9-2020

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Maximum Likelihood Estimation for the Fractional Vasicek Model

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Received: 9 November 2019; Accepted: 7 August 2020; Published: 12 August 2020

Abstract: This paper estimates the drift parameters in the fractional Vasicek model from a continuous record of observations via maximum likelihood (ML). The asymptotic theory for the ML estimates (MLE) is established in the stationary case, the explosive case, and the boundary case for the entire range of the Hurst parameter, providing a complete treatment of asymptotic analysis. It is shown that changing the sign of the persistence parameter changes the asymptotic theory for the MLE, including the rate of convergence and the limiting distribution. It is also found that the asymptotic theory depends on the value of the Hurst parameter.

Keywords: maximum likelihood estimate; fractional Vasicek model; asymptotic distribution; stationary process; explosive process; boundary process

JEL Classification: C15; C22; C32

1. Introduction

Since Vasicek (1977) introduced a model to describe the evolution of short-term interest rates, the so-called Vasicek model has enjoyed a wide range of applications. Jamshidian (1989) used it to price bond options. Scott (1987) used it to model the evolution of the instantaneous volatility of stock price and to price European call options.

Many extensions have been made to generalize the specification of Vasicek. For example, motivated by the phenomenon of long-range dependence found in the data of hydrology, geophysics, climatology, telecommunication, economics, and finance, the Brownian motion in the Vasicek model has been replaced by a fractional Brownian motion (fBm), leading to the following fractional Vasicek model (fVm):

$$dX_t = \kappa\left(\mu - X_t\right)dt + \sigma dB_t^H, \tag{1}$$

where σ is a positive constant, $\mu, \kappa \in \mathbb{R}$, B_t^H is an fBm with $H \in (0,1)$ being the Hurst parameter, and $X_0 = o_p(\sqrt{T})$ with o_p defined at the end of this section. An fBm B_t^H is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following covariance function:

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{2}$$

The process B_t^H is self-similar in the sense that $\forall a \in \mathbb{R}^+$, $B_{at}^H \overset{d}{=} a^H B_t^H$. It becomes the standard Brownian motion W_t when $H = 1/2$ and can be represented as a stochastic integral with respect to the standard Brownian motion. It is negatively correlated when $0 < H < 1/2$. When $1/2 < H < 1$, it has long-range dependence in the sense that $\sum_{n=1}^{\infty} \mathbb{E}\left(B_1^H(B_{n+1}^H - B_n^H)\right) = \infty$. In this case, the positive (negative) increments are likely to be followed by positive (negative) increments. The parameter H, which is also called the self-similarity parameter, measures the intensity of the long-range dependence.

Parameter κ is often referred to as the persistence parameter. When $\kappa > 0$, X_t is stationary and ergodic. In this case, μ is the unconditional mean of X_t and κ is the mean-reversion parameter. When $\kappa < 0$, X_t is explosive and hence non-ergodic. When $\kappa = 0$, X_t is the boundary case, and the drift term $\kappa(\mu - X_t)\,dt$ disappears. Therefore, μ is superfluous in this case. The ergodic fVm was used to model the evolution of instantaneous volatility in Comte and Renault (1998), the evolution of quadratic variation in Aït-Sahalia and Mancini (2008), and the evolution of realized variance in Gatheral et al. (2018). The explosive fVm was used to model the NASDAQ index in Lui et al. (2020). The explosive OU was used to model the log real estate price in Chen et al. (2017).

An alternative to and perhaps slightly more general specification than Model (1) is:

$$dX_t = (\alpha - \kappa X_t)\,dt + \sigma dB_t^H. \tag{3}$$

In Model (3), even when $\kappa = 0$, the drift term does not vanish, and it is αdt. This alternative specification for the drift term was used in Chan et al. (1992) and Yu and Phillips (2001). When α in (3) is known (without loss of generality, it is assumed to be zero), (3) becomes the fractional Ornstein–Uhlenbeck (fOU) process. A unique path-wise solution to the stochastic differential equation in (3) is:

$$X_t = e^{-\kappa t}X_0 + \frac{\alpha}{\kappa}\left(1 - e^{-\kappa t}\right) + \sigma\int_0^t e^{-\kappa(t-s)}dB_s^H, \tag{4}$$

where the stochastic integral, $\int_0^t e^{-\kappa(t-s)}dB_s^H$, is the path-wise Riemann–Stieltjes integral, and the solution is unique (Proposition A.1 in Cheridito et al. 2003).

Assuming that a continuous record of observations is available for X_t with $t \in [0, T]$, a number of studies have introduced methods to estimate κ and α (or μ) and developed asymptotic distributions for the proposed estimators under the scheme of $T \to \infty$.[1] When $H > 1/2$ and $\kappa > 0$, borrowing the idea of Hu and Nualart (2010) and Hu et al. (2019), Xiao and Yu (2019a) considered the ergodic-type estimates of κ and μ. Xiao and Yu (2019b) extended the results of Xiao and Yu (2019a) from the case where $H \in (1/2, 1)$ to where $H \in (0, 1/2)$. Assuming $\kappa > 0$ and $H > 1/2$, Nourdin and Tran (2019) extended the results of Xiao and Yu (2019a, 2019b) to a model where the fBm is replaced with a Hermite process. Using the Malliavin calculus, Es-Sebaiy and Viens (2019) studied the estimation problem for the drift parameter for some stochastic differential equations driven by fBm. Lohvinenko and Ralchenko (2017) considered the maximum likelihood (ML) estimates of κ and α when $\kappa > 0$ and $H \in (1/2, 1)$. Moreover, Lohvinenko and Ralchenko (2019) considered the maximum likelihood (ML) estimates of κ and α when $\kappa < 0$ and $H \in (1/2, 1)$.

Our paper also focuses on the MLE of κ and α. We aim to develop the asymptotic distributions for the MLE of κ and α under the following scenarios: (1) $\kappa > 0$ and $H \in (0, 1/2]$; (2) $\kappa = 0$ and $H \in (0, 1)$; (3) $\kappa < 0$ and $H \in (0, 1)$. Therefore, together with Lohvinenko and Ralchenko (2017), a complete coverage of asymptotic theory for all possible cases is provided to the MLE of κ and α.

Other estimation methods and alternative sampling schemes are possible. A recent study by Ng and Wirjanto (2019) investigated the bias property of the least squares estimator (LS) of κ based on discrete-sampled data. It is shown that the bias depends on the Hurst parameter and the true value of κ. While the assumption of a continuous-time record is practically too strong, it allows us to obtain the ML estimates in closed-form. Moreover, the results obtained in our paper will serve as the benchmark for those based on discrete-sampled data.

[1] When a continuous record of observations is available, H and σ can be recovered without estimation errors.

Several recent applications of Model (1) can be found in economics and finance. Gatheral et al. (2018) assumed $\kappa = 0$ and found the evidence of $H < 1/2$ in the log realized volatility (RV) of a DAX contract, the Bund futures contract, the S&P 500 index, and the NASDAQ index. Bennedsen et al. (2017) documented the evidence of $H < 1/2$ in the log RV of a large number of U.S. equities. Wang et al. (2019) found the evidence of $\kappa > 0$ and $H < 1/2$ in the log RV, the realized kernel and bipower variation of the S&P500, NASDAQ, and DJIA, and in the log RV of three exchange rates. Fukasawa et al. (2019) found the log RV of several stock indices indicating that H is smaller than 0.1. Lui et al. (2020) reported the evidence of $\kappa < 0$ and $H < 1/2$ in the S&P500 in the 1990s. Unfortunately, none of these empirical studies used the ML method to estimate κ.

The rest of the paper is organized as follows. Section 2 introduces the MLE of κ and α. Section 3 is devoted to the asymptotic theory for the stationary case (i.e., $\kappa > 0$), but $H \in (0, 1/2]$. Section 4 studies the asymptotic properties of the MLE in the boundary case (i.e., $\kappa = 0$) and for the entire range for the Hurst parameter $H \in (0, 1)$. In Section 5, we establish the asymptotic behaviors of the MLE for the non-ergodic case (i.e., $\kappa < 0$) and for the entire range for the Hurst parameter $H \in (0, 1)$. Section 6 contains some concluding remarks and gives directions for further research. All the proofs are collected in Appendix A.

We use the following notations throughout the paper: O, o, O_p, o_p, $\xrightarrow{p}$, $\xrightarrow{a.s}$, $\xrightarrow{d}$, and $\sim$ denote the same order, the smaller order, the same order in probability, the smaller order in probability, convergence in probability, convergence almost surely, convergence in the distribution, and asymptotic equivalence, respectively, as $T \to \infty$. Throughout this paper, the constant C only depends on H, whose values can differ at different places.

2. ML Estimation

Following Kleptsyna et al. (2000), by applying the Girsanov theorem for the fBm developed in Norros et al. (1999), one can get the expression for the continuous-record log-likelihood function for Model (3) as follows:

$$\ell(\kappa, \alpha) = \int_0^T Q_H(t) dM_t^H + \frac{1}{2} \int_0^T (Q_H(t))^2 \, d\omega_t^H,$$

where:

$$Q_H(t) = \frac{1}{\sigma} \frac{d}{d\omega_t^H} \int_0^t k_H(t,s) (\alpha - \kappa X_s) \, ds, \tag{5}$$

$$k_H(t,s) = \frac{1}{k_H} (s(t-s))^{\frac{1}{2}-H}, \; k_H = 2H\Gamma\left(\frac{3}{2} - H\right) \Gamma\left(H + \frac{1}{2}\right), \tag{6}$$

$$\omega_t^H = \frac{1}{\lambda_H} t^{2-2H}, \tag{7}$$

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}, \tag{8}$$

$$M_t^H = \int_0^t k_H(t,s) dB_s^H. \tag{9}$$

Taking the derivatives of the log-likelihood function with respect to κ and α and setting them to zero, Lohvinenko and Ralchenko (2017) obtained the following expressions for the MLE of α and κ:

$$\tilde{\alpha}_T = \frac{S_T \int_0^T P_H^2(t) \, d\omega_t^H - \int_0^T P_H(t) \, dS_t \int_0^T P_H(t) \, d\omega_t^H}{\omega_T^H \int_0^T P_H^2(t) \, d\omega_t^H - \left(\int_0^T P_H(t) \, d\omega_t^H\right)^2} \sigma, \tag{10}$$

$$\tilde{\kappa}_T = \frac{S_T \int_0^T P_H(t) \, d\omega_t^H - \omega_T^H \int_0^T P_H(t) \, dS_t}{\omega_T^H \int_0^T P_H^2(t) \, d\omega_t^H - \left(\int_0^T P_H(t) \, d\omega_t^H\right)^2}, \tag{11}$$

where:

$$S_t = \frac{1}{\sigma}\int_0^t k_H(t,s)\,dX_s\,, \tag{12}$$

$$P_H(t) = \frac{1}{\sigma}\frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\,X_s ds\,, \tag{13}$$

Combining (3) and (6) with (13), we deduce that:

$$P_H(t) = \frac{1}{\sigma}\frac{\alpha}{\kappa} + \frac{1}{\sigma}\left(X_0 - \frac{\alpha}{\kappa}\right)V_H(t) + \tilde{P}_H(t)\ , \tag{14}$$

where:

$$V_H(t) = \frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\,e^{-\kappa s}ds\,, \tag{15}$$

$$\tilde{P}_H(t) = \frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\,U_s ds\,, \tag{16}$$

$$U_t = \int_0^t e^{-\kappa(t-s)}dB_s^H\,. \tag{17}$$

Using the idea of Kleptsyna and Le Breton (2002), Lohvinenko and Ralchenko (2017) obtained the following results:

$$Q_H(t) = \frac{\alpha}{\sigma} - \kappa P_H(t)\ , \tag{18}$$

$$S_t = \int_0^t Q_H(s)\,d\omega_s^H + M_t^H = \frac{\alpha}{\sigma}\omega_t^H - \kappa\int_0^t P_H(s)\,d\omega_s^H + M_t^H\,, \tag{19}$$

$$dS_t = \frac{\alpha}{\sigma}d\omega_t^H - \kappa P_H(t)\,d\omega_t^H + dM_t^H\,. \tag{20}$$

The process M_t^H, the so-called fundamental martingale, is a Gaussian martingale with the variance function being ω_t^H. Moreover, the natural filtration of the martingale M^H coincides with the natural filtration of the fBm. Based on (19) and (20), the MLE of α and κ can be represented as:

$$\tilde{\alpha}_T = \alpha + \frac{M_T^H\int_0^T P_H^2(t)\,d\omega_t^H - \int_0^T P_H(t)\,dM_t^H\int_0^T P_H(t)\,d\omega_t^H}{\omega_T^H\int_0^T P_H^2(t)\,d\omega_t^H - \left(\int_0^T P_H(t)\,d\omega_t^H\right)^2}\sigma\,, \tag{21}$$

$$\tilde{\kappa}_T = \kappa + \frac{M_T^H\int_0^T P_H(t)\,d\omega_t^H - \omega_T^H\int_0^T P_H(t)\,dM_t^H}{\omega_T^H\int_0^T P_H^2(t)\,d\omega_t^H - \left(\int_0^T P_H(t)\,d\omega_t^H\right)^2}\,, \tag{22}$$

When a continuous record of observations of X_t is available, Lohvinenko and Ralchenko (2017) studied the consistency and the asymptotic normality of the MLE defined by (10) and (11) when $H > 1/2$ and $\kappa > 0$. The goal of the present paper is to establish the asymptotic theory for the MLE of α and κ for all the other cases, including $H < 1/2$ and $\kappa > 0$, $H \in (0,1)$ and $\kappa = 0$, and $H \in (0,1)$ and $\kappa < 0$.

