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Takashi KUNIMOTO
Singapore Management University, tkunimoto@smu.edu.sg

Takuro YAMASHITA
University of Toulouse I

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Takashi Kunimoto† Takuro Yamashita‡

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Abstract

This paper introduces a novel concept of orders on types by which the so-called monotone comparative statics is valid in *all* supermodular games with incomplete information. We fully characterize this order in terms of what we call *common optimism*, providing a sense in which our order has a sharp epistemic interpretation. We say that type $t_i^{'}$ is higher than type t_i in the order of the common optimism if $t_i^{'}$ is more optimistic about state than t_i; $t_i^{'}$ is more optimistic that all players are more optimistic about state than t_i; and so on, ad infinitum. First, we show that whenever the common optimism holds, monotone comparative statics hold in all supermodular games. Second, we show the converse. We construct an "optimism-elicitation game" as a single supermodular game with the property that whenever the common optimism fails, monotone comparative statics fails as well.
JEL Classification: C72, D78, D82.

Keywords: common optimism, least equilibrium, greatest equilibrium, interim correlated rationalizaibility, monotone comparative statics, supermodularity, universal type space.

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†School of Economics, Singapore Management University; `tkunimoto@smu.edu.sg`
‡Toulouse School of Economics, University of Toulouse Capitole; `takuro.yamashita@tse-fr.eu`

1 Introduction

In many economic problems, we are often interested in studying the effects of changes in certain variables ("parameters") on the behaviors of economic agents. This is well known as *comparative statics*. The comparative statics analysis is ubiquitous in economics, such as in the analyses of bidding strategies in auctions, portfolio choices in financial markets, optimal taxation policy, and so on. The literature on *supermodular games* shows that, given certain conditions on the way the parameters enter the players' payoff functions, monotone changes in those parameters affect the players' equilibria in a monotonic way, a property called *monotone comparative statics*.[1]

Although many papers in the literature study supermodular games with *complete information*, it is sometimes the case that players do not observe some of the parameters of the game they play, or they have different information about them. The main goal of this paper is to find a necessary and sufficient condition under which monotone comparative statics is conducted in any supermodular games with *incomplete information*. Athey (2001, 2002), McAdams (2003), Van Zandt (2010), and Van Zandt and Vives (2007) consider supermodular games with incomplete information. Their main motivation is the existence of equilibria (with nice properties such as in pure and monotone strategies), while our main focus is purely on monotone comparative statics. In addition, these papers consider different assumptions with respect to the players' information structure, reflecting different levels of generality in this respect. In this sense, the setting in our paper is closest to that of Van Zandt and Vives (henceforth, VZV, 2007) in that we make no restrictions on each player's belief and higher-order beliefs (except for certain topological and order structures). Specifically, we allow not only standard common-prior type spaces, but also non-common-prior cases where the players are allowed to enjoy arbitrary heterogeneous beliefs and higher-order beliefs. Indeed, in our setup, any type in the universal type space (and hence any belief hierarchy) is allowed.[2]

Including non-common-prior environments in our analysis is not just for technical generality. In some economic problems, it is well recognized that assuming a common prior may be too demanding. For example, the cerebrated no-trade theorem (Milgrom and Stokey (1982)) shows that, in certain trading contexts with common-value assets and

[1] In particular, certain forms of complementarity (such as supermodularity, increasing difference, and single-crossing conditions) among economic variables and players' actions are shown to be important. See, for example, Milgrom and Roberts (1990), Milgrom and Shannon (1994), Topkis (1998), and Vives (1990) for the usefulness of monotone comparative statics in both applied and theoretical works.

[2] See Mertens and Zamir (1985) and Brandenberger and Dekel (1993).

a common prior,[3] all equilibria exhibit no trading. However, this does not correspond to our conventional understanding that many traders appear to be involved in speculative trading. As another example, the behavioral economics literature propose a number of ways in which real economic agents "wrongly" process information and provides the evidence for this.[4]

With strategic interaction, such heterogeneity in (high-order) beliefs makes comparative statics much more subtle. Even if a trader becomes "more optimistic" about the fundamentals, which makes him eager to "trade more" *ceteris paribus*, he may not want to do so if he believes that the other traders' beliefs change in the way that his "trading more" would hurt himself. Our result could be useful in the analysis of such situations. Our result suggests that, even in such heterogeneous-belief environments, monotone comparative statics can still be conducted. For example, imagine that investors agree that certain news is "good news" for a startup even though they do not agree on "how good the news is" (because they may believe different underlying distributions). We show that this qualitative agreement may be sufficient to drive up the stock price of this startup. In Section 6, we also consider a trading environment where it is common knowledge that the asset has pure common value, but the traders enjoy heterogeneous (high-order) beliefs about the exact value. Monotone comparative statics for the volume of trade is conducted with respect to the size of the belief divergence between players.

To introduce our order on types more formally, imagine an incomplete information supermodular game with a parameter space X^0. Each player's interim belief is identified by his *type*, which induces his *belief hierarchy*, that is, his first-order belief over X^0, his second-order belief (i.e., the joint belief over X^0 and the other players' first-order beliefs), and so on, ad infinitum. We order two types of each player based on their belief hierarchies. Namely, we say that type $\hat{t}_i$ of player i is higher than t_i in the sense of *common optimism (henceforth, CO)* if $\hat{t}_i$'s first-order belief on X^0 first-order stochastically dominates t_i's first-order belief; $\hat{t}_i$'s second-order belief (jointly about X^0 and the first-order beliefs of the other players) first-order stochastically dominates t_i's second-order belief; and so on ad infinitum. That is, $\hat{t}_i$ is more "optimistic" about the realization of $x^0 \in X^0$ than t_i; $\hat{t}_i$ is more "optimistic" about the "optimism" of the other players, and so on.

In Theorem 1, we show that the common optimism is sufficient for monotone comparative statics to hold in all supermodular games. More specifically, we show that if type $\hat{t}_i$

[3]To be precise, their no-trade result holds with *concordant* beliefs, and a common-prior environment is a special case.

[4]For example, see the cursed equilibrium of Eyster and Rabin (2005) and the analogy-based expectation equilibrium of Jehiel (2005).

is higher than type t_i in the CO sense, then, $\hat{t}_i$'s action in the least (greatest) equilibrium is higher than t_i's action in the least (greatest) equilibrium in all supermodular games.[5]

Theorem 2 shows its converse. Namely, we construct a supermodular game, which we refer to as an "optimism-elicitation game" such that, if type $\hat{t}_i$ is *not* higher than t_i in the CO sense, $\hat{t}_i$'s action in the least (greatest) equilibrium of this game is *not* higher than t_i's action in the least (greatest) equilibrium. In other words, the CO order and monotone comparative statics in this game are equivalent.

One of the advantages of our CO order is its sharp epistemic interpretation. It is purely based on the players' first and higher-order beliefs with respect to the state variables, and does not depend on what the underlying type space is or which game we consider, as long as the game is supermodular. Therefore, our results may be useful in the context of mechanism design, where a game is not fixed but rather endogenously constructed. The reader is referred to Mathevet (2010) for the study of designing supermodular mechanisms, motivated by the desirable features of supermodular games in terms of learning and bounded rationality in certain senses. We consider the application of our paper to mechanism design as one promising direction for future work.[6]

As briefly mentioned above, this paper is closest to VZV who also investigate supermodular games with incomplete information. VZV consider an implicit (Harsanyi) type space endowed with an *exogenously given* partial order and then introduce each type's belief map that is consistent with this exogenous order. VZV establish the existence of the least and greatest equilibria that are monotone in types as well as the following monotone comparative statics result: the greatest and least equilibria are higher if there is a first-order stochastic dominant shift in the interim belief. In what follows, we refer to any order of two types that can be considered as a first-order stochastic dominant shift from one type to another as a *VZV order*. Naturally, our CO order and their VZV order are quite related, which we discuss in detail in Section 5. Modulo some technical caveats, our Theorem 3 shows that our CO order is a VZV order and the finest among all possible VZV orders. Besides, VZV just assumes any exogenously given order that satisfies their requirement, and hence is silent as to how one can find such an order, while the CO order is constructive. In this sense, our approach would be advantageous in relatively

[5]The set of equilibria in supermodular games is a complete lattice, and in particular, admits the least and greatest equilibria. Establishing monotone comparative statics for those extremal equilibria, monotone comparative statics for the set of equilibria (in an appropriate set-order sense) is established.

[6]For example, our result shows that one type of a player must play a higher action than another type of that player in any supermodular game. Viewing a game as a mechanism that implements certain allocation rule, such a condition may imply natural monotonicity structures on implementable social choice rules.

complex environments where it is not immediately clear how to introduce a "rich" order on types useful for monotone comparative statics. We illustrate this advantage of our constructive approach in Example 3 in Section 5. In this example, except for a trivial vacuous partial order (i.e., no order), it may not be immediate which orders are VZV orders. By applying our constructive approach, we fully characterize the CO order in this environment (which is, again, the finest possible VZV order). Depending on the parameters, this could be either no order or a complete order.

The rest of the paper is organized as follows. In Section 2, we introduce the basic setup and definitions and identify the least and greatest equilibria via the iterated elimination of never best responses. Section 3 establishes the common optimism (CO) as a sufficient condition for monotone comparative statics to hold in all supermodular games. In Section 4, we establish the CO order as a necessary condition for monotone comparative statics. In this section, we only illustrate how this result can be established in a heuristic manner and refer the reader to the Appendix for its formal proof. Section 5 provides a detailed discussion about the relationship with VZV. In Section 6, we provide an application of our CO order on types in the context of the no-trade result of Milgrom and Stokey (1982). Section 7 concludes the paper with some remarks. In the Appendix, we formally prove that our CO order is necessary for monotone comparative statics and include the proofs omitted from the main body of the paper.

2 Preliminaries

We shall prepare the preliminary materials needed throughout the paper. Section 2.1 introduces the concept of first-order stochastic dominance. Although this section is important to formally establish the subsequent results, the reader might skip it at a first reading. Section 2.2 introduces belief hierarchies and defines the concept of common optimism. We define supermodular games in Section 2.3 and their Bayesian equilibria in Section 2.4. It is well-known that, in a supermodular game, the set of Bayesian equilibria has a lattice structure, and hence, admits the least and greatest equilibria. As is standard in the literature, monotone comparative statics are about those extremal equilibria.

2.1 First Order Stochastic Dominance

Let X be a separable, complete metric space.[7] Consider two Borel probability measures, b and b', on X. Let $\Delta(X)$ denote the set of all Borel-measurable probability distributions

[7] Examples include any finite set, $[0, 1]$, $\mathbb{R}^d$, and $L^p(\mathbb{R}^d)$.

over X endowed with the weak* topology.[8] We say that a partial order $\succeq$ on X is *closed* if, for any pair of sequences $\{x_n\}, \{y_n\} \in X$, whenever $x_n \succeq y_n$ for each n and $x_n \to x$, and $y_n \to y$ as $n \to \infty$, we have $x \succeq y$. We endow X with such a closed partial order $\succeq$.

We say that b' *first-order stochastically dominates* b, denoted $b' \succeq_{SD} b$, if, for any non-decreasing, measurable, and bounded function $f : X \to \mathbb{R}$,

$$\int_x f(x)db' \geq \int_x f(x)db.$$

In this case, we also say that b' is more optimistic than b. We say that b' *strictly* first-order stochastically dominates b if b' first-order stochastically dominates b and, in addition, the inequality is strict for at least some f that is non-decreasing, measurable, and bounded.

Under the following two assumptions on X and $\succeq$, we have an alternative representation of first-order stochastic dominance, which is used in Section 4. We believe that they are mild requirements. For example, a Euclidean space with the usual component-wise partial order satisfies them.

Assumption 1. There exists a countable dense subset $X_0 \subseteq X$ for which for each $x \in X$ and $\varepsilon > 0$, there is $y \in X_0$ such that $y \geq x$ and $y \in B_\varepsilon(x)$.[9]

Because X is separable, it has a countable dense subset. However, our assumption requires an additional condition, which is a sort of "local non-satiation".[10]

To introduce the second assumption, for each $x \in X$, let $up(x) \subseteq X$ be the smallest upper set that contains x: that is, $up(x) = \{y \in X | y \geq x\}$. For each $Y \subseteq X$, let $up(Y) \subseteq X$ be the smallest upper set that contains Y: that is, $up(Y) = \bigcup_{y \in Y} up(y)$. The second assumption says that the upper-set correspondence is continuous.

Assumption 2. For each $Y \subseteq X$ and $\varepsilon > 0$, there exists $\delta(Y, \varepsilon) > 0$ such that, for any $Z \subseteq X$ with $d(Y, Z) < \delta(Y, \varepsilon)$,[11] we have $d(up(Y), up(Z)) < \varepsilon$.

[8] In Sections 4 and A.1, we introduce capacities (non-additive measures) over X endowed with a finer topology than the weak* topology. In such a case, we avoid the use of the notation like $\Delta(X)$, which usually means the set of all probability measures.

[9] $B_\varepsilon(x)$ denotes the open ball around x with radius ε.

[10] In their analysis of revealed preference theory, Chambers, Echenique, and Lambert (2017) use the *countable order property*, which is similar to our Assumption 1. See also Proposition 13 of their paper for two prominent cases where the countable order property is satisfied.

[11] By abuse of notation, we let

$$d(Y, Z) = \max\{\sup_{y \in Y} \inf_{z \in Z} d(y, z), \sup_{z \in Z} \inf_{y \in Y} d(y, z)\}$$

denote the Hausdorff metric between $Y, Z \subseteq X$.

For each $Y \subseteq X$, let $clup(Y)$ denote the closure of $up(Y)$. The next result shows that, under the assumptions above, we can show that we do not need to check all non-decreasing, measurable, and bounded functions to determine whether b first-order stochastically dominates b'. We only need to check a countable subclass of "clup" sets. The proof is in the Appendix, but it is worth mentioning that the proposition (in particular, Lemma 1 as its intermediate step) requires that X be Polish (i.e., separable and completely metrizable) and $\succeq$ be a closed partial order.

Proposition 1. Let $b, b^{'} \in \Delta(X)$. b (first-order) stochastically dominates b' if and only if, for any $Y_0 \subseteq X_0$, $b(clup(Y_0)) \geq b'(clup(Y_0))$. In addition, b strictly stochastically dominates b' if and only if the inequality holds for any $Y_0 \subseteq X_0$ and it is strict at least for some $Y_0 \subseteq X_0$.

Proof. First, we state the following intermediate result, due to Kamae, Krengel, and O'Brien (1977). Its proof is omitted.

We define $U(X) = \{up(Y) |\ Y \subseteq X\}$ as the collection of upper sets $up(Y)$ such that $Y \subseteq X$.

Lemma 1. Let $b, b^{'} \in \Delta(X)$. $b^{'}$ first-order *stochastically dominates* b (denoted $b^{'} \succeq_{SD} b$) if and only if $b^{'}(Y) \geq b(Y)$ for any $Y \in U(X)$. In addition, $b^{'}$ *strictly* first-order stochastically dominates b if and only if $b^{'} \succeq_{SD} b$ and $b^{'}(Y) > b(Y)$ for some $Y \in U(X)$.

Lemma 1 states that it is enough to consider all the closed upper sets (instead of all the increasing, measurable, and bounded functions) to establish a first-order stochastic dominance relation (and its strict variant) between two probability measures. The rest of the proof is in the Appendix. □

2.2 Belief Hierarchies

Throughout this paper, let I denote the set of (finitely many) players, and let Θ denote the payoff-relevant state space. We assume that (i) Θ is a separable, complete metric space; (ii) there exists a closed partial order over Θ, denoted by $\succeq_\Theta$[12]; and (iii) Assumptions 1 and 2 are satisfied for $X = \Theta$ and $\succeq = \succeq_\Theta$,[13] where the corresponding countable dense subset is denoted by Θ_0.

[12] In Section 6, we introduce a player specific order on Θ, $\succeq_{\Theta,i}$, for each $i \in I$. Note that the whole argument of this paper is not affected by this generalization after some necessary adjustments are made.

[13] Note that these assumptions are not used for the sufficiency for monotone comparative statics in Section 3.

In practice, often the players only partially and asymmetrically observe θ before they play a particular game. We represent their beliefs over θ and over each other's beliefs by *types*. Let $(T_i, \mathscr{T}_i, \pi_i)_{i \in I}$ be a *type space* where each T_i represents player i's set of types; each $\mathscr{T}_i$ represents a sigma-algebra over T_i and $\mathscr{T}$ and $\mathscr{T}_{-i}$ represent the product sigma-algebra over T and T_{-i}, respectively; and a $\mathscr{T}_i$-measurable $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$ is player i's interim belief map about the parameter and the other players' types.

Given any type space $(T_i, \mathscr{T}_i, \pi_i)_{i \in I}$, we can deduce the *belief hierarchy* of each type t_i of each player i as follows. Define $Z_j^1 = \Delta(\Theta)$ for $j \in I$, and endow it with the partial order induced by first-order stochastic dominance and with the weak* topology. Define $Z_{-i}^1 = \prod_{j \neq i} Z_j^1$, and endow it with the product order and the product topology. Then, for each $k \geq 1$, inductively define $Z_j^{k+1} = \Delta(\Theta \times Z_{-j}^1 \times \cdots \times Z_{-j}^k)$ for $j \in I$ endowed with the stochastic-dominance partial order and weak* topology; and $Z_{-i}^{k+1} = \prod_{j \neq i} Z_j^{k+1}$ endowed with the product order and the product topology. Then, (i) his *first-order belief* is defined by $h_i^1(t_i) = \text{marg}_\Theta \pi_i(t_i) \in Z_i^1$: that is, for each measurable $\tilde{\Theta} \subseteq \Theta$,

$$h_i^1(t_i)[\tilde{\Theta}] = \pi_i(t_i)[\tilde{\Theta} \times T_{-i}];$$

and inductively for each $k \geq 1$, (ii) his $(k+1)$*th-order belief* is defined by $h_i^{k+1}(t_i) = \text{marg}_{\Theta \times Z_{-i}^1 \times \cdots \times Z_{-i}^k} \pi_i(t_i) \in Z_i^{k+1}$. The *belief hierarchy* of t_i is then defined by $(h_i^k(t_i))_{k=1}^\infty$.

