1.18	12.32	7.14	1.52
	50	T_{SAC}^{*}	13.70	6.86	0.82	13.30	5.68	0.54	13.24	5.56	0.74	5.82	3.40	0.78
	50	T_{SAC}^{r*}	11.54	5.22	0.60	10.54	3.58	0.22	10.00	3.82	0.44	4.34	1.76	0.32
	100	T_{SAC}	17.82	10.90	3.06	15.62	9.00	2.58	17.48	10.10	2.80	16.76	10.00	3.08
	100	T_{SAC}^{r}	11.60	5.70	1.36	8.10	3.80	0.82	10.56	5.36	1.12	11.44	6.06	1.44
	100	T_{SAC}^{*}	12.66	5.86	0.90	10.44	4.62	0.72	10.88	5.48	0.96	12.10	5.84	0.92
	100	T_{SAC}^{r*}	11.14	4.78	0.56	9.28	3.96	0.60	9.80	4.44	0.76	9.86	4.50	0.52
	200	T_{SAC}	13.44	7.06	1.82	13.66	7.76	1.98	14.28	7.96	1.94	13.80	7.50	1.76
	200	T_{SAC}^{r}	9.50	4.78	0.92	10.20	5.30	1.06	11.08	5.76	1.00	10.68	5.38	1.02
	200	T_{SAC}^{*}	10.56	5.06	0.74	8.66	4.56	1.06	9.38	4.76	0.94	10.22	4.68	0.70
	200	T_{SAC}^{r*}	10.08	4.58	0.70	8.08	4.02	0.96	8.74	4.32	0.70	9.16	4.08	0.44
	500	T_{SAC}	11.20	6.28	1.40	12.02	6.64	1.64	11.78	6.26	1.84	11.14	5.72	1.50
	500	T_{SAC}^{r}	10.16	5.20	1.12	10.36	5.34	1.18	10.78	5.46	1.30	9.98	5.06	1.10
	500	T_{SAC}^{*}	10.20	5.08	0.90	10.44	5.20	1.06	10.58	5.06	1.22	9.78	4.74	0.92
	500	T_{SAC}^{r*}	9.92	4.76	0.84	10.12	4.88	1.00	10.30	4.90	1.12	9.34	4.46	0.76
(0.8,0.2)	50	T_{SAC}	22.20	14.00	4.64	22.98	14.92	4.50	21.34	13.42	4.56	20.44	12.80	4.58
	50	T_{SAC}^{r}	10.14	5.40	1.20	9.82	5.40	1.18	10.96	5.84	1.24	6.98	3.70	0.86
	50	T_{SAC}^{*}	13.54	6.10	0.80	13.94	6.40	0.88	12.94	5.66	0.72	11.82	5.96	1.06
	50	T_{SAC}^{r*}	11.54	4.92	0.48	11.18	4.60	0.52	9.82	4.41	0.34	10.16	4.96	0.58
	100	T_{SAC}	16.22	9.64	2.88	17.10	10.02	3.14	16.88	10.20	3.16	17.28	10.46	2.42
	100	T_{SAC}^{r}	8.10	4.32	0.90	11.18	5.50	1.16	11.46	5.94	1.32	8.46	3.88	0.80
	100	T_{SAC}^{*}	11.44	5.28	0.86	11.04	4.98	1.04	10.58	5.12	0.92	10.68	5.42	1.36
	100	T_{SAC}^{r*}	10.08	4.58	0.68	9.60	4.36	0.78	9.26	4.38	0.66	9.80	4.94	1.18
	200	T_{SAC}	13.24	7.12	1.86	14.68	8.20	2.20	14.08	8.04	2.00	13.48	7.54	1.98
	200	T_{SAC}^{r}	8.82	4.10	0.92	9.36	4.72	0.98	9.80	4.86	0.96	8.18	4.04	0.94
	200	T_{SAC}^{*}	10.26	4.60	0.78	11.10	5.46	0.78	10.72	5.00	0.62	10.62	5.18	1.00
	200	T_{SAC}^{r*}	9.92	4.32	0.74	10.36	4.92	0.74	9.98	4.52	0.52	10.20	4.78	0.92
	500	T_{SAC}	11.70	6.46	1.64	11.96	6.02	1.28	12.06	6.20	1.34	12.00	6.10	1.36
	500	T_{SAC}^{r}	10.28	5.36	1.30	8.42	4.20	0.72	8.70	4.48	0.86	10.76	5.22	1.12
	500	T_{SAC}^{*}	10.34	5.50	1.18	10.20	4.88	0.94	10.60	5.22	0.98	10.24	4.78	0.88
	500	T_{SAC}^{r*}	10.26	5.38	1.10	9.86	4.74	0.90	10.24	5.14	0.90	10.04	4.66	0.84
(0.8,0.8)	50	T_{SAC}	21.88	14.20	5.20	21.16	13.60	4.24	21.76	13.34	3.76	23.36	14.82	5.34
	50	T_{SAC}^{r}	12.56	7.12	1.66	13.72	7.38	1.58	12.98	6.32	1.08	14.56	8.10	1.96
	50	T_{SAC}^{*}	13.10	6.70	1.64	11.38	5.28	0.52	10.78	4.76	0.64	14.78	7.30	1.00
	50	T_{SAC}^{r*}	10.36	4.80	0.96	8.38	3.44	0.22	7.66	3.02	0.34	10.54	4.88	0.44
	100	T_{SAC}	16.98	10.32	2.52	16.86	10.08	3.16	17.02	10.44	3.06	17.22	10.42	3.02
	100	T_{SAC}^{r}	10.16	5.24	1.14	12.50	6.76	1.68	12.72	7.18	1.76	13.28	7.00	1.82
	100	T_{SAC}^{*}	11.76	5.42	0.86	11.24	5.36	1.10	11.38	5.76	1.02	12.42	6.02	1.16
	100	T_{SAC}^{r*}	10.56	4.60	0.70	9.94	4.50	0.86	9.94	4.76	0.76	10.48	4.60	0.72
	200	T_{SAC}	14.42	8.28	2.30	14.22	7.70	2.16	13.66	7.32	2.08	13.56	7.30	1.96
	200	T_{SAC}^{r}	10.66	5.44	1.30	11.74	5.84	1.46	11.40	5.76	1.40	10.44	5.40	1.18
	200	T_{SAC}^{*}	11.00	5.20	1.08	10.78	5.04	0.96	10.26	4.84	0.82	10.78	5.28	0.78
	200	T_{SAC}^{r*}	10.48	4.82	0.88	10.02	4.72	0.76	9.68	4.50	0.64	9.86	4.72	0.64
	500	T_{SAC}	12.32	6.52	1.62	12.06	6.12	1.12	12.14	6.06	1.52	11.50	5.68	1.46
	500	T_{SAC}^{r}	11.10	5.88	1.30	11.02	5.40	0.84	10.96	5.36	1.28	10.18	5.06	1.04
	500	T_{SAC}^{*}	10.40	5.44	1.10	10.16	4.78	0.76	10.64	5.08	1.08	10.18	4.94	0.88
	500	T_{SAC}^{r*}	10.08	5.26	1.02	9.90	4.60	0.70	10.30	4.84	1.06	10.02	4.72	0.76

Table 4 – Empirical Sizes of the Tests for SAC Model, **DGP2 – Non-normal Disturbances**

			Circular world			Rook contiguity			Queen contiguity			Group Interaction		
(λ_1,λ_2)	n	Tests	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(0.2,0.2)	50	T_{SAC}	40.44	31.82	17.94	41.88	33.00	18.04	41.50	32.78	17.86	39.94	32.04	17.26
	50	T^r_{SAC}	18.50	11.86	3.84	16.86	11.00	3.96	17.26	10.84	3.98	15.32	9.72	3.44
	50	T^*_{SAC}	19.58	11.06	2.28	19.74	11.70	3.44	19.30	10.64	3.50	19.84	10.68	2.18
	50	T^{r*}_{SAC}	14.48	7.54	1.30	15.48	8.40	2.22	14.58	7.90	2.02	14.10	7.26	1.02