3. Asymptotic Theory When $\kappa > 0$

In this section, inspired by Lohvinenko and Ralchenko (2017), we extend the asymptotic properties of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ from the case of $H \in (1/2, 1)$ to the case of $H \in (0, 1/2]$. For the sake of comparison, we first introduce the main result of Lohvinenko and Ralchenko (2017). When $H > 1/2$, Lohvinenko and Ralchenko (2017) obtained the asymptotic normality for the MLE of α and κ, i.e.,

$$T^{1-H}\left(\tilde{\alpha}_T - \alpha\right) \xrightarrow{d} \mathcal{N}\left(0, \lambda_H \sigma^2\right), \tag{23}$$

$$\sqrt{T}\left(\tilde{\kappa}_T - \kappa\right) \xrightarrow{d} \mathcal{N}\left(0, 2\kappa\right). \tag{24}$$

Remark 1. *We can use the ergodic property to estimate α and κ (denoted by $\widehat{\alpha}_{HN}$ and $\widehat{\kappa}_{HN}$, respectively). Following the idea of Equation (1.3) in Nourdin and Tran (2019) or Equations (2.9) and (2.10) in Xiao and Yu (2019a), we can easily obtain the following results by the ergodic theorem:*

$$\begin{cases} \frac{1}{T}\int_0^T X_t dt \xrightarrow{a.s.} \frac{\alpha}{\kappa}, \\ \frac{1}{T}\int_0^T X_t^2 dt \xrightarrow{a.s.} \sigma^2 \kappa^{-2H} H\Gamma\left(2H\right) + \left(\frac{1}{T}\int_0^T X_t dt\right)^2. \end{cases}$$

Solving these two equations for α and κ, we obtain the ergodic-type estimators of κ and μ as:

$$\widehat{\alpha}_{HN} = \frac{1}{T}\int_0^T X_t dt \left(\frac{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}{\sigma^2 H\Gamma\left(2H\right)}\right)^{-1/(2H)}, \tag{25}$$

$$\widehat{\kappa}_{HN} = \left(\frac{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}{\sigma^2 H\Gamma\left(2H\right)}\right)^{-1/(2H)}. \tag{26}$$

with $H > 1/2$.

Using Theorem 1.3 in Nourdin and Tran (2019) or Theorem 3.3 in Xiao and Yu (2019a), we can easily obtain:

$$T^{1-H}\left(\widehat{\alpha}_{HN} - \alpha\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right), \tag{27}$$

for $1/2 < H < 1$ and:

$$\sqrt{T}\left(\widehat{\kappa}_{HN} - \kappa\right) \xrightarrow{d} \mathcal{N}\left(0, \kappa\phi_H\right), \tag{28}$$

for $1/2 < H < 3/4$ and $\phi_H = \frac{4H-1}{4H^2}\left[1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)}\right]$.

Similarly, using Theorem 1.3 in Nourdin and Tran (2019) or Theorem 3.3 in Xiao and Yu (2019a), we can obtain:

$$\frac{\sqrt{T}}{\log(T)}\left(\widehat{\kappa}_{HN} - \kappa\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{16\kappa}{9\pi}\right), \tag{29}$$

for $H = 3/4$.

Using Theorem 1.3 in Nourdin and Tran (2019) or Theorem 3.3 in Xiao and Yu (2019a), again, for $H \in (3/4, 1)$, we have:

$$T^{2-2H}\left(\widehat{\kappa}_{HN} - \kappa\right) \xrightarrow{\mathcal{L}} \frac{-\kappa^{2H-1}}{H\Gamma(2H+1)} R(H),$$

where $R(H)$ is the Rosenblatt distribution with $\mathbb{E}\left[R\left(H\right)\right] = 2H^2\left(2H-1\right)/\left(4H-3\right)$.

Comparing (23) with (27), we can see that the convergence rate of $\tilde{\alpha}_T$ is the same as that of $\widehat{\alpha}_{HN}$. However, $\tilde{\alpha}_T$ is more efficient than $\widehat{\alpha}_{HN}$ because of $\lambda_H < 1$ for $H \in (1/2, 1)$. Similarly, comparing (24) with (28), we can see that the convergence rate of $\tilde{\kappa}_T$ is the same as that of $\widehat{\kappa}_{HN}$ when $H \in (1/2, 3/4)$. In this case, the asymptotic variance of $\widehat{\kappa}_{HN}$ depends on H while the asymptotic variance of $\tilde{\kappa}_T$ is always 2κ. Since $\phi_H > 2$, $\tilde{\kappa}_T$ is more efficient than $\widehat{\kappa}_{HN}$. When $H = 3/4$, the convergence rate of $\widehat{\kappa}_{HN}$ is slower than that of $\tilde{\kappa}_T$. Let us also mention that $\widehat{\kappa}_{HN}$ is asymptotically more efficient than the LS estimator of κ when $H \in (1/2, 1)$; see Xiao and Yu (2019a) *for details about the LS estimator of κ. When $H = 1/2$, $\widehat{\kappa}_{HN}$ and the LS estimator of κ are asymptotically equivalent. For more comparison of the LS estimator and MLE, see Tanaka (2020).*

The objective of this section is to obtain the consistency and the asymptotic normality of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ when $H \in (0, 1/2]$. Since the asymptotic laws of $\tilde{\alpha}_T$ are different when $H \in (0, 1/2)$ from those when $H = 1/2$, we need to treat them separately.

3.1. Asymptotic Theory When $H \in (0, 1/2)$

Before presenting the asymptotic properties of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ for $H \in (0, 1/2)$, we first state a useful technical lemma.

Lemma 1. *For $\kappa > 0$ and $H \in (0, 1)$ in Model (3), as $T \to \infty$, we have:*

$$
\begin{aligned}
V_H(T) &= O\left(T^{H-\frac{3}{2}}\right), && (30)\\
\int_0^T V_H(t)\, d\omega_t^H &= O\left(T^{\frac{1}{2}-H}\right), && (31)\\
\int_0^T \tilde{P}_H(t)\, dM_t^H &= O_p\left(\sqrt{T}\right), && (32)\\
\int_0^T \tilde{P}_H^2(t)\, d\omega_t^H &= O_p(T), && (33)\\
\int_0^T V_H^2(t)\, d\omega_t^H &= O(1), && (34)\\
\int_0^T \tilde{P}_H(t)\, d\omega_t^H &= O_p\left(T^{1-H}\right), && (35)\\
\int_0^T V_H(t)\, \tilde{P}_H(t)\, d\omega_t^H &= O_p\left(\sqrt{T}\right) && (36)\\
\int_0^T V_H(t)\, dM_t^H &= O_p(1). && (37)
\end{aligned}
$$

We can now describe the asymptotic laws of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ as $T \to \infty$.

Theorem 1. *For $\kappa > 0$ and $H \in (0, 1/2)$ in Model (3), as $T \to \infty$, we have:*

$$
\begin{aligned}
\sqrt{T}\left(\tilde{\alpha}_T - \alpha\right) &\xrightarrow{d} \mathcal{N}\left(0, \frac{2\alpha^2}{\kappa}\right), && (38)\\
\sqrt{T}\left(\tilde{\kappa}_T - \kappa\right) &\xrightarrow{d} \mathcal{N}(0, 2\kappa). && (39)
\end{aligned}
$$

Remark 2. *Comparing the asymptotic theory with that obtained in Lohvinenko and Ralchenko (2017), the asymptotic normality continues to hold for both estimators. Moreover, comparing (39) with (24), we can see that the asymptotic theory for $\tilde{\kappa}_T$ is the same regardless of $H \in (0, 1/2)$ or $H \in (1/2, 1)$. Comparing (38) with (23), we can see that the asymptotic variance of $\tilde{\alpha}_T$ depends on H. The asymptotic variance is $\lambda_H \sigma^2$ with the consistency order T^{1-H} if $H \in (1/2, 1)$, whereas it does not depend on H with the consistency order $\sqrt{T}$ as T becomes large if $H \in (0, 1/2)$.*

Remark 3. *The asymptotic theory for the MLE of κ in the fOU when $H \in (0, 1/2)$ has been developed in the literature; see, for example, Theorem 2 in Brouste and Kleptsyna (2010). It is the same as in (39). Therefore, having to estimate an additional parameter α, there is no efficiency loss in estimating κ asymptotically.*

Remark 4. *Following Hu et al. (2019), Xiao and Yu (2019b) considered the ergodic-type estimate of κ defined in (25) when $0 < H < 1/2$ and showed that:*

$$\sqrt{T}\left(\widehat{\kappa}_{HN} - \kappa\right) \xrightarrow{d} \mathcal{N}\left(0, \kappa\delta_{HN}^2\right), \tag{40}$$

where $\delta_{HN}^2 = \frac{1}{4H^2}\left[(4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}\right]$. Figure 1 compares the asymptotic variances of $\tilde{\kappa}_T$ and $\widehat{\kappa}_{HN}$ by plotting δ_{HN}^2 against twofor $H \in (0, 1/2)$. It can be seen that $\tilde{\kappa}_T$ is more efficient than $\widehat{\kappa}_{HN}$. The smaller H is, the larger the relative asymptotic efficiency in $\tilde{\kappa}_T$. The two estimators have the same asymptotic variance when $H = 1/2$.

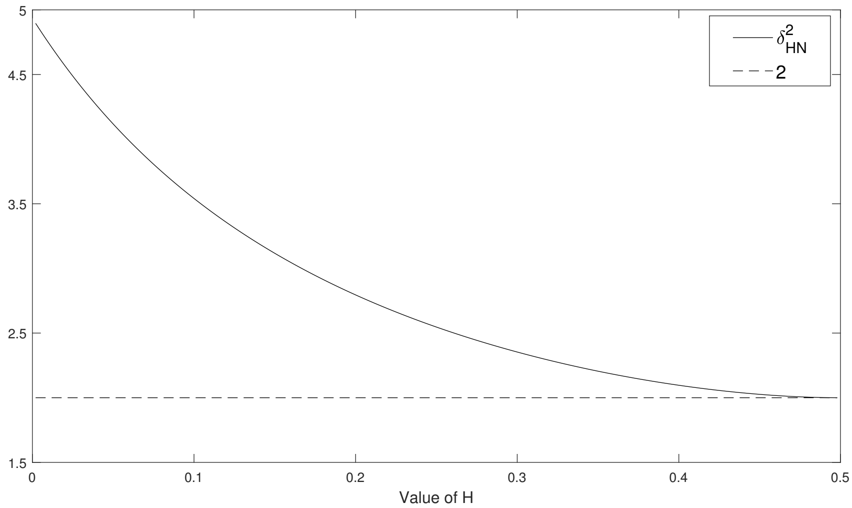

Figure 1. Plots of δ_{HN}^2 against two as functions of H.

Remark 5. *Using the Taylor expansion and Theorem 3.2 of Xiao and Yu (2019b), we can easily obtain that for $0 < H < 1/2$:*

$$\sqrt{T}\left(\widehat{\alpha}_{HN} - \alpha\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\alpha^2}{\kappa}\delta_{HN}^2\right). \tag{41}$$

Comparing (38) with (41), we can see that $\tilde{\alpha}_T$ and $\widehat{\alpha}_{HN}$ share the same convergence rate. However, $\tilde{\alpha}_T$ is more efficient than $\widehat{\alpha}_{HN}$ because $\delta_{HN}^2 < 2$ for $H \in (0, 1/2)$.

3.2. Asymptotic Theory When $H = 1/2$

When $H = 1/2$, $B_t^{1/2} = W_t$, which is a standard Brownian motion, and the fVm becomes the standard Vasicek model. In this case, it can be shown that fundamental martingale M_t^H becomes a standard Brownian motion. Consequently, the MLE reduces to the LS estimates and can be rewritten as:

$$\tilde{\alpha}_T = \frac{X_T \int_0^T X_t^2 dt - \int_0^T X_t dt \int_0^T X_t dX_t}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \tag{42}$$

$$\tilde{\kappa}_T = \frac{X_T \int_0^T X_t dt - T\int_0^T X_t dX_t}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \tag{43}$$

where the stochastic integration $\int_0^T X_t dX_t$ is interpreted as an Itô integral. The asymptotic theory in (42) and (43) has been studied in the literature; see, for example, Kubilius et al. (2018) and Kutoyants (2004). From page 64 of Kutoyants (2004), we can easily show that $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ are consistent and asymptotically normally distributed. Hence, we only provide the asymptotic laws of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ for $H = 1/2$ without proof.