We now introduce this paper's fundamental concept of *common optimism*. Let t_i and $t_i^{'}$ be two types of player i. Suppose that (i) $t_i^{'}$ is more optimistic about Θ than t_i; (ii) $t_i^{'}$ is more optimistic that all players are more optimistic about Θ than t_i; (iii) $t_i^{'}$ is more optimistic that all players are more optimistic that all players are more optimistic about Θ than t_i; and so on ad infinitum. In such a case, we say that $t_i^{'}$ is at least as high as t_i in the order of common optimism. We formally define this as follows:

Definition 1. $t_i^{'}$ is at least as high as t_i in the order of *common optimism* (denoted by $t_i^{'} \succeq_{CO} t_i$) if $h_i^k(t_i^{'}) \succeq_{SD} h_i^k(t_i)$ for each $k \in \mathbb{N}$.

Using our alternative representation of first-order stochastic dominance (Proposition 1), we mean by $h_i^k(t_i^{'}) \succeq_{SD} h_i^k(t_i)$ that $h_i^k(t_i^{'})[Y] \geq h_i^k(t_i)[Y]$ for any $Y \in U(Z_i^k)$, where $Z_i^1 = \Delta(\Theta)$ for $k = 1$ and $Z_i^k = \Delta(\Theta \times Z_{-i}^1 \times \cdots \times Z_{-i}^{k-1})$ for $k \geq 2$. In what follows, we refer to this order $\succeq_{CO}$ as the *CO order*.

2.3 Supermodular Games

The players in the set I play the following game. For each player $i \in I$, let A_i denote his action space, and let $u_i : A \times \Theta \to \mathbb{R}$ denote his payoff function, where $A = \prod_{j \in I} A_j$. Recall that Θ is the payoff-relevant state space introduced in the previous subsection.

Let X be a complete lattice and a partial order $\geq$. For each $Y \subseteq X$, let $\bigvee Y \in X$ denote the least upper bound ("join") of Y, and $\bigwedge Y \in X$ denote the greatest lower bound ("meet") of Y.[14] That X is a complete lattice means that the join and meet exist for any $Y \subseteq X$. In case Y is a binary set of the form $\{x, y\}$ with $x, y \in X$, following the standard notation, we denote its join by $x \vee y$ and its meet by $x \wedge y$.

We consider *supermodular games* based on Θ as a domain of games, defined as follows. First, A_i is a complete lattice endowed with a partial order $\succeq_{A_i}$. Second, each $u_i(\cdot)$ is *supermodular* on A_i and has *increasing difference* in both (a_i, a_{-i}) and (a_i, θ). That is, for each $a_i, a_i' \in A_i$, $a_{-i}, a_{-i}^{'} \in A_{-i}$, and $\theta, \theta^{'} \in \Theta$, whenever $a_{-i} \succeq_{A_{-i}} a_{-i}'$ and $\theta \succeq_{\Theta,i} \theta'$, it follows that

$$u_i((a;\theta) \vee (a';\theta')) + u_i((a;\theta) \wedge (a';\theta')) \geq u_i(a;\theta) + u_i(a';\theta'),$$

or equivalently,

$$u_i(a_i \vee a_i', a_{-i};\theta) + u_i(a_i \wedge a_i', a_{-i}';\theta') \geq u_i(a_i, a_{-i};\theta) + u_i(a_i', a_{-i}';\theta').$$

A tuple $G = (I, \Theta, (A_i, u_i, T_i, \pi_i)_{i \in I})$ comprises an (incomplete-information) supermodular game.

2.4 Equilibria

In an incomplete-information supermodular game G, we denote a *pure* strategy of each player i by a $\mathscr{T}_i$-measurable function $\sigma_i : T_i \to A_i$. We first define a pure strategy Bayesian equilibrium.

Definition 2. A strategy profile $\sigma^* = (\sigma_i^*)_{i \in I}$ is a (pure-strategy) *Bayesian equilibrium* if, for each $i \in I,\ t_i \in T_i$, and $a_i \in A_i$,

$$\int_{\Theta \times T_{-i}} u_i(\sigma_i^*(t_i), \sigma_{-i}^*(t_{-i}), \theta) d\pi_i(t_i)[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(a_i, \sigma_{-i}^*(t_{-i}), \theta) d\pi_i(t_i)[\theta, t_{-i}].$$

Let Σ^* denote the set of "all" Bayesian equilibria of an incomplete information supermodular game $G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$. It may well be the case that Σ^* is empty. The interested reader should be referred to Van Zandt and Vives (2007) for a sufficient condition for Σ^* to be nonempty.[15] In what follows, we simply assume that Σ^* is nonempty.

[14] $z \in X$ is an upper (a lower) bound of $Y \subseteq X$ if $z \succeq y$ ($z \preceq y$) for all $y \in Y$. $z \in X$ is the least upper bound of $Y \subseteq X$ if is an upper bound of Y, and moreover, we have $z' \succeq z$ for any upper bound z' of Y. Analogously, $z \in X$ is the greatest lower bound of $Y \subseteq X$ if is a lower bound of Y, and moreover, we have $z \succeq z^{'}$ for any lower bound z' of Y.

[15] See also our Proposition 2 and Remark 1 right after that proposition.

We call $\underline{\sigma} \in \Sigma^*$ the *least equilibrium* if, for each $\sigma^* \in \Sigma^*$, $i \in I$, and $t_i \in T_i$, we have $\sigma_i^*(t_i) \succeq_{A_i} \underline{\sigma}_i(t_i)$, and similarly, call $\bar{\sigma} \in \Sigma^*$ the *greatest equilibrium* if, for each $\sigma^* \in \Sigma^*$, $i \in I$, and $t_i \in T_i$, we have $\bar{\sigma}_i(t_i) \succeq_{A_i} \sigma_i^*(t_i)$. As is usually the case for monotone comparative statics, this paper focuses on the least and greatest Bayesian equilibria in supermodular games. The following is our definition of monotone comparative statics with respect to the CO order.

Definition 3. We say that *monotone comparative statics holds in a supermodular game G with respect to the CO order* if, for each $i \in I$ and $t_i, t_i' \in T_i$, if $t_i \succeq_{CO} t_i'$, then we have $\underline{\sigma}_i(t_i) \succeq_{A_i} \underline{\sigma}_i(t_i')$ and $\bar{\sigma}_i(t_i) \succeq_{A_i} \bar{\sigma}_i(t_i')$.

3 Sufficiency for Monotone Comparative Statics

In this section, with technical regularity conditions guaranteeing the existence of the least and greatest equilibria, we show that if type $t_i^{'}$ is higher than type t_i in the CO order, $t_i^{'}$'s action in the least (greatest) equilibrium is higher than t_i's action in the least (greatest) equilibrium in all supermodular games (Theorem 1). The key observation for the proof of this theorem is that the least and greatest equilibria of an incomplete-information supermodular game (under certain regularity conditions) coincide with the game's least and greatest rationalizable strategy profiles, which are fully identified by its infinite belief hierarchy. Of course, different orders on these types may be induced if different games are considered. Theorem 1 shows that our CO order is a "robust" order on types in the sense that, if a type of a player is higher than another in this order, then the former plays a higher (least and greatest) equilibrium action than the latter in *any* supermodular game.[16]

In what follows, we focus only on the least equilibrium of a supermodular game G, because the logic for the greatest equilibrium is similar.

The key observation is that, for each type t_i of each player i, his least equilibrium action is characterized by his part of the least *interim correlated rationalizability* (ICR) of Dekel, Fudenberg, and Morris (2007).

The least ICR is identified by iterative elimination of never best responses "from below". First, for each $i \in I$, $t_i \in T_i$, let $A_i^0(t_i) = A_i$ and $\underline{a}_i^0(t_i) = \bigwedge A_i^0(t_i)$, and then, let

$$\underline{A}_i^1(t_i) = \arg\max_{a_i \in A_i^0(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}_{-i}^0(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

[16] Of course, this may not be the *only* interesting order. For example, one may have a specific supermodular game in mind, and desire to conduct comparative statics only in this game. Although we provide some preliminary argument on this issue in Section 7, we leave this for future research.

and $\underline{a}_i^1(t_i) = \bigwedge \underline{A}_i^1(t_i)$. Later we assume that $\underline{a}_i^1(\cdot)$ is a measurable mapping and that $\underline{A}_i^1(t_i)$ is a complete sublattice, which implies that $\underline{a}_i^1(t_i) \in \underline{A}_i^1(t_i)$. Note that, by supermodularity, any a_i that does not satisfy $a_i \succeq_{A_i} \underline{a}_i^1(t_i)$ is never a best re, his least equilibrium action is characterized by his part of the least *interim correlated rationalizability* (ICR) of Dekel, Fudenberg, and Morris (2007).

The least ICR is identified by iterative elimination of never best responses "from below". First, for each $i \in I$, $t_i \in T_i$, let $A_i^0(t_i) = A_i$ and $\underline{a}_i^0(t_i) = \bigwedge A_i^0(t_i)$, and then, let

$$\underline{A}_i^1(t_i) = \arg \max_{a_i \in A_i^0(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}_{-i}^0(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^1(t_i) = \bigwedge \underline{A}_i^1(t_i)$. Later we assume that $\underline{a}_i^1(\cdot)$ is a measurable mappinsponse.

By an induction argument, for each $k \geq 1$, $i \in I$, and $t_i \in T_i$, let

$$\underline{A}_i^{k+1}(t_i) = \arg \max_{a_i \in A_i^k(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, \underline{a}_{-i}^k(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^{k+1}(t_i) = \bigwedge \underline{A}_i^{k+1}(t_i)$. Again, later we assume that $\underline{A}_i^{k+1}(t_i)$ is a complete sublattice, implying that $\underline{a}_i^{k+1}(t_i) \in \underline{A}_i^{k+1}(t_i)$, and that $\underline{a}_i^{k+1}(\cdot)$ is a measurable mapping. Note that by supermodularity, any a_i that does not satisfy $a_i \succeq_{A_i} \underline{a}_i^{k+1}(t_i)$ is never a best response.

Finally, for each $i \in I$ and $t_i \in T_i$, define

$$\underline{a}_i^\infty(t_i) \quad = \quad \bigvee \{\underline{a}_i^1(t_i), \underline{a}_i^2(t_i), \ldots\}.$$

Since A_i is a complete lattice, we have $\underline{a}_i^\infty(t_i) \in A_i$. Thus, if $\underline{a}_i^\infty(t_i)$ is a best response to $\underline{a}_{-i}^\infty(\cdot)$ (given his belief $\pi_i(t_i)$ over $\Theta \times T_{-i}$), then $\underline{\sigma}$ defined by $\underline{\sigma}_i(t_i) = \underline{a}_i^\infty(t_i)$ for each $i \in I$ and t_i constitutes an equilibrium. By construction, $\underline{\sigma}$ must be the least equilibrium of the game, because in each step k of the induction, any action a_i that does not satisfy $a_i \succeq_{A_i} \underline{a}_i^k(t_i)$ is shown to be a never-best response to the lowest selection of the others' actions from $\underline{A}_i^{k-1}(\cdot)$, and hence, a never-best response to any other strategy profile σ_{-i} of the other players such that $\sigma_{-i}(t_{-i}) \succeq_{A_{-i}} \underline{a}_{-i}^k(t_{-i})$. We note this result as a proposition.

Proposition 2. Assume that, for each $i \in I$, t_i, and $k \geq 1$, (i) $\underline{A}_i^k(t_i)$ is a complete sublattice, (ii) $\underline{a}_i^k(\cdot) = \bigwedge \underline{A}_i^k(\cdot)$ is a measurable mapping, and (iii) $\underline{a}_i^\infty(t_i)$ is a best response to $\underline{a}_{-i}^\infty(\cdot)$. Then, $\underline{\sigma}$ defined by $\underline{\sigma}_i(t_i) = \underline{a}_i^\infty(t_i)$ for each $i \in I$ and t_i constitutes the least equilibrium.

Remark 1. Interested readers are referred to Van Zandt and Vives (2007) and Van Zandt (2010) for more primitive assumptions on the environment that guarantee the existence of the least (and analogously, greatest) equilibrium. Specifically, they assume that (i) A_i

is a compact metric lattice;[17] (ii) u_i is bounded, continuous in a_i and measurable in θ; and (iii) $\pi_i(\cdot)$ is measurable (as a mapping from T_i to $\Delta(\Theta \times T_{-i})$).

Now we prove that monotone comparative statics holds in any supermodular game G with respect to the CO order.

Theorem 1. Let $G = (g, (T_i), (\mathscr{T}_i), (\pi_i))_{i \in I}$ be an incomplete information supermodular game that satisfies (as in Proposition 2): for each $i \in I$, $t_i \in T_i$, and $k \geq 1$, (i) $\underline{A}_i^k(t_i)$ is a complete sublattice; (ii) $\underline{a}_i^k(\cdot) = \bigwedge \underline{A}_i^k(\cdot)$ is a measurable mapping; and (iii) $\underline{a}_i^\infty(t_i)$ is a best response to $\underline{a}_{-i}^\infty$. Let t_i and $t_i^{'}$ be two types of player i such that $t_i^{'} \succeq_{CO} t_i$. Then, for the least equilibrium of the game G, $\underline{\sigma}$, we have $\underline{\sigma}_i(t_i^{'}) \succeq_{A_i} \underline{\sigma}_i(t_i)$.

Proof. In the previous proposition, we show that the least equilibrium is fully characterized by the iterated elimination of never-best responses of interim correlated rationalizability "from below." Thus, it suffices to show that, for each $i \in I$, $k \geq 1$, and t_i, t_i' such that $t_i \succeq_{CO} t_i'$, we have $\underline{a}_i^k(t_i) \succeq_{A_i} \underline{a}_i^k(t_i')$.

First, because $\underline{a}_i^1(t_i) \in \underline{A}_i^1(t_i)$ and $\underline{a}_i^1(t_i') \in \underline{A}_i^1(t_i')$, we have

$$\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i'), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \vee \underline{a}_i^1(t_i^{'}), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}]$$

and

$$\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] \geq \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \wedge \underline{a}_i^1(t_i^{'}), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}]$$

Since $h_i^1(t_i^{'}) \succeq_{SD} h_i^1(t_i)$ and $\underline{a}_{-i}^0(t_{-i})$ does not depend on t_{-i}, the distribution over $\Theta \times A_{-i}$ induced by $\pi_i(t_i')$ first-order stochastically dominates that induced by $\pi_i(t_i)$. Therefore, by the supermodularity of the game, we have

$$\begin{aligned} &\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \vee \underline{a}_i^1(t_i^{'}), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] - \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i'), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] \\ \geq &\int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] - \int_{\Theta \times T_{-i}} u_i(\underline{a}_i^1(t_i) \wedge \underline{a}_i^1(t_i'), \underline{a}_{-i}^0(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}]. \end{aligned}$$

Because the left-hand side of the above inequality is nonpositive and the right-hand side is non-negative, we must have both equal to zero. In particular, this implies that $\underline{a}_i^1(t_i) \wedge \underline{a}_i^1(t_i^{'}) \in \underline{A}_i^1(t_i)$. However, because $\underline{a}_i^1(t_i) = \bigwedge \underline{A}_i^1(t_i) \in \underline{A}_i^1(t_i)$, we must have $\underline{a}_i^1(t_i^{'}) \succeq_{A_i} \underline{a}_i^1(t_i)$.

[17] Compactness is used not for the existence of best replies (thanks to the supermodularity), but for guaranteeing that $\underline{a}_i^k(\cdot)$ is a measurable mapping. For this point, see Footnote 3 of Van Zandt (2010) who mentions that compactness can be replaced by sigma-compactness for this measurable selection argument.

We move on to the next step. Let

$$\underline{A}_i^2(t_i) = \arg\max_{a_i \in A_i^1(t_i)} \int_{\Theta\times T_{-i}} u_i(a_i, \underline{a}_{-i}^1(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $\underline{a}_i^2(t_i) = \bigwedge \underline{A}_i^2(t_i)$. Again, we assume that $\underline{A}_i^2(t_i)$ is a complete sublattice, implying that $\underline{a}_i^2(t_i) \in A_i^2(t_i)$, and that $\underline{a}_i^2(\cdot)$ is a measurable mapping.

It follows from supermodularity that any a_i not satisfying $a_i \succeq_{A_i} \underline{a}_i^2(t_i)$ does not survive the iterative elimination of never-best responses. Recall that for any $j \neq i$ and $t_j, t_j^{'}$, if $h_j^1(t_j^{'}) \succeq_{SD} h_j^1(t_j)$, then $\underline{a}_j^1(t_j^{'}) \succeq_{A_j} \underline{a}_j^1(t_j)$. Since we assume that $t_i^{'} \succeq_{CO} t_i$, we also have $h_i^2(t_i^{'}) \succeq_{SD} h_i^2(t_i)$. Define

$$\tilde{Y} = \left\{ (\theta, a_{-i}^1) \in \Theta \times A_{-i} \middle| \ \exists (\hat{\theta}, \hat{t}_{-i}) \text{ s.t. } \theta \succeq_{\Theta} \hat{\theta}, \ a_{-i}^1 \succeq_{A_i} \underline{a}_{-i}^1(\hat{t}_{-i})) \right\}.$$

Clearly, $\tilde{Y} \in U(\Theta \times A_{-i})$ where $U(\Theta \times A_{-i})$ denotes the set of all upper events of $\Theta \times A_{-i}$. By Lemma 1, we can conclude that the weight $h_i^2(t_i^{'})$ assigns to the event $\tilde{Y}$ is at least as high as the weight $h_i^2(t_i)$ does.