	100	T_{SAC}	37.86	29.70	15.86	36.54	27.54	14.86	37.48	28.80	15.42	38.10	29.18	16.24
	100	T^r_{SAC}	14.42	8.36	2.46	15.50	9.48	3.00	12.94	7.70	2.38	14.00	8.28	2.74
	100	T^*_{SAC}	16.04	7.78	1.76	15.02	9.38	3.18	15.04	8.74	3.08	16.00	8.36	1.78
	100	T^{r*}_{SAC}	12.50	5.66	1.26	12.60	7.28	2.42	12.62	7.24	2.26	12.38	6.04	1.20
	200	T_{SAC}	33.46	25.10	13.22	33.52	24.64	12.36	33.88	25.48	12.94	33.28	24.24	12.94
	200	T^r_{SAC}	12.06	6.74	1.84	12.08	6.88	1.78	13.20	7.44	2.32	14.06	8.24	2.32
	200	T^*_{SAC}	13.46	6.50	1.10	13.26	6.26	1.10	13.22	6.82	1.24	13.66	6.98	1.26
	200	T^{r*}_{SAC}	11.28	5.02	0.88	10.94	4.72	0.74	10.78	5.42	0.92	11.38	4.70	0.90
	500	T_{SAC}	28.78	20.42	9.74	28.98	21.66	11.16	30.10	21.78	10.36	28.96	20.50	9.08
	500	T^r_{SAC}	10.22	5.22	1.50	12.76	7.28	1.80	12.22	6.60	1.76	11.82	6.34	1.76
	500	T^*_{SAC}	11.98	5.82	1.02	12.36	6.54	1.36	11.98	6.14	1.44	11.04	5.52	1.20
	500	T^{r*}_{SAC}	10.96	5.04	0.76	11.32	6.06	1.22	11.20	5.62	1.30	10.04	4.66	1.00
(0.2,0.8)	50	T_{SAC}	40.36	31.14	17.26	41.90	31.76	16.02	40.82	31.20	16.26	40.04	31.30	16.96
	50	T^r_{SAC}	16.22	10.34	3.26	16.36	9.82	3.46	17.36	10.84	3.00	20.40	12.70	3.82
	50	T^*_{SAC}	19.18	10.32	2.02	18.06	9.22	1.58	18.16	9.14	1.34	20.16	12.14	3.54
	50	T^{r*}_{SAC}	13.18	6.42	0.98	12.00	5.78	0.74	12.50	5.74	0.80	13.18	6.70	1.22
	100	T_{SAC}	37.28	28.80	15.26	36.78	28.78	16.08	38.14	29.46	16.44	37.00	28.50	15.78
	100	T^r_{SAC}	17.18	10.02	2.88	14.52	8.24	2.50	17.62	10.28	3.10	18.44	11.26	3.88
	100	T^*_{SAC}	16.10	7.92	1.42	14.98	8.46	1.70	15.48	8.22	1.66	17.40	9.38	1.92
	100	T^{r*}_{SAC}	11.78	5.28	0.92	12.14	5.74	0.96	11.92	6.04	1.10	11.72	5.76	0.88
	200	T_{SAC}	31.62	22.96	11.66	32.72	24.14	12.44	33.22	24.58	13.04	34.12	25.36	13.18
	200	T^r_{SAC}	12.60	6.86	2.00	14.30	7.64	1.94	14.70	8.34	2.56	15.92	8.52	2.56
	200	T^*_{SAC}	13.84	6.44	1.12	13.24	7.86	2.24	13.62	7.58	2.40	15.68	7.98	1.60
	200	T^{r*}_{SAC}	10.54	4.24	0.70	11.28	6.48	1.56	11.60	6.28	1.78	10.50	4.76	0.90
	500	T_{SAC}	26.72	18.94	8.44	28.78	20.46	9.38	28.76	20.32	8.94	29.26	21.22	10.06
	500	T^r_{SAC}	11.90	6.42	1.48	12.68	6.86	1.58	12.32	6.38	1.38	13.20	6.96	1.60
	500	T^*_{SAC}	12.52	6.56	1.18	12.50	6.36	1.20	11.52	5.66	1.02	12.00	5.90	1.28
	500	T^{r*}_{SAC}	10.04	4.56	0.70	10.78	5.36	0.84	10.14	4.84	0.76	9.36	4.18	0.76
(0.8,0.2)	50	T_{SAC}	42.04	33.36	18.66	41.08	32.50	17.16	40.64	31.46	17.04	41.82	33.30	18.26
	50	T^r_{SAC}	19.44	12.82	5.08	17.60	11.24	3.74	18.72	11.58	4.30	15.10	9.40	3.36
	50	T^*_{SAC}	19.30	10.88	2.30	18.16	9.72	1.46	18.06	8.56	1.30	19.62	10.72	2.34
	50	T^{r*}_{SAC}	14.40	7.28	1.08	12.58	6.10	0.60	12.34	5.34	0.52	14.26	6.84	1.06
	100	T_{SAC}	38.68	29.92	16.10	37.86	28.80	15.66	38.92	30.54	17.18	38.72	29.34	16.22
	100	T^r_{SAC}	15.94	9.58	3.02	16.72	10.40	3.56	18.02	10.90	3.78	15.12	8.76	2.84
	100	T^*_{SAC}	15.98	9.60	2.92	14.10	7.32	1.94	15.20	8.40	2.42	16.38	10.02	2.98
	100	T^{r*}_{SAC}	13.46	7.54	2.04	10.62	5.42	1.32	11.88	6.42	1.58	13.82	7.88	2.04
	200	T_{SAC}	34.24	25.26	12.70	34.32	25.84	13.16	35.94	27.62	14.04	35.26	25.86	12.80
	200	T^r_{SAC}	12.92	7.28	2.06	13.12	7.66	2.26	14.40	8.00	2.60	13.22	7.26	1.82
	200	T^*_{SAC}	13.28	6.50	1.28	13.60	7.18	1.16	14.54	7.46	1.26	13.66	6.58	1.50
	200	T^{r*}_{SAC}	11.06	5.16	0.88	11.10	5.62	0.86	12.10	5.80	0.92	11.56	5.38	1.06
	500	T_{SAC}	29.88	21.60	9.92	28.90	20.72	9.80	28.50	20.82	9.14	30.22	21.64	10.20
	500	T^r_{SAC}	12.72	6.98	1.58	10.88	5.54	1.18	10.38	4.90	1.32	13.34	6.96	1.64
	500	T^*_{SAC}	11.14	5.94	0.92	11.48	5.86	1.08	10.72	5.28	0.94	11.92	5.88	0.88
	500	T^{r*}_{SAC}	10.38	5.10	0.68	10.66	5.12	0.98	9.74	4.74	0.76	10.68	4.94	0.74
(0.8,0.8)	50	T_{SAC}	40.72	32.32	18.42	40.36	31.92	18.32	42.50	32.94	18.30	39.80	30.90	17.10
	50	T^r_{SAC}	19.32	12.28	4.58	21.90	13.72	5.12	21.82	13.58	4.66	19.20	12.36	4.62
	50	T^*_{SAC}	18.18	10.04	2.62	15.36	8.20	2.02	16.84	8.78	1.94	20.46	11.70	3.18
	50	T^{r*}_{SAC}	11.94	5.98	1.34	10.06	5.16	1.06	10.96	5.34	1.12	13.44	6.72	1.12
	100	T_{SAC}	37.84	29.28	15.28	38.42	29.84	16.94	36.94	28.84	16.10	36.28	28.32	14.92
	100	T^r_{SAC}	16.56	9.62	3.02	19.18	12.08	4.26	18.38	11.56	4.02	17.28	10.08	2.70
	100	T^*_{SAC}	17.10	8.90	1.66	16.72	9.04	2.12	15.66	8.68	2.48	19.00	10.06	1.68
	100	T^{r*}_{SAC}	11.72	5.56	0.96	12.14	6.30	1.44	12.00	6.26	1.52	11.08	4.86	0.54
