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A Taxonomy of Non-dictatorial Domains*

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Abstract

We provide an exhaustive classification of *all* preference domains that allow the design of unanimous social choice functions (henceforth, rules) that are non-dictatorial and strategy-proof. This taxonomy is based on a richness assumption and employs a simple property of two-voter rules called invariance. The preference domains that form the classification are semi-single-peaked domains (introduced by Chatterji et al. (2013)) and semi-hybrid domains (introduced here) which are two appropriate weakenings of the single-peaked domains, and which, more importantly, are shown to allow strategy-proof rules to depend on non-peak information of voters' preferences. As a refinement of the classification, single-peaked domains and hybrid domains emerge as the only preference domains that force strategy-proof rules to be determined completely by the peaks of voters' preferences. We also provide characterization results for strategy-proof rules on these domains.

Keywords: Strategy-proofness; invariance; path-connectedness; (semi)-single-peaked preference; (semi)-hybrid preference

JEL Classification: D71.

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1 Introduction

An overarching theme in the theory of incentives is that unanimous social choice functions (henceforth, *rules*) that are non-manipulable are dictatorial, and hence unsuitable for social decisions, unless preferences of voters are restricted in particular ways so as to yield non-dictatorial domains, i.e., domains of preferences which allow the design of rules that are strategy-proof and non-dictatorial. Indeed, single-peaked preference domains in the classical voting model (Moulin, 1980) and quasi-linear preferences in models with monetary compensations (Roberts, 1979) are leading instances of non-dictatorial domains. In this paper, we restrict attention to the voting model, where the large literature notwithstanding[1], a comprehensive classification of *all* non-dictatorial domains in terms of the design opportunities they afford has remain elusive, in part due to the fact that not much is known about the structure of preferences domains that allow strategy-proof rules to vary with non-peak information on preferences: We identify and incorporate such preference domains in our analysis and classify non-dictatorial domains based on whether they admit two-voter strategy-proof rules that (i) vary with non-peak information, and (ii) satisfy a simple property called invariance. We show that under a richness assumption on preferences, the resulting classification turns out to be an exhaustive one for non-dictatorial domains.

The classification follows from the analysis of two-voter rules on a "rich" domain of preferences, i.e., a domain that satisfies a form of connectedness, a mild property called extreme-vertex symmetry, and the existence of a pair of preferences that are complete reversals of each other.[2] We first identify two weakenings of single-peaked domains, respectively semi-single-peaked and semi-hybrid domains, that allow the design of non-tops-only and strategy-proof rules, i.e., rules that utilize non-peak

[1]The literature is taken up in Section 4.

[2]Some form of richness is needed to study the implications of strategy-proofness, for if the domain contains only few preferences, strategy-proofness becomes trivial. We postulate the condition of path-connectedness which has been used in the earlier study of Chatterji et al. (2013) and is closely related to the idea of connectedness investigated by Grandmont (1978), Monjardet (2009), Sato (2013) and Puppe (2018).

information from preferences. We next show that rich non-dictatorial domains are one of these two varieties and that which of these configurations prevails is, somewhat surprisingly, completely determined by the behavior of the rule at the two preferences profiles where the two voters are endowed with the completely reversed preferences: if there exists a two-voter tops-only and strategy-proof rule which is invariant, that is, selects the same social outcome at the these two test profiles, the domain must be semi-single-peaked, and otherwise, the domain must be a semi-hybrid domain. Finally, we specialize to tops-only domains, i.e., domains where strategy-proof rules are endogenously completely determined by voters' preference peaks, and show that the existence of an invariant rule leads us to the classical single-peaked domain while its non-existence, to a recently introduced variant of it called the hybrid domain. In particular, our analysis highlights the role of "critical spots" embedded in the gap between semi-single-peakedness and single-peakedness (respectively, between semi-hybridness and hybridness) that display a curious and seemingly paradoxical phenomenon, namely, that adding preferences to a single-peaked domain may allow non-tops-only rules to emerge in a strategy-proof way and simultaneously shrink the scope for tops-only rules.[3]

To put this analysis in perspective, we note that earlier work has shown that while the Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) is robust and survives on restricted domains with enough connectedness (see for instance Aswal et al., 2003; Sato, 2010; Pramanik, 2015), semi-single-peaked domains are implied by the existence of a tops-only and anonymous strategy-proof rule on a rich domain (see Chatterji et al., 2013; Chatterji and Massó, 2018). In order to obtain a more complete picture of non-dictatorial domains, we dispense with the axioms of tops-onlyness and anonymity, and show (by strengthening mildly the richness condition) that allowing for non-tops-only, non-anonymous rules adds exactly one domain, semi-hybrid domain, as a non-dictatorial domain.

[3]The seeming paradox is of course not a paradox: the non-tops-only rules for a semi-single-peaked domain continue to be strategy-proof for the single-peaked domain, except that they become tops-only when restricted to the single-peaked domain and are thus subsumed in the usual known class of strategy-proof rules for single-peaked domains.

Since a semi-hybridness is also a weakening of single-peakedness that is in a sense complementary to semi-single-peakedness (see 1.1 below), our result showing that these two domains taken together exhaust all non-dictatorial domains may be seen as demonstrating that appropriate weakenings of single-peakedness characterize non-dictatorial domains, addressing thereby a long standing conjecture in this field (see Barberà, 2011; Barberà et al., 2020). Our principal focus however is on distinguishing between these two regimes by showing that semi-hybrid domains, while being more permissive, are not consistent with the existence of tops-only and anonymous strategy-proof rules (since invariance is implied by anonymity) while semi-single-peaked domains are. We contend that this analysis is more than just a theoretical curiosity since it explains tradeoffs, for instance, between allowing more permissive preference domains (which is desirable for applications of mechanism design) that however turn out to admit critical spots and hence non-tops-only and strategy-proof rules, and more restrictive ones that admit rules that treat all voters symmetrically and are easier to operationalize, both from the perspective of a planner and voters by virtue of requiring only peak information on preferences.[4]

The paper is organized as follows. In Section 1.1 we provide a heuristic presentation of our classification. In Section 2 we specify the model. The main results are presented in Section 3. Section 4 contains a review of the literature, examples and suggestions for future work. All proofs are gathered in an Appendix.

1.1 A heuristic description

We begin with an informal description of semi-single-peaked and semi-hybrid preferences. Assume that a finite set of alternatives $a_1, \ldots, a_m$ are located on a line according to the natural order. Select an alternative $a_{\bar{k}}$, call it a threshold and assume that every preference with the peak distinct from $a_{\bar{k}}$ is single-peaked in

[4] For instance, as suggested by Bartholdi et al. (1989), it would be computationally hard to detect a manipulation in a voting mechanism that depends too much on information of preferences, whereas the imposition of tops-onlyness on the voting mechanism helps detect a voter's manipulation within a polynomial time via their Greedy-Manipulation algorithm.

the conventional sense up to the threshold and alternatives beyond $a_{\bar{k}}$ are ranked lower than it. A typical semi-single-peaked preference is illustrated in Figure 1(a).

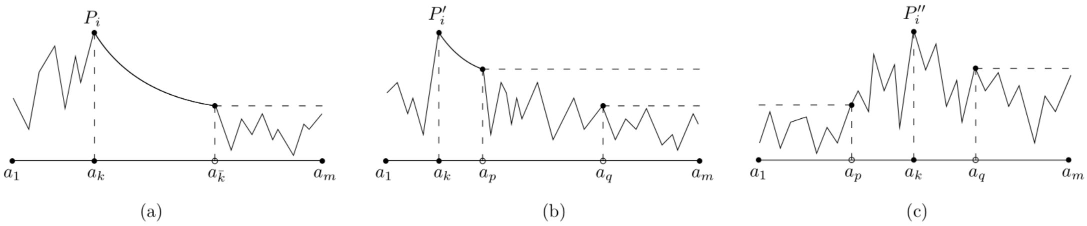

Figure 1: A semi-single-peaked preference and two semi-hybrid preferences

The description of a semi-hybrid preference in Figure 1(b) requires us to fix two distinct thresholds a_p and a_q, which separate the line into three intervals: $\langle a_1, a_p \rangle$, $\langle a_p, a_q \rangle$ and $\langle a_q, a_m \rangle$. If the peak lies in $\langle a_1, a_p \rangle$ (an analogous condition holds if the peak lies in $\langle a_q, a_m \rangle$), the single-peakedness condition only prevails in the interval between the peak and the threshold a_p, and the two thresholds a_p and a_q are required to be top-ranked within $\langle a_p, a_q \rangle$ and $\langle a_q, a_m \rangle$ respectively. In Figure 1(c), when the preference peak is located between two thresholds, a semi-hybrid preference preserves no restriction of single-peakedness, but only keeps a_p and a_q top-ranked within their intervals $\langle a_1, a_p \rangle$ and $\langle a_q, a_m \rangle$.

The remarkable feature of these two weakenings of single-peakedness that has been hitherto unnoticed is that while they are much less structured than single-peaked preferences, they nonetheless allow the design of strategy-proof rules that are considerably more nuanced than, say a phantom voter rule of Moulin (1980) and Border and Jordan (1983), in that they remain strategy-proof in spite of depending on non-peak information.

We adopt the domain of semi-single-peaked preferences to exemplify this feature.[5] Given the line of Figure 1(a), we fix the threshold $a_{\bar{k}}$ with $\bar{k} > 2$, and consider the domain of *all* semi-single-peaked preferences with respect to this choice of threshold. We next make an observation that we refer to as (#): There exist two preferences that have the same peak located in $\langle a_2, a_m \rangle$, and which disagree on the relative ranking of a_1 and a_2. Finally, we cut the edge between a_1 and a_2 to separate the line into two parts $\langle a_1, a_1 \rangle = \{a_1\}$ and $\langle a_2, a_m \rangle$ and construct an

[5] A similar example can constructed on the domain of semi-hybrid preferences.

SCF $f : \mathbb{D}^2 \to A$ which selects voter 1's most preferred alternative when it lies in $\langle a_2, a_m \rangle$, the alternative a_1 when both voters unanimously prefer it the most, and voter 2's preferred alternative between a_1 and a_2 when voter 1's preference peak is a_1 and voter 2's peak is located in $\langle a_2, a_m \rangle$, i.e.,

$$f(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } r_1(P_1) \in \langle a_2, a_m \rangle, \\ a_1 & \text{if } r_1(P_1) = a_1 = r_1(P_2), \\ \max^{P_2}(\{a_1, a_2\}) & \text{if } r_1(P_1) = a_1 \text{ and } r_1(P_2) \in \langle a_2, a_m \rangle. \end{cases}$$

Clearly, f is unanimous by construction, and observation (#) above implies that f violates the tops-only property. The strategy-proofness of f can be easily verified.[6]

Of course the phantom voter rule where the phantom is located on the threshold $a_{\bar{k}}$ in Figure 1(a) is strategy-proof, and is moreover a tops-only rule that obeys invariance. This is the key feature that sets the semi-single-peaked domain apart from semi-hybrid domains, for we prove that any rich non-dictatorial domain that admits an invariant, tops-only and strategy-proof rule must be semi-single-peaked (and vice versa), while the non-existence of such a rule implies a semi-hybrid domain (and vice versa). Next we focus on domains that do not allow the design of strategy-proof rules that are non-tops-only, and show that the classification of non-dictatorial domains is refined respectively to single-peaked domains and hybrid domains.

2 The Model

Let $A = \{a, b, c, \ldots\}$ be a finite set of alternatives with $|A| = m \geq 3$. Let $N = \{1, \ldots, n\}$ be a finite set of voters with $|N| = n \geq 2$. Each voter i has a (strict) preference order P_i over A which is the asymmetric part of a linear order. For any $a, b \in A$, aP_ib is interpreted as "a is strictly preferred to b according to

[6]Rule f can be further simplified: $f(P_1, P_2) = r_1(P_1)$ if $r_1(P_1) \in \langle a_2, a_m \rangle$, and $f(P_1, P_2) = \max^{P_2}(\{a_1, a_2\})$ otherwise. Next, since all preferences are semi-single-peaked w.r.t. $a_{\bar{k}}$ where $\bar{k} > 2$, observe that a_2 is second ranked in every preference with the peak a_1. This observation immediately ensures the strategy-proofness of the simplified version of f. We intentionally avoid the simplified configuration of f in the main text so that the reader can easily compare f to its generalization in Section 3.1.

P_i".[7] Let $r_k(P_i)$ denote the kth ranked alternative in P_i for all $k = 1, \ldots, m$. Given a subset $B \subset A$,[8] let $\max^{P_i}(B)$ and $\min^{P_i}(B)$ respectively denote the most and the least preferred alternatives in B according to P_i. Two preferences P_i and P'_i are **completely reversed** if for all $a, b \in A$, $[aP_ib] \Leftrightarrow [bP'_ia]$. Let $\mathbb{P}$ denote the set containing *all* linear orders over A. The set of all admissible orders is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the **preference domain**.[9] For notational convenience, let $\mathbb{D}^a = \{P_i \in \mathbb{D} : r_1(P_i) = a\}$ denote the set of preferences with the peak a, and $\mathcal{S}(\mathbb{D}^a) = \{b \in A : b = r_2(P_i) \text{ for some } P_i \in \mathbb{D}^a\}$ collect all alternatives that are second ranked in the preferences of $\mathbb{D}^a$. Accordingly, a domain $\mathbb{D}$ is *minimally rich* if $\mathbb{D}^a \neq \emptyset$ for every $a \in A$. A preference profile $P = (P_1, \ldots, P_n) = (P_i, P_{-i}) \in \mathbb{D}^n$ is an n-tuple of orders where P_{-i} represents a collection of $n-1$ voters' preferences without considering voter i.

A **Social Choice Function** (or SCF) is a map $f : \mathbb{D}^n \to A$. At every profile $P \in \mathbb{D}^n$, $f(P)$ is referred to as the "socially desirable" outcome associated to this preference profile. An SCF $f : \mathbb{D}^n \to A$ is **unanimous** if for all $a \in A$ and $P \in \mathbb{D}^n$, we have $[r_1(P_i) = a \text{ for all } i \in N] \Rightarrow [f(P) = a]$. Henceforth, we call a unanimous SCF a **rule**. An SCF $f : \mathbb{D}^n \to A$ is **strategy-proof** if for all $i \in N$, $P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, we have either $f(P_i, P_{-i}) = f(P'_i, P_{-i})$ or $f(P_i, P_{-i})P_if(P'_i, P_{-i})$. In particular, an SCF $f : \mathbb{D}^n \to A$ is **anonymous** if for all $(P_1, \ldots, P_n) \in \mathbb{D}^n$ and permutations $\sigma : N \to N$, we have $f(P_1, \ldots, P_n) = f(P_{\sigma(1)}, \ldots, P_{\sigma(n)})$. A prominent class of SCFs is the class of tops-only SCFs. The value of these SCFs at every preference profile depends only on voters' peaks. Formally, an SCF $f : \mathbb{D}^n \to A$ satisfies **the tops-only property** if for all $P, P' \in \mathbb{D}^n$, we have $[r_1(P_i) = r_1(P'_i)$ for all $i \in N] \Rightarrow [f(P) = f(P')]$.

Dictatorships are rules that are tops-only and strategy-proof on arbitrary domains. Formally, an SCF $f : \mathbb{D}^n \to A$ is a **dictatorship** if there exists $i \in N$ such that $f(P) = r_1(P_i)$ for all $P \in \mathbb{D}^n$. In particular, given a non-empty subset

[7] In a table, we specify a preference "vertically". In a sentence, we specify a preference "horizontally". For instance, $P_i = (a\, b\, c \cdots)$ represents that a is the top, b is the second best, c is the third ranked alternative while the rest of rankings in P_i are arbitrary.

[8] Throughout the paper, $\subset$ and $\subseteq$ denote the strict and weak inclusions respectively.

[9] We call $\mathbb{P}$ *the universal domain.* When $\mathbb{D} \neq \mathbb{P}$, $\mathbb{D}$ is referred to as a *restricted domain.*

$B \subset A$, we say that an SCF $f : \mathbb{D}^n \to A$ **behaves like a dictatorship** on B if there exists $i \in N$ such that $f(P_1, \ldots, P_n) = r_1(P_i)$ for all $(P_1, \ldots, P_n) \in \mathbb{D}^n$ with $r_1(P_1), \ldots, r_1(P_n) \in B$. The Gibbard-Satterthewaite Theorem shows that on the universal domain, an SCF $f : \mathbb{P}^n \to A$, $n \geq 2$, is a strategy-proof rule if and only if it is a dictatorship. The same dictatorship characterization result also holds on some restricted domains (see the literature listed in Section 4.1). We call a domain $\mathbb{D}$ a **dictatorial domain** if every strategy-proof rule $f : \mathbb{D}^n \to A$, $n \geq 2$, is a dictatorship, and call any domain that admits a non-dictatorial strategy-proof rule a **non-dictatorial domain**. It is clear that a domain that admits an anonymous and strategy-proof rule is a non-dictatorial domain. Conversely, a non-dictatorial domain may not admit an anonymous and strategy-proof rule. Last, a domain $\mathbb{D}$ is a **tops-only domain** if every strategy-proof rule $f : \mathbb{D}^n \to A$, $n \geq 2$, satisfies the tops-only property. Clearly, the set of tops-only domains includes all dictatorial domains and many non-dictatorial domains.

2.1 Graphs

Let $G^A = \langle A, \mathcal{E}^A \rangle$ denote a *undirected graph* where A is the vertex set and $\mathcal{E}^A$ is the set of edges.[10] A **path** in G^A is a sequence of *non-repeated* vertices $(x_1, \ldots, x_t)$ such that $(x_k, x_{k+1}) \in \mathcal{E}^A$ for all $k = 1, \ldots, t-1$. The graph G^A is **connected** if for every pair of distinct vertices, there exists a path connecting them. Given $a \in A$, let $\mathcal{N}^A(a) = \{b \in A : (a, b) \in \mathcal{E}^A\}$ denote the set of alternatives that are neighbor to a in the graph G^A. Let $Ext(G^A) = \big\{x \in A : |\mathcal{N}^A(x)| = 1\big\}$ denote the set of **extreme vertices**. Given a subset $B \subset A$, let $G^B = \langle B, \mathcal{E}^B \rangle$ denote the subgraph of G^A where the vertex set is B and the edge set is $\mathcal{E}^B = \{(a, b) \in \mathcal{E}^A : a, b \in B\}$.

A **tree** $\mathcal{T}^A = \langle A, \mathcal{E}^A \rangle$ is a connected graph where each pair of distinct vertices is connected by a *unique* path. Given $x, y \in A$, let $\langle x, y | \mathcal{T}^A \rangle$ denote the unique path connecting x and y in $\mathcal{T}^A$.[11] Fix a subset $B \subset A$ such that the path between any two alternatives of B is also included in B, i.e., $[a, b \in B] \Rightarrow \big[\langle a, b | \mathcal{T}^A \rangle \subseteq B\big]$.

[10] If $(a, b) \in \mathcal{E}^A$, then $a \neq b$ and $(b, a) \in \mathcal{E}^A$.

[11] For notational convenience, we also use $\langle x, y | \mathcal{T}^A \rangle$ to denote the set of alternatives in the path between x and y. We also call $\langle x, y | \mathcal{T}^A \rangle$ the *interval* between x and y in $\mathcal{T}^A$.

Then, the subgraph $\mathcal{T}^B = \langle B, \mathcal{E}^B \rangle$ is also a tree. Furthermore, given $a \in A$, if $a \in B$, it is evident that the projection of a on $\mathcal{T}^B$ is itself; otherwise, there exists a unique $a' \in B$ such that $a' \in \langle a, b|\mathcal{T}^A \rangle$ for all $b \in B$, which can be viewed as the projection of a on $\mathcal{T}^B$. Accordingly, let $\text{Proj}(a, \mathcal{T}^B)$ denote the **projection** of a on the subtree $\mathcal{T}^B$. A **line** is a particular tree which has exactly two extreme vertices. Throughout the paper, we *fix* $\mathcal{L}^A = (a_1, \ldots, a_m)$ to be the line where (a_k, a_{k+1}) is an edge for all $k = 1, \ldots, m-1$. Given a tree $\mathcal{T}^A$ and two distinct alternatives $x, y \in A$, we fix the set $A^{x \rightharpoonup y} = \{z \in A : x \in \langle z, y|\mathcal{T}^A \rangle\}$ to include every alternative whose path to y always goes through x. Therefore, $\mathcal{T}^{A^{x \rightharpoonup y}}$ is a subtree nested in $\mathcal{T}^A$. We use the diagram of Figure 2 to illustrate.

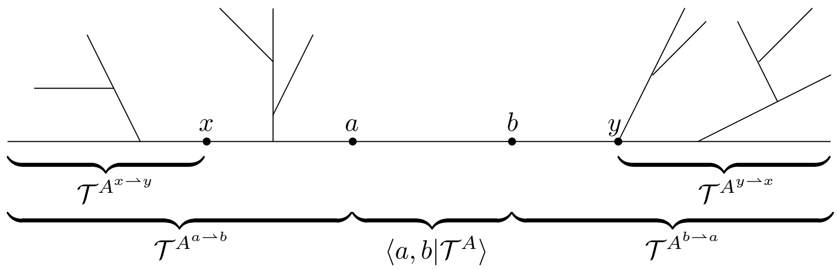

Figure 2: Four subtrees $\mathcal{T}^{A^{x \rightharpoonup y}}$, $\mathcal{T}^{A^{y \rightharpoonup x}}$, $\mathcal{T}^{A^{a \rightharpoonup b}}$, $\mathcal{T}^{A^{b \rightharpoonup a}}$ and an interval $\langle a, b|\mathcal{T}^A \rangle$

2.2 Richness condition

Fix a domain $\mathbb{D}$. First, two alternatives $a, b \in A$ are **adjacent**, denoted $a \sim b$, if there exist $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) = r_2(P_i') = a$, $r_1(P_i') = r_2(P_i) = b$ and $r_k(P_i) = r_k(P_i')$ for all $k = 3, \ldots, m$. According to $\mathbb{D}$, we construct a graph $G_\sim^A = \langle A, \mathcal{E}_\sim^A \rangle$ where the vertex set is A and two alternatives $a, b \in A$ form an edge if and only if $a \sim b$, i.e., $\mathcal{E}_\sim^A = \{(a, b) \in A^2 : a \sim b\}$. We call $G_\sim^A$ an *adjacency graph*. Then, domain $\mathbb{D}$ is said **path-connected** if $G_\sim^A$ is a connected graph. Clearly, path-connectedness implies minimal richness. We further require $\mathbb{D}$ satisfy **extreme-vertex symmetry**, that is, given $x \in Ext(G_\sim^A)$ and $(x, y) \in \mathcal{E}_\sim^A$, if $|\mathcal{S}(\mathbb{D}^x)| > 1$, there exists $z \in \mathcal{S}(\mathbb{D}^x)$ such that $x \in \mathcal{S}(\mathbb{D}^z)$ and $z \neq y$.[12] *Throughout the paper, we assume that the domain in question is path-connected and satisfies*

[12] If $Ext(G_\sim^A) = \emptyset$, or $Ext(G_\sim^A) \neq \emptyset$ and $|\mathcal{S}(\mathbb{D}^x)| = 1$ for all $x \in Ext(G_\sim^A)$, domain $\mathbb{D}$ satisfies extreme-vertex symmetry vacuously.

extreme-vertex symmetry. Moreover, we say that a domain $\mathbb{D}$ is **rich** if it is path-connected, satisfies extreme-vertex symmetry, and in addition includes a pair of completely reversed preferences.

2.3 Single-peaked and hybrid preferences

In this section, we introduce two important preference restrictions, single-peakedness and hybridness. Single-peaked preferences were discovered by Black (1948) as a way of avoiding the Condorcet paradox. Demange (1982) generalized Black's notion to single-peakedness on a tree and showed that majority voting continued to deliver a Condorcet winner.

Definition 1 *Fixing a tree $\mathcal{T}^A$, a preference P_i is **single-peaked** on $\mathcal{T}^A$ if for all distinct $a, b \in A$, we have $\big[a \in \langle r_1(P_i), b|\mathcal{T}^A\rangle\big] \Rightarrow [aP_ib]$. Let $\mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ denote **the single-peaked domain** which includes all single-peaked preferences on $\mathcal{T}^A$. A domain $\mathbb{D}$ is called **a single-peaked domain** if there exists a tree $\mathcal{T}^A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SP}}(\mathcal{T}^A)$.*

If a single-peaked domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ is path-connected, its adjacency graph is identical to the underlying tree $\mathcal{T}^A$, i.e., $G_{\sim}^A = \mathcal{T}^A$. The single-peaked domain $\mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ is naturally a path-connected domain, vacuously satisfies extreme-vertex symmetry,[13] and it includes a pair of completely reversed preferences if and only if $\mathcal{T}^A$ is a line.

A hybrid preference is a generalization of a single-peaked preference which allows some freedom on the rankings of certain alternatives. Given a tree $\mathcal{T}^A$, we consider two distinct alternatives a and b that *completely* separate $\mathcal{T}^A$ into the interval $\langle a, b|\mathcal{T}^A\rangle$ and the two subtrees $\mathcal{T}^{A^{a\rightharpoonup b}}$ and $\mathcal{T}^{A^{b\rightharpoonup a}}$ (recall Figure 2).[14] Thus, we have $\text{Proj}(c, \langle a, b|\mathcal{T}^A\rangle) \in \{a, b\}$ for all $c \in A\backslash\langle a, b|\mathcal{T}^A\rangle$. We fix such a pair a and b and call them **thresholds** in $\mathcal{T}^A$.[15] The interval $\langle a, b|\mathcal{T}^A\rangle$ can be viewed

[13] For all $x \in Ext(G_{\sim}^A) = Ext(\mathcal{T}^A)$, $|\mathcal{S}(\mathbb{D}_{\text{SP}}(\mathcal{T}^A)^x)| = 1$.

[14] If we refer to x and y in Figure 2, the combination of the interval $\langle x, y|\mathcal{T}^A\rangle$ and subtrees $\mathcal{T}^{A^{x\rightharpoonup y}}$ and $\mathcal{T}^{A^{y\rightharpoonup x}}$ does *not* recover $\mathcal{T}^A$ as the branch attached to the interior of $\langle x, y|\mathcal{T}^A\rangle$ is not covered.

[15] Note that if a and b form an edge in $\mathcal{T}^A$, they are naturally thresholds.

as a "free zone" for a hybrid preference. A hybrid preference preserves single-peakedness on the two subtrees $\mathcal{T}^{A^{a\rightharpoonup b}}$ and $\mathcal{T}^{A^{b\rightharpoonup a}}$, but allows the alternatives in the free zone to be arbitrarily ranked with respect to each other, subject to the maximum ranking of a (respectively, b) when the preference peak is located in $\mathcal{T}^{A^{a\rightharpoonup b}}$ (respectively, $\mathcal{T}^{A^{b\rightharpoonup a}}$).[16]

DEFINITION **2** *Fixing a tree $\mathcal{T}^A$ and two thresholds $a, b \in A$, a preference P_i is (a,b)-**hybrid** on $\mathcal{T}^A$ if it satisfies the following two conditions:*

(i) *P_i is single-peaked on $\mathcal{T}^{A^{a\rightharpoonup b}}$ and $\mathcal{T}^{A^{b\rightharpoonup a}}$, i.e., for all distinct $y, z \in A^{a\rightharpoonup b}$ or $y, z \in A^{b\rightharpoonup a}$, $\big[y \in \langle r_1(P_i), z|\mathcal{T}^A\rangle\big] \Rightarrow [yP_iz]$, and*

(ii) *$[r_1(P_i) \in A^{a\rightharpoonup b}\backslash\{a\}] \Rightarrow \big[\max^{P_i}\big(\langle a, b|\mathcal{T}^A\rangle\big) = a\big]$ and $[r_1(P_i) \in A^{b\rightharpoonup a}\backslash\{b\}] \Rightarrow \big[\max^{P_i}\big(\langle a, b|\mathcal{T}^A\rangle\big) = b\big]$.*

*Let $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$ denote **the hybrid domain** which includes all (a,b)-hybrid preferences on $\mathcal{T}^A$. A domain $\mathbb{D}$ is called **an** (a,b)**-hybrid domain on** $\mathcal{T}^A$ if $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$, $\big|\langle a, b|\mathcal{T}^A\rangle\big| \geq 3$,[17] and there exist no tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$.[18] In particular, $\mathbb{D}$ is said to be **non-degenerate** if either $A^{a\rightharpoonup b} \neq \{a\}$ or $A^{b\rightharpoonup a} \neq \{b\}$ holds, and **degenerate** otherwise.*

We simply call $\mathbb{D}$ "a hybrid domain" if there exist a tree $\mathcal{T}^A$ and thresholds $a, b \in A$ such that $\mathbb{D}$ is an (a,b)-hybrid domain on $\mathcal{T}^A$.[19]

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[16]The idea of a hybrid preference originates from the multiple single-peaked domain of Reffgen (2015). Achuthankutty and Roy (2020) and Chatterji et al. (2020) establish the formal definition of a hybrid preference on a line, and study strategy-proof rules and strategy-proof random SCFs respectively on the hybrid domain.

[17]If $|\langle a, b|\mathcal{T}^A\rangle| = 2$, then $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b) = \mathbb{D}_{\mathrm{SP}}(\mathcal{T}^A)$. In order to separate the definitions of the hybrid domain and the single-peaked domain, we impose $\big|\langle a, b|\mathcal{T}^A\rangle\big| \geq 3$.

[18]The notation $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$ concerns with the inclusion relation between the two subsets of alternatives, not the inclusion relation between the two graphs of intervals.

[19]Evidently, the hybrid domain $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$, where $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$, is an (a,b)-semi-hybrid domain on $\mathcal{T}^A$. Conversely, in most cases, an (a,b)-semi-hybrid domain on $\mathcal{T}^A$ is *strictly* included in the hybrid domain $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$.

The hybrid domain $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$, where $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$, is naturally a path-connected domain, vacuously satisfies extreme-vertex symmetry,[20] and it includes a pair of completely reversed preferences if and only if $\mathcal{T}^A$ is a line.

We provides three diagrams to illustrate a single-peaked preference and two hybrid preferences on the line $\mathcal{L}^A$.

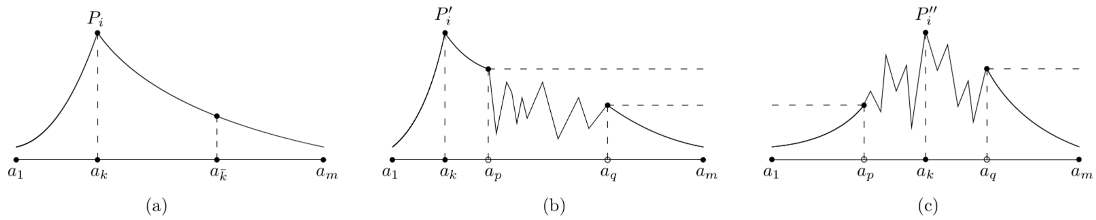

Figure 3: A single-peaked preference and two hybrid preferences on $\mathcal{L}^A$

2.4 Semi-single-peaked and semi-hybrid preferences

Next, we weaken single-peakedness and hybridness to the notions of semi-single-peakedness introduced by Chatterji et al. (2013) and semi-hybridness respectively. One observes immediately the weakening by comparing the three diagrams of Figure 1 to their counterparts in Figure 3.

Definition 3 *Fixing a tree $\mathcal{T}^A$ and an alternative $\bar{x} \in A$ which is called a threshold, a preference P_i is* ***semi-single-peaked*** *on $\mathcal{T}^A$ w.r.t. $\bar{x}$ if it satisfies the following two conditions:*

(i) *for all distinct $a, b \in \langle r_1(P_i), \bar{x}|\mathcal{T}^A\rangle$, $\big[a \in \langle r_1(P_i), b|\mathcal{T}^A\rangle\big] \Rightarrow [aP_ib]$, and*

(ii) *for all $a \notin \langle r_1(P_i), \bar{x}|\mathcal{T}^A\rangle$, $\big[\mathrm{Proj}\big(a, \langle r_1(P_i), \bar{x}|\mathcal{T}^A\rangle\big) = a'\big] \Rightarrow [a'P_ia]$.*

Let $\mathbb{D}_{\mathrm{SSP}}(\mathcal{T}^A, \bar{x})$ denote ***the semi-single-peaked domain*** *which includes all semi-single-peaked preferences on $\mathcal{T}^A$ w.r.t. $\bar{x}$. A domain $\mathbb{D}$ is called* ***a semi-single-peaked domain*** *if there exist a tree $\mathcal{T}^A$ and a threshold $\bar{x} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(\mathcal{T}^A, \bar{x})$.*

[20] Let $Ext(G^A_\sim) \neq \emptyset$. Then $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$ must be non-degenerate. Given an arbitrary $x \in Ext(G^A_\sim)$, it is true that $x \in Ext(\mathcal{T}^{A^{a\rightharpoonup b}})\backslash\{a\}$ or $x \in Ext(\mathcal{T}^{A^{b\rightharpoonup a}})\backslash\{b\}$ which implies $x \in Ext(\mathcal{T}^A)$ and $x \notin \{a, b\}$. Consequently, we have $\big|\mathcal{S}\big(\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)^x\big)\big| = 1$.

The semi-single-peaked domain $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ is naturally a path-connected domain, and it includes a pair of completely reversed preferences if and only if $|\mathcal{N}^A(\bar{x})| \leq 2$.[21] Given a path-connected domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$, it is true that $G_\sim^A = \mathcal{T}^A$, and moreover $\mathbb{D}$ satisfies extreme-vertex symmetry if and only if either $\bar{x} \notin Ext(\mathcal{T}^A)$, or $\bar{x} \in Ext(\mathcal{T}^A)$ and $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) \cap \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, x)$ where $\mathcal{N}^A(\bar{x}) = \{x\}$.[22] Clearly, $\mathbb{D}_{\text{SP}}(\mathcal{T}^A) = \cap_{\bar{x}\in A} \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$.

Definition 4 *Fixing a tree $\mathcal{T}^A$ and two thresholds $a, b \in A$, a preference P_i is (a,b)-**semi-hybrid** on $\mathcal{T}^A$ if it satisfies one of the following three conditions:*

(i) *given $r_1(P_i) \in A^{a\rightharpoonup b}\backslash\{a\}$,*

- *P_i is semi-single-peaked on $\mathcal{T}^{A^{a\rightharpoonup b}}$ w.r.t. a, i.e., for all distinct $x, y \in \langle r_1(P_i), a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle$, $\big[x \in \langle r_1(P_i), y|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle\big] \Rightarrow [xP_iy]$, and for all $x \in A^{a\rightharpoonup b}\backslash\langle r_1(P_i), a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle$, $\big[\text{Proj}\,\big(x, \langle r_1(P_i), a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle\big) = x'\big] \Rightarrow [x'P_ix]$.*
- *$\max^{P_i}\big(\langle a, b|\mathcal{T}^A\rangle\big) = a$ and $\max^{P_i}(A^{b\rightharpoonup a}) = b$.*

(ii) *given $r_1(P_i) \in A^{b\rightharpoonup a}\backslash\{b\}$,*

- *P_i is semi-single-peaked on $\mathcal{T}^{A^{b\rightharpoonup a}}$ w.r.t. b, i.e., for all distinct $x, y \in \langle r_1(P_i), b|\mathcal{T}^{A^{b\rightharpoonup a}}\rangle$, $\big[x \in \langle r_1(P_i), y|\mathcal{T}^{A^{b\rightharpoonup a}}\rangle\big] \Rightarrow [xP_iy]$, and for all $x \in A^{b\rightharpoonup a}\backslash\langle r_1(P_i), b|\mathcal{T}^{A^{b\rightharpoonup a}}\rangle$, $\big[\text{Proj}\,\big(x, \langle r_1(P_i), b|\mathcal{T}^{A^{b\rightharpoonup a}}\rangle\big) = x'\big] \Rightarrow [x'P_ix]$.*
- *$\max^{P_i}\big(\langle a, b|\mathcal{T}^A\rangle\big) = b$ and $\max^{P_i}(A^{a\rightharpoonup b}) = a$.*

(iii) *given $r_1(P_i) \in \langle a, b|\mathcal{T}^A\rangle$, $\max^{P_i}(A^{a\rightharpoonup b}) = a$ and $\max^{P_i}(A^{b\rightharpoonup a}) = b$.*

*Let $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ denote **the semi-hybrid domain** which includes all (a,b)-semi-hybrid preferences on $\mathcal{T}^A$.*[23] *A domain $\mathbb{D}$ is called **an (a,b)-semi-hybrid domain on $\mathcal{T}^A$** if the following three conditions are satisfied:*

[21] See Clarification 1 of Appendix G.

[22] See Clarification 2 of Appendix G.

[23] When we write the notation $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, both $|\langle a, b|\mathcal{T}^A\rangle| = 2$ or $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$ are admissible.

(1) $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$ *and* $\big|\langle a, b|\mathcal{T}^A\rangle\big| \geq 3$,[24]

(2) *there exist no tree* $\widehat{\mathcal{T}}^A$ *and thresholds* $\hat{a}, \hat{b} \in A$ *such that* $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ *and* $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$, *and*

(3) *there exists no* $\widehat{\mathcal{T}}^A$ *such that* $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(\widehat{\mathcal{T}}^A, a)$ *or* $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(\widehat{\mathcal{T}}^A, b)$.[25]

In particular, $\mathbb{D}$ *is said to be* ***non-degenerate*** *if either* $A^{a \rightharpoonup b} \neq \{a\}$ *or* $A^{b \rightharpoonup a} \neq \{b\}$ *holds, and* ***degenerate*** *otherwise.*

We simply call $\mathbb{D}$ "a semi-hybrid domain" if there exist a tree $\mathcal{T}^A$ and thresholds $a, b \in A$ such that $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$.[26]

The semi-hybrid domain $\mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$, where $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$, is naturally a path-connected domain, vacuously satisfies extreme-vertex symmetry, and it includes a pair of completely reversed preferences if and only if we have $[A^{a \rightharpoonup b} \neq \{a\}] \Rightarrow [a \in Ext(\mathcal{T}^{A^{a \rightharpoonup b}})]$ and $[A^{b \rightharpoonup a} \neq \{b\}] \Rightarrow [b \in Ext(\mathcal{T}^{A^{b \rightharpoonup a}})]$.[27] Clearly, $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b) \subseteq \mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$.

Given an arbitrary path-connected domain $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$, note that $G_\sim^{A^{a \rightharpoonup b}} = \mathcal{T}^{A^{a \rightharpoonup b}}$, $G_\sim^{A^{b \rightharpoonup a}} = \mathcal{T}^{A^{b \rightharpoonup a}}$, and that the adjacency subgraph $G_\sim^{\langle a, b|\mathcal{T}^A\rangle}$ may be significantly different from the interval $\langle a, b|\mathcal{T}^A\rangle$ in $\mathcal{T}^A$.[28] We provide an example to illustrate.

EXAMPLE **1** Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. We specify 14 preferences of a domain $\mathbb{D} \subset \mathbb{D}_{\mathrm{SH}}(\mathcal{L}^A, a_2, a_6)$ in Table 1. The line $\mathcal{L}^A$, interval $\langle a_2, a_6|\mathcal{L}^A\rangle$, adjacency graph $G_\sim^A$ and adjacency subgraph $G_\sim^{\langle a_2, a_6|\mathcal{L}^A\rangle}$ are all specified in Figure 4, respectively.

[24] If $\big|\langle a, b|\mathcal{T}^A\rangle\big| = 2$, then $\mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b) = \mathbb{D}_{\mathrm{SSP}}(\mathcal{T}^A, a) \cap \mathbb{D}_{\mathrm{SSP}}(\mathcal{T}^A, b)$.

[25] In particular, if domain $\mathbb{D}$ is path-connected, to verify this condition, it suffices to show that either $G_\sim^A$ is not a tree, or $G_\sim^A$ is a tree and neither $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(G_\sim^A, a)$ nor $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(G_\sim^A, b)$ holds. Example 1 below provides an instance where this condition is violated.

[26] Evidently, the semi-hybrid domain $\mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$, where $\big|\langle a, b|\mathcal{T}^A\rangle\big| \geq 3$, is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$. Conversely, in most cases, an (a, b)-semi-hybrid domain on $\mathcal{T}^A$ is strictly contained in the semi-hybrid domain $\mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$.

[27] For instance, in Figure 2, we have $A^{a \rightharpoonup b} \neq \{a\}$ and $a \in Ext(\mathcal{T}^{A^{a \rightharpoonup b}})$, whereas $A^{y \rightharpoonup x} \neq \{y\}$ and $y \notin Ext(\mathcal{T}^{A^{y \rightharpoonup x}})$. The detailed verification is put in Clarification 3 of Appendix G.