Theorem 2. *For $\kappa > 0$ and $H = 1/2$ in Model (3), as $T \to \infty$, we have:*

$$\sqrt{T}\left(\tilde{\alpha}_T - \alpha\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2 + \frac{2\alpha^2}{\kappa}\right), \tag{44}$$

$$\sqrt{T}\left(\tilde{\kappa}_T - \kappa\right) \xrightarrow{d} \mathcal{N}\left(0, 2\kappa\right). \tag{45}$$

Remark 6. *When $\alpha \neq 0$, we can summarize the three sets of asymptotic theory for the MLE of α as follows:*

$$\begin{aligned} \textit{If } H \in (0,1/2),\ \sqrt{T}\left(\tilde{\alpha}_T - \alpha\right) &\xrightarrow{d} \mathcal{N}\left(0, \frac{2\alpha^2}{\kappa}\right), \\ \textit{If } H = 1/2,\ \sqrt{T}\left(\tilde{\alpha}_T - \alpha\right) &\xrightarrow{d} \mathcal{N}\left(0, \sigma^2 + \frac{2\alpha^2}{\kappa}\right), \\ \textit{If } H \in (1/2,1),\ T^{1-H}\left(\tilde{\alpha}_T - \alpha\right) &\xrightarrow{d} \mathcal{N}\left(0, \lambda_H \sigma^2\right), \end{aligned}$$

where the last asymptotic theory was obtained in Theorem 3.4 of Lohvinenko and Ralchenko (2017). While the three sets of asymptotic theory for $\tilde{\kappa}_T$ are identical, the three sets of asymptotic theory for $\tilde{\alpha}_T$ are different. When H changes from a value in $(0, 1/2)$ to $1/2$, while the rate of convergence stays the same (i.e., $\sqrt{T}$), the asymptotic variance changes from $\frac{2\alpha^2}{\kappa}$ to $\sigma^2 + \frac{2\alpha^2}{\kappa}$. When H changes from a value in $(0, 1/2]$ to $(1/2, 1)$, both the rate of convergence and the asymptotic variance change.

Remark 7. *When α is known and assumed to be zero and $H = 1/2$, the asymptotic theory for the MLE of κ was obtained in Brown and Hewitt (1975) and in Feigin (1976). The two sets of asymptotic theory are the same, suggesting that there is no efficiency loss in estimating κ when α is estimated or not.*

4. Asymptotic Theory When $\kappa = 0$

In this section, we consider the asymptotic laws of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ for the entire range for the Hurst parameter, i.e., $H \in (0, 1)$. Note that when $\kappa = 0$, we have:

$$X_t = X_0 + \alpha t + \sigma B_t^H. \tag{46}$$

For the model $dU_t = -\kappa U_t dt + dB_t^H$, it is well known that the MLE of κ can be expressed as:

$$\hat{\kappa}_T - \kappa = \frac{-\int_0^T \hat{P}_H(t)\, dM_t^H}{\int_0^T \hat{P}_H^2(t)\, d\omega_t^H}, \tag{47}$$

where $\hat{P}_H(t) = \frac{d}{d\omega_t^H} \int_0^t k_H(t,s)\, B_s^H ds$.

Before considering the asymptotic properties of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$, we first introduce a lemma, which will be used to derive the asymptotic theory.

Lemma 2. *For $\kappa = 0$ and $H \in (0, 1)$ in Model (1), as $T \to \infty$, we have:*

$$\begin{aligned}
\int_0^T \hat{P}_H(t)\, dM_t^H &= O_p(T)\,, && (48)\\
\int_0^T \hat{P}_H^2(t)\, d\omega_t^H &= O_p\left(T^2\right)\,, && (49)\\
\int_0^T t d\omega_t^H &= \frac{1}{\lambda_H}\frac{2-2H}{3-2H}T^{3-2H}\,, && (50)\\
\int_0^T t^2 d\omega_t^H &= \frac{1}{\lambda_H}\frac{1-H}{2-H}T^{4-2H}\,, && (51)\\
\int_0^T \hat{P}_H(t)\, d\omega_t^H &= O_p\left(T^{2-H}\right)\,, && (52)\\
\int_0^T t\hat{P}_H(t)\, d\omega_t^H &= O_p\left(T^{3-H}\right)\,, && (53)\\
V_H(T) &= \frac{\lambda_H}{k_H} B\left(\frac{3}{2}-H, \frac{3}{2}-H\right)\,, && (54)\\
\frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\, s ds &= a_H t\,, && (55)
\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function, λ_H is defined by (8), and $a_H = \frac{3-2H}{4(1-H)}$.

We can now describe the asymptotic behavior of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ as $T \to \infty$.

Theorem 3. *For $\kappa = 0$, $\alpha \neq 0$, and $H \in (0,1)$ in Model (1), as $T \to \infty$, we have:*

$$\begin{aligned}
T^{1-H}\left(\tilde{\alpha}_T - \alpha\right) &\xrightarrow{d} \mathcal{N}\left(0, \sigma^2 \rho_H\right)\,, && (56)\\
T^{2-H}\left(\tilde{\kappa}_T - \kappa\right) &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\alpha^2}\phi_H\right)\,, && (57)
\end{aligned}$$

where $\rho_H = \lambda_H (3-2H)^2$, $\phi_H = \frac{32H(1-H)(2-H)\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$ and λ_H is defined by (8).

Remark 8. *In the case of $H = 1/2$ and $\alpha \neq 0$, we can see that $X_t = X_0 + \alpha t + \sigma W_t$. A straightforward algebraic calculation shows $\omega_t^H = t$, $P_H(t) = \frac{1}{\sigma}X_t$, $M_T^H = W_t$, $\hat{P}_H(t) = W_t$ and that:*

$$\begin{aligned}
\frac{1}{T^3}\int_0^T X_t^2 dt &= \frac{\alpha^2}{3} + o_p(1)\,, && (58)\\
\frac{1}{T^2}\int_0^T X_t dt &= \frac{\alpha}{2} + o_p(1)\,, && (59)\\
\frac{1}{T\sqrt{T}}\int_0^T X_t dW_t &= \frac{\alpha}{T\sqrt{T}}\int_0^T t dW_t + o_p(1)\,. && (60)
\end{aligned}$$

Then, by the scaling properties of the Brownian motion, (21) and (58)–(60), we deduce that:

$$\begin{aligned}
\sqrt{T}\left(\tilde{\alpha}_T-\alpha\right) &= \sqrt{T}\left(\frac{W_T\int_0^T X_t^2dt-\int_0^T X_tdW_t\int_0^T X_tdt}{T\int_0^T X_t^2dt-\left(\int_0^T X_tdt\right)^2}\sigma\right) \\
&= \sqrt{T}\left(\frac{1}{\sqrt{T}}\frac{\frac{1}{\sqrt{T}}W_T\frac{1}{T^3}\int_0^T X_t^2dt-\frac{1}{T\sqrt{T}}\int_0^T X_tdW_t\frac{1}{T^2}\int_0^T X_tdt}{\frac{1}{T^3}\int_0^T X_t^2dt-\left(\frac{1}{T^2}\int_0^T X_tdt\right)^2}\sigma\right) \\
&= \frac{\frac{\alpha^2}{3}\frac{W_T}{\sqrt{T}}-\frac{\alpha^2}{2}\frac{1}{T\sqrt{T}}\int_0^T tdW_t+o_p(1)}{\frac{\alpha^2}{3}-\left(\frac{\alpha}{2}\right)^2+o_p(1)}\sigma \\
&= 12\sigma\left(\frac{W_T}{3\sqrt{T}}-\frac{1}{2T\sqrt{T}}\int_0^T tdW_t\right)+o_p(1) \\
&\xrightarrow{d} \mathcal{N}\left(0,4\sigma^2\right),
\end{aligned}$$

which is identical to (56) with $H=1/2$. Moreover, using (22) and (58)–(60), we can write:

$$\begin{aligned}
T\sqrt{T}\left(\tilde{\kappa}_T-\kappa\right) &= T\sqrt{T}\left[\frac{\left(W_T\int_0^T X_tdt-T\int_0^T X_tdW_t\right)\sigma}{T\int_0^T X_t^2dt-\left(\int_0^T X_tdt\right)^2}\right] \\
&= \frac{\left[\frac{W_T}{\sqrt{T}}\frac{1}{T^2}\int_0^T X_tdt-\frac{1}{T\sqrt{T}}\int_0^T X_tdW_t\right]\sigma}{\frac{1}{T^3}\int_0^T X_t^2dt-\left(\frac{1}{T^2}\int_0^T X_tdt\right)^2} \\
&\xrightarrow{d} \mathcal{N}\left(0,\frac{12\sigma^2}{\alpha^2}\right),
\end{aligned}$$

which is identical to (57) with $H=1/2$ being assumed.

Remark 9. *In the case of $H=1/2$ and $\alpha=0$, with α and κ being estimated, by the scaling properties of the Brownian motion, we have:*

$$\begin{aligned}
\sqrt{T}\left(\tilde{\alpha}_T-\alpha\right) &\xrightarrow{d} \frac{W_1\left(\int_0^1 W_tdt\right)^2-\int_0^1 W_tdt\int_0^1 W_tdW_t}{\int_0^1 W_t^2dt-\left(\int_0^1 W_tdt\right)^2}\sigma, \\
T\left(\tilde{\kappa}_T-\kappa\right) &\xrightarrow{d} \frac{W_1\int_0^1 W_tdt-\int_0^1 W_tdW_t}{\int_0^1 W_t^2dt-\left(\int_0^1 W_tdt\right)^2}.
\end{aligned}$$

Thus, the limiting distributions of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ are not normal. In particular, the asymptotic distribution of $\tilde{\kappa}_T$ is a Dickey–Fuller–Phillips-type distribution with the rate of convergence being T. Hence, when $\kappa=0$ is unknown, the value of α plays an important role in the study of asymptotic laws for the MLE.

5. Asymptotic Theory When $\kappa<0$

When $\kappa<0$, the model given by (3) is non-ergodic or explosive. Since the proofs of the asymptotic theory of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ when $H=1/2$ are different from those when $H\in(0,1/2)\cup(1/2,1)$, we first consider the case of $H=1/2$. For the sake of notational simplicity, we introduce the process $\xi_t=\sigma\int_0^t e^{\kappa s}dW_s\sim\mathcal{N}\left(0,\frac{\sigma^2\left(e^{2\kappa t}-1\right)}{2\kappa}\right)$ for $t\geq 0$. Obviously, $\xi_\infty\sim\mathcal{N}\left(0,-\frac{\sigma^2}{2\kappa}\right)$. Moreover, using (17) and the definition of ξ_t, we can easily obtain:

$$\sigma e^{\kappa T} \int_0^T U_t dt = e^{\kappa T} \int_0^T e^{-\kappa t} \xi_t dt \xrightarrow{p} -\frac{\xi_\infty}{\kappa}, \tag{61}$$

$$\sigma e^{2\kappa T} \int_0^T e^{-\kappa t} U_t dt \xrightarrow{p} -\frac{\xi_\infty}{2\kappa}. \tag{62}$$

Since $\mathbb{E}\left[\left(\int_0^T e^{\kappa(T-t)} dW_t\right)^2\right] = e^{2\kappa T} \int_0^T e^{-2\kappa t} dt = \frac{e^{2\kappa T}-1}{2\kappa}$, we can obtain:

$$\int_0^T e^{\kappa(T-t)} dW_t = O_p(1). \tag{63}$$

Moreover, let $\eta_t = \sigma \int_0^t e^{\kappa(t-s)} dW_s$ be a zero mean Gaussian process. Since $\mathbb{E}\left[\eta_t^2\right] = \sigma^2 \int_0^t e^{2\kappa(t-s)} ds = -\frac{\sigma^2}{2\kappa} + \frac{\sigma^2}{2\kappa} e^{2\kappa t}$, we have $\eta_\infty \sim \mathcal{N}\left(0, -\frac{\sigma^2}{2\kappa}\right)$. Consequently, we can obtain:

$$\sigma \int_0^T e^{\kappa(T-t)} \xi_t dW_t \xrightarrow{p} \xi_\infty \eta_\infty, \tag{64}$$

where ξ_∞ and η_∞ are two independent $N(0, -\sigma^2/(2\kappa))$ random variables.

5.1. Asymptotic Theory When $H = 1/2$

Now, we can state the key results of the asymptotic theory for $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ when $H = 1/2$.

Theorem 4. *For $\kappa < 0$, $H = 1/2$ in Model (1), as $T \to \infty$, we have:*

$$\sqrt{T}\left(\tilde{\alpha}_T - \alpha\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\right), \tag{65}$$

$$\frac{e^{-\kappa T}}{2\kappa}\left(\tilde{\kappa}_T - \kappa\right) \xrightarrow{d} \frac{\eta_\infty}{X_0 - \frac{\alpha}{\kappa} + \xi_\infty}, \tag{66}$$

where ξ_∞ and η_∞ are two independent $\mathcal{N}(0, -\sigma^2/(2\kappa))$ random variables.

Remark 10. *In (66), if we set $X_0 = \frac{\alpha}{\kappa}$, the limiting distribution of $\frac{e^{-\kappa T}}{2\kappa}\left(\tilde{\kappa}_T - \kappa\right)$ becomes a standard Cauchy variate. This limiting distribution is the same as that in the Vasicek model driven by a standard Brownian motion (see, e.g., Feigin 1976). The asymptotic theory in (66) is similar to that in the explosive discrete-time and continuous-time models when discretely-sampled data are available (see, e.g., White 1958; Anderson 1959; Phillips and Magdalinos 2007; Wang and Yu 2015, 2016).*

5.2. Asymptotic Theory When $H \in (0, 1/2) \cup (1/2, 1)$

We now turn to the case when $H \in (0, 1/2) \cup (1/2, 1)$ assuming $\kappa < 0$. The limiting theory is the most difficult to derive in our paper and, hence, is the main technical contribution to the literature. First, we can have the following lemma.