	200	T_{SAC}	34.38	25.68	13.02	34.42	25.06	12.68	34.48	26.02	13.50	33.60	25.66	12.90
	200	T^r_{SAC}	14.20	7.74	2.26	15.28	8.88	2.36	15.10	8.86	2.56	14.26	8.56	2.78
	200	T^*_{SAC}	14.68	7.70	1.90	12.98	6.42	1.32	12.90	6.46	1.36	16.94	9.46	2.54
	200	T^{r*}_{SAC}	10.88	5.22	1.10	10.00	4.70	0.88	10.54	4.96	0.96	11.24	5.76	1.20
	500	T_{SAC}	28.60	20.02	9.10	28.28	20.54	9.02	29.80	21.38	9.94	29.48	21.26	9.90
	500	T^r_{SAC}	12.32	6.58	2.00	12.30	6.62	1.36	13.22	7.04	1.76	12.88	6.54	1.64
	500	T^*_{SAC}	12.86	6.72	1.44	12.60	6.12	1.14	12.60	6.24	1.18	13.86	7.02	1.40
	500	T^{r*}_{SAC}	10.28	5.00	0.96	10.34	4.96	0.72	11.16	5.30	0.80	10.88	4.84	0.62

Table 5 – Empirical Sizes of the Tests for FESAC Model, **DGP1 – Normal Disturbances**

			Circular world			Rook contiguity			Queen contiguity			Group interaction		
(λ_1, λ_2)	n	Tests	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(0.2,0.2)	50	T_{FESAC}	13.76	7.78	2.00	13.24	7.58	2.06	13.08	7.02	1.74	13.58	7.68	2.02
	50	T_{FESAC}^{r}	9.88	5.22	1.18	8.04	4.24	0.96	7.48	3.84	0.76	10.24	5.16	1.08
	50	T_{FESAC}^{rr}	9.62	4.78	0.64	7.78	3.84	0.60	7.20	3.54	0.60	9.80	4.60	0.64
	50	T_{FESAC}^{*}	11.08	5.60	0.92	10.66	5.48	0.94	10.68	5.18	0.88	11.14	5.50	1.04
	50	T_{FESAC}^{r*}	10.56	5.24	0.70	10.00	4.92	0.84	10.12	4.76	0.74	10.56	5.06	0.80
	50	T_{FESAC}^{rr*}	10.70	5.00	0.62	10.02	4.88	0.68	9.82	4.64	0.52	10.50	4.80	0.66
	100	T_{FESAC}	11.20	6.18	1.42	12.38	6.58	1.24	11.12	5.90	1.22	11.46	6.28	1.32
	100	T_{FESAC}^{r}	10.00	5.12	1.02	11.14	5.30	0.92	8.92	4.70	0.70	9.76	5.20	0.94
	100	T_{FESAC}^{rr}	10.24	5.16	0.90	11.06	5.14	0.96	9.24	4.50	0.80	9.98	5.10	0.84
	100	T_{FESAC}^{*}	10.04	4.88	0.80	10.94	5.08	0.72	9.50	4.76	0.80	10.06	5.18	0.84
	100	T_{FESAC}^{r*}	9.72	4.74	0.76	10.54	4.96	0.66	9.32	4.64	0.80	9.84	4.82	0.78
	100	T_{FESAC}^{rr*}	9.86	4.94	0.76	10.74	4.66	0.78	9.64	4.54	0.84	10.08	4.82	0.78
	200	T_{FESAC}	10.86	5.40	1.26	10.76	5.72	1.20	10.68	5.66	1.02	10.04	5.00	1.10
	200	T_{FESAC}^{r}	10.28	4.92	1.04	10.12	5.18	1.04	10.04	5.04	0.80	9.46	4.48	0.92
	200	T_{FESAC}^{rr}	10.00	5.04	0.92	10.34	5.16	0.98	10.08	5.06	0.78	9.38	4.32	0.86
	200	T_{FESAC}^{*}	10.04	4.68	0.84	10.26	5.10	0.94	10.00	4.88	0.82	9.36	4.38	0.74
	200	T_{FESAC}^{r*}	9.90	5.54	0.80	10.10	5.02	0.90	9.88	4.80	0.80	9.12	4.34	0.74
	200	T_{FESAC}^{rr*}	9.66	4.52	0.76	10.12	5.02	0.84	9.98	4.94	0.70	9.06	4.16	0.64
	500	T_{FESAC}	10.74	5.42	0.98	11.28	5.66	0.94	10.38	5.18	1.16	11.12	5.62	1.16
	500	T_{FESAC}^{r}	10.48	5.16	0.88	11.08	5.40	0.82	9.96	4.90	1.08	10.42	5.00	1.06
	500	T_{FESAC}^{rr}	10.44	5.14	0.88	11.04	5.52	0.80	9.90	5.12	1.02	10.28	5.00	1.00
	500	T_{FESAC}^{*}	10.48	5.10	0.80	10.96	5.28	0.76	9.94	4.82	1.08	10.80	5.34	1.10
	500	T_{FESAC}^{r*}	10.40	5.02	0.80	10.86	5.24	0.74	9.84	4.74	1.08	10.76	5.30	1.06
	500	T_{FESAC}^{rr*}	10.38	5.08	0.84	10.82	5.38	0.72	9.86	5.00	1.00	10.68	5.24	0.88
(0.2,0.8)	50	T_{FESAC}	12.52	7.08	1.76	12.92	6.70	1.66	13.80	7.42	1.92	13.00	7.44	2.00
	50	T_{FESAC}^{r}	10.68	5.64	1.26	8.56	4.16	0.80	10.88	5.36	1.24	11.08	5.96	1.40
	50	T_{FESAC}^{rr}	10.48	5.26	1.00	8.56	4.22	0.62	10.70	5.26	0.94	11.28	5.82	1.22
	50	T_{FESAC}^{*}	9.50	4.88	1.14	10.30	4.88	0.70	11.22	5.12	0.98	9.78	5.08	1.22
	50	T_{FESAC}^{r*}	8.82	4.50	0.98	9.60	4.56	0.60	10.52	4.60	0.76	8.84	4.50	0.90
	50	T_{FESAC}^{rr*}	8.58	4.48	0.84	10.02	4.64	0.60	10.74	5.00	0.70	8.36	3.84	0.50
	100	T_{FESAC}	10.74	5.60	1.34	12.64	6.78	1.52	12.02	6.94	1.82	11.76	5.84	1.08
	100	T_{FESAC}^{r}	9.82	4.98	1.04	10.16	5.06	0.98	10.68	5.82	1.44	10.38	5.16	0.82
	100	T_{FESAC}^{rr}	9.68	4.80	0.90	10.22	4.78	0.84	10.90	5.54	1.16	10.32	4.78	0.90
	100	T_{FESAC}^{*}	9.62	4.72	0.68	11.58	5.62	1.00	10.84	5.66	1.34	10.28	4.76	0.64
	100	T_{FESAC}^{r*}	9.24	4.40	0.62	11.28	5.44	0.92	10.60	5.54	1.28	9.44	4.26	0.42
	100	T_{FESAC}^{rr*}	9.08	4.48	0.70	11.38	5.02	0.82	10.92	5.28	1.04	9.20	3.98	0.52
	200	T_{FESAC}	10.90	5.56	1.38	10.60	5.56	1.22	10.24	5.04	1.00	11.10	5.82	1.18
	200	T_{FESAC}^{r}	10.36	5.14	1.22	9.58	4.76	0.98	9.56	4.60	0.88	10.68	5.36	1.08
	200	T_{FESAC}^{rr}	10.36	5.10	1.08	9.70	4.78	0.78	9.54	4.62	0.80	10.56	5.16	1.04
	200	T_{FESAC}^{*}	9.94	4.78	1.02	10.10	4.88	0.92	9.62	4.48	0.80	10.10	5.22	0.92
	200	T_{FESAC}^{r*}	9.82	4.70	0.92	9.90	4.66	0.90	9.32	4.36	0.80	9.58	4.72	0.80