[28] The graph $G_\sim^{\langle a, b|\mathcal{T}^A\rangle}$ has the vertex set $\langle a, b|\mathcal{T}^A\rangle$, and therefore is a subgraph of $G_\sim^A$.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
a_1	a_1	a_1	a_2	a_2	a_3	a_4	a_4	a_4	a_5	a_5	a_6	a_6	a_7
a_2	a_2	a_2	a_1	a_4	a_4	a_2	a_3	a_5	a_4	a_6	a_5	a_7	a_6
a_3	a_5	a_4	a_4	a_1	a_2	a_1	a_2	a_3	a_3	a_4	a_4	a_5	a_5
a_4	a_4	a_3	a_3	a_3	a_1	a_3	a_1	a_2	a_2	a_3	a_3	a_4	a_4
a_5	a_3	a_5	a_5	a_6	a_5	a_6	a_5	a_1	a_1	a_2	a_2	a_3	a_3
a_6	a_6	a_6	a_6	a_5	a_6	a_5	a_6	a_6	a_6	a_1	a_1	a_2	a_2
a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_7	a_1	a_1

Table 1: Domain $\mathbb{D}$

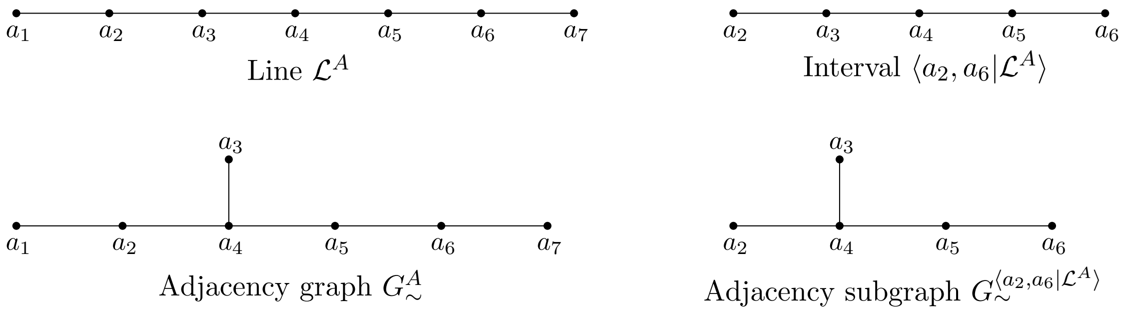

Figure 4: Line, interval, adjacency graph and adjacency subgraph

Domain $\mathbb{D}$ is path-connected according to the adjacency graph $G_\sim^A$ of Figure 4. Preferences P_1 and P_{14} are complete reversals in $\mathbb{D}$. Domain $\mathbb{D}$ vacuously satisfies extreme-vertex symmetry since $|\mathcal{S}(\mathbb{D}^x)| = 1$ for all $x \in Ext(G_\sim^A) = \{a_1, a_3, a_7\}$. Hence, $\mathbb{D}$ is a rich domain. One immediately notices the difference between the interval $\langle a_2, a_6 | \mathcal{L}^A \rangle$ and the adjacency subgraph $G_\sim^{\langle a_2, a_6 | \mathcal{L}^A \rangle}$ in Figure 4.

Next, we show in Clarification 4 of Appendix G that there exist no tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \subset \langle a_2, a_6 | \mathcal{L}^A \rangle$. However, $\mathbb{D}$ violates condition (3) of Definition 4 since it is also semi-single-peaked on the tree $G_\sim^A$ of Figure 4 w.r.t. a_2. If we add two preferences: $P_{15} = (a_3\, a_6\, a_4\, a_5\, a_7\, a_2\, a_1)$ and $P_{16} = (a_6\, a_3\, a_2\, a_1\, a_4\, a_5\, a_7)$, which are (a_2, a_6)-semi-hybrid on $\mathcal{L}^A$ as well, the new domain $\widehat{\mathbb{D}} = \mathbb{D} \cup \{P_{15}, P_{16}\}$ turns to meet condition (3) of Definition 4, and hence becomes a rich (a_2, a_6)-semi-hybrid domain on $\mathcal{L}^A$.[29] $\square$

[29] The adjacency graph of $\widehat{\mathbb{D}}$ remains to be $G_\sim^A$ of Figure 4. Domain $\widehat{\mathbb{D}}$ continues to satisfy extreme-vertex symmetry: (i) $Ext(G_\sim^A) = \{a_1, a_3, a_7\}$, (ii) $|\mathcal{S}(\widehat{\mathbb{D}}^{a_1})| = 1$ and $|\mathcal{S}(\widehat{\mathbb{D}}^{a_7})| = 1$, and (iii) given $(a_3, a_4) \in \mathcal{E}_\sim^A$, we have $\mathcal{S}(\widehat{\mathbb{D}}^{a_3}) = \{a_4, a_6\}$, $a_3 \in \mathcal{S}(\widehat{\mathbb{D}}^{a_6})$ and $a_6 \neq a_4$. Preference P_{15} (or P_{16}) indicates that $\widehat{\mathbb{D}}$ is never semi-single-peaked on the tree $G_\sim^A$ w.r.t. a_2, while P_1 indicates that $\widehat{\mathbb{D}}$ is never semi-single-peaked on the tree $G_\sim^A$ w.r.t. a_6.

2.5 An Auxiliary Proposition

We identify here a condition on a semi-hybrid domain which is necessary and sufficient for all strategy-proof rules to behave like dictatorships on the "free zone".

Definition 5 *Fixing a tree $\mathcal{T}^A$ and two thresholds $a, b \in A$, a domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ is* ***non-trivial*** *on $\langle a, b|\mathcal{T}^A\rangle$ if either $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \emptyset$, or $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) \neq \emptyset$ and the following three conditions are satisfied: given $x \in Ext\big(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}\big)$ and $(x, y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$,*

(i) *if $x \notin \{a, b\}$, there exists $P_i \in \mathbb{D}^x$ such that $r_2(P_i) \neq y$,*

(ii) *if $x = a$, there exists $P_i \in \mathbb{D}$ such that $r_1(P_i) \in A^{a \rightharpoonup b}$ and $\max^{P_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big) \neq y$, and*

(iii) *if $x = b$, there exists $P_i \in \mathbb{D}$ such that $r_1(P_i) \in A^{b \rightharpoonup a}$ and $\max^{P_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{b\}\big) \neq y$.*

It is evident that the semi-hybrid domain $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, where $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$, is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$ since any two distinct alternatives of $\langle a, b|\mathcal{T}^A\rangle$ are adjacent and hence $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \emptyset$. Next, we recall domain $\widehat{\mathbb{D}} = \mathbb{D} \cup \{P_{15}, P_{16}\} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_2, a_6)$ of Example 1 to illustrate the non-trivialness condition. According to the adjacency subgraph $G_{\sim}^{\langle a_2,a_6|\mathcal{L}^A\rangle}$ of Figure 4, note that $Ext\big(G_{\sim}^{\langle a_2,a_6|\mathcal{L}^A\rangle}\big) = \{a_2, a_3, a_6\}$. First, condition (i) of Definition 5 is satisfied by preference P_{15}, i.e., given $(a_3, a_4) \in \mathcal{E}_{\sim}^{\langle a_2,a_6|\mathcal{L}^A\rangle}$, we have $r_1(P_{15}) = a_3$ and $r_2(P_{15}) = a_6 \neq a_4$. Next, preference P_1 meets condition (ii) of Definition 5, i.e., given $(a_2, a_4) \in \mathcal{E}_{\sim}^{\langle a_2,a_6|\mathcal{L}^A\rangle}$, we have $r_1(P_1) = a_1 \in A^{a_2 \rightharpoonup a_6}$, $\max^{P_1}\big(\langle a_2, a_6|\mathcal{L}^A\rangle \backslash \{a_2\}\big) = a_3 \neq a_4$. Last, preference P_{16} satisfies condition (iii) of Definition 5, i.e., given $(a_6, a_5) \in \mathcal{E}_{\sim}^{\langle a_2,a_6|\mathcal{L}^A\rangle}$, we have $r_1(P_{16}) = a_6 \in A^{a_6 \rightharpoonup a_2}$ and $\max^{P_{16}}\big(\langle a_2, a_6|\mathcal{L}^A\rangle \backslash \{a_6\}\big) = a_3 \neq a_5$.

An Auxiliary Proposition *Fixing a tree $\mathcal{T}^A$ and two thresholds $a, b \in A$, let domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ be path-connected and satisfy extreme-vertex symmetry. Then, every strategy-proof rule $f : \mathbb{D}^n \to A$, $n \geq 2$, behaves likes a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$ if and only if $\mathbb{D}$ is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$.*

The proof of the Auxiliary Proposition is contained in Appendix A.

Remark 1 Let domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ be path-connected and satisfy extreme-vertex symmetry. If $\mathbb{D}$ is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$, it is easy to show that $\mathbb{D}$ meets conditions (1) and (3) of Definition 4, while the sufficiency part of the Auxiliary Proposition implies that $\mathbb{D}$ satisfies condition (2) of Definition 4.[30] Therefore, $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$. Conversely, a rich semi-hybrid domain may not be non-trivial on its "free zone".[31] □

Remark 2 Aswal et al. (2003) introduced a domain condition called **the unique seconds property**, which says that given a domain $\mathbb{D}$, there exists $x \in A$ such that $|\mathcal{S}(\mathbb{D}^x)| = 1$, and showed that it is sufficient for a domain to be a non-dictatorial domain.[32] A semi-single-peaked domain by definition satisfies the unique seconds property.[33] Given a tree $\mathcal{T}^A$ and two thresholds $a, b \in A$, let $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ be path-connected. If $A^{a\rightharpoonup b} \neq \{a\}$ or $A^{b\rightharpoonup a} \neq \{b\}$, then $\mathbb{D}$ satisfies the unique seconds property by the definition of (a, b)-semi-hybridness on $\mathcal{T}^A$.[34] If $A^{a\rightharpoonup b} = \{a\}$ and $A^{b\rightharpoonup a} = \{b\}$, then $\mathbb{D}$ satisfies the unique seconds property if and only if it violates the non-trivialness condition on $\langle a, b|\mathcal{T}^A\rangle$.[35] □

As mentioned earlier, the unique seconds property is sufficient for non-dictatorial domains. Conversely, consider a non-dictatorial domain $\mathbb{D}$ which is assumed to be path-connected and satisfy extreme-vertex symmetry. Note that $\mathbb{D} \subseteq \mathbb{P} = \mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_1, a_m)$. Thus, as a non-dictatorial domain, there exists a strategy-proof rule which does not behave like a dictatorship on $A = \langle a_1, a_m|\mathcal{L}^A\rangle$. Then, the Auxiliary Proposition implies that $\mathbb{D}$ violates the non-trivialness condition on

[30]See the detailed verification in Clarification 5 of Appendix G.

[31]For instance, domain $\mathbb{D}$ of Example 1, as an (a_2, a_6)-semi-hybrid domain on the line $\mathcal{L}^A$, violates the non-trivialness condition on $\langle a_2, a_6|\mathcal{L}^A\rangle$: given $Ext(G_{\sim}^{\langle a_2,a_6|\mathcal{T}^A\rangle}) = \{a_2, a_3, a_6\}$, we have (1) $(a_3, a_4) \in \mathcal{E}_{\sim}^{\langle a_2,a_6|\mathcal{T}^A\rangle}$ and $\mathcal{S}(\mathbb{D}^{a_3}) = \{a_4\}$, and (2) $(a_6, a_5) \in \mathcal{E}_{\sim}^{\langle a_2,a_6|\mathcal{T}^A\rangle}$ and $\max^{P_i}\big(\langle a_2, a_6|\mathcal{T}^A\rangle\backslash\{a_6\}\big) = a_5$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in \{a_6, a_7\} = A^{a_6\rightharpoonup a_2}$.

[32]Also see the *inseparable top-pair property* introduced by Kalai and Ritz (1980).

[33]Given a domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$, it is clear that $|Ext(\mathcal{T}^A)| \geq 2$. If $\bar{x} \notin Ext(\mathcal{T}^A)$, then $|\mathcal{S}(\mathbb{D}^x)| = 1$ for all $x \in Ext(\mathcal{T}^A)$. If $\bar{x} \in Ext(\mathcal{T}^A)$, then $|\mathcal{S}(\mathbb{D}^x)| = 1$ for all $x \in Ext(\mathcal{T}^A)\backslash\{\bar{x}\}$.

[34]Since $A^{a\rightharpoonup b} \neq \{a\}$ or $A^{b\rightharpoonup a} \neq \{b\}$, we have $Ext(\mathcal{T}^A)\backslash\{a, b\} \neq \emptyset$ and $|\mathcal{S}(\mathbb{D}^x)| = 1$ for all $x \in Ext(\mathcal{T}^A)\backslash\{a, b\}$.

[35]See Clarification 6 of Appendix G. In the end of Clarification 6 of Appendix G, we provide an example of a rich degenerate semi-hybrid domain that satisfies the unique seconds property.

$\langle a_1, a_m | \mathcal{L}^A \rangle$, and hence satisfies the unique seconds property by Remark 2. Thus, the Auxiliary Proposition helps us identify the exact boundary between dictatorial and non-dictatorial domains in our framework; the boundary is defined by the unique seconds property.

Corollary 1 *Let domain $\mathbb{D}$ be path-connected and satisfy extreme-vertex symmetry. Then, $\mathbb{D}$ is a non-dictatorial domain if and only if it satisfies the unique seconds property.*

We demonstrate Corollary 1 using the diagram of Figure 5.

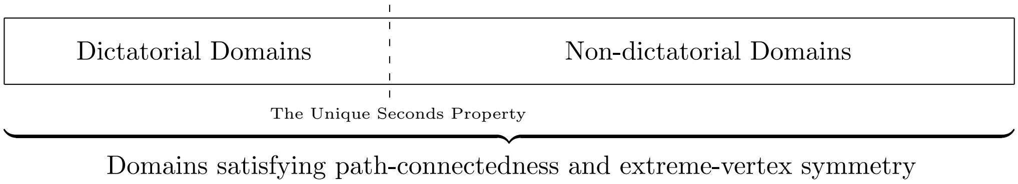

Figure 5: A characterization of non-dictatorial domains

Remark 3 Roy and Storcken (2019) provide another domain richness assumption which contains three conditions, and show that the unique seconds property is necessary and sufficient for non-dictatorial domains.[36] Path-connectedness implies their first and third conditions, and is easier to verify. extreme-vertex symmetry significantly weakens their second condition as it only concerns the extreme vertices in the adjacency graph. This weakening is meaningful and is critical to our analysis as it accommodates the class of semi-single-peaked domains, which however is ruled out by their second condition. □

[36] Let $\mathbb{D}$ be a rich domain of Roy and Storcken (2019). To meet their first condition, for all distinct $a, b \in A$, there exists a sequence of alternatives $\{x_1, \ldots, x_v\}$ such that $x_1 = a$, $x_v = b$, and for all $1 \leq k < v$, $x_k \in \mathcal{S}(\mathbb{D}^{x_{k+1}})$ and $x_{k+1} \in \mathcal{S}(\mathbb{D}^{x_k})$. The second condition requires that for all $a, b \in A$, $[a \in \mathcal{S}(\mathbb{D}^b)] \Leftrightarrow [b \in \mathcal{S}(\mathbb{D}^a)]$. The third condition says that given $a, b, c \in A$ with $a \in \mathcal{S}(\mathbb{D}^b)$, $b \in \mathcal{S}(\mathbb{D}^a) \cap \mathcal{S}(\mathbb{D}^c)$ and $c \in \mathcal{S}(\mathbb{D}^b)$, there exist two preferences $P_i \in \mathbb{D}^a$ and $P_i' \in \mathbb{D}^c$ such that given $d \in A$ with dP_ic and $dP_i'a$, we have $dP_i''a$ or $dP_i''c$ at some preference $P_i'' \in \mathbb{D}^b$.

3 Results

3.1 Non-tops-only rules and critical spots

We introduce a rule that on a semi-single-peaked (respectively semi-hybrid) domain can extract non-peak information from some preference profiles while remaining strategy-proof, and identify critical spots as configurations that allow such rules to arise. These critical spots vanish if and only if the domain is refined to be single-peaked (respectively hybrid).

Fix a tree $\mathcal{T}^A$. Given a preference profile P, we construct the following set $\Gamma(P) = \big\{a \in A : a \in \langle r_1(P_i), r_1(P_j)|\mathcal{T}^A\rangle \text{ for some } i,j \in N\big\}$ which includes all voters' preference peaks and alternatives that are located between voters' preference peaks. Thus, $\mathcal{T}^{\Gamma(P)}$ is the *minimal* subtree nested in $\mathcal{T}^A$ that covers all voters' preference peaks. Then, we construct a rule on $\mathcal{T}^A$ according to one edge (x,y) which separates $\mathcal{T}^A$ into two subtrees $\mathcal{T}^{A^{x\rightharpoonup y}}$ and $\mathcal{T}^{A^{y\rightharpoonup x}}$. We fix two distinct voters $i,j \in N$. First, at each preference profile, the social outcome equals voter i's most preferred alternative if it belongs to $A^{y\rightharpoonup x}$. Next, if both voters i and j have preference peaks in $A^{x\rightharpoonup y}$, the social outcome is the projection of x on the minimal subtree of the preference profile. Last, when the two most preferred alternatives of voters i and j lie respectively in $A^{x\rightharpoonup y}$ and $A^{y\rightharpoonup x}$, the social outcome varies according to voter j's preference over x and y.

Definition 6 *An SCF $f : \mathbb{D}^n \to A$ is a* ***Possibly Non-Tops-only*** *(or* ***PNT****) SCF on a tree $\mathcal{T}^A$ w.r.t. an edge (x,y) if there exist distinct $i,j \in N$ such that*

$$f(P) = \begin{cases} r_1(P_i) & \text{if } r_1(P_i) \in A^{y\rightharpoonup x}, \\ \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right) & \text{if } r_1(P_i) \in A^{x\rightharpoonup y} \text{ and } r_1(P_j) \in A^{x\rightharpoonup y}, \\ \max^{P_j}(\{x,y\}) & \text{if } r_1(P_i) \in A^{x\rightharpoonup y} \text{ and } r_1(P_j) \in A^{y\rightharpoonup x}. \end{cases}$$

By construction, a PNT SCF is unanimous and will henceforth be referred to as a PNT rule. A PNT rule defined on a minimally rich domain by construction is non-dictatorial and generalizes the constructed non-dictatorial rule associated with the unique seconds property.[37] Moreover, the following fact generalizes the

[37]Fix a domain $\mathbb{D}$ which satisfies the unique seconds property, say $\mathcal{S}(\mathbb{D}^x) = \{y\}$. We construct

heuristic example in Section 1.1 and pins down the necessary and sufficient condition for PNT rules to be strategy-proof and non-tops-only.

Fact **1** *Fix a minimally rich domain* $\mathbb{D}$*, a tree* $\mathcal{T}^A$ *and an edge* (x, y)*. For all* $n \geq 2$*, the PNT rule* $f : \mathbb{D}^n \to A$ *on* $\mathcal{T}^A$ *w.r.t.* (x, y) *is strategy-proof if and only if the following two conditions are satisfied: for all* $P_i \in \mathbb{D}$*,*

(i) *given* $r_1(P_i) \in A^{x \rightharpoonup y}$*,* P_i *is semi-single-peaked on* $\mathcal{T}^A$ *w.r.t.* y*, and*

(ii) *given* $r_1(P_i) \in A^{y \rightharpoonup x}$*,* $\max^{P_i}(A^{x \rightharpoonup y}) = x$*.*

Moreover, the PNT rule $f : \mathbb{D}^n \to A$ *on* $\mathcal{T}^A$ *w.r.t.* (x, y) *violates the tops-only property if and only if an additional condition is satisfied:*

(iii) *there exist* $P_i, P_i' \in \mathbb{D}$ *such that* $r_1(P_i) = r_1(P_i') \in A^{y \rightharpoonup x}$*,* yP_ix *and* $xP_i'y$*.*

The proof of Fact 1 is contained in Appendix B.

Given a domain $\mathbb{D}$ and a tree $\mathcal{T}^A$, we call an edge (x, y) a **critical spot**, if all conditions (i), (ii) and (iii) of Fact 1 are satisfied. The proposition below shows that the existence of a critical spot is necessary and sufficient for distinguishing a semi-single-peaked domain from a single-peaked domain (respectively distinguishing a semi-hybrid domain from a hybrid domain), and therefore each critical spot supports a strategy-proof PNT rule that violates the tops-only property.

Proposition **1** *Fixing a path-connected domain* $\mathbb{D}$ *and a tree* $\mathcal{T}^A$*, the following two statements hold:*

(i) *Given* $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ *for some threshold* $\bar{x} \in A$*, we have* $\mathbb{D} \not\subseteq \mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ *if and only if there exists a critical spot. Therefore, if* $\mathbb{D} \not\subseteq \mathbb{D}_{\text{SP}}(\mathcal{T}^A)$*, it admits a non-tops-only and strategy-proof rule.*

(ii) *Given* $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ *for some thresholds* $a, b \in A$*, we have* $\mathbb{D} \not\subseteq \mathbb{D}_{\text{H}}(\mathcal{T}^A, a, b)$ *if and only if there exists a critical spot in* $\mathcal{T}^{A^{a \rightharpoonup b}}$ *or* $\mathcal{T}^{A^{b \rightharpoonup a}}$*. Therefore, if* $\mathbb{D} \not\subseteq \mathbb{D}_{\text{H}}(\mathcal{T}^A, a, b)$*, it admits a non-tops-only and strategy-proof rule.*

The proof of Proposition 1 is contained in Appendix C.

a line $\mathcal{L} = (x, y, \dots)$ over A. Then, the PNT rule on $\mathcal{L}$ w.r.t. the edge (x, y) is identical to rule $(*)$ in Appendix A, which is constructed according to the unique seconds property.

3.2 A classification of non-dictatorial domains

In this section, we establish a classification of non-dictatorial domains using the notions of semi-single-peakedness and semi-hybridness. We do so by introducing a pair of completely reversed preferences to the domain and introducing a new axiom on a two-voter SCF, *invariance*, which requires that the SCF choose the same alternative at the two profiles where the voters are endowed with the two completely reversed preferences. We next refine the classification to single-peaked domains and hybrid domains on the line $\mathcal{L}^A$ by requiring additionally that the domains in question be tops-only domains. Finally, we investigate *all* tops-only and strategy-proof rules (either invariant or not) on rich non-dictatorial domains.

Henceforth, we fix $\underline{P}_i$ and $\overline{P}_i$ as two completely reversed preferences included in every rich domain we hencefort investigate, and moreover we *fix* $\underline{P}_i = (a_1 \cdots a_k\, a_{k+1} \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_{k+1}\, a_k \cdots a_1)$ by relabelling alternatives as necessary.[38]

Definition **7** *Given a domain $\mathbb{D}$, let $\underline{P}_i, \overline{P}_i \in \mathbb{D}$. Then, an SCF $f : \mathbb{D}^2 \to A$ is* ***invariant*** *if we have $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2)$.*

Clearly, invariance is weaker than anonymity in a two-voter SCF.

The following is the main result of the paper.

Theorem *Let $\mathbb{D}$ be a rich non-dictatorial domain. Then, the following two statements hold:*

(i) *There exists an invariant, tops-only and strategy-proof rule if and only if $\mathbb{D}$ is a semi-single-peaked domain.*

(ii) *There exists no invariant, tops-only and strategy-proof rule if and only if $\mathbb{D}$ is a semi-hybrid domain and satisfies the unique seconds property.*

The proof of the Theorem is contained in Appendix D.

Remark **4** The Theorem refines the characterization of non-dictatorial domains obtained in Corollary 1 by showing that all non-dictatorial rich domains can be classified into one of the three variants illustrated in Figure 6 below:

[38] In preferences $\underline{P}_i$ and $\overline{P}_i$, we have $a_k \underline{P}_i a_{k+1}$ and $a_{k+1} \overline{P}_i a_k$ for all $k = 1, \ldots, m-1$.

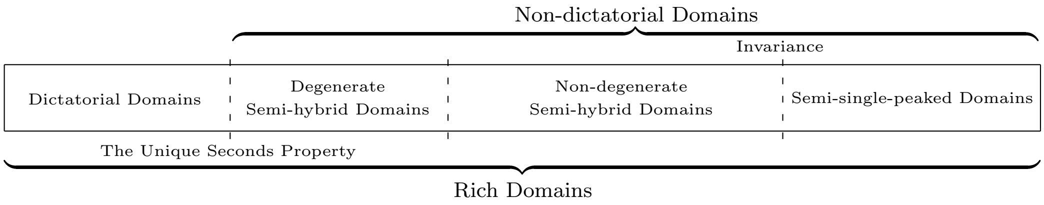

Figure 6: A classification of rich non-dictatorial domains

First, semi-single-peaked domains are sorted out according to Statement (i) of the Theorem as the unique ones that admit an invariant, tops-only and strategy-proof rule, while every other rich non-dictatorial domain is shown by Statement (ii) of the Theorem to be an (a, b)-semi-hybrid domain on a tree $\mathcal{T}^A$ that satisfies the unique seconds property. Moreover, the proof of Statement (ii) shows that the rich semi-hybrid domain in question must force every tops-only and strategy-proof rule to behave like a dictatorship on the "free zone" $\langle a, b|\mathcal{T}^A\rangle$. Such a semi-hybrid domain can be either non-degenerate or degenerate. If it is non-degenerate, then it admits a tops-only and strategy-proof rule that is non-dictatorial (see the rule specified in the verification of Claim 4, Lemma 24, Appendix D). If it is degenerate, then we have $\langle a, b|\mathcal{T}^A\rangle = A$ (see for instance Example 5 in Clarification 6 of Appendix G), and hence every tops-only and strategy-proof rule is a dictatorship; however the unique seconds property ensures the existence of a non-dictatorial strategy-proof rule. More specifically, in this latter case, all non-dictatorial strategy-proof rules inevitably violate the tops-only property, and hence a non-tops-only and strategy-proof rule is called upon to satisfy the non-dictatorial-domain hypothesis, e.g., the PNT rule associated with the unique seconds property (see footnote 37). □

Remark 5 The Theorem and its proof imply that given a rich domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, every tops-only and strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$ if and only if $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$.[39] □

Remark 6 Chatterji et al. (2013) have shown that on a path-connected domain, semi-single-peakedness is implied by the existence of an anonymous, tops-only and strategy-proof rule with an even number of voters. On our rich domain, the Theorem strengthens their result as it implies that regardless of the number

[39] See the detailed verification in Clarification 7 of Appendix G.

of voters, semi-single-peakedness is necessary for the existence of an anonymous, tops-only and strategy-proof rule.[40] □

If we further restrict the rich non-dictatorial domains in question to be tops-only domains, the classification is refined in the following three ways: (i) degenerate semi-hybrid domains that exogenously satisfy the unique seconds property are explicitly excluded from the classification as they admit non-tops-only and strategy-proof rules (recall Remark 4), (ii) non-degenerate semi-hybrid domains are refined to non-degenerate hybrid domain on the line $\mathcal{L}^A$ (by statement (ii) of Proposition 1 and by the existence of the two completely reversed preferences), and are strengthened to satisfy the non-trivialness condition on their own "free zones" by the Auxiliary Proposition, and (iii) semi-single-peaked domains are refined to be single-peaked on the line $\mathcal{L}^A$ (by statement (i) of Proposition 1 and by the existence of the two completely reversed preferences). Furthermore, we show that single-peakedness and non-trivial hybridness on $\mathcal{L}^A$ are also sufficient for a rich domain to be a tops-only domain.

Corollary **2** *Let $\mathbb{D}$ be a rich non-dictatorial tops-only domain $\mathbb{D}$. Then, the following two statements hold:*

(i) *There exists an invariant and strategy-proof rule if and only if $\mathbb{D}$ is a single-peaked domain on $\mathcal{L}^A$.*

(ii) *There exists no invariant and strategy-proof rule if and only if $\mathbb{D}$ is a non-trivial and non-degenerate hybrid domain on $\mathcal{L}^A$.*

Moreover, given a rich domain $\mathbb{D}$, it is a tops-only domain if and only if it is single-peaked or non-trivially hybrid on $\mathcal{L}^A$.

[40] Let a rich domain $\mathbb{D}$ admit an anonymous, tops-only and strategy-proof rule. Clearly, $\mathbb{D}$ is a non-dictatorial domain. Suppose that $\mathbb{D}$ is not semi-single-peaked. Statement (i) of the Theorem first implies non-existence of an invariant, tops-only and strategy-proof rule. Then, statement (ii) implies that $\mathbb{D}$ is an (a, b)-semi-hybrid domain on a tree $\mathcal{T}^A$, and Remark 5 implies that every tops-only and strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. This contradicts the hypothesis that $\mathbb{D}$ admits an anonymous, tops-only and strategy-proof rule.

The proof of Corollary 2 is contained in Appendix E.

We use another diagram to illustrate the classification of rich non-dictatorial domains refined by Corollary 2.

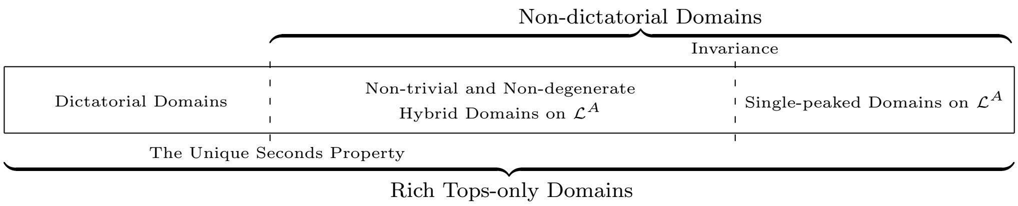

Figure 7: A classification of rich non-dictatorial tops-only domains

REMARK **7** In the literature, specific restricted domains have been verified to be tops-only domains, and general sufficient conditions have been introduced for establishing tops-only domains (see the literature review in Section 4.1). Apart from providing a sufficient condition for tops-only domains, Corollary 2 more importantly utilizes the classification result to justify the *necessity* of single-peakedness and hybridness in establishing tops-only domains; this in turn demonstrates the salience of critical spots in supporting non-tops-only and strategy-proof rules. □

3.3 TOPS-ONLY RULES AND CHARACTERIZATIONS

In this section, we utilize our classification results to investigate the structure of non-dictatorial tops-only and strategy-proof rules[41] on the semi-single-peaked domains and the semi-hybrid domains.[42]

[41]We have introduced one class of non-tops-only rules in Section 3.1, and identified the critical spot on semi-single-peaked and semi-hybrid domains to support its strategy-proofness. Characterizing all non-tops-only and strategy-proof rules on a rich non-dictatorial domains requires a more thorough investigation of the domain and we leave it for future work.

[42]On the single-peaked domain $\mathbb{D}_{\text{SP}}(\mathcal{L}^A)$, Moulin (1980) characterized all strategy-proof rules to be *generalized median voter rules*. On the hybrid domain $\mathbb{D}_{\text{H}}(\mathcal{L}^A, a_p, a_q)$, where $q - p > 1$, Theorem 1 of Achuthankutty and Roy (2020) implies that each strategy-proof rule is a *restricted generalized median voter rule* where one voter dictates on the interval $\langle a_p, a_q | \mathcal{L}^A \rangle$. The same characterization results still hold on a rich single-peaked domain and a rich non-trivial hybrid domain. The detailed proof can be provided on request.

3.3.1 *Projection rule*

First, the fact below introduces a specific anonymous SCF, called the *projection rule* (also see Thomson (1993) and Vohra (1999)), and states that semi-single-peakedness is necessary and sufficient for its strategy-proofness.

Fact 2 *Fixing a minimally rich domain* $\mathbb{D}$, *a tree* $\mathcal{T}^A$ *and an alternative* $\bar{x} \in A$, *let SCF* $f: \mathbb{D}^n \to A$ *be a* ***projection rule*** *w.r.t.* $\bar{x}$, *i.e.,* $f(P) = \mathrm{Proj}\left(\bar{x}, \mathcal{T}^{\Gamma(P)}\right)$ *for all* $P \in \mathbb{D}^n$. *Then,* f *is a strategy-proof rule if and only if we have* $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(\mathcal{T}^A, \bar{x})$.

The sufficiency part of Fact 2 follows exactly from the Theorem of Chatterji et al. (2013), while the necessity part is implied by strategy-proofness of a two-voter projection rule (see Lemma 14 in Appendix D), which can be directly induced from an n-voter projection rule by separating all voters into two non-empty groups and cloning all voters in the same group.

Corollary 1 of Bonifacio and Massó (2020) implies that the projection rule is the unique anonymous, tops-only and strategy-proof rule on *almost* all semi-single-peaked domains.[43] This is formally stated below.

Proposition 2 *Fixing the semi-single-peaked domain* $\mathbb{D}_{\mathrm{SSP}}(\mathcal{T}^A, \bar{x})$, *let* $\bar{x} \notin \mathcal{N}^A(x)$ *for any* $x \in Ext(\mathcal{T}^A)$. *Then, an SCF* $f: \left[\mathbb{D}_{\mathrm{SSP}}(\mathcal{T}^A, \bar{x})\right]^n \to A$ *is an anonymous, tops-only and strategy-proof rule only if it is a projection rule w.r.t.* $\bar{x}$.

3.3.2 *Hybrid rule*

Next, the fact below introduces another specific non-dictatorial SCF, called the *hybrid rule*, which is a hybrid of a dictatorship and two projection rules, and shows that semi-hybridness is necessary and sufficient for its strategy-proofness.

[43] The detailed verification is contained in Clarification 8 of Appendix G.

Fact **3** *Fixing a minimally rich domain* $\mathbb{D}$*, a tree* $\mathcal{T}^A$ *and two thresholds* $a, b \in A$*, let SCF* $f: \mathbb{D}^n \to A$ *be a* ***hybrid rule*** *w.r.t.* a *and* b*, i.e., there exist a voter* $i \in N$ *and two winning coalitions* $\mathcal{W}^{a \rightharpoonup b}, \mathcal{W}^{b \rightharpoonup a} \subseteq N$ *with* $i \in \mathcal{W}^{a \rightharpoonup b} \cap \mathcal{W}^{b \rightharpoonup a}$ *such that*

$$f(P) = \begin{cases} r_1(P_i) & \text{if } r_1(P_i) \in \langle a, b | \mathcal{T}^A \rangle, \\ \text{Proj}\left(a, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{a \rightharpoonup b}}\right)}\right) & \text{if } r_1(P_i) \in A^{a \rightharpoonup b} \backslash \{a\}, \\ \text{Proj}\left(b, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b \rightharpoonup a}}\right)}\right) & \text{if } r_1(P_i) \in A^{b \rightharpoonup a} \backslash \{b\}.^{44} \end{cases}$$

Then, given $\mathcal{W}^{a \rightharpoonup b} \neq \{i\}$ *and* $\mathcal{W}^{b \rightharpoonup a} \neq \{i\}$*,* f *is a strategy-proof rule if and only if we have* $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$*.*

The verification of the sufficiency part of Fact 3 is similar to the verification of Claim 1 in the proof of Lemma 24 in Appendix D, while the necessity part is implied by the strategy-proofness of a two-voter hybrid rule (see Lemma 21 in Appendix D), which can be induced from an n-voter strategy-proof hybrid rule by cloning all voters other than the one who dictates on the interval $\langle a, b | \mathcal{T}^A \rangle$.

Last, the proposition below provides a characterization of the hybrid rule on the semi-hybrid domain.

Proposition **3** *Fixing the semi-hybrid domain* $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$*, let* $|\langle a, b | \mathcal{T}^A \rangle| \geq 3$ *and* $\underline{P}_i, \overline{P}_i \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$*. Then, an SCF* $f: \left[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\right]^n \to A$ *is a tops-only and strategy-proof rule only if it is a hybrid rule w.r.t.* a *and* b*.*

The proof of Proposition 3 is contained in Appendix F.

Remark **8** On the one hand, we know by Proposition 1 that semi-single-peaked domains and semi-hybrid domains admit non-tops-only and strategy-proof rules, whereas the refinement to single-peaked and hybrid domains in Corollary 2 eliminates all non-tops-only and strategy-proof rules. On the other hand, such a refinement enlarges the scope for designing tops-only and strategy-proof rules, as the projection rule uniquely characterized in Proposition 2 is generalized to the

[44] The notation $P_{\mathcal{W}^{a \rightharpoonup b}}$ is a preference profile which only contains the preferences of the voters in the winning coalition $\mathcal{W}^{a \rightharpoonup b}$. If $\mathcal{W}^{a \rightharpoonup b} = \{i\}$ (respectively, $\mathcal{W}^{b \rightharpoonup a} = \{i\}$), then voter i dictates on $A^{a \rightharpoonup b} \cup \langle a, b | \mathcal{T}^A \rangle$ (respectively, $\langle a, b | \mathcal{T}^A \rangle \cup A^{b \rightharpoonup a}$). If $\mathcal{W}^{a \rightharpoonup b} = \mathcal{W}^{b \rightharpoonup a} = \{i\}$, then the hybrid rule degenerates to a dictatorship of voter i.

class of phantom voter rules on the single-peaked domain, and the hybrid rule of Proposition 3 is expanded to the whole family of restricted generalized median voter rules on the hybrid domain. □

Remark **9** According to Theorem 4 of Schummer and Vohra (2002), we know that each strategy-proof rule defined on the hybrid domain $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$ is an *extended generalized median voter rule* which behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Compared to $\mathbb{D}_{\mathrm{H}}(\mathcal{T}^A, a, b)$, the semi-hybrid domain $\mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$ is more permissive on the relative rankings of alternatives in $A^{a\rightharpoonup b}$ and $A^{b\rightharpoonup a}$. To accommodate these more permissive rankings while maintaining strategy-proofness, additional restrictions have to be imposed on the extended generalized median voter rule. Proposition 3 identifies the additional restrictions (see Claims 2 and 3 in Appendix F) and shows that these drive the extended generalized median voter rule to an explicit configuration, a hybrid rule w.r.t. a and b, which is more transparent from the design point of view. □

4 Literature Review and Final Remarks

4.1 A review of the literature

Following the seminal Gibbard-Satterthwaite Theorem, domain restrictions have received much attention in the literature on strategic voting. One stream of the literature examines the robustness of the Gibbard-Satterthwaite Theorem by showing that some sparse restricted domains, see for instance, FPT (Free Pair at the Top) domains of Sen (2001), linked domains of Aswal et al. (2003), circular domains of Sato (2010) and the $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ domains of Pramanik (2015), are in fact dictatorial domains. These papers use richness assumptions on the domain variously to construct connectedness relations between alternatives, while the violation of these richness conditions appear, somewhat surprisingly, to lead to the unique seconds property in the sense that if the unique seconds property holds, all the aforementioned richness conditions that precipitate dictatorship are violated.[45] Recently,

[45]This assertion can be made more precise by observing that in the case $|A| = 3$, any domain other than the universal domain satisfies the unique seconds property.

Roy and Storcken (2019) have concluded the of the unique seconds property in characterizing non-dictatorial domains. Our Corollary 1 is in the same vein. But more importantly, our focus on the *classification* of non-dictatorial domains uncovers more meaningful non-dictatorial strategy-proof rules (recall the projection rule in Proposition 2 and the hybrid rule in Proposition 3), compared to the specific non-dictatorial strategy-proof rule associated with the unique seconds property.

Another stream of the literature starts with a specific restricted domain that not only helps escape the Gibbard-Satterthwaite impossibility, but also accommodates the design of various well-behaved strategy-proof rules. Almost all such domains are variants of the notion of single-peakedness. On the single-peaked domain, the seminal paper of Moulin (1980) characterized all anonymous, tops-only and strategy-proof rules as phantom voter rules, and all tops-only and strategy-proof rules as generalized median voter rules. In the past four decades, several key variants of single-peakedness have been developed and non-dictatorial strategy-proof rules have been explored: Demange (1982) introduced single-peakedness on a tree and Schummer and Vohra (2002) investigated all corresponding strategy-proof rules and extended Moulin's generalized median voter rules; Barberà et al. (1993) generalized single-peakedness from a one-dimensional underlying line to a multidimensional grid, and discovered an important class of strategy-proof rules: multidimensional generalized median voter rules on the multidimensional single-peaked domain; Nehring and Puppe (2007) adopted a ternary relation to generally address the geometric relation among alternatives, invented the notion of generalized single-peakedness, and characterized all strategy-proof rules;[46] and recently, Reffgen (2015) provided a transition from the single-peaked domain to the uni-

[46] Using the terminology of Nehring and Puppe (2007), the (inclusion/non-inclusion) separable domain of Barberà et al. (1991), the multidimensional single-peaked domain of Barberà et al. (1993) and the separable domain of Le Breton and Sen (1999) can be equivalently translated to generalized single-peaked domains according to three analogous ternary relations respectively. Moreover, the important strategy-proof rules, voting by committees of Barberà et al. (1991) and component-wise dictatorship of Le Breton and Sen (1999) can be translated to two specific multidimensional generalized median voter rules of Barberà et al. (1993).

versal domain by taking unions of multiple single-peaked domains which are constructed according to different underlying lines, established the notion of a multiple single-peaked domain and characterized all strategy-proof rules as a specific subset of the generalized median voter rules which simultaneously preserve features of a dictatorship and of a median voter rule. Two comprehensive survey papers, Sprumont (1995) and Barberà (2011), provided more detailed discussions on the development of single-peakedness restrictions and non-dictatorial strategy-proof rules. All aforementioned restricted preference domains are in fact tops-only domains. Therefore all non-tops-only and strategy-proof rules are implicitly excluded from the investigation. We depart from this literature by considering non-tops-only rules; Proposition 1 here identifies a critical spot in a restricted domain that supports a non-tops-only and strategy-proof rule, and Corollary 2 illustrates its necessity by showing that non-tops-only and strategy-proof rules disappear as the critical spot vanishes.