Lemma 3. *For $\kappa < 0$ and $H \in (0, 1)$ in Model (1), we have:*

$$
\begin{aligned}
V_H(t) &= O\left(t^{H-\frac{1}{2}}e^{-\kappa t}\right), &(67)\\
\int_0^T \tilde{P}_H^2(t)\,d\omega_t^H &= O_p\left(e^{-2\kappa T}\right), &(68)\\
\int_0^T \tilde{P}_H(t)\,dM_t^H &= O_p\left(e^{-\kappa T}\right), &(69)\\
\int_0^T V_H(t)\,d\omega_t^H &= O\left(T^{\frac{1}{2}-H}e^{-\kappa T}\right), &(70)\\
\int_0^T \tilde{P}_H(t)\,d\omega_t^H &= O_p\left(e^{-\kappa T}T^{\frac{1}{2}-H}\right), &(71)\\
\int_0^T V_H^2(t)\,d\omega_t^H &= O\left(e^{-2\kappa T}\right), &(72)\\
\int_0^T V_H(t)\,\tilde{P}_H(t)\,d\omega_t^H &= O_p\left(e^{-2\kappa T}\right), &(73)\\
\int_0^T V_H(t)\,dM_t^H &= O_p\left(e^{-\kappa T}\right). &(74)
\end{aligned}
$$

Now, we can state the asymptotic theory for $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ for $\kappa < 0$ and $H \in (0,1/2) \cup (1/2,1)$.

Theorem 5. *When $\kappa < 0$, $H \in (0,1/2) \cup (1/2,1)$, and $X_0 = \frac{\alpha}{\kappa}$ in Model (1), as $T \to \infty$, we have:*

$$
\begin{aligned}
T^{1-H}\left(\tilde{\alpha}_T - \alpha\right) &\xrightarrow{d} \mathcal{N}\left(0, \lambda_H \sigma^2\right), &(75)\\
\frac{e^{-\kappa T}}{2\kappa}\left(\tilde{\kappa}_T - \kappa\right) &\xrightarrow{d} \frac{X\sqrt{\sin(\pi H)}}{Y}, &(76)
\end{aligned}
$$

where X and Y are two independent $\mathcal{N}(0,1)$ random variables.

Remark 11. *For the entire range of $H \in (0,1)$, the asymptotic distribution of $\tilde{\alpha}_T$ is normal with the rate of convergence of T^{1-H} and variance σ^2. This asymptotic distribution is the same as that of the LS estimate (see Theorem 3.5 in Xiao and Yu (2019a) and Section 3 in Xiao and Yu (2019b)).*

Remark 12. *According to (76), the asymptotic law of $\frac{e^{-\kappa T}}{2\kappa}\left(\tilde{\kappa}_T - \kappa\right)$ is the standard Cauchy times $\sqrt{\sin(\pi H)}$. For $H \in (0,1/2) \cup (1/2,1)$, $\sqrt{\sin(\pi H)} \in (0,1)$, suggesting that as H draws further away from $1/2$, κ is estimated with higher accuracy. Moreover, with $X_0 = \frac{\alpha}{\kappa}$, from Theorem 3.5 in Xiao and Yu (2019a, 2019b), we can see that the LS estimator of κ, which is denoted by $\widehat{\kappa}_{LS}$, has the asymptotic law $\frac{e^{-\kappa T}}{2\kappa}\left(\widehat{\kappa}_{LS} - \kappa\right) \xrightarrow{d} C$, where C is the standard Cauchy distribution. Since the second moment of the Cauchy distribution is infinite, we cannot use the variances to measure the asymptotic relative efficiency. From Theorem 2 in Tanaka (2020) and based on the asymptotic concentration probability, we can see that the MLE is always more efficient asymptotically than the LS estimator for $H \in (0,0.5) \cup (0.5,1)$. For $H = 1/2$, the MLE is asymptotically the same as the LS estimator.*

Remark 13. *When $X_0 \neq \frac{\alpha}{\kappa}$, using Lemma 3, we can obtain:*

$$
\frac{e^{-\kappa T}}{2\kappa}\left(\tilde{\kappa}_T - \kappa\right) = \frac{-2\kappa e^{\kappa T}\int_0^T \left[\frac{1}{\sigma}\left(X_0 - \frac{\alpha}{\kappa}\right)V_H(t) + \tilde{P}_H(t)\right] dM_t^H + o_p(1)}{4\kappa^2 e^{2\kappa T}\int_0^T \left[\frac{1}{\sigma}\left(X_0 - \frac{\alpha}{\kappa}\right)V_H(t) + \tilde{P}_H(t)\right]^2 d\omega_t^H + o_p(1)}.
$$

In this case, to obtain the asymptotic distribution of $e^{-\kappa T}\left(\tilde{\kappa}_T - \kappa\right)/(2\kappa)$, one needs to calculate the Laplace transform of $\int_0^T \left[\frac{1}{\sigma}\left(X_0 - \frac{\alpha}{\kappa}\right)V_H(t) + \tilde{P}_H(t)\right]^2 d\omega_t^H$. On the other hand, for $1/2 < H < 1$, using the moment generating function of S_T, $\int_0^T P_H^2(t)\,d\omega_t^H$, $\int_0^T P_H(t)\,dS_t$, and $\int_0^T P_H(t)\,d\omega_t^H$, *Lohvinenko and Ralchenko (2019) provided the joint asymptotic normality of MLE of the vector parameter (α,κ):*

$$\begin{pmatrix} T^{1-H}(\tilde{\alpha}_T-\alpha) \\ e^{-\kappa T}(\tilde{\kappa}_T-\kappa) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \vartheta_1 \\ \vartheta_2/\vartheta_3 \end{pmatrix},$$

where $\vartheta_1 \xrightarrow{d} \mathcal{N}\left(0,\lambda_H\sigma^2\right)$, $\vartheta_2 \xrightarrow{d} \mathcal{N}(0,1)$, *and* $\vartheta_3 \xrightarrow{d} \mathcal{N}\left(\frac{\left(x_0-\frac{\alpha}{\kappa}\right)\sqrt{\lambda_H}\rho_H(-\kappa)^{H-1}}{\sqrt{\pi(4-4H)}}, \frac{1}{4\kappa^2\sin(\pi H)}\right)$, *which generalizes (76) for particular* $X_0=\alpha/\kappa$.

6. Concluding Remarks and Future Directions

The fVm has found more and more applications in practice. In this paper, we consider the MLE of parameters in the drift term when a continuous record of observations is available. The ML estimation is made possible due to the presence of the fundamental martingale and the generalized Girsanov theorem. The asymptotic theory is based on the assumption that $T\to\infty$.

It is shown that the MLE of α is asymptotically normal regardless of the sign of κ. However, the asymptotic law of the MLE of κ critically depends on the sign of κ. More precisely, when $\kappa>0$ and $H\in(0,1)$, we have shown that the asymptotic distribution of the MLE of κ is normal with the rate of convergence being $\sqrt{T}$. The asymptotic variance is 2κ, which is independent of H. When $\kappa=0$ and $\alpha\neq 0$, the asymptotic distribution of the MLE of κ is normal with the rate of convergence being T^{2-H}. The asymptotic variance depends on H. When $\kappa=0$ and $\alpha=0$, the asymptotic distribution of the MLE of κ is a Dickey–Fuller–Phillips distribution with the rate of convergence being T. When $\kappa<0$, it is shown that the limiting distribution is a Cauchy-type with the rate of convergence being $e^{-\kappa T}$. If one further assumes that $X_0=\alpha/\kappa$, the limiting distribution becomes a standard Cauchy variate multiplied by $\sqrt{\sin(\pi H)}$. Table 1 summarizes the asymptotic laws of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ for different ranges of H and κ, where $\lambda_H=\frac{2H\Gamma(3-2H)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}$, $\rho_H=\lambda_H(3-2H)^2$, $\phi_H=\frac{32H(1-H)(2-H)\Gamma(3-2H)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}$, ξ_∞, and η_∞ are two independent $\mathcal{N}(0,-\sigma^2/(2\kappa))$ random variables and X and Y are two independent $\mathcal{N}(0,1)$ random variables. Moreover, we assume $X_0=\alpha/\kappa$ for $\kappa<0$ and $H\in(0,1/2)\cup(1/2,1)$.

Table 1. Summary of the asymptotic laws of $\tilde{\alpha}_T$ and $\tilde{\kappa}_T$ for different ranges of H and κ.

	$0<H<1/2$	**$H=1/2$**	**$1/2<H<1$**
$\kappa>0$	$\sqrt{T}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\frac{2\alpha^2}{\kappa}\right)$ $\sqrt{T}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\mathcal{N}(0,2\kappa)$	$\sqrt{T}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\sigma^2+\frac{2\alpha^2}{\kappa}\right)$ $\sqrt{T}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\mathcal{N}(0,2\kappa)$	$T^{1-H}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\lambda_H\sigma^2\right)$ $\sqrt{T}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\mathcal{N}(0,2\kappa)$
$\kappa=0$	$T^{1-H}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\sigma^2\rho_H\right)$ $T^{2-H}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\mathcal{N}\left(0,\frac{\sigma^2}{\alpha^2}\phi_H\right)$	$\sqrt{T}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,4\sigma^2\right)$ $T^{3/2}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\mathcal{N}\left(0,\frac{12\sigma^2}{\alpha^2}\right)$	$T^{1-H}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\sigma^2\rho_H\right)$ $T^{2-H}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\mathcal{N}\left(0,\frac{\sigma^2}{\alpha^2}\phi_H\right)$
$\kappa<0$	$T^{1-H}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\lambda_H\sigma^2\right)$ $\frac{e^{-\kappa T}}{2\kappa}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\frac{X\sqrt{\sin(\pi H)}}{Y}$	$\sqrt{T}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\sigma^2\right)$ $\frac{e^{-\kappa T}}{2\kappa}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\frac{\eta_\infty}{X_0-\frac{\alpha}{\kappa}+\xi_\infty}$	$T^{1-H}(\tilde{\alpha}_T-\alpha)\xrightarrow{d}\mathcal{N}\left(0,\lambda_H\sigma^2\right)$ $\frac{e^{-\kappa T}}{2\kappa}(\tilde{\kappa}_T-\kappa)\xrightarrow{d}\frac{X\sqrt{\sin(\pi H)}}{Y}$

This study also suggests several important directions for future research. First, it is worth investigating generalizing the results in this paper to nonlinear stochastic differential equations driven by the fBm. The ergodic theorem, fractional calculus, and Malliavin calculus will be employed for obtaining the asymptotic properties of both the MLE and the LS estimators.

Second, in this paper, H and σ are assumed to be known. In practice, both H and σ are almost always unknown. Although many approaches have been proposed to estimate the Hurst coefficient and the volatility parameter from discrete time observations, how to estimate H and σ in fVm with a continuous record of observations is an open question. It is interesting to realize that we can use the generalized quadratic variation to estimate both the Hurst parameter and the volatility parameter in fVm. For $T > 0$ and any $\epsilon \neq \xi$,

$$H = \lim_{\epsilon \downarrow 0, \xi \downarrow 0} \frac{1}{2} \log\left(\frac{\epsilon}{\xi}\right) \log\left(\frac{\int_0^T (X_{t+\xi} - X_t)^2 dt}{\int_0^T (X_{t+\epsilon} - X_t)^2 dt}\right), \quad \sigma^2 = \frac{\lim_{\epsilon \downarrow 0} \int_0^T (X_{t+\epsilon} - X_t)^2 dt}{\epsilon^{2H} T}.$$

It would be interesting to study the asymptotic properties of these estimators mentioned above, which will be reported in later work.

Third, this paper assumes that a continuous record of an increasing time span is available for the development of asymptotic theory. In practice, data are typically observed at discrete time points with $(0, h, 2h, \ldots, Nh(:= T))$ where h is the sampling interval and T is the time span. When high frequency data over a long spanning time period are available, one may consider using a double asymptotic scheme by assuming $h \to 0$ and $T \to \infty$. The discretized model corresponding to (3) is given by:

$$y_{th} = \mu + \exp(-\kappa h)\left(y_{(t-1)h} - \mu\right) + u_t, \ (1 - L)^d u_t = \varepsilon_t, \ t = 1, \ldots, N,$$

where L is the lag operator, $d = H - 1/2$. As shown in Wang and Yu (2016), under the double asymptotic scheme, $\exp(-\kappa h) = \exp\{-\kappa/k_N\} = 1 - \kappa/k_N + O(k_N^{-2}) \to 1$ where $k_N := 1/h \to \infty$ as $h \to 0$ and $k_N/N = 1/T \to 0$ as $T \to \infty$. This implies an autoregressive (AR) model with the AR root moderately deviating from unity and with a fractionally integrated error term with $d \in (-1/2, 0)$. This model is closely related to a model considered in Magdalinos (2012) where it is assumed that $d \in (0, 1/2)$. Developing double asymptotic theory based on discretely-sampled data will allow one to extend the results of Magdalinos (2012) to the case where $d \in (-1/2, 1/2)$. The development of the MLE and the asymptotic theory is beyond the scope of this paper and will be reported in later work.