Due to the definition of $\underline{a}_i^2(t_i) \in \underline{A}_i^2(t_i)$ and $\underline{a}_i^2(t_i') \in \underline{A}_i^2(t_i')$, we have

$$\int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i'), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] \geq \int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i) \vee \underline{a}_i^2(t_i'), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}],$$

and

$$\int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] \geq \int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i) \wedge \underline{a}_i^2(t_i'), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}].$$

Since $h_i^2(t_i^{'}) \succeq_{SD} h_i^2(t_i)$, by the supermodularity of the game, we have

$$\begin{aligned} & \int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i) \vee \underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] - \int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i'), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i')[\theta, t_{-i}] \\ \geq \ & \int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}] - \int_{\Theta\times T_{-i}} u_i(\underline{a}_i^2(t_i) \wedge \underline{a}_i^2(t_i^{'}), \underline{a}_{-i}^1(t_{-i}); \theta) d\pi(t_i)[\theta, t_{-i}]. \end{aligned}$$

Because the left-hand side of the above inequality is nonpositive and the right-hand side is nonnegative, we must have both equal to zero. In particular, this implies that $\underline{a}_i^2(t_i) \wedge \underline{a}_i^2(t_i^{'}) \in \underline{A}_i^2(t_i)$. However, because $\underline{a}_i^2(t_i) = \bigwedge \underline{A}_i^2(t_i) \in \underline{A}_i^2(t_i)$, we must have $\underline{a}_i^2(t_i^{'}) \succeq_{A_i} \underline{a}_i^2(t_i)$.

By an induction argument, we can show that $\underline{a}_i^k(t_i^{'}) \succeq_{A_i} \underline{a}_i^k(t_i)$ for each $k \in \mathbb{N}$, which implies that $\underline{a}_i^\infty(t_i^{'}) \succeq_{A_i} \underline{a}_i^\infty(t_i)$.

Since the least equilibrium $\underline{\sigma}$ is defined as $\underline{\sigma}_i(t_i) = \underline{a}_i^\infty(t_i)$ for every $i \in I$ and t_i, we complete the proof. □

4 Necessity for Monotone Comparative Statics

The main result of this section is the converse to Theorem 1, that is, the CO order is indeed the finest possible order for monotone comparative statics to hold in all supermodular games. Specifically, we construct a supermodular game, to which we refer as the "optimism-elicitation game" such that, if type $t_i^{'}$'s action in the least (greatest) equilibrium of this game is higher than t_i's action in the least (greatest) equilibrium, then $t_i^{'}$ is higher than t_i in the CO order. Formally, we establish this as the following theorem.

Theorem 2. There is a supermodular game with the property that, for any player $i \in I$ and two types $t_i, t_i^{'}$, we have that $t_i^{'} \succeq_{CO} t_i$ if and only if $\underline{\sigma}_i(t_i^{'}) \succeq_{A_i} \underline{\sigma}_i(t_i)$ and $\bar{\sigma}_i(t_i^{'}) \succeq_{A_i} \bar{\sigma}_i(t_i)$, where $\underline{\sigma}$ and $\bar{\sigma}$ are the least and greatest equilibria of this supermodular game, respectively.[18]

The heart of the proof of our Theorem 2 lies in the construction of the optimism-elicitation game. In the rest of this section, while we relegate the formal proof of Theorem 2 to the Appendix, we illustrate how the construction of the optimism-elicitation game is executed in a heuristic manner.

Section 4.1 assumes a single player. Appealing to an analogy with the belief-elicitation literature,[19] we show that, by setting his action space to be the space of his (first-order) beliefs and with an appropriate "scoring rule", we can elicit his belief. Section 4.2 assumes multiple players. In this case, we construct a belief-elicitation *game* where not only the players' first-order beliefs but their higher-order beliefs are elicited. For this purpose, the action space is set to be a universal type space. Then, at this stage, we observe that this game itself is not a supermodular game in the sense that the action space is not a lattice. Hence, in Section 4.3, we explain how we enlarge the action space so that it has a lattice structure, while taking care of some countable structures that are crucial in establishing our result.

4.1 The Single Agent Case

We construct a decision problem with a parameter space X^0 and its associated partial order $\succeq_0$ such that type t is higher than type $t^{'}$ if and only if the action taken by type t in the least (greatest) equilibrium is higher than the action taken by type $t^{'}$ in the least (greatest) equilibrium.

[18] To be more precise, what we show here is that for any parameter space Θ, there is a supermodular game based on Θ with the desired property.

[19] For example, Savage (1971) proposes the *proper scoring rule* to elicit an individual's belief.

Step 1: Let X^0 be a separable, complete metric space. Then, its (Borel) σ-algebra is countably generated, that is, there exists a collection of countably many measurable subsets, $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$, which generates the σ-algebra.[20] In particular, this means that any probability measure on X^0 can be identified by a mapping $\beta : \mathcal{U} \to [0, 1]$.

Each $\beta(U_n)$ may be interpreted as a probability assignment for event U_n. Obviously, any probability measure assigns a probability for each $\beta(U_n)$, but what is said here is that a collection of $\{\beta(U_n)\}_{n=1}^{\infty}$ is also *sufficient* to pin down the corresponding probability measure. In other words, two distinct probability measures do not induce the same mapping $\beta : \mathcal{U} \to [0, 1]$.

Step 2: Let X^0 be endowed with a partial order.[21] Let B denote the set of all mappings $\beta : \mathcal{U} \to [0, 1]$ that correspond to the set all probability measures over X^0. Then, B inherits the natural order $\succeq$ in the sense that $\beta \succeq \beta'$ if β first-order stochastically dominates (FOSDs) β'.

Step 3: Consider a single-person decision problem where the player's action space is B, and his payoff is, given each state realization $x^0 \in X^0$,

$$u(\beta, x^0) = \sum_{n=1}^{\infty} \left[\beta(U_n) \mathbf{1}_{U_n}(x^0) - \frac{\beta(U_n)^2}{2} \right] \mu(U_n),$$

where $\mathbf{1}_{U_n}$ denotes the indicator function such that for each $x^0 \in X^0$,

$$\mathbf{1}_{U_n}(x^0) = \begin{cases} 1 & \text{if } x^0 \in U_n \\ 0 & \text{otherwise} \end{cases}$$

and μ is a full support distribution over $\{U_n\}_{n=1}^{\infty}$.[22]

The construction is based on a standard quadratic scoring rule used in the belief-elicitation literature. This decision problem may be interpreted as follows. First, the player chooses $\beta \in B$, and then, state x^0 is realized. Note that each U_n is chosen with probability $\mu(U_n) > 0$. The player's payoff is $\beta(U_n) - \frac{\beta(U_n)^2}{2}$ if $x^0 \in U_n$, while $-\frac{\beta(U_n)^2}{2}$ if $x^0 \notin U_n$. Then, based on the standard result in the belief elicitation literature, the uniquely optimal $\beta(U_n)$ coincides with the player's belief for U_n. Because this is true for every U_n, by the argument in Step 1, this means that his belief is fully revealed in this

[20]Letting $\tilde{X}^0$ denote a countable dense subset of X^0, a typical choice of $\mathcal{U}$ is to set each U_n as an open ball with a rational radius around each point in $\tilde{X}^0$.

[21]More precisely, we consider a *closed* partial order with other technical conditions. See Section 2.1 for the detail.

[22]For example, we set $\mu(U_n) = 2^{-n}$ for each $n \in \mathbb{N}$.

decision problem. We stress that the fact that there exist *countably* many "test sets" $\{U_n\}_{n=1}^{\infty}$ plays a crucial role here to make every $\mu(U_n)$ strictly positive.[23]

4.2 An Extension to the Multiple-Players Case

We extend the previous argument to a game with two players and the same parameter space X^0, endowed with a partial order $\succeq_0$.[24] Now our goal is to construct a game where each player's entire belief hierarchy is elicited (recall that the CO order is based on the entire belief hierarchy). So, in "Step 0" below, we first construct the space of belief hierarchies based on Brandenberger and Dekel (1993).

Step 0: Set $X^1 = \Delta(X^0)$ as the set of first-order beliefs; $X^2 = \Delta(X^0 \times X^1)$ as the set of second-order beliefs; and $X^k = \Delta(X^0 \times \cdots \times X^{k-1})$ for each $k \geq 3$ as the set of k-th order beliefs. Finally, define $X^\infty = \prod_{k=1}^{\infty} X^k$ as the set of all infinite hierarchies of beliefs. In the case of multiple players, we want to elicit not only first-order beliefs but also second and all higher-order beliefs of each player. By Brandenberger and Dekel (1993), if X^0 is itself a separable, complete metric space, then X^∞ above can also be made a separable, complete metric space.

Step 1: Analogous to Step 1 in the single-player case, the σ-algebra of each X^k is generated by a collection of countably many measurable subsets, $\mathcal{U}^{(k)} = \{U_n^{(k)}\}_{n=1}^{\infty}$. In particular, this means that any belief and higher-order belief of each player over X^0 can be identified by a mapping $\beta = (\beta^{(1)}, \beta^{(2)}, \ldots)$ where each $\beta^{(k)} : \mathcal{U}^{(k-1)} \to [0,1]$. Intuitively, each $\beta^{(k)}(U_n^{(k-1)})$ may be interpreted as a probability assignment for event $U_n^{(k-1)}$, and hence, $\beta^{(k)}$ as this player's k-th order belief.

Step 2: Let B_k denote the set of all mappings $\beta^{(k)} : \mathcal{U}^{(k-1)} \to [0,1]$ that correspond to the set of all k-th order beliefs. Let B_k inherit the natural order $\succeq^k$ in the sense that $\beta^{(k)} \succeq^k \tilde{\beta}^{(k)}$ if $\beta^{(k)}$ first-order stochastically dominates (FOSDs) $\tilde{\beta}^{(k)}$. Then, let $B = \times_{k=1}^{\infty} B_k$ denote the set of all β, corresponding to the set of all belief hierarchies. By endowing B with a product order $\succeq$, $\beta \succeq \tilde{\beta}$ if β first-order stochastically dominates (FOSDs) $\tilde{\beta}$. Though technically just analogous to Step 2 in the single-agent case, the interpretation of this first-order dominance order is more elaborate, because now we consider X^∞ as the space for which each β assigns probabilities. That is, each $\mathcal{U}^{(0)}$ is the collection of test sets which identifies each player's first-order belief (i.e., a belief over X^0), each $\mathcal{U}^{(1)}$ is the collection of test sets which identifies each player's second-order

[23] For example, Chambers and Lambert (2018) use a similar approach, though in a dynamic decision environment, for belief elicitation with many small test sets.

[24] It is straightforward to extend this argument to the case with more than two players.

belief, and so on, ad infinitum, which, as a whole, identifies his belief hierarchy.

Based on the above order $\succeq$ on belief hierarchies, $\beta \succeq \tilde{\beta}$ means that $\beta^{(1)}$ FOSDs $\tilde{\beta}^{(1)}$ (i.e., β is "more optimistic" about X^0), $\beta^{(2)}$ FOSDs $\tilde{\beta}^{(2)}$ (i.e., β is "more optimistic" about the optimism of the other players about X^0), and so on, ad infinitum. This first-order dominance order is precisely our CO order.

Step 3: Consider a game where each player i's action space is B, and player i's payoff given each state realization $x^0 \in X^0$ and the other player's strategy β_{-i} is:

$$u(\beta_i, \beta_{-i}, x^0) = \sum_{k=1}^{\infty} \delta^{k-1} \left[\sum_{n=1}^{\infty} \left[\beta_i^k(U_n^{(k-1)}) \mathbf{1}_{U_n^{(k-1)}}(\beta_{-i}^{(k-1)}, x^0) - \frac{(\beta_i^k(U_n^{(k-1)}))^2}{2} \right] \mu^k(U_n^{(k)}) \right],$$

where $0 < \delta < 1$; μ^k is a full support distribution over $\{U_n^{(k-1)}\}_{n=1}^{\infty}$; and $\mathbf{1}_{U_n^{(k-1)}}(\beta_{-i}^{(k-1)}, x^0) = 1$ if $(x^0, \beta_{-i}^{(k-1)}) \in U_n^{(k-1)}$ and 0 otherwise.

Analogous to the single-person case, $\beta_i^{(k)}(U_n^{(k-1)})$ is shown to coincide with i's belief for $U_n^{(k-1)}$. More specifically, for $k = 1$, it is shown to be i's dominant action to set $\beta_i^{(1)}(U_n^{(0)})$ to be the same as i's first-order belief over $U_n^{(0)} (\subseteq X^0)$. For $k = 2$, note that $U_n^{(1)} (\subseteq X^0 \times X^1)$ is an event jointly about the realization of the state and $-i$'s first-order belief. Hence, *conditional on* $-i$'s truth-telling of his first-order belief, it is i's dominant action to set $\beta_i^{(2)}(U_n^{(1)})$ to be the same as i's second-order belief over $U_n^{(1)}$. We can then continue this procedure to conclude that each $\beta_i^k(U_n^{(k-1)})$ coincides with i's k-th order belief for $U_n^{(k-1)}$, in the unique equilibrium of the game.

Therefore, given the inductive hypothesis that all agents' $(k-1)$st-order beliefs are truthfully elicited, each player faces essentially the same decision problem as the single-person one so that announcing his k-th order belief truthfully is a strictly dominant strategy. We consider this as a natural adaptation of our argument for the case of single-person case to multiple players.

Finally, analogous to the single-person case, it is crucial that there exist countably many "test sets" $\{U_n^{(k)}\}_{n,k}$ so that each test set has a strictly positive weight for each player. Otherwise, truth-telling may not be the unique equilibrium.

4.3 What Makes Our Theorem 2 Nontrivial

The constructions so far in Sections 4.1 and 4.2 are, although complicated, conceptually a straightforward application of the standard belief-elicitation idea. Now we explain several points where our construction in Theorem 2 substantially departs from it.

The first point is that the belief-elicitation game considered above is *not* a supermodular game. Even in a simple single-person case, the action space B with the corresponding partial order (based on the first-order stochastic dominance) is not rich enough to constitute a lattice, that is, it is not closed with respect to the joins and meets operators. To illustrate this point, we consider the following example.

Example 1 (Kamae, Krengel, and O'Brien (1977)). [25]
Let $\Theta = \{0,1\}^2$, and let $\succeq_\Theta$ denote its component-wise partial order. Consider two probability measures $P, P' \in \Delta(\Theta)$ such that $P(0,0) = P(1,1) = 1/2$ and $P'(1,0) = P'(0,1) = 1/2$. Then, two probability measures $Q, Q' \in \Delta(\Theta)$ are upper bounds of $\{P, P'\}$: $Q(1,0) = Q(1,1) = 1/2$ and $Q'(0,1) = Q'(1,1) = 1/2$. Suppose that there exists a least upper bound Q''. Then, we have $Q''(1,1) = 1/2$ because we need $P(1,1) \leq Q''(1,1) \leq Q(1,1)$. Moreover, we have $Q''(0,1) = 0$ (or $Q''(0,1) + Q''(1,1) = 1/2$) because we need $Q''(0,1) + Q''(1,1) \leq Q(0,1) + Q(1,1)$. Similarly, we have $Q''(1,0) = 0$. However, then, such Q'' is equivalent to P, which does not first-order stochastically dominate P'. This contradicts the fact that Q'' is an upper bound of $\{P, P'\}$.

Therefore, if we consider (the single-person version of) the belief-elicitation game where an individual chooses his probability measure over Θ, the corresponding game is "not" a supermodular game because his action space, the set of all probability measures over Θ (endowed with a stochastic dominance partial order), does not constitute a lattice.

The problem illustrated in the above example is that the set of all probability distributions (over Θ) is not closed in the meet and join operators. To elaborate on this point, we revisit the same example.

Example 2. We consider the same example as above, but now the agent chooses a function

$$\alpha : U(\Theta) \to [0,1],$$

where $U(\Theta) \subseteq 2^\Theta$ denotes the set of all subsets of Θ that are upper sets (recall that $Y \subseteq \Theta$ is an upper set if $[x \in Y$ and $y \geq x]$ implies $y \in Y$). In the current context, we have

$$\begin{aligned} U(\Theta) \;=\; & \Big\{\emptyset, \{(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \\ & \{(0,1),(1,0),(1,1)\}, \underbrace{\{(0,0),(0,1),(1,0),(1,1)\}}_{=\Theta}\Big\}. \end{aligned}$$

We may interpret each $\alpha(\Theta)$ as the agent's "belief" regarding the event Θ, and in fact, each belief corresponds to some mapping α.[26] However, such α may not correspond

[25] To be precise, it is a slight variant of their example.

[26] For example, P in the previous example is equivalent to α_P such that (i) $\alpha_P(\emptyset) = 0$, (ii) $\alpha_P(Y) = 1/2$ for any non-empty $Y \in U(\Theta)$ with $(0,0) \notin Y$, and (iii) $\alpha_P(Y) = 1$ for any $Y \in U(\Theta)$ with $(0,0) \in Y$.

to any probability measure. For example, let α be defined in such a way that, for each $Y \in U(\Theta)$,

$$\alpha(Y) = \max\{P(Y), P'(Y)\}.$$

That is,

$$\alpha(Y) = \begin{cases} 0 & \text{if } Y = \emptyset \\ 1/2 & \text{if } Y = \{(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(1,1)\}, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

If it corresponds to a probability measure Q^* over Θ, then $\alpha(\{1,1\}) = 1/2$ implies $Q^*(1,1) = 1/2$, which, together with $\alpha(\{(0,1),(1,1)\}) = \alpha(\{(1,0),(1,1)\}) = 1/2$, implies $Q^*(0,1) = Q^*(1,0) = 0$. However, $\alpha(\{(0,1),(1,0),(1,1)\}) = 1$ implies that $Q^*(0,1) + Q^*(1,0) = 1$, which is a contradiction. Therefore, this α does not correspond to any probability measure.