A third stream of the literature poses the following natural "converse" question: is single-peakedness a consequence of the existence of a "well-behaved" strategy-proof rule? Earlier literature Barberà et al. (1993) showed that if a minimally rich domain admits the median voter rule as a strategy-proof rule, then the domain must be single-peaked on a line. Instead of considering a specific rule, Chatterji et al. (2013) established that on a path-connected domain, semi-single-peakedness, rather than single-peakedness, is necessary for the existence of an anonymous, tops-only and strategy-proof rule, and Chatterji and Massó (2018) showed that semilattice single-peakedness, a generalization of semi-single-peakedness, arises as a consequence of the existence of an anonymous, tops-only and strategy-proof rule on a rich domain (where the richness condition is formulated relative to the particular rule that is assumed to exist). Recently, Barberà et al. (2020) provide an insightful survey explaining the single-peakedness restriction, its various weakenings and the frontier between dictatorial and non-dictatorial domains. This literature too restricts attention to the class of strategy-proof rules that in addition satisfy the tops-only property and anonymity, and is therefore silent on domains where all tops-only and strategy-proof rules are dictatorships; also excluded are domains that admit tops-only and strategy-proof rules that violate anonymity but remain non-dictatorial. Our classification theorem essentially demonstrates that appropriate weakenings of single-peakedness characterize all non-dictatorial domains. It emphasizes the role of the unique seconds property and the associated non-tops-only and strategy-proof rules for a non-dictatorial domain where all tops-only and strategy-proof rules are dictatorships, retrieves the salience of semi-single-peakedness in allowing a tops-only and strategy-proof rule that is invariant (a weakening of anonymity), and discovers non-degenerate semi-hybrid domains as the ones that allow the existence of a non-dictatorial, tops-only and strategy-proof rule that fails to meet the full requirement of anonymity. For simplicity, our analysis restricts attention to the case of two voters. We claim that there is no loss of generality in doing so, and it helps us to avoid the exogenous assumption on the number of voters in the literature. First, Remark 6 shows that semi-single-peakedness is necessary for the existence of an anonymous, tops-only and strategy-proof rule, regardless of the number of voters. Second, given semi-hybridness revealed from all two-voter non-dictatorial, tops-only and strategy-proof rules, the Second Ramification Theorem specified in the Supplementary Material essentially suggests that no preference restriction beyond the revealed semi-hybridness can be elicited via any n-voter counterpart rule.[47] Chatterji and Zeng (2019) introduced a richness condition that endogenously ensures all strategy-proof rules are tops-only in a multidimensional setting and showed that the existence of a well-behaved strategy-proof rule implies full multidimensional single-peakedness. We do not follow this approach here as our intention is to uncover the role of semi-single-peakedness and semi-hybridness in allowing the design of non-tops-only and

[47]Via one n-voter non-dictatorial, tops-only and strategy-proof rule, one may refine the preference restriction on the two subtrees attached to the two thresholds, and hence push the domain closer to a hybrid domain. Clearly, such a refinement is still accommodated by the notion of semi-hybridness. More importantly, we conjecture that such a refinement can also be achieved by aggregating the preference restrictions revealed from multiple two-voter non-dictatorial, tops-only and strategy-proof rules.

strategy-proof rules.

Lastly, our refinement of the classification of non-dictatorial domains provided in Corollary 2 is also related to the literature on tops-only domains. In this literature, various restricted domains have been shown to be tops-only domains (see for instance Barberà et al., 1991, 1993; Ching, 1997; Le Breton and Sen, 1999; Le Breton and Weymark, 1999; Nehring and Puppe, 2007; Weymark, 2008; Reffgen, 2015; Achuthankutty and Roy, 2020). Chatterji and Sen (2011) provided two general sufficient conditions for tops-only domains. The non-trivial rich hybrid domains studied in this paper (see for instance domain $\widehat{\mathbb{D}}$ of Example 1), viewed as tops-only domains, are not covered by the literature. In particular, Corollary 2, to our knowledge, is the first result that fully *characterizes* tops-only domains, and therefore reveals the important role of the full single-peakedness requirement, embedded in either the whole line, or on both the left and right parts of the line, in establishing a tops-only domain.

4.2 Indispensability

To conclude this paper, we provide three examples to show the indispensability of our richness condition in establishing the classification of non-dictatorial domains. In the three examples, we drop the completely reversed preferences, path-connectedness and extreme-vertex symmetry in turn and provide a non-dictatorial domain beyond the classification. We believe that these three examples also suggest some directions on future investigation of non-dictatorial domains and non-dictatorial strategy-proof rules.

Example **2 (Indispensability of the completely reversed preferences)** Let $A = \{a, b, c, d\}$ be allocated on a star-shape tree $\mathcal{T}^A$ of Figure 8.

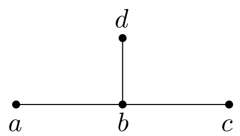

Figure 8: The star-shape tree $\mathcal{T}^A$

The corresponding single-peaked domain $\mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ is path-connected, satisfies extreme-vertex symmetry vacuously, but does not contain a pair of completely reversed preferences. Clearly, $\mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ is a non-dictatorial domain, since the majority voting rule continues to deliver a Condorcet winner and preserves unanimity, anonymity and strategy-proofness. Althrough $\mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ is covered by the characterization of non-dictatorial domains in Corollary 1, it is exogenously excluded by our classification in the Theorem. □

Example 3 **(Indispensability of path-connectedness)**

Let $A = \{a, b, c, d\}$ be allocated on a box specified in Figure 9, which can be reinterpreted as a Cartesian product structure $\{0,1\} \times \{0,1\}$.

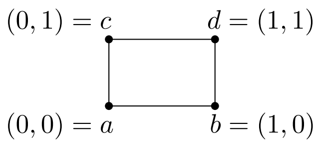

Figure 9: A graph of a box

According to the Cartesian product structure $\{0,1\} \times \{0,1\}$, we specify the *multidimensional single-peaked domain* $\mathbb{D}_{\text{MSP}}$ in Table 2.[48]

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a	a	b	b	c	c	d	d
b	c	a	d	a	d	b	c
c	b	d	a	d	a	c	b
d	d	c	c	b	b	a	a

Table 2: The multidimensional single-peaked domain $\mathbb{D}_{\text{MSP}}$

Domain $\mathbb{D}_{\text{MSP}}$ violates path-connectedness as no pair of alternatives are adjacent. Hence, the adjacency graph $G^A_\sim$ has an empty edge set, and $\mathbb{D}_{\text{MSP}}$ then satisfies extreme-vertex symmetry vacuously. Indeed, domain $\mathbb{D}_{\text{MSP}}$ contains four pairs of completely reversed preferences. By Theorem 1 of Barberà et al. (1991), a particular non-dictatorial strategy-proof rule can be constructed on $\mathbb{D}_{\text{MSP}}$, *voting by committees*. Therefore, $\mathbb{D}_{\text{MSP}}$ is a non-dictatorial domain. Last, we notice that $\mathbb{D}_{\text{MSP}}$ violates the unique seconds property as $|\mathcal{S}(\mathbb{D}^x_{\text{MSP}})| > 1$ for all $x \in A$. Therefore, $\mathbb{D}_{\text{MSP}}$ is not covered by our classification. □

[48] The domain specified in Table 2 was initially introduced by Barberà et al. (1991) in a non-Cartesian-product formulation.

EXAMPLE 4 **(Indispensability of extreme-vertex symmetry)**
Let $A = \{a, b, c, d\}$ be allocated on the star-shape tree $\mathcal{T}^A$ of Figure 8. We specify a domain $\mathbb{D}$ of 9 preferences in Table 3.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a	b	b	b	c	d	a	c	d
b	a	c	d	b	b	d	a	c
c	c	d	a	d	a	b	b	b
d	d	a	c	a	c	c	d	a

Table 3: Domain $\mathbb{D}$

Notice that the first 6 preferences of $\mathbb{D}$ are single-peaked on $\mathcal{T}^A$ and imply path-connectedness. In particular, $G^A_\sim = \mathcal{T}^A$. Domain $\mathbb{D}$ has a pair of completely reversed preferences, P_1 and P_9. However, $\mathbb{D}$ violates extreme-vertex symmetry: given $a \in Ext(G^A_\sim)$ and $(a, b) \in \mathcal{E}^A_\sim$, we have $\mathcal{S}(\mathbb{D}^a) = \{b, d\}$ but $a \notin \mathcal{S}(\mathbb{D}^d)$.

We construct the following non-tops-only, anonymous and strategy-proof rule to illustrate that $\mathbb{D}$ is a non-dictatorial domain: for all $(P_i, P_j) \in \mathbb{D}^2$,

$$f(P_i, P_j) = \begin{cases} d & \text{if } P_i = P_7 \text{ and } P_j \in \mathbb{D}^d, \text{ or } P_i \in \mathbb{D}^d \text{ and } P_j = P_7, \\ a & \text{if } P_i = P_8 \text{ and } P_j \in \mathbb{D}^a, \text{ or } P_i \in \mathbb{D}^a \text{ and } P_j = P_8, \\ c & \text{if } P_i = P_9 \text{ and } P_j \in \mathbb{D}^c, \text{ or } P_i \in \mathbb{D}^c \text{ and } P_j = P_9, \\ \text{Proj}\left(b, \langle r_1(P_i), r_1(P_j) | \mathcal{T}^A \rangle\right) & \text{otherwise.}^{49} \end{cases}$$

Last, we observe $|\mathcal{S}(\mathbb{D}^x)| \geq 2$ for all $x \in A$ which suggests the failure of the unique seconds property. Therefore, $\mathbb{D}$ is not covered by our classification. □

[49] The verification of unanimity, anonymity, non-tops-onlyness and strategy-proofness is simple, and we leave it to the reader.

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Appendix

A Proof of the Auxiliary Proposition

(**Necessity**) Let domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ be path-connected and satisfy extreme-vertex symmetry, where $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$. Suppose that every strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. We show that $\mathbb{D}$ is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$.

By the definition of (a, b)-semi-hybridness on $\mathcal{T}^A$ and path-connectedness, we know that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a connected graph. Clearly, either $Ext\big(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}\big) = \emptyset$ or $Ext\big(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}\big) \neq \emptyset$ holds. If $Ext\big(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}\big) = \emptyset$, $\mathbb{D}$ is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$ by definition. Henceforth, we assume $Ext\big(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}\big) \neq \emptyset$. We fix $x \in Ext\big(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}\big)$ and $(x, y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$. First, we consider the case that $x \notin \{a, b\}$, and show that there exists $P_i^* \in \mathbb{D}^x$ such that $r_2(P_i^*) \neq y$. Suppose not, i.e., $r_2(P_i) = y$ for all $P_i \in \mathbb{D}^x$. We construct the following SCF:

$$f(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } r_1(P_1) \neq x, \\ \max^{P_2}(\{x, y\}) & \text{otherwise.} \end{cases} \qquad (*)$$

By construction, f satisfies unanimity and hence is a rule. Moreover, by the proof of Theorem 5.1 of Aswal et al. (2003), we know that f is strategy-proofness. Clearly, f does not behave like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$, since we have $x, y \in \langle a, b|\mathcal{T}^A\rangle$, $[r_1(P_1) \neq x] \Rightarrow [f(P_1, P_2) = r_1(P_1)]$ and $[r_1(P_1') = x$ and $r_1(P_2') = y] \Rightarrow [f(P_1', P_2') = y \neq r_1(P_1')]$. This contradicts the hypothesis of the necessity part, and hence proves condition (i) of Definition 5.

Next, let $x = a$. We show that there exists $P_i^* \in \mathbb{D}$ such that $r_1(P_i^*) \in A^{a \rightharpoonup b}$ and $\max^{P_i^*}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big) \neq y$. Suppose by contradiction that $\max^{P_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big) = y$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{a \rightharpoonup b}$.

Now, we keep the subtree $\mathcal{T}^{A^{a \rightharpoonup b}}$, rearrange all alternatives of $\langle a, b|\mathcal{T}^A\rangle \cup A^{b \rightharpoonup a}$ on a line $(z_1, z_2, \ldots, z_v)$ such that $z_1 = a$ and $z_2 = y$, and combine the subtree $\mathcal{T}^{A^{a \rightharpoonup b}}$ and the line $(z_1, z_2, \ldots, z_v)$ to construct a new tree $\widehat{\mathcal{T}}^A$. Note that a and y are naturally two thresholds of $\widehat{\mathcal{T}}^A$. Accordingly, let $\hat{A}^{a \rightharpoonup y} = \big\{z \in A : a \in \langle z, y|\widehat{\mathcal{T}}^A\rangle\big\}$ and $\hat{A}^{y \rightharpoonup a} = \big\{z \in A : y \in \langle z, a|\widehat{\mathcal{T}}^A\rangle\big\}$. Clearly, $\hat{A}^{a \rightharpoonup y} = A^{a \rightharpoonup b}$, $\hat{A}^{y \rightharpoonup a} = \{z_2, \ldots, z_v\}$, and $\widehat{\mathcal{T}}^A$ is a combination of the subtree $\widehat{\mathcal{T}}^{\hat{A}^{a \rightharpoonup y}} = \mathcal{T}^{A^{a \rightharpoonup b}}$, the edge

$(a, y) = (z_1, z_2)$ and the line $\widehat{\mathcal{T}}^{\hat{A}^{y \rightharpoonup a}} = (z_2, \ldots, z_v)$. Thus, by the restriction of (a, b)-semi-hybridness on $\mathcal{T}^A$ and the contradictory hypothesis, one would easily notice that for every preference $P_i \in \mathbb{D}$ with $r_1(P_i) \in \hat{A}^{a \rightharpoonup y}$, the following two conditions hold:

(1) P_i is semi-single-peaked on the subtree $\widehat{\mathcal{T}}^{\hat{A}^{a \rightharpoonup y} \cup \{y\}}$ w.r.t. y, and

(2) $\max^{P_i}\left(\hat{A}^{y \rightharpoonup a}\right) = y$.

Claim 1: According to $\widehat{\mathcal{T}}^A$ and the thresholds a and y, the following two conditions hold: for all $P_i \in \mathbb{D}$,

(i) if $r_1(P_i) \in \hat{A}^{a \rightharpoonup y}$, then P_i is semi-single-peaked on $\widehat{\mathcal{T}}^A$ w.r.t. y, and

(ii) if $r_1(P_i) \in \hat{A}^{y \rightharpoonup a}$, then $\max^{P_i}(\hat{A}^{a \rightharpoonup y}) = a$.

Evidently, condition (i) is implied by conditions (1) and (2) above. Next, given an arbitrary $P_i \in \mathbb{D}$ with $r_1(P_i) \in \hat{A}^{y \rightharpoonup a}$, (a, b)-semi-hybridness on $\mathcal{T}^A$ implies $a = \max^{P_i}(A^{a \rightharpoonup b}) = \max^{P_i}(\hat{A}^{a \rightharpoonup y})$. This completes the verification of the claim.

Now, according to $\widehat{\mathcal{T}}^A$, we construct an SCF:

$$f(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } r_1(P_1) \in \hat{A}^{y \rightharpoonup a}, \\ \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right) & \text{if } r_1(P_1) \in \hat{A}^{a \rightharpoonup y} \text{ and } r_1(P_2) \in \hat{A}^{a \rightharpoonup y}, \\ \max^{P_2}(\{a, y\}) & \text{if } r_1(P_1) \in \hat{A}^{a \rightharpoonup y} \text{ and } r_1(P_2) \in \hat{A}^{y \rightharpoonup a}. \end{cases}$$

It is evident that f is unanimous and hence is a rule. Next, we show that f does not behave like a dictatorship on $\langle a, b | \mathcal{T}^A \rangle$. Note that $\langle a, b | \mathcal{T}^A \rangle \backslash \{a\} \subseteq \hat{A}^{y \rightharpoonup a}$, $a, y \in \langle a, b | \mathcal{T}^A \rangle$, $|\langle a, b | \mathcal{T}^A \rangle| \geq 3$ and $G_{\sim}^{\langle a, b | \mathcal{T}^A \rangle}$ is a connected graph. Therefore, we have $z \in \langle a, b | \mathcal{T}^A \rangle \backslash \{a, y\}$ such that $z \sim y$. Then, there exists $P_2 \in \mathbb{D}^z$ such that $r_2(P_2) = y$. Given $P_1 \in \mathbb{D}^a$, the construction of f implies $f(P_1, P_2) = \max^{P_2}(\{a, y\}) = y \notin \{r_1(P_1), r_1(P_2)\}$. This indicates that f does not behave like a dictatorship on $\langle a, b | \mathcal{T}^A \rangle$. Therefore, to complete the proof, it suffices to show that f is strategy-proof.

First, we consider voter 1. Given $P = (P_1, P_2)$ and $P' = (P_1', P_2)$, there are three possible manipulations:

(1) $f(P) = \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right)$ and $f(P') = \text{Proj}\left(a, \langle r_1(P'_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right)$,

(2) $f(P) = \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right)$ and $f(P') = r_1(P'_1)$, and

(3) $f(P) = \max^{P_2}(\{a, y\})$ and $f(P') = r_1(P'_1)$.

In each case, we show that either $f(P) = f(P')$ or $f(P)P_1f(P')$ holds. In case (1), $r_1(P_1) \in \hat{A}^{a \rightharpoonup y}$, $r_1(P_2) \in \hat{A}^{a \rightharpoonup y}$ and $r_1(P'_1) \in \hat{A}^{a \rightharpoonup y}$. Consequently, we have either $f(P) = f(P')$ or $f(P)P_1f(P')$ by the first condition of Claim 1. In case (2), $r_1(P_1) \in \hat{A}^{a \rightharpoonup y}$, $r_1(P_2) \in \hat{A}^{a \rightharpoonup y}$ and $r_1(P'_1) \in \hat{A}^{y \rightharpoonup a}$. Then, by construction, we have $f(P) = \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right) \in \langle r_1(P_1), a | \widehat{\mathcal{T}}^A \rangle$ and $f(P') = r_1(P'_1) \in \hat{A}^{y \rightharpoonup a}$. The first condition of Claim 1 implies $a = \min^{P_1}\left(\langle r_1(P_1), a | \widehat{\mathcal{T}}^A \rangle\right)$, aP_1y and $y = \max^{P_1}(\hat{A}^{y \rightharpoonup a})$. Therefore, $f(P)P_1f(P')$. In case (3), $r_1(P_1) \in \hat{A}^{a \rightharpoonup y}$, $r_1(P_2) \in \hat{A}^{y \rightharpoonup a}$ and $r_1(P'_1) \in \hat{A}^{y \rightharpoonup a}$. Thus, $f(P') = r_1(P'_1) \in \hat{A}^{y \rightharpoonup a}$. Since the first condition of Claim 1 implies aP_1y and $y = \max^{P_1}(\hat{A}^{y \rightharpoonup a})$, we have either $f(P) = f(P') = y$ or $f(P)P_1f(P')$. Therefore, voter 1 has no incentive to manipulate.

Last, we consider voter 2. Given $P = (P_1, P_2)$ and $P' = (P_1, P'_2)$, there are three possible manipulations:

(1) $f(P) = \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right)$ and $f(P') = \text{Proj}\left(a, \langle r_1(P_1), r_1(P'_2) | \widehat{\mathcal{T}}^A \rangle\right)$,

(2) $f(P) = \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right)$ and $f(P') = \max^{P'_2}(\{a, y\})$, and

(3) $f(P) = \max^{P_2}(\{a, y\})$ and $f(P') = \text{Proj}\left(a, \langle r_1(P_1), r_1(P'_2) | \widehat{\mathcal{T}}^A \rangle\right)$.

In each case, we show that either $f(P) = f(P')$ or $f(P)P_2f(P')$ holds. The verification of case (1) is symmetric to the verification of case (1) for voter 1. In case (2), $r_1(P_1) \in \hat{A}^{a \rightharpoonup y}$, $r_1(P_2) \in \hat{A}^{a \rightharpoonup y}$ and $r_1(P'_2) \in \hat{A}^{y \rightharpoonup a}$. Then, by construction, $f(P) = \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right) \in \langle r_1(P_2), a | \widehat{\mathcal{T}}^A \rangle$. Since the first condition of Claim 1 implies $a = \min^{P_2}\left(\langle r_1(P_2), a | \widehat{\mathcal{T}}^A \rangle\right)$ and aP_2y, we have either $f(P) = f(P')$ or $f(P)P_2f(P')$. In case (3), $r_1(P_1) \in \hat{A}^{a \rightharpoonup y}$, $r_1(P_2) \in \hat{A}^{y \rightharpoonup a}$ and $r_1(P'_2) \in \hat{A}^{a \rightharpoonup y}$. Thus, by construction, $f(P') = \text{Proj}\left(a, \langle r_1(P_1), r_1(P'_2) | \widehat{\mathcal{T}}^A \rangle\right) \in \langle r_1(P'_2), a | \widehat{\mathcal{T}}^A \rangle \subseteq \hat{A}^{a \rightharpoonup y}$. If yP_2a, we have $f(P) = \max^{P_2}(\{a, y\}) = y$, and the second condition of Claim 1 implies yP_2z for all $z \in \hat{A}^{a \rightharpoonup y}$. Therefore, $f(P)P_2f(P')$. If aP_2y, we have $f(P) = \max^{P_2}(\{a, y\}) = a$, and furthermore by the second condition of Claim

1, either $f(P) = f(P') = a$ or $f(P)P_2f(P')$ holds. Therefore, voter 2 has no incentive to manipulate. Hence, f is strategy-proof, as required.

In conclusion, we have a strategy-proof rule f which does not behave like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. This contradicts the hypothesis that every strategy-proof rule defined on $\mathbb{D}$ behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Therefore, $\mathbb{D}$ must satisfy condition (ii) of Definition 5. Last, by a symmetric argument, we can also show that $\mathbb{D}$ satisfies condition (iii) of Definition 5. This hence completes the verification of the necessity part of the Auxiliary Proposition.

(**Sufficiency**) The proof of the sufficiency part of the Auxiliary Proposition consists of three steps. First, we have established a ramification result in the Supplementary Material (the First Ramification Theorem), which says that given a domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ that is assumed to be path-connected and satisfy extreme-vertex symmetry, if every two-voter strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$, then every n-voter strategy-proof rule, $n \geq 2$, behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Therefore, we here can restrict attention to the two-voter strategy-proof rules. The rest of the proof consists of two steps. In the first step, we provide 4 important independent lemmas (Lemmas 1 - 4) on a path-connected domain and a two-voter strategy-proof rule, which will be repeatedly referred to. In the second step, we move to a domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ where $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$, assume that $\mathbb{D}$ is path-connected, satisfies extreme-vertex symmetry and is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$, and show that every two-voter strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$ (see Lemmas 5 - 8).

In the first step, for Lemmas 1 - 4, we fix $N = \{1, 2\}$, a path-connected domain $\mathbb{D}$ and a strategy-proof rule $f : \mathbb{D}^2 \to A$. For notational convenience, let $\big((x\cdots), (y\cdots)\big)$ denote a preference profile where voter 1 reports an *arbitrary* preference with the peak x and voter 2 reports an *arbitrary* preference with the peak y. More importantly, let $f\big((x\cdots), (y\cdots)\big) = a$ denote that "for all $P_1 \in \mathbb{D}^x$ and $P_2 \in \mathbb{D}^y$, $f(P_1, P_2) = a$."

Lemma 1 *Given a path $\pi = (x_1, \ldots, x_v)$ in $G^A_\sim$, the following four statements hold:*

(i) *If $f(P_1, P_2) = x_1$ for some $P_1 \in \mathbb{D}^{x_1}$ and $P_2 \in \mathbb{D}^{x_2}$, then $f\big((x_k \cdots), (x_{k'} \cdots)\big) = x_k$ for all $1 \leq k \leq k' \leq v$.*

(ii) *If $f(P_1, P_2) = x_1$ for some $P_1 \in \mathbb{D}^{x_2}$ and $P_2 \in \mathbb{D}^{x_1}$, then $f\big((x_{k'} \cdots), (x_k \cdots)\big) = x_k$ for all $1 \leq k \leq k' \leq v$.*

(iii) *If $f(P_1, P_2) = x_v$ for some $P_1 \in \mathbb{D}^{x_v}$ and $P_2 \in \mathbb{D}^{x_{v-1}}$, then $f\big((x_k \cdots), (x_{k'} \cdots)\big) = x_k$ for all $1 \leq k' \leq k \leq v$.*

(iv) *If $f(P_1, P_2) = x_v$ for some $P_1 \in \mathbb{D}^{x_{v-1}}$ and $P_2 \in \mathbb{D}^{x_v}$, then $f\big((x_{k'} \cdots), (x_k \cdots)\big) = x_k$ for all $1 \leq k' \leq k \leq v$.*

Proof: First, note that the first two statements are symmetric, the last two statements are symmetric, and the third statement is analogous to the first one.[50] Therefore, we focus on the verification of the first statement. Let $f(P_1, P_2) = x_1$ for some $P_1 \in \mathbb{D}^{x_1}$ and $P_2 \in \mathbb{D}^{x_2}$. Since $x_1 \sim x_2$, by strategy-proofness, it is easy to show $f\big((x_1 \cdots), (x_2 \cdots)\big) = x_1$.[51]

Next, we show $f\big((x_2 \cdots), (x_3 \cdots)\big) = x_2$. Since $x_2 \sim x_3$, it is easy to show that either $f\big((x_2 \cdots), (x_3 \cdots)\big) = x_2$ or $f\big((x_2 \cdots), (x_3 \cdots)\big) = x_3$ holds. Suppose $f\big((x_2 \cdots), (x_3 \cdots)\big) = x_3$. Since $x_1 \sim x_2$ and $x_3 \sim x_2$, we have preferences $\hat{P}_1 \in \mathbb{D}^{x_2}$, $\hat{P}'_1 \in \mathbb{D}^{x_1}$, $\check{P}_2 \in \mathbb{D}^{x_3}$ and $\check{P}'_2 \in \mathbb{D}^{x_2}$ such that $r_2(\hat{P}_1) = x_1$, $r_2(\hat{P}'_1) = x_2$, $r_k(\hat{P}_1) = r_k(\hat{P}'_1)$ for all $k = 3, \ldots, m$, $r_2(\check{P}_2) = x_2$, $r_2(\check{P}'_2) = x_3$, and $r_k(\check{P}_2) = r_k(\check{P}'_2)$ for all $k = 3, \ldots, m$. Thus, $f(\hat{P}_1, \check{P}_2) = x_3$. Then, strategy-proofness implies $f(\hat{P}'_1, \check{P}_2) = f(\hat{P}_1, \check{P}_2) = x_3$. Meanwhile, we also have $f(\hat{P}'_1, \check{P}'_2) = x_1$. Then, voter 2 will manipulate at $(\hat{P}'_1, \check{P}'_2)$ via $\check{P}_2$, i.e., $f(\hat{P}'_1, \check{P}_2) = x_3 \check{P}'_2 x_1 = f(\hat{P}'_1, \check{P}'_2)$.

[50] Given the first statement, to prove the third statement, we can relabel alternatives of π as follows: $y_k = x_{v+1-k}$ for all $k = 1, \ldots, v$. Thus, we have $P_1 \in \mathbb{D}^{x_v} = \mathbb{D}^{y_1}$, $P_2 \in \mathbb{D}^{x_{v-1}} = \mathbb{D}^{y_2}$ and $f(P_1, P_2) = x_v = y_1$. Then, the first statement on the path $(y_1, \ldots, y_v)$ implies that $f\big((y_k \cdots), (y_{k'} \cdots)\big) = y_k$ for all $1 \leq k \leq k' \leq v$, which is equivalent to $f\big((x_k \cdots), (x_{k'} \cdots)\big) = x_k$ for all $1 \leq k' \leq k \leq v$.

[51] See the proof of Claims A and B of Sen (2001).

Therefore, it must be the case that $f\big((x_2\cdots),(x_3\cdots)\big) = x_2$. Applying the same argument along the path π from x_3 to x_v step by step, we eventually have $f\big((x_k\cdots),(x_{k+1}\cdots)\big) = x_k$ for all $k = 1,\ldots,v-1$.

Last, we show $f\big((x_k\cdots),(x_{k'}\cdots)\big) = x_k$ for all $1 \leq k \leq k' \leq v$. We provide an induction hypothesis: given $2 < l \leq v$, for all $1 \leq k \leq k' < l$, we have $f\big((x_k\cdots),(x_{k'}\cdots)\big) = x_k$. To prove the induction hypothesis, it suffices to show $f\big((x_k\cdots),(x_l\cdots)\big) = x_k$ for all $1 \leq k \leq l$. Fixing an arbitrary $P_2 \in \mathbb{D}^{x_l}$, we first know $f\big((x_k\cdots),P_2\big) = x_k$ for both $k \in \{l-1,l\}$. We next show $f\big((x_{l-2}\cdots),P_2\big) = x_{l-2}$. Since $x_{l-2} \sim x_{l-1}$, we have preferences $\hat{P}_1 \in \mathbb{D}^{x_{l-1}}$ and $\hat{P}'_1 \in \mathbb{D}^{x_{l-2}}$ such that $r_2(\hat{P}_1) = x_{l-2}$, $r_2(\hat{P}'_1) = x_{l-1}$ and $r_k(\hat{P}_1) = r_k(\hat{P}'_1)$ for all $k = 3,\ldots,m$. Since $f(\hat{P}_1,P_2) = x_{l-1}$, strategy-proofness implies $f(\hat{P}'_1,P_2) \in \{x_{l-1},x_{l-2}\}$. Suppose $f(\hat{P}'_1,P_2) = x_{l-1}$. Then, strategy-proofness implies $f\big(\hat{P}'_1,(x_{l-1}\cdots)\big) = x_{l-1}$, which contradicts the fact $f\big((x_{l-2}\cdots),(x_{l-1}\cdots)\big) = x_{l-2}$. Therefore, $f(\hat{P}'_1,P_2) = x_{l-2}$, and then strategy-proofness implies $f\big((x_{l-2}\cdots),P_2\big) = x_{l-2}$. Applying the same argument along the path π from x_{l-2} to x_1 step by step, we eventually have $f\big((x_k\cdots),P_2\big) = x_k$ for all $k = 1,\ldots,l$. This completes the verification of the induction hypothesis, and hence proves the lemma. ■

Lemma **2** *Given two subsets $\bar{A},\hat{A} \subseteq A$ with $|\bar{A}| > 1$ and $|\hat{A}| > 1$, let $G^{\bar{A}}_{\sim}$ and $G^{\hat{A}}_{\sim}$ be two connected graphs. Given a path $\pi = (x_1,\ldots,x_v)$ in $G^{A}_{\sim}$, let $x_1 \in \bar{A}$ and $x_v \in \hat{A}$. If f behaves like a dictatorship on $\bar{A}$ and $\hat{A}$ respectively, then f behaves like a dictatorship on $\bar{A} \cup \pi \cup \hat{A}$.*

Proof: Since f behaves like a dictatorship on $\bar{A}$, we assume w.l.o.g. that voter 1 dictates on $\bar{A}$, i.e., $f(P_1,P_2) = r_1(P_1)$ for all $P_1,P_2 \in \mathbb{D}$ with $r_1(P_1),r_1(P_2) \in \bar{A}$. The first claims shows that voter 1 also dictates on $\hat{A}$.

Claim 1: For all $P_1,P_2 \in \mathbb{D}$ with $r_1(P_1),r_1(P_2) \in \hat{A}$, $f(P_1,P_2) = r_1(P_1)$.

We first consider the case $\bar{A} \cap \hat{A} \neq \emptyset$. Let $x \in \bar{A} \cap \hat{A}$. Since $G^{\bar{A}}_{\sim}$ is a connected graph and $|\bar{A}| > 1$, there exists $y \in \bar{A}$ such that $y \sim x$. Clearly, $f\big((y\cdots),(x\cdots)\big) = y$. Symmetrically, since $G^{\hat{A}}_{\sim}$ is a connected graph and $|\hat{A}| > 1$, there exists $z \in \hat{A}$ such that $z \sim x$. Note that either $y = z$ or $y \neq z$ holds. If $y = z$, we have $f\big((z\cdots),(x\cdots)\big)=z$, which, by the dictatorship of f in $\hat{A}$, implies that voter 1 dictates on $\hat{A}$. If $y\neq z$, we consider the path (y,x,z). Then, by statement (i) of Lemma 1, $f\big((y\cdots),(x\cdots)\big)=y$ implies $f\big((x\cdots),(z\cdots)\big)=x$. Then, by the dictatorship of f in $\hat{A}$, we infer that voter 1 dictates on $\hat{A}$. Overall, voter 1 dictates on $\hat{A}$ when $\bar{A}\cap\hat{A}\neq\emptyset$.

Next, we assume $\bar{A}\cap\hat{A}=\emptyset$. Given $x_1\in\bar{A}$ and $x_v\in\hat{A}$, we can identify $1\leq k\leq k'\leq v$ such that $x_k\in\bar{A}$, $x_{k'}\in\hat{A}$ and $x_l\notin\bar{A}\cup\hat{A}$ for all $k<l<k'$. Since $G_{\sim}^{\bar{A}}$ is a connected graph and $|\bar{A}|>1$, there exists $x\in\bar{A}$ such that $x\sim x_k$. Symmetrically, there exists $y\in\hat{A}$ such that $y\sim x_{k'}$. Thus, we have a path $(x,x_k,\ldots,x_{k'},y)$. Since voter 1 dictates on $\bar{A}$, we have $f\big((x\cdots),(x_k\cdots)\big)=x$. Then, according to the path $(x,x_k,\ldots,x_{k'},y)$, statement (i) of Lemma 1 implies $f\big((x_{k'}\cdots),(y\cdots)\big)=x_{k'}$. Furthermore, since f behaves like a dictatorship on $\hat{A}$, we infer that voter 1 dictates on $\hat{A}$. In conclusion, voter 1 dictates on $\hat{A}$. This completes the verification of the claim.

The next claim shows that that voter 1 dictates on the path π.

Claim 2: For all $P_1,P_2\in\mathbb{D}$ with $r_1(P_1),r_1(P_2)\in\pi$, $f(P_1,P_2)=r_1(P_1)$.

If $x_2\in\bar{A}$, we have $f\big((x_1\cdots),(x_2\cdots)\big)=x_1$ by voter 1's dictatorship on $\bar{A}$. If $x_2\notin\bar{A}$, we identify $x_0\in\bar{A}$ such that $x_0\sim x_1$ according to the connected graph $G_{\sim}^{\bar{A}}$. Clearly, $x_0\neq x_2$. Thus, we have $f\big((x_0\cdots),(x_1\cdots)\big)=x$ by voter 1's dictatorship on $\bar{A}$. Then, according to the path (x_0,x_1,x_2), statement (i) of Lemma 1 implies $f\big((x_1\cdots),(x_2\cdots)\big)=x_1$. Overall, we have $f\big((x_1\cdots),(x_2\cdots)\big)=x_1$. Then, according to the path π, statement (i) of Lemma 1 implies $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_k$ for all $1\leq k\leq k'\leq v$. Symmetrically, by voter 1's dictatorship on $\hat{A}$ and statement (iii) of Lemma 1 on the path π, we also induce $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_k$ for all $1\leq k'\leq k\leq v$. This completes the verification of the claim.

Last, we show that voter 1 dictates on $\bar{A}\cup\pi\cup\hat{A}$. We first show that voter 1 dictates on $\bar{A}\cup\pi$. Given arbitrary preferences $P_1,P_2\in\mathbb{D}$, let $r_1(P_1)=x\in\bar{A}\cup\pi$ and $r_1(P_2)=y\in\bar{A}\cup\pi$. If $x=y$, unanimity implies $f(P_1,P_2)=x=r_1(P_1)$. Next, assume $x\neq y$. Evidently, if $x,y\in\bar{A}$ or $x,y\in\pi$, we have $f(P_1,P_2)=x$ by voter 1's dictatorship on $\bar{A}$ and π respectively. Last, we consider two cases: (i)

$x \in \bar{A}\backslash\pi$ and $y \in \pi\backslash\bar{A}$, and (ii) $x \in \pi\backslash\bar{A}$ and $y \in \bar{A}\backslash\pi$. Note that the two cases are symmetric, and we hence focus on the verification of the first one. In the first case, it is clear that $x \in \bar{A}\backslash\{x_1\}$ and $y = x_k$ for some $1 < k \leq v$. Since $G_{\sim}^{\bar{A}}$ is a connected graph, we have a path $(z_1, \ldots, z_l)$ in $G_{\sim}^{\bar{A}}$ connecting x and x_1. Now, according to the paths $(z_1, \ldots, z_l)$ and $(x_1, \ldots, x_k)$, since $z_l = x_1$, $z_1 = x \in \bar{A}\backslash\pi$ and $x_k = y \in \pi\backslash\bar{A}$, we can identify $1 < s \leq l$ and $1 \leq t < k$ such that $z_s = x_t$ and $\{z_1, \ldots, z_{s-1}\} \cap \{x_{t+1}, \ldots, x_k\} = \emptyset$. Then, the concatenated path $(z_1, \ldots, z_s = x_t, \ldots, x_k)$ connects x and y. First, we have $f\big((z_1 \cdots), (z_2 \cdots)\big) = z_1$ by voter 1's dictatorship on $\bar{A}$. Next, according to the path $(z_1, \ldots, z_s = x_t, \ldots, x_k)$, statement (i) of Lemma 1 implies $f\big((z_1 \cdots), (x_k \cdots)\big) = z_1$, and hence, $f(P_1, P_2) = x$, as required. Therefore, voter 1 dictates on $\bar{A} \cup \pi$. Furthermore, note that both $G_{\sim}^{\bar{A}\cup\pi}$ and $G_{\sim}^{\hat{A}}$ are connected graphs, $[\bar{A} \cup \pi] \cap \hat{A} \neq \emptyset$ and voter 1 dictates on $\bar{A} \cup \pi$ and $\hat{A}$ respectively. By the same argument, we can infer that voter 1 dictates on $\big[\bar{A} \cup \pi\big] \cup \hat{A} = \bar{A} \cup \pi \cup \hat{A}$. ■

Lemma 3 *Given a path $\pi = (x_1, \ldots, x_v)$, $v \geq 3$, in $G_{\sim}^{A}$, if there exist $\hat{P}_i \in \mathbb{D}^{x_1}$ and $\check{P}_i \in \mathbb{D}^{x_v}$ such that $r_2(\hat{P}_i) = x_v$ and $r_2(\check{P}_i) = x_1$, then f behaves like a dictatorship on π.*

Proof: We first establish a claim to show that if f behaves like a dictatorship on $\{x_1, x_2\}$, then f behaves like a dictatorship on π.

Claim 1: If f behaves like a dictatorship on $\{x_1, x_2\}$, then f behaves like a dictatorship on π.

We assume w.l.o.g. that voter 1 dictates on $\{x_1, x_2\}$, i.e., $f(P_1, P_2) = r_1(P_1)$ for all $P_1, P_2 \in \mathbb{D}$ with $r_1(P_1), r_1(P_2) \in \{x_1, x_2\}$. By Lemma 2, it suffices to show that f behaves like a dictatorship on $\{x_{v-1}, x_v\}$. Since $x_{v-1} \sim x_v$, one of the following three cases occurs:

(1) f behaves like a dictatorship on $\{x_{v-1}, x_v\}$,

(2) $f\big((x_{v-1} \cdots), (x_v \cdots)\big) = f\big((x_v \cdots), (x_{v-1} \cdots)\big) = x_v$, and

(3) $f\big((x_{v-1} \cdots), (x_v \cdots)\big) = f\big((x_v \cdots), (x_{v-1} \cdots)\big) = x_{v-1}$.

To complete the verification, we rule out the last two cases. First, we have $f\big((x_1\cdots),(x_2\cdots)\big) = x_1$ by voter 1's dictatorship on $\{x_1, x_2\}$. Then, according to the path π, statement (i) of Lemma 1 implies $f\big((x_{v-1}\cdots),(x_v\cdots)\big) = x_{v-1}$, which rules out case (2). Suppose that case (3) occurs. First, note that $f\big((x_2\cdots),(x_1\cdots)\big) = x_2$ and $f\big((x_v\cdots),(x_{v-1}\cdots)\big) = x_{v-1}$. To be consistent to statements (ii) and (iii) of Lemma 1, there must exist $1 < \overline{k} < v$ such that $f\big((x_k\cdots),(x_{k'}\cdots)\big) = x_k$ for all $1 \leq k' \leq k \leq \overline{k}$ and $f\big((x_s\cdots),(x_{s'}\cdots)\big) = x_{s'}$ for all $\overline{k} \leq s' \leq s \leq v$.[52] Therefore, we have $f\big((x_v\cdots),(x_{\overline{k}}\cdots)\big) = x_{\overline{k}}$. Meanwhile, since $f\big((x_1\cdots),(x_2\cdots)\big) = x_1$, according to the path π, statement (i) of Lemma 1 implies $f\big((x_1\cdots),(x_{\overline{k}}\cdots)\big) = x_1$. Then, comparing $f\big((x_v\cdots),(x_{\overline{k}}\cdots)\big) = x_{\overline{k}}$ and $f\big((x_1\cdots),(x_{\overline{k}}\cdots)\big) = x_1$, strategy-proofness implies $x_{\overline{k}} P_i x_1$ for all $P_i \in \mathbb{D}^{x_v}$, which contradicts the hypothesis that $\check{P}_i \in \mathbb{D}^{x_v}$ and $r_2(\check{P}_i) = x_1$. Hence, case (3) is ruled out, as required. This completes the verification of the claim.