	200	T_{FESAC}^{rr*}	9.50	4.68	0.88	9.96	4.76	1.78	9.46	4.50	0.74	9.40	4.30	0.74
	500	T_{FESAC}	10.14	5.48	1.18	10.52	5.62	1.06	10.86	5.98	1.06	10.42	5.42	1.22
	500	T_{FESAC}^{r}	9.94	5.28	1.14	10.08	5.32	1.00	10.56	5.80	0.98	10.18	5.26	1.16
	500	T_{FESAC}^{rr}	9.90	5.22	1.00	10.06	5.36	1.02	10.22	5.66	1.04	10.12	5.36	1.18
	500	T_{FESAC}^{*}	9.88	5.14	1.08	10.12	5.32	0.94	10.40	5.68	0.90	10.20	5.14	1.16
	500	T_{FESAC}^{r*}	9.72	5.06	1.08	10.06	5.28	0.94	10.36	5.66	0.90	10.00	5.04	1.10
	500	T_{FESAC}^{rr*}	9.74	4.96	0.92	10.12	5.26	0.94	10.06	5.50	0.96	9.96	5.04	0.98
(0.8,0.8)	50	T_{FESAC}	13.14	7.40	1.82	12.94	6.92	1.66	12.68	6.70	1.80	14.44	7.70	2.00
	50	T_{FESAC}^{r}	11.28	6.06	1.44	10.86	5.66	1.16	10.80	5.48	1.28	12.64	6.14	1.48
	50	T_{FESAC}^{rr}	11.06	4.92	0.80	10.92	5.28	0.82	10.64	5.32	0.72	12.62	5.76	0.96
	50	T_{FESAC}^{*}	10.58	5.52	1.12	10.18	4.86	0.78	1.00	4.96	0.84	11.92	5.62	1.00
	50	T_{FESAC}^{r*}	9.92	4.86	0.92	9.48	4.24	0.64	9.32	4.54	0.64	10.76	5.00	0.86
	50	T_{FESAC}^{rr*}	10.06	4.18	0.70	9.72	4.22	0.60	9.88	4.52	0.46	10.70	4.64	0.62
	100	T_{FESAC}	12.52	6.34	1.42	11.72	6.12	1.24	11.24	5.96	1.16	11.94	6.56	1.62
	100	T_{FESAC}^{r}	11.34	5.54	1.12	10.94	5.58	1.06	10.24	5.16	0.90	11.10	5.84	1.38
	100	T_{FESAC}^{rr}	11.58	5.48	0.98	11.20	5.48	0.88	10.60	5.16	0.84	11.00	5.72	1.14
	100	T_{FESAC}^{*}	10.84	5.00	0.88	10.66	5.18	0.80	10.02	4.82	0.74	10.54	5.26	1.04
	100	T_{FESAC}^{r*}	10.30	4.82	0.78	10.20	4.94	0.74	9.80	4.58	0.68	9.68	4.60	0.86
	100	T_{FESAC}^{rr*}	10.56	4.76	0.70	10.42	4.90	0.64	10.04	4.64	0.70	9.80	4.44	0.68
	200	T_{FESAC}	11.12	5.74	1.40	10.10	5.18	0.98	10.34	5.18	0.92	10.72	5.70	1.36
	200	T_{FESAC}^{r}	10.66	5.38	1.30	9.64	4.94	0.88	9.62	4.74	0.74	10.16	5.24	1.24
	200	T_{FESAC}^{rr}	10.78	5.20	1.20	9.58	4.90	0.80	9.92	4.82	0.76	9.96	5.44	1.24
	200	T_{FESAC}^{*}	10.38	5.12	1.08	9.24	4.74	0.70	9.28	4.54	0.66	10.00	4.98	0.98
	200	T_{FESAC}^{r*}	10.30	5.04	0.98	9.02	4.64	0.64	9.08	4.32	0.60	9.54	4.70	0.90
	200	T_{FESAC}^{rr*}	10.42	4.86	0.88	9.26	4.40	0.58	9.64	4.48	0.66	9.32	4.50	0.86
	500	T_{FESAC}	11.34	5.38	1.08	10.22	5.44	1.04	11.30	5.88	1.20	10.92	5.66	1.22
	500	T_{FESAC}^{r}	11.12	5.28	0.98	10.10	5.34	1.00	11.02	5.68	1.12	10.70	5.54	1.10
	500	T_{FESAC}^{rr}	10.84	5.20	0.88	10.28	5.20	0.98	11.30	5.76	1.08	10.80	5.38	1.22
	500	T_{FESAC}^{*}	11.00	5.18	0.92	10.10	5.30	0.96	11.00	5.58	1.06	10.58	5.32	1.10
	500	T_{FESAC}^{r*}	10.90	5.12	0.88	10.00	5.18	0.96	10.90	5.56	1.06	10.30	5.14	1.06
	500	T_{FESAC}^{rr*}	10.60	5.04	0.82	10.14	5.06	0.94	11.12	5.68	1.04	10.32	5.14	1.06

Table 6 – Empirical Sizes of the Tests for FESAC Model, **DGP1 – Non-normal Disturbances**

			Circular world			Rook contiguity			Queen contiguity			Group interaction		
(λ_1, λ_2)	n	Tests	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(0.2,0.2)	50	T_{FESAC}	24.34	16.10	6.20	24.44	16.48	7.16	24.56	16.82	7.20	24.22	15.80	6.50
	50	T^{r}_{FESAC}	17.54	10.46	3.08	16.38	9.68	3.16	15.58	9.38	3.30	18.28	10.48	3.00
	50	T^{rr}_{FESAC}	10.92	5.38	0.70	10.46	5.22	0.92	9.94	4.94	0.94	11.02	5.32	0.82
	50	T^{*}_{FESAC}	18.16	9.66	1.68	18.18	10.16	2.34	18.72	10.52	2.44	18.20	9.92	1.66
	50	T^{r*}_{FESAC}	17.00	8.76	1.36	16.92	9.20	2.00	17.10	9.40	2.14	16.94	8.86	1.28
	50	T^{rr*}_{FESAC}	9.44	3.40	0.24	9.62	3.96	0.50	9.48	4.08	0.54	9.32	3.56	0.30
	100	T_{FESAC}	22.50	14.36	5.22	22.74	14.78	5.38	21.62	14.06	5.40	22.38	14.60	5.36
	100	T^{r}_{FESAC}	19.32	11.14	3.60	19.52	11.38	3.64	17.20	10.68	3.70	19.12	11.10	3.56
	100	T^{rr}_{FESAC}	11.44	5.90	1.28	11.54	5.60	1.00	10.74	5.32	1.00	11.54	5.76	1.28
	100	T^{*}_{FESAC}	18.74	9.84	2.74	18.44	10.52	2.70	17.30	10.34	2.82	18.78	10.30	2.64
	100	T^{r*}_{FESAC}	18.02	9.26	2.38	17.86	9.84	2.42	16.56	9.74	2.60	18.02	9.60	2.44
	100	T^{rr*}_{FESAC}	9.66	4.50	0.54	9.80	4.14	0.52	9.42	4.22	0.60	9.70	4.28	0.68
	200	T_{FESAC}	19.52	12.70	4.90	20.54	13.64	4.90	21.22	13.42	4.98	20.52	13.44	5.26
	200	T^{r}_{FESAC}	17.86	10.96	3.72	18.44	11.42	3.68	19.42	11.48	3.98	18.66	11.78	4.16
	200	T^{rr}_{FESAC}	10.72	5.56	1.20	10.92	5.64	1.14	11.40	6.16	1.04	11.68	6.44	1.18
	200	T^{*}_{FESAC}	17.12	10.16	3.08	18.14	10.52	3.06	18.78	10.28	3.32	17.94	11.06	3.22
	200	T^{r*}_{FESAC}	16.58	9.92	2.84	17.62	10.18	2.88	18.34	10.28	3.14	17.66	10.74	3.12
	200	T^{rr*}_{FESAC}	9.38	4.34	0.74	9.96	4.66	0.88	10.24	5.38	0.72	10.36	5.34	0.74