To overcome this issue, the action space is "enlarged" to be the set of all *capacities*, that is, the measures that are not necessarily additive.[27] We can then show that this enlarged action space constitutes a (complete) lattice. We therefore consider a "modified" game with a single player who chooses any $\alpha : U(\Theta) \to [0,1]$. Let A^* denote the set of all such α. The player has a strictly larger strategy space than in the original belief-elicitation game because some "non-additive" measures are allowed. Moreover, A^* is now a lattice (associated with the first-order stochastic dominance partial order), because, for any α, α', we have α'', α''' such that $\alpha''(Y) = \max\{\alpha(Y), \alpha'(Y)\}$ and $\alpha'''(Y) = \min\{\alpha(X), \alpha'(Y)\}$ for any $Y \in U(\Theta)$. In fact, it is even a complete lattice, because for any non-empty subset $A \subseteq A^*$, there are $\alpha', \alpha'' \in A^*$ such that $\alpha'(Y) = \sup_{\alpha \in A} \alpha(Y)$ and $\alpha''(Y) = \inf_{\alpha \in A} \alpha(Y)$ for any $Y \in U(\Theta)$.

To explain the next challenge, recall that the σ-algebra of X^0 (and also that of each X^k with multiple players) is countably generated, and hence, any probability measure over it can be identified by some mapping $\beta : \mathcal{U} \to [0,1]$. However, once the action space is enlarged to include all the *capacities* (i.e., not-necessarily additive measures), it is possible that two distinct capacities correspond to the same mapping β, that is, we lose identification. Therefore, we must consider different test sets. Moreover, as underlined in Sections 4.1 and 4.2, the countability of the test sets $\mathcal{U}$ plays a crucial role for the *uniqueness* of the truth-telling equilibrium.

To overcome this issue, in the game constructed later, we set each U_n (and each $U_n^{(k)}$ with multiple players) as a (closed) *upper set*.[28] The countablity of the collection of

[27] See Schmeidler (1986, 1989) for its application to decision theory.
[28] See Section 2.1 for the definition of upper sets.

the upper sets is guaranteed by continuity-type assumption on X^0 in Section 2 (which are satisfied in some popular cases in applications: for example, if X^0 is Euclidean). In principle, it is possible that two distinct capacities (or even two distinct probability measures) induce the same mapping β, and in this sense, our game does not necessarily fully elicit the players' beliefs. Still, whenever two distinct capacities are ordered in the sense of first-order stochastic dominance, they never induce the same β, and in this sense, we can always detect such a dominance relation. For this reason, we call our game an *optimism*-elicitation game rather than a belief-elicitation game.[29]

5 Relation to Van Zandt and Vives (2007)

In this section, we discuss the relationship with Van Zandt and Vives (VZV, 2007). Both VZV and our paper represent the supermodular games with a (possibly non-common-prior) general type space, and discuss monotone comparative statics with respect to the orders on types. However, there are several differences:

- VZV consider an implicit (Harsanyi) type space endowed with a partial order and then introduce each type's belief map that is consistent with those implicitly given structures. On the other hand, our order on types is based on belief hierarchies constructed from the fundamentals space Θ, and in this sense, our order is based on the given order on Θ (rather than we give an order directly on a type space);

- Both papers order types based on the first-order stochastic dominance relation, but their formal relationship is not clear because they have different constructions of type spaces and their orders;

- VZV and our paper make different assumptions on the primitives: We consider a Polish space Θ so that each belief hierarchy is a Borel probability measure on a Polish space (thanks to the universal-type-space construction of Brandenburger and Dekel (1993)). VZV make no topological assumption on their type space;

- These two papers consider somewhat different classes of games. We consider a class of games where only θ is payoff-relevant information, and the players' (first and higher-order) beliefs are not directly payoff-relevant. In VZV, however, the agents' types can be directly payoff-relevant.

[29] All of those issues are relevant even if there is a single player. However, if there are multiple players, additional care is required in order to make sure that, *for every order k*, the same argument as above goes through. See the Appendix for the detail.

Despite these differences, both papers provide (their versions of) monotone comparative statics. Therefore, it seems natural to conjecture that one obtains some formal relationship between the two approaches. In order to rigorously discuss the connection, we first explain VZV's approach in the context of our setup.

Let Θ be given with a partial order $\succeq_\Theta$, and let $\mathcal{T} = (T_i, \mathscr{T}_i, \pi_i)_{i\in\mathcal{I}}$ be a Harsanyi type space, i.e., $(T_i, \mathscr{T}_i)$ is a measurable space and $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$ denotes i's belief map. Each T_i is exogenously given a partial order, $\succeq_i^{VZV}$, where $t_i \succeq_i^{VZV} t_i'$ if $\pi_i(t_i)$ first-order stochastically dominates $\pi_i(t_i')$ with respect to $\succeq_\Theta \times \succeq_{-i}^{VZV}$. In what follows, any partial order satisfying this condition is called a VZV order.

Notice that this definition is endogenous, because the first-order stochastic dominance (FOSD) requirement on $\succeq_i^{VZV}$ depends on $\succeq_{-i}^{VZV}$, while the FOSD requirement on $\succeq_{-i}^{VZV}$ depends on $\succeq_i^{VZV}$ as well. This endogeneity in the definition raises two potential issues: first, especially if $\mathcal{T}$ is not a simple type space, it is not even obvious if there exists at least one non-trivial VZV order.[30] Second, there can be multiple VZV orders on a Harsanyi type space $\mathcal{T}$. In this case, even if we find one VZV order, it is not necessarily the one that is the "finest" across all possible VZV orders (and whether such a finest order exists is not obvious at this stage). If a given VZV order is "not" the finest possible one, some possible comparative statics is not captured by that order. Also, given an arbitrary Harsanyi type space $\mathcal{T}$, the framework of VZV is silent about how to construct non-trivial VZV orders.

Our CO order $\succeq_i^{CO}$ is related to the VZV orders in the following way.

Theorem 3. Let $\mathcal{T} = (T_i, \mathscr{T}_i, \pi_i)_{i\in\mathcal{I}}$ be a Harsanyi type space. For each $i \in I$, let T_i be endowed with $\succeq_i^{CO}$. Then, we have the following:

1. If $t_i \succeq_i^{CO} t_i'$, then $\pi_i(t_i)$ first-order stochastically dominates $\pi_i(t_i')$ with respect to $\succeq_\Theta$ and $\succeq_{-i}^{CO}$. That is, the CO order is a VZV order.
2. Given any VZV order $\succeq_i^{VZV}$ on T, if $t_i \succeq_i^{VZV} t_i'$, then $t_i \succeq_i^{CO} t_i'$. That is, the CO order is the finest partial order among all possible VZV orders.

Proof. The first property holds by definition, and so we omit the proof.

For the second property, let $t_i \succeq_i^{VZV} t_i'$. First, by definition of $t_i \succeq_i^{VZV} t_i'$, we have that $\pi_i(t_i)$ first-order stochastically dominates $\pi_i(t_i')$ with respect to $\succeq_\Theta \times \succeq_{-i}^{VZV}$. This implies that type t_i's first-order belief, $h_i^1(t_i)$, first-order stochastically dominates $h_i^1(t_i')$. Then, we claim that type t_i's second-order belief, $h_i^2(t_i)$, first-order stochastically dominates $h_i^2(t_i')$, because otherwise, there exists a subset $\tilde{T}_{-i} \subseteq T_{-i}$, which is (i) an upper

[30] Of course, a vacuous partial order (i.e., no order) is always admissible.

set, i.e., for any $t_{-i} \in \tilde{T}_{-i}$ and $t'_{-i} \succeq^{VZV}_{-i} t_{-i}$ implies $t'_{-i} \in \tilde{T}_{-i}$, and (ii) $\pi_i(t_i)[\tilde{T}_{-i}] < \pi_i(t_i)[\tilde{T}_{-i}]$. However, this contradicts the hypothesis that $t_i \succeq^{VZV}_i t'_i$.

By the same reasoning, for any $k \geq 1$, we conclude that type t_i's k-th order belief, $h^k_i(t_i)$, first-order stochastically dominates $h^k_i(t'_i)$. That is, $t_i \succeq^{CO}_i t'_i$. □

The theorem has the following implications. First, given any Harsanyi type space $\mathcal{T}$, our CO order is a VZV order. Therefore, unless the CO order is a trivial "empty order", the result provides an existence result of a non-trivial VZV order.[31] It is also important to note that there exists a VZV order other than the CO order if and only if the CO order is not trivial. Moreover, we provide an explicit construction of this CO order. This explicit construction would be useful in the environment where the type space of our interest is "not so simple", and hence it is not obvious to identify the desired partial order. Given our infinite-level iteration procedure, it is not necessarily easy to construct our CO order. Nevertheless, at least in principle, we provide a systematic way of constructing a partial order which is useful in conducting monotone comparative statics. We make this point explicitly in Example 3 below.

The second result means that our CO ordering captures the maximum possible monotone comparative statics that can be implied by the VZV approach. Moreover, our Theorem 2 shows that any partial order that is strictly richer than the CO order (and hence richer than any VZV order) is not consistent with monotone comparative statics if we consider "all" supermodular games based on parameter space Θ.

To highlight these points, we consider the following example.

Example 3. Consider the following type space with two players and $\Theta = \{1, 2\}$. Each player $i \in \{1, 2\}$ commonly believes that: (i) each state $\theta \in \{1, 2\}$ is equally likely; and (ii) each player receives a signal s_i which, conditional on θ, follows a cumulative distribution function (CDF) F_θ, which is independent of the other's signal. Thus, in this example, the players enjoy a common prior, and each player i's type is essentially identified by his signal realization s_i.

Now, imagine that an "analyst" is interested in monotone comparative statics (for an arbitrarily given supermodular game). Her first step would be to determine a partial order over each player's type space so that monotone comparative statics is obtained by either VZV or our approach. This would be relatively straightforward if F_1 is first-order

[31] The following information structure, which is often considered in the literature (e.g., global games, auctions), is a sufficient condition for the existence of nontrivial CO order: There exists a common prior over $\Theta \times \prod_i T_i$; Θ and each T_i is one-dimensional (with its natural total order); a type profile $t \in \prod_{i \in I} T_i$ is independently chosen conditional on $\theta \in \Theta$; and the first-order belief of t_i over Θ first-order stochastically dominates that of t'_i if $t_i > t'_i$. In this case, we have $t_i \succeq^{CO}_i t'_i$ if and only if $t_i \geq t'_i$.

stochastically dominated by F_2. However, in some contexts, such a stochastic ordering may be violated. For example, consider an example with $F_\theta = N(\theta, \theta^2)$, that is, higher θ implies *both* higher (conditional) mean of s_i and its (conditional) variance, then it is no longer clear to see which partial order over the signal realizations is the right order in terms of monotone comparative statics.[32] Note that the result of VZV is not instructive in how to construct such an order, because a (right) partial order is exogenously given and simply a primitive there.

We find our Theorem 3 useful because it provides an explicit construction of such a partial order with which monotone comparative statics holds, and furthermore, that order is the finest possible among all VZV orders. We proceed as follows: first, compare (only) each type's first-order beliefs over Θ and identify the partial order $\succeq_i^0$; next, compare each type's second-order beliefs to identify $\succeq_i^1$; and so on ad infinitum.

We close this section by explicitly characterizing the CO order in the example where F_θ is a normal CDF with mean $\theta - 1$ and variance θ^2, truncated on the interval $[\underline{s}, \bar{s}] \subseteq \mathbb{R}$. More specifically, F_θ is identified by its density f_θ:

$$f_\theta(s_i) = \frac{1}{c_\theta} \exp\left(-\frac{(s_i - (\theta - 1))^2}{2\theta^2}\right),$$

for each $s_i \in [\underline{s}, \bar{s}]$ (and 0 otherwise), where c_θ is a constant which makes it integrate to 1.

First, we identify a partial order for s_i with respect to its first-order belief, denoted by $\succeq_i^0$:

$$s_i \succeq_i^0 s_i' \;\Leftrightarrow\; \Pr(\theta = 2|s_i) \geq \Pr(\theta = 2|s_i').$$

Observe that

$$\begin{aligned}\Pr(\theta = 2|s_i) &= \frac{\frac{1}{c_2}\exp\left(-\frac{(s_i-1)^2}{8}\right)}{\frac{1}{c_2}\exp\left(-\frac{(s_i-1)^2}{8}\right) + \frac{1}{c_1}\exp\left(-\frac{s_i^2}{2}\right)} \\ &= \frac{\exp\left(\frac{3}{8}s_i^2 + \frac{1}{4}s_i - \frac{1}{8}\right)}{\frac{c_2}{c_1} + \exp\left(\frac{3}{8}s_i^2 + \frac{1}{4}s_i - \frac{1}{8}\right)},\end{aligned}$$

and thus,

$$s_i \succeq_i^0 s_i' \;\Leftrightarrow\; 3s_i^2 + 2s_i - 1 \geq 3(s_i')^2 + 2s_i' - 1.$$

[32]The problem here is analogous to a well-known fact that two normal distributions with different variances cannot be ordered by first-order stochastic dominance (although their truncated versions may still be ordered).

In particular, for $s_i, s_i' > -1/3$, we have $s_i \succeq_i^0 s_i' \Leftrightarrow s_i \geq s_i'$, while for $s_i, s_i' < -1/3$, we have $s_i \succeq_i^0 s_i' \Leftrightarrow s_i \leq s_i'$. Intuitively, this reversal of the order occurs because, for a very low (or "large negative") realization of s_i, it is more likely that such s_i comes from $N(1,4)$ than from $N(0,1)$ due to the higher variance in $N(1,4)$.

Now we identify a partial order for s_i with respect to its second-order belief, denoted by $\succeq_i^1$. By definition, $s_i \succeq_i^1 s_i'$ implies $\Pr(\theta = 2|s_i) \geq \Pr(\theta = 2|s_i')$. For s_i to be higher than $s_i^{'}$ in terms of second-order belief, we further demand that s_i assigns a weakly higher probability on any upper set induced by $\succeq_j^0$ than $s_i^{'}$ does. Any upper set of this kind is given by the following form: for each $x, y \in \mathbb{R}$ such that $x \geq -\frac{1}{3} \geq y$,

$$E_{x,y} = \{s_j | s_j \geq x\} \cup \{s_j | s_j \leq y\}.$$

Hence, by conditional independence, the probability that s_i assigns on $E_{x,y}$ is given by:

$$\begin{aligned} P_{x,y}(s_i) \quad = \quad & \Pr(\theta = 2|s_i)(\Pr(N(1,4) \geq x) + \Pr(N(1,4) \leq y)) \\ & +(1 - \Pr(\theta = 2|s_i))(\Pr(N(0,1) \geq x) + \Pr(N(0,1) \leq y)), \end{aligned}$$

where $N(\mu, \sigma^2)$ denotes a normal random variable with mean μ and variance σ^2. Therefore,

$$\begin{aligned} s_i \succeq_i^1 s_i' \quad \Leftrightarrow \quad & \Pr(\theta = 2|s_i) \geq \Pr(\theta = 2|s_i'), \text{ and} \\ & P_{x,y}(s_i) \geq P_{x,y}(s_i') \quad \forall x \geq -\frac{1}{3} \geq y. \end{aligned}$$

We check this by considering the following three sub-cases.

Case (i): $\underline{s} \geq -1/3$.

In this case, we have $\Pr(\theta = 2|s_i) \geq \Pr(\theta = 2|s_i')$ if and only if $s_i \geq s_i'$. In addition, because $E_{x,y} = \{s_j | s_j \geq x\}$ (for $x \geq -1/3$), we have

$$P_{x,y}(s_i) = \Pr(\theta = 2|s_i)(\Pr(N(1,4) \geq x) + (1 - \Pr(\theta = 2|s_i))(\Pr(N(0,1) \geq x).$$

Observe that

$$\Pr(N(1,4) \geq x) \geq \Pr(N(0,1) \geq x)$$

for any $x \geq -1$ (and hence, in particular, for any $x \geq -1/3$). Therefore,

$$P_{x,y}(s_i) - P_{x,y}(s_i') = \left(\Pr(\theta = 2|s_i) - \Pr(\theta = 2|s_i')\right)\left(\Pr(N(1,4) \geq x) - \Pr(N(0,1) \geq x)\right) \geq 0$$

if and only if $s_i \geq s_i'$.

In this case, we thus establish that, for $s_i \succ_i^{CO} s_i'$, it is necessary to have $s_i \geq s_i'$. Note also that this order by $\geq$ is a VZV order on each player's types. As the CO order is a VZV order, this order by $\geq$ is the CO order.

Case (ii): $\bar{s} \leq -1$.

In this case, we have $\Pr(\theta = 2|s_i) \geq \Pr(\theta = 2|s_i')$ if and only if $s_i \leq s_i'$. Also, because $E_{x,y} = \{s_j|s_j \leq y\}$ (for $y \leq -1/3$), we have

$$P_{x,y}(s_i) = \Pr(\theta = 2|s_i)(\Pr(N(1,4) \leq y) + (1 - \Pr(\theta = 2|s_i))(\Pr(N(0,1) \leq y).$$

Observe that

$$\Pr(N(1,4) \leq y) \geq \Pr(N(0,1) \leq y)$$

for any $y \leq -1$. Therefore,

$$P_{x,y}(s_i) - P_{x,y}(s_i') = (\Pr(\theta = 2|s_i) - \Pr(\theta = 2|s_i'))(\Pr(N(1,4) \leq y) - \Pr(N(0,1) \leq y)) \geq 0$$

if and only if $s_i \leq s_i'$.

In this case, we thus establish that, for $s_i \succ_i^{CO} s_i'$, it is necessary to have $s_i \leq s_i'$. Note also that this order by $\leq$ is a VZV order on each player's types. As the CO order is a VZV order, this order by $\leq$ is the CO order.

Case (iii): Any other case.