Symmetrically, we can show that if f behaves like a dictatorship on $\{x_{v-1}, x_v\}$, then f behaves like a dictatorship on π.

Last, we show that f behaves like a dictatorship on either $\{x_1, x_2\}$ or $\{x_{v-1}, x_v\}$. Suppose that it is not true. Then, $x_1 \sim x_2$ implies that either $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_1$, or $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_2$ holds. Suppose $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_1$. Then, according to the path π, by statements (i) and (ii) of Lemma 1, we have $f\big((x_k\cdots),(x_{k'}\cdots)\big) = f\big((x_{k'}\cdots),(x_k\cdots)\big) = x_k$ for all $1 \leq k \leq k' \leq v$. For each $1 \leq k < v$, since $f\big((x_v\cdots),(x_{k+1}\cdots)\big) = x_{k+1}$ and $f\big((x_k\cdots),(x_{k+1}\cdots)\big) = x_k$, strategy-proofness

[52] The index $\overline{k}$ can be identified in the following way. First, consider the profile $\big((x_3\cdots),(x_2\cdots)\big)$. Since $x_2 \sim x_3$, it is true that either $f\big((x_3\cdots),(x_2\cdots)\big) = x_2$ or $f\big((x_3\cdots),(x_2\cdots)\big) = x_3$ holds. If $f\big((x_3\cdots),(x_2\cdots)\big) = x_2$, let $\overline{k} = 2$ and then according to the subpath $(x_2, x_3, \ldots, x_v)$, statement (ii) of Lemma 1 implies $f\big((x_s\cdots),(x_{s'}\cdots)\big) = x_{s'}$ for all $\overline{k} \leq s' \leq s \leq v$. If $f\big((x_3\cdots),(x_2\cdots)\big) = x_3$, we have $f\big((x_k\cdots),(x_{k'}\cdots)\big) = x_k$ for all $1 \leq k' \leq k \leq 3$ by statement (iii) of Lemma 1 on the subpath (x_1, x_2, x_3), and then move to the profile $\big((x_4\cdots),(x_3\cdots)\big)$. Next, by repeatedly applying the argument above step by step, bounded by the fact $f\big((x_v\cdots),(x_{v-1}\cdots)\big) = x_{v-1}$, we eventually can identify $1 < \overline{k} < v$ such that $f\big((x_k\cdots),(x_{k'}\cdots)\big) = x_k$ for all $1 \leq k' \leq k \leq \overline{k}$ and $f\big((x_s\cdots),(x_{s'}\cdots)\big) = x_{s'}$ for all $\overline{k} \leq s' \leq s \leq v$.

implies $x_{k+1} P_i x_k$ for all $P_i \in \mathbb{D}^{x_v}$. This contradicts the hypothesis that $\check{P}_i \in \mathbb{D}^{x_v}$ and $r_2(\check{P}_i) = x_1$. Therefore, we have $f\big((x_1 \cdots), (x_2 \cdots)\big) = f\big((x_2 \cdots), (x_1 \cdots)\big) = x_2$. Symmetrically, since f does not behave like a dictatorship on $\{x_{v-1}, x_v\}$ by the contradictory hypothesis, according to the preference $\hat{P}_i$ in the hypothesis of the lemma, we can show $f\big((x_{v-1} \cdots), (x_v \cdots)\big) = f\big((x_v \cdots), (x_{v-1} \cdots)\big) = x_{v-1}$.

Recall the preferences $\hat{P}_i$ and $\check{P}_i$ in the hypothesis of the lemma. Since $r_1(\hat{P}_i) = r_2(\check{P}_i) = x_1$ and $r_1(\check{P}_i) = r_2(\hat{P}_i) = x_v$, it is true that either $f\big((x_1 \cdots), (x_v \cdots)\big) = x_1$ or $f\big((x_1 \cdots), (x_v \cdots)\big) = x_v$ holds. We will induce a contradiction in each case. First, let $f\big((x_1 \cdots), (x_v \cdots)\big) = x_1$. Since $x_v \sim x_{v-1}$, we have $P_2, P_2' \in \mathbb{D}$ such that $r_1(P_2) = r_2(P_2') = x_v$, $r_1(P_2') = r_2(P_2) = x_{v-1}$ and $r_k(P_2) = r_k(P_2')$ for all $k = 3, \ldots, m$. Let $\hat{P}_1 = \hat{P}_i$ and $\check{P}_1 = \check{P}_i$. Thus, $f(\hat{P}_1, P_2) = x_1$, and then strategy-proofness implies $f(\hat{P}_1, P_2') = f(\hat{P}_1, P_2) = x_1$. Furthermore, since $x_v \sim x_{v-1}$, strategy-proofness implies $f(\check{P}_1, P_2') \in \{x_v, x_{v-1}\}$. Therefore, we by strategy-proofness infer $f(\check{P}_1, P_2') \check{P}_1 f(\hat{P}_1, P_2')$, which implies $f(\check{P}_1, P_2') = x_v$. This contradicts the induced fact $f\big((x_v \cdots), (x_{v-1} \cdots)\big) = x_{v-1}$. Last, let $f\big((x_1 \cdots), (x_v \cdots)\big) = x_v$. Since $x_1 \sim x_2$, we have $P_1, P_1' \in \mathbb{D}$ such that $r_1(P_1) = r_2(P_1') = x_1$, $r_1(P_1') = r_2(P_1) = x_2$ and $r_k(P_1) = r_k(P_1')$ for all $k = 3, \ldots, m$. Let $\check{P}_2 = \check{P}_i$ and $\hat{P}_2 = \hat{P}_i$. Thus, $f(P_1, \check{P}_2) = x_v$, and then strategy-proofness implies $f(P_1', \check{P}_2) = f(P_1, \check{P}_2) = x_v$. Furthermore, since $x_1 \sim x_2$, strategy-proofness implies $f(P_1', \hat{P}_2) \in \{x_1, x_2\}$. Therefore, we by strategy-proofness infer $f(P_1', \hat{P}_2) \hat{P}_2 f(P_1', \check{P}_2)$, which implies $f(P_1', \hat{P}_2) = x_1$. This contradicts the induced fact $f\big((x_2 \cdots), (x_1 \cdots)\big) = x_2$. Therefore, f must behave like a dictatorship on either $\{x_1, x_2\}$ or $\{x_{v-1}, x_v\}$, as required. This proves the lemma. ■

According to Lemma 3, one would easily observe that f behaves like a dictatorship on a cycle in $G^A_\sim$.

Observation 1 Given a cycle $\mathcal{C} = (x_1, \ldots, x_v, x_1)$, $v \geq 3$, in $G^A_\sim$, f behaves like a dictatorship on $\mathcal{C}$.[53] □

[53] A cycle $\mathcal{C} = (x_1, \ldots, x_v, x_1)$ is a sequence such that $x_1, \ldots, x_v$ are pairwise distinct, $v \geq 3$, and $x_k \sim x_{k+1}$ for all $k = 1, \ldots, v$, where $x_{v+1} = x_1$.

Lemma 4 *Fixing a subset $B \subseteq A$ with $|B| \geq 3$, let $G^B_\sim$ be a connected graph. Then, the following two statements hold:*

(i) *If $Ext(G^B_\sim) = \emptyset$, then f behaves likes a dictatorship on B.*

(ii) *Given $Ext(G^B_\sim) \neq \emptyset$, if f behaves like a dictatorship on $\{x, y\}$ for all $x \in Ext(G^B_\sim)$ and $(x, y) \in \mathcal{E}^B_\sim$, then f behaves likes a dictatorship on B.*

Proof: First, given $Ext(G^B_\sim) = \emptyset$, note that for each $x \in B$, x is included in either a cycle or a path that connects two distinct cycles. Therefore, by Observation 1 and Lemma 2, we infer that f behaves like a dictatorship on B.

Next, let $Ext(G^B_\sim) \neq \emptyset$ and assume that f behaves like a dictatorship on $\{x, y\}$ for all $x \in Ext(G^B_\sim)$ and $y \in B$ such that $(x, y) \in \mathcal{E}^B_\sim$. For notational convenience, let $Ext(G^B_\sim) = \{x_1, x_2, \ldots, x_t\}$, $t \geq 2$, and moreover, let $(x_k, y_k) \in \mathcal{E}^B_\sim$ for all $k = 1, \ldots, t$. Thus, f behaves like a dictatorship on $\{x_k, y_k\}$ for all $k = 1, \ldots, t$. We consider two cases: $G^B_\sim$ is not a tree and $G^B_\sim$ is a tree.

In the first case, $G^B_\sim$ must include a cycle $\mathcal{C}$. Then, we can identify a subset $\bar{B} \subseteq B$ such that $G^{\bar{B}}_\sim$ is a connected graph and $Ext(G^{\bar{B}}_\sim) = \emptyset$. Clearly, $\mathcal{C}$ is included in $G^{\bar{B}}_\sim$ and $Ext(G^B_\sim) \cap \bar{B} = \emptyset$. Then, by statement (i), we know that f behaves like a dictatorship on $\bar{B}$. For each $1 \leq k \leq t$, since $G^B_\sim$ is a connected graph there exist $z_k \in \bar{B}$ and a path $\pi_k = (x_1, \ldots, x_{v-1}, x_v)$ in $G^B_\sim$ that connects z_k and x_k. Clearly, $z_{v-1} = y_k$. Then, Lemma 2 implies that f behaves like a dictatorship $\bar{B} \cup \pi_k$. Moreover, note that $G^B_\sim$ in fact is a combination of $G^{\bar{B}}_\sim$ and paths $\pi_1, \ldots, \pi_t$. Then, by repeatedly applying Lemma 2, we can conclude that f behaves like a dictatorship on B.

Last, we assume that $G^B_\sim$ is a tree. Evidently, $G^B_\sim$ has at least two extreme vertices, i.e., $t \geq 2$. Note that for any two distinct $x_p, x_q \in Ext(G^B_\sim)$, there exists a unique path $\pi_{p,q} = (z_1, z_2, \ldots, z_{v-1}, z_v)$ in $G^B_\sim$ connecting x_p and x_q. Clearly, $z_2 = y_p$ and $z_{v-1} = y_q$ (it is possible that $y_p = y_q$). Then, Lemma 2 implies that f behaves like a dictatorship on π. Moreover, note that $G^B_\sim$ in fact is a combination of all paths $\{\pi_{p,q} : 1 \leq p < q \leq t\}$. Then, by repeatedly applying Lemma 2, we conclude that f behaves like a dictatorship on B. ■

This completes the first step of the proof.

Now, we turn to the second step. Henceforth, we fix a domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, where $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$ and assume that $\mathbb{D}$ is path-connected, satisfies extreme-vertex symmetry and is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$. Moreover, for Lemmas 5 - 8, we fix $N = \{1, 2\}$ and a strategy-proof rule $f : \mathbb{D}^2 \to A$.

By statement (i) of Lemma 4, if $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \emptyset$, then f behaves likes a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Henceforth, we assume $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) \neq \emptyset$. By statement (ii) of Lemma 4, to prove that f behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$, it suffices to show that for each $x \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$, f behaves like a dictatorship on x and its unique neighbor in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$. The lemma below first considers an extreme vertex of $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ other than a and b.

Lemma 5 *Given $x \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$, let $x \notin \{a, b\}$ and $(x, y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$. Then f behaves like a dictatorship on $\{x, y\}$.*

Proof: First, since $x \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$, $x \notin \{a, b\}$ and $(x, y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$, by the non-trivialness condition, we have some $P_i \in \mathbb{D}^x$ such that $r_2(P_i) \neq y$, and hence $|\mathcal{S}(\mathbb{D}^x)| > 1$. Next, note that by the definition of (a, b)-semi-hybridness and path-connectedness, $x \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ and $x \notin \{a, b\}$ imply $x \in Ext(G_{\sim}^A)$. Then, extreme-vertex symmetry implies that there exist $\hat{P}_i, \check{P}_i \in \mathbb{D}$ such that $r_1(\hat{P}_i) = r_2(\check{P}_i) = x$ and $r_1(\check{P}_i) = r_2(\hat{P}_i) \equiv z \neq y$. Moreover, since $\hat{P}_i$ is (a, b)-semi-hybrid on $\mathcal{T}^A$ and $r_1(\hat{P}_i) = x \in \langle a, b|\mathcal{T}^A\rangle$, $z = r_2(\hat{P}_i)$ implies $z \in \langle a, b|\mathcal{T}^A\rangle$. Then, there exists a path $\pi = (x_1, x_2, \ldots, x_v)$ in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ connecting x and z. Furthermore, since $r_1(\hat{P}_i) = r_2(\check{P}_i) = x_1$ and $r_1(\check{P}_i) = r_2(\hat{P}_i) = x_v$, Lemma 3 implies that f behaves like a dictatorship on π. Last, since $(x, y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$, it is clear that $x_2 = y$. Therefore, f behaves like a dictatorship on $\{x, y\}$. ■

In the rest of the proof, we turn to consider the possibility that $a \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ or $b \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$. We first provide two lemmas as a preparation (see Lemmas 6 and 7). Then, Lemma 8 concludes that if $a \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ (respectively, $b \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$), then f behaves like a dictatorship on a (respectively, b) and its unique neighbor in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$.

LEMMA **6** *Given* $x \in \langle a, b|\mathcal{T}^A\rangle\backslash\{a\}$, *let* $\pi = (x_1, \ldots, x_v)$, $v \geq 3$, *be a path in* $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ *connecting* a *and* x. *Given* $\bar{a} \in A^{a \rightharpoonup b}$, *let* $\hat{P}_i \in \mathbb{D}^{\bar{a}}$ *be such that* $\max^{\hat{P}_i}(\langle a, b|\mathcal{T}^A\rangle\backslash\{a\}) = a' \in \{x_3, \ldots, x_v\}$. *Then, one of the following two statements must hold:*

(i) f *behaves like a dictatorship on* π, *and*

(ii) $f\big((x_{v-1}\cdots), (x_v\cdots)\big) = f\big((x_v\cdots), (x_{v-1}\cdots)\big) = x_{v-1}$.

PROOF: For notational convenience, let $a' = x_{k^*}$ for some $3 \leq k^* \leq v$. Clearly, we have a path $\bar{\pi} = (\bar{x}_1, \ldots, \bar{x}_u)$, $u \geq 1$, in $G_{\sim}^{A^{a \rightharpoonup b}}$ that connects $\bar{a}$ and a. If $\bar{a} = a$, then $\bar{\pi}$ is a null path. Since $x_{v-1} \sim x_v$, one of the following three cases must occur:

(1) f behaves like a dictatorship on $\{x_{v-1}, x_v\}$,

(2) $f\big((x_{v-1}\cdots), (x_v\cdots)\big) = f\big((x_v\cdots), (x_{v-1}\cdots)\big) = x_v$, and

(3) $f\big((x_{v-1}\cdots), (x_v\cdots)\big) = f\big((x_v\cdots), (x_{v-1}\cdots)\big) = x_{v-1}$.

We first rule out case (2). Suppose, by contradiction, that case (2) occurs. Then, according to the concatenated path $(\bar{\pi}, \pi)$ which connects $\bar{a}$ and x in $G_{\sim}^{A}$, statements (iii) and (iv) of Lemma 1 imply

- $f\big((x_k\cdots), (x_{k'}\cdots)\big) = f\big((x_{k'}\cdots), (x_k\cdots)\big) = x_{\max(k,k')}$ for all $1 \leq k, k' \leq v$,
- $f\big((\bar{x}_k\cdots), (\bar{x}_{k'}\cdots)\big) = f\big((\bar{x}_{k'}\cdots), (\bar{x}_k\cdots)\big) = \bar{x}_{\max(k,k')}$ for all $1 \leq k, k' \leq u$, and
- $f\big((x_k\cdots), (\bar{x}_{k'}\cdots)\big) = f\big((\bar{x}_{k'}\cdots), (x_k\cdots)\big) = x_k$ for all $1 \leq k \leq v$ and $1 \leq k' \leq u$.

Thus, given $\hat{P}_1 = \hat{P}_i$, $P_2 \in \mathbb{D}^{x_2}$ and $P_1' \in \mathbb{D}^{x_{k^*}}$, we have $f(\hat{P}_1, P_2) = x_2$ and $f(P_1', P_2) = x_{k^*}$. Then, voter 1 will manipulate at $(\hat{P}_1, P_2)$ via P_1', i.e., $f(P_1', P_2) = x_{k^*}\hat{P}_1 x_2 = f(\hat{P}_1, P_2)$. Therefore, either case (1) or (3) occurs. Case (3) is identical to statement (ii) of Lemma 6. Therefore, to complete the proof, we show that statement (i) holds in case (1).

Henceforth, let case (1) occur. By Lemma 2, to establish statement (i), it suffices to show that f behaves like a dictatorship on $\{x_1, x_2\}$. Suppose, by contradiction, that f does not behave like a dictatorship on $\{x_1, x_2\}$. Since

$x_1 \sim x_2$, it is true that either $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_1$, or $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_2$ holds. We will induce a contradiction in each situation.

First, let $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_1$. Then, according to the path π, statements (i) and (ii) of Lemma 1 imply $f\big((x_v\cdots),(x_{v-1}\cdots)\big) = f\big((x_{v-1}\cdots),(x_v\cdots)\big) = x_{v-1}$, which contradicts the hypothesis of case (1).

Next, let $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_2$. According to the path $(x_2, x_1 = \bar{x}_u, \ldots, \bar{x}_1)$, statements (i) and (ii) of Lemma 1 imply $f\big((\bar{x}_k\cdots),(\bar{x}_{k'}\cdots)\big) = f\big((\bar{x}_{k'}\cdots),(\bar{x}_k\cdots)\big) = \bar{x}_{\max(k,k')}$ for all $1 \le k, k' \le u$. Furthermore, according to case (1), we consider two subcases: (1') voter 1 dictates on $\{x_{v-1}, x_v\}$ and (2') voter 2 dictates on $\{x_{v-1}, x_v\}$. The verification of these two subcases are symmetric, and we hence focus on the verification of the subcase (1').

In the subcase (1'), we have $f\big((x_{v-1}\cdots),(x_v\cdots)\big) = x_{v-1}$ and $f\big((x_v\cdots),(x_{v-1}\cdots)\big) = x_v$. First, according to the path $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_u = x_1, \ldots, x_{v-1}, x_v)$, since $f\big((\bar{x}_1\cdots),(\bar{x}_2\cdots)\big) = \bar{x}_2$ and $f\big((x_{v-1}\cdots),(x_v\cdots)\big) = x_{v-1}$, to be consistent to statements (i) and (iv) of Lemma 1, one of the two statements below must hold:[54]

- There exists $1 \le \eta < v$ such that
  - $f\big((\bar{x}_{k'}\cdots),(\bar{x}_k\cdots)\big) = \bar{x}_k$ for all $1 \le k' \le k \le u$,
  - $f\big((\bar{x}_{k'}\cdots),(x_k\cdots)\big) = x_k$ for all $1 \le k' \le u$ and $1 \le k \le \eta$,
  - $f\big((x_{k'}\cdots),(x_k\cdots)\big) = x_k$ for all $1 \le k' \le k \le \eta$, and
  - $f\big((x_{k'}\cdots),(x_k\cdots)\big) = x_{k'}$ for all $\eta \le k' \le k \le v$.
- There exists $1 < \omega < u$ such that
  - $f\big((\bar{x}_{k'}\cdots),(\bar{x}_k\cdots)\big) = \bar{x}_k$ for all $1 \le k' \le k \le \omega$,
  - $f\big((\bar{x}_{k'}\cdots),(\bar{x}_k\cdots)\big) = \bar{x}_{k'}$ for all $\omega \le k' \le k \le u$,
  - $f\big((\bar{x}_{k'}\cdots),(x_k\cdots)\big) = \bar{x}_{k'}$ for all $\omega \le k' \le u$ and $1 \le k \le v$, and
  - $f\big((x_{k'}\cdots),(x_k\cdots)\big) = x_{k'}$ for all $1 \le k' \le k \le v$.

[54]The detailed verification is similar to the argument in footnote 52.

Note that the second statement above contradicts the induced fact $f\big((\bar{x}_{u-1}\cdots),(\bar{x}_u\cdots)\big) = \bar{x}_u$. Therefore, the first statement must hold. Furthermore, recall $f\big((x_v\cdots),(x_{v-1}\cdots)\big) = x_v$ by voter 1's dictatorship on $\{x_{v-1}, x_v\}$. Then, according to the path $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_u = x_1, \ldots, x_{v-1}, x_v)$, statement (iii) of Lemma 1 implies

- $f\big((x_k\cdots),(x_{k'}\cdots)\big) = x_k$ for all $1 \leq k' \leq k \leq v$,
- $f\big((\bar{x}_k\cdots),(\bar{x}_{k'}\cdots)\big) = \bar{x}_k$ for all $1 \leq k' \leq k \leq u$, and
- $f\big((x_k\cdots),(\bar{x}_{k'}\cdots)\big) = x_k$ for all $1 \leq k' \leq u$ and $1 \leq k \leq v$.

Overall, on the path $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_u = x_1, \ldots, x_\eta, \ldots, x_{v-1}, x_v)$, we summarize that

- $f\big((\bar{x}_k\cdots),(\bar{x}_{k'}\cdots)\big) = f\big((\bar{x}_{k'}\cdots),(\bar{x}_k\cdots)\big) = \bar{x}_{\max(k,k')}$ for all $1 \leq k, k' \leq u$,
- $f\big((x_k\cdots),(\bar{x}_{k'}\cdots)\big) = f\big((\bar{x}_{k'}\cdots),(x_k\cdots)\big) = x_k$ for all $1 \leq k \leq \eta$ and $1 \leq k' < u$,
- $f\big((x_k\cdots),(x_{k'}\cdots)\big) = f\big((x_{k'}\cdots),(x_k\cdots)\big) = x_{\max(k,k')}$ for all $1 \leq k, k' \leq \eta$, and
- $f\big((x_{k'}\cdots),(x_k\cdots)\big) = x_{k'}$ and $f\big((x_k\cdots),(x_{k'}\cdots)\big) = x_k$ for all $\eta \leq k' \leq k \leq v$, which imply that voter 1 dictates on $(x_\eta, \ldots, x_v)$.[55]

Recall the hypothesis that $\hat{P}_i \in \mathbb{D}^{\bar{a}} = \mathbb{D}^{\bar{x}_1}$ and $\max^{\hat{P}_i}(\langle a, b|\mathcal{T}^A\rangle\backslash\{a\}) = a' = x_{k^*}$ for some $3 \leq k^* \leq v$. We know that either $3 \leq k^* \leq \eta$ or $\eta < k^* \leq v$ holds. If $3 \leq k^* \leq \eta$, according to $f\big((\bar{x}_1\cdots),(x_2\cdots)\big) = x_2$ and $f\big((x_{k^*}\cdots),(x_2\cdots)\big) = x_{k^*}$, strategy-proofness implies $x_2 P_i x_{k^*}$ for all $P_i \in \mathbb{D}^{\bar{x}_1}$, which contradicts the hypothesis that $\hat{P}_i \in \mathbb{D}^{\bar{x}_1}$ and $\max^{\hat{P}_i}(\langle a, b|\mathcal{T}^A\rangle\backslash\{a\}) = x_{k^*}$. If $\eta < k^* \leq v$, according to $f\big((\bar{x}_1\cdots),(x_\eta\cdots)\big) = x_\eta$ and $f\big((x_{k^*}\cdots),(x_\eta\cdots)\big) = x_{k^*}$,[56] strategy-proofness implies $x_\eta P_i x_{k^*}$ for all $P_i \in \mathbb{D}^{\bar{x}_1}$, which contradicts the hypothesis that $\hat{P}_i \in \mathbb{D}^{\bar{x}_1}$ and $\max^{\hat{P}_i}(\langle a, b|\mathcal{T}^A\rangle\backslash\{a\}) = x_{k^*}$. Therefore, subcase (1') never occurs. In conclusion, the situation $f\big((x_1\cdots),(x_2\cdots)\big) = f\big((x_2\cdots),(x_1\cdots)\big) = x_2$ never occurs. Therefore, f must behave like a dictatorship on $\{x_1, x_2\}$, as required. This completes the verification of the lemma. ■

[55]If subcase (2') occurs, this conditions becomes that voter 2 dictates on $(x_\eta, \ldots, x_v)$, while the first three conditions remain unchanged.

[56]If subcase (2') occurs, we refer to $f\big((x_\eta\cdots),(\bar{x}_1\cdots)\big) = x_\eta$ and $f\big((x_\eta\cdots),(x_{k^*}\cdots)\big) = x_{k^*}$.

LEMMA 7 *Given* $\bar{a} \in A^{a \rightharpoonup b}$, $\bar{b} \in A^{b \rightharpoonup a}$, *let* $\pi = (x_1, \ldots, x_p, \ldots, x_q, \ldots, x_v)$ *be a path in* $G_{\sim}^A$ *connecting* $\bar{a}$ *and* $\bar{b}$, $1 \leq p < q \leq v$, $q - p > 1$, $x_p = a$ *and* $x_q = b$. *Let* $\hat{P}_i \in \mathbb{D}^{\bar{a}}$ *and* $\check{P}_i \in \mathbb{D}^{\bar{b}}$ *be such that* $\max^{\hat{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle\backslash\{a\}\big) = a' \in \{x_{p+2}, \ldots, x_q\}$ *and* $\max^{\check{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle\backslash\{b\}\big) = b' \in \{x_p, \ldots, x_{q-2}\}$. *Then,* f *behaves like a dictatorship on the subpath* $(x_p, \ldots, x_q)$.

PROOF: For notational convenience, let $a' = x_s$ for some $p + 2 \leq s \leq q$ and $b' = x_t$ for some $p \leq t \leq q - 2$.

Suppose that f does not behave like a dictatorship on $(x_p, \ldots, x_q)$. Then, according to the subpath $(x_1, \ldots, x_p, \ldots, x_{q-1}, x_q)$, statement (ii) of Lemma 6 must hold, i.e., $f\big((x_{q-1}\cdots), (x_q\cdots)\big) = f\big((x_q\cdots), (x_{q-1}\cdots)\big) = x_{q-1}$. Consequently, according to the subpath $(x_{q-1}, x_q, \ldots, x_{v-1}, x_v)$, by statements (i) and (ii) of Lemma 1, we have $f\big((x_{v-1}\cdots), (x_v\cdots)\big) = f\big((x_v\cdots), (x_{v-1}\cdots)\big) = x_{v-1}$. Symmetrically, since f does not behave like a dictatorship on $(x_p, \ldots, x_q)$, according to the subpath $(x_v, \ldots, x_q, \ldots, x_{p+1}, x_p)$, the analogy of statement (ii) of Lemma 6 must hold, i.e., $f\big((x_{p+1}\cdots), (x_p\cdots)\big) = f\big((x_p\cdots), (x_{p+1}\cdots)\big) = x_{p+1}$. Then, according to the subpath $(x_1, x_2, \ldots, x_p, x_{p+1})$, statements (iii) and (iv) of Lemma 1 imply $f\big((x_1\cdots), (x_2\cdots)\big) = f\big((x_2\cdots), (x_1\cdots)\big) = x_2$. Therefore, on the subpath $(x_p, x_{p+1}, \ldots, x_{q-1}, x_q)$, according to $f\big((x_p\cdots), (x_{p+1}\cdots)\big) = x_{p+1}$ and $f\big((x_{q-1}\cdots), (x_q\cdots)\big) = x_{q-1}$, to be consistent to statements (i) and (iv) of Lemma 1, there exists $p < \underline{k} < q$ such that $f\big((x_k\cdots), (x_{k'}\cdots)\big) = x_{k'}$ for all $p \leq k \leq k' \leq \underline{k}$ and $f\big((x_k\cdots), (x_{k'}\cdots)\big) = x_k$ for all $\underline{k} \leq k \leq k' \leq q$. Furthermore, according to the path $(x_1, \ldots, x_p, \ldots, x_{\underline{k}}, \ldots, x_q, \ldots, x_v)$, by statements (iv) and (i) of Lemma 1, we can infer that $f\big((x_k\cdots), (x_{k'}\cdots)\big) = x_{k'}$ for all $1 \leq k \leq k' \leq \underline{k}$ and $f\big((x_k\cdots), (x_{k'}\cdots)\big) = x_k$ for all $\underline{k} \leq k \leq k' \leq v$. Symmetrically, on the subpath $(x_p, x_{p+1}, \ldots, x_{q-1}, x_q)$, according to $f\big((x_{p+1}\cdots), (x_p\cdots)\big) = x_{p+1}$ and $f\big((x_q\cdots), (x_{q-1}\cdots)\big) = x_{q-1}$, to be consistent to statements (ii) and (iii) of Lemma 1, there exists $p < \overline{k} < q$ such that $f\big((x_{k'}\cdots), (x_k\cdots)\big) = x_{k'}$ for all $p \leq k \leq k' \leq \overline{k}$ and $f\big((x_{k'}\cdots), (x_k\cdots)\big) = x_k$ for all $\overline{k} \leq k \leq k' \leq q$. Furthermore, according to the path $(x_1, \ldots, x_p, \ldots, x_{\overline{k}}, \ldots, x_q, \ldots, x_v)$, by statements (iii) and (ii) of Lemma 1, we infer that $f\big((x_{k'}\cdots), (x_k\cdots)\big) = x_{k'}$ for all $1 \leq k \leq k' \leq \overline{k}$

and $f\big((x_{k'}\cdots),(x_k\cdots)\big)=x_k$ for all $\overline{k}\leq k\leq k'\leq v$. Last, according to $\underline{k}$ and $\overline{k}$, one of the following three cases must occur:

$$(1)\ \ \big[\underline{k}<\overline{k}\,\big]\Rightarrow\left[f\big((x_k\cdots),(x_{k'}\cdots)\big)=\begin{cases}x_k & \text{if } \underline{k}\leq k\leq\overline{k},\\ x_{\max(k,k')} & \text{if } 1\leq k,k'\leq\underline{k},\\ x_{\min(k,k')} & \text{if } \overline{k}\leq k,k'\leq v.\end{cases}\right],$$ [57]

$$(2)\ \ \big[\underline{k}>\overline{k}\,\big]\Rightarrow\left[f\big((x_k\cdots),(x_{k'}\cdots)\big)=\begin{cases}x_{k'} & \text{if } \overline{k}\leq k'\leq\underline{k},\\ x_{\max(k,k')} & \text{if } 1\leq k,k'\leq\overline{k},\\ x_{\min(k,k')} & \text{if } \underline{k}\leq k,k'\leq v.\end{cases}\right],\ \text{and}$$

$$(3)\ \ \big[\underline{k}=\overline{k}\equiv k^*\big]\Rightarrow\left[f\big((x_k\cdots),(x_{k'}\cdots)\big)=\begin{cases}x_{\max(k,k')} & \text{if } 1\leq k,k'\leq k^*,\\ x_{\min(k,k')} & \text{if } k^*\leq k,k'\leq v.\end{cases}\right].$$ [58]

In each case, we will induce a contradiction. We first identify the vertices $a'=x_s$ and $b'=x_t$ on the path π in all three cases.

Claim 1: We have $\max(\underline{k},\overline{k})\leq s\leq q$ and $p\leq t\leq\min(\underline{k},\overline{k})$.

In case (1), we show $\overline{k}<s\leq q$ and $p\leq t<\underline{k}$. By a symmetric argument, we can show $\underline{k}<s\leq q$ and $p\leq t<\overline{k}$ in case (2), and $k^*\leq s\leq q$ and $p\leq t\leq k^*$ in case (3). We focus on showing $\overline{k}<s\leq q$ and $p\leq t<\underline{k}$ in case (1).

We first show that for every $P_i\in\mathbb{D}^{x_1}$, P_i is single-peaked on the subpath $(x_1,\ldots,x_{\underline{k}})$ and $\max^{P_i}\big(\{x_{\underline{k}},\ldots,x_{\overline{k}}\}\big)=x_{\underline{k}}$. Given an arbitrary $1\leq k<\underline{k}$, since $f\big((x_1\cdots),(x_k\cdots)\big)=x_k$ and $f\big((x_{k+1}\cdots),(x_k\cdots)\big)=x_{k+1}$, strategy-proofness implies $x_kP_ix_{k+1}$ for all $P_i\in\mathbb{D}^{x_1}$. Therefore, every $P_i\in\mathbb{D}^{x_1}$ is single-peaked on

[57] Given $\underline{k}\leq k\leq\overline{k}$, if $k'=k$, it is evident that $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_k$; if $k>k'$, it is true that $1\leq k'<k\leq\overline{k}$, and then according to the path $(x_1,\ldots,x_{k'},\ldots,x_k,\ldots,x_{\overline{k}},\ldots,x_v)$, we have $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_k$; and if $k<k'$, it is true that $\underline{k}\leq k<k'\leq v$, and then according to the path $(x_1,\ldots,x_{\underline{k}},\ldots,x_k,\ldots,x_{k'},\ldots,x_v)$, we have $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_k$. Given $1\leq k,k'\leq\underline{k}$, if $k'=k$, it is evident that $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_k$; if $k>k'$, it is true that $1\leq k'<k\leq\underline{k}<\overline{k}$, and then according to the path $(x_1,\ldots,x_{k'},\ldots,x_k,\ldots,x_{\overline{k}},\ldots,x_v)$, we have $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_k=x_{\max(k,k')}$; and if $k<k'$, it is true that $1\leq k<k'\leq\underline{k}$, and then according to the path $(x_1,\ldots,x_k,\ldots,x_{k'},\ldots,x_{\underline{k}},\ldots,x_v)$, we have $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_{k'}=x_{\max(k,k')}$. Symmetrically, given $\overline{k}\leq k,k'\leq v$, we can infer $f\big((x_k\cdots),(x_{k'}\cdots)\big)=x_{\min(k,k')}$.

[58] Note that all three cases are just partial characterizations of f on the path π.

$(x_1, \dots, x_{\underline{k}})$. Next, given an arbitrary $\underline{k} < k \leq \overline{k}$, since $f\big((x_1 \cdots), (x_{\underline{k}} \cdots)\big) = x_{\underline{k}}$ and $f\big((x_k \cdots), (x_{\underline{k}} \cdots)\big) = x_k$, strategy-proofness implies $x_{\underline{k}} P_i x_k$ for all $P_i \in \mathbb{D}^{x_1}$. Hence, we have $\max^{P_i}\big(\{x_{\underline{k}}, \dots, x_{\overline{k}}\}\big) = x_{\underline{k}}$ for all $P_i \in \mathbb{D}^{x_1}$. Consequently, the hypothesis $r_1(\hat{P}_i) = x_1$, $x_s = \max^{\hat{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big)$ and $p+2 \leq s \leq q$ imply $\overline{k} < s \leq q$. Symmetrically, we can show that for every $P_i \in \mathbb{D}^{x_v}$, P_i is single-peaked on the subpath $(x_v, \dots, x_{\overline{k}})$ and $\max^{P_i}\big(\{x_{\underline{k}}, \dots, x_{\overline{k}}\}\big) = x_{\overline{k}}$. Then, the hypothesis $r_1(\check{P}_i) = x_v$, $x_t = \max^{\check{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big)$ and $p \leq t \leq q-2$ imply $p \leq t < \underline{k}$. This completes the verification of the claim.

Now, we have the path π specified in each case below:

Case (1) : $\pi = (x_1, \dots, x_p, \dots, x_t, \dots, x_{\underline{k}}, \dots, x_{\overline{k}}, \dots, x_s, \dots, x_q, \dots, x_v)$,

Case (2) : $\pi = (x_1, \dots, x_p, \dots, x_t, \dots, x_{\overline{k}}, \dots, x_{\underline{k}}, \dots, x_s, \dots, x_q, \dots, x_v)$, and

Case (3) : $\pi = (x_1, \dots, x_p, \dots, x_t, \dots\dots, x_{k^*}, \dots\dots, x_s, \dots, x_q, \dots, x_v)$.

Given $\hat{P}_1 = \hat{P}_i$ and $\check{P}_2 = \check{P}_i$, by Claim 1, each case implies $f\big((x_s \cdots), \check{P}_2\big) = x_s$ and $f\big(\hat{P}_1, (x_t \cdots)\big) = x_t$. First, since $f\big((x_s \cdots), \check{P}_2\big) = x_s$, strategy-proofness implies $f(\hat{P}_1, \check{P}_2) \in \{x_s\} \cup \{x \in A : x\hat{P}_1 x_s\}$. Next, since $f\big(\hat{P}_1, (x_t \cdots)\big) = x_t$, strategy-proofness implies $f(\hat{P}_1, \check{P}_2) \in \{x_t\} \cup \{x \in A : x\check{P}_2 x_t\}$. Therefore, it must be the case that $\big[\{x_s\} \cup \{x \in A : x\hat{P}_1 x_s\}\big] \cap \big[\{x_t\} \cup \{x \in A : x\check{P}_2 x_t\}\big] \neq \emptyset$. Given $x_s = a' = \max^{\hat{P}_1}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big)$ and $x_t = b' = \max^{\check{P}_2}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{b\}\big)$, by the definition of (a, b)-semi-hybridness on $\mathcal{T}^A$, we know $\{x_s\} \cup \{x \in A : x\hat{P}_1 x_s\} = \{x_s\} \cup \{x \in A : x\hat{P}_1 a \text{ or } x = a\} \subseteq \{x_s\} \cup A^{a \rightharpoonup b}$ and $\{x_t\} \cup \{x \in A : x\check{P}_2 x_t\} = \{x_t\} \cup \{x \in A : x\check{P}_2 b \text{ or } x = b\} \subseteq \{x_t\} \cup A^{b \rightharpoonup a}$, which however imply $\big[\{x_s\} \cup \{x \in A : x\hat{P}_1 x_s\}\big] \cap \big[\{x_t\} \cup \{x \in A : x\check{P}_2 x_t\}\big] \subseteq \big[\{x_s\} \cup A^{a \rightharpoonup b}\big] \cap \big[\{x_t\} \cup A^{b \rightharpoonup a}\big] = \emptyset$. Contradiction! Therefore, f must behave like a dictatorship on $(x_p, \dots, x_q)$. ■

According to Lemma 7, we make the following observation.

Observation 2 If $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a tree and $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \{a, b\}$, then f behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$.[59] □

[59]Since $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a tree and $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \{a, b\}$, $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ indeed is a line and has exactly two extreme vertices a and b. Then, the hypothesis of Lemma 7 is implied by path-connectedness and the non-trivialness condition on $\langle a, b|\mathcal{T}^A\rangle$. Hence, Lemma 7 implies that f behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$.

Lemma 8 *The following two statements hold:*

(i) *Given* $a \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ *and* $(a,y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$, f *behaves like a dictatorship on* $\{a,y\}$.

(ii) *Given* $b \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ *and* $(b,y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$, f *behaves like a dictatorship on* $\{b,y\}$.

Proof: The two statements are symmetric, and we focus on showing the first one. According to Observation 2, we only need to consider the following two cases: (i) $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is not a tree, and (ii) $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a tree and $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) \neq \{a,b\}$. In each case, we show that f behaves like a dictatorship on $\{a,y\}$.

First, we assume that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is not a tree. Since $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a connected graph, $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ must contain a cycle. Then, we can identify a subset $\hat{A} \subseteq \langle a,b|\mathcal{T}^A\rangle$ with $|\hat{A}| \geq 3$ such that $G_{\sim}^{\hat{A}}$ is a connected graph and has no extreme vertex. Furthermore, we can assume that $G_{\sim}^{\hat{A}}$ is the maximum connected subgraph of $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ that has no extreme vertex, i.e., for any non-empty subset $B \subseteq \langle a,b|\mathcal{T}^A\rangle$ with $\hat{A} \subset B$, either $G_{\sim}^{B}$ is not a connected graph, or $G_{\sim}^{B}$ is a connected graph and has an extreme vertex. Next, we identify a subset of $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$: $\hat{B} = \big\{x \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) : x \notin \{a,b\}\big\}$. Note that either $\hat{B} = \emptyset$ or $\hat{B} \neq \emptyset$ holds. In particular, if $\hat{B} \neq \emptyset$, note that for each $x \in \hat{B}$, there must exist a unique $z \in \hat{A}$ and a unique path π in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ connecting z and x such that $\pi \cap \hat{A} = \{z\}$. Clearly, $\hat{A} \cap \hat{B} = \emptyset$. Furthermore, we define the subset below:

$$\hat{C} = \left\{ c \in \langle a,b|\mathcal{T}^A\rangle : \begin{array}{l} \text{there exist } z \in \hat{A}, x \in \hat{B} \text{ and a path } \pi \text{ in } G_{\sim}^{\langle a,b\rangle} \text{ connecting} \\ z \text{ and } x \text{ such that } \pi \cap \hat{A} = \{z\} \text{ and } c \in \pi \backslash \{z,x\} \end{array} \right\}.$$

Clearly, if $\hat{B} = \emptyset$, then $\hat{C} = \emptyset$. Last, let $\bar{A} = \hat{A} \cup \hat{B} \cup \hat{C}$. It is evident that $G_{\sim}^{\bar{A}}$ is a connected subgraph of $G_{\sim}^{\langle a,b\rangle}$.