Author Contributions: Three authors make the equal contribution to the paper. All authors have read and agreed to the published version of the manuscript.

Funding: Xiao gratefully acknowledges the financial support of the National Natural Science Foundation of China (No. 71871202); Yu wishes to thank the Singapore Ministry of Education for Academic Research Fund under grant number MOE2013-T3-1-009 and acknowledges the financial support from the Lee Foundation.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Appendix A.1. Proof of Lemma 1

We first consider (30). Let us introduce the modified Bessel functions of the first kind (see, e.g., Abramowitz and Stegun 1972), which are defined as:

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu + k + 1)}, \text{ where } \nu \in \mathbb{R}\,. \tag{A1}$$

From page 377 in Abramowitz and Stegun (1972), we can see that the asymptotic behavior of $I_\nu(z)$ is:

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}[1 + O(1/z)], \text{ when } z \to \infty, |\arg z| < \frac{\pi}{2}\,. \tag{A2}$$

Using (6), (15) and (A2), we get:

$$\begin{aligned} V_H(T) &= \frac{d}{d\omega_T^H}\int_0^T k_H(T,s)\,e^{-\kappa s}ds \\ &= \frac{d}{d\omega_T^H}\left[\frac{\sqrt{\pi}\kappa^{H-1}\Gamma\left(\frac{3}{2}-H\right)}{k_H}T^{1-H}e^{-\frac{\kappa T}{2}}I_{1-H}\left(\frac{\kappa T}{2}\right)\right] \\ &= \frac{\kappa^{H-\frac{3}{2}}\Gamma(2-2H)}{\Gamma\left(\frac{1}{2}-H\right)}T^{H-\frac{3}{2}}+O\left(T^{H-\frac{5}{2}}\right), \end{aligned} \tag{A3}$$

which is (30).

Then, as $T\to\infty$, using arguments similar to those in Lemma 4.2 of Lohvinenko and Ralchenko (2017), we can obtain:

$$\int_0^T V_H(t)\,d\omega_t^H = \int_0^T k_H(T,s)\,e^{-\kappa s}ds = O\left(T^{\frac{1}{2}-H}\right),$$

which yields (31).

By the proof of Theorem 3 in Tanaka (2013), we can easily obtain (32) and (33). The result of (34) follows directly from $\int_0^1 V_H^2(t)\,d\omega_t^H<\infty$ and $\int_1^T V_H^2(t)\,d\omega_t^H<\infty$ (see the proof of Lemma 4.7 in Lohvinenko and Ralchenko 2017).

Next, we consider (35). Let $\tilde{J}_T=\int_0^T\tilde{P}_H(t)d\omega_t^H$. Then, using Lemma 2 in Lohvinenko and Ralchenko (2019) and replacing J_T with $\tilde{J}_T$, we have:

$$\mathbb{E}\left[e^{\theta\tilde{J}_T}\right]=\tilde{m}_1^{(\alpha_1,\beta_1)}\left(-\frac{\alpha_1}{\sigma},-\kappa+\beta_1\right)\exp\left\{\frac{\alpha_1^2}{2\sigma^2}\omega_T^H\right\}, \tag{A4}$$

where $\alpha_1=-\sigma\theta/\kappa$, $\beta_1=-\kappa$, $\tilde{m}_1^{(\alpha_1,\beta_1)}(\xi_1,\xi_2)$ is the moment generating function defined by Lemma 1 in Lohvinenko and Ralchenko (2019). Using Lemma 1 in Lohvinenko and Ralchenko (2019) again, we can obtain:

$$\begin{aligned} \tilde{m}_1^{(\alpha_1,\beta_1)}\left(-\frac{\alpha_1}{\sigma},-\kappa+\beta_1\right) &= \left(D^{(\alpha_1,\beta_1)}(-\kappa+\beta_1)\right)^{-1/2}\exp\left[\frac{\frac{\alpha_1^2}{\sigma^2}c_3T^{2-2H}e^{-\beta_1 T}}{8D^{(\alpha_1,\beta_1)}(-\kappa+\beta_1)}\right. \\ &\quad \left.\times I_{1-H}\left(-\frac{\beta_1 T}{2}\right)I_{H-1}\left(-\frac{\beta_1 T}{2}\right)-\frac{-\kappa+\beta_1}{2}T\right], \end{aligned} \tag{A5}$$

where $c_3=4(1-H)\Gamma(H)\Gamma(1-H)/\lambda_H$ and $D^{(\alpha_1,\beta_1)}(\xi_2)$ is defined by Equation (10) in Lohvinenko and Ralchenko (2019).

Consequently, using the fact $D^{(\alpha_1,\beta_1)}(-\kappa+\beta_1)=e^{2\kappa T}$, $\alpha_1=-\sigma\theta/\kappa$, $\beta_1=-\kappa$, (A4) and (A5), we have:

$$\begin{aligned} \mathbb{E}\left[e^{\theta\tilde{J}_T}\right] &= e^{-\kappa T}\exp\left[\frac{\frac{\theta^2}{\kappa^2}c_3T^{2-2H}e^{-\beta_1 T}}{8e^{2\kappa T}}\frac{e^{\kappa T}}{\pi\kappa T}\times\left(1+O\left(T^{-1}\right)\right)-\frac{-\kappa+\beta_1}{2}T+\frac{\theta^2\omega_T^H}{2\kappa^2}\right] \\ &= \exp\left[\frac{\theta^2}{2\kappa^2}\omega_T^H+\theta^2 O\left(T^{1-2H}\right)\right], \end{aligned} \tag{A6}$$

which yields:

$$\mathbb{E}\left[\tilde{J}_T\right] = \frac{d}{d\theta}\mathbb{E}\left[e^{\theta\tilde{J}_T}\right]\Big|_{\theta=0} = 0\,, \tag{A7}$$

$$\mathbb{E}\left[\tilde{J}_T^2\right] = \frac{d^2}{d\theta^2}\mathbb{E}\left[e^{\theta\tilde{J}_T}\right]\Big|_{\theta=0} = \frac{\omega_T^H}{\kappa^2} + O\left(T^{1-2H}\right). \tag{A8}$$

Using (7), (A5), (A7) and (A8), we can obtain:

$$\begin{aligned}\tilde{J}_T &= O\left(T^{1-H}\right),\\ \frac{\tilde{J}_T}{T^{1-H}} &\xrightarrow{d} \mathcal{N}\left(0, \frac{1}{\kappa^2\lambda_H}\right),\end{aligned}$$

which implies (35).

Now, we consider (36). Using the Cauchy–Schwarz inequality, (33) and (34), we obtain:

$$\begin{aligned}\int_0^T V_H(t)\,\tilde{P}_H(t)\,d\omega_t^H &\leq \sqrt{\int_0^T V_H^2(t)\,d\omega_t^H \int_0^T \tilde{P}_H^2(t)\,d\omega_t^H}\\ &= \sqrt{O(1)O_p(T)}\,,\end{aligned}$$

which implies (36).

Finally, we are left with (37). Using (34), we can obtain:

$$\mathbb{E}\left[\left(\int_0^T V_H(t)\,dM_t^H\right)^2\right] = \int_0^T V_H^2(t)\,d\omega_t^H = O(1)\,,$$

which implies (37) directly.

Appendix A.2. Proof of Theorem 1

To simplify the notations, let $\widetilde{X}_0 := X_0 - \frac{\alpha}{\kappa}$. Using (14), we have:

$$\begin{aligned}\int_0^T P_H^2(t)\,d\omega_t^H &= \int_0^T \left[\frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma}\widetilde{X}_0 V_H(t) + \tilde{P}_H(t)\right]^2 d\omega_t^H\\ &= \frac{\alpha^2}{\sigma^2\kappa^2}\omega_T^H + \frac{1}{\sigma^2}\widetilde{X}_0^2\int_0^T V_H^2(t)\,d\omega_t^H + \int_0^T \tilde{P}_H^2(t)\,d\omega_t^H\\ &\quad + \frac{2\alpha}{\sigma^2\kappa}\widetilde{X}_0\int_0^T V_H(t)\,d\omega_t^H + \frac{2\alpha}{\sigma\kappa}\int_0^T \tilde{P}_H(t)\,d\omega_t^H\\ &\quad + \frac{2}{\sigma}\widetilde{X}_0\int_0^T V_H(t)\,\tilde{P}_H(t)\,d\omega_t^H\,.\end{aligned} \tag{A9}$$

Using (14) again, we obtain:

$$\begin{aligned}\left(\int_0^T P_H(t)\,d\omega_t^H\right)^2 &= \left[\int_0^T \left(\frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma}\widetilde{X}_0 V_H(t) + \tilde{P}_H(t)\right) d\omega_t^H\right]^2\\ &= \frac{\alpha^2}{\sigma^2\kappa^2}\left(\omega_T^H\right)^2 + \frac{1}{\sigma^2}\widetilde{X}_0^2\left(\int_0^T V_H(t)\,d\omega_t^H\right)^2 + \left(\int_0^T \tilde{P}_H(t)\,d\omega_t^H\right)^2\\ &\quad + \frac{2\alpha}{\sigma\kappa}\omega_T^H\int_0^T \tilde{P}_H(t)\,d\omega_t^H + \frac{2\alpha}{\sigma^2\kappa}\omega_T^H\widetilde{X}_0\int_0^T V_H(t)\,d\omega_t^H\\ &\quad + \frac{2}{\sigma}\widetilde{X}_0\int_0^T V_H(t)\,d\omega_t^H\int_0^T \tilde{P}_H(t)\,d\omega_t^H\,.\end{aligned} \tag{A10}$$

Using (A9), (A10) and Lemma 1, we deduce that:

$$\begin{aligned}
& \omega_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \left(\int_0^T P_H(t)\, d\omega_t^H \right)^2 \\
= \; & \frac{\omega_T^H}{\sigma^2} \widetilde{X}_0^2 \int_0^T V_H^2(t)\, d\omega_t^H + \omega_T^H \int_0^T \tilde{P}_H^2(t)\, d\omega_t^H + \omega_T^H \frac{2\alpha}{\sigma^2\kappa} \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H \\
& + \omega_T^H \frac{2\alpha}{\sigma\kappa} \int_0^T \tilde{P}_H(t)\, d\omega_t^H + \omega_T^H \frac{2}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, \tilde{P}_H(t)\, d\omega_t^H \\
& - \frac{1}{\sigma^2} \widetilde{X}_0^2 \left(\int_0^T V_H(t)\, d\omega_t^H \right)^2 - \left(\int_0^T \tilde{P}_H(t)\, d\omega_t^H \right)^2 - \frac{2\alpha}{\sigma^2\kappa} \omega_T^H \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H \\
& - \frac{2\alpha}{\sigma\kappa} \omega_T^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H - \frac{2}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
= \; & \omega_T^H \int_0^T \tilde{P}_H^2(t)\, d\omega_t^H + o_p(T^{3-2H}) .
\end{aligned} \tag{A11}$$

Moreover, using (14), we get:

$$\begin{aligned}
& \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H \\
= \; & \left[\frac{\alpha}{\sigma\kappa} M_T^H + \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, dM_t^H + \int_0^T \tilde{P}_H(t)\, dM_t^H \right] \times \\
& \left[\frac{\alpha}{\sigma\kappa} \omega_t^H + \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H + \int_0^T \tilde{P}_H(t)\, d\omega_t^H \right] \\
= \; & \frac{\alpha^2}{\sigma^2\kappa^2} M_T^H \omega_T^H + \frac{\alpha}{\sigma\kappa} M_T^H \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H + \frac{\alpha}{\sigma\kappa} M_T^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
& + \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, dM_t^H \frac{\alpha}{\sigma\kappa} \omega_T^H + \frac{1}{\sigma^2} \widetilde{X}_0^2 \int_0^T V_H(t)\, dM_t^H \int_0^T V_H(t)\, d\omega_t^H \\
& + \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, dM_t^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H + \frac{\alpha}{\sigma\kappa} \omega_T^H \int_0^T \tilde{P}_H(t)\, dM_t^H \\
& + \frac{1}{\sigma} \widetilde{X}_0 \int_0^T \tilde{P}_H(t)\, dM_t^H \int_0^T V_H(t)\, d\omega_t^H + \int_0^T \tilde{P}_H(t)\, dM_t^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H .
\end{aligned} \tag{A12}$$