Let $s_i \neq s_i'$ be such that $s_i \succ_i^0 s_i'$, or equivalently, $\Pr(\theta = 2|s_i) > \Pr(\theta = 2|s_i')$. In what follows, we show that we cannot have $s_i \succeq_i^1 s_i'$, and therefore, we cannot have $s_i \succ_i^{CO} s_i'$ either. That is, in this case, no order is possible. To show this, it suffices to find an upper set $E_{x,y}$ such that the probability that s_i assigns on $E_{x,y}$ is strictly lower than $s_i^{'}$ does. Take $y \in (-1, -1/3)$ such that

$$\Pr(N(1,4) \leq y) < \Pr(N(0,1) \leq y).$$

This is possible because

$$\Pr(N(1,4) \leq y) \geq \Pr(N(0,1) \leq y),\ \forall y \geq -1/3 \ \text{ and } \ \Pr(N(1,4) \leq y) \leq \Pr(N(0,1) \leq y),\ \forall y \leq -1.$$

If we take $x = \bar{s}$, we have $\Pr(\{s_j|s_j \geq \bar{s}\}) = 0$. Therefore, we compute the following:

$$P_{x,y}(s_i) - P_{x,y}(s_i') = (\Pr(\theta = 2|s_i) - \Pr(\theta = 2|s_i'))(\Pr(N(1,4) \leq y) - \Pr(N(0,1) \leq y)) < 0,$$

which means that we cannot have $s_i \succeq_i^1 s_i'$.

In conclusion, depending on the support of s_i, a non-trivial ordering over types is possible. Our procedure of constructing the CO order is shown to be useful to identify the finest VZV order. □

6 Application

To show the usefulness of our CO order, we consider a simple trading game between two parties with pure common values. According to the celebrated *no-trade theorem* of Milgrom and Stokey (1982), the parties never trade if the initial allocation of the goods is Pareto efficient (which is trivially satisfied with pure common values), they are strictly risk averse, and share a common belief about how the prices of the goods are determined ex post, that is, the rational expectations hypothesis is satisfied. However, in practice, traders may enjoy heterogeneous beliefs about how the prices of the goods are determined, and some traders may be systematically more optimistic than others. That is, the rational expectations hypothesis may be violated. Such belief divergence may admit some possibility of trading.[33] Then, a natural question arises as to the relationship between belief divergence and volume of trading. We introduce a partial order with respect to the size of belief divergence, and show that this order corresponds to our CO order that admits monotone comparative statics.

There are two traders, a seller ($i = 1$) and a buyer ($i = 2$). Let $I = \{1, 2\}$. The seller has an asset whose common value is $v \in \mathbb{R}$. Due to this common value assumption, the initial allocation of the asset is trivially Pareto efficient. Each trader chooses the volume of trade. Let $A_i = [0, 1]$ with its natural order, where $a_i \in A_i$ denotes the volume of trade that trader i wants to execute.

Unless both traders choose positive volume of trade, there is no trade. If each trader i chooses a positive volume of trade $a_i > 0$, they trade $a_1 \cdot a_2$ amount of the asset at per-unit price $p \in \mathbb{R}$. This guarantees voluntary participation because each agent i secures his no-trade payoff by choosing $a_i = 0$. For simplicity, we assume that the players have CARA (constant absolute risk aversion) utilities: The seller's (ex post) payoff is $-\exp(-(p - v)a_1a_2)$, and the buyer's (ex post) payoff is $-\exp(-(v - p)a_1a_2)$.

At the timing of the (simultaneous) trade decision, imagine that v and p could be uncertain for the players. Let $\Theta = [-1, 1]^2$ represent the payoff-state space so that, given $\theta = (\theta_1, \theta_2) \in \Theta$, θ_i denotes i's trading payoff from one unit of the asset. That is, $\theta_1 = p - v$ and $\theta_2 = v - p$. It is assumed to be common knowledge that $\theta_1 + \theta_2 = 0$ (i.e., the asset has a pure common value), but the players may not agree on the exact value of θ_1 (and hence that of θ_2). Player i's ex post payoff can be written as follows: for any $\theta = (\theta_1, \theta_2) \in \Theta$ and $a \in A_1 \times A_2 = [0, 1]^2$,

$$u_i(a, \theta) = -\exp\left(-\theta_i a_1 a_2\right).$$

[33] See, for example, Feinberg (2000), for a formal connection between the common-prior assumption and no-betting condition.

For each $i \in I$, we introduce a (natural) player-specific partial order $\succeq_{\Theta,i}$ over Θ so that $\theta \succeq_{\Theta,i} \theta'$ if $\theta_i \geq \theta'_i$.[34] Observe that the constructed game $g = (I, \prod_{i \in I} A_i, \Theta, (u_i)_{i \in I})$ is a (complete-information) supermodular game.

Let $\mathcal{T} = (T_i, \mathscr{T}_i, \pi_i)_{i \in I}$ denote a Harsanyi type space, where a measurable space T_i denotes player i's set of types, and a measurable map $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$ denotes his belief map. We assume that, for any $i \in I$ and $t_i \in T_i$,

$$\pi_i(t_i)\left[\{\theta \in \Theta | \theta_1 + \theta_2 = 0\} \times T_{-i}\right] = 1,$$

that is, it is common knowledge that the asset has a pure common value.

First, consider the case where the players share a common prior. That is, there exists $\mu \in \Delta(\Theta \times T_1 \times T_2)$ such that each i's belief map π_i is a system of conditional probabilities induced by μ in the following sense: for all $i \in I$ and measurable events $\tilde{\Theta} \subseteq \Theta$, $\tilde{T}_1 \subseteq T_1$, and $\tilde{T}_2 \subseteq T_2$, we have

$$\mu(\tilde{\Theta} \times \tilde{T}_1 \times \tilde{T}_2) = \int_{\tilde{T}_i} \pi_i(t_i) \left[\tilde{\Theta} \times \tilde{T}_{-i}\right] d\mu_i(t_i),$$

where $\mu_i \in \Delta(T_i)$ is the marginal of μ on T_i. We further assume that each type t_i is never certain of the true value of θ_i so that there remains the residual uncertainty of his ex post payoff, even conditional on t_i. That is, $\pi_i(t_i)[\{\theta_i\} \times T_{-i}] < 1$ for any θ_i.

Proposition 3. Suppose that the players share a common prior in a Harsanyi type space $\mathcal{T} = (T_i, \mathscr{T}_i, \pi_i)_{i \in \{1,2\}}$. Then, the ex ante expected volume of trade is zero for any Bayesian (Nash) equilibrium $\sigma = (\sigma_i(t_i))_{i \in I,\ t_i \in T_i}$.

Proof. The proof is in the Appendix. □

Next, imagine an alternative situation where the players enjoy heterogeneous beliefs. For each player $i \in I$ and types $t_i, t'_i \in T_i$, we write $t'_i \succeq_{CO} t_i$ if $h^k(t'_i) \succeq_{SD} h^k(t_i)$ for any $k \in \mathbb{N}$. That is, t'_i is more optimistic about θ_i than t_i; t'_i is more optimistic that all players are more optimistic about θ_i than t_i, and so on, ad infinitum.[35]

As a corollary to our Theorem 1, we establish monotone comparative statics with respect to this partial order. We state the result without its proof.

Corollary 1. If $t'_i \succeq_{CO} t_i$, then $\underline{\sigma}_i(t'_i) \geq \underline{\sigma}_i(t_i)$ and $\bar{\sigma}_i(t'_i) \geq \bar{\sigma}_i(t_i)$, where $\underline{\sigma}$ and $\bar{\sigma}$ denote the least and greatest equilibrium of the game, respectively.

[34] In the rest of the paper, we have a common order over Θ across all players. We emphasize that all the analyses in this paper can be extended to the case where each player has a specific order $\Theta_{i,\theta}$, as we do here.

[35] Recall that any type t_i believes that $\theta_1 + \theta_2 = 0$ throughout the entire belief hierarchies.

However, the above monotonicity result is weak and thus it is interesting to investigate when the volume of trade goes "strictly" up. To establish such a result, we first introduce a strict CO order: $t_i^{'} \succ_{CO} t_i$ if $h_i^k(t_i^{'})$ strictly stochastically dominates $h_i^k(t_i)$ for each $k \in \mathbb{N}$. For each $i \in I$, let $\bar{\sigma}_i : T_i \to [0,1]$ denote agent i's action in the greatest equilibrium. For each $i \in I$, $a_i \in [0,1]$, and $t_i \in T_i$, we define

$$\begin{aligned} f(a_i, t_i) &= \int_{\Theta \times T_{-i}} u_i(a_i, \bar{\sigma}_{-i}(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}] \\ &= \int_{\Theta \times T_{-i}} -\exp\left(-\theta_i a_i \bar{\sigma}_{-i}(t_{-i})\right) d\pi_i(t_i)[\theta, t_{-i}]. \end{aligned}$$

Thus, we obtain

$$\bar{\sigma}_i(t_i) \in \arg \max_{a_i \in [0,1]} f(a_i, t_i).$$

We also define

$$\frac{\partial f(a_i, t_i)}{\partial a_i} = \int_{\Theta \times T_{-i}} \theta_i \bar{\sigma}_{-i}(t_{-i}) \exp\left(-\theta_i a_i \bar{\sigma}_{-i}(t_{-i})\right) d\pi_i(t_i)[\theta, t_{-i}].$$

Proposition 4. There exists a Harsanyi type space $\mathcal{T} = (T_i, \mathscr{T}_i, \pi_i)_{i \in \{1,2\}}$ such that, for any $t_i, t_i^{'} \in T_i$, if $\bar{\sigma}_i(t_i), \bar{\sigma}_i(t_i^{'}) \in (0,1)$ and $t_i^{'} \succ_{CO} t_i$, then $\bar{\sigma}_i(t_i^{'}) > \bar{\sigma}_i(t_i)$, i.e., type $t_i^{'}$ plays a strictly higher action than type t_i under the greatest equilibrium $\bar{\sigma}$.

Proof: The proof boils down into the adaptation of "Strict Monotonicity Theorem 1" of Edlin and Shannon (1998) to our setup. To use this theorem, it suffices to construct a Harsany type space $\mathcal{T} = (T_i, \mathscr{T}_i, \pi_i)_{i \in \{1,2\}}$ such that

$$t_i^{'} \succ_{CO} t_i \Rightarrow \frac{\partial f(a_i, t_i^{'})}{\partial a_i} > \frac{\partial f(a_i, t_i)}{\partial a_i}.$$

Suppose that there exist $t_i, t_i^{'}$ such that $t_i^{'} \succ_{CO} t_i$ and $\bar{\sigma}_i(t_i), \bar{\sigma}_i(t_i^{'}) \in (0,1)$. Since $t_i^{'} \succ_{CO} t_i$, there exists a non-empty upper event Y in T_{-i}^* where T^* denote the universal type space such that $h_i^k(t_i^{'})[Y] > h_i^k(t_i)[Y]$ for each $k \in \mathbb{N}$. We construct a type space $(T_i, \mathscr{T}_i, \pi_i)_{i \in \{1,2\}}$ such that $\{t_i, t_i^{'}\} \subseteq T_i$; $Y \subseteq \Theta \times T_{-i}$; and $\pi_i : T_i \to \Delta(\Theta \times T_{-i})$ such that

$$\int_{(\theta, t_{-i}) \in Y} d\pi_i(t_i)[\theta, t_{-i}] > 0 \text{ and } \int_{(\theta, t_{-i}) \in Y} d\pi_i(t_i^{'})[\theta, t_{-i}] > 0.$$

Then, by Corollary 1, if $t_i^{'} \succ_{CO} t_i$, we obtain

$$\int_{\Theta \times T_{-i}} \theta_i \bar{\sigma}_{-i}(t_{-i}) \exp\left(-\theta_i a_i \bar{\sigma}_{-i}(t_{-i})\right) d\left[\pi_i(t_i^{'})[\theta, t_{-i}] - \pi_i(t_i)[\theta, t_{-i}]\right] > 0,$$

where $\mathscr{T}_{-i}$ is defined to make sense of the above integration. This implies that $\partial f(a_i, t_i^{'})/\partial a_i > \partial f(a_i, t_i)/\partial a_i$, which is the desired inequality. This completes the proof. ■

Recall that it is common knowledge that $\theta_1 + \theta_2 = 0$. Nevertheless, the positive volume of trade can sometimes occur, because the players do not agree on the exact level of $\theta_1(= -\theta_2)$. The CO order introduced in this paper captures the connection between the size of the belief divergence and the trading volume.

7 Final Remarks

In this paper, we introduce an order on types over a universal type space. We consider our CO order as a natural order in the sense that monotone comparative statics is valid in a class of supermodular games with incomplete information. We fully characterize this order in terms of common optimism, that is, type $t_i^{'}$ is higher than type t_i if $t_i^{'}$ is more optimistic about the state than t_i; more optimistic that all players are more optimistic about the state than t_i; and so on ad infinitum. Thus, our CO order has a sharp epistemic interpretation. First, we show that whenever the common optimism holds, monotone comparative statics holds in all supermodular games (Theorem 1). Second, as its converse, we construct an "optimism-elicitation game" as a single supermodular game with the property that whenever the common optimism fails, monotone comparative statics fails (Theorem 2). Third, we show that our CO order is an order considered by VZV (i.e., a VZV order) and the finest one among all VZV orders (Theorem 3). We also argue by example (Example 3) that the constructive nature of our CO order allows us to identify a nontrivial order, whereas VZV assume an exogenous order and therefore their approach is silent about this. We consider this as our advantage over VZV.

Although our CO order characterizes monotone comparative statics across *all* supermodular games, in some cases, one may be more interested in monotone comparative statics in a *fixed* supermodular game. In such a case, the CO order continues to be a sufficient condition for monotone comparative statics of that game, but may not be necessary. That is, monotone comparative statics may hold even between types that are not ordered in the CO sense. !!Using an example, we illustrate this point in the Appendix.!!

Establishing a possibly finer order on types that is both necessary and sufficient for monotone comparative statics in a given supermodular game is interesting but challenging. Although we leave it as a future research question, here we briefly explain our conjecture, on which we are currently working.[36] The basic idea is to introduce "indiffer-

[36]We thank Takashi Ui because this conjecture stems from a discussion with him.

ence" relations on types of each player as follows. Consider the first round of elimination of the never best responses. If we have $\underline{a}_i^1(t_i) \succeq_{A_i} \underline{a}_i^1(t_i')$, then we let $t_i \succeq_i^1 t_i'$.[37] Here $\underline{a}_i^1(t_i)$ denotes the least action for type t_i chosen after deleting the never best responses (See also Section 3 for this notation).

This order is richer than the first-order stochastic dominance order: if t_i first-order stochastically dominates t_i' in terms of their first-order beliefs, then we have $t_i \succeq_i^1 t_i'$, but the converse may not be true.

As the next step, if we have $\underline{a}_i^2(t_i) \succeq_{A_i} \underline{a}_i^2(t_i')$, then we let $t_i \succeq_i^2 t_i'$. Here $\underline{a}_i^2(t_i)$ denotes the least action for type t_i chosen after two times iteratively deleting never best responses (See also Section 3 for this notation). We conjecture that this order is richer than the first-order stochastic dominance order: if t_i first-order stochastically dominates t_i' in terms of their second-order beliefs, then we have $t_i \succeq_i^2 t_i'$. If this logic goes through up to any level of iterative elimination, then in the limit, we conjecture that this alternative order is (i) richer than the CO order; (ii) is implied by monotone comparative statics in this game; and (iii) implies monotone comparative statics in this game.

[37] Technically, we may not be able to interpret $\succeq_i^1$ as a partial order because both $t_i \succeq_i^1 t_i'$ and $t_i \preceq_i^1 t_i'$ are possible even if $t_i \neq t_i'$. In that case, we can interpret those types as equivalent (in the sense of $\succeq_i^1$) and can consider a quotient space based on this equivalence class. Then, $\succeq_i^1$ is a partial order on this quotient space.

A Necessity for Monotone Comparative Statics: The Single Person Case

In this section, we formally prove that common optimism is necessary for monotone comparative statics to be valid in all supermodular games. We establish this result by constructing a specific supermodular game, which we call an *optimism-elicitation game*, which satisfies the following: for each player i and his types t_i and t'_i, if $\underline{\sigma}_i(t_i) \succeq_{A_i} \underline{\sigma}_i(t'_i)$ where $\underline{\sigma}$ denotes the least equilibrium of this optimism-elicitation game, then $t_i \succeq_{CO} t'_i$. Together with the previous theorem, we thus conclude that the CO order is necessary and sufficient for monotone comparative statics in all supermodular games.

A.1 A Single-Person Game

We first consider the single-person environment to explain the key technical issue and the main intuition how we treat it. The restriction to the single-person case simplifies our analysis significantly because there is no need to consider interactive beliefs so that we lose nothing to focus on the first-order beliefs only. Thus, a naive candidate for our optimism-elicitation game is a so-called *scoring rule*, which is essentially a single-person decision problem where the decision maker reveals his belief over Θ (and his payoff function is defined in such a way that the truthful revelation is uniquely optimal). That is, his action space is the set of all probability measures over Θ. Monotone comparative statics is obtained in a straightforward manner by endowing this action space with a partial order based on the first-order stochastic dominance.

However, as we observed in Example 1 (Section 4.3), this decision problem is not a (single-person) supermodular game, because the action space, $\Delta(\Theta)$, is not a lattice, even if the parameter space Θ itself is. This means that we need a more careful choice of the action space. We illustrated this point in Example 2 (Section 4.3).

The key for the construction of our optimism-elicitation game is two-fold. First, the action space of our game is based on non-additive beliefs such as α discussed in Example 2 of Section 4.3, in order to make it a complete lattice. Second, as we see below in the formal construction, the action space of our game essentially comprises only countably many "test sets" to (partially) identify the agent's belief. We explain these features more in detail after formally introducing our optimism-elicitation game.