Claim 1: Rule f behaves like a dictatorship on $\bar{A}$.

By statement (i) of Lemma 4, f behaves like a dictatorship on $\hat{A}$. Given $x \in \hat{B}$, there exists a unique $x' \in \hat{C}$ such that $(x,x') \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$. Then, Lemma 5 implies f behaves like a dictatorship on $\{x,x'\}$. Moreover, there exists a unique $z \in \hat{A}$

and a unique path $\pi = (x_1, x_2, \ldots, x_{v-1}, x_v)$ in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ connecting z and x such that $x_2, \ldots, x_{v-1} \in \hat{C}$. Clearly, $x' = x_{v-1}$. Then, Lemma 2 implies that f behaves like a dictatorship on $\hat{A} \cup \pi$. Last, note that $G_{\sim}^{\bar{A}}$ is a combination of $G_{\sim}^{\hat{A}}$ and paths between $\hat{A}$ and all extreme vertices of $\hat{B}$ which cover vertices of $\hat{C}$. Therefore, by Lemma 2, f must behave like a dictatorship on $\bar{A}$. This completes the verification of the claim.

Since $a \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$, it is clear that $a \notin \bar{A}$. Then, there must exist a unique vertex $z^* \in \bar{A}$ and a unique path $\pi^* = (z_1, \ldots, z_t)$ in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ connecting z^* and a such that $z_2, \ldots, z_t \notin \bar{A}$. Clearly, $z_{t-1} = y$. By the non-trivialness condition, there exists $\hat{P}_i \in \mathbb{D}$ such that $r_1(\hat{P}_i) \equiv \bar{a} \in A^{a \rightharpoonup b}$ and $\max^{\hat{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big) \equiv a' \neq y$. We then identify a path $\bar{\pi}$ in $G_{\sim}^{A^{a \rightharpoonup b}}$ that connects $\bar{a}$ and a. There are two situations: $a' \in \pi^* \cup \bar{A}$ and $a' \notin \pi^* \cup \bar{A}$. In the first situation, we can construct a path $\pi = (x_1, \ldots, x_v)$ in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ connecting a and some $x \in \bar{A}$ such that $y = x_2$, $a' \in \{x_3, \ldots, x_v\}$ and $x_{v-1}, x_v \in \bar{A}$. Since f behaves like a dictatorship on $\{x_{v-1}, x_v\}$ by Claim 1, statement (i) of Lemma 6 must hold, i.e., f behaves like a dictatorship on π. Therefore, f behaves like a dictatorship on $\{x_1, x_2\} = \{a, y\}$. Henceforth, we assume $a' \notin \pi^* \cup \bar{A}$.

Claim 2: We have $b \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$.

By the definition of $\hat{B}$, we know either $b \notin Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ and $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \hat{B} \cup \{a\}$, or $b \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ and $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \hat{B} \cup \{a, b\}$ hold. Accordingly, we know that either $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is combination of $G_{\sim}^{\bar{A}}$ and π^*, or $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is combination of $G_{\sim}^{\bar{A}}$, π^* and one path connecting b to some vertex of $\bar{A}$. Suppose $b \notin Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$. Then, $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a combination of $G_{\sim}^{\bar{A}}$ and π^*, i.e., $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle} = \pi^* \cup G_{\sim}^{\bar{A}}$, and hence we have $a' \in \langle a, b|\mathcal{T}^A\rangle = \pi^* \cup \bar{A}$, which contradicts the hypothesis $a' \notin \pi^* \cup \bar{A}$. This completes the verification of the claim.

Now, let $(b, y') \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$. Thus, there must exist a unique $z' \in \bar{A}$ and a unique path $\pi' = (z'_1, \ldots, z'_s)$ in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ connecting z' and b such that $z'_2, \ldots, z'_s \notin \bar{A}$. Clearly, $y' = z'_{s-1}$ Moreover, by the non-trivialness condition, there exists $\check{P}_i \in \mathbb{D}$ such that $r_1(\check{P}_i) \equiv \bar{b} \in A^{b \rightharpoonup a}$ and $\max^{\check{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{b\}\big) \equiv b' \neq y'$. Let $\check{\pi}$ denote a path in $G_{\sim}^{A^{b \rightharpoonup a}}$ that connects b and $\bar{b}$. Furthermore, we have to consider two subcases: $\pi^* \cap \pi' = \emptyset$ and $\pi^* \cap \pi' \neq \emptyset$.

Claim 3: If $\pi^* \cap \pi' = \emptyset$, then f behaves like a dictatorship on $\{a, y\}$.

Since $z^*, z' \in \bar{A}$ and $G_{\sim}^{\bar{A}}$ is a connected graph, there exists a path $\pi'' = (z_1'', \dots, z_r'')$ in $G_{\sim}^{\bar{A}}$ connecting z^* and z'. We then have a concatenated path $\tilde{\pi} = (z_t, \dots, z_1 = z_1'', \dots, z_r'' = z_1', \dots, z_s')$ in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ that connects a and b. Moreover, since $\pi^* \cap \pi' = \emptyset$, it must be the case that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a combination of π^*, $G_{\sim}^{\bar{A}}$ and π', i.e., $G_{\sim}^{\langle a,b\rangle} = \pi^* \cup G_{\sim}^{\bar{A}} \cup \pi'$ (see Figure 10).

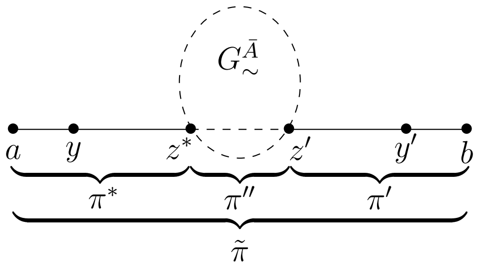


Figure 10: Adjacency graph $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$

Then, $a' \notin \pi^* \cup \bar{A}$ implies $a' \in \{z_2', \dots, z_s'\}$. We know that either $b' \in \pi' \cup \bar{A}$ or $b' \notin \pi' \cup \bar{A}$ holds. If $b' \in \pi' \cup \bar{A}$, similar to a', we can easily infer that f behaves like a dictatorship on $\{b, y'\}$, which indicates the failure of statement (ii) of Lemma 6 on the path $\tilde{\pi}$. Consequently, according to the path $\tilde{\pi}$ in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ that connects a and b, and the preference $\hat{P}_i \in \mathbb{D}$ with $r_1(\hat{P}_i) = \bar{a} \in A^{a \rightharpoonup b}$ and $\max^{\hat{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big) = a' \in \{z_2', \dots, z_s'\}$, statement (i) of Lemma 6 must hold, i.e., f behaves like a dictatorship on $\tilde{\pi}$. Therefore, f behaves like a dictatorship on $\{z_t, z_{t-1}\} = \{a, y\}$. If $b' \notin \pi' \cup \bar{A}$, it is evident that $b' \in \{z_2, \dots, z_t\}$. Then, according to the concatenated path $(\bar{\pi}, \tilde{\pi}, \breve{\pi})$ in $G_{\sim}^{A}$ that connects $\bar{a}$ and $\bar{b}$, and the preferences $\hat{P}_i$ and $\check{P}_i$, we know that the hypothesis of Lemma 7 is satisfied, and then Lemma 7 implies that f behaves like a dictatorship on $\tilde{\pi}$. Therefore, f behaves like a dictatorship on $\{z_t, z_{t-1}\} = \{a, y\}$. This completes the verification of the claim.

Claim 4: If $\pi^* \cap \pi' \neq \emptyset$, then f behaves like a dictatorship on $\{a, y\}$.

We first show that there exists a unique $1 \leq k^* < \min(t, s)$ such that $z_k = z_k'$ for all $k = 1, \dots, k^*$ and $(z_t, \dots, z_{k^*} = z_{k^*}', \dots, z_s')$ is the unique path in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$

that connects a and b. Let $z_{\underline{k}} = z'_{\overline{k}}$ for some $1 \le \underline{k} < t$ and $1 \le \overline{k} < s$. Since $z^*, z' \in \bar{A}$ and $G^{\bar{A}}_{\sim}$ is a connected graph, there exists a path $\pi'' = (z''_1, \dots, z''_r)$ in $G^{\bar{A}}_{\sim}$ connecting z^* and z'.[60] Thus, we have three paths $(z_1, \dots, z_{\underline{k}})$, $\pi'' = (z''_1, \dots, z''_r)$ and $(z'_1, \dots, z'_{\overline{k}})$ in $G^{\langle a,b \rangle}_{\sim}$. If $(z_1, \dots, z_{\underline{k}}) \neq (z'_1, \dots, z'_{\overline{k}})$, we can identify a cycle $\mathcal{C}$ from these three paths such that either $|\mathcal{C} \cap \{z_1, \dots, z_{\underline{k}}\}| \ge 2$ or $|\mathcal{C} \cap \{z'_1, \dots, z'_{\overline{k}}\}| \ge 2$ holds. Clearly, $\mathcal{C} \subseteq \hat{A} \subseteq \bar{A}$. This contradicts the fact that $z_2, \dots, z_t \notin \bar{A}$ and $z'_2, \dots, z'_s \notin \bar{A}$. Therefore, it must be the case $(z_1, \dots, z_{\underline{k}}) = (z'_1, \dots, z'_{\overline{k}})$, i.e., $\underline{k} = \overline{k}$ and $z_k = z'_k$ for all $k = 1, \dots, \underline{k}$. Last, since $z_t \neq z_s$, we can identify $1 \le k^* < \min(t, s)$ such that $z_k = z'_k$ for all $k = 1, \dots, k^*$ and $\{z_{k^*+1}, \dots, z_t\} \cap \{z'_{k^*+1}, \dots, z'_s\} = \emptyset$. Then, we have the path $\hat{\pi} = (z_t, \dots, z_{k^*+1}, z_{k^*} = z'_{k^*}, z'_{k^*+1}, \dots, z'_s)$ in $G^{\langle a,b|\mathcal{T}^A \rangle}_{\sim}$ connecting a and b. Since $z_2, \dots, z_t \notin \bar{A}$ and $z'_2, \dots, z'_s \notin \bar{A}$, it is true that either $k^* > 1$ and $\hat{\pi} \cap \bar{A} = \emptyset$ (see Figure 11(a)), or $k^* = 1$ and $\hat{\pi} \cap \bar{A} = \{z_1\}$ (see Figure 11(b)). Therefore, $|\hat{\pi} \cap \bar{A}| \le 1$.

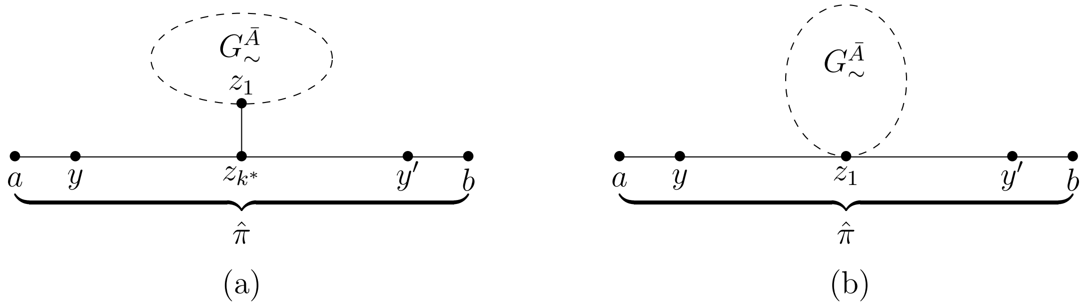


Figure 11: Adjacency graph $G^{\langle a,b|\mathcal{T}^A \rangle}_{\sim}$

Last, if $\hat{\pi}$ is not the unique path in $G^{\langle a,b|\mathcal{T}^A \rangle}_{\sim}$ that connects a and b, we can identify at least two distinct vertices of $\hat{\pi}$ that are included in a cycle, and hence are contained in $\hat{A}$. This contradicts the fact $|\hat{\pi} \cap \bar{A}| \le 1$. Therefore, $\hat{\pi}$ is the unique path in $G^{\langle a,b|\mathcal{T}^A \rangle}_{\sim}$ connecting a and b. Thus, we can conclude that when $k^* > 1$, $G^{\langle a,b|\mathcal{T}^A \rangle}_{\sim}$ is a combination of $G^{\bar{A}}_{\sim}$, the path $\hat{\pi}$ and the path $(z_1, \dots, z_{k^*})$ (see Figure 11(a)), and when $k^* = 1$, $G^{\langle a,b|\mathcal{T}^A \rangle}_{\sim}$ is a combination of $G^{\bar{A}}_{\sim}$ and the path $\hat{\pi}$ (see Figure 11(b)).

Now, we start to show that f behaves like a dictatorship on $\{a, y\}$. First,

[60] If $z^* = z'$, then π'' is a null path.

recall that we are in the case $a' \notin \pi^* \cup \bar{A}$, which implies $a' \in \{z'_{k^*+1}, \dots, z'_s\}$. We know that either $b' \in \pi' \cup \bar{A}$ or $b' \notin \pi' \cup \bar{A}$ holds. If $b' \in \pi' \cup \bar{A}$, similar to a', we can easily infer that f behaves like a dictatorship on $\{b, y'\}$, which indicates the failure of statement (ii) of Lemma 6 on the path $\hat{\pi}$. Then, according to the path $\hat{\pi}$, and the preference $\hat{P}_i \in \mathbb{D}$ with $r_1(\hat{P}_i) = \bar{a} \in A^{a \rightharpoonup b}$ and $\max^{\hat{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}\big) = a' \in \{z'_{k^*+1}, \dots, z'_s\}$, statement (i) of Lemma 6 must hold, i.e., f behaves like a dictatorship on $\hat{\pi}$. Therefore, f behaves like a dictatorship on $\{z_t, z_{t-1}\} = \{a, y\}$. If $b' \notin \pi' \cup \bar{A}$, then $b' \in \{z_{k^*+1}, \dots, z_t\}$. Then, according to the concatenated path $(\bar{\pi}, \tilde{\pi}, \check{\pi})$ in $G^A_\sim$ that connects $\bar{a}$ and $\bar{b}$, and the preferences $\hat{P}_i$ and $\check{P}_i$, Lemma 7 implies that f behaves like a dictatorship on $\tilde{\pi}$. Therefore, f behaves like a dictatorship on $\{z_t, z_{t-1}\} = \{a, y\}$. This completes the verification of the claim.

Overall, if $G^{\langle a,b|\mathcal{T}^A\rangle}_\sim$ is not a tree, f behaves like a dictatorship on $\{a, y\}$.

Next, we assume that $G^{\langle a,b|\mathcal{T}^A\rangle}_\sim$ is a tree and $Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim) \neq \{a, b\}$. Clearly, $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)| \geq 2$ and $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)\backslash\{a, b\}| \geq 1$. Also, recall $a \in Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)$. Therefore, there are three subcases: (i) $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)| = 2$ and $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)\backslash\{a, b\}| = 1$, which imply $Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim) = \{a, x\}$ and $x \neq b$, (ii) $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)| = 3$ and $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)\backslash\{a, b\}| = 1$ which imply $Ext(G^{\langle a,b\rangle}_\sim) = \{a, b, x\}$, and (iii) $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)| \geq 3$ and $|Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)\backslash\{a, b\}| \geq 2$.

In the first subcase, $G^{\langle a,b|\mathcal{T}^A\rangle}_\sim$ is a line which has the extreme vertices a and x. Let $\pi = (x_1, \dots, x_v)$ be such a line. Thus, $a = x_1$, $y = x_2$, $x = x_v$ and $b = x_s$ for some $1 < s < v$. By the non-trivialness condition, we have $\bar{a} \in A^{a \rightharpoonup b}$ and $\hat{P}_i \in \mathbb{D}^{\bar{a}}$ such that $\max^{\hat{P}_i}\big(\langle a, b|\mathcal{T}^A\rangle\backslash\{a\}\big) = \max^{\hat{P}_i}\big(\{x_2, \dots, x_v\}\big) \equiv a' \in \{x_3, \dots, x_v\}$. Moreover, Lemma 5 implies that f behaves like a dictatorship on $\{x_{v-1}, x_v\}$, which indicates the failure of statement (ii) of Lemma 6 on the path π. Therefore, statement (i) of Lemma 6 must hold, i.e., f behaves like a dictatorship on π. Hence, f behaves like a dictatorship on $\{x_1, x_2\} = \{a, y\}$.

In the second subcase, we know that x is uniquely adjacent to some $x' \in \langle a, b|\mathcal{T}^A\rangle$. Let $\bar{A} = \{x, x'\}$. First, Lemma 5 implies that f behaves like a dictatorship on $\bar{A}$, which is analogous to Claim 1 above. Second, note that the hypothesis $b \in Ext(G^{\langle a,b|\mathcal{T}^A\rangle}_\sim)$ is analogous to Claim 2 above. Third, recall the vertices $z^*, z' \in \bar{A}$ and the paths π^*, π' in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ that we identify for the proof in the case that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is not a tree. Note that x' is analogous to z^*, x' is analogous to z', the unique path connecting x' and a in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is analogous to π^*, and the unique path connecting x' and b in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is analogous to π'. Last, we can adopt an argument analogous to the proof of Claim 4 to show that f behaves like a dictatorship on $\{a, y\}$.

In the third subcase, there exists a subset $\bar{A} \subseteq \langle a, b|\mathcal{T}^A\rangle$ such that $G_{\sim}^{\bar{A}}$ is the minimum subtree of $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ that includes all extreme vertices of $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})\backslash\{a, b\}$. By Lemmas 5 and 2, we know that f behaves like a dictatorship on $\bar{A}$, which is analogous to Claim 1 above. Then, by the same verifications of Claims 2, 3 and 4 above,[61] we can show that f behaves like a dictatorship on $\{a, y\}$. In conclusion, we have shown that if $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a tree and $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) \neq \{a, b\}$, then f behaves like a dictatorship on $\{a, y\}$. This completes the proof of the lemma. ■

This completes the second-step proof, and shows that every two-voter strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$, as required. This proves the sufficiency part of the Auxiliary Proposition.

B Proof of Fact 1

(**Necessity**) Fixing $n > 2$ and the PNT rule $f : \mathbb{D}^n \to A$ on $\mathcal{T}^A$ w.r.t. (x, y), let f be strategy-proof and violate the tops-only property. For notational convenience, we let $i = 1$ and $j = 2$ in Definition 6.

We fix an arbitrary preference $P_1 \in \mathbb{D}$ with $r_1(P_1) = z \in A^{x\rightharpoonup y}$, and show that P_1 satisfies condition (i) of Fact 1 in two steps. In the first step, we show that P_1 is semi-single-peaked on $\mathcal{T}^{A^{x\rightharpoonup y}}$ w.r.t. x. Given $a, b \in \langle z, x|\mathcal{T}^{A^{x\rightharpoonup y}}\rangle = \langle z, x|\mathcal{T}^A\rangle$, let $a \in \langle z, b|\mathcal{T}^{A^{x\rightharpoonup y}}\rangle$. By minimal richness, we have $P_1' \in \mathbb{D}^b$ and $P_\nu \in \mathbb{D}^a$ for all $\nu \neq 1$. By construction, we have $f(P_1, P_{-1}) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P_1,P_{-1})}\right) = \text{Proj}\left(x, \langle z, a|\mathcal{T}^A\rangle\right) =$

[61]Although Claims 2, 3 and 4 are established in the case that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ contains a cycle, the verifications of all these three claims do not rely on the presence of cycles, but depends on the dictatorship of f on $\bar{A}$. Therefore, these three claims continue to be applicable in the case that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a tree.

a and $f(P_1', P_{-1}) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P_1', P_{-1})}\right) = \text{Proj}(x, \langle b, a|\mathcal{T}^A\rangle) = b$, which by strategy-proofness imply aP_1b. Given $a \in A^{x\rightharpoonup y}\backslash\langle z, x|\mathcal{T}^{A^{x\rightharpoonup y}}\rangle$, let $a' = \text{Proj}\left(a, \langle z, x|\mathcal{T}^{A^{x\rightharpoonup y}}\rangle\right)$. Thus, $a' = \text{Proj}\left(a, \langle z, x|\mathcal{T}^{A^{x\rightharpoonup y}}\rangle\right) = \text{Proj}(a, \langle z, x|\mathcal{T}^A\rangle) = \text{Proj}(x, \langle z, a|\mathcal{T}^A\rangle)$. By minimal richness, we have $P_1' \in \mathbb{D}^a$ and $\hat{P}_\nu \in \mathbb{D}^a$ for all $\nu \neq 1$. By construction, we have $f(P_1, \hat{P}_{-1}) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P_1, \hat{P}_{-1})}\right) = \text{Proj}\left(x, \langle z, a|\mathcal{T}^A\rangle\right) = a'$ and $f(P_1', \hat{P}_{-1}) = a$, which by strategy-proofness imply $a'P_1a$. Therefore, P_1 is semi-single-peaked on $\mathcal{T}^{A^{x\rightharpoonup y}}$ w.r.t. x.

In the second step, we show that xP_1y and $\max^{P_1}(A^{y\rightharpoonup x}) = y$. We construct two profiles $P = (P_1, P_2, P_{-\{1,2\}})$ and $P' = (P_1', P_2, P_{-\{1,2\}})$, where $P_1' \in \mathbb{D}^y$, $r_1(P_2) \in \mathbb{D}^x$ and $r_1(P_\nu) \in A^{x\rightharpoonup y}$ for all $\nu = 3, \ldots, n$. Then, by construction, we have $f(P) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right) = x$ and $f(P') = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P')}\right) = y$, which by strategy-proofness imply xP_1y. Next, given an arbitrary $a \in A^{y\rightharpoonup x}\backslash\{y\}$, we construct two other profiles $P = (P_1, P_2, P_{-\{1,2\}})$ and $P' = (P_1', P_2, P_{-\{1,2\}})$, where $P_1' \in \mathbb{D}^a$, $P_2 \in \mathbb{D}^y$ and P_ν is arbitrary for all $\nu = 3, \ldots, n$. By construction, we have $f(P) = \max^{P_2}(\{x, y\}) = y$ and $f(P') = a$, which by strategy-proofness imply yP_1a. Therefore, $\max^{P_1}(A^{y\rightharpoonup x}) = y$. Now, by combining the results in the two steps, we infer that for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{x\rightharpoonup y}$, P_i is semi-single-peaked on $\mathcal{T}^A$ w.r.t. y. This proves condition (i) of Fact 1.

To verify condition (ii) of Fact 1, we fix $i \in N\backslash\{1, 2\}$ and an arbitrary preference $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{y\rightharpoonup x}$, and show $\max^{P_i}(A^{x\rightharpoonup y}) = x$. Given an arbitrary $z \in A^{x\rightharpoonup y}\backslash\{x\}$, we construct two profiles $P = (P_1, P_2, P_i, P_{-\{1,2,i\}})$ and $P' = (P_1, P_2, P_i', P_{-\{1,2,i\}})$, where $P_i' \in \mathbb{D}^z$ and $P_\nu \in \mathbb{D}^z$ for all $\nu \neq i$. By construction, we have $f(P) = \text{Proj}(x, \mathcal{T}^{\Gamma(P)}) = \text{Proj}\left(x, \langle r_1(P_i), z|\mathcal{T}^A\rangle\right) = x$ and $f(P') = z$, which by strategy-proofness imply xP_iz. Therefore, $\max^{P_i}(A^{x\rightharpoonup y}) = x$. This proves condition (ii) of Fact 1.

Last, since f violates the tops-only property, by construction, there must exist two profiles $P = (P_1, P_2, P_{-\{1,2\}})$ and $P' = (P_1, P_2', P_{-\{1,2\}})$, where $r_1(P_1) \in A^{x\rightharpoonup y}$ and $r_1(P_2) = r_1(P_2') \in A^{y\rightharpoonup x}$ such that $f(P) = \max^{P_2}(\{x, y\}) = x \neq y = \max^{P_2'}(\{x, y\}) = f(P')$. Therefore, xP_2y and $yP_2'x$. This proves condition (iii) of Fact 1, and hence completes the verification of the necessity part.

(**Sufficiency**) Let domain $\mathbb{D}$ satisfy conditions (i), (ii) and (iii) of Fact 1. Fix the PNT rule $f : \mathbb{D}^n \to A$, $n \geq 2$, on $\mathcal{T}^A$ w.r.t. (x, y). For notational convenience, let $i = 1$ and $j = 2$ in Definition 6. Clearly, condition (iii) of Fact 1 implies that f violates the tops-only property. Henceforth, we show strategy-proofness of f.

We first consider voters other than 1 and 2 in the case $n > 2$. Given an voter $i \in N\backslash\{1,2\}$ and two profiles $P = (P_1, P_2, P_i, P_{-\{1,2,i\}})$ and $P' = (P_1, P_2, P'_i, P_{-\{1,2,i\}})$, let $f(P) \neq f(P')$. By construction, it must be the case that $r_1(P_1) \in A^{x \rightharpoonup y}$, $r_1(P_2) \in A^{x \rightharpoonup y}$, $f(P) = \text{Proj}(x, \mathcal{T}^{\Gamma(P)})$ and $f(P') = \text{Proj}(x, \mathcal{T}^{\Gamma(P')})$. Furthermore, we can infer that $r_1(P_j) \in A^{x \rightharpoonup y}$ for all $j \in N\backslash\{1, 2, i\}$. Otherwise, x is included in both $\Gamma(P)$ and $\Gamma(P')$, and we induce the contradiction $f(P) = x = f(P')$. Symmetrically, we can infer that it is impossible that $r_1(P_i) \in A^{y \rightharpoonup x}$ and $r_1(P'_i) \in A^{y \rightharpoonup x}$. Therefore, either $r_1(P_i) \in A^{x \rightharpoonup y}$ and $r_1(P'_i) \in A^{y \rightharpoonup x}$, or $r_1(P_i) \in A^{y \rightharpoonup x}$ and $r_1(P'_i) \in A^{x \rightharpoonup y}$, or $r_1(P_i) \in A^{x \rightharpoonup y}$ and $r_1(P'_i) \in A^{x \rightharpoonup y}$ occur. If $r_1(P_i) \in A^{x \rightharpoonup y}$ and $r_1(P'_i) \in A^{y \rightharpoonup x}$, we by construction have $f(P) = \text{Proj}(x, \mathcal{T}^{\Gamma(P)}) \in \langle r_1(P_i), x | \mathcal{T}^A \rangle$ and $f(P') = \text{Proj}(x, \mathcal{T}^{\Gamma(P')}) = x$. Then, condition (i) of Fact 1 implies $f(P) P_i f(P')$. If $r_1(P_i) \in A^{y \rightharpoonup x}$ and $r_1(P'_i) \in A^{x \rightharpoonup y}$, we by construction have $f(P) = \text{Proj}(x, \mathcal{T}^{\Gamma(P)}) = x$ and $f(P') = \text{Proj}(x, \mathcal{T}^{\Gamma(P')}) \in A^{x \rightharpoonup y}$. Then, condition (ii) of Fact 1 implies $f(P) P_i f(P')$. Last, if $r_1(P_i) \in A^{x \rightharpoonup y}$ and $r_1(P'_i) \in A^{x \rightharpoonup y}$, we by construction have $f(P) = \text{Proj}(x, \mathcal{T}^{\Gamma(P)}) \in A^{x \rightharpoonup y}$ and $f(P') = \text{Proj}(x, \mathcal{T}^{\Gamma(P')}) \in A^{x \rightharpoonup y}$. Note that condition (i) of Fact 1 implies that P_i is also semi-single-peaked on $\mathcal{T}^{A^{x \rightharpoonup y}}$ w.r.t. x. Then, by the sufficiency part of the Theorem of Chatterji et al. (2013), we have $f(P) P_i f(P')$. Overall, voter i has no incentive to manipulate. Henceforth, we focus on the possible manipulations of voters 1 and 2.

First, given two profiles $P = (P_1, P_2, P_{-\{1,2\}})$ and $P' = (P'_1, P_2, P_{-\{1,2\}})$, there are three possible manipulations of voter 1:

(1) $f(P) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right)$ and $f(P') = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P')}\right)$,

(2) $f(P) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right)$ and $f(P') = r_1(P'_1) \in A^{y \rightharpoonup x}$, and

(3) $f(P) = \max^{P_2}(\{x, y\})$ and $f(P') = r_1(P'_1) \in A^{y \rightharpoonup x}$.

In each case, we show either $f(P) = f(P')$ or $f(P) P_1 f(P')$ holds.

In case (1), $r_1(P_1) \in A^{x \rightharpoonup y}$, $r_1(P_2) \in A^{x \rightharpoonup y}$ and $r_1(P'_1) \in A^{x \rightharpoonup y}$. If there exists $i \in \{3, \dots, n\}$ such that $r_1(P_i) \in A^{y \rightharpoonup x}$, we have $f(P) = y = f(P')$ by construction. Next, assume $r_1(P_i) \in A^{x \rightharpoonup y}$ for all $i \in \{3, \dots, n\}$. Note that condition (i) of Fact 1 implies that P_1 is also semi-single-peaked on $\mathcal{T}^{A^{x \rightharpoonup y}}$ w.r.t. x. Therefore, by the sufficiency part of the Theorem of Chatterji et al. (2013), we have either $f(P) = f(P')$ or $f(P)P_1 f(P')$, as required.

In case (2), $r_1(P_1) \in A^{x \rightharpoonup y}$ and $r_1(P_2) \in A^{x \rightharpoonup y}$. By construction, we have $f(P) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right) \in \langle r_1(P_1), x | \mathcal{T}^A \rangle$. By condition (i) of Fact 1, we know $x = \min^{P_1}\left(\langle r_1(P_1), x | \mathcal{T}^A \rangle\right)$, $xP_1 y$ and $y = \max^{P_1}(A^{y \rightharpoonup x})$. Then, $f(P) \in \langle r_1(P_1), x | \mathcal{T}^A \rangle$ and $f(P') \in A^{y \rightharpoonup x}$ imply $f(P)P_1 f(P')$, as required.

In case (3), $r_1(P_1) \in A^{x \rightharpoonup y}$ and $r_1(P_2) \in A^{y \rightharpoonup x}$. By condition (i) of Fact 1, we have $xP_1 y$ and $y = \max^{P_1}(A^{y \rightharpoonup x})$. Then, $f(P) = \max^{P_2}(\{x, y\})$ and $f(P') \in A^{y \rightharpoonup x}$ imply either $f(P) = f(P') = y$ or $f(P)P_1 f(P')$, as required.

Last, given two profiles $P = (P_1, P_2, P_{-\{1,2\}})$ and $P' = (P_1, P'_2, P_{-\{1,2\}})$, there are three possible manipulations of voter 2:

(1) $f(P) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right)$ and $f(P') = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P')}\right)$,

(2) $f(P) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right)$ and $f(P') = \max^{P'_2}(\{x, y\})$, and

(3) $f(P) = \max^{P_2}(\{x, y\})$ and $f(P') = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P')}\right)$.

In each case, we show either $f(P) = f(P')$ or $f(P)P_2 f(P')$ holds. Clearly, the verification of case (1) is similar to that of case (1) for voter 1.

In case (2), $r_1(P_1) \in A^{x \rightharpoonup y}$, $r_1(P_2) \in A^{x \rightharpoonup y}$ and $r_1(P'_2) \in A^{y \rightharpoonup x}$. Hence, $f(P) = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P)}\right) \in \langle r_1(P_2), x | \mathcal{T}^A \rangle$. By condition (i) of Fact 1, we know $x = \min^{P_2}\left(\langle r_1(P_2), x | \mathcal{T}^A \rangle\right)$ and $xP_2 y$. Therefore, $f(P) \in \langle r_1(P_2), x | \mathcal{T}^A \rangle$ and $f(P') = \max^{P'_2}(\{x, y\})$ imply either $f(P) = f(P') = x$ or $f(P)P_2 f(P')$, as required.

In case (3), $r_1(P_1) \in A^{x \rightharpoonup y}$, $r_1(P_2) \in A^{y \rightharpoonup x}$ and $r_1(P'_2) \in A^{x \rightharpoonup y}$. Thus, by construction, $f(P') = \text{Proj}\left(x, \mathcal{T}^{\Gamma(P')}\right) \in \langle r_1(P'_2), x | \mathcal{T}^A \rangle \subseteq A^{x \rightharpoonup y}$. If $xP_2 y$, we have $f(P) = \max^{P_2}(\{x, y\}) = x$. Then condition (ii) of Fact 1 implies either $f(P) = f(P') = x$ or $f(P)P_2 f(P')$. If $yP_2 x$, we have $f(P) = \max^{P_2}(\{x, y\}) = y$. Then, condition (ii) of Fact 1 implies $f(P)P_2 f(P')$, as required.

In conclusion, the PNT rule f is strategy-proof. This completes the verification of sufficiency part of Fact 1.

C Proof of Proposition 1

We first show statement (i) of Proposition 1. Let $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ be a path-connected domain. Because of Fact 1, we only need to show that $\mathbb{D} \nsubseteq \mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ if and only if there exists a critical spot. According to condition (iii) of Fact 1, it is evident that the existence of a critical spot ensures $\mathbb{D} \nsubseteq \mathbb{D}_{\text{SP}}(\mathcal{T}^A)$. Henceforth, we focus on showing the necessity part.

Since $\mathbb{D}$ is semi-single-peaked, we identify the set of all thresholds $\mathcal{Z} \subseteq A$ w.r.t. which $\mathbb{D}$ is semi-single-peaked on $\mathcal{T}^A$. Note that either $\mathcal{Z} = \{\bar{x}\}$, or $\mathcal{Z}$ contains multiple alternatives and $\mathcal{T}^{\mathcal{Z}}$ is a subtree nested in $\mathcal{T}^A$.[62] Since $\mathbb{D} \nsubseteq \mathbb{D}_{\text{SP}}(\mathcal{T}^A)$, it is true that some extreme vertex of $\mathcal{T}^A$ and its unique neighbor must not be contained in $\mathcal{Z}$. Furthermore, we can identify a threshold $x' \in \mathcal{Z}$ such that $x' = \bar{x}$ if $\mathcal{Z} = \{\bar{x}\}$, or $x' \in Ext(\mathcal{T}^{\mathcal{Z}})$ otherwise, and an edge $(x, y) \in \mathcal{E}^A$ with $x, y \notin \mathcal{Z}$ and $y \in \langle x, x' | \mathcal{T}^A \rangle$ such that the following two conditions are satisfied:

(1) every preference $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{y \rightharpoonup x}$ is single-peaked on $\mathcal{T}^{A^{x \rightharpoonup y}}$, i.e., for all distinct $a, b \in A^{x \rightharpoonup y}$, $\big[a \in \langle b, x | \mathcal{T}^A \rangle\big] \Rightarrow [aP_i b]$, and

(2) some preference $P_i^* \in \mathbb{D}$ with $r_1(P_i^*) \in A^{y \rightharpoonup x}$ is not single-peaked on the subtree $\mathcal{T}^{A^{x \rightharpoonup y} \cup \{y\}}$.[63]

[62] Given a tree $\mathcal{T}^A$ and two distinct alternatives $a, b \in A$, if domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, a)$ and $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, b)$, then $\mathbb{D}$ is semi-single-peaked on $\mathcal{T}^A$ w.r.t. every alternative of $\langle a, b | \mathcal{T}^A \rangle$.

[63] The subtree $\mathcal{T}^{A^{x \rightharpoonup y} \cup \{y\}}$ is a combination of the subtree $\mathcal{T}^{A^{x \rightharpoonup y}}$ and the edge (x, y). We adopt a simple instance of the line $\mathcal{L}^A$ to exemplify how conditions (1) and (2) are specified. Fixing a path-connected domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{L}^A, a_k)$ for some $1 \leq k \leq m$, let $\mathbb{D} \nsubseteq \mathbb{D}_{\text{SP}}(\mathcal{L}^A)$. Clearly, we have $\mathcal{Z} = \langle a_p, a_q | \mathcal{L}^A \rangle$ for some $1 \leq p \leq q \leq m$. Hence, $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{L}^A, a_k)$ for all $p \leq k \leq q$, and $\mathbb{D} \nsubseteq \mathbb{D}_{\text{SSP}}(\mathcal{L}^A, a_l)$ for all $l < p$ and $q < l$. Clearly, $\mathbb{D} \nsubseteq \mathbb{D}_{\text{SP}}(\mathcal{L}^A)$ implies either $p > 2$ or $q < m - 1$. We assume w.l.o.g. that $p > 2$. First, according to a_1, it is natural that every preference with the peak located in $\langle a_2, a_m | \mathcal{L}^A \rangle$ is single-peaked on $\langle a_1, a_1 | \mathcal{L}^A \rangle$. Second, since $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{L}^A, a_p)$ and $\mathbb{D} \nsubseteq \mathbb{D}_{\text{SSP}}(\mathcal{L}^A, a_{p-1})$ it must be the case that some preference with the peak located in $\langle a_{p-1}, a_m | \mathcal{L}^A \rangle$ is not single-peaked on $\langle a_1, a_{p-1} | \mathcal{L}^A \rangle$. Therefore, searching from a_1 to a_{p-1}, we can identify $1 \leq s < p - 1$ such that (i) every preference with the peak located in $\langle a_{s+1}, a_m | \mathcal{L}^A \rangle$ is single-peaked on $\langle a_1, a_s | \mathcal{L}^A \rangle$, and (ii) some preference with the peak located in $\langle a_{s+1}, a_m | \mathcal{L}^A \rangle$ is not single-peaked on $\langle a_1, a_{s+1} | \mathcal{L}^A \rangle$. The two conditions here are analogous to conditions (1) and (2) above respectively.

Since condition (1) implies $\max^{P_i}(A^{x \rightharpoonup y}) = x$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{y \rightharpoonup x}$, we know that condition (ii) of Fact 1 is satisfied. We next show that $\mathbb{D}$ satisfies condition (i) of Fact 1.

Claim 1: For all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{x \rightharpoonup y}$, P_i is semi-single-peaked on $\mathcal{T}^A$ w.r.t. y.

Fix an arbitrary preference $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{x \rightharpoonup y}$. Note that $x' \in A^{y \rightharpoonup x}$, $x, y \in \langle r_1(P_i), x' | \mathcal{T}^A \rangle$ and $y \in \langle x, x' | \mathcal{T}^A \rangle$. First, we show P_i is semi-single-peaked on the subtree $\mathcal{T}^{A^{x \rightharpoonup y} \cup \{y\}}$ w.r.t. y. Since $r_1(P_i) \in A^{x \rightharpoonup y}$ and P_i is semi-single-peaked on $\mathcal{T}^A$ w.r.t. x', it is true that P_i is also semi-single-peaked on $\mathcal{T}^{A^{x \rightharpoonup y}}$ w.r.t. x, and xP_iy, which imply that P_i is semi-single-peaked on $\mathcal{T}^{A^{x \rightharpoonup y} \cup \{y\}}$ w.r.t. y. Next, we show $\max^{P_i}(A^{y \rightharpoonup x}) = y$. Given an arbitrary $z \in A^{y \rightharpoonup x} \backslash \{y\}$, we show yP_iz. We know that either $z \in \langle y, x' | \mathcal{T}^A \rangle$, or $z \notin \langle y, x' | \mathcal{T}^A \rangle$ which further implies $\notin \langle r_1(P_i), x' | \mathcal{T}^A \rangle$. If $z \in \langle y, x' | \mathcal{T}^A \rangle$, we have $y, z \in \langle r_1(P_i), x' | \mathcal{T}^A \rangle$ and then the first condition of semi-single-peakedness on $\mathcal{T}^A$ w.r.t. x' implies yP_iz. If $z \notin \langle y, x' | \mathcal{T}^A \rangle$, let $\text{Proj}\big(z, \langle r_1(P_i), x' | \mathcal{T}^A \rangle\big) = z'$. Clearly, $z' \in \langle y, x' | \mathcal{T}^A \rangle$. Then, by semi-single-peakedness on $\mathcal{T}^A$ w.r.t. x', we know $z'P_iz$ and yP_iz' (if $y \neq z'$), which hence imply yP_iz by transitivity. Therefore, $\max^{P_i}(A^{y \rightharpoonup x}) = y$. Last, since $\mathcal{T}^A$ is a combination of the subtree $\mathcal{T}^{A^{x \rightharpoonup y}}$, the edge (x, y) and the subtree $\mathcal{T}^{A^{y \rightharpoonup x}}$, it must be the case that P_i is semi-single-peaked on $\mathcal{T}^A$ w.r.t. y. This completes the verification of the claim.

We last show that $\mathbb{D}$ satisfies condition (iii) of Fact 1. First, recall the preference P_i^* in condition (2) above. Since P_i^* is single-peaked on $\mathcal{T}^{A^{x \rightharpoonup y}}$ by condition (1) and violates single-peakedness on $\mathcal{T}^{A^{x \rightharpoonup y} \cup \{y\}}$ by condition (2) above, it must be the case that $\max^{P_i^*}(A^{x \rightharpoonup y}) = x$ and xP_i^*y.

Claim 2: There exist $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) = r_1(P_i') \in A^{y \rightharpoonup x}$, yP_ix and $xP_i'y$.