By combining (A9), (A12) and Lemma 1, we have:

$$\begin{aligned}
& M_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H \\
= \; & \frac{\alpha^2}{\sigma^2\kappa^2} M_T^H \omega_T^H + \frac{M_T^H}{\sigma^2} \widetilde{X}_0^2 \int_0^T V_H^2(t)\, d\omega_t^H + M_T^H \int_0^T \tilde{P}_H^2(t)\, d\omega_t^H \\
& + M_T^H \frac{2\alpha}{\sigma^2\kappa} \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H + M_T^H \frac{2\alpha}{\sigma\kappa} \int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
& + M_T^H \frac{2}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, \tilde{P}_H(t)\, d\omega_t^H - \frac{\alpha^2}{\sigma^2\kappa^2} M_T^H \omega_T^H \\
& - \frac{\alpha}{\sigma\kappa} M_T^H \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H - \frac{\alpha}{\sigma\kappa} M_T^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
& - \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, dM_t^H \frac{\alpha}{\sigma\kappa} \omega_T^H - \frac{1}{\sigma^2} \widetilde{X}_0^2 \int_0^T V_H(t)\, dM_t^H \int_0^T V_H(t)\, d\omega_t^H \\
& - \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, dM_t^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H - \frac{\alpha}{\sigma\kappa} \omega_T^H \int_0^T \tilde{P}_H(t)\, dM_t^H \\
& - \frac{1}{\sigma} \widetilde{X}_0 \int_0^T \tilde{P}_H(t)\, dM_t^H \int_0^T V_H(t)\, d\omega_t^H - \int_0^T \tilde{P}_H(t)\, dM_t^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
= \; & - \frac{\alpha}{\sigma\kappa} \omega_T^H \int_0^T \tilde{P}_H(t)\, dM_t^H + o_p\left(T^{\frac{5}{2}-2H} \right) .
\end{aligned} \tag{A13}$$

According to Corollary 5.2 of Jost (2006), for $H < 1/2$, we have the relationship between B_t^H and B_t^{1-H},

$$B_t^H = \left(\frac{2H}{\Gamma(2H)} \Gamma(3-2H) \right)^{\frac{1}{2}} \int_0^t (t-s)^{2H-1} \, dB_s^{1-H}. \tag{A14}$$

Using (A14), following Equation (34) in Brouste and Kleptsyna (2010), we can transform Model (1) to the following model,

$$d\widehat{X}_t = \kappa \left(\mu - \widehat{X}_t \right) dt + \sigma dB_t^{1-H},$$

where $\widehat{X}_t = \left(\frac{2H}{\Gamma(2H)} \Gamma(3-2H) \right)^{\frac{1}{2}} \int_0^t (t-s)^{1-2H} dX_s$. When $H < 1/2$ and $1 - H > 1/2$, hence, the results in (4.4) and (4.5) of Lohvinenko and Ralchenko (2017) are valid for all $H \in (0,1)$.

Now, combining (21), (A13), (A11) and (4.4), (4.5) in Lohvinenko and Ralchenko (2017) with Slutsky's theorem, we obtain:

$$\begin{aligned}
\sqrt{T}\left(\tilde{\alpha}_T - \alpha\right) &= \frac{\sqrt{T}\left[M_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \int_0^T P_H(t)\, dM_t^H \int_0^T P_H(t)\, d\omega_t^H \right]}{\omega_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \left(\int_0^T P_H(t)\, d\omega_t^H \right)^2} \sigma \\
&= \frac{-\frac{\alpha}{\sigma\kappa}\omega_T^H \frac{1}{\sqrt{T}} \int_0^T \tilde{P}_H(t)\, dM_t^H + o_p\left(T^{2-2H}\right)}{\omega_T^H \frac{1}{T} \int_0^T \tilde{P}_H^2(t)\, d\omega_t^H + o_p(T^{2-2H})} \sigma \\
&\xrightarrow{d} \mathcal{N}\left(0, \frac{2\alpha^2}{\kappa}\right).
\end{aligned}$$

Now, we consider (39). Using (14) and (A10), we have:

$$\begin{aligned}
& M_T^H \int_0^T P_H(t)\, d\omega_t^H - \omega_T^H \int_0^T P_H(t)\, dM_t^H \\
= \;& M_T^H \frac{\alpha}{\sigma\kappa} \omega_T^H + M_T^H \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H + M_T^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
& - \left[\omega_T^H \frac{\alpha}{\sigma\kappa} M_T^H + \omega_T^H \frac{1}{\sigma} \widetilde{X}_0 \int_0^T V_H(t)\, dM_t^H + \omega_T^H \int_0^T \tilde{P}_H(t)\, dM_t^H \right] \\
= \;& -\omega_T^H \int_0^T \tilde{P}_H(t)\, dM_t^H + o_p\left(T^{\frac{5}{2}-2H}\right).
\end{aligned} \tag{A15}$$

Finally, combining (22), (A15), (A11) and (4.4), (4.5) in Lohvinenko and Ralchenko (2017) with Slutsky's theorem, we have:

$$\sqrt{T}\left(\tilde{\kappa}_T - \kappa\right) = \frac{-\omega_T^H \frac{1}{\sqrt{T}} \int_0^T \tilde{P}_H(t)\, dM_t^H + o_p(T^{2-2H})}{\omega_T^H \frac{1}{T} \int_0^T \tilde{P}_H^2(t)\, d\omega_t^H + o_p(T^{2-2H})} \xrightarrow{d} \mathcal{N}(0, 2\kappa).$$

Appendix A.3. Proof of Lemma 2

From the proof of Theorem 2 in Tanaka (2013), we can easily obtain (48) and (49). A simple calculation shows that:

$$\int_0^T t d\omega_t^H = \int_0^T t \frac{1}{\lambda_H} (2-2H)\, t^{1-2H} dt = \frac{1}{\lambda_H} \frac{2-2H}{3-2H} T^{3-2H}.$$

Similarly, a standard calculation yields:

$$\int_0^T t^2 d\omega_t^H = \int_0^T t^2 \frac{1}{\lambda_H} (2-2H)\, t^{1-2H} dt = \frac{1}{\lambda_H} \frac{1-H}{2-H} T^{4-2H}.$$

Combining (49) with the Cauchy–Schwarz inequality, we have:

$$\int_0^T \hat{P}_H(t)\, d\omega_t^H \leq \sqrt{\omega_T^H \int_0^T \hat{P}_H^2(t)\, d\omega_t^H} = O_p\left(T^{2-H}\right).$$

Using (49), (51) and the Cauchy–Schwarz inequality, we obtain:

$$\int_0^T t\hat{P}_H(t)\, d\omega_t^H \leq \sqrt{\int_0^T t^2 d\omega_t^H \int_0^T \hat{P}_H^2(t)\, d\omega_t^H} = O_p\left(T^{3-H}\right).$$

Now, we consider (54). Form the definition of $V_H(T)$, we conclude that:

$$\begin{aligned}
V_H(T) &= \frac{d}{d\omega_T^H}\int_0^T k_H(T,s)\, e^{-\kappa s} ds = \frac{d}{d\omega_T^H}\int_0^T k_H(T,s)\, ds \\
&= \frac{d}{d\omega_T^H}\left[\frac{1}{k_H}\int_0^T (s(T-s))^{\frac{1}{2}-H} ds\right] \\
&= \frac{d}{d\omega_T^H}\left[\frac{1}{k_H}\int_0^1 T^{1-2H}(u(1-u))^{\frac{1}{2}-H} T du\right] \\
&= \frac{d}{dT}\left[\frac{1}{k_H}T^{2-2H}B\left(\frac{3}{2}-H,\frac{3}{2}-H\right)\right] \Big/ \frac{d\omega_T^H}{dT} \\
&= 1.
\end{aligned}$$

Finally, we deal with (55). A standard calculation yields:

$$\begin{aligned}
\frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\, s ds &= \frac{d}{d\omega_t^H}\left[\frac{1}{k_H}\int_0^t (s(t-s))^{\frac{1}{2}-H} s ds\right] \\
&= \frac{d}{d\omega_t^H}\left[\frac{1}{k_H}\int_0^t s^{\frac{3}{2}-H}(t-s)^{\frac{1}{2}-H} ds\right] \\
&= \frac{d}{d\omega_t^H}\left[\frac{1}{k_H}\int_0^1 (vt)^{\frac{3}{2}-H}(t-vt)^{\frac{1}{2}-H} t dv\right] \\
&= \frac{d}{d\omega_t^H}\left[\frac{1}{k_H}t^{3-2H}\int_0^1 v^{\frac{3}{2}-H}(1-v)^{\frac{1}{2}-H} dv\right] \\
&= \frac{d}{d\omega_t^H}\left[\frac{1}{k_H}t^{3-2H}B\left(\frac{5}{2}-H,\frac{3}{2}-H\right)\right] \\
&= a_H t,
\end{aligned}$$

where $a_H = (3-2H)/(4-4H)$, and the proof of this lemma is complete.

Appendix A.4. Proof of Theorem 3

Using (13), (46) and (55), we have:

$$\begin{aligned}
P_H(t) &= \frac{1}{\sigma}\frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\, X_s ds \\
&= \frac{1}{\sigma}\frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\left[X_0 + \alpha s + \sigma B_s^H\right] ds \\
&= \frac{X_0}{\sigma} + \frac{\alpha}{\sigma}\frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\, s ds + \hat{P}_H(t) \\
&= \frac{X_0}{\sigma} + \frac{\alpha}{\sigma}a_H t + \hat{P}_H(t),
\end{aligned} \tag{A16}$$

where $\hat{P}_H(t) = \frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\, B_s^H ds$. Using (49)–(53) and (A16), we have:

$$\begin{aligned}
\int_0^T P_H^2(t)\, d\omega_t^H &= \int_0^T \left[\frac{X_0}{\sigma} + \frac{\alpha}{\sigma}a_H t + \hat{P}_H(t)\right]^2 d\omega_t^H \\
&= \frac{X_0^2}{\sigma^2}\omega_T^H + \frac{\alpha^2}{\sigma^2}a_H^2\int_0^T t^2 d\omega_t^H + \int_0^T \hat{P}_H^2(t)\, d\omega_t^H \\
&\quad +2\frac{X_0}{\sigma^2}\alpha a_H\int_0^T t d\omega_t^H + \frac{2X_0}{\sigma}\int_0^T \hat{P}_H(t)\, d\omega_t^H \\
&\quad +\frac{2\alpha}{\sigma}a_H\int_0^T t\hat{P}_H(t)\, d\omega_t^H \\
&= \frac{\alpha^2}{\sigma^2}a_H^2\int_0^T t^2 d\omega_t^H + o_p(T^{4-2H}). \qquad \text{(A17)}
\end{aligned}$$

Similarly, combining (50) with (52) leads to:

$$\begin{aligned}
\int_0^T P_H(t)\, d\omega_t^H &= \int_0^T \left[\frac{X_0}{\sigma} + \frac{\alpha}{\sigma}a_H t + \hat{P}_H(t)\right] d\omega_t^H \\
&= \frac{X_0}{\sigma}\omega_T^H + \frac{\alpha}{\sigma}a_H\int_0^T t d\omega_t^H + \int_0^T \hat{P}_H(t)\, d\omega_t^H \\
&= \frac{\alpha}{\sigma}a_H\int_0^T t d\omega_t^H + o_p(T^{3-2H}). \qquad \text{(A18)}
\end{aligned}$$

Moreover, using (48) and (A16), we have:

$$\begin{aligned}
\int_0^T P_H(t)\, dM_t^H &= \int_0^T \left[\frac{X_0}{\sigma} + \frac{\alpha}{\sigma}a_H t + \hat{P}_H(t)\right] dM_t^H \\
&= \frac{X_0}{\sigma}M_T^H + \frac{\alpha}{\sigma}a_H\int_0^T t dM_t^H + \int_0^T \hat{P}_H(t)\, dM_t^H \\
&= \frac{X_0}{\sigma}M_T^H + \frac{\alpha}{\sigma}a_H\int_0^T t dM_t^H + O_p(T). \qquad \text{(A19)}
\end{aligned}$$

According to (A17) and (A18), we get:

$$\begin{aligned}
&\omega_T^H\int_0^T P_H^2(t)\, d\omega_t^H - \left(\int_0^T P_H(t)\, d\omega_t^H\right)^2 \\
={}& \frac{1}{\lambda_H}T^{2-2H}\frac{\alpha^2}{\sigma^2}a_H^2\int_0^T t^2 d\omega_t^H - \frac{\alpha^2}{\sigma^2}a_H^2\frac{1}{\lambda_H^2}\frac{(2-2H)^2}{(3-2H)^2}T^{6-4H} + o_p(T^{6-4H}) \\
={}& \frac{T^{6-4H}}{\lambda_H^2}\frac{\alpha^2}{\sigma^2}a_H^2\left(\frac{1-H}{2-H} - \frac{(2-2H)^2}{(3-2H)^2}\right) + o_p(T^{6-4H}) \\
={}& \frac{T^{6-4H}}{\sigma^2\lambda_H^2}\alpha^2 a_H^2\frac{1-H}{(2-H)(3-2H)^2} + o_p\left(T^{6-4H}\right). \qquad \text{(A20)}
\end{aligned}$$