	500	T_{FESAC}	19.18	12.04	4.04	18.44	11.88	4.26	19.00	11.82	3.94	19.10	12.26	4.10
	500	T^{r}_{FESAC}	18.00	11.18	3.40	17.46	10.66	3.68	18.04	11.08	3.28	17.82	10.94	3.38
	500	T^{rr}_{FESAC}	10.64	5.30	1.14	10.12	5.42	1.08	10.54	5.44	0.94	10.74	4.98	1.10
	500	T^{*}_{FESAC}	17.76	10.90	3.02	17.20	10.26	3.24	17.78	10.44	2.84	17.94	11.14	3.22
	500	T^{r*}_{FESAC}	17.60	10.78	2.90	17.08	10.14	3.22	17.54	10.22	2.78	17.86	11.04	3.12
	500	T^{rr*}_{FESAC}	10.24	4.82	0.90	9.48	5.08	0.84	9.64	4.90	0.74	10.54	4.80	0.88
(0.2,0.8)	50	T_{FESAC}	24.42	16.54	7.14	25.24	16.86	7.34	24.30	16.52	7.22	25.36	17.00	7.14
	50	T^{r}_{FESAC}	20.10	12.58	4.48	17.56	11.24	3.68	19.20	12.32	4.18	20.40	12.90	4.82
	50	T^{rr}_{FESAC}	12.98	6.94	1.32	11.16	5.64	1.04	12.40	6.12	1.04	13.56	7.10	1.60
	50	T^{*}_{FESAC}	17.50	10.34	2.96	19.08	10.90	2.62	18.00	10.84	2.42	18.40	10.52	3.42
	50	T^{r*}_{FESAC}	15.94	9.34	2.46	17.48	9.84	1.94	16.92	9.58	1.98	16.12	8.68	2.44
	50	T^{rr*}_{FESAC}	9.16	4.46	0.60	9.88	4.18	0.28	9.76	3.70	0.36	9.20	4.46	0.64
	100	T_{FESAC}	23.40	15.86	6.16	22.84	14.54	5.06	22.28	14.44	5.12	22.34	14.92	5.78
	100	T^{r}_{FESAC}	21.08	13.10	4.38	18.04	10.58	2.88	19.24	11.76	3.36	19.64	12.06	3.88
	100	T^{rr}_{FESAC}	13.10	6.78	1.34	10.54	5.08	1.00	11.58	5.46	1.14	11.94	5.94	1.14
	100	T^{*}_{FESAC}	19.68	11.60	2.98	18.86	10.72	2.52	18.84	10.78	2.54	19.06	10.92	2.84
	100	T^{r*}_{FESAC}	18.80	10.98	2.66	18.12	9.96	2.18	17.98	9.84	2.28	17.44	9.24	2.28
	100	T^{rr*}_{FESAC}	10.68	4.86	0.66	9.92	3.94	0.46	9.82	4.20	0.64	9.48	4.02	0.44
	200	T_{FESAC}	20.12	13.06	4.70	19.60	12.70	4.44	20.46	12.56	4.46	20.42	12.60	4.36
	200	T^{r}_{FESAC}	18.44	11.18	3.76	17.36	10.58	3.42	18.38	10.96	3.56	18.68	11.00	3.28
	200	T^{rr}_{FESAC}	11.04	5.70	1.32	10.32	5.06	0.98	10.68	5.76	1.12	11.12	5.28	1.02
	200	T^{*}_{FESAC}	17.60	10.30	2.92	17.58	10.22	2.94	18.28	10.16	2.92	17.78	10.12	2.72
	200	T^{r*}_{FESAC}	17.10	9.90	2.86	17.12	9.84	2.72	17.68	9.92	2.74	16.74	9.10	2.12
	200	T^{rr*}_{FESAC}	9.58	4.42	0.84	9.54	4.42	0.50	9.54	4.80	0.56	8.78	3.76	0.40
	500	T_{FESAC}	18.96	12.02	4.08	17.82	11.06	3.86	18.64	11.58	3.92	19.14	12.00	4.44
	500	T^{r}_{FESAC}	18.32	11.14	3.66	16.88	9.96	3.44	17.82	10.88	3.58	18.30	11.00	3.80
	500	T^{rr}_{FESAC}	10.70	5.48	1.28	9.84	4.96	0.96	10.40	5.48	1.00	10.56	5.62	1.32
	500	T^{*}_{FESAC}	17.98	10.76	3.30	16.70	9.66	3.02	17.38	10.44	3.28	18.06	10.58	3.42
	500	T^{r*}_{FESAC}	17.82	10.54	3.22	16.58	9.50	2.94	17.14	10.30	3.02	17.66	10.28	3.16
	500	T^{rr*}_{FESAC}	10.06	5.02	0.96	9.20	4.60	0.68	9.90	4.78	0.60	9.92	4.94	0.98
(0.8,0.8)	50	T_{FESAC}	25.20	17.06	7.04	24.44	16.40	6.82	25.32	16.98	6.52	23.38	16.06	6.22
	50	T^{r}_{FESAC}	21.10	12.94	4.18	20.30	12.76	4.02	20.98	12.80	3.48	19.74	12.22	3.86
	50	T^{rr}_{FESAC}	13.28	6.24	1.22	12.92	6.24	0.96	12.68	6.04	0.82	12.64	5.96	1.12
	50	T^{*}_{FESAC}	19.70	10.56	2.70	18.40	10.00	2.28	18.48	9.84	2.00	17.86	10.12	2.24
	50	T^{r*}_{FESAC}	17.52	9.52	2.22	17.10	9.10	1.78	17.22	8.86	1.60	16.10	8.44	1.62
	50	T^{rr*}_{FESAC}	9.54	3.92	0.34	9.06	3.64	0.38	9.28	3.56	0.28	8.88	3.18	0.38
	100	T_{FESAC}	21.96	14.58	5.64	23.06	14.90	5.84	22.42	14.90	6.02	23.36	15.60	6.24
	100	T^{r}_{FESAC}	19.28	12.04	3.86	20.30	12.36	4.00	19.70	12.32	4.18	20.52	13.08	4.12
	100	T^{rr}_{FESAC}	12.26	6.10	1.22	12.34	6.42	1.14	12.50	6.70	1.32	12.56	6.68	1.50
	100	T^{*}_{FESAC}	18.04	10.34	2.60	19.00	10.58	2.66	18.74	10.94	2.90	19.40	11.54	2.86
	100	T^{r*}_{FESAC}	17.26	9.70	2.44	18.26	9.94	2.38	17.88	10.44	2.54	17.20	9.86	2.22
	100	T^{rr*}_{FESAC}	9.92	4.14	0.48	9.78	4.58	0.62	10.44	4.52	0.58	9.70	3.98	0.46
	200	T_{FESAC}	20.24	13.24	4.56	19.80	12.88	4.54	19.52	12.54	4.54	20.58	13.68	5.06
	200	T^{r}_{FESAC}	18.60	11.80	3.62	17.66	11.32	3.46	17.88	10.62	3.36	18.92	12.40	3.94
	200	T^{rr}_{FESAC}	11.62	5.64	1.10	11.00	5.40	1.10	10.50	5.54	1.14	11.92	6.36	1.48
	200	T^{*}_{FESAC}	18.08	10.86	3.00	17.24	10.56	2.68	17.12	9.62	2.70	18.30	11.46	3.26
	200	T^{r*}_{FESAC}	17.72	10.36	2.90	16.68	10.06	2.52	16.64	9.16	2.54	17.04	10.24	2.80
	200	T^{rr*}_{FESAC}	10.34	4.56	0.70	9.60	4.26	0.62	9.18	4.52	0.50	9.60	4.50	0.80
	500	T_{FESAC}	18.66	11.74	4.16	19.82	12.14	3.72	18.64	11.72	3.60	18.90	11.86	4.00
	500	T^{r}_{FESAC}	17.90	10.94	3.74	18.78	11.26	3.08	17.76	10.96	3.22	17.94	11.20	3.44