Formally, the optimism-elicitation game for the single agent case is defined as follows: (i) the agent chooses an action $\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0, 1]$ (recall that Θ_0 denotes the countable dense subset of Θ where Assumption 1 is satisfied) where

- $F(\Theta_0)$ denotes the set of all *finite* subsets of Θ_0,
- $\mathbb{Q}_+$ denotes the set of nonnegative rational numbers, and
- β is non-decreasing (i.e., for any (γ, q) and (γ', q') with $clup(B_q(\gamma)) \subseteq clup(B_{q'}(\gamma'))$, we have $\beta(\gamma, q) \leq \beta(\gamma', q')$), where $clup(B_q(\gamma))$ denotes the closure of upper set $B_q(\gamma)$.

and (ii) given any realization $\theta \in \Theta$, the agent's payoff is given:

$$u(\beta, \theta) = \sum_{(\gamma,q)\in F(\Theta_0)\times\mathbb{Q}_+} \left[\beta(\gamma, q) I_{\{clup(B_q(\gamma))\}}(\theta) - \frac{\beta(\gamma, q)^2}{2}\right] \mu(\gamma, q),$$

where

- $B_q(\gamma) = \bigcup_{y\in\gamma} B_q(y)$;
- μ is a full-support distribution over a countable set $F(\Theta_0) \times \mathbb{Q}_+$[38]; and
- The indicator function is defined as:

$$I_{\{clup(B_q(\gamma))\}}(x) = \begin{cases} 1 & \text{if } x \in clup(B_q(\gamma)) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{B} = \left\{\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0, 1] \middle| \beta \text{ is nonderecasing}\right\}$$

denote the space of the agent's strategies. Note that $\mathcal{B}$ constitutes the set of capacities (i.e., not-necessarily additive measures) for closed upper sets generated by $F(\Theta_0)\times\mathbb{Q}_+$.[39] As we argue in Section 4.3, the space of capacities has an advantageous feature that it is a complete lattice.

As we mention above, another feature of our construction is that the action space of our game essentially comprises only countably many "test sets" to (partially) identify the agent's belief. Countability enables us to have a full-support distribution over the test

[38] We can set $h : F(\Theta_0) \times \mathbb{Q}_+ \to \mathbb{N}$ as an injection mapping because $F(\Theta_0) \times \mathbb{Q}_+$ is countable. Specifically, we define the full-support distribution μ by $\mu(\gamma, q) = (1/2)^{h(\gamma,q)} > 0$.

[39] A capacity is usually defined as a monotone set function as above, but with additional normalization conditions that it assigns probability zero (one) on the null (entire) set. Redefining $\mathcal{B}$ by adding these normalization conditions does not change our arguments, and hence, we adopt the current definition to simplify the notation. In this sense, the definition above is without loss of generality. The same comment applies to the definitions of $\mathcal{B}^m$ and $\mathcal{B}^\infty$ in Section B.1.

sets, which makes the agent's incentive to tell the truth strict (and hence, the optimal decision is unique).[40]

Indeed, if a player has a probabilistic belief $b \in \Delta(\Theta)$,[41] then his unique optimal action is $\beta^*(b) : F(\Theta_0) \times \mathbb{Q}_+ \to [0,1]$ such that

$$\beta^*(\gamma, q|b) = b(clup(B_q(\gamma)))$$

for each $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$.

Endowing $\mathcal{B}$ with a natural product order, we show that the game is a supermodular game. First, we claim that $\mathcal{B}$ is a complete lattice: for each $C \subseteq \mathcal{B}$, define two functions, $\bigvee(C)$ and $\bigwedge(C)$, so that

$$\begin{aligned} \bigvee(C)(\gamma, q) &= \sup_{\beta \in C} \beta(\gamma, q), \\ \bigwedge(C)(\gamma, q) &= \inf_{\beta \in C} \beta(\gamma, q), \end{aligned}$$

which makes both $\bigvee(C)$ and $\bigwedge(C)$ elements of $\mathcal{B}$, because they both take values in $[0,1]$ for any (γ, q), and they are both monotonic. Suppose, on the contrary, that $\bigvee(C)$ is not monotonic for some C. Then, there exist (γ, q), (γ', q') such that $clup(B_q(\gamma)) \subseteq clup(B_{q'}(\gamma'))$ and $\bigvee(C)(\gamma, q) > \bigvee(C)(\gamma', q')$. By definition, there exists $\beta \in C$ such that $\beta(\gamma, q)$ is close to $\bigvee(C)(\gamma, q)$, and in particular, $\beta(\gamma, q) > \bigvee(C)(\gamma', q') \geq \beta(\gamma', q')$. This contradicts the hypothesis that β is monotonic.

Second, the payoff function $u(\cdot)$ is supermodular on $\mathcal{B}$ and has increasing difference in (β, x): for any $\beta, \beta' \in \mathcal{B}$, $x, x' \in \Theta$ with $x \geq x'$, we have

$$\begin{aligned} & u(\beta \vee \beta', x) - u(\beta, x) + u(\beta \wedge \beta', x') - u(\beta', x') \\ = & \int_{(\gamma,q):\beta'(\gamma,q)>\beta(\gamma,q)} \left[(\beta'(\gamma,q) - \beta(\gamma,q)) I_{\{clup(B_q(\gamma))\}}(x) - \beta'(\gamma,q)^2 + \beta(\gamma,q)^2\right] d\mu \\ & - \int_{(\gamma,q):\beta'(\gamma,q)>\beta(\gamma,q)} \left[(\beta(\gamma,q) - \beta'(\gamma,q)) I_{\{clup(B_q(\gamma))\}}(x') - \beta(\gamma,q)^2 + \beta'(\gamma,q)^2\right] d\mu \\ = & \int_{(\gamma,q):\beta'(\gamma,q)>\beta(\gamma,q)} \left[(\beta'(\gamma,q) - \beta(\gamma,q)) I_{\{clup(B_q(\gamma))\}}(x)(1 - I_{\{clup(B_q(\gamma))\}}(x'))\right] d\mu \\ \geq & \ 0. \end{aligned}$$

[40]Moreover, as we see in Section B to extend this construction to the case of multiple players, this countability plays another crucial role. There, we consider the "higher-order belief" version of the current construction as each player's action space (in order to elicit his belief hierarchy). Countability at each level of hierarchy (and certain continuity) is crucial to make the next level of hierarchy (and hence at any level of hierarchy) stay countable.

[41]Recall that $\Delta(\Theta)$ denotes the set of all probability measures over Θ.

We now examine monotone comparative statics for this supermodular game. The first result establishes the sufficiency of first-order stochastic dominance for monotone comparative statics in this supermodular game (as should be expected).

Proposition 5. Let $b, b' \in \Delta(\Theta)$. If $b^{'}$ first-order stochastically dominates b, then, for any $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$, we have $\beta^*(\gamma, q|b^{'}) \geq \beta^*(\gamma, q|b)$.

Proof. While it is a corollary to Theorem 1, the proof is also straightforward once we notice that $\beta^*(\gamma, q|b) = b(clup(B_q(\gamma)))$ and $\beta^*(\gamma, q|b') = b'(clup(B_q(\gamma)))$. □

Next, we show the desired necessity of first-order stochastic dominance for monotone comparative statics in this supermodular game.

Proposition 6. Let $b, b' \in \Delta(\Theta)$. If $\beta^*(\gamma, q|b^{'}) \geq_{\mathcal{B}} \beta^*(\gamma, q|b)$ for each $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$, then $b^{'} \succeq_{SD} b$.

Proof. We take the contrapositive of the statement. Then what we want to show is that if b' does not stochastically dominate b, then $\beta^*(\gamma, q|b')$ cannot be higher than $\beta^*(\gamma, q|b)$ in the sense of the partial order on $\mathcal{B}$. Thus, the rest of the proof is completed by the following lemma.

Lemma 2. Let $b, b' \in \Delta(\Theta)$. If b' does not first-order stochastically dominate b, then there exists $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$ such that $\beta^*(\gamma, q|b) > \beta^*(\gamma, q|b')$.

Remark 2. Suppose that there exists some closed upper set $Y \subseteq X$ such that $b(Y) > b'(Y)$. By Proposition 1, there exists some $Y_0 \subseteq X_0$ such that $Y = clup(Y_0)$. If this Y_0 is finite, that is, $Y_0 \in F(X_0)$, then we trivially have $\beta^*(Y_0, 0|b) > \beta^*(Y_0, 0|b')$. Thus, the subtlety of the proof of Lemma 2 lies in the possibility that Y_0 is (countably) infinite.

Proof. Suppose that there exists some closed upper set $Y \subseteq X$ such that $b(Y) > b^{'}(Y)$. Then, we fix $\varepsilon \in (0, (b(Y) - b'(Y))/2)$. First, by the inner-regularity property, there exist two compact sets $Z_1, Z_2 \subseteq X$ such that $b(Z_1) \geq 1 - \varepsilon$ and $b'(Z_2) \geq 1 - \varepsilon$. Let $Z = Z_1 \cup Z_2$. This Z is again compact, and we have that $b(Z) \geq 1 - \varepsilon$ and $b'(Z) \geq 1 - \varepsilon$.

Let $\{\eta_j\}_{j=1}^{\infty}$ be a decreasing sequence such that $\eta_j > 0$ for each $j \in \mathbb{N}$ and $\eta_j \to 0$ as $j \to \infty$. For each j, define

$$\delta_j = \frac{\delta(Y \cap Z, \eta_j)}{2},$$

where $\delta(Y \cap Z, \eta_j)$ is given as $\delta(Y, \varepsilon)$ in Assumption 2. By construction, we have that $\delta_j \to 0$ as $j \to \infty$.

Because Y is closed, $Y \cap Z$ is compact. Fix $j \in \mathbb{N}$. Let $\{B_{\delta_j}(x)\}_{x \in Y \cap Z}$ be an open cover of $Y \cap Z$. Since $Y \cap Z$ is compact, we can take a finite subcover $\{B_{\delta_j}(x_n)\}_{n=1}^{N_j}$ such that $x_n \in Y \cap Z$ for each $n = 1, \ldots, N_j$. Since X_0 is dense in X, for each $n = 1, \ldots N_j$, we can take $y_n \in X_0$ so that $y_n \in B_{\delta_j}(x_n)$.

Define $\gamma_j = \{y_1, \ldots, y_{N_j}\} \in F(X_0)$. Then, for each $n = 1, \ldots, N_j$, we have $d(y_n, B_{\delta_j}(x_n)) < 2\delta_j$. This implies that $B_{2\delta_j}(y_n) \supseteq B_{\delta_j}(x_n)$. Therefore,

$$B_{2\delta_j}(\gamma_j) = \bigcup_{n=1}^{N_j} B_{2\delta_j}(y_n) \supseteq \bigcup_{n=1}^{N_j} B_{\delta_j}(x_n) \supseteq Y \cap Z.$$

Define also

$$D_j = \bigcup_{k=j}^{\infty} B_{2\delta_k}(\gamma_k).$$

By construction, we observe that $D_j \supseteq B_{2\delta_j}(\gamma_j)$ for each $j \in \mathbb{N}$, and $D_1 \supseteq D_2 \supseteq \cdots \supseteq Y \cap Z$. Moreover, we have that $d(Y \cap Z, D_j) < 2\delta_j$. Since Assumption 2 guarantees that the upper set correspondence is continuous with respect to the Hausdorff metric, we obtain

$$d(clup(Y \cap Z), clup(D_j)) = d(up(Y \cap Z), up(D_j)) < \eta_j.$$

Fix $x \notin clup(Y \cap Z)$ arbitrarily. Then, we have $d(x, clup(Y \cap Z)) > 0$ because $clup(Y \cap Z)$ is closed. Let $j(x) \in \mathbb{N}$ be defined in such a way that $d(x, clup(Y \cap Z)) \geq \eta_{j(x)}$. Then we have that $x \notin clup(D_j)$ for any $j \geq j(x)$, implying that $x \notin \bigcap_{j=1}^{\infty} clup(D_j)$. Therefore, we have $\bigcap_{j=1}^{\infty} clup(D_j) \subseteq clup(Y \cap Z)$. However, because $clup(D_j) \supseteq clup(Y \cap Z)$ for any $j \in \mathbb{N}$, we obtain $\bigcap_{j=1}^{\infty} clup(D_j) = clup(Y \cap Z)$. Thus, we have $\lim_{j \to \infty} b(clup(D_j)) = b(clup(Y \cap Z))$.

Now, recall that, for each $j \in \mathbb{N}$,

$$clup(Y \cap Z) \subseteq clup(B_{2\delta_j}(\gamma_j)) \subseteq clup(D_j),$$

and thus,

$$b(clup(B_{2\delta_j}(\gamma_j))) \in [b(clup(Y \cap Z)), b(clup(D_j))],$$
$$b'(clup(B_{2\delta_j}(\gamma_j))) \in [b'(clup(Y \cap Z)), b'(clup(D_j))].$$

Regarding b', first observe that

$$\lim_{j \to \infty} b'(clup(B_{2\delta_j}(\gamma_j))) = b'(clup(Y \cap Z)).$$

Thus, by our hypothesis, there must exist $J \in \mathbb{N}$ such that $b'(clup(B_{2\delta_J}(\gamma_J))) \leq b'(clup(Y \cap Z)) + \varepsilon$. Define $\gamma = \gamma_J \in F(X_0)$ and $q \in \mathbb{Q}_+$ such that $q \in (0, 2\delta_J]$. Then, we deduce the following implication:

$$\begin{aligned}\beta^*(\gamma, q|b') &= b'(clup(B_q(\gamma))) \text{ (by the optimality of } \beta^* \text{ given } b') \\ &\leq b'(clup(Y \cap Z)) + \varepsilon \text{ (by our hypothesized inequality)} \\ &\leq b'(clup(Y)) + \varepsilon \ (\because \ Y \cap Z \subseteq Y) \\ &= b'(Y) + \varepsilon \ (\because \ Y \text{ is a closed upper set}).\end{aligned}$$

Regarding b, we have

$$\begin{aligned}\beta^*(\gamma, q|b) &= b(clup(B_q(\gamma))) \text{ (by the optimality of } \beta^* \text{ given } b) \\ &\geq b(clup(Y \cap Z)) \ (\because \ clup(Y \cap Z) \subseteq clup(B_q(\gamma))) \\ &\geq b(Y) - \varepsilon,\end{aligned}$$

where the last inequality is obtained because

$$\begin{aligned}b(Y) &= b(Y \cap Z) + b(Y \setminus Z) \ (\because \ b \text{ is a probability measure}) \\ &\leq b(clup(Y \cap Z)) + \varepsilon \\ &\quad (\because \ Y \cap Z \subseteq clup(Y \cap Z) \text{ and } b(Z) \geq 1 - \varepsilon \Rightarrow b(Y \backslash Z) \leq \varepsilon).\end{aligned}$$

Because $0 < \varepsilon < (b(Y) - b'(Y))/2$, we conclude that $\beta^*(\gamma, q|b) > \beta^*(\gamma, q|b')$. □

With this lemma, we complete the proof of Proposition 4. □

A.2 Properties of $\mathcal{B}$

Recall the definition of $\mathcal{B} = \{\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0,1] | \ \beta \text{ is nonderecasing}\}$. Let $\overline{\mathcal{B}} = \{\beta : F(\Theta_0) \times \mathbb{Q}_+ \to [0,1]\}$ be the superset of $\mathcal{B}$ in which we only drop the property that β is non-decreasing from $\mathcal{B}$. In this subsection, we first introduce a metric for $\overline{\mathcal{B}}$, inducing a topology with respect to which $\overline{\mathcal{B}}$ is shown to be a compact metric space. Second, we claim that $\mathcal{B}$ is a closed subset of $\overline{\mathcal{B}}$ so that $\mathcal{B}$ is also a compact metric space. Note that every compact metric space is complete and separable. Thus, $\mathcal{B}$ has a countable dense subset $\mathcal{B}_0$. Finally, we will establish that $\mathcal{B}$ possesses its closed partial order, and satisfies Assumptions 1 and 2 (with replacement of X by $\mathcal{B}$ and X_0 with $\mathcal{B}_0$ in the statements). These properties are exploited in the next section when we consider the multi-player case.

First, we introduce a norm over $\overline{\mathcal{B}}$ to make it a subspace of a normed space (and accordingly, its metric is induced by this norm).[42] For each $\beta \in \overline{\mathcal{B}}$, its norm is given by

$$\|\beta\| = \sum_{(\gamma,q)\in F(\Theta_0)\times\mathbb{Q}_+} |\beta(\gamma,q)|\mu(\gamma,q),$$

where μ is a full-support probability distribution over $F(\Theta_0)\times\mathbb{Q}_+$ such that we set $h: F(\Theta_0)\times\mathbb{Q}_+ \to \mathbb{N}$ as an injection map and $\mu(\gamma,q) = (1/2)^{h(\gamma,q)}$ for each $(\gamma,q) \in F(\Theta_0)\times\mathbb{Q}$. Because $\beta(\gamma,q)\in[0,1]$ for any (γ,q), we have $\|\beta\|\in[0,1]$ for any $\beta\in\overline{\mathcal{B}}$.

Lemma 3. $\mathcal{B}$ is a compact metric space.

Proof. Since $\overline{\mathcal{B}}$ is made isomorphic to Hilbert cube, we confirm that $\overline{\mathcal{B}}$ is a compact metric space. Thus, it suffices to show that $\mathcal{B}$ is a closed subset of $\overline{\mathcal{B}}$. Therefore, our task here reduces to showing that $\overline{\mathcal{B}}\backslash\mathcal{B}$ is open. Fix $\beta\in\overline{\mathcal{B}}\backslash\mathcal{B}$ arbitrarily. Then, we know that there exist $(\gamma',q'),(\gamma'',q'')\in F(X_0)\times\mathbb{Q}_+$ such that $clup(B_{q'}(\gamma'))\subseteq clup(B_{q''}(\gamma''))$ and $\beta(\gamma',q') > \beta(\gamma'',q'')$. What we want to show is that there exists an open ball containing β that does not intersect with $\mathcal{B}$.