Similarly, applying (A17) and (A19), we have:

$$\begin{aligned}
& M_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \int_0^T P_H(t)\, d\omega_t^H \int_0^T P_H(t)\, dM_t^H \\
= \; & M_T^H \frac{\alpha^2}{\sigma^2} a_H^2 \int_0^T t^2 d\omega_t^H - \frac{\alpha}{\sigma} a_H \int_0^T t d\omega_t^H \frac{\alpha}{\sigma} a_H \int_0^T t dM_t^H + o_p(T^{5-3H}) \\
= \; & \frac{\alpha^2}{\sigma^2} a_H^2 \left[M_T^H \frac{T^{4-2H}}{\lambda_H} \frac{1-H}{2-H} - \frac{T^{3-2H}}{\lambda_H} \frac{2-2H}{3-2H} \int_0^T t dM_t^H \right] + o_p(T^{5-3H}) \\
= \; & \frac{\alpha^2 a_H^2 T^{5-3H}}{\lambda_H \sigma^2} \left[\frac{1-H}{2-H} \frac{M_T^H}{T^{1-H}} - \frac{2-2H}{3-2H} \frac{1}{T^{2-H}} \int_0^T t dM_t^H \right] + o_p(T^{5-3H}) .
\end{aligned} \tag{A21}$$

Using (A21), we can see that:

$$\begin{aligned}
& \frac{1}{T^{5-3H}} \left[M_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \int_0^T P_H(t)\, d\omega_t^H \int_0^T P_H(t)\, dM_t^H \right] \\
= \; & \frac{\alpha^2 a_H^2}{\lambda_H \sigma^2} \left[\frac{1-H}{2-H} \frac{M_T^H}{T^{1-H}} - \frac{2-2H}{3-2H} \frac{1}{T^{2-H}} \int_0^T t dM_t^H \right] + o_p(1) \\
\xrightarrow{d} \; & \mathcal{N}\left(0, \frac{(1-H)^2}{(2-H)^2} \frac{1}{\lambda_H} - \frac{1-H}{2-H} \frac{(2-2H)^2}{(3-2H)^2} \frac{1}{\lambda_H}\right) .
\end{aligned} \tag{A22}$$

Consequently, combining (21) and (A20)–(A22) with Slutsky's theorem, we have:

$$\begin{aligned}
T^{1-H}(\tilde{\alpha}_T - \alpha) \; = \; & \frac{\frac{1}{T^{5-3H}} \left[M_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \int_0^T P_H(t)\, dM_t^H \int_0^T P_H(t)\, d\omega_t^H \right]}{\frac{1}{T^{6-4H}} \left[\omega_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \left(\int_0^T P_H(t)\, d\omega_t^H \right)^2 \right]} \sigma \\
\xrightarrow{d} \; & \mathcal{N}\left(0, \sigma^2 \rho_H\right) .
\end{aligned}$$

By (7), (A18), (A19) and the fact that $M_T^H = O_p(T^{1-H})$, we obtain:

$$\begin{aligned}
& M_T^H \int_0^T P_H(t)\, d\omega_t^H - \omega_T^H \int_0^T P_H(t)\, dM_t^H \\
= \; & M_T^H \frac{\alpha}{\sigma} a_H \int_0^T t d\omega_t^H - \omega_T^H \frac{\alpha}{\sigma} a_H \int_0^T t dM_t^H + o_p\left(T^{4-3H}\right) \\
= \; & \frac{\alpha}{\sigma} a_H \left[M_T^H \frac{1}{\lambda_H} \frac{2-2H}{3-2H} T^{3-2H} - \frac{1}{\lambda_H} T^{2-2H} \int_0^T t dM_t^H \right] + o_p\left(T^{4-3H}\right) \\
= \; & \frac{\alpha}{\sigma \lambda_H} a_H T^{4-3H} \left[\frac{M_T^H}{T^{1-H}} \frac{2-2H}{3-2H} - \frac{1}{T^{2-H}} \int_0^T t dM_t^H \right] + o_p\left(T^{4-3H}\right) .
\end{aligned} \tag{A23}$$

Using (22), (A20), (A23) and Slutsky's theorem, we can see that:

$$\begin{aligned}
T^{2-H}(\tilde{\kappa}_T - \kappa) \; = \; & \frac{\frac{1}{T^{4-3H}} \left[M_T^H \int_0^T P_H(t)\, d\omega_t^H - \omega_T^H \int_0^T P_H(t)\, dM_t^H \right]}{\frac{1}{T^{6-4H}} \left[\omega_T^H \int_0^T P_H^2(t)\, d\omega_t^H - \left(\int_0^T P_H(t)\, d\omega_t^H \right)^2 \right]} \\
\xrightarrow{d} \; & \mathcal{N}\left(0, \frac{\sigma^2}{\alpha^2} \phi_H\right) .
\end{aligned}$$

Appendix A.5. Proof of Theorem 4

Using (17), (61) and (62), we can obtain:

$$\begin{aligned}
\int_0^T X_t^2 dt &= \int_0^T \left[\frac{\alpha}{\kappa} + e^{-\kappa t}\widetilde{X}_0 + \sigma U_t\right]^2 dt \\
&= \frac{\alpha^2}{\kappa^2} T + \widetilde{X}_0^2 \int_0^T e^{-2\kappa t} dt + \sigma^2 \int_0^T U_t^2 dt + \frac{2\alpha}{\kappa}\widetilde{X}_0 \int_0^T e^{-\kappa t} dt \\
&\quad + \frac{2\alpha\sigma}{\kappa}\int_0^T U_t dt + 2\widetilde{X}_0\sigma \int_0^T e^{-\kappa t} U_t dt \\
&= \widetilde{X}_0^2 \int_0^T e^{-2\kappa t} dt + \int_0^T e^{-2\kappa t}\xi_t^2 dt + 2\widetilde{X}_0 \int_0^T e^{-2\kappa t}\xi_t dt + o_p\left(e^{-2\kappa T}\right) \\
&= \int_0^T e^{-2\kappa t}\left(\widetilde{X}_0 + \xi_t\right)^2 dt + o_p(e^{-2\kappa T}). \qquad \text{(A24)}
\end{aligned}$$

Similarly, using (17), (61) and (62) again, we can easily have:

$$\begin{aligned}
\int_0^T X_t dt &= \int_0^T \left[\frac{\alpha}{\kappa} + e^{-\kappa t}\widetilde{X}_0 + \sigma U_t\right] dt \\
&= \frac{\alpha}{\kappa} T + \widetilde{X}_0 \frac{1}{\kappa}\left(1 - e^{-\kappa T}\right) + \sigma \int_0^T U_t dt \\
&= O_p\left(e^{-\kappa T}\right). \qquad \text{(A25)}
\end{aligned}$$

A straightforward calculation shows:

$$\begin{aligned}
\int_0^T X_t dW_t &= \int_0^T \left[\frac{\alpha}{\kappa} + e^{-\kappa t}\widetilde{X}_0 + \sigma U_t\right] dW_t \\
&= \frac{\alpha}{\kappa} W_T + \widetilde{X}_0 \int_0^T e^{-\kappa t} dW_t + \sigma \int_0^T U_t dW_t \\
&= O_p\left(e^{-\kappa T}\right). \qquad \text{(A26)}
\end{aligned}$$

From the definition of ξ_t, we can rewrite X_t as $X_t = \frac{\alpha}{\kappa} + e^{-\kappa t}\widetilde{X}_0 + e^{-\kappa t}\xi_t$. As a consequence, using (63), (64) and (A24), we can see that:

$$e^{2\kappa T}\int_0^T X_t^2 dt = \frac{\int_0^T e^{-2\kappa t}\left(\widetilde{X}_0 + \xi_t\right)^2 dt}{e^{-2\kappa T}} + o_p(1), \qquad \text{(A27)}$$

$$\sigma e^{\kappa T}\int_0^T X_t dW_t = \widetilde{X}_0\sigma \int_0^T e^{\kappa(T-t)} dW_t + \sigma \int_0^T e^{\kappa(T-t)}\xi_t dW_t + o_p(1). \qquad \text{(A28)}$$

Now, applying (21) and (A25)–(A27) and Slutsky's theorem, we deduce:

$$\begin{aligned}
\sqrt{T}\left(\tilde{\alpha}_T - \alpha\right) &= \frac{\frac{W_T}{\sqrt{T}} e^{2\kappa T}\int_0^T X_t^2 dt - \frac{e^{2\kappa T}}{\sqrt{T}}\int_0^T X_t dW_t \int_0^T X_t dt}{e^{2\kappa T}\left(\int_0^T X_t^2 dt - \frac{1}{T}\left(\int_0^T X_t dt\right)^2\right)}\sigma \\
&= \frac{\frac{W_T}{\sqrt{T}} e^{2\kappa T}\int_0^T X_t^2 dt + o_p(1)}{e^{2\kappa T}\int_0^T X_t^2 dt + o_p(1)}\sigma = \frac{\sigma W_T}{\sqrt{T}} + o_p(1),
\end{aligned}$$

which implies (65).

Finally, using (22), (A25), (A27), (A28) and Slutsky's theorem, we have:

$$\begin{aligned} e^{-\kappa T}\left(\tilde{\kappa}_T-\kappa\right) &= e^{-\kappa T}\left(\frac{\frac{W_T}{T}\int_0^T X_t dt-\int_0^T X_t dW_t}{\int_0^T X_t^2 dt-\frac{1}{T}\left(\int_0^T X_t dt\right)^2}\sigma\right) \\ &= \frac{-\sigma e^{\kappa T}\int_0^T X_t dW_t+o_p(1)}{e^{2\kappa T}\int_0^T X_t^2 dt+o_p(1)} \xrightarrow{d} \frac{-\left(X_0-\frac{\alpha}{\kappa}+\xi_\infty\right)\eta_\infty}{-\frac{1}{2\kappa}\left(X_0-\frac{\alpha}{\kappa}+\xi_\infty\right)^2}, \end{aligned}$$

which yields (66), and the proof is done.

Appendix A.6. Proof of Lemma 3

We first consider (67). Applying (14) and (15), we can obtain:

$$\begin{aligned} V_H(t) &= \frac{d}{d\omega_t^H}\int_0^t k_H(t,s)\,e^{-\kappa s}ds \\ &= \frac{\frac{d}{dt}\int_0^t k_H(t,s)e^{-\kappa s}ds}{\frac{d\omega_t^H}{dt}} \\ &= \frac{\lambda_H\sqrt{\pi}(-\kappa)^{H-1}\Gamma(3/2-H)}{k_H(2-2H)t^{1-2H}}\Bigg[(1-H)\,t^{-H}e^{-\frac{\kappa t}{2}}I_{1-H}\left(-\tfrac{\kappa t}{2}\right) \\ &\quad -\tfrac{\kappa}{2}t^{1-H}e^{-\frac{\kappa t}{2}}I_{1-H}\left(-\tfrac{\kappa t}{2}\right)+t^{1-H}e^{-\frac{\kappa t}{2}}\tfrac{1}{2}\left(-\tfrac{\kappa}{2}\right)\left(I_{2-H}\left(-\tfrac{\kappa t}{2}\right)+I_{-H}\left(-\tfrac{\kappa t}{2}\right)\right)\Bigg] \\ &= \frac{\lambda_H\sqrt{\pi}(-\kappa)^{H-1}\Gamma\left(\frac{3}{2}-H\right)}{k_H(2-2H)}\Bigg[(1-H)\,t^{-1+H}e^{-\frac{\kappa t}{2}}I_{1-H}\left(-\tfrac{\kappa t}{2}\right) \\ &\quad -\tfrac{\kappa}{2}t^{H}e^{-\frac{\kappa t}{2}}I_{1-H}\left(-\tfrac{\kappa t}{2}\right)-\tfrac{\kappa}{4}t^{H}e^{-\frac{\kappa t}{2}}I_{2-H}\left(-\tfrac{\kappa t}{2}\right)-\tfrac{\kappa}{4}t^{H}e^{-\frac{\kappa t}{2}}I_{-H}\left(-\tfrac{\kappa t}{2}\right)\Bigg] \\ &= \frac{\lambda_H\sqrt{\pi}(-\kappa)^{H-1}\Gamma\left(\frac{3}{2}-H\right)}{k_H(2-2H)}\Bigg[(1-H)\,t^{-1+H}\frac{e^{-\kappa t}}{\sqrt{-\pi\kappa t}}\left(1+O\left(t^{-1}\right)\right) \\ &\quad -\tfrac{\kappa}{2}t^{H}\frac{e^{-\kappa t}}{\sqrt{-\pi\kappa t}}\left(1+O\left(t^{-1}\right)\right)-\tfrac{\kappa}{4}t^{H}\frac{e^{-\kappa t}}{\sqrt{-\pi\kappa t}}\left(1+O\left(t^{-1}\right)\right) \\ &\quad -\tfrac{\kappa}{4}t^{H}\frac{e^{-\kappa t}}{\sqrt{-\pi\kappa t}}\left(1+O\left(t^{-1}\right)\right)\Bigg] \\ &= \frac{\lambda_H(-\kappa)^{H-1}\Gamma\left(\frac{3}{2}-1\right)}{k_H(2-2H)}e^{-\kappa t}\Bigg[(1-H)\,(-\kappa)^{-\frac{1}{2}}t^{-\frac{3}{2}+H}\left(1+O\left(t^{-1}\right)\right) \\ &\quad +\tfrac{1}{2}(-\kappa)^{\frac{1}{2}}t^{H-\frac{1}{2}}\left(1+O\left(t^{-1}\right)\right) \\ &\quad +\tfrac{1}{4}t^{H-\frac{1}{2}}(-\kappa)^{\frac{1}{2}}\left(1+O\left(t^{-1}\right)\right)+\tfrac{1}{4}(-\kappa)^{\frac{1}{2}}t^{H-\frac{1}{2}}\left(1+O\left(t^{-1}\right)\right)\Bigg] \\ &= O\left(t^{H-\frac{1}{2}}e^{-\kappa t}\right). \end{aligned} \tag{A29}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind defined in (A1) and we used the asymptotic behavior of (A2).