	500	T^{rr}_{FESAC}	10.46	5.42	1.12	10.66	5.20	0.84	10.32	4.92	0.86	10.54	5.08	1.00
	500	T^{*}_{FESAC}	17.70	10.50	3.24	18.60	10.76	2.72	17.54	10.60	2.82	17.74	10.86	2.98
	500	T^{r*}_{FESAC}	17.44	10.40	3.20	18.32	10.58	2.66	17.32	10.42	2.80	17.30	10.28	2.70
	500	T^{rr*}_{FESAC}	9.94	5.00	0.82	10.12	4.64	0.60	9.82	4.30	0.70	9.80	4.38	0.72

Table 7 – Empirical Sizes of the Tests for FESAC Model, **DGP2 – Normal Disturbances**

			Circular world			Rook contiguity			Queen contiguity			Group interaction		
(λ_1,λ_2)	n	Tests	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(0.2,0.2)	50	T_{FESAC}	13.16	7.28	1.92	13.20	7.80	2.02	13.36	7.80	2.02	13.18	7.20	1.84
	50	T_{FESAC}^{r}	10.98	5.82	1.26	11.10	6.26	1.30	11.24	6.14	1.14	10.72	5.58	1.22
	50	T_{FESAC}^{rr}	11.06	5.20	1.00	11.22	5.78	0.92	11.08	5.60	0.92	10.60	5.16	1.08
	50	T_{FESAC}^{*}	10.92	5.18	0.98	10.78	5.66	0.74	10.68	5.62	0.66	10.78	5.40	0.96
	50	T_{FESAC}^{r*}	10.20	4.84	0.82	10.00	5.08	0.64	10.10	5.18	0.56	9.86	4.98	0.80
	50	T_{FESAC}^{rr*}	10.46	4.60	0.66	10.38	5.16	0.58	10.12	4.86	0.52	10.28	4.62	0.64
	100	T_{FESAC}	11.98	6.14	1.34	11.82	6.24	1.54	11.92	6.06	1.22	11.96	6.12	1.36
	100	T_{FESAC}^{r}	10.74	5.42	1.00	10.82	5.42	1.18	10.76	5.08	0.80	10.74	5.52	1.06
	100	T_{FESAC}^{rr}	11.16	5.18	0.96	10.82	5.38	0.86	10.70	5.10	0.68	11.00	5.14	1.18
	100	T_{FESAC}^{*}	10.54	5.00	0.78	10.28	5.12	0.86	10.32	4.92	0.70	10.34	5.14	0.84
	100	T_{FESAC}^{r*}	10.22	4.76	0.66	10.04	4.86	0.76	10.02	4.68	0.64	9.96	4.84	0.76
	100	T_{FESAC}^{rr*}	10.64	4.80	0.72	10.40	4.94	0.70	10.36	4.48	0.56	10.44	4.66	0.82
	200	T_{FESAC}	11.04	5.76	1.36	10.70	6.08	0.96	11.64	6.22	1.00	10.68	5.30	1.22
	200	T_{FESAC}^{r}	10.44	5.26	1.14	10.20	5.52	0.88	11.02	5.74	0.88	10.18	4.92	1.10
	200	T_{FESAC}^{rr}	10.54	5.38	1.08	10.50	5.34	0.92	11.28	5.72	0.94	9.86	5.00	1.06
	200	T_{FESAC}^{*}	10.40	5.08	0.96	9.90	5.08	0.80	10.82	5.58	0.74	10.12	4.86	0.98
	200	T_{FESAC}^{r*}	10.26	4.94	0.90	9.78	4.96	0.80	10.70	5.40	0.70	9.90	4.72	0.94
	200	T_{FESAC}^{rr*}	10.32	5.30	0.82	9.94	4.76	0.70	10.64	5.14	0.66	9.42	4.80	0.94
	500	T_{FESAC}	10.04	5.40	1.04	9.48	4.62	1.04	10.02	5.20	0.98	10.18	5.36	1.24
	500	T_{FESAC}^{r}	9.74	5.22	0.98	9.26	4.48	1.02	9.66	5.02	0.88	9.90	5.20	1.16
	500	T_{FESAC}^{rr}	9.82	4.94	0.92	9.20	4.58	1.00	9.58	4.90	0.84	10.00	5.14	0.96
	500	T_{FESAC}^{*}	9.44	5.02	0.84	9.26	4.40	0.96	9.60	4.98	0.82	9.58	4.88	1.00
	500	T_{FESAC}^{r*}	9.44	4.98	0.84	9.20	4.38	0.92	9.56	4.92	0.82	9.50	4.80	0.98
	500	T_{FESAC}^{rr*}	9.30	4.66	0.82	9.08	4.44	0.92	9.44	4.86	0.78	9.26	4.84	0.84
(0.2,0.8)	50	T_{FESAC}	13.32	7.08	1.76	13.44	6.96	1.50	11.66	6.42	1.74	12.46	7.14	1.82
	50	T_{FESAC}^{r}	11.06	5.62	1.18	10.44	4.86	0.98	9.38	4.72	1.16	10.78	6.06	1.44
	50	T_{FESAC}^{rr}	11.22	5.46	0.64	9.86	4.68	0.82	9.52	4.48	0.82	10.78	5.46	1.22
	50	T_{FESAC}^{*}	10.48	5.04	0.82	10.72	4.48	0.58	9.48	4.48	0.68	9.72	4.94	0.92
	50	T_{FESAC}^{r*}	9.90	4.56	0.66	10.02	3.96	0.52	8.84	3.86	0.52	7.98	3.50	0.58
	50	T_{FESAC}^{rr*}	9.98	4.62	0.42	9.68	4.18	0.58	9.18	4.04	0.58	7.50	3.24	0.50
	100	T_{FESAC}	11.26	6.24	1.34	11.96	6.52	1.48	11.30	6.10	1.60	11.60	5.84	1.48
	100	T_{FESAC}^{r}	10.46	5.42	1.00	10.62	5.64	1.18	11.32	5.24	1.34	10.58	5.30	1.16
	100	T_{FESAC}^{rr}	10.86	4.96	0.84	10.58	5.48	0.98	10.58	5.12	0.10	10.76	5.02	1.06
	100	T_{FESAC}^{*}	10.00	4.98	0.82	10.44	5.24	0.98	9.88	4.78	1.02	9.90	4.60	0.88
	100	T_{FESAC}^{r*}	9.64	4.62	0.68	10.12	5.04	0.94	9.58	4.56	0.98	9.00	3.90	0.60
	100	T_{FESAC}^{rr*}	10.10	4.34	0.62	10.26	5.00	0.76	9.70	4.74	0.82	8.74	3.52	0.64
	200	T_{FESAC}	10.68	5.50	1.40	10.58	5.22	1.00	10.24	5.50	1.18	10.34	5.28	1.02
	200	T_{FESAC}^{r}	10.14	5.10	1.20	9.90	4.80	0.84	9.70	5.10	1.02	9.88	4.96	0.94
	200	T_{FESAC}^{rr}	10.38	5.16	1.18	9.76	4.76	0.78	9.82	5.24	0.94	9.88	4.88	0.74
	200	T_{FESAC}^{*}	9.84	4.62	1.08	9.90	4.70	0.72	9.40	4.90	0.94	9.64	4.72	0.88
	200	T_{FESAC}^{r*}	9.70	4.50	0.94	9.74	4.48	0.70	9.28	4.78	0.84	9.14	4.06	0.74
	200	T_{FESAC}^{rr*}	9.68	4.54	0.96	9.58	4.54	0.70	9.46	5.04	0.86	8.92	3.94	0.50
	500	T_{FESAC}	10.60	5.04	1.26	10.18	5.38	1.10	10.86	5.44	1.12	10.58	5.16	1.06
	500	T_{FESAC}^{r}	10.30	4.92	1.16	9.92	5.20	1.02	10.54	5.24	1.06	10.28	4.94	1.00
	500	T_{FESAC}^{rr}	10.44	5.04	1.14	10.10	5.22	1.00	10.54	5.12	1.10	10.14	4.98	0.98