Define

$$\varepsilon = (\beta(\gamma',q') - \beta(\gamma'',q''))\min\{\mu(\gamma',q'),\mu(\gamma'',q'')\}.$$

By our hypothesis, we have $\varepsilon > 0$. It then suffices to show that an open ball $B_\varepsilon(\beta) = \{\beta'\in\overline{\mathcal{B}} : \|\beta-\beta'\| < \varepsilon\}$ does not intersect with $\mathcal{B}$. Suppose, on the contrary, that there is $\beta'\in B_\varepsilon(\beta)\cap\mathcal{B}$. Then,

$$\begin{aligned}
\|\beta-\beta'\| &= \sum_{(\gamma,q)\in F(X_0)\times\mathbb{Q}_+} |\beta(\gamma,q)-\beta'(\gamma,q)|\mu(\gamma,q) \\
&\geq |\beta(\gamma',q')-\beta'(\gamma',q')|\mu(\gamma',q') + |\beta(\gamma'',q'')-\beta'(\gamma'',q'')|\mu(\gamma'',q'') \\
&= |\beta(\gamma',q')-\beta'(\gamma',q')|\mu(\gamma',q') + |\beta'(\gamma'',q'')-\beta(\gamma'',q'')|\mu(\gamma'',q'') \\
&\geq \Big\{|\beta(\gamma',q')-\beta'(\gamma',q')| + |\beta'(\gamma'',q'')-\beta(\gamma'',q'')|\Big\}\min\{\mu(\gamma',q'),\mu(\gamma'',q'')\} \\
&\geq |\beta(\gamma',q')-\beta'(\gamma',q')+\beta'(\gamma'',q'')-\beta(\gamma'',q'')|\min\{\mu(\gamma',q'),\mu(\gamma'',q'')\} \\
&\geq (\beta(\gamma',q')-\beta(\gamma'',q''))\min\{\mu(\gamma',q'),\mu(\gamma'',q'')\} \\
&\quad (\because\ \beta'\in\mathcal{B} \text{ and } clup(B_{q'}(\gamma'))\subseteq clup(B_{q''}(\gamma''))\Rightarrow \beta'(\gamma',q')\leq\beta'(\gamma'',q'')) \\
&= \varepsilon,
\end{aligned}$$

which contradicts that $\|\beta-\beta'\| < \varepsilon$. □

[42] A standard topology for the set of probability distributions is a weak* topology (e.g., Brandenburger and Dekel (1993)), but note that $\mathcal{B}$ is not a set of probability distributions. In particular, some $\beta\in\mathcal{B}$ does not necessarily correspond to any probability measure over Θ. The norm above and its induced topology on $\mathcal{B}$ are well-defined despite this "non-probabilistic" nature of $\mathcal{B}$. The same comment applies when we discuss the objects like $\mathcal{B}^m$ and $\mathcal{B}^\infty$ later in Section B.

Remark 3. The lemma implies that $\mathcal{B}$ is a separable and complete metric space.

Next, we show that $\mathcal{B}$ satisfies Assumption 1. First, for each $K \in \mathbb{N}$, we define $\mathcal{B}^{1,K} \subseteq \mathcal{B}$ as follows: $\beta \in \mathcal{B}^{1,K}$ if and only if there exists a K-element subset of Θ_0, say $X_K = \{x_1 \dots, x_K\} \in F(\Theta_0)$, such that for any $(\gamma, q) \in F(\Theta_0) \times \mathbb{Q}_+$, we have

$$\beta(\gamma, q) = \begin{cases} \min_{q' \in Q_K} \ \beta(X_K \cap \gamma, q') \text{ sub. to } B_{q'}(X_K \cap \gamma) \supseteq B_q(\gamma) & \text{if} \quad X_K \cap \gamma \neq \emptyset, \\ 1 & \text{if} \quad X_K \cap \gamma = \emptyset. \end{cases}$$

where $Q_K = \{k/K | k = 0, 1, \dots, K\}$.

Note that such β is fully identified by $(\beta(\tilde{X}, q))_{\tilde{X} \subseteq X_K, q \in Q_K}$. This implies that $\mathcal{B}^{1,K}$ contains countably many elements, and thus $\mathcal{B}_0 = \bigcup_{K \in \mathbb{N}} \mathcal{B}^{1,K}$ contains countably many elements. The next lemma shows that Assumption 1 is satisfied for $\mathcal{B}$, where in the statement, X is replaced by $\mathcal{B}$ and X_0 is replaced by $\mathcal{B}_0$.

Lemma 4. For any $\beta \in \mathcal{B}$ and $\varepsilon > 0$, there exists $\beta_0 \in \mathcal{B}_0$ such that $\|\beta_0 - \beta\| < \varepsilon$ and $\beta_0 \geq \beta$.

Proof. Fix $\beta \in \mathcal{B}$ and $\varepsilon > 0$. For each $N \in \mathbb{N}$, let $\Gamma_N = \bigcup_{h(\gamma,q) \leq N} \gamma$ (recall that $h : F(X_0) \times \mathbb{Q}_+ \to \mathbb{N}$ is an injection). Because each γ is a finite subset of X_0, so is Γ_N. Hence, we denote Γ_N by $\{x_1, \dots, x_{|\Gamma_N|}\}$.

We first construct $\beta_0^N \in \mathcal{B}_0$ as an approximation of $\beta \in \mathcal{B}$ such that β_0^N approaches β as $N \to \infty$.[43] For each $\tilde{X} \subseteq \Gamma_N$ and $q \in Q_N$, we set $n \in \mathbb{N}$ with the following three properties: (i) $\beta(\tilde{X}, q) \in ((n-1)/N, n/N]$; (ii) $\beta_0^N(\tilde{X}, q) = n/N$; and (iii) for each $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$,

$$\begin{aligned} \beta_0^N(\gamma, q) \quad = \quad & \inf_{q' \in Q_N} \quad \beta_0^N(\Gamma_N \cap \gamma, q') \\ & \text{subject to} \quad B_{q'}(\Gamma_N \cap \gamma) \supseteq B_q(\gamma). \end{aligned}$$

Second, we have that $\beta_0^N \in \mathcal{B}_0^1$ because $\beta_0^N(\gamma, q) \in Q_N$ and β_0^N is nondecreasing. Third, we claim that $\beta_0^N \geq \beta$. For any $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$, we have

$$\begin{aligned} \beta_0^N(\gamma, q) \quad \geq \quad & \inf_{q' \in Q_N} \quad \beta(\Gamma_N \cap \gamma, q') \\ & \text{subject to} \quad B_{q'}(\Gamma_N \cap \gamma) \supseteq B_q(\gamma), \end{aligned}$$

while, because β is nondecreasing, we have

$$\beta(\gamma, q) \leq \beta(\Gamma_N \cap \gamma, q'),$$

[43] To be precise, $\beta_0^N \in \mathcal{B}^{1,M}$, where $M = N \times |\Gamma_N|$.

for any $q' \in Q_N$ satisfying $B_{q'}(\Gamma_N \cap \gamma) \supseteq B_q(\gamma)$. The above two inequalities together imply $\beta_0^N(\gamma, q) \geq \beta(\gamma, q)$.

Finally, we show that there exists $N \in \mathbb{N}$ such that $\|\beta - \beta_0^N\| < \varepsilon$. For each $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$, whenever $h(\gamma, q) \leq |\Gamma_N|$, we have $\gamma \subseteq \Gamma_N$, and hence, $0 \leq (\beta_0^N(\gamma, q) - \beta(\gamma, q)) \leq 1/N$. Thus,

$$\begin{aligned} \|\beta - \beta_0^N\| &\leq \frac{1}{N} + \sum_{n=N+1}^{\infty} \mu(h^{-1}(n)) \\ &= \frac{1}{N} + \frac{1}{2^N} \quad (\because\ \mu(\gamma, q) = (1/2)^{h(\gamma,q)}). \end{aligned}$$

By taking N large enough so that $N > \max\{2/\varepsilon, 1+\log_2(1/\varepsilon)\}$, we obtain $\|\beta-\beta_0^N\| < \varepsilon$. This completes the proof. $\square$

The next lemma shows that Assumption 2 is also satisfied for $\mathcal{B}$.

Lemma 5. For each $C \subseteq \mathcal{B}$ and $\varepsilon > 0$, there exists $\delta(C, \varepsilon) > 0$ such that, for any $D \subseteq \mathcal{B}$ with $d(C, D) < \delta(C, \varepsilon)$, we have $d(up(C), up(D)) < \varepsilon$.

Proof. Fix $C \subseteq \mathcal{B}$, $\varepsilon > 0$, and $D \subseteq \mathcal{B}$ with $d(C, D) < \varepsilon$. We show that $d(up(C), up(D)) < \varepsilon$ (i.e., we show that $\delta(C, \varepsilon) = \varepsilon$ works for any C).

Take any $\beta \in C$ and $\beta' \geq \beta$. Because $d(C, D) < \varepsilon$, there exists $\beta'' \in D$ such that $d(\beta, \beta'') < \varepsilon$.

Let $\beta^* = \beta' \vee \beta'' \in up(D)$. Then we have

$$\begin{aligned} d(\beta^*, \beta') &= \|\beta^* - \beta'\| \\ &= \sum_{(\gamma,q)} (\beta^*(\gamma, q) - \beta'(\gamma, q))\mu(\gamma, q) \\ &= \sum_{(\gamma,q)|\beta'(\gamma,q)<\beta''(\gamma,q)} (\beta''(\gamma, q) - \beta'(\gamma, q))\mu(\gamma, q) \\ &\leq \sum_{(\gamma,q)|\beta'(\gamma,q)<\beta''(\gamma,q)} (\beta''(\gamma, q) - \beta(\gamma, q))\mu(\gamma, q) \\ &\leq \sum_{(\gamma,q)} |\beta''(\gamma, q) - \beta(\gamma, q)|\mu(\gamma, q) \\ &= d(\beta, \beta'') \\ &< \varepsilon. \end{aligned}$$

By a symmetric argument, taking any $\beta \in D$ and $\beta' \geq \beta$, there exists $\beta'' \in C$ such that $d(\beta, \beta'') < \varepsilon$, and we have $d(\beta' \vee \beta'', \beta') < \varepsilon$.

Therefore, we conclude that $d(up(C), up(D)) < \varepsilon$. $\square$

Finally, we show that the partial order on $\mathcal{B}$ is a closed partial order.

Lemma 6. Let $\mathcal{B}$ be endowed with a natural product order $\geq_{\mathcal{B}}$. Then, $\geq_{\mathcal{B}}$ is a closed order.

Proof. Consider two sequences $\{\beta_n\}$ and $\{\beta'_n\}$ in $\mathcal{B}$, such that $\beta_n \to \beta$ and $\beta'_n \to \beta'$ as $n \to \infty$. Then, due to the continuity of β and $\beta^{'}$, for each $(\gamma, q) \in F(X_0) \times \mathbb{Q}_+$, we have $\beta_n(\gamma, q) \to \beta(\gamma, q)$ and $\beta'_n(\gamma, q) \to \beta'(\gamma, q)$ as $n \to \infty$. Now suppose that $\beta_n(\gamma, q) \geq_{\mathcal{B}} \beta'_n(\gamma, q)$ for any n. Then, we must have $\beta(\gamma, q) \geq_{\mathcal{B}} \beta'(\gamma, q)$ for each (γ, q). This means that the partial order on $\mathcal{B}$ is a closed partial order. □

B Necessity for Monotone Comparative Statics: The Multi-Player Case

With multiple agents, we need an optimism-elicitation game where the equilibrium reflects each player's belief hierarchy (not only his first-order belief). Although one may think that the situation becomes prohibitively more complicated, we show in this section that the same technique as in the single-person case can be extended appropriately.

The goal of this section is to construct a (multi-player) supermodular game such that common optimism holds if and only if monotone comparative statics holds in this game (Theorem 2 in Section B.2). The crucial step lies in the construction of each player's action space where each player bets not only on the realization of $\theta \in \Theta$ but also on each other's betting behavior, reflecting his higher-order beliefs (Section B.1).

B.1 Preliminary

Let $X^1 = \Theta$, $X^1_0 = \Theta_0$, and $\mathcal{B}^1 = \mathcal{B}$. For $m \geq 2$, we inductively construct supermodular games where each player's m-th order belief is relevant. Specifically, for $m \geq 2$, assume that (i) X^{m-1} is a separable, complete metric space with a countable dense subset X^{m-1}_0, (ii) X^{m-1} satisfies Assumptions 1 and 2 (with replacement of X by X^{m-1} and X_0 by X^{m-1}_0 in the statements) with the corresponding closed partial order, (iii) $\mathcal{B}^{m-1}$ is a compact metric space with a countable dense subset $\mathcal{B}^{m-1}_0$, and (iv) $\mathcal{B}^{m-1}$ satisfies Assumptions 1 and 2 (with replacement of X by $\mathcal{B}^{m-1}$ and X_0 by $\mathcal{B}^{m-1}_0$ in the statements) with the corresponding closed partial order.[44]

[44] Note that the corresponding partial orders vary across players. Although those sets, formally speaking, vary with i, we omit the i-subscript for brevity. The same comment applies to $X^m, \mathcal{B}^m, X^\infty$, and $\mathcal{B}^\infty$ below.

We define $X^m = X^{m-1} \times (\mathcal{B}^{m-1})^{I-1}$, endowed with the product topology and closed partial order. Because both X^{m-1} and $\mathcal{B}^{m-1}$ are separable, complete metric spaces and satisfy Assumptions 1 and 2, X^m also satisfies the same properties:

Lemma 7. *X^m is a separable, complete metric space with a countable dense subset X_0^m such that Assumptions 1 and 2 are satisfied with replacement of X by X^m and X_0 by X_0^m in the statements.*

Next, we define

$$\mathcal{B}^m = \left\{ \beta : F(X_0^m) \times \mathbb{Q}_+ \to [0,1] \,\middle|\, \beta \text{ is nonderecasing } \right\}.$$

Then, applying the same logic in Section A.2, we obtain the following (the proof omitted):

Lemma 8. *$\mathcal{B}^m$ is a compact metric space with a countable dense subset $\mathcal{B}_0^m$ such that Assumptions 1 and 2 are satisfied with replacement of X by $\mathcal{B}^m$ and X_0 by $\mathcal{B}_0^m$ in the statements.*

Therefore, for any $m \geq 1$, properties (i)-(iv) are satisfied: (i) X^m is a separable, complete metric space with its countable dense subset X_0^m, (ii) X^m satisfies Assumptions 1 and 2 (with replacement of X by X^m and X_0 by X_0^m in the statements) with the corresponding closed partial order, (iii) $\mathcal{B}^m$ is a compact metric space with a countable dense subset $\mathcal{B}_0^m$, and (iv) $\mathcal{B}^m$ satisfies Assumptions 1 and 2 (with replacement of X by $\mathcal{B}^m$ and X_0 by $\mathcal{B}_0^m$ in the statements) with the corresponding closed partial order.

Finally, let $X^\infty = \prod_{m=1}^\infty X^m$. Then, we obtain the analogous properties for X^∞.

Lemma 9. *X^∞ is a separable, complete metric space with a countable dense subset X_0^∞ such that Assumptions 1 and 2 are satisfied with replacement of X by X^∞ and X_0 by X_0^∞ in the statements.*

Similarly, let

$$\mathcal{B}^\infty = \left\{ \beta : F(X_0^\infty) \times \mathbb{Q}_+ \to [0,1] \,\middle|\, \beta \text{ is nonderecasing } \right\},$$

and we obtain the following analogous properties for $\mathcal{B}^\infty$. Its proof is omitted.

Lemma 10. *$\mathcal{B}^\infty$ is a compact metric space with a countable dense subset $\mathcal{B}_0^\infty$ such that Assumptions 1 and 2 are satisfied with replacement of X by $\mathcal{B}^\infty$ and X_0 by $\mathcal{B}_0^\infty$ in the statements.*

B.2 Optimism-Elicitation Game: The Multi-Player Case

Now we show that the necessity of the CO order for monotone comparative statics.

Theorem 2 There is a supermodular game with the property that, for any player $i \in I$ and two types $t_i, t_i^{'}$, we have that $t_i^{'} \succeq_{CO} t_i$ if and only if $\underline{\sigma}_i(t_i^{'}) \succeq_{A_i} \underline{\sigma}_i(t_i)$ and $\bar{\sigma}_i(t_i^{'}) \succeq_{A_i} \bar{\sigma}_i(t_i)$, where $\underline{\sigma}$ is the least and $\bar{\sigma}$ is the greatest equilibrium of this supermodular game, respectively.[45]

Proof. We construct an optimism-elicitation game such that: (i) each player i chooses an action from $\mathcal{B}_i = \mathcal{B}^\infty$ and (ii) given any realization $x \in X^\infty$ and action $\beta \in \mathcal{B}_i$, each player i's payoff is given:

$$u_i(\beta, x) = \sum_{(\gamma,q)\in F(X_0^\infty)\times\mathbb{Q}_+} \left[\sum_{m=1}^{\infty} \delta^{m-1} \left\{\beta(\gamma,q) I^m_{\{clup(B_q(\gamma))\}}(x) - \frac{\beta(\gamma,q)^2}{2}\right\}\right] \mu(\gamma,q),$$

where

- $0 < \delta < 1$;
- $B_q(\gamma) = \bigcup_{y\in\gamma} B_q(y)$;
- μ is a full-support distribution over a countable set $F(X_0^\infty) \times \mathbb{Q}_+$[46]; and
- The indicator function is defined as:

$$I^m_{\{clup(B_q(\gamma))\}}(x) = \begin{cases} 1 & \text{if } x^m \in clup(B_q(\gamma)) \cap X^m \\ 0 & \text{otherwise,} \end{cases}$$

  where x^m denotes the truncation of x to X^m.

We can establish the following result by mimicking the argument for the case of single-person optimism-elicitation game. So, we only state the result.