Let us observe that (68) can be obtained easily from Theorem 2 in Tanaka (2015), and the details are omitted here. For (69), using (68), we have:

$$\mathbb{E}\left[\left(\int_0^T \tilde{P}_H(t)\,dM_t^H\right)^2\right]=\int_0^T \tilde{P}_H^2(t)\,d\omega_t^H=O(e^{-2\kappa T}),$$

which implies (69) directly.

Let ${}_1F_1(\cdot,\cdot,\cdot)$ be the confluent hypergeometric function of the first kind. From (A29) and the well-known result of the confluent hypergeometric function (see, for example, Equation 3.383 (1) in Gradshteyn and Ryzhik 2007; Bateman 1953, p. 278), we have:

$$\begin{aligned}
\frac{\int_0^T V_H(t)\,d\omega_t^H}{T^{\frac{1}{2}-H}e^{-\kappa T}} &\le C\int_0^T \left(\frac{t}{T}\right)^{\frac{1}{2}-H} e^{\kappa(T-t)}dt \\
&= CT\int_0^1 u^{\frac{1}{2}-H}e^{\kappa T(1-u)}du \\
&= CT\int_0^1 (1-v)^{\frac{1}{2}-H}e^{\kappa T v}dv \\
&= CT\,{}_1F_1\left(1,\frac{5}{2}-H,\kappa T\right) \\
&= O(1),
\end{aligned}$$

which yields (70).

We now deal with (71). Let $\zeta_t = \sigma\int_0^t e^{\kappa s}dB_s^H$. Then, using Lemma 2.2 of El Machkouri et al. (2016), as $T\to\infty$, we have:

$$\zeta_T \xrightarrow{p} \zeta_\infty \sim \mathcal{N}\left(0,\mathbb{E}\left[\zeta_\infty^2\right]\right). \tag{A30}$$

Using (6), (17), (A30) and the property of the confluent hypergeometric function (see, for example, Equation 3.383 (1) in Gradshteyn and Ryzhik 2007; Bateman 1953, p. 278), we have:

$$\begin{aligned}
\int_0^T \tilde{P}_H(t)\,d\omega_t^H &= \int_0^T k_H(T,t)\,U_t dt \\
&= \frac{1}{k_H}\int_0^T (t(T-t))^{\frac{1}{2}-H}\frac{e^{-\kappa t}}{\sigma}\zeta_t dt \\
&= CT^{2-2H}\int_0^1 (u(1-u))^{\frac{1}{2}-H}e^{-\kappa T u}\zeta_{Tu}du \\
&= O_p(1)T^{2-2H}\int_0^1 (u(1-u))^{\frac{1}{2}-H}e^{-\kappa T u}du \\
&= O_p(1)T^{2-2H}\,{}_1F_1\left(\frac{3}{2}-H,3-2H,-\kappa T\right) \\
&= O_p(1)T^{2-2H}O_p(T^{H-\frac{3}{2}}e^{-\kappa T}) \\
&= O_p(T^{\frac{1}{2}-H}e^{-\kappa T}),
\end{aligned}$$

which implies (71).

We now turn to the term (72). Using (A29), we can easily obtain:

$$\int_0^T V_H^2(t)\,d\omega_t^H \le C\int_0^T t^{2H-1}e^{-2\kappa t}t^{1-2H}dt = O(e^{-2\kappa T}),$$

which yields (72).

Using the Cauchy–Schwarz inequality, (68) and (72), we obtain:

$$\left(\int_0^T V_H(t)\,\tilde{P}_H(t)\,d\omega_t^H\right)^2 \le \int_0^T V_H^2(t)\,d\omega_t^H\int_0^T \tilde{P}_H^2(t)\,d\omega_t^H = O_p\left(e^{-4\kappa T}\right),$$

which implies (73).

Similarly, using (72), we have:

$$\mathbb{E}\left[\left(\int_0^T V_H(t)\,dM_t^H\right)^2\right] = \int_0^T V_H^2(t)\,d\omega_t^H = O\left(e^{-2\kappa T}\right),$$

which yields (74), and we complete the proof.

Appendix A.7. Proof of Theorem 5

Using (14), (68) and (70)–(73), we can obtain:

$$\begin{aligned}
\int_0^T P_H^2(t)\, d\omega_t^H &= \int_0^T \left[\frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma}\widetilde{X}_0 V_H(t) + \tilde{P}_H(t)\right]^2 d\omega_t^H \\
&= \frac{\alpha^2}{\sigma^2\kappa^2}\omega_T^H + \frac{1}{\sigma^2}\widetilde{X}_0^2 \int_0^T V_H^2(t)\, d\omega_t^H + \int_0^T \tilde{P}_H^2(t)\, d\omega_t^H \\
&\quad + \frac{2\alpha}{\sigma^2\kappa}\widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H + \frac{2\alpha}{\sigma\kappa}\int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
&\quad + \frac{2}{\sigma}\widetilde{X}_0 \int_0^T V_H(t)\, \tilde{P}_H(t)\, d\omega_t^H \\
&= \frac{1}{\sigma^2}\widetilde{X}_0^2 \int_0^T V_H^2(t)\, d\omega_t^H + \int_0^T \tilde{P}_H^2(t)\, d\omega_t^H \\
&\quad + \frac{2}{\sigma}\widetilde{X}_0 \int_0^T V_H(t)\, \tilde{P}_H(t)\, d\omega_T^H + o_p\left(e^{-2\kappa T}\right) \\
&= \int_0^T \left(\frac{1}{\sigma}\widetilde{X}_0 V_H(t) + \tilde{P}_H(t)\right)^2 d\omega_t^H + o_p(e^{-2\kappa T}) .
\end{aligned} \tag{A31}$$

According to (14), (68), (70) and (71), we obtain:

$$\begin{aligned}
\frac{1}{\omega_T^H}\left(\int_0^T P_H(t)\, d\omega_t^H\right)^2 &= \frac{1}{\omega_T^H}\left[\int_0^T \left(\frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma}\widetilde{X}_0 V_H(t) + \tilde{P}_H(t)\right) d\omega_t^H\right]^2 \\
&= \frac{1}{\omega_T^H}\left[\frac{\alpha}{\sigma\kappa}\omega_T^H + \frac{1}{\sigma}\widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H + \int_0^T \tilde{P}_H(t)\, d\omega_t^H\right]^2 \\
&= \frac{1}{\omega_T^H}\Bigg[\frac{\alpha^2}{\sigma^2\kappa^2}\left(\omega_T^H\right)^2 + \frac{1}{\sigma^2}\widetilde{X}_0^2\left(\int_0^T V_H(t)\, d\omega_t^H\right)^2 \\
&\quad + \left(\int_0^T \tilde{P}_H(t)\, d\omega_t^H\right)^2 + \frac{2\alpha}{\sigma^2\kappa}\omega_T^H \int_0^T V_H(t)\, d\omega_t^H \\
&\quad + \frac{2\alpha}{\sigma\kappa}\omega_T^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H + \frac{2}{\sigma}\widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H \int_0^T \tilde{P}_H(t)\, d\omega_t^H\Bigg] \\
&= o_p\left(e^{-2\kappa T}\right) .
\end{aligned} \tag{A32}$$

From (14), (69) and (74), we can see that:

$$\begin{aligned}
\int_0^T P_H(t)\, dM_t^H &= \int_0^T \left[\frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma}\widetilde{X}_0 V_H(t) + \tilde{P}_H(t)\right] dM_t^H \\
&= \frac{\alpha}{\sigma\kappa}M_T^H + \frac{1}{\sigma}\widetilde{X}_0 \int_0^T V_H(t)\, dM_t^H + \int_0^T \tilde{P}_H(t)\, dM_t^H \\
&= \frac{\widetilde{X}_0}{\sigma}\int_0^T V_H(t)\, dM_t^H + \int_0^T \tilde{P}_H(t)\, dM_t^H + o_p(e^{-\kappa T}).
\end{aligned} \tag{A33}$$

From (14) and the definition of ω_t^H, we can obtain:

$$\begin{aligned}
\int_0^T P_H(t)\, d\omega_t^H &= \int_0^T \left[\frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma}\widetilde{X}_0 V_H(t) + \tilde{P}_H(t)\right] d\omega_t^H \\
&= \frac{\alpha}{\sigma\kappa}\omega_T^H + \frac{1}{\sigma}\widetilde{X}_0 \int_0^T V_H(t)\, d\omega_t^H + \int_0^T \tilde{P}_H(t)\, d\omega_t^H \\
&= \frac{\widetilde{X}_0}{\sigma}\int_0^T V_H(t)\, d\omega_t^H + \int_0^T \tilde{P}_H(t)\, d\omega_t^H + O(T^{2-2H}) .
\end{aligned} \tag{A34}$$

Using (A31), we have:

$$
\begin{aligned}
& \frac{M_T^H}{\omega_T^H}\int_0^T P_H^2(t)\,d\omega_t^H - \frac{1}{\omega_T^H}\int_0^T P_H(t)\,dM_t^H \int_0^T P_H(t)\,d\omega_t^H \\
= {} & \frac{M_T^H}{\omega_T^H}\int_0^T \left(\frac{1}{\sigma}\left(X_0-\frac{\alpha}{\kappa}\right)V_H(t)+\tilde{P}_H(t)\right)^2 d\omega_t^H \\
& -\frac{1}{\omega_T^H}\left[\frac{1}{\sigma}\left(X_0-\frac{\alpha}{\kappa}\right)\int_0^T V_H(t)\,dM_t^H+\int_0^T \tilde{P}_H(t)\,dM_t^H\right]\left[\int_0^T \tilde{P}_H(t)\,d\omega_t^H\right]+o_p\left(\frac{e^{-2\kappa T}}{T^{1-H}}\right).
\end{aligned}
$$

Now, combining the above result, (21), (A31), (A32) and Lemma 3 with Slutsky's theorem, we have:

$$
\begin{aligned}
T^{1-H}\left(\tilde{\alpha}_T-\alpha\right) &= \frac{T^{1-H}\frac{M_T^H}{\omega_T^H}\int_0^T P_H^2(t)\,d\omega_t^H - \frac{T^{1-H}}{\omega_T^H}\int_0^T P_H(t)\,dM_t^H\int_0^T P_H(t)\,d\omega_t^H}{\int_0^T P_H^2(t)\,d\omega_t^H-\frac{1}{\omega_T^H}\left(\int_0^T P_H(t)\,d\omega_t^H\right)^2}\sigma \\
&\xrightarrow{d} \mathcal{N}\left(0,\sigma^2\right).
\end{aligned}
$$

Moreover, from Equation (31) in Tanaka (2015), replacing H with $1-H$ leads to the same moment generating function. This suggests that the distribution of $\kappa - \int_0^T \tilde{P}_H(t)\,dM_t^H / \int_0^T \tilde{P}_H^2(t)\,d\omega_t^H$ is symmetric around $H=1/2$. Hence, Equation (33) in Tanaka (2015) holds true for all $H\in(0,1)$.

Now, let X and Y be two independent $\mathcal{N}(0,1)$ random variables. Then, using (14), (22), (A31)–(A34), Lemma 3, Slutsky's theorem and Equation (33) in Tanaka (2015), we can see that:

$$
\begin{aligned}
\frac{e^{-\kappa T}}{2\kappa}\left(\tilde{\kappa}_T-\kappa\right) &= \frac{\frac{e^{-\kappa T}}{2\kappa}\left[\frac{M_T^H}{\omega_T^H}\int_0^T P_H(t)\,d\omega_t^H-\int_0^T P_H(t)\,dM_t^H\right]}{\int_0^T P_H^2(t)\,d\omega_t^H-\frac{1}{\omega_T^H}\left(\int_0^T P_H(t)\,d\omega_t^H\right)^2} \\
&= \frac{-2\kappa e^{\kappa T}\int_0^T P_H(t)\,dM_t^H+o_p(1)}{4\kappa^2 e^{2\kappa T}\int_0^T P_H^2(t)\,d\omega_t^H+o_p(1)} \\
&= \frac{-2\kappa e^{\kappa T}\int_0^T\left[\frac{\widetilde{X}_0}{\sigma}V_H(t)+\tilde{P}_H(t)\right]dM_t^H+o_p(1)}{4\kappa^2 e^{2\kappa T}\int_0^T\left[\frac{\widetilde{X}_0}{\sigma}V_H(t)+\tilde{P}_H(t)\right]^2 d\omega_t^H+o_p(1)} \\
&= \frac{-2\kappa e^{\kappa T}\int_0^T \tilde{P}_H(t)\,dM_t^H+o_p(1)}{4\kappa^2 e^{2\kappa T}\int_0^T \tilde{P}_H^2(t)\,d\omega_t^H+o_p(1)} \\
&\xrightarrow{d} \frac{X\sqrt{\sin(\pi H)}}{Y},
\end{aligned}
$$

with $\widetilde{X}_0=0$.

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