	500	T_{FESAC}^{*}	10.06	4.58	1.02	9.92	5.14	0.96	10.54	5.12	1.04	10.12	4.94	0.92
	500	T_{FESAC}^{r*}	9.94	4.48	0.98	9.88	5.06	0.92	10.40	5.06	1.02	9.92	4.80	0.90
	500	T_{FESAC}^{rr*}	9.88	4.74	0.96	9.90	5.07	1.02	10.44	4.92	1.00	9.80	4.72	0.86
(0.8,0.8)	50	T_{FESAC}	13.70	7.72	1.52	13.26	7.22	1.84	13.14	7.08	1.58	13.20	6.90	1.60
	50	T_{FESAC}^{r}	12.04	6.32	1.06	11.42	6.12	1.34	11.32	5.80	1.16	11.50	5.60	0.96
	50	T_{FESAC}^{rr}	11.64	5.88	0.76	10.50	5.62	0.82	11.04	5.00	0.76	10.94	5.54	0.84
	50	T_{FESAC}^{*}	11.14	5.42	0.74	10.80	5.32	0.98	10.68	5.22	0.76	10.84	4.90	0.72
	50	T_{FESAC}^{r*}	10.28	4.76	0.60	10.06	5.04	0.84	9.88	4.68	0.66	9.68	4.06	0.50
	50	T_{FESAC}^{rr*}	10.48	4.98	0.42	9.78	4.60	0.70	10.18	4.18	0.52	9.42	4.02	0.36
	100	T_{FESAC}	11.62	6.06	1.38	12.28	6.30	1.46	12.22	6.58	1.60	11.74	6.16	1.40
	100	T_{FESAC}^{r}	10.62	5.38	1.16	11.00	5.60	1.20	11.42	5.74	1.30	10.74	5.46	1.06
	100	T_{FESAC}^{rr}	10.94	5.48	0.94	10.86	5.48	1.06	11.26	5.54	1.12	10.72	5.50	1.00
	100	T_{FESAC}^{*}	10.22	5.12	0.74	10.78	5.20	1.04	11.10	5.30	1.00	10.58	5.12	0.82
	100	T_{FESAC}^{r*}	9.80	4.80	0.68	10.46	4.92	0.92	10.68	5.12	0.96	9.06	3.92	0.46
	100	T_{FESAC}^{rr*}	10.22	4.80	0.66	10.40	4.94	0.76	10.68	5.08	0.90	8.32	3.72	0.46
	200	T_{FESAC}	11.26	5.56	1.22	10.40	5.28	1.26	10.88	5.46	1.08	11.10	5.92	1.46
	200	T_{FESAC}^{r}	10.84	5.24	1.06	9.98	4.92	1.12	10.58	5.10	0.94	10.56	5.60	1.30
	200	T_{FESAC}^{rr}	10.70	5.06	0.84	10.14	5.28	0.94	10.24	5.02	0.90	10.90	5.34	1.16
	200	T_{FESAC}^{*}	10.10	4.80	0.86	9.84	4.80	1.06	10.30	4.76	0.82	10.14	5.26	1.22
	200	T_{FESAC}^{r*}	9.96	4.68	0.82	9.70	4.70	0.94	10.16	4.68	0.80	9.54	4.62	0.96
	200	T_{FESAC}^{rr*}	9.80	4.52	0.66	9.74	4.76	0.84	9.90	4.72	0.76	9.28	4.34	0.76
	500	T_{FESAC}	10.48	5.22	1.10	10.14	5.16	1.20	10.12	4.92	0.90	10.86	5.70	1.26
	500	T_{FESAC}^{r}	10.16	5.02	1.04	9.98	4.90	1.14	9.98	4.60	0.90	10.70	5.64	1.06
	500	T_{FESAC}^{rr}	10.16	5.02	1.06	9.88	5.04	1.10	9.92	4.86	0.94	10.66	5.48	1.16
	500	T_{FESAC}^{*}	10.08	4.92	0.94	9.74	4.72	1.04	9.92	4.52	0.84	10.62	5.48	1.04
	500	T_{FESAC}^{r*}	9.98	4.80	0.90	9.70	4.66	1.04	9.88	4.42	0.82	10.46	5.26	1.02
	500	T_{FESAC}^{rr*}	9.92	4.82	1.02	9.70	4.86	1.06	9.72	4.64	0.90	10.32	5.12	1.00

Table 8 – Empirical Sizes of the Tests for FESAC Model, **DGP2 – Non-normal Disturbances**

			Circular world			Rook contiguity			Queen contiguity			Group interaction		
(λ_1,λ_2)	n	Tests	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(0.2,0.2)	50	T_{FESAC}	25.38	16.94	6.84	24.70	17.40	7.36	24.66	17.24	7.14	25.36	16.78	6.84
	50	T_{FESAC}^{r}	20.78	12.56	3.60	20.42	12.70	4.42	20.18	12.74	4.34	20.42	12.26	3.88
	50	T_{FESAC}^{rr}	12.78	6.42	1.04	12.82	6.64	1.28	12.78	6.30	1.34	12.66	6.30	1.06
	50	T_{FESAC}^{*}	19.16	10.14	1.98	19.02	10.26	2.64	18.90	10.56	2.66	19.32	10.10	2.18
	50	T_{FESAC}^{r*}	17.78	8.86	1.62	17.28	9.42	2.20	17.20	9.34	2.22	17.62	8.80	1.72
	50	T_{FESAC}^{rr*}	9.36	3.52	0.42	9.64	4.40	0.44	9.40	4.28	0.48	9.44	3.66	0.38
	100	T_{FESAC}	22.78	14.98	5.80	22.64	14.76	5.42	21.38	13.82	5.42	23.40	15.78	6.02
	100	T_{FESAC}^{r}	20.06	12.24	3.92	19.86	11.92	3.26	18.50	11.20	3.50	20.54	12.46	4.12
	100	T_{FESAC}^{rr}	11.88	6.04	1.56	12.10	5.62	0.92	11.34	5.74	1.10	12.72	6.46	1.58
	100	T_{FESAC}^{*}	18.90	10.78	2.76	18.70	10.48	2.22	17.60	10.24	2.48	19.32	11.18	3.12
	100	T_{FESAC}^{r*}	18.28	10.02	2.54	17.98	9.78	1.92	16.90	9.78	2.14	18.40	10.08	2.80
	100	T_{FESAC}^{rr*}	9.70	4.36	0.80	9.52	3.86	0.32	9.30	3.94	0.56	10.30	4.72	0.90
	200	T_{FESAC}	20.60	12.94	4.34	20.70	13.32	4.94	20.46	13.04	4.78	20.14	12.90	4.70
	200	T_{FESAC}^{r}	18.96	11.22	3.58	19.04	11.70	3.84	18.68	11.34	3.36	18.62	11.44	3.80
	200	T_{FESAC}^{rr}	11.04	5.48	1.06	11.68	6.10	1.00	10.92	5.78	0.90	10.98	5.64	1.36
	200	T_{FESAC}^{*}	18.24	10.54	2.90	17.94	10.82	3.12	17.64	10.36	2.54	18.08	10.88	3.22
	200	T_{FESAC}^{r*}	17.92	10.18	2.78	17.40	10.30	2.98	17.42	9.88	2.42	17.48	10.34	3.06
	200	T_{FESAC}^{rr*}	9.72	4.56	0.60	10.14	4.90	0.62	9.50	4.38	0.70	9.60	4.80	0.72
	500	T_{FESAC}	19.14	12.10	4.00	20.02	12.64	4.32	19.46	12.22	3.70	18.74	11.40	3.68
	500	T_{FESAC}^{r}	18.42	11.08	3.62	19.14	11.92	3.68	18.66	11.32	3.14	18.06	10.68	3.26
	500	T_{FESAC}^{rr}	10.56	5.28	1.20	11.24	5.54	1.02	11.06	4.94	0.72	10.48	5.02	1.00
	500	T_{FESAC}^{*}	17.44	10.34	3.10	18.96	11.76	3.34	18.48	11.00	2.90	17.40	10.22	2.80