Lemma 11. We obtain the following results:

1. $\mathcal{B}_i$ is a complete lattice;
2. $u(\cdot)$ is supermodular on $\mathcal{B}_i$; and
3. $u(\cdot)$ has increasing difference in (β, x).

[45] To be more precise, what we show here is that for any parameter space Θ, there is a supermodular game based on Θ with the desired property.

[46] We can set $h : F(X_0^\infty) \times \mathbb{Q}_+ \to \mathbb{N}$ as an injection mapping because $F(X_0^\infty) \times \mathbb{Q}_+$ is countable. Specifically, we define the full-support distribution μ by $\mu(\gamma, q) = (1/2)^{h(\gamma,q)} > 0$.

Therefore, the game constructed above is indeed a supermodular game. The proposition below shows that player i reveals his probability assessment for each upper event (those generated by $F(X_0^\infty \times \mathbb{Q}_+)$) truthfully in this game, as his unique interim correlated rationalizable (ICR) action.

Proposition 7. For each player i with type t_i, we have $A_i^\infty(t_i) = \{\beta^*\}$, where for each $m \in \mathbb{N}$ and each $(\gamma, q) \in F(X_0^m) \times \mathbb{Q}_+$, we have

$$\beta^*(\gamma, q) = h^m(t_i)[clup(B_q(\gamma))],$$

where $h^m(t_i)$ is type t_i's probabilistic belief on X^m.

Proof. Fix $m = 1$. Then, player i effectively plays a single-person game in which he reveals his first-order belief only. Suppose by way of contradiction that there is $\beta \in \mathcal{B}_i$ such that $\beta(\hat{\gamma}, \hat{q}) \neq h^1(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$ for some $(\hat{\gamma}, \hat{q}) \in F(X_0^1) \times \mathbb{Q}_+$. Then, β is strictly dominated by another $\beta' \in \mathcal{B}_i$, where $\beta'(\gamma, q) = \beta(\gamma, q)$ for any $(\gamma, q) \neq (\hat{\gamma}, \hat{q})$ and $\beta'(\hat{\gamma}, \hat{q}) = h^1(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$. Note that such β' is feasible because we impose no coherency condition among across different orders of beliefs. Thus, β^* must satisfy the truth-telling condition:

$$\beta^*(\gamma, q) = h^1(t_i)[clup(B_q(\gamma))].$$

The rest of the proof is completed by induction. Fix $m \geq 2$, and assume that, up to $(m-1)$th order, each type of each agent behaves truthfully. Assume by way of contradiction that there is an action $\beta \in \mathcal{B}_i$ such that $\beta(\hat{\gamma}, \hat{q}) \neq h^m(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$ for some $(\hat{\gamma}, \hat{q}) \in F(X_0^m) \times \mathbb{Q}_+$. Then, β is strictly dominated by another $\beta' \in \mathcal{B}_i$, where $\beta'(\gamma, q) = \beta(\gamma, q)$ for any $(\gamma, q) \neq (\hat{\gamma}, \hat{q})$ and $\beta'(\hat{\gamma}, \hat{q}) = h^m(t_i)[clup(B_{\hat{q}}(\hat{\gamma}))]$. Again, such β' is feasible because we impose no coherency condition among across different orders of beliefs. Therefore, for any m and $(\gamma, q) \in F(X_0^m) \times \mathbb{Q}_+$ we have

$$\beta^*(\gamma, q) = h^m(t_i)[clup(B_q(\gamma))].$$

□

This means that "any" interim correlated rationalizable strategy of each player i induces his true belief about any upper event $U(X^\infty)$. We now examine monotone comparative statics for this supermodular game. The first result establishes the sufficiency for monotone comparative statics in this supermodular game (as should be expected).

Proposition 8. For any $i \in I$ and types t_i, t_i' such that $t_i' \succeq_{CO} t_i$, we have $\beta' \succeq_{\mathcal{B}_i} \beta$ where β and β' satisfy $A_i^\infty(t_i) = \{\beta\}$ and $A_i^\infty(t_i') = \{\beta'\}$, respectively.

Proof. Formally, this is a corollary to Theorem 1. However, the proof is straightforward once we notice that, by Proposition 5, for each $m \in \mathbb{N}$ and $(\gamma, q) \in F(X_0^\infty) \times \mathbb{Q}_+$, we obtain

$$\beta(\gamma, q) = h^m(t_i)[clup(B_q(\gamma))] \text{ and } \beta'(\gamma, q) = h^m(t_i')[clup(B_q(\gamma))].$$

□

Next, we establish the desired necessity for monotone comparative statics in this supermodular game.

Proposition 9. For any $i \in I$ and types t_i, t_i' such that $A_i^\infty(t_i) = \{\beta\}$ and $A_i^\infty(t_i') = \{\beta'\}$, if $\beta \succeq_{\mathcal{B}_i} \beta'$, then $t_i' \succeq_{CO} t_i$.

Proof. We take the contrapositive of the statement: if there is some $m \in \mathbb{N}$ such that $h^m(t_i')$ does "not" stochastically dominate $h^m(t_i)$ (so that $h^m(t_i)[Y] > h^m(t_i')[Y]$ for some closed upper set $Y \subseteq X^m$), then β' "cannot" be higher than β in the sense of the partial order on $\mathcal{B}_i = \mathcal{B}^\infty$. This can be shown quite analogously as in Lemma 2, by replacing (i) X by X^m; (ii) X_0 by X_0^m; (iii) b with $h^m(t_i)$; and (iv) b' with $h^m(t_i')$, respectively. This completes the proof. □

Propositions 6 and 7 together complete the proof of Theorem 2. □

C Proof of Proposition 1

Proposition 1: Let $b, b' \in \Delta(X)$. b (first-order) stochastically dominates b' if and only if, for any $Y_0 \subseteq X_0$, $b(clup(Y_0)) \geq b'(clup(Y_0))$. In addition, b strictly stochastically dominates b' if and only if the inequality holds for any $Y_0 \subseteq X_0$ and it is strict at least for some $Y_0 \subseteq X_0$.

Proof. ($\Leftarrow$) First, suppose that b does not stochastically dominate b'. Then, there exists a closed upper set Y such that $b(Y) < b'(Y)$. We show that, in such a case, there exists $Y_0 \subseteq X_0$ such that $clup(Y_0) = Y$. Then this implies that $b(clup(Y_0)) < b'(clup(Y_0))$. To show this, we establish the following result:

Lemma 12. For any $Y \subseteq X$, $up(Y) \cap X_0$ is dense in $up(Y)$, i.e., the closure of $up(Y) \cap X_0$ is $up(Y)$. In particular, if Y is itself an upper set, then $Y \cap X_0$ is dense in Y.

Proof. Fix $Y \subseteq X$. The lemma is trivially true if $up(Y)$ is empty. So let us assume not. Let $y \in up(Y)$. Then, by Assumption 1, for any $\varepsilon > 0$, there is $x \in X_0$ such that $x \geq y$ (and hence $x \in up(Y)$) and $x \in B_\varepsilon(y)$. This shows that $up(Y)$ is dense. □

($\Rightarrow$) Next, suppose that b stochastically dominates b'. Fix $Y_0 \subseteq X_0$. If $clup(Y_0)$ is a closed upper set, then we have $b(clup(Y_0)) \geq b'(clup(Y_0))$ by the previous lemma. Since $clup(Y_0)$ is closed by definition, it remains to show in the next lemma that $clup(Y_0)$ is an upper set, which completes the proof. In fact, we can further show that, for any $Y \subseteq X$ (not only for any $Y_0 \subseteq X_0$), $clup(Y)$ is a closed upper set, which turns out to be useful later.

Lemma 13. For any $Y \subseteq X$, $clup(Y)$ is a closed upper set.

Proof. Suppose, on the contrary, that $clup(Y)$ is not an upper set. Then, there exist $x \in clup(Y)$ and $y \geq x$ such that $y \notin clup(Y)$. Since $clup(Y)$ is closed, one can find $\varepsilon > 0$ such that $d(y, clup(Y)) \geq \varepsilon$.

By the previous lemma, $up(Y) \cap X_0$ is dense in $up(Y)$, and hence, $up(Y) \cap X_0$ is dense in $clup(Y)$. Thus, for any $\delta > 0$, there is $z \in up(Y) \cap X_0$ such that $d(x, z) < \delta$. By Assumption 2, we can set $\delta = \delta(x, \varepsilon)$ so that we have $d(up(x), up(z)) < \varepsilon$. This contradicts our hypothesis that $d(y, clup(Y)) \geq \varepsilon$ because we can deduce the following implication:

$$\begin{aligned}
\varepsilon &\leq d(y, clup(Y)) \\
&= \inf_{y_0 \in clup(Y)} d(y, y_0) \\
&\leq \inf_{y_0 \in up(z)} d(y, y_0) \ \ (\because \ up(z) \subseteq clup(Y)) \\
&= d(y, up(z)) \ \ \text{(due to the definition of the Hausdorff metric)} \\
&\leq \sup_{y' \in up(x)} d(y', up(z)) \ \ (\because \ y \in up(x)) \\
&\leq d(up(x), up(z)) \ \ \text{(due to the definition of the Hausdorff metric)} \\
&< \varepsilon,
\end{aligned}$$

a contradiction. This completes the proof of Lemma 13. □

With Lemmas 12 and 13, we thus complete the proof of Proposition 1.

□

D Proof of Proposition 3

We need a separate proof for this result because Milgrom and Stokey (1982) assume that the agents write contracts contingent on θ, which is not assumed here.

Proof. Suppose, on the contrary, that the ex ante expected volume of trade is strictly positive for some equilibrium σ. In what follows, we only consider the case where σ is a pure strategy equilibrium.

Let $\hat{T}_i \subseteq T_i$ denote the set of all types of player i who choose a positive volume of trade. Focus on player $i = 1$. For any $t_1 \in \hat{T}_1$, we have

$$E[-\exp\left(-\theta_1 !!\sigma_1(t_1)\sigma_2(t_2)!!\right)|\theta \in \Theta,\ t \in \{t_1\} \times \hat{T}_2] \geq -\exp(0) = -1.$$

!!Takashi's comment: We left the old specification for the volume of trade involving the minimum operator. Here I simply replace it with the product of two agents' demand for trade.!! Recall that each type t_i is never certain of the true value of θ_i so that there remains the residual uncertainty of his ex post payoff, even conditional on t_i. Since $-\exp(-x)$ is strictly concave, we have

$$\begin{aligned} & E[-\exp\left(-\theta_1 !!\sigma_1(t_1)\sigma_2(t_2)!!\right)|\theta \in \Theta,\ t \in \{t_1\} \times \hat{T}_2] \\ = \ & E\left[E[-\exp\left(-\theta_1 !!\sigma_1(t_1)\sigma_2(t_2)!!\right)|t_1, t_2]\Big|\theta \in \Theta, t \in \{t_1\} \times \hat{T}_2\right] \\ < \ & E\left[-\exp\left(-E[\theta_1 !!\sigma_1(t_1)\sigma_2(t_2)!!]|t_1, t_2]\right)\Big|\theta \in \Theta, t \in \{t_1\} \times \hat{T}_2\right], \end{aligned}$$

where the equality comes from the law of iterated expectation and the strict inequality comes from Jensen's inequality. Therefore, we obtain

$$E\left[-\exp\left(-E[\theta_1 !!\sigma_1(t_1)\sigma_2(t_2)!!]|t_1, t_2]\right)\Big|\theta \in \Theta, t \in \{t_1\} \times \hat{T}_2\right] > -1.$$

This further implies

$$E\left[-\exp\left(-E[\theta_1 \min\{\sigma_1(t_1), \sigma_2(t_2)\}]|t_1, t_2]\right)\Big|\theta \in \Theta, t \in \hat{T}_1 \times \hat{T}_2\right] > -1.$$

To ease the complexity of the derivation to follow, we introduce the following pieces of notation:

$$\begin{aligned} y_1(\theta, t) &= \theta_1 !!\sigma_1(t_1)\sigma_2(t_2)!! \\ y_2(\theta, t) &= \theta_2 !!\sigma_1(t_1)\sigma_2(t_2)!! \\ f(x) &= -\exp(-x). \end{aligned}$$

We rewrite the inequalities we obtained above as follows:

$$\begin{aligned} E\left[f(y_1(\theta, t)))\Big|\theta \in \Theta, t \in \hat{T}_1 \times \hat{T}_2\right] > f(0), \\ E\left[f(y_2(\theta, t))\Big|\theta \in \Theta, t \in \hat{T}_1 \times \hat{T}_2\right] > f(0). \end{aligned}$$

Therefore, we also have

$$E\left[\frac{f(y_1(\theta,t))+f(y_2(\theta,t))]}{2}\Big|\theta\in\Theta, t\in\hat{T}_1\times\hat{T}_2\right] > f(0).$$

By Jensen's inequality,

$$\begin{aligned} E\left[\frac{f(y_1(\theta,t))+f(y_2(\theta,t))]}{2}\Big|\theta\in\Theta, t\in\hat{T}_1\times\hat{T}_2\right] &< f\left(E\left[\frac{y_1(\theta,t)+y_2(\theta,t)]}{2}\Big|\theta\in\Theta, t\in\hat{T}_1\times\hat{T}_2\right]\right) \\ &= f(0), \end{aligned}$$

where the last equality follows from the pure-common value assumption:

$$E\left[\frac{y_1(\theta,t)+y_2(\theta,t)]}{2}\Big|\theta\in\Theta, t\in\hat{T}_1\times\hat{T}_2\right] = E\left[\frac{\theta_1+\theta_2}{2}\Big|\theta\in\Theta, t\in\hat{T}_1\times\hat{T}_2\right] = 0.$$

This is a contradiction. □

E An Example Omitted from Section 7

Example 4. Consider a technology adoption game as in VZV (Example 1, pp.344-346), but for simplicity, assume the following:

- there are two players, $I = \{1, 2\}$;
- each player i has binary actions: either to "adopt a new technology" ($a_i = 1$) or "not" ($a_i = 0$); and
- each player i's payoff is: $u_i(a_i, a_{-i}, \theta) = \theta a_i$, where $\theta \in \{-1, 1\}$.

It is common knowledge that each state $\theta \in \{-1, 1\}$ occurs equally likely, and that each player i receives a signal $t_i \in \{-1, 1\}$ before taking his action, where $t = (t_1, t_2)$ conditional on θ is distributed as follows:

$\theta = -1$	$t_2 = -1$	$t_2 = 1$
$t_1 = -1$	1/3	1/3
$t_1 = 1$	1/3	0

$\theta = 1$	$t_2 = -1$	$t_2 = 1$
$t_1 = -1$	0	1/3
$t_1 = 1$	1/3	1/3

Notice that $t = (t_1, t_2)$ exhibits some type of negative correlation. For example, we have $\Pr(t_2 = 1|\theta = -1, t_1 = -1) = 1/3 > 0 = \Pr(t_2 = 1|\theta = -1, t_1 = 1)$, that is, the probability that $t_2 = 1$ conditional on $\theta = -1$ and t_1 is *decreasing* in t_1.

In this simple setup, any player i with type t_i has a strict dominant action, $a_i^D(t_i) = 1_{\{E[\theta|t_i]\geq 0\}}$, which identifies the unique equilibrium action. As a natural comparative statics, higher t_i plays a (weakly) higher equilibrium action.

Because of this observation, one may be tempted to imagine that the order given by VZV and our paper captures this monotone comparative statics. However, this is not the case. Due to the negative correlation of type profiles, only the trivial (or "empty") partial order is consistent with these orders. First, fix player $i \in \{1,2\}$ and two types $(t_i, t_i') = (1, -1)$. We compute each type's marginal distribution over $\Theta = \{-1, 1\}$ induced by π_i: for each θ,

$$\text{marg}_\Theta \pi_i(t_i)[\theta] = \begin{cases} 1/3 & \text{if } \theta = -1 \\ 2/3 & \text{if } \theta = 1 \end{cases} \quad \text{and} \quad \text{marg}_\Theta \pi_i(t_i')[\theta] = \begin{cases} 2/3 & \text{if } \theta = -1 \\ 1/3 & \text{if } \theta = 1 \end{cases}$$

This implies that $\text{marg}_\Theta \pi_i(t_i)$ first-order stochastically dominates $\text{marg}_\Theta \pi_i(t_i')$. Therefore, we conclude that $t_i' \not\succeq_i t_i$ where $\succeq_i$ may be interpreted either as our CO order or as that of VZV. We next compute player 1's belief π_1 over $\Theta \times T_2$: for each (θ, t_2),

$$\pi_1(t_1)[\theta, t_2] \;=\; \begin{cases} 1/3 & \text{if } \theta = -1 \text{ and } t_2 = -1 \\ 1/3 & \text{if } \theta = 1 \text{ and } t_2 = -1 \\ 1/3 & \text{if } \theta = 1 \text{ and } t_2 = 1 \end{cases}$$

$$\pi_1(t_1')[\theta, t_2] \;=\; \begin{cases} 1/3 & \text{if } \theta = -1 \text{ and } t_2 = -1 \\ 1/3 & \text{if } \theta = -1 \text{ and } t_2 = 1 \\ 1/3 & \text{if } \theta = 1 \text{ and } t_2 = 1 \end{cases}$$

However, in order to obtain $t_1 \succeq_1 t_1'$ for $(t_1, t_1') = (1, -1)$, we must have $\pi_1(t_1)$ first-order stochastically dominates $\pi_1(t_1')$. This implies that we must have $t_2'(= -1) \succeq_2 t_2(= 1)$, which contradicts the previously obtained conclusion that $t_i' \not\succeq_i t_i$ for each $i \in \{1, 2\}$.

This example suggests that, if a game of interest is fixed, in general, a strictly finer order than the partial orders suggested by us (and VZV) may be obtained. Note that this observation itself is not in conflict with our results, because our CO order is the richest partial order that induces monotone comparative statics if *all* supermodular games are considered.

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