Clearly, there exists a preference $\hat{P}_i \in \mathbb{D}^y$ by path-connectedness. Thus, we have $\hat{P}_i, P_i^* \in \mathbb{D}$ such that $r_1(\hat{P}_i) \in A^{y \rightharpoonup x}$, $r_1(P_i^*) \in A^{y \rightharpoonup x}$, $y\hat{P}_ix$ and xP_i^*y. We separate the subdomain $\overline{\mathbb{D}} = \{P_i \in \mathbb{D} : r_1(P_i) \in A^{y \rightharpoonup x}\}$ into two parts: $\overline{\mathbb{D}}_1 = \{P_i \in \overline{\mathbb{D}} : yP_ix\}$ and $\overline{\mathbb{D}}_2 = \{P_i \in \overline{\mathbb{D}} : xP_iy\}$. Clearly, $\overline{\mathbb{D}}_1 \neq \emptyset$ and $\overline{\mathbb{D}}_2 \neq \emptyset$. Since

$\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$, it is true that $G^A_\sim = \mathcal{T}^A$. Recall that $\mathcal{T}^A$ is a combination of the subtree $\mathcal{T}^{A^{x \rightharpoonup y}}$, the edge (x, y) and the subtree $\mathcal{T}^{A^{y \rightharpoonup x}}$. Hence, $G^{A^{y \rightharpoonup x}}_\sim$ is a subtree.

Now, suppose that the claim is not true, i.e., for all distinct $P_i, P_i' \in \overline{\mathbb{D}}$ with $r_1(P_i) = r_1(P_i')$, they agree on the relative ranking of x and y. Consequently, for all $z \in A^{y \rightharpoonup x}$, either $\mathbb{D}^z \subseteq \overline{\mathbb{D}}_1$ or $\mathbb{D}^z \subseteq \overline{\mathbb{D}}_2$ holds. Furthermore, given two arbitrary distinct $z, z' \in A^{y \rightharpoonup x}$, let $\mathbb{D}^z \subseteq \overline{\mathbb{D}}_1$ and $\mathbb{D}^{z'} \subseteq \overline{\mathbb{D}}_2$. Thus, we have zP_iz' and yP_ix for all $P_i \in \mathbb{D}^z$, and $z'P_i'z$ and $xP_i'y$ for all $P_i' \in \mathbb{D}^{z'}$, which imply that z and z' are never adjacent. This contradicts the fact that $G^{A^{y \rightharpoonup x}}_\sim$ is a subtree. This completes the verification of the claim, and hence proves the necessity part of statement (i).

Next, we show statement (ii) of Proposition 1. Let $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ be a path-connected domain. Symmetrically, because of Fact 1, we only need to show that $\mathbb{D} \not\subseteq \mathbb{D}_{\text{H}}(\mathcal{T}^A, a, b)$ if and only if there exists a critical spot. According to condition (iii) of Fact 1, it is evident that the existence of a critical spot ensures $\mathbb{D} \not\subseteq \mathbb{D}_{\text{H}}(\mathcal{T}^A, a, b)$. Henceforth, we focus on showing the necessity part. Since $\mathbb{D} \not\subseteq \mathbb{D}_{\text{H}}(\mathcal{T}^A, a, b)$, there must exist a preference that is not single-peaked on either $\mathcal{T}^{A^{a \rightharpoonup b}}$ or $\mathcal{T}^{A^{b \rightharpoonup a}}$. We assume w.l.o.g. that there exists a preference that is not single-peaked in $\mathcal{T}^{A^{a \rightharpoonup b}}$. Therefore, $A^{a \rightharpoonup b} \neq \{a\}$. We separate $\mathbb{D}$ into two parts: $\mathbb{D}_1 = \{P_i \in \mathbb{D} : r_1(P_i) \in A^{a \rightharpoonup b}\}$ and $\mathbb{D}_2 = \{P_i \in \mathbb{D} : r_1(P_i) \notin A^{a \rightharpoonup b}\}$. Thus, we know either $\mathbb{D}_1$ or $\mathbb{D}_2$ is not single-peaked on $\mathcal{T}^{A^{a \rightharpoonup b}}$. Note that all preferences of $\mathbb{D}_1$ are semi-single-peaked on $\mathcal{T}^{A^{a \rightharpoonup b}}$ w.r.t. a. If $\mathbb{D}_1$ is not single-peaked on $\mathcal{T}^{A^{a \rightharpoonup b}}$, we can adopt the same proof of the necessity part of statement (i) to identify a critical spot in $\mathcal{T}^{A^{a \rightharpoonup b}}$. Next, assume that $\mathbb{D}_1$ is single-peaked on $\mathcal{T}^{A^{a \rightharpoonup b}}$, and $\mathbb{D}_2$ is not single-peaked on $\mathcal{T}^{A^{a \rightharpoonup b}}$. Note that for every preference $P_i \in \mathbb{D}_2$, (i) P_i vacuously satisfies the first condition of semi-single-peakedness on $\mathcal{T}^{A^{a \rightharpoonup b}}$ w.r.t. a, and (ii) (a, b)-semi-hybridness on $\mathcal{T}^A$ implies $\max^{P_i}(A^{a \rightharpoonup b}) = a$, which implies that P_i satisfies the second condition of semi-single-peakedness on $\mathcal{T}^{A^{a \rightharpoonup b}}$ w.r.t. a. Therefore, loosely speaking, $\mathbb{D}_2$ is semi-single-peaked on $\mathcal{T}^{A^{a \rightharpoonup b}}$ w.r.t. a. Then, by an argument similar to the proof of the necessity part of statement (i), we can also identify a critical spot in $\mathcal{T}^{A^{a \rightharpoonup b}}$. This completes the verification of the necessity part, and hence proves Proposition 1.

D Proof of the Theorem

We first provide two independent lemmas (Lemmas 9 and 10) which will be repeatedly applied in the proof of the Theorem. For Lemmas 9 and 10, we fix $N = \{1, 2\}$ and a two-voter tops-only and strategy-proof rule $f : \mathbb{D}^2 \to A$. Since f satisfies the tops-only property, we slightly abuse the notation $f(a, b)$ to represent a social outcome at a preference profile where voter 1 reports a preference with the peak a and voter 2 reports a preference with the peak b. Also, $f(a, P_2)$ represents a social outcome at a preference profile where voter 1 reports a preference with the peak a and voter 2 reports preference P_2.

Lemma **9** *Fixing a path $\pi = (x_1, \dots, x_v)$ in $G^A_{\sim}$, the following three statements hold:*

(i) *$f(x_s, x_t) \in \{x_s, \dots, x_t\}$ and $f(x_t, x_s) \in \{x_s, \dots, x_t\}$ for all $1 \leq s < t \leq v$.*

(ii) *If $f(P_1, x_s) = x \notin \pi$ where $x_s \in \pi$, then $f(P_1, x_k) = x$ for all $k = 1, \dots, v$. Symmetrically, if $f(x_s, P_2) = x \notin \pi$ where $x_s \in \pi$, then $f(x_k, P_2) = x$ for all $k = 1, \dots, v$.*

(iii) *If $f(P_1, x_s) = x_s$ where $x_s \in \pi$, then $f(P_1, x_p) \in \{x_p, \dots, x_s\}$ for all $1 \leq p < s$, and $f(P_1, x_q) \in \{x_s, \dots, x_q\}$ for all $s < q \leq v$. Symmetrically, if $f(x_s, P_2) = x_s$ where $x_s \in \pi$, then $f(x_p, P_2) \in \{x_p, \dots, x_s\}$ for all $1 \leq p < s$, and $f(x_q, P_2) \in \{x_s, \dots, x_q\}$ for all $s < q \leq v$.*

Proof: Fix $1 \leq s < t \leq v$. First, since $x_s \sim x_{s+1}$, by the proof of Claim A of Sen (2001), unanimity and strategy-proofness imply $f(x_s, x_{s+1}) \in \{x_s, x_{s+1}\}$. Next, we provide an induction hypothesis: given $s + 1 < l \leq t$, for all $s + 1 \leq l' < l$, we have $f(x_s, x_{l'}) \in \{x_s, \dots, x_{l'}\}$. We show $f(x_s, x_l) \in \{x_s, \dots, x_l\}$. By the induction hypothesis, we first have $f(x_s, x_{l-1}) \in \{x_s, \dots, x_{l-2}, x_{l-1}\}$. Next, since $x_{l-1} \sim x_l$, we have $P_2, P_2' \in \mathbb{D}$ such that $r_1(P_2) = r_2(P_2') = x_{l-1}$, $r_1(P_2') = r_2(P_2) = x_l$ and $r_k(P_2) = r_k(P_2')$ for all $k = 3, \dots, m$. If $f(x_s, P_2) = f(x_s, x_{l-1}) \in \{x_s, \dots, x_{l-2}\}$, then the tops-only property and strategy-proofness imply $f(x_s, x_l) = f(x_s, P_2') = f(x_s, P_2) \in \{x_s, \dots, x_{l-2}\} \subset \{x_s, \dots, x_l\}$. If $f(x_s, P_2) = f(x_s, x_{l-1}) = x_{l-1}$, then the tops-only property and strategy-proofness imply $f(x_s, x_l) = f(x_s, P_2') \in$

$\{x_{l-1}, x_l\} \subset \{x_s, \ldots, x_l\}$. Overall, $f(x_s, x_l) \in \{x_s, \ldots, x_l\}$. This completes the verification of the induction hypothesis. Therefore, $f(x_s, x_t) \in \{x_s, \ldots, x_t\}$. Symmetrically, $f(x_t, x_s) \in \{x_s, \ldots, x_t\}$. This completes the verification of statement (i).

Next, we show statement (ii). By symmetry, we focus on $f(P_1, x_k)$. Given $1 \leq k \leq v$, we first assume $s < k$. We consider the path $(x_s, \ldots, x_k)$. Since $x_s \sim x_{s+1}$, we have $P_2, P_2' \in \mathbb{D}$ such that $r_1(P_2) = r_2(P_2') = x_s$, $r_1(P_2') = r_2(P_2) = x_{s+1}$ and $r_k(P_2) = r_k(P_2')$ for all $k = 3, \ldots, m$. Since $x \notin \{x_s, x_{s+1}\}$, the tops-only property and strategy-proofness imply $f(P_1, x_{s+1}) = f(P_1, P_2') = f(P_1, P_2) = f(P_1, x_s) = x$. Following the path $(x_s, \ldots, x_k)$ from x_{s+1} to x_k and repeatedly applying the same argument step by step, we eventually have $f(P_1, x_k) = x$. Similarly, if $s > k$, we also induce $f(P_1, x_k) = x$. Therefore, $f(P_1, x_k) = x$ for all $k = 1, \ldots, v$. This completes the verification of statement (ii).

Last, we prove statement (iii). By symmetry, we focus on $f(P_1, x_p)$ and $f(P_1, x_q)$. Given $1 \leq p < s$, suppose $f(P_1, x_p) = x \notin \{x_p, \ldots, x_s\}$. Then, according to the path $(x_p, \ldots, x_s)$, statement (ii) implies $f(P_1, x_s) = x \neq x_s$. Contradiction! Therefore, $f(P_1, x_p) \in \{x_p, \ldots, x_s\}$. Similarly, $f(P_1, x_q) \in \{x_s, \ldots, x_q\}$ for all $s < q \leq v$. This completes the verification of statement (iii). ■

LEMMA **10** *Fixing a path $\pi = (x_1, \ldots, x_v)$, $v \geq 3$, in $G_{\sim}^{A}$, let $f(x_1, x_v) = x_{\underline{k}}$ and $f(x_v, x_1) = x_{\overline{k}}$. The following three statements hold:*

$$\text{(i)} \quad \left[\underline{k} < \overline{k}\,\right] \Rightarrow \left[f(x_s, x_t) = \begin{cases} x_s & \text{if } \underline{k} \leq s \leq \overline{k}, \\ x_{\text{med}(s,t,\underline{k})} & \text{if } s < \underline{k}, \\ x_{\text{med}(s,t,\overline{k})} & \text{if } s > \overline{k}. \end{cases} \right]$$

$$\text{(ii)} \quad \left[\underline{k} > \overline{k}\,\right] \Rightarrow \left[f(x_s, x_t) = \begin{cases} x_t & \text{if } \overline{k} \leq t \leq \underline{k}, \\ x_{\text{med}(s,t,\overline{k})} & \text{if } t < \overline{k}, \\ x_{\text{med}(s,t,\underline{k})} & \text{if } t > \underline{k}. \end{cases} \right]$$

$$\text{(iii)} \quad \left[\underline{k} = \overline{k} \equiv k^*\right] \Rightarrow \left[f(x_s, x_t) = x_{\text{med}(s,t,k^*)} \text{ for all } 1 \leq s, t \leq v\right].$$

PROOF: First, according to $f(x_1, x_v) = x_{\underline{k}}$, we establish the following claim.

CLAIM 1: We have
$$f(x_k, x_{k'}) = \begin{cases} x_{k'} & \text{if } 1 \le k \le k' \le \underline{k}, \\ x_k & \text{if } \underline{k} \le k \le k' \le v, \\ x_{\underline{k}} & \text{if } 1 \le k < \underline{k} < k' \le v. \end{cases}$$

Since $f(x_1, x_v) = x_{\underline{k}}$, strategy-proofness implies $f(x_1, x_{\underline{k}}) = x_{\underline{k}}$. If $\underline{k} = 1$, then it is evident by unanimity that $f(x_k, x_{k'}) = x_{k'}$ for all $1 \le k \le k' \le \underline{k}$. Next, assume $\underline{k} > 1$. According to the subpath $(x_1, \ldots, x_{\underline{k}})$, by statement (iv) of Lemma 1, to show $f(x_k, x_{k'}) = x_{k'}$ for all $1 \le k \le k' \le \underline{k}$, it suffices to show $f(x_{\underline{k}-1}, x_{\underline{k}}) = x_{\underline{k}}$. By statement (i) of Lemma 9, we first know $f(x_{\underline{k}-1}, x_{\underline{k}}) \in \{x_{\underline{k}-1}, x_{\underline{k}}\}$. Suppose $f(x_{\underline{k}-1}, x_{\underline{k}}) = x_{\underline{k}-1}$. Then, according to the subpath $(x_1, \ldots, x_{\underline{k}-1})$, statement (iii) of Lemma 9 implies $f(x_1, x_{\underline{k}}) \in \{x_1, \ldots, x_{\underline{k}-1}\}$. Contradiction! Therefore, $f(x_{\underline{k}-1}, x_{\underline{k}}) = x_{\underline{k}}$, as required.

Symmetrically, since $f(x_1, x_v) = x_{\underline{k}}$ implies $f(x_{\underline{k}}, x_v) = x_{\underline{k}}$ by strategy-proofness, we can refer to the subpath $(x_{\underline{k}}, \ldots, x_v)$ and show that $f(x_k, x_{k'}) = x_k$ for all $\underline{k} \le k \le k' \le v$.

Last, given $1 \le k < \underline{k} < k' \le v$, according to the subpath $(x_1, \ldots, x_k)$, by statement (ii) of Lemma 9, $f(x_1, x_v) = x_{\underline{k}}$ implies $f(x_k, x_v) = x_{\underline{k}}$. Furthermore, according to the subpath $(x_{k'}, \ldots, x_v)$, by statement (ii) of Lemma 9, $f(x_k, x_v) = x_{\underline{k}}$ implies $f(x_k, x_{k'}) = x_{\underline{k}}$. This completes the verification of the claim.

Symmetrically, according to $f(x_v, x_1) = x_{\overline{k}}$, we can establish the claim below.

CLAIM 2: We have
$$f(x_{k'}, x_k) = \begin{cases} x_{k'} & \text{if } 1 \le k \le k' \le \overline{k}, \\ x_k & \text{if } \overline{k} \le k \le k' \le v, \\ x_{\overline{k}} & \text{if } 1 \le k < \overline{k} < k' \le v. \end{cases}$$

Last, we combine two claims to prove the lemma. Note that the verifications of the three cases $\underline{k} < \overline{k}$, $\underline{k} > \overline{k}$ and $\underline{k} = \overline{k}$ are symmetric. Hence, we focus on the verification of the first case $\underline{k} < \overline{k}$. Let $\underline{k} < \overline{k}$ and fix an arbitrary profile (x_s, x_t).

First, let $\underline{k} \le s \le \overline{k}$. If $s \le t$, we have $\underline{k} \le s < t \le v$ and Claim 1 implies $f(x_s, x_t) = x_s$. If $t < s$, we have $1 \le t \le s \le \overline{k}$ and Claim 2 implies $f(x_s, x_t) = x_s$. Overall, $f(x_s, x_t) = x_s$, as required.

Second, let $s < \underline{k}$. If $t \le s$, we have $1 \le t < s < \underline{k} < \overline{k}$ and Claim 2 implies $f(x_s, x_t) = x_s = x_{\text{med}(s,t,\underline{k})}$. If $s < t \le \underline{k}$, we have $1 \le s < t \le \underline{k}$ and Claim 1 implies $f(x_s, x_t) = x_t = x_{\text{med}(s,t,\underline{k})}$. If $\underline{k} < t$, we have $1 \leq s < \underline{k} < t \leq v$ and Claim 1 implies $f(x_s, x_t) = x_{\underline{k}} = x_{\text{med}(s,t,\underline{k})}$. Overall, $f(x_s, x_t) = x_{\text{med}(s,t,\underline{k})}$, as required.

Last, let $s > \overline{k}$. If $t < \overline{k}$, we have $1 \leq t < \overline{k} < s \leq v$ and Claim 2 implies $f(x_s, x_t) = x_{\overline{k}} = x_{\text{med}(s,t,\overline{k})}$. If $\overline{k} \leq t \leq s$, we have $\overline{k} \leq t < s \leq v$ and Claim 2 implies $f(x_s, x_t) = x_t = x_{\text{med}(s,t,\overline{k})}$. If $s < t$, we have $\underline{k} < \overline{k} < s < t \leq v$ and Claim 1 implies $f(x_s, x_t) = x_s = x_{\text{med}(s,t,\overline{k})}$. Overall, $f(x_s, x_t) = x_{\text{med}(s,t,\overline{k})}$, as required. This proves the lemma. ■

Proof of Statement (i). The sufficiency part of Statement (i) holds, since by the proof of the sufficiency part of the Theorem of Chatterji et al. (2013), a semi-single-peaked domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ admits the following two-voter anonymous, tops-only and strategy-proof rule: $f(P_1, P_2) = \text{Proj}\left(\bar{x}, \langle r_1(P_1), r_1(P_2) | \mathcal{T}^A \rangle\right)$ for all $P_1, P_2 \in \mathbb{D}$, which hence satisfies invariance.

Henceforth, we focus on the necessity part. Let $\mathbb{D}$ be a rich domain and $f : \mathbb{D}^2 \to A$ be an invariant, tops-only and strategy-proof rule. The proof consists of two steps: (1) we show that $G^A_\sim$ is a tree (see Lemmas 11 and 12), and (2) we completely characterize SCF f (see Lemma 13), which by strategy-proofness implies that $\mathbb{D}$ is semi-single-peaked on $G^A_\sim$ w.r.t. the threshold which equals the social outcome $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2)$ (see Lemma 14).

Lemma 11 *There exists a unique path in $G^A_\sim$ connecting a_1 and a_m.*

Proof: Since $G^A_\sim$ is a connected graph, there exists a path in $G^A_\sim$ connecting a_1 and a_m. Suppose that the lemma is not correct, i.e., there are two distinct paths $\pi = (x_1, \ldots, x_p)$ and $\pi' = (y_1, \ldots, y_q)$ in $G^A_\sim$ connecting a_1 and a_m. Then, we can identify $1 \leq s < t \leq p$ and $1 \leq s' < t' \leq q$ with either $t - s > 1$ or $t' - s' > 1$ such that $x_s = y_{s'}$, $x_t = y_{t'}$ and $\{x_{s+1}, \ldots, x_{t-1}\} \cap \{y_{s'+1}, \ldots, y_{t'-1}\} = \emptyset$. Consequently, we construct a cycle $\mathcal{C} = (x_s, \ldots, x_t = y_{t'}, \ldots, y_{s'+1}, y_{s'} = x_s)$. Note that $(x_1, \ldots, x_s)$ is a path in $G^A_\sim$ connecting a_1 to the cycle $\mathcal{C}$, and $(x_p, \ldots, x_t)$ is a path in $G^A_\sim$ connecting a_m to the cycle $\mathcal{C}$.

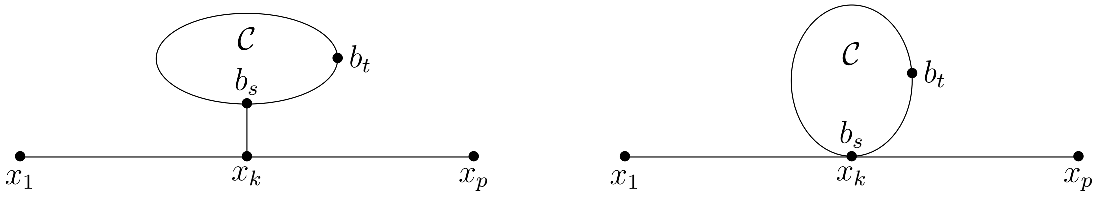
By Observation 1, f behaves like a dictatorship on $\mathcal{C}$. We assume w.l.o.g. that voter 1 dictates on $\mathcal{C}$. Thus, $f(x_s, x_t) = x_s$ and $f(x_t, x_s) = x_t$. According to the subpaths $(x_t, \ldots, x_p)$ and $(x_s, \ldots, x_1)$, by statements (ii) of Lemma 9, $f(x_s, x_t) = x_s$ implies $f(x_s, x_p) = x_s$, and $f(x_t, x_s) = x_t$ implies $f(x_t, x_1) = x_t$. Furthermore, according to the subpaths $(x_s, \ldots, x_1)$ and $(x_t, \ldots, x_p)$, by statements (iii) of Lemma 9, $f(x_s, x_p) = x_s$ implies $f(x_1, x_p) \in \{x_1, \ldots, x_s\}$ and $f(x_t, x_1) = x_t$ implies $f(x_p, x_1) \in \{x_t, \ldots, x_p\}$. Consequently, $f(x_1, x_p) \neq f(x_p, x_1)$, which contradicts invariance. ■

Let $\pi^* = (x_1, \ldots, x_p)$ be the unique path connecting a_1 and a_m in $G^A_\sim$. Note that this path may not include all alternatives of A.

Lemma **12** *The graph $G^A_\sim$ is a tree.*

Proof: Suppose not, i.e., there exists a cycle $\mathcal{C} = (b_1, \ldots, b_v, b_1)$, $v \geq 3$. By Observation 1, f behaves like a dictatorship on $\mathcal{C}$. We assume w.l.o.g. that voter 1 dictates on $\mathcal{C}$, i.e., $f(P_1, P_2) = r_1(P_1)$ for all $P_1, P_2 \in \mathbb{D}$ with $r_1(P_1), r_1(P_2) \in \mathcal{C}$.

We know either $\mathcal{C} \cap \pi^* = \emptyset$ or $\mathcal{C} \cap \pi^* \neq \emptyset$. If $\mathcal{C} \cap \pi^* = \emptyset$, we can identify $b_s \in \mathcal{C}$, $x_k \in \pi^*$ and a path $(y_1, \ldots, y_u)$ in $G^A_\sim$ connecting b_s and x_k such that $\{y_2, \ldots, y_{u-1}\} \cap [\mathcal{C} \cup \pi^*] = \emptyset$ (see Figure 12(a)). If $\mathcal{C} \cap \pi^* \neq \emptyset$, it must be the case that $|\mathcal{C} \cap \pi^*| = 1$, for otherwise, we can identify two distinct paths in $G^A_\sim$ connecting a_1 and a_m, which then contradicts Lemma 11. Accordingly, let $b_s = x_k \in \pi^*$ be the unique alternative of π^* contained in $\mathcal{C}$ (see Figure 12(b)). Overall, we have the cycle $\mathcal{C} = (b_1, \ldots, b_v, b_1)$, the path $\pi^* = (x_1, \ldots, x_p)$ and the path $(y_1, \ldots, y_u)$ which may be a null path when $b_s = x_k$. We consider three cases: (i) $1 < k < p$, (ii) $k = 1$ and (iii) $k = p$, and induce a contradiction in each case.



(a) (b)

Figure 12: The relation between the cycle $\mathcal{C}$ and the path π^*

In the first case, fixing $b_t \in \mathcal{C} \backslash \{b_s\}$, we have $f(b_t, b_s) = b_t$ and $f(b_s, b_t) = b_s$ by voter 1's dictatorship on $\mathcal{C}$. According to the paths $(b_s = y_1, \ldots, y_u = x_k, \ldots, x_1)$

and $(b_s = y_1, \ldots, y_u = x_k, \ldots, x_p)$, by statement (ii) of Lemma 9, $f(b_t, b_s) = b_t$ implies $f(b_t, x_1) = b_t$ and $f(b_t, x_p) = b_t$. Furthermore, according to the paths $(b_t, \ldots, b_s = y_1, \ldots, y_u = x_k, \ldots, x_p)$ and $(b_t, \ldots, b_s = y_1, \ldots, y_u = x_k, \ldots, x_1)$, statement (iii) of Lemma 9 implies $f(x_p, x_1) \in \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_k, \ldots, x_p\}$ and $f(x_1, x_p) \in \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_k, \ldots, x_1\}$. Then, invariance implies $f(x_p, x_1) = f(x_1, x_p) \in \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_k, \ldots, x_p\} \cap \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_k, \ldots, x_1\} = \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_k\}$. Moreover, according to the path π^*, invariance and statement (i) of Lemma 9 imply $f(x_p, x_1) = f(x_1, x_p) \in \{x_1, \ldots, x_p\}$. Therefore, $f(x_p, x_1) = f(x_1, x_p) \in \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_k\} \cap \{x_1, \ldots, x_p\} = \{x_k\}$, and hence $f(x_p, x_1) = f(x_1, x_p) = x_k$. Last, given $\overline{P}_1 = \overline{P}_2 = \overline{P}_i$ and $\underline{P}_1 = \underline{P}_2 = \underline{P}_i$, the tops-only property implies $f(\overline{P}_1, \underline{P}_2) = f(x_p, x_1) = x_k$ and $f(\underline{P}_1, \overline{P}_2) = f(x_1, x_p) = x_k$. Recall $f(b_t, \underline{P}_2) = f(b_t, x_1) = b_t$ and $f(b_t, \overline{P}_2) = f(b_t, x_p) = b_t$. Then, strategy-proofness implies $f(\overline{P}_1, \underline{P}_2) = x_k \overline{P}_1 b_t = f(b_t, \underline{P}_2)$ and $f(\underline{P}_1, \overline{P}_2) = x_k \underline{P}_1 b_t = f(b_t, \overline{P}_2)$. This contradicts the hypothesis that $\overline{P}_1$ and $\underline{P}_1$ are completely reversed. This completes the verification in the first situation.

The second and third case are symmetric. We focus on the verification on the second case $k = 1$. Fixing $b_t \in \mathcal{C} \backslash \{b_s\}$, we have $f(b_t, b_s) = b_t$. According to the path $(b_s = y_1, \ldots, y_u = x_1, \ldots, x_p)$, by statement (ii) of Lemma 9, $f(b_t, b_s) = b_t$ implies $f(b_t, x_p) = b_t$ and $f(b_t, x_1) = b_t$. Furthermore, according to the path $(b_t, \ldots, b_s = y_1, \ldots, y_u = x_1)$, by statement (iii) of Lemma 9, $f(b_t, x_p) = b_t$ implies $f(x_1, x_p) \in \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_1\}$. Meanwhile, according to the path π^*, statement (i) of Lemma 9 implies $f(x_1, x_p) \in \{x_1, \ldots, x_p\}$. Therefore, $f(x_1, x_p) \in \{b_t, \ldots, b_s = y_1, \ldots, y_u = x_1\} \cap \{x_1, \ldots, x_p\} = \{x_1\}$, and hence, $f(x_1, x_p) = x_1$. Then, given $\overline{P}_1 = \overline{P}_i$ and $\underline{P}_2 = \underline{P}_i$, the tops-only property and invariance imply $f(\overline{P}_1, \underline{P}_2) = f(x_p, x_1) = f(x_1, x_p) = x_1$. Recall $f(b_t, \underline{P}_2) = f(b_t, x_1) = b_t$. Since $\overline{P}_i$ and $\underline{P}_i$ are completely reversed, $x_1 = a_1$ is the bottom-ranked alternative in $\overline{P}_i$, and hence $b_t \overline{P}_1 x_1$. Consequently, voter 1 will manipulate at $(\overline{P}_1, \underline{P}_2)$, i.e., $f(b_t, \underline{P}_2) = b_t \overline{P}_1 x_1 = f(\overline{P}_1, \underline{P}_2)$. This completes the verification in the second situation, and hence proves the lemma. ■

According to the path π^*, by statement (i) of Lemma 9, we fix $f(x_1, x_p) = f(x_p, x_1) = x_{\bar{k}}$ for some $1 \leq \bar{k} \leq p$. The next lemma completely characterizes f.

Lemma 13 *According to the tree $G^A_\sim$, we have $f(y,z) = f(z,y) = \text{Proj}\left(x_{\bar{k}}, \langle y, z | G^A_\sim\rangle\right)$ for all $y, z \in A$.*

Proof: We first consider the path $\pi^* = (x_1, \dots, x_p)$. Clearly, statement (iii) of Lemma 10 implies $f(x_s, x_t) = f(x_t, x_s) = x_{\text{med}(s,t,\bar{k})} = \text{Proj}\left(x_{\bar{k}}, \langle x_s, x_t | G^A_\sim\rangle\right)$ for all $1 \leq s, t \leq p$.

Claim 1: Given $y \notin \pi^*$, we have $f(y, x_{\bar{k}}) = f(x_{\bar{k}}, y) = x_{\bar{k}} = \text{Proj}\left(x_{\bar{k}}, \langle x_{\bar{k}}, y | G^A_\sim\rangle\right)$.

We focus on showing $f(y, x_{\bar{k}}) = x_{\bar{k}}$. By a symmetric proof, one would immediately conclude that $f(x_{\bar{k}}, y) = x_{\bar{k}}$. First, let $\langle x_{\bar{k}}, y | G^A_\sim\rangle = (y_1, \dots, y_u)$ be the unique path connecting $x_{\bar{k}}$ and y in the tree $G^A_\sim$. We first show $f(y_2, y_1) = y_1$. If $y_2 \in \pi^*$, we have $y_2 = x_k$ for some $1 \leq k < \bar{k}$ or $\bar{k} < k \leq p$, and hence $f(y_2, y_1) = f(x_k, x_{\bar{k}}) = x_{\bar{k}} = y_1$. Next, assume $y_2 \notin \pi^*$. Since $y_1 \sim y_2$, statement (i) of Lemma 9 implies $f(y_2, y_1) \in \{y_1, y_2\}$. Suppose $f(y_2, y_1) = y_2$. Consequently, according to the paths $(y_1 = x_{\bar{k}}, \dots, x_1)$ and $(y_1 = x_{\bar{k}}, \dots, x_p)$, statement (ii) of Lemma 9 implies $f(y_2, x_1) = y_2$ and $f(y_2, x_p) = y_2$. Meanwhile, given $\overline{P}_1 = \overline{P}_i$ and $\underline{P}_1 = \underline{P}_i$, since $f(\overline{P}_1, x_1) = f(x_p, x_1) = x_{\bar{k}} = y_1$ and $f(\underline{P}_1, x_p) = f(x_1, x_p) = x_{\bar{k}} = y_1$, strategy-proofness implies $f(\overline{P}_1, x_1) = y_1 \overline{P}_1 y_2 = f(y_2, x_1)$ and $f(\underline{P}_1, x_p) = y_1 \underline{P}_1 y_2 = f(y_2, x_p)$. This contradicts the hypothesis that $\overline{P}_1$ and $\underline{P}_1$ are completely reversed. Therefore, $f(y_2, y_1) = y_1$. Then, according to the path $(y_2, \dots, y_u)$, statement (ii) of Lemma 9 implies $f(y, x_{\bar{k}}) = f(y_u, y_1) = y_1 = x_{\bar{k}} = \text{Proj}\left(x_{\bar{k}}, \langle x_{\bar{k}}, y | G^A_\sim\rangle\right)$. This completes the verification of the claim.

Henceforth, we fix arbitrary $y, z \in A$ and let $\langle y, z | G^A_\sim\rangle = (y_1, \dots, y_u)$. There are three situations: (1) $\text{Proj}\left(x_{\bar{k}}, \langle y, z | G^A_\sim\rangle\right) = y$, (2) $\text{Proj}\left(x_{\bar{k}}, \langle y, z | G^A_\sim\rangle\right) = z$ and (3) $\text{Proj}\left(x_{\bar{k}}, \langle y, z | G^A_\sim\rangle\right) = y_l$ for some $1 < l < u$. In each situation, we show $f(y,z) = f(z,y) = \text{Proj}\left(x_{\bar{k}}, \langle y, z | G^A_\sim\rangle\right)$. Note that the first two situations are symmetric, and we hence focus on situation (1).

Claim 2: In situation (1), $f(y,z) = f(z,y) = y = \text{Proj}\left(x_{\bar{k}}, \langle y, z | G^A_\sim\rangle\right)$.

If $y = x_{\bar{k}}$, this claim is implied by Claim 1. Next, assume $y \neq x_{\bar{k}}$. Since $\text{Proj}\left(x_{\bar{k}}, \langle y, z|G_{\sim}^{A}\rangle\right) = y$, we can construct a path $\pi = (z = y_u, \ldots, y_1 = y = z_1, \ldots, z_{v-1}, z_v = x_{\bar{k}})$, where $v \geq 2$. By Claim 1, we first have $f(z_v, z_{v-1}) = f(z_{v-1}, z_v) = z_v$. Then, according to the path π, by statements (iii) and (iv) of Lemma 1, $f(z_v, z_{v-1}) = z_v$ implies $f(y, z) = y$ and $f(z_{v-1}, z_v) = z_v$ implies $f(z, y) = y$. Therefore, we have $f(y, z) = f(z, y) = y = \text{Proj}\left(x_{\bar{k}}, \langle y, z|G_{\sim}^{A}\rangle\right)$. This completes the verification of the claim.

Claim 3: In situation (3), $f(y, z) = f(z, y) = y_l = \text{Proj}\left(x_{\bar{k}}, \langle y, z|G_{\sim}^{A}\rangle\right)$.

First, Let $(z_1, \ldots, z_q)$ denote the unique path connecting $x_{\bar{k}}$ and y_l in $G_{\sim}^{A}$, which may be a null path if $x_{\bar{k}} = y_l$. We focus on showing $f(y, z) = y_l$. By the symmetric argument, one would immediately conclude that $f(z, y) = y_l$. First, according to the path $\langle y, z|G_{\sim}^{A}\rangle = (y_1, \ldots, y_u)$, statement (i) of Lemma 9 implies $f(y, z) = y_k$ for some $1 \leq k \leq u$. Suppose $k \neq l$. Thus, either $1 \leq k < l$ or $l < k \leq u$ holds. If $1 \leq k < l$, then according to the path $\pi = (z = y_u, \ldots, y_l = z_q, \ldots, z_1 = x_{\bar{k}})$. by statement (ii) of Lemma 9, $f(y, z) = y_k \notin \pi$ implies $f(y, x_{\bar{k}}) = y_k \neq x_{\bar{k}}$, which contradicts Claim 1. Symmetrically, if $l < k \leq u$, according to the path $\pi' = (y = y_1, \ldots, y_l = z_q, \ldots, z_1 = x_{\bar{k}})$, by statement (ii) of Lemma 9, $f(y, z) = y_k \notin \pi'$ implies $f(x_{\bar{k}}, z) = y_k \neq x_{\bar{k}}$, which contradicts Claim 1. Therefore, $f(y, z) = y_l = \text{Proj}\left(x_{\bar{k}}, \langle y, z|G_{\sim}^{A}\rangle\right)$. This completes the verification of the claim.

In conclusion, for all $y, z \in A$, $f(y, z) = f(z, y) = \text{Proj}\left(x_{\bar{k}}, \langle y, z|G_{\sim}^{A}\rangle\right)$. ■

Lemma **14** *Domain $\mathbb{D}$ is a semi-single-peaked domain on the tree $G_{\sim}^{A}$ w.r.t. $x_{\bar{k}}$.*

Proof: Fixing an arbitrary preference $P_i \in \mathbb{D}$, let $r_1(P_i) = x$. First, given $a, b \in \langle x, x_{\bar{k}}|G_{\sim}^{A}\rangle$, let $a \in \langle x, b|G_{\sim}^{A}\rangle$. By Lemma 13, given $P_1 = P_i$, we have $f(P_1, a) = \text{Proj}\left(x_{\bar{k}}, \langle x, a|G_{\sim}^{A}\rangle\right) = a$ and $f(b, a) = \text{Proj}\left(x_{\bar{k}}, \langle b, a|G_{\sim}^{A}\rangle\right) = b$. Then, strategy-proofness implies aP_1b, and we hence have aP_ib, which proves the first condition of Definition 3. Last, given $a \notin \langle x, x_{\bar{k}}|G_{\sim}^{A}\rangle$, let $\text{Proj}\left(a, \langle x, x_{\bar{k}}|G_{\sim}^{A}\rangle\right) = a'$. Given $P_1 = P_i$, we have $f(P_1, a) = \text{Proj}\left(x_{\bar{k}}, \langle x, a|G_{\sim}^{A}\rangle\right) = \text{Proj}\left(a, \langle x, x_{\bar{k}}|G_{\sim}^{A}\rangle\right) = a'$ by Lemma 13, and $f(a, a) = a$ by unanimity. Then, strategy-proofness implies

$a'P_1a$, and we hence have $a'P_ia$, which confirms the second condition of Definition 3. Hence, $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(G_{\sim}^A, x_{\bar{k}})$. This proves Statement (i) of the Theorem. ■

Proof of Statement (ii): (**Sufficiency**) Let $\mathbb{D}$ be a rich (a,b)-semi-hybrid domain on a tree $\mathcal{T}^A$. By Statement (i), to show that there exists no invariant, tops-only and strategy-proof rule, we show that $\mathbb{D}$ is never semi-single-peaked.

Suppose by contradiction that $\mathbb{D}$ is semi-single-peaked, i.e., there exist a tree $\widetilde{\mathcal{T}}^A$ and a threshold $\bar{x}$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, \bar{x})$. Immediately, since $\mathbb{D}$ is path-connected, $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, \bar{x})$ implies $G_{\sim}^A = \widetilde{\mathcal{T}}^A$. Meanwhile, since $\mathbb{D}$ is path-connected and (a,b)-semi-hybrid on $\mathcal{T}^A$, it is also true that $G_{\sim}^{A^{a\rightharpoonup b}} = \mathcal{T}^{A^{a\rightharpoonup b}}$ and $G_{\sim}^{A^{b\rightharpoonup a}} = \mathcal{T}^{A^{b\rightharpoonup a}}$. Thus, according to both $\widetilde{\mathcal{T}}^A$ and $\mathcal{T}^A$ and their relations to $G_{\sim}^A$, we can infer that $\mathcal{T}^A$ and $\widetilde{\mathcal{T}}^A$ induce the same subtrees on the subsets $A^{a\rightharpoonup b}$ and $A^{b\rightharpoonup a}$ respectively, i.e., $\widetilde{\mathcal{T}}^{A^{a\rightharpoonup b}} = \mathcal{T}^{A^{a\rightharpoonup b}}$ and $\widetilde{\mathcal{T}}^{A^{b\rightharpoonup a}} = \mathcal{T}^{A^{b\rightharpoonup a}}$.[64] Furthermore, since $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a connected subgraph and $G_{\sim}^A = \widetilde{\mathcal{T}}^A$, it must be the case that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle} = \widetilde{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle}$ is a subtree nested in $\widetilde{\mathcal{T}}^A$. Thus, $\widetilde{\mathcal{T}}^A$ is a union of the subtrees $\mathcal{T}^{A^{a\rightharpoonup b}}$, $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ and $\mathcal{T}^{A^{b\rightharpoonup a}}$. Since $G_{\sim}^A = \widetilde{\mathcal{T}}^A$ is a tree, we have a unique path $\langle a_1, a_m | G_{\sim}^A\rangle = (x_1, \dots, x_v)$ connecting a_1 and a_m.

Claim 1: Alternative $\bar{x}$ is included in $\langle a_1, a_m | G_{\sim}^A\rangle$, and has at most two neighbors in $\widetilde{\mathcal{T}}^A$.

Claim 2: If $A^{a\rightharpoonup b} \neq \{a\}$, then a has a unique neighbor in the subtree $\mathcal{T}^{A^{a\rightharpoonup b}}$. Symmetrically, if $A^{b\rightharpoonup a} \neq \{b\}$, then b has a unique neighbor in the subtree $\mathcal{T}^{A^{b\rightharpoonup a}}$.

The verification of the two claims follows from Clarifications 1 and 3 in Appendix G respectively.

Henceforth, let $\bar{x} = x_{k^*}$ for some $1 \leq k^* \leq v$, $\bar{a}$ be the unique neighbor of a in $\mathcal{T}^{A^{a\rightharpoonup b}}$, provided $A^{a\rightharpoonup b} \neq \{a\}$, and let $\bar{b}$ be the unique neighbor of b in $\mathcal{T}^{A^{b\rightharpoonup a}}$, provided $A^{b\rightharpoonup a} \neq \{b\}$. There are four cases: (1) $A^{a\rightharpoonup b} \neq \{a\}$ and $A^{b\rightharpoonup a} \neq \{b\}$, (2) $A^{a\rightharpoonup b} = \{a\}$ and $A^{b\rightharpoonup a} \neq \{b\}$, (3) $A^{a\rightharpoonup b} \neq \{a\}$ and $A^{b\rightharpoonup a} = \{b\}$, and (4) $A^{a\rightharpoonup b} = \{a\}$ and $A^{b\rightharpoonup a} = \{b\}$. In each case, we will induce a contradiction.