	500	T_{FESAC}^{r*}	17.22	10.14	3.04	18.78	11.52	3.24	18.24	10.90	2.84	17.20	10.06	2.78
	500	T_{FESAC}^{rr*}	9.48	4.72	0.86	10.80	5.06	0.84	10.50	4.30	0.56	9.62	4.46	0.68
(0.2,0.8)	50	T_{FESAC}	25.38	16.82	7.22	24.14	16.14	6.76	24.14	15.96	6.54	24.18	16.22	7.56
	50	T_{FESAC}^{r}	20.76	13.30	4.60	18.14	11.12	4.02	18.98	11.84	3.92	19.52	12.00	4.56
	50	T_{FESAC}^{rr}	13.82	6.74	1.48	11.26	5.54	1.18	12.02	6.22	0.98	12.72	6.72	1.34
	50	T_{FESAC}^{*}	18.18	10.16	3.02	17.68	9.78	2.64	18.04	10.06	2.18	18.20	10.56	3.34
	50	T_{FESAC}^{r*}	16.40	9.06	2.38	16.08	8.62	2.08	16.66	8.68	1.64	14.68	7.22	1.64
	50	T_{FESAC}^{rr*}	9.82	4.28	0.56	9.16	3.70	0.42	9.20	3.76	0.14	7.36	2.74	0.22
	100	T_{FESAC}	21.80	14.58	5.56	23.00	14.94	6.26	22.44	14.88	5.64	22.02	14.58	5.02
	100	T_{FESAC}^{r}	19.44	12.04	3.88	19.88	11.64	4.16	19.48	12.42	3.76	19.68	11.90	3.52
	100	T_{FESAC}^{rr}	12.12	6.46	1.36	11.70	6.40	1.28	11.90	6.08	0.96	12.00	5.82	1.26
	100	T_{FESAC}^{*}	17.80	10.50	2.68	18.68	10.54	2.88	17.88	10.14	2.54	18.08	10.06	2.28
	100	T_{FESAC}^{r*}	16.88	9.28	2.40	17.68	9.60	2.48	17.02	9.48	2.24	16.26	8.50	1.48
	100	T_{FESAC}^{rr*}	9.22	4.10	0.70	9.68	4.38	0.58	9.46	3.92	0.46	7.94	3.06	0.42
	200	T_{FESAC}	19.38	12.34	4.82	20.24	12.48	4.32	20.08	12.80	4.26	19.22	12.18	4.76
	200	T_{FESAC}^{r}	17.66	10.88	3.84	18.14	10.84	3.04	18.40	11.04	3.18	17.14	10.82	3.86
	200	T_{FESAC}^{rr}	10.64	5.80	1.16	10.52	5.06	0.94	10.72	4.98	0.98	10.62	5.62	1.12
	200	T_{FESAC}^{*}	17.24	10.12	3.14	17.72	10.28	2.68	17.92	10.16	2.50	16.84	10.26	3.26
	200	T_{FESAC}^{r*}	16.74	9.92	2.90	17.28	9.88	2.44	17.54	9.80	2.30	15.44	9.06	2.48
	200	T_{FESAC}^{rr*}	9.36	4.58	0.76	9.22	4.24	0.46	9.56	3.86	0.48	8.38	4.10	0.42
	500	T_{FESAC}	20.24	12.58	4.36	19.66	12.10	3.94	19.20	12.02	4.32	19.22	12.14	4.04
	500	T_{FESAC}^{r}	19.24	11.76	3.64	18.74	11.44	3.52	18.26	11.12	3.78	18.46	11.42	3.56
	500	T_{FESAC}^{rr}	11.16	5.54	0.72	10.78	5.46	1.34	10.60	5.52	1.18	10.92	5.40	1.18
	500	T_{FESAC}^{*}	18.78	11.40	3.00	18.50	11.08	3.20	18.16	10.74	3.34	18.20	11.06	3.24
	500	T_{FESAC}^{r*}	18.64	11.16	2.98	18.42	10.84	3.10	17.92	10.60	3.32	17.68	10.60	2.98
	500	T_{FESAC}^{rr*}	10.52	4.94	0.50	10.22	5.00	1.02	10.06	5.06	0.96	9.88	4.58	0.88
(0.8,0.8)	50	T_{FESAC}	25.08	17.24	7.42	24.40	15.48	6.00	25.10	17.58	7.46	25.52	16.86	7.28
	50	T_{FESAC}^{r}	21.26	13.38	4.90	19.98	12.18	3.80	21.46	13.46	4.90	21.48	13.24	4.54
	50	T_{FESAC}^{rr}	13.38	7.04	1.62	12.08	6.12	0.98	13.40	7.18	1.18	13.48	7.06	1.20
	50	T_{FESAC}^{*}	18.92	10.34	2.60	18.38	9.84	1.94	19.66	11.20	2.50	19.28	10.76	2.32
	50	T_{FESAC}^{r*}	17.24	9.22	2.08	16.76	8.68	1.54	18.10	10.10	1.92	17.24	8.58	1.68
	50	T_{FESAC}^{rr*}	10.04	4.06	0.50	8.98	3.94	0.42	10.14	4.40	0.34	9.26	3.48	0.42
	100	T_{FESAC}	21.88	14.10	5.38	22.70	15.12	5.74	22.20	14.36	5.92	21.70	14.34	5.28
	100	T_{FESAC}^{r}	19.46	11.74	3.86	20.06	12.50	4.02	19.34	11.74	3.96	19.06	11.86	3.92
	100	T_{FESAC}^{rr}	12.00	5.76	1.08	12.50	6.56	1.42	11.64	6.06	1.18	11.82	5.88	1.22
	100	T_{FESAC}^{*}	17.96	10.06	2.40	19.38	10.98	2.86	18.26	10.38	2.52	18.06	10.34	2.80
	100	T_{FESAC}^{r*}	17.20	9.58	2.16	18.42	10.22	2.48	17.38	9.68	2.22	15.38	7.74	1.64
	100	T_{FESAC}^{rr*}	9.42	3.86	0.50	10.46	4.52	0.74	9.14	3.70	0.46	7.50	3.14	0.38
	200	T_{FESAC}	21.60	13.68	5.00	20.24	13.10	4.66	20.68	13.26	4.56	20.84	13.34	4.98
	200	T_{FESAC}^{r}	19.64	12.04	3.78	18.56	11.46	3.30	18.82	11.66	3.80	18.80	11.10	3.72
	200	T_{FESAC}^{rr}	11.88	5.78	0.94	11.20	5.54	1.08	11.28	5.86	1.12	11.02	5.62	1.04
	200	T_{FESAC}^{*}	18.58	10.90	3.20	18.08	10.68	2.88	18.44	10.62	2.90	17.82	10.68	3.22
	200	T_{FESAC}^{r*}	18.14	10.38	2.98	17.64	10.32	2.78	18.04	10.26	2.80	16.36	9.26	2.40
	200	T_{FESAC}^{rr*}	10.00	4.86	0.74	9.90	4.34	0.72	10.00	4.80	0.74	8.86	4.12	0.54
	500	T_{FESAC}	19.04	11.74	4.04	18.96	11.58	3.82	19.02	12.30	3.90	18.36	11.68	4.28
	500	T_{FESAC}^{r}	18.30	11.22	3.56	18.08	10.74	3.36	18.20	11.44	3.44	17.48	10.94	3.70
	500	T_{FESAC}^{rr}	10.50	5.20	0.92	10.10	5.40	1.04	10.84	5.28	1.14	10.48	5.62	1.22
	500	T_{FESAC}^{*}	17.98	10.76	3.06	17.70	10.24	2.92	17.76	10.88	3.04	17.44	10.60	3.24
	500	T_{FESAC}^{r*}	17.68	10.54	3.00	17.42	10.14	2.84	17.58	10.74	3.02	16.76	10.08	2.92
	500	T_{FESAC}^{rr*}	9.84	4.82	0.78	9.54	4.92	0.80	10.12	4.62	0.84	9.46	4.72	0.84