[64]Note that $A^{a\rightharpoonup b}$ and $A^{b\rightharpoonup a}$ are two subsets identified according to $\mathcal{T}^A$. The notation $\widetilde{\mathcal{T}}^{A^{a\rightharpoonup b}}$ (respectively, $\widetilde{\mathcal{T}}^{A^{b\rightharpoonup a}}$) represents a subgraph of $\widetilde{\mathcal{T}}^A$ induced on the alternatives of $A^{a\rightharpoonup b}$ (respectively, $A^{b\rightharpoonup a}$).

Let case (1) occur. Since $\mathbb{D}$ includes the completely reversed preferences $\underline{P}_i$ and $\overline{P}_i$, it must be the case that either $a_1 \in A^{a \rightharpoonup b}\backslash\{a\}$ and $a_m \in A^{b \rightharpoonup a}\backslash\{b\}$, or $a_m \in A^{a \rightharpoonup b}\backslash\{a\}$ and $a_1 \in A^{b \rightharpoonup a}\backslash\{b\}$ hold. We assume w.l.o.g. that $a_1 \in A^{a \rightharpoonup b}\backslash\{a\}$ and $a_m \in A^{b \rightharpoonup a}\backslash\{b\}$. Then, we have a's unique neighbor $\bar{a}$ in $\mathcal{T}^{A^{a \rightharpoonup b}}$ and b's unique neighbor $\bar{b}$ in $\mathcal{T}^{b \rightharpoonup a}$ by Claim 2. Moreover, since $\mathbb{D}$ is rich (a,b)-semi-hybrid on $\mathcal{T}^A$, it must be the case that all $\bar{a}, a, b$ and $\bar{b}$ are included in the path $\langle a_1, a_m | G_{\sim}^A \rangle = (x_1, \ldots, x_v)$ such that $a = x_s$ and $b = x_t$ for some $1 < s < t < v$, $\bar{a} = x_{s-1}$ and $\bar{b} = x_{t+1}$. Since $\mathbb{D}$ is (a,b)-semi-hybrid on $\mathcal{T}^A$ and semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. x_{k^*}, condition (3) of Definition 4 implies $x_{k^*} \notin \{a, b\}$. Thus, we have three situations $s < k^* < t$, $1 \leq k^* < s$ and $t < k^* \leq v$. In the first situation, we induce a statement that contradicts condition (2) of Definition 4, while in each of the last two situations, we induce a statement that contradicts condition (3) of Definition 4.

Claim 3: The following three statements hold:

(i) If $s < k^* < t$, there exists a tree $\widehat{\mathcal{T}}^A$ such that x_{k^*} and b are two thresholds, $\langle x_{k^*}, b | \widehat{\mathcal{T}}^A \rangle \subset \langle a, b | \mathcal{T}^A \rangle$ and $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, x_{k^*}, b)$.

(ii) If $1 \leq k^* < s$, then $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, a)$.

(iii) If $t < k^* \leq v$, then $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, b)$.

The last two statements are symmetric, and we hence focus on verifying statements (i) and (ii).

First, let $s < k^* < t$. Thus, x_{k^*} and x_{k^*+1} are the exactly two neighbors of x_{k^*} in $\widetilde{\mathcal{T}}^A$ by Claim 1. According to the paths $\pi = (x_{k^*-1}, x_{k^*}, \ldots, x_t, x_{t+1})$ and $\pi' = (x_{k^*}, \ldots, x_t)$ in $G_{\sim}^A = \widetilde{\mathcal{T}}^A$, we identify the following subset of alternatives: $\widetilde{M} = \{x \in A : \text{Proj}(x, \pi) \in \pi'\}$. Clearly, $x_{k^*}, x_{k^*+1}, b \in \widetilde{M}$. Then, we construct a line $(z_1, z_2, \ldots, z_\eta)$ over $\widetilde{M}$, where $z_1 = x_{k^*}$, $z_2 = x_{k^*+1}$ and $z_\eta = b$. Moreover, recall that $\widetilde{\mathcal{T}}^A$ is a union of the subtrees $\mathcal{T}^{A^{a \rightharpoonup b}}$, $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ and $\mathcal{T}^{A^{b \rightharpoonup a}}$. Then, the definition of $\widetilde{M}$ implies $\widetilde{M} \subset \langle a, b | \mathcal{T}^A \rangle$. Next, since (x_{k^*}, x_{k^*+1}) is an edge in $\widetilde{\mathcal{T}}^A$, x_{k^*} and x_{k^*+1} are naturally two thresholds in $\widetilde{\mathcal{T}}^A$. We then identify the subset $\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}} = \left\{x \in A : x_{k^*} \in \langle x, x_{k^*+1} | \widetilde{\mathcal{T}}^A \rangle\right\}$ and the subtree $\widetilde{\mathcal{T}}^{\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}}$.

Last, we construct a new tree $\widehat{\mathcal{T}}^A$ by combining the subtree $\widetilde{\mathcal{T}}^{\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}}$, the line $(z_1, z_2, \dots, z_\eta)$ and subtree $\mathcal{T}^{A^{b \rightharpoonup a}}$.[65] Clearly, z_1 and z_η are two thresholds of $\widehat{\mathcal{T}}^A$. Let $\hat{A}^{z_1 \rightharpoonup z_\eta} = \big\{x \in A : z_1 \in \langle x, z_\eta | \widehat{\mathcal{T}}^A \rangle\big\}$ and $\hat{A}^{z_\eta \rightharpoonup z_1} = \big\{x \in A : z_\eta \in \langle x, z_1 | \widehat{\mathcal{T}}^A \rangle\big\}$. Note that $\hat{A}^{z_1 \rightharpoonup z_\eta} = \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}$, $\widehat{\mathcal{T}}^{\hat{A}^{z_1 \rightharpoonup z_\eta}} = \widetilde{\mathcal{T}}^{\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}}$, $\langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle = \widetilde{M} \subset \langle a, b | \mathcal{T}^A \rangle$, $\hat{A}^{z_\eta \rightharpoonup z_1} = A^{b \rightharpoonup a}$ and $\widehat{\mathcal{T}}^{\hat{A}^{z_\eta \rightharpoonup z_1}} = \mathcal{T}^{A^{b \rightharpoonup a}}$.

Now, we show $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_1, z_\eta)$. Fix an arbitrary preference $P_i \in \mathbb{D}$. We know that either $r_1(P_i) \in \hat{A}^{z_1 \rightharpoonup z_\eta} \backslash \{z_1\}$, or $r_1(P_i) \in \hat{A}^{z_\eta \rightharpoonup z_1} \backslash \{z_\eta\}$, or $r_1(P_i) \in \langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle$ holds. First, let $r_1(P_i) \in \hat{A}^{z_1 \rightharpoonup z_\eta} \backslash \{z_1\}$. It is clear that $r_1(P_i) \notin \hat{A}^{z_\eta \rightharpoonup z_1} \backslash \{z_\eta\} = A^{b \rightharpoonup a} \backslash \{b\}$. Since P_i is semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. $x_{k^*} = z_1$ by the contradictory hypothesis, it is true that P_i is semi-single-peaked on $\widetilde{\mathcal{T}}^{\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}} = \widehat{\mathcal{T}}^{\hat{A}^{z_1 \rightharpoonup z_\eta}}$ w.r.t. z_1, and $z_1 P_i z$ for all $z \in A \backslash \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}} = A \backslash \hat{A}^{z_1 \rightharpoonup z_\eta}$ which further implies $\max^{P_i} \left(\langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle\right) = z_1$. Moreover, since P_i is (a, b)-semi-hybrid on $\mathcal{T}^A$, $r_1(P_i) \notin A^{b \rightharpoonup a} \backslash \{b\}$ implies $\max^{P_i} \left(\hat{A}^{z_\eta \rightharpoonup z_1}\right) = \max^{P_i}(A^{b \rightharpoonup a}) = b = z_\eta$. Therefore, P_i is (z_1, z_η)-semi-hybrid on $\widehat{\mathcal{T}}^A$. Second, let $r_1(P_i) \in \hat{A}^{z_\eta \rightharpoonup z_1} \backslash \{z_\eta\}$. Since P_i is (a, b)-semi-hybrid on $\mathcal{T}^A$, $r_1(P_i) \in \hat{A}^{z_\eta \rightharpoonup z_1} \backslash \{z_\eta\} = A^{b \rightharpoonup a} \backslash \{b\}$ implies that P_i is semi-single-peaked on $\mathcal{T}^{A^{b \rightharpoonup a}} = \widehat{\mathcal{T}}^{\hat{A}^{z_\eta \rightharpoonup z_1}}$ w.r.t. $b = z_\eta$, and $z_\eta P_i z$ for all $z \in A \backslash A^{b \rightharpoonup a} = A \backslash \hat{A}^{z_\eta \rightharpoonup z_1}$ which further implies $\max^{P_i} \left(\langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle\right) = z_\eta$. Moreover, recall that $\widetilde{\mathcal{T}}^A$ is a union of the subtrees $\mathcal{T}^{A^{a \rightharpoonup b}}$, $\underset{\sim}{G}^{\langle a, b | \mathcal{T}^A \rangle}$ and $\mathcal{T}^{A^{b \rightharpoonup a}}$, and $z_1 = x_{k^*} \in \langle a, b | \widetilde{\mathcal{T}}^A \rangle$. Then, $r_1(P_i) \in A^{b \rightharpoonup a} \backslash \{b\} \subset \tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}}$ implies $z_1 = x_{k^*} \in \langle x, r_1(P_i) | \widetilde{\mathcal{T}}^A \rangle$ for all $x \in \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}$. Consequently, since P_i is semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. x_{k^*} by the contradictory hypothesis, we have $\max^{P_i} \left(\hat{A}^{z_1 \rightharpoonup z_\eta}\right) = \max^{P_i} \left(\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}\right) = x_{k^*} = z_1$. Therefore, P_i is (z_1, z_η)-semi-hybrid on $\widehat{\mathcal{T}}^A$. Last, let $r_1(P_i) \in \langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle$. Since P_i is (a, b)-semi-hybrid on $\mathcal{T}^A$, $r_1(P_i) \in \langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle \subset \langle a, b | \mathcal{T} \rangle$ implies $\max^{P_i}(\hat{A}^{z_\eta \rightharpoonup z_1}) = \max^{P_i}(A^{b \rightharpoonup a}) = b = z_\eta$. Meanwhile, $r_1(P_i) \in \langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle \subset \{x_{k^*}\} \cup \tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}}$ implies $z_1 = x_{k^*} \in \langle x, r_1(P_i) | \widetilde{\mathcal{T}}^A \rangle$ for all $x \in \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}$. Consequently, since P_i is semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. x_{k^*} by the contradictory hypothesis, we have $\max^{P_i} \left(\hat{A}^{z_1 \rightharpoonup z_\eta}\right) = \max^{P_i} \left(\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}\right) = x_{k^*} = z_1$. Therefore, P_i is (z_1, z_η)-semi-hybrid on $\widehat{\mathcal{T}}^A$. Overall, we have $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_1, z_\eta) = \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, x_{k^*}, b)$. This completes the

[65] It is easy to show that $A = \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}} \cup \{z_1, z_2, \dots, z_\eta\} \cup A^{b \rightharpoonup a}$, $\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}} \cap \{z_1, z_2, \dots, z_\eta\} = \{z_1\}$, $\{z_1, z_2, \dots, z_\eta\} \cap A^{b \rightharpoonup a} = \{z_\eta\}$ and $\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}} \cap A^{b \rightharpoonup a} = \emptyset$.

verification of statement (i).

Second, let $1 \le k^* < s$. Recall that $\bar{a}$ is the unique neighbor of a in the subtree $\mathcal{T}^{A^{a\rightharpoonup b}} = G_{\sim}^{A^{a\rightharpoonup b}}$. Since $(\bar{a}, a)$ is an edge in $G_{\sim}^{A} = \widetilde{\mathcal{T}}^A$, $\bar{a}$ and a are naturally two thresholds in $\widetilde{\mathcal{T}}^A$. We then define $\tilde{A}^{\bar{a}\rightharpoonup a} = \big\{x \in A : \bar{a} \in \langle x, a|\widetilde{\mathcal{T}}^A\rangle\big\}$ and $\tilde{A}^{a\rightharpoonup \bar{a}} = \big\{x \in A : a \in \langle x, \bar{a}|\widetilde{\mathcal{T}}^A\rangle\big\}$. According to $\mathcal{T}^A$ and $\tilde{\mathcal{T}}^A$ and their relations to $G_{\sim}^A$, it is true that $A^{a\rightharpoonup b} = \tilde{A}^{\bar{a}\rightharpoonup a} \cup \{a\}$ and $\mathcal{T}^{A^{a\rightharpoonup b}} = G_{\sim}^{A^{a\rightharpoonup b}} = G_{\sim}^{\tilde{A}^{\bar{a}\rightharpoonup a}\cup\{a\}} = \widetilde{\mathcal{T}}^{\tilde{A}^{\bar{a}\rightharpoonup a}\cup\{a\}}$. We show $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, a)$. Fix an arbitrary preference $P_i \in \mathbb{D}$. We know that either $r_1(P_i) \in \tilde{A}^{\bar{a}\rightharpoonup a} \cup \{a\}$, or $r_1(P_i) \in \tilde{A}^{a\rightharpoonup \bar{a}}\backslash\{a\}$ holds. First, let $r_1(P_i) \in \tilde{A}^{\bar{a}\rightharpoonup a} \cup \{a\}$. Since P_i is (a, b)-semi-hybrid on $\mathcal{T}^A$, we know that P_i is semi-single-peaked on $\mathcal{T}^{A^{a\rightharpoonup b}} = \widetilde{\mathcal{T}}^{\tilde{A}^{\bar{a}\rightharpoonup a}\cup\{a\}}$ w.r.t. a, and aP_ix for all $x \in A\backslash A^{a\rightharpoonup b} = \tilde{A}^{a\rightharpoonup \bar{a}}\backslash\{a\}$. This implies that P_i is semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. a. Second, let $r_1(P_i) \in \tilde{A}^{a\rightharpoonup \bar{a}}\backslash\{a\}$. Since $1 \le k^* < s$, we know $a = x_s \in \langle x_{k^*}, r_1(P_i)|\widetilde{\mathcal{T}}^A\rangle$. Since P_i is semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. x_{k^*} by the contradictory hypothesis, $a \in \langle x_{k^*}, r_1(P_i)|\widetilde{\mathcal{T}}^A\rangle$ implies that P_i is also semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. a. Overall, $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, a)$. This completes the verification of statement (ii), and proves the claim.

In conclusion, we induce a contradiction for case (1).

Next, let case (2): $A^{a\rightharpoonup b} = \{a\}$ and $A^{b\rightharpoonup a} \neq \{b\}$ occur. Since $\mathbb{D}$ contains the completely reversed preferences $\underline{P}_i$ and $\overline{P}_i$, either $a_1 \in A^{b\rightharpoonup a}\backslash\{b\}$ or $a_m \in A^{b\rightharpoonup a}\backslash\{b\}$ holds. We assume w.l.o.g. that $a_m \in A^{b\rightharpoonup a}\backslash\{b\}$. Then, it must be the case that $a_1 \in \langle a, b|\mathcal{T}^A\rangle\backslash\{b\}$. Recall the path $\langle a_1, a_m|G_{\sim}^A\rangle = (x_1, \dots, x_v)$. Since $\mathbb{D}$ is (a, b)-semi-hybrid on $\mathcal{T}^A$, it is true that b is included in $\langle a_1, a_m|G_{\sim}^A\rangle$, i.e., $b = x_t$ for some $1 < t < v$.[66] Meanwhile, since $\mathbb{D}$ is semi-single-peaked on $\widetilde{\mathcal{T}}^A$ w.r.t. x_{k^*}, condition (3) of Definition 4 implies $t \neq k^*$. Then, one of the following three situations must hold: $1 < k^* < t$, $t < k^* \le v$ and $k^* = 1$. In fact, by the proof of Claim 3, we can induce two statements that respectively contradicts conditions (2) and (3) of Definition 4 in the first two situations here. Hence, we focus on the third situation and induce a statement that contradicts condition (2) of Definition 4.

[66] It is not clear whether a is included in the path $\langle a_1, a_m|G_{\sim}^A\rangle$.

Claim 4: If $k^* = 1$, there exists a tree $\widehat{\mathcal{T}}^A$ such that x_2 and b are two thresholds, $\langle x_2, b|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$, and $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, x_2, b)$.

We first show $a_1 \in Ext(G^A_\sim)$. Suppose by contradiction $x_1 = a_1 \notin Ext(G^A_\sim)$. Since $x_1 \sim x_2$, $x_1 \notin Ext(G^A_\sim)$ implies that there exists $x \in A$ such that $x \neq x_2$ and $x \sim a_1$. Since $G^A_\sim = \widetilde{\mathcal{T}}^A$ is a tree, x must not be included in the path $\langle a_1, a_m|G^A_\sim\rangle = \langle a_1, a_m|\widetilde{\mathcal{T}}^A\rangle$. Hence, $(x, x_1, \ldots, x_v)$ is the unique path in $\widetilde{\mathcal{T}}^A$ that connects x and a_m. Consequently, the contradictory hypothesis $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, x_1)$ implies $x_1 P_i x$ for all $P_i \in \mathbb{D}^{a_m}$, which contradicts the fact that $x_1 = a_1$ is bottom ranked in the preference $\overline{P}_i \in \mathbb{D}^{a_m}$. Therefore, $a_1 \in Ext(G^A_\sim)$. Then, x_2 is unique neighbor of a_1 in $G^A_\sim = \widetilde{\mathcal{T}}^A$. Consequently, extreme-vertex symmetry and the contradictory hypothesis $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, x_1)$ together imply $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, x_2)$.[67]

Now, we construct a line $(z_1, z_2 \ldots, z_\eta)$ over the alternatives of $\langle a, b|\mathcal{T}^A\rangle$ such that $z_1 = a_1$, $z_2 = x_2$ and $z_\eta = b$. Furthermore, we construct a tree $\widehat{\mathcal{T}}^A$ by combining the line $(z_1, z_2, \ldots, z_\eta)$ and the subtree $\mathcal{T}^{A^{b\rightharpoonup a}}$. Clearly, z_2 and z_η are two thresholds of $\widehat{\mathcal{T}}^A$ and $\langle z_2, z_\eta|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$. Let $\hat{A}^{z_2\rightharpoonup z_\eta} = \{x \in A : z_2 \in \langle x, z_\eta|\widehat{\mathcal{T}}^A\rangle\}$ and $\hat{A}^{z_\eta\rightharpoonup z_2} = \{x \in A : z_\eta \in \langle x, z_2|\widehat{\mathcal{T}}^A\rangle\}$. It is evident that $\hat{A}^{z_2\rightharpoonup z_\eta} = \{a_1, z_2\}$, $\hat{A}^{z_\eta\rightharpoonup z_2} = A^{b\rightharpoonup a}$ and $\widehat{\mathcal{T}}^{\hat{A}^{z_\eta\rightharpoonup z_2}} = \mathcal{T}^{A^{b\rightharpoonup a}}$. We last show $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_2, z_\eta)$. Fix an arbitrary preference $P_i \in \mathbb{D}$. First, let $r_1(P_i) \in \hat{A}^{z_2\rightharpoonup z_\eta}\backslash\{z_2\}$. Then, $r_1(P_i) = a_1 \in \langle a, b|\mathcal{T}^A\rangle$, $P_i \in \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, z_2)$ implies $r_2(P_i) = z_2$, and $P_i \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ implies $\max^{P_i}(\hat{A}^{z_\eta\rightharpoonup z_2}) = \max^{P_i}(A^{b\rightharpoonup a}) = b$. Therefore, $P_i \in \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_2, z_\eta)$. Second, let $r_1(P_i) \in \hat{A}^{z_\eta\rightharpoonup z_2}\backslash\{z_\eta\}$. Then, $r_1(P_i) \neq a_1$, $r_1(P_i) \in A^{b\rightharpoonup a}\backslash\{b\}$, $P_i \in \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, z_2)$ implies $z_2 P_i a_1$ and hence $\max^{P_i}(\hat{A}^{z_2\rightharpoonup z_\eta}) = z_2$, and $P_i \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ implies that P_i is semi-single-peaked on $\mathcal{T}^{A^{b\rightharpoonup a}} = \widehat{\mathcal{T}}^{A^{z_\eta\rightharpoonup z_2}}$ w.r.t. $b = z_\eta$ and $\max^{P_i}(\langle a, b|\mathcal{T}^A\rangle) = \max^{P_i}(\{z_1, z_2, \ldots, z_\eta\}) = b = z_\eta$ which implies $\max^{P_i}(\langle z_2, z_\eta|\widehat{\mathcal{T}}^A\rangle) = z_\eta$. Therefore, $P_i \in \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_2, z_\eta)$. Last, let $r_1(P_i) \in \langle z_2, z_\eta|\widehat{\mathcal{T}}^A\rangle$. Then, $r_1(P_i) \neq a_1$, $r_1(P_i) \in \langle a, b|\mathcal{T}^A\rangle$, $P_i \in \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, z_2)$ implies $z_2 P_i a_1$ and hence $\max^{P_i}(\hat{A}^{z_2\rightharpoonup b}) = z_2$, and $P_i \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ implies $\max^{P_i}(\hat{A}^{z_\eta\rightharpoonup z_2}) = \max^{P_i}(A^{b\rightharpoonup a}) = b = z_\eta$. Therefore, $P_i \in \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_2, z_\eta)$. Overall, $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_2, z_\eta) = \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, x_2, b)$. This completes the verification of the claim.

[67] The detailed verification follows from Clarification 2 in Appendix G.

In conclusion, we induce a contradiction for case (2).

Symmetrically, we can induce a contradiction for case (3).

Last, let case (4): $A^{a \rightharpoonup b} = \{a\}$ and $A^{b \rightharpoonup a} = \{b\}$ occur. Thus, $\langle a, b | \mathcal{T}^A \rangle$ is a line which contains all alternatives of A. Recall the path $\langle a_1, a_m | G^A_{\sim} \rangle = \langle a_1, a_m | \widetilde{\mathcal{T}}^A \rangle = (x_1, \ldots, x_v)$ and the contradictory hypothesis $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, x_{k^*})$.

By the proof of statement (iii) of Claim 4, we infer that $x_{k^*} \neq a_1$ and analogously $x_{k^*} \neq a_m$. Hence, $1 < k^* < v$, which by Claim 1 implies that x_{k^*-1} and x_{k^*+1} are the exactly two neighbors of x_{k^*} in $G^A_{\sim} = \widetilde{\mathcal{T}}^A$. Since (x_{k^*}, x_{k^*+1}) is an edge in $\widetilde{\mathcal{T}}^A$, x_{k^*} and x_{k^*+1} are two thresholds in $\widetilde{\mathcal{T}}^A$. Let $\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}} = \big\{x \in A : x_{k^*} \in \langle x, x_{k^*+1} | \widetilde{\mathcal{T}}^A \rangle\big\}$ and $\tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}} = \big\{x \in A : x_{k^*+1} \in \langle x, x_{k^*} | \widetilde{\mathcal{T}}^A \rangle\big\}$. We construct a line $(z_1, z_2, \ldots, z_\eta)$ over $\{x_{k^*}\} \cup \tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}}$ where $z_1 = x_{k^*}$ and $z_2 = x_{k^*+1}$. Furthermore, we construct a new tree $\widehat{\mathcal{T}}^A$ by combining the subtree $\widetilde{\mathcal{T}}^{\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}}$ and the line $(z_1, z_2, \ldots, z_\eta)$. Clearly, z_1 and z_η are two thresholds in $\widehat{\mathcal{T}}^A$. Let $\hat{A}^{z_1 \rightharpoonup z_\eta} = \big\{x \in A : z_1 \in \langle x, z_\eta | \widehat{\mathcal{T}}^A \rangle\big\}$ and $\hat{A}^{z_\eta \rightharpoonup z_1} = \big\{x \in A : z_\eta \in \langle x, z_1 | \widehat{\mathcal{T}}^A \rangle\big\}$. Clearly, $\hat{A}^{z_1 \rightharpoonup z_\eta} = \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}$, $\widehat{\mathcal{T}}^{\hat{A}^{z_1 \rightharpoonup z_\eta}} = \widetilde{\mathcal{T}}^{\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}}$, $\langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle = \{x_{k^*}\} \cup \tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}} \subset A = \langle a, b | \mathcal{T}^A \rangle$ and $\hat{A}^{z_\eta \rightharpoonup z_1} = \{z_\eta\}$. Then, we induce $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_1, z_\eta)$, which contradicts condition (2) of Definition 4.

Fix $P_i \in \mathbb{D}$. Clearly, $\hat{A}^{z_\eta \rightharpoonup z_1} = \{z_\eta\}$ implies $\max^{P_i}(\hat{A}^{z_\eta \rightharpoonup z_1}) = z_\eta$ and $\hat{A}^{z_\eta \rightharpoonup z_1} \backslash \{z_\eta\} = \emptyset$. First, let $r_1(P_i) \in \hat{A}^{z_1 \rightharpoonup z_\eta} \backslash \{z_1\}$. Thus, $r_1(P_i) \in \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}} \backslash \{x_{k^*}\}$, which further implies $x_{k^*} \in \langle x, r_1(P_i) | \widetilde{\mathcal{T}}^A \rangle$ for all $x \in \tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}}$. Since $P_i \in \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, x_{k^*})$ by the contradictory hypothesis, we know that P_i is semi-single-peaked on the subtree $\widetilde{\mathcal{T}}^{\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}} = \widehat{\mathcal{T}}^{\hat{A}^{z_1 \rightharpoonup z_\eta}}$ w.r.t. $x_{k^*} = z_1$, and $x_{k^*} P_i x$ for all $x \in \tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}} = \{z_2, \ldots, z_\eta\}$ which implies $z_1 = \max^{P_i}\big(\langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle\big)$. Therefore, $P_i \in \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_1, z_\eta)$. Second, let $r_1(P_i) \in \langle z_1, z_\eta | \widehat{\mathcal{T}}^A \rangle$. Thus, $r_1(P_i) \in \{x_{k^*}\} \cup \tilde{A}^{x_{k^*+1} \rightharpoonup x_{k^*}}$, which further implies $x_{k^*} \in \langle x, r_1(P_i) | \widetilde{\mathcal{T}}^A \rangle$ for all $x \in \tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}$. Then, the contradictory hypothesis $P_i \in \mathbb{D}_{\text{SSP}}(\widetilde{\mathcal{T}}^A, x_{k^*})$ implies $z_1 = x_{k^*} = \max^{P_i}\big(\tilde{A}^{x_{k^*} \rightharpoonup x_{k^*+1}}\big) = \max^{P_i}\big(\hat{A}^{z_1 \rightharpoonup z_\eta}\big)$. Therefore, $P_i \in \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_1, z_\eta)$. Overall, we have shown $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, z_1, z_\eta)$, as required.

In conclusion, we have induced a contradiction for each one of the four cases. This implies that the contradictory hypothesis that $\mathbb{D}$ is semi-single-peaked cannot hold. Hence, $\mathbb{D}$ is never semi-single-peaked, as required. This completes the verification of the sufficiency part of Statement (ii).

(**Necessity**) Now, we turn to the verification of the necessity part of Statement (ii). Let $\mathbb{D}$ be a rich domain, and admit no invariant, tops-only and strategy-proof rule. Since $\mathbb{D}$ is a rich non-dictatorial domain, Corollary 1 implies that $\mathbb{D}$ satisfies the unique seconds property. Hence, we force on showing that $\mathbb{D}$ is semi-hybrid.

The proof consists of five steps: (1) we construct a line over all alternatives that are involved in the path(s) connecting a_1 and a_m in the adjacency graph $G_{\sim}^{A}$ (see all proofs before Lemma 15), fix an arbitrary two-voter tops-only and strategy-proof rule $f : \mathbb{D}^2 \to A$, and partially characterize f according to the constructed line (see Lemma 15), (2) we construct a tree $\mathcal{T}_f^A$ using $G_{\sim}^{A}$ and the partial characterization of f in the first step (see Lemmas 16, 17 and 18), (3) we completely characterize f using the constructed tree $\mathcal{T}_f^A$ (see Lemmas 19 and 20) and elicit preference restrictions via strategy-proofness of f, (4) we apply the second and third steps to all two-voter tops-only and strategy-proof rules, elicit all preference restrictions via strategy-proofness of all rules (see Observation 3), identify two particular rules and specify their relations to other rules and to each other (see Lemmas 22 and 23), and (5) we aggregate the trees associated to the two identified rules to formulate a tree $\mathcal{T}^A$, identify two thresholds a and b in $\mathcal{T}^A$, and aggregate the elicited preference restrictions from these two rules to show that $\mathbb{D}$ is semi-hybrid on $\mathcal{T}^A$ w.r.t. a and b (see Lemma 24).

Consider the set of paths connecting a_1 and a_m in $G_{\sim}^{A}$, denoted by $\Pi(a_1, a_m)$. Since $G_{\sim}^{A}$ is a connected graph, $\Pi(a_1, a_m) \neq \emptyset$. There are two cases: $|\Pi(a_1, a_m)| = 1$ and $|\Pi(a_1, a_m)| > 1$.

First, assume $|\Pi(a_1, a_m)| = 1$. Thus, let $\mathcal{L} = (x_1, \dots, x_v)$ be the unique path in $G_{\sim}^{A}$ that connects a_1 and a_m.

Next, assume $|\Pi(a_1, a_m)| > 1$. Since all paths of $\Pi(a_1, a_m)$ start from a_1 and end at a_m, we can identify two distinct alternatives $x, y \in A$ such that the following four conditions are satisfied: (i) $x, y \in \pi$ for all $\pi \in \Pi(a_1, a_m)$, (ii) for all $\pi \in \Pi(a_1, a_m)$, $x \in \langle a_1, y|\pi\rangle$ and $y \in \langle x, a_m|\pi\rangle$, (iii) for all distinct $\pi, \pi' \in \Pi(a_1, a_m)$,

$\langle a_1, x|\pi\rangle = \langle a_1, x|\pi'\rangle$ and $\langle y, a_m|\pi\rangle = \langle y, a_m|\pi'\rangle$, and (iv) there exists a path $\pi \in \Pi(a_1, a_m)$ such that $|\langle x, y|\pi\rangle| \geq 3$. We collect all alternatives that are involved in the paths of $\Pi(a_1, a_m)$, i.e., let $\hat{A} = \{a \in A : a \in \pi \text{ for some } \pi \in \Pi(a_1, a_m)\}$. Note that $\hat{A}$ may not include all alternatives of A, and all paths of $\Pi(a_1, a_m)$ are included in $G_{\sim}^{\hat{A}}$. More important, we make three important observations on the adjacency graph $G_{\sim}^{\hat{A}}$: (i) $G_{\sim}^{\hat{A}}$ is a connected graph, (ii) there exists a unique path, denoted π^L, in $G_{\sim}^{\hat{A}}$ connecting a_1 and x, and there exists a unique path, denoted π^R, in $G_{\sim}^{\hat{A}}$ connecting y and a_m, and (iii) the set $\mathcal{O} = \{a \in \hat{A} : a \notin \pi^L \cup \pi^R\} \cup \{x, y\}$ contains at least three alternatives, $G_{\sim}^{\mathcal{O}}$ is a connected graph and has no extreme vertex (see the first diagram of Figure 14). Furthermore, we arrange all alternatives of $\mathcal{O}$ on a line, denoted $(x, \dots, y)$, and combine π^L, $(x, \dots, y)$ and π^R to construct a line $\mathcal{L} = (x_1, \dots, x_s, \dots, x_t, \dots, x_v)$, where $v = |\hat{A}|$, $1 \leq s < t \leq v$, $t - s > 1$, $x_1 = a_1$, $x_v = a_m$, $x_s = x$, $x_t = y$, $(x_1, \dots, x_s) = \pi^L$, $(x_s, \dots, x_t) = (x, \dots, y)$ and $(x_t, \dots, x_v) = \pi^R$ (see the second diagram of Figure 14).[68]

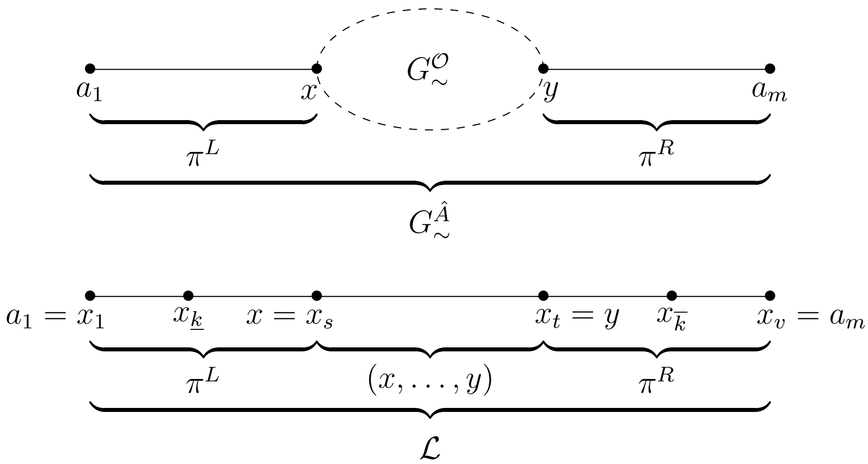


Figure 13: Adjacency graph $G_{\sim}^{\hat{A}}$ and the constructed line $\mathcal{L}$

Note that the set of two-voter tops-only and strategy-proof rules defined on $\mathbb{D}$ is not empty as the dictatorships are always included. We fix an arbitrary two-

[68] We intentionally make the notation of the constructed line $\mathcal{L}$ in the case $|\Pi(a_1, a_m)| > 1$ identical to the line $\mathcal{L}$ in the case $|\Pi(a_1, a_m)| = 1$. This helps us unify the henceforth proof for both cases, and does not create any loss of generality. Note that the line $\mathcal{L}$ in the case $|\Pi(a_1, a_m)| = 1$ is also a path in $G_{\sim}^{A}$, while the constructed line $\mathcal{L}$ in the case $|\Pi(a_1, a_m)| > 1$ *may not be a path in* $G_{\sim}^{A}$. Note that the line $\mathcal{L}$ is independent of the rule f.

voter tops-only and strategy-proof rule $f : \mathbb{D}^2 \to A$. Since $\mathbb{D}$ does not admit an invariant, tops-only and strategy-proof rule, we have $f(\underline{P}_1, \overline{P}_2) \neq f(\overline{P}_1, \underline{P}_2)$. In the case $|\Pi(a_1, a_m)| = 1$, by statement (i) of Lemma 9, we know $f(x_1, x_v) = x_{\underline{k}}$ and $f(x_v, x_1) = x_{\overline{k}}$ for some distinct $1 \leq \underline{k}, \overline{k} \leq v$. Since $f(\underline{P}_1, \overline{P}_2) \neq f(\overline{P}_1, \underline{P}_2)$, it is clear that $\underline{k} \neq \overline{k}$. We assume w.l.o.g. that $\underline{k} < \overline{k}$, which then by statement (i) of Lemma 10 implies that voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$.

In the case $|\Pi(a_1, a_m)| > 1$, since $G_{\sim}^{\mathcal{O}}$ is a connected graph and has no extreme vertex, statement (i) of Lemma 4 implies that f behaves like a dictatorship on $\mathcal{O} = \langle x_s, x_t | \mathcal{L}\rangle$. We assume that voter 1 dictates on $\langle x_s, x_t | \mathcal{L}\rangle$, i.e., $f(x_k, x_{k'}) = x_k$ for all $s \leq k, k' \leq t$.[69] Furthermore, we show $f(x_1, x_v) = x_{\underline{k}}$ for some $1 \leq \underline{k} \leq s$ and $f(x_v, x_1) = x_{\overline{k}}$ for some $t \leq \overline{k} \leq v$. By voter 1's dictatorship on $\langle x_s, x_t | \mathcal{L}\rangle$, we have $f(x_s, x_t) = x_s$ and $f(x_t, x_s) = x_t$. Then, according to the paths $(x_t, \dots, x_v)$ and $(x_1, \dots, x_s)$, statement (ii) of Lemma 9 implies $f(x_s, x_v) = x_s$ and $f(x_t, x_1) = x_t$. Moreover, according to the paths $(x_1, \dots, x_s)$ and $(x_t, \dots, x_v)$, by statement (iii) of Lemma 9, $f(x_s, x_v) = x_s$ implies $f(x_1, x_v) = x_{\underline{k}}$ for some $1 \leq \underline{k} \leq s$, and $f(x_t, x_1) = x_t$ implies $f(x_v, x_1) = x_{\overline{k}}$ for some $t \leq \overline{k} \leq v$.

The next lemma provide a unified characterization of f on the line $\mathcal{L}$ in both cases of $|\Pi(a_1, a_m)| = 1$ and $|\Pi(a_1, a_m)| > 1$.

Lemma **15** *According to the line $\mathcal{L}$, for all $1 \leq k, k' \leq v$, we have*

$$f(x_k, x_{k'}) = \begin{cases} x_k & \text{if } \underline{k} \leq k \leq \overline{k}, \\ x_{\mathrm{med}(k,k',\underline{k})} & \text{if } k < \underline{k}, \\ x_{\mathrm{med}(k,k',\overline{k})} & \text{if } k > \overline{k}. \end{cases}$$

Proof: If $|\Pi(a_1, a_m)| = 1$, the lemma follows exactly from statement (i) of Lemma 10. Henceforth, we assume $|\Pi(a_1, a_m)| > 1$.

Given an arbitrary path $\pi \in \Pi(a_1, a_m)$, since $f(a_1, a_m) = f(x_1, x_v) = x_{\underline{k}} \in \langle a_1, x | \pi\rangle$ and $f(a_m, a_1) = f(x_v, x_1) = x_{\overline{k}} \in \langle y, a_m | \pi\rangle$, it is true that statement (i) of Lemma 10 holds on π. To prove the lemma, we fix an arbitrary profile $(x_k, x_{k'})$.

[69] In the case $|\Pi(a_1, a_m)| = 1$, we assume that voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$. Here, we also assume that voter 1 dictates on $\langle x_s, x_t | \mathcal{L}\rangle$. This helps us unify the henceforth proof for both cases, and does not create any loss of generality.

First, let $\underline{k} \leq k \leq \overline{k}$. If $\underline{k} \leq k' \leq \overline{k}$, then by voter 1's dictatorship on $\langle x_k, x_{k'} | \mathcal{L} \rangle$, we have $f(x_k, x_{k'}) = x_k$. If $k' < \underline{k}$ or $k' > \overline{k}$, we know that there exists a path $\pi \in \Pi(a_1, a_m)$ which includes both x_k and $x_{k'}$. Clearly, $x_k \in \langle x_{\underline{k}}, x_{\overline{k}} | \pi \rangle$. Then, statement (i) of Lemma 10 on the path π implies $f(x_k, x_{k'}) = x_k$. Overall, $f(x_k, x_{k'}) = x_k$, as required.

Second, let $k < \underline{k}$. Then, there exists a path $\pi \in \Pi(a_1, a_m)$ which includes both x_k and $x_{k'}$. Clearly, $x_k \in \langle x_1, x_{\underline{k}} | \pi \rangle$. Then, statement (i) of Lemma 10 the path π implies $f(x_k, x_{k'}) = x_{\text{med}(k, k', \underline{k})}$, as required.

Last, let $k > \overline{k}$. Then, there exists a path $\pi \in \Pi(a_1, a_m)$ which includes both x_k and $x_{k'}$. Clearly, $x_k \in \langle x_{\overline{k}}, x_v | \pi \rangle$. Then, statement (i) of Lemma 10 the path π implies $f(x_k, x_{k'}) = x_{\text{med}(k, k', \overline{k})}$, as required. This proves the lemma. ■

Lemma **16** *Fixing an alternative $z \in A$ and a path $\pi = (z_1, \ldots, z_{s-1}, z_s)$ in $G^A_\sim$, the following two statements hold:*

(i) *If $z_1 = z$, $z_{s-1} = x_{\underline{k}-1}$ and $z_s = x_{\underline{k}}$, then π is the unique path in $G^A_\sim$ connecting z and $x_{\underline{k}}$.*

(ii) *If $z_1 = z$, $z_{s-1} = x_{\overline{k}+1}$ and $z_s = x_{\overline{k}}$, then π is the unique path in $G^A_\sim$ connecting z and $x_{\overline{k}}$.*

Proof: The two statements are symmetric, and we hence focus on the verification of the first one. Suppose that there exists another path $\pi' = (y_1, \ldots, y_t)$ in $G^A_\sim$ connecting z and $x_{\underline{k}}$. Then, we can identify a cycle $\mathcal{C}$ in $G^A_\sim$ such that (i) $\mathcal{C} \subseteq \pi \cup \pi'$, (ii) $\pi \cap \mathcal{C} \neq \emptyset$ and (iii) every edge in $\mathcal{C}$ is an edge in π or π'. Clearly, by Observation 1, f behaves like a dictatorship on $\mathcal{C}$. Given $\pi \cap \mathcal{C} \neq \emptyset$, we identify the alternative in $\pi \cap \mathcal{C}$ that has the maximum index, i.e., $z_{k^*} \in \pi \cap \mathcal{C}$ and $z_k \notin \mathcal{C}$ for all $k > k^*$. We consider two cases: $k^* = s$ and $k^* < s$.

In the first case, we first show that $x_{\underline{k}-1}$ is also included in $\mathcal{C}$. On the one hand, since $x_{\underline{k}} = z_{k^*} \in \mathcal{C}$, $x_{\underline{k}}$ has two distinct neighbors in $\mathcal{C}$. On the other hand, note that $x_{\underline{k}}$ has a unique neighbor in π, which is $x_{\underline{k}-1}$, and a unique neighbor in π'. Moreover, since every edge in $\mathcal{C}$ is an edge in π or π', it must be true that

$x_{\underline{k}-1}$ is included in $\mathcal{C}$. Recall that voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$ by Lemma 15. Note that both $\mathcal{C}$ and $G_{\sim}^{\langle x_{\underline{k}}, x_{\overline{k}}|\mathcal{L}\rangle}$ are connected graphs and each has at least two alternatives. Then, by $x_{\underline{k}} \in \mathcal{C} \cap \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$, the proof of Lemma 2 implies that voter 1 dictates on $\mathcal{C} \cup \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$. Thus, we have $f(x_{\underline{k}-1}, x_{\underline{k}}) = x_{\underline{k}-1} \neq x_{\text{med}(\underline{k}-1, \underline{k}, \underline{k})}$ which contradicts Lemma 15. In the second case, we have the path $(z_{k^*}, \dots, z_{s-1}, z_s)$ which connects the cycle $\mathcal{C}$ and the connected graph $G_{\sim}^{\langle x_{\underline{k}}, x_{\overline{k}}|\mathcal{L}\rangle}$. Since voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$ by Lemma 15, the proof of Lemma 2 implies that voter 1 dictates on $\{z_{k^*}, \dots, z_{s-1}, z_s\}$. Thus, we have $f(x_{\underline{k}-1}, x_{\underline{k}}) = f(z_{s-1}, z_s) = z_{s-1} = x_{\underline{k}-1} \neq x_{\text{med}(\underline{k}-1, \underline{k}, \underline{k})}$, which contradicts Lemma 15. Therefore, π is the unique path in $G_{\sim}^{A}$ connecting z and $x_{\underline{k}}$. ■

We construct the following five sets:

$$\underline{B} = \left\{ z \in A : \begin{array}{l} \text{there exists a path } (z_1, \dots, z_{s-1}, z_s) \text{ in } G_{\sim}^{A} \\ \text{connecting } z \text{ and } x_{\underline{k}} \text{ such that } z_{s-1} = x_{\underline{k}-1} \end{array} \right\},$$

$$\overline{B} = \left\{ z \in A : \begin{array}{l} \text{there exists a path } (z_1, \dots, z_{s-1}, z_s) \text{ in } G_{\sim}^{A} \\ \text{connecting } z \text{ and } x_{\overline{k}} \text{ such that } z_{s-1} = x_{\overline{k}+1} \end{array} \right\},$$

$$\underline{A} = \underline{B} \cup \{x_{\underline{k}}\}, \ \overline{A} = \overline{B} \cup \{x_{\overline{k}}\} \text{ and } M = \left\{z \in A : z \notin \underline{B} \cup \overline{B}\right\}.$$

Lemma 17 *We have* $G_{\sim}^{A} = G_{\sim}^{\underline{A}} \cup G_{\sim}^{M} \cup G_{\sim}^{\overline{A}}$.

Proof: First, it is clear that $A = \underline{A} \cup M \cup \overline{A}$ and $\mathcal{E}_{\sim}^{A} \supseteq \mathcal{E}_{\sim}^{\underline{A}} \cup \mathcal{E}_{\sim}^{M} \cup \mathcal{E}_{\sim}^{\overline{A}}$. To complete the proof, we show $\mathcal{E}_{\sim}^{A} = \mathcal{E}_{\sim}^{\underline{A}} \cup \mathcal{E}_{\sim}^{M} \cup \mathcal{E}_{\sim}^{\overline{A}}$. It suffices to show that in $G_{\sim}^{A}$, no alternative of $\underline{B}$ is adjacent to any alternative not in $\underline{A}$, and no alternative of $\overline{B}$ is adjacent to any alternative not in $\overline{A}$.

Given $z \in \underline{B}$ and $y \notin \underline{A}$, we show $(z, y) \notin \mathcal{E}_{\sim}^{A}$. Suppose not, i.e., $z \sim y$. By Lemma 16, let $(z_1, \dots, z_{s-1}, z_s)$ denote the unique path in $G_{\sim}^{A}$ that connects z and $x_{\underline{k}}$. Thus, $z_1, \dots, z_{s-1} \in \underline{B}$ and hence $y \notin \{z_1, \dots, z_{s-1}\}$. Clearly, $y \notin \underline{A}$ implies $y \neq x_{\underline{k}} = z_s$. Therefore, we can construct a path $(y, z_1, \dots, z_{s-1}, z_s)$ in $G_{\sim}^{A}$ that connects y and $x_{\underline{k}}$. Since $z_{s-1} = x_{\underline{k}-1}$, the definition of $\underline{B}$ implies $y \in \underline{B} \subset \underline{A}$. Contradiction! Therefore, $(z, y) \notin \mathcal{E}_{\sim}^{A}$. Symmetrically, given $z' \in \overline{B}$ and $y' \notin \overline{A}$, we have $(z', y') \notin \mathcal{E}_{\sim}^{A}$. ■

Lemma **18** *Adjacency graphs* $G^{\underline{A}}_{\sim}$ *and* $G^{\overline{A}}_{\sim}$ *are two trees. Moreover, we have* $\big[\underline{A} \neq \{x_{\underline{k}}\}\big] \Rightarrow \big[x_{\underline{k}} \in Ext\big(G^{\underline{A}}_{\sim}\big)\big]$ *and* $\big[\overline{A} \neq \{x_{\overline{k}}\}\big] \Rightarrow \big[x_{\overline{k}} \in Ext\big(G^{\overline{A}}_{\sim}\big)\big]$.

Proof: If $\underline{A} = \{x_{\underline{k}}\}$ is a singleton set, and vacuously $G^{\underline{A}}_{\sim}$ is a tree. Next, assume $\underline{A} \neq \{x_{\underline{k}}\}$. Given an arbitrary alternative $z \in \underline{B}$, by statement (i) of Lemma 16, we have a unique path $(z_1, \ldots, z_{s-1}, z_s)$ in $G^{A}_{\sim}$ connecting z and $x_{\underline{k}}$. Moreover, we know $\{z_1, \ldots, z_{s-1} = x_{\underline{k}-1}\} \subseteq \underline{B}$ by the definition of $\underline{B}$. Therefore, there exists a unique path in $G^{\underline{A}}_{\sim}$ connecting z and $x_{\underline{k}}$. Hence, $G^{\underline{A}}_{\sim}$ is a tree. Furthermore, given $\underline{A} \neq \{x_{\underline{k}}\}$, we notice that $x_{\underline{k}-1} \sim x_{\underline{k}}$ which implies $x_{\underline{k}-1} \in \underline{B}$, and furthermore by the definition of $\underline{B}$ and statement (i) of Lemma 16, there exists no $z \in \underline{B}\backslash\{x_{\underline{k}-1}\}$ such that $z \sim x_{\underline{k}}$. Therefore, $x_{\underline{k}} \in Ext\big(G^{\underline{A}}_{\sim}\big)$. Symmetrically, $G^{\overline{A}}_{\sim}$ is also a tree and $\big[\overline{A} \neq \{x_{\overline{k}}\}\big] \Rightarrow \big[x_{\overline{k}} \in Ext\big(G^{\overline{A}}_{\sim}\big)\big]$. ■

Now, we arrange all alternatives of M on a line, denoted $(x_{\underline{k}}, \ldots, x_{\overline{k}})$. Then, by Lemmas 17 and 18, we combine the tree $G^{\underline{A}}_{\sim}$, the line $(x_{\underline{k}}, \ldots, x_{\overline{k}})$ and the tree $G^{\overline{A}}_{\sim}$ to construct a tree $\mathcal{T}^{A}_{f}$. Clearly, $\mathcal{T}^{\underline{A}}_{f} = G^{\underline{A}}_{\sim}$, $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}^{A}_{f}\rangle = (x_{\underline{k}}, \ldots, x_{\overline{k}})$ and $\mathcal{T}^{\overline{A}}_{f} = G^{\overline{A}}_{\sim}$. By construction, $x_{\underline{k}}$ and $x_{\overline{k}}$ are two thresholds in $\mathcal{T}^{A}_{f}$. Hence, we have $A^{x_{\underline{k}} \rightharpoonup x_{\overline{k}}} = \underline{A}$ and $A^{x_{\overline{k}} \rightharpoonup x_{\underline{k}}} = \overline{A}$. In the rest of proof, for notational convenience, we use the notation $\underline{A}$ and $\overline{A}$, instead of $A^{x_{\underline{k}} \rightharpoonup x_{\overline{k}}}$ and $A^{x_{\overline{k}} \rightharpoonup x_{\underline{k}}}$.

The next lemma shows that voter 1 dictates on $M = \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}^{A}_{f}\rangle$.

Lemma **19** *We have* $f(z, z') = z$ *for all* $z, z' \in \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}^{A}_{f}\rangle$.

Proof: We first show $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle \subseteq M$. Suppose that there exists $z \in \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle \cap \underline{B}$. By definition, $x_{\underline{k}-1} \notin \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$ and $z \neq x_{\underline{k}}$. On the one hand, by the definition of $\underline{B}$, statement (i) of Lemma 16 and Lemma 17, we have a path $\pi = (z_1, \ldots, z_{s-1}, z_s)$ in $G^{\underline{A}}_{\sim}$ that connects z and $x_{\underline{k}}$. Moreover, we have $z_{s-1} = x_{\underline{k}-1}$. On the other hand, since $G^{\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle}_{\sim}$ is a connected graph, there exists a path $\pi' = (z'_1, \ldots, z'_{t-1}, z'_t)$ in $G^{\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle}_{\sim}$ connecting z and $x_{\underline{k}}$. Clearly, $z_{t-1'} \neq x_{\underline{k}-1}$. Thus, according to paths π and π', we identify two distinct neighbors for $x_{\underline{k}}$. Consequently, since both paths connect z and $x_{\underline{k}}$, we infer that there exists a cycle $\mathcal{C}$ such that $\mathcal{C} \subseteq \pi \cup \pi'$ and $x_{\underline{k}-1}, x_{\underline{k}} \in \mathcal{C}$. Since voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L}\rangle$ by

Lemma 15, by $x_{\underline{k}} \in \mathcal{C} \cap \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle$, the proof of Lemma 2 implies that voter 1 dictates on $\mathcal{C}$. Consequently, we have $f(x_{\underline{k}-1}, x_{\underline{k}}) = x_{\underline{k}-1} \neq x_{\text{med}(\underline{k}-1, \underline{k}, \underline{k})}$ which contradicts Lemma 15. Therefore, $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cap \underline{B} = \emptyset$. Symmetrically, $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cap \overline{B} = \emptyset$. Therefore, it is true that $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \subseteq M$.

Since $G_{\sim}^{A}$ is a connected graph and both $G_{\sim}^{\underline{A}}$ and $G_{\sim}^{\overline{A}}$ are trees, Lemma 17 implies that $G_{\sim}^{M} = G_{\sim}^{\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle}$ must be a connected graph. Thus, for each $z \in \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle$, there exists a path in $G_{\sim}^{\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle}$ that connects $x_{\underline{k}}$ and z. Recall that voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle$ by Lemma 15 and $G_{\sim}^{\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle}$ is a connected graph. To complete the proof, by Lemma 2, it suffices to show that given an arbitrary $z \in \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle \backslash \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle$ and a path $(z_1, \ldots, z_t)$ in $G_{\sim}^{\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle}$ that connects $x_{\underline{k}}$ and z, voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_t\}$.

Since $z_1 = x_{\underline{k}}$, we have $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1\} = \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle$. Therefore, by Lemma 15, voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1\}$. Next, we provide an induction hypothesis: given $1 < l \leq t$, for all $1 \leq l' < l$, voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_{l'}\}$. We show that voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_l\}$. If $z_l \in \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle$, then $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_l\} = \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_{l-1}\}$, and hence by the induction hypothesis, voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_l\}$. Henceforth, we assume $z_l \notin \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle$. Note that voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_{l-1}\}$ by the induction hypothesis and the adjacency graph over $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_{l-1}\}$ is a connected graph. Therefore, if we show that voter 1 dictates on $\{z_{l-1}, z_l\}$, then the proof of Lemma 2 implies that voter 1 dictates on $\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{L} \rangle \cup \{z_1, \ldots, z_{l-1}, z_l\}$, which hence proves the induction hypothesis. Hence, in the rest of the proof, we show $f(z_{l-1}, z_l) = z_{l-1}$ and $f(z_l, z_{l-1}) = z_l$.

Since $z_{l-1} \sim z_l$, statement (i) of Lemma 9 implies $f(z_{l-1}, z_l) \in \{z_{l-1}, z_l\}$ and $f(z_l, z_{l-1}) \in \{z_{l-1}, z_l\}$. Suppose $f(z_{l-1}, z_l) = z_l$. Then, according to the path $(z_{l-1}, \ldots, z_1)$, statement (ii) of Lemma 9 implies $f(z_1, z_l) = z_l$. Given $\underline{P}_2 = \underline{P}_i$ and $\overline{P}_2 = \overline{P}_i$, Lemma 15 implies $f(z_1, \underline{P}_2) = f(x_{\underline{k}}, x_1) = x_{\underline{k}} = z_1$ and $f(z_1, \overline{P}_2) = f(x_{\underline{k}}, x_v) = x_{\underline{k}} = z_1$. Then, strategy-proofness implies $z_1 \underline{P}_2 z_l$ and $z_1 \overline{P}_2 z_l$ which contradicts the hypothesis that $\underline{P}_2$ and $\overline{P}_2$ are completely reversed. Therefore, $f(z_{l-1}, z_l) = z_{l-1}$, as required.

Next, we show $f(z_l, z_{l-1}) = z_l$. Suppose not, i.e., $f(z_l, z_{l-1}) = z_{l-1}$. We first fix a path $\pi \in \Pi(a_1, a_m) = \Pi(x_1, x_v)$. We know that $x_{\underline{k}}, x_{\overline{k}} \in \pi$, $\langle x_1, x_{\underline{k}} | \pi \rangle = (x_1, \ldots, x_{\underline{k}})$, $\langle x_{\underline{k}}, x_v | \pi \rangle = (x_{\overline{k}}, \ldots, x_v)$, $x_{\underline{k}} \in \langle x_1, x_{\overline{k}} | \pi \rangle$ and $x_{\overline{k}} \in \langle x_{\underline{k}}, x_v | \pi \rangle$. According to the paths $(z_{l-1}, \ldots, z_1 = x_{\underline{k}})$ and $\langle x_{\underline{k}}, x_{\overline{k}} | \pi \rangle$, by statement (iii) of Lemma 9, $f(z_l, z_{l-1}) = z_{l-1}$ implies $f(z_l, z_1) \in \{z_1, \ldots, z_{l-1}\}$ and $f(z_l, x_{\overline{k}}) \in \{z_1, \ldots, z_{l-1}\} \cup \langle x_{\underline{k}}, x_{\overline{k}} | \pi \rangle$. Meanwhile, by the induction hypothesis, we have $f(z_{l-1}, z_1) = z_{l-1}$ and $f(z_{l-1}, x_{\overline{k}}) = z_{l-1}$. Then, according to $z_l \sim z_{l-1}$, statement (iii) of Lemma 9 implies $f(z_l, z_1) \in \{z_{l-1}, z_l\}$ and $f(z_l, x_{\overline{k}}) \in \{z_{l-1}, z_l\}$. Therefore, $f(z_l, z_1) \in \{z_1, \ldots, z_{l-1}\} \cap \{z_{l-1}, z_l\} = \{z_{l-1}\}$ and $f(z_l, x_{\overline{k}}) \in \big[\{z_1, \ldots, z_{l-1}\} \cup \langle x_{\underline{k}}, x_{\overline{k}} | \pi \rangle\big] \cap \{z_{l-1}, z_l\} = \{z_{l-1}\}$. Hence, $f(z_l, x_{\underline{k}}) = f(z_l, z_1) = z_{l-1}$ and $f(z_l, x_{\overline{k}}) = z_{l-1}$. We will show $f(z_l, x_1) = z_{l-1}$ and $f(z_l, x_v) = z_{l-1}$. There are three cases: (i) $z_{l-1} \notin \{x_{\underline{k}}, x_{\overline{k}}\}$, (ii) $z_{l-1} = x_{\underline{k}}$ and (iii) $z_{l-1} = x_{\overline{k}}$.

In case (i), according to the paths $\langle x_1, x_{\underline{k}} | \pi \rangle$ and $\langle x_{\overline{k}}, x_v | \pi \rangle$, by statement (ii) of Lemma 9, $f(z_l, x_{\underline{k}}) = z_{l-1}$ implies $f(z_l, x_1) = z_{l-1}$, and $f(z_l, x_{\overline{k}}) = z_{l-1}$ implies $f(z_l, x_v) = z_{l-1}$. In case (ii), we first refer to the path $\langle x_{\overline{k}}, x_v | \pi \rangle$. Then, by statement (ii) of Lemma 9, $f(z_l, x_{\overline{k}}) = z_{l-1}$ implies $f(z_l, x_v) = z_{l-1}$. We next claim $f(z_l, x_1) = z_{l-1}$. Given $f(z_l, x_{\underline{k}}) = z_{l-1} = x_{\underline{k}}$, according to the path $\langle x_1, x_{\underline{k}} | \pi \rangle$, statement (iii) of Lemma 9 implies $f(z_l, x_1) \in \langle x_1, x_{\underline{k}} | \pi \rangle = (x_1, \ldots, x_{\underline{k}})$. Suppose $f(z_l, x_1) = x_k$ for some $1 \leq k < \underline{k}$. Then, we combine $(z_l, z_{l-1} = x_{\underline{k}})$ and $\langle x_{\underline{k}}, x_v | \pi \rangle$ to construct a path from z_l to x_v, which clearly excludes x_k. According to this path, by statement (ii) of Lemma 9, $f(z_l, x_1) = x_k$ implies $f(x_v, x_1) = x_k$, which contradicts the fact $f(x_v, x_1) = x_{\overline{k}}$. Therefore, it must be the case that $f(z_l, x_1) = x_{\underline{k}} = z_{l-1}$. Symmetrically, in case (iii), we have $f(z_l, x_1) = z_{l-1}$ and $f(z_l, x_v) = x_{\overline{k}} = z_{l-1}$. Overall, given $\underline{P}_2 = \underline{P}_i$ and $\overline{P}_2 = \overline{P}_i$, we have $f(z_l, \underline{P}_2) = f(z_l, x_1) = z_{l-1}$ and $f(z_l, \overline{P}_2) = f(z_l, x_v) = z_{l-1}$. Then, strategy-proofness implies $f(z_l, \underline{P}_2) = z_{l-1} \underline{P}_2 z_l = f(z_l, z_l)$ and $f(z_l, \overline{P}_2) = z_{l-1} \overline{P}_2 z_l = f(z_l, z_l)$ which contradict the hypothesis that $\underline{P}_2$ and $\overline{P}_2$ are completely reversed. Therefore, $f(z_l, z_{l-1}) = z_l$, as required. This completes the proof of the lemma. ■

Lemma **20** *According to the tree $\mathcal{T}_f^A$ and the two thresholds $x_{\underline{k}}$ and $x_{\overline{k}}$, we have*

$$f(x,y)=\begin{cases} x & \text{if } x\in\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle,\\ \text{Proj}\left(x_{\underline{k}},\langle x,y|\mathcal{T}_f^A\rangle\right) & \text{if } x\in\underline{A}\backslash\{x_{\underline{k}}\},\\ \text{Proj}\left(x_{\overline{k}},\langle x,y|\mathcal{T}_f^A\rangle\right) & \text{if } x\in\overline{A}\backslash\{x_{\overline{k}}\}.\end{cases}$$

Proof: We first know that voter 1 dictates on $\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle$ by Lemma 19. Next, given $x\in\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle$ and $y\in\underline{A}\backslash\{x_{\underline{k}}\}$ or $y\in\overline{A}\backslash\{x_{\overline{k}}\}$, we show $f(x,y)=x$. We assume w.l.o.g. that $y\in\underline{A}\backslash\{x_{\underline{k}}\}$. The verification for the case $y\in\overline{A}\backslash\{x_{\overline{k}}\}$ is symmetric. Since $y\in\underline{A}\backslash\{x_{\underline{k}}\}$, we know $\underline{A}\neq\{x_{\underline{k}}\}$ and hence $x_{\underline{k}-1}\in\underline{A}\backslash\{x_{\underline{k}}\}$ by the construction of $\mathcal{T}_f^A$. Then, we have $f(x_{\underline{k}},x_{\underline{k}-1})=x_{\underline{k}}$ by Lemma 15. In $G_{\sim}^{\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle}$, there exists a path $(z_1,\ldots,z_s)$ connecting $x_{\underline{k}}$ and x. Then, by statement (iii) of Lemma 9, $f(x_{\underline{k}},x_{\underline{k}-1})=x_{\underline{k}}$ implies $f(x,x_{\underline{k}-1})\in\{z_1,\ldots,z_s\}$. Suppose $f(x,x_{\underline{k}-1})=z_k$ for some $1\leq k<s$. Then, strategy-proofness implies $f(x,z_k)=z_k$ which contradicts the fact that voter 1 dictates on $\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle$. Therefore, $f(x,x_{\underline{k}-1})=z_s=x$. Furthermore, in the tree $G_{\sim}^{\underline{A}}$, we have a path $(y_1,\ldots,y_t)$ that connects $x_{\underline{k}-1}$ and y, and excludes x. Then, by statement (ii) of Lemma 9, $f(x,x_{\underline{k}-1})=x$ implies $f(x,y)=x$, as required.

Second, given $x\in\underline{A}\backslash\{x_{\underline{k}}\}$ and $y\in A$, we show $f(x,y)=\text{Proj}\left(x_{\underline{k}},\langle x,y|\mathcal{T}_f^A\rangle\right)$. Since $x\in\underline{A}\backslash\{x_{\underline{k}}\}$, we know $\underline{A}\neq\{x_{\underline{k}}\}$ and hence $x_{\underline{k}-1}\in\underline{A}\backslash\{x_{\underline{k}}\}$ by the construction of $\mathcal{T}_f^A$. We consider two cases: $y\in\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle\cup\overline{A}$ and $y\in\underline{A}\backslash\{x_{\underline{k}}\}$. First, let $y\in\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle\cup\overline{A}$. By Lemma 15, we know $f(x_{\underline{k}-1},x_{\underline{k}})=x_{\underline{k}}$. In the adjacency graph over $\langle x_{\underline{k}},x_{\overline{k}}|\mathcal{T}_f^A\rangle\cup\overline{A}$, we have a path $(y_1,\ldots,y_s)$ connecting $x_{\underline{k}}$ and y. Thus, by statement (iii) of Lemma 9, $f(x_{\underline{k}-1},x_{\underline{k}})=x_{\underline{k}}$ implies $f(x_{\underline{k}-1},y)\in\{y_1,\ldots,y_s\}$. Meanwhile, since $f(x_{\underline{k}},y)=x_{\underline{k}}$ and $x_{\underline{k}-1}\sim x_{\underline{k}}$, statement (iii) of Lemma 9 implies $f(x_{\underline{k}-1},y)\in\{x_{\underline{k}},x_{\underline{k}-1}\}$. Therefore, $f(x_{\underline{k}-1},y)\in\{x_{\underline{k}}=y_1,\ldots,y_s\}\cap\{x_{\underline{k}},x_{\underline{k}-1}\}=\{x_{\underline{k}}\}$, and hence $f(x_{\underline{k}-1},y)=x_{\underline{k}}$. Furthermore, we have a path $(z_1,\ldots,z_t)$ in the tree $G_{\sim}^{\underline{A}}$ that connects $x_{\underline{k}-1}$ and x, and excludes $x_{\underline{k}}$. Then, by statement (ii) of Lemma 9, $f(x_{\underline{k}-1},y)=x_{\underline{k}}$ implies $f(x,y)=x_{\underline{k}}=\text{Proj}\left(x_{\underline{k}},\langle x,y|\mathcal{T}_f^A\rangle\right)$, as required. Next, let $y\in\underline{A}\backslash\{x_{\underline{k}}\}$. We have a path π in the tree $G_{\sim}^{\underline{A}}$ connecting x and y. Then, statement (i) of Lemma 9 implies $f(x,y)\in\pi$. Meanwhile, we have $f(x_{\underline{k}},y)=x_{\underline{k}}$ and $f(x,x_{\underline{k}})=\text{Proj}\left(x_{\underline{k}},\langle x,x_{\underline{k}}|\mathcal{T}_f^A\rangle\right)=x_{\underline{k}}$ by the first case.

Note that in the tree $G_{\sim}^{\underline{A}}$, there exist a path π' connecting $x_{\underline{k}}$ and x and a path π'' connecting $x_{\underline{k}}$ and y. Then, by statement (iii) of Lemma 9, $f(x_{\underline{k}}, y) = x_{\underline{k}}$ implies $f(x, y) \in \pi'$, and $f(x, x_{\underline{k}}) = x_{\underline{k}}$ implies $f(x, y) \in \pi''$. Last, since $G_{\sim}^{\underline{A}}$ is a tree, it is true that $f(x, y) \in \pi \cap \pi' \cap \pi'' = \big\{ \text{Proj}(x_{\underline{k}}, \langle x, y | \mathcal{T}_f^A \rangle) \big\}$. Hence, $f(x, y) = \text{Proj}\left(x_{\underline{k}}, \langle x, y | \mathcal{T}_f^A \rangle\right)$, as required.

Symmetrically, we can show $f(x, y) = \text{Proj}\left(x_{\overline{k}}, \langle x, y | \mathcal{T}_f^A \rangle\right)$ when $x \in \overline{A} \backslash \{x_{\overline{k}}\}$. This completes the characterization of f. ■

Lemma **21** *We have* $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}_f^A, x_{\underline{k}}, x_{\overline{k}})$.

Proof: To prove $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}_f^A, x_{\underline{k}}, x_{\overline{k}})$, we fix an arbitrary preference $P_i \in \mathbb{D}$, let $r_1(P_i) = z$ and show that the three conditions of Definition 4 are satisfied on the tree $\mathcal{T}_f^A$ w.r.t. the thresholds $x_{\underline{k}}$ and $x_{\overline{k}}$.

First, we show the first condition of Definition 4, i.e., given $z \in \underline{A} \backslash \{x_{\underline{k}}\}$, P_i is semi-single-peaked on $\mathcal{T}_f^{\underline{A}}$ w.r.t. $x_{\underline{k}}$, $\max^{P_i}\left(\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle\right) = x_{\underline{k}}$ and $\max^{P_i}(\overline{A}) = x_{\overline{k}}$. By the characterization of f in Lemma 20, we infer $f(x, y) = \text{Proj}\left(x_{\underline{k}}, \langle x, y | \mathcal{T}_f^A \rangle\right) = \text{Proj}\left(x_{\underline{k}}, \langle x, y | \mathcal{T}_f^{\underline{A}} \rangle\right)$ for all $x, y \in \underline{A}$. Then, by the proof of Lemma 14, strategy-proofness of f implies that P_i is semi-single-peaked on $\mathcal{T}_f^{\underline{A}}$ w.r.t. $x_{\underline{k}}$, as required. Fixing an arbitrary alternative $x \in \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle \backslash \{x_{\underline{k}}\}$, given $P_1 = P_i$, we have $f(P_1, x) = \text{Proj}\left(x_{\underline{k}}, \langle z, x | \mathcal{T}_f^A \rangle\right) = x_{\underline{k}}$ by Lemma 20. Then, strategy-proofness implies $f(P_1, x) = x_{\underline{k}} P_1 x = f(x, x)$. Therefore, we have $\max^{P_i}\left(\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle\right) = x_{\underline{k}}$, as required. Symmetrically, fixing an arbitrary alternative $y \in \overline{A} \backslash \{x_{\overline{k}}\}$, given $P_2 = P_i$, we have $f(y, P_2) = \text{Proj}\left(x_{\overline{k}}, \langle y, z | \mathcal{T}_f^A \rangle\right) = x_{\overline{k}}$ by Lemma 20. Then, strategy-proofness implies $f(y, P_2) = x_{\overline{k}} P_2 y = f(y, y)$. Therefore, we have $\max^{P_i}(\overline{A}) = x_{\overline{k}}$, as required.

Symmetrically, given $z \in \overline{A} \backslash \{x_{\overline{k}}\}$, we can show that P_i is semi-single-peaked on $\mathcal{T}_f^{\overline{A}}$ w.r.t. $x_{\overline{k}}$, $\max^{P_i}\left(\langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle\right) = x_{\overline{k}}$ and $\max^{P_i}(\underline{A}) = x_{\overline{k}}$. This confirms the second condition of Definition 4.

Last, we show the third condition of Definition 4, i.e., given $z \in \langle x_{\underline{k}}, x_{\overline{k}} | \mathcal{T}_f^A \rangle$, we show $\max^{P_i}(\underline{A}) = x_{\underline{k}}$ and $\max^{P_i}(\overline{A}) = x_{\overline{k}}$. Fixing arbitrary alternatives $x \in \underline{A} \backslash \{x_{\underline{k}}\}$ and $y \in \overline{A} \backslash \{x_{\overline{k}}\}$, given $P_2 = P_i$, we have $f(x, P_2) = \text{Proj}\left(x_{\underline{k}}, \langle x, z | \mathcal{T}_f^A \rangle\right) =$

$x_{\underline{k}}$ and $f(y, P_2) = \text{Proj}\left(x_{\overline{k}}, \langle y, z|\mathcal{T}_f^A\rangle\right) = x_{\overline{k}}$ by Lemma 20. Then, strategy-proofness implies $f(x, P_2) = x_{\underline{k}} P_2 x = f(x, x)$ and $f(y, P_2) = x_{\overline{k}} P_2 y = f(y, y)$. Therefore, we have $\max^{P_i}(\underline{A}) = x_{\underline{k}}$ and $\max^{P_i}(\overline{A}) = x_{\overline{k}}$, as required.

In conclusion, we have $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}_f^A, x_{\underline{k}}, x_{\overline{k}})$. ■

Let Ω denote the set of two-voter tops-only and strategy-proof rules defined on $\mathbb{D}$. Clearly, Ω is a finite and non-empty set, and we hence label $\Omega = \{f^1, \dots, f^\eta\}$. We apply the aforementioned characterization of f on every rule of Ω and summarize all results in the following observation.

Observation 3 For each $k = 1, \dots, \eta$, we construct a tree $\mathcal{T}_k^A$ and identify two thresholds $a^k, b^k \in A$ such that the following six conditions are satisfied:

(i) thresholds a^k and b^k are contained in the line $\mathcal{L} = (x_1, \dots, x_v)$, where $a^k = x_{\underline{k}}$ and $b^k = x_{\overline{k}}$ for some $1 \leq \underline{k} < \overline{k} \leq v$,

(ii) given $\underline{A}^k = A^{a^k \rightharpoonup b^k}$, $G_\sim^{\underline{A}^k} = \mathcal{T}_k^{\underline{A}^k}$, $\{x_1, \dots, x_{\underline{k}}\} \subseteq \underline{A}^k$, $(x_1, \dots, x_{\underline{k}})$ is the unique path in $G_\sim^A$ connecting a_1 and a^k, and $\left[\underline{A}^k \neq \{a^k\}\right] \Rightarrow \left[a^k \in Ext(\mathcal{T}_k^{\underline{A}^k})\right]$,

(iii) given $\overline{A}^k = A^{b^k \rightharpoonup a^k}$, $G_\sim^{\overline{A}^k} = \mathcal{T}_k^{\overline{A}^k}$, $\{x_{\overline{k}}, \dots, x_v\} \subseteq \overline{A}^k$, $(x_{\overline{k}}, \dots, v_v)$ is the unique path in $G_\sim^A$ connecting b^k and a_m, and $\left[\overline{A}^k \neq \{b^k\}\right] \Rightarrow \left[b^k \in Ext(\mathcal{T}_k^{\overline{A}^k})\right]$,

(iv) set $\{x_{\underline{k}}, \dots, x_{\overline{k}}\} \subseteq \langle a^k, b^k|\mathcal{T}_k^A\rangle$, $\langle a^k, b^k|\mathcal{T}_k^A\rangle$ is a line constructed over all alternatives of $\left[A\backslash(\underline{A}^k \cup \overline{A}^k)\right] \cup \{a^k, b^k\}$ where a^k and b^k are the two extreme vertices, and $G_\sim^{\langle a^k, b^k|\mathcal{T}_k^A\rangle}$ is a connected graph,

(v) the adjacency graph $G_\sim^A = G_\sim^{\underline{A}^k} \cup G_\sim^{\langle a^k, b^k|\mathcal{T}_k^A\rangle} \cup G_\sim^{\overline{A}^k}$, and

(vi) $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}\left(\mathcal{T}_k^A, a^k, b^k\right)$, and f^k behaves like a dictatorship on $\langle a^k, b^k|\mathcal{T}_k^A\rangle$. □

According to condition (i) of Observation 3, among $a^1, \dots, a^\eta$, we can identify a^s such that $a^k \in \langle x_1, a^s|\mathcal{L}\rangle$ for all $k = 1, \dots, \eta$, and symmetrically, among $b^1, \dots, b^\eta$, we can identify b^t such that $b^k \in \langle b^t, x_v|\mathcal{L}\rangle$ for all $k = 1, \dots, \eta$. Thus, we refer to the rules f^s and f^t. Accordingly, we have the pair of thresholds a^s and b^s and the pair of thresholds a^t and b^t.

Lemma **22** *The following two statements hold:*

(i) *For all $k \in \{1, \ldots, \eta\}$ with $a^k \neq a^s$, we have $\mathcal{T}_k^{\underline{A}^k} \subset \mathcal{T}_s^{\underline{A}^s}$.*

(ii) *For all $k \in \{1, \ldots, \eta\}$ with $b^k \neq b^t$, we have $\mathcal{T}_k^{\overline{A}^k} \subset \mathcal{T}_t^{\overline{A}^t}$.*

Proof: The two statements are symmetric. We focus on verifying the first one.

Given $k \in \{1, \ldots, \eta\}$ with $a^k \neq a^s$, by Observation 3(i) on both f^k and f^s, let $a^k = x_p \in \mathcal{L}$ and $a^s = x_q \in \mathcal{L}$. Since $a^k \neq a^s$, the definition of a^s implies $1 \leq p < q$. Then, Observation 3(ii) on both f^k and f^s implies $a^k = x_p \in \underline{A}^s$ and $a^s = x_q \notin \underline{A}^k$.

Next, we show $\mathcal{T}_k^{\underline{A}^k} \subset \mathcal{T}_s^{\underline{A}^s}$. Given an arbitrary alternative $z \in \underline{A}^k \backslash \{a^k\}$, since $G_{\sim}^{\underline{A}^k} = \mathcal{T}_k^{\underline{A}^k}$ by Observation 3(ii) on f^k, there exists a unique path $(z_1, \ldots, z_w)$ in $G_{\sim}^{\underline{A}^k}$ connecting z and $a^k = x_p$. Moreover, by items (iii) and (iv) of Observation 3 on f^k, we know $x_{p+1}, \ldots, x_{q-1}, x_q \notin \underline{A}^k$. Therefore, $\{z_1, \ldots, z_w\} \cap \{x_{p+1}, \ldots, x_{q-1}, x_q\} = \emptyset$. Since $x_p \sim x_{p+1}$, we have a concatenated path $(z_1, \ldots, z_w = x_p, x_{p+1}, \ldots, x_{q-1}, x_q)$ in $G_{\sim}^{A}$ that connects z and $a^s = x_q$ and includes x_{q-1}. Consequently, by the construction of $\mathcal{T}_s^A$, z must be included in $\underline{A}^s$. Therefore, $\underline{A}^k \subseteq \underline{A}^s$. More precisely, since $a^s \in \underline{A}^s$ and $a^s \notin \underline{A}^k$, we have $\underline{A}^k \subset \underline{A}^s$ and $\underline{A}^k \subseteq \underline{A}^s \backslash \{a^s\}$. Since the proof also implies that for each $z \in \underline{A}^k \backslash \{a^k\}$, its unique path to a^k in $\mathcal{T}_k^A$ is contained in $\mathcal{T}_s^A$, $\underline{A}^k \subset \underline{A}^s$ implies $\mathcal{E}_k^{\underline{A}^k} \subset \mathcal{E}_s^{\underline{A}^s}$, and hence $\mathcal{T}_k^{\underline{A}^k} \subset \mathcal{T}_s^{\underline{A}^s}$. ■

Given $k \in \{1, \ldots, \eta\}$, if $a^k = a^s$, it is true by the construction of $\mathcal{T}_k^A$ and $\mathcal{T}_s^A$ that $\mathcal{T}_k^{\underline{A}^k} = \mathcal{T}_s^{\underline{A}^s}$. Symmetrically, if $b^k = b^t$, then $\mathcal{T}_k^{\overline{A}^k} = \mathcal{T}_t^{\overline{A}^t}$.

Lemma **23** *We have $\underline{A}^s \cap \overline{A}^t = \emptyset$.*

Proof: The proof consists of four claims.

Claim 1: If $a^s \notin \overline{A}^t$ and $b^s \notin \underline{A}^t$, we have $\underline{A}^s \cap \overline{A}^t = \emptyset$.

Suppose $z \in \underline{A}^s \cap \overline{A}^t$. Since $a^s \notin \overline{A}^t$ and $b^t \notin \underline{A}^s$, it is clear that $z \notin \{a^s, b^t\}$. Now, according to Observation 3(vi) on f^s, by the definition of (a^s, b^s)-semi-hybridness on $\mathcal{T}_s^A$, $z \in \underline{A}^s$ and $b^t \notin \underline{A}^s$ imply $a^s P_i b^t$ for all $P_i \in \mathbb{D}^z$. However, according to Observation 3(vi) on f^t, by the definition of (a^t, b^t)-semi-hybridness on $\mathcal{T}_t^A$, $z \in \overline{A}^t$ and $a^s \notin \overline{A}^t$ imply $b^l P_i a^k$ for all $P_i \in \mathbb{D}^z$. Contradiction! Therefore, $\underline{A}^s \cap \overline{A}^t = \emptyset$. This completes the verification of the claim.

By Claim 1, to complete the verification, it suffices to show $a^s \notin \overline{A}^t$ and $b^s \notin \underline{A}^t$. We focus on showing $a^s \notin \overline{A}^t$. By a symmetric argument, one would immediately prove $b^s \notin \underline{A}^t$. By Observation 3(i) on both f^t and f^s, let $a^t = x_p$ and $b^t = x_q$ for some $1 \leq p < q \leq v$, and let $a^s = x_r$ for some $1 \leq r < v$.

Suppose $a^s \in \overline{A}^t$ by contradiction. By items (ii) and (iv) of Observation 3 on f^t, it is clear that $x_1, \ldots, x_{q-1} \notin \overline{A}^t$. Therefore, $x_r = a^s \in \overline{A}^t$ implies $r \geq q$. By Observation 3(ii) on f^s, we know that $\{x_1, \ldots, x_p, \ldots, x_q, \ldots, x_r\} \subseteq \underline{A}^s$ and $(x_1, \ldots, x_p, \ldots, x_q, \ldots, x_r)$ is the unique path in $G_{\sim}^{\underline{A}^s}$ connecting a_1 and a^s. Thus, $a^t = x_p \in \underline{A}^s$ and $a^t \neq x_r = a^s$. Then, Lemma 22 implies $\mathcal{T}_t^{\underline{A}^t} \subset \mathcal{T}_s^{\underline{A}^s}$ and hence $\underline{A}^t \subset \underline{A}^s$. Next, we show $\langle a^t, b^t | \mathcal{T}_t^A \rangle \subseteq \underline{A}^s$. There are two cases: $b^t \neq a^s$ and $b^t = a^s$.

Claim 2: Given $b^t \neq a^s$, we have $\langle a^t, b^t | \mathcal{T}_t^A \rangle \subseteq \underline{A}^s$.

Clearly, $x_q = b^t \neq a^s = x_r$ implies $q < r$. Fixing an arbitrary alternative $z \in \langle a^t, b^t | \mathcal{T}_t^A \rangle$, we show $z \in \underline{A}^s$. By Observation 3(iv) on f^t, we know that there exists a path $(z_1, \ldots, z_w)$ in $G_{\sim}^{\langle a^t, b^t | \mathcal{T}_t^A \rangle}$ connecting z and $b^t = x_q$. Since $x_r = a^s \in \overline{A}^t$ by the contradictory hypothesis, we infer, by Observation 3(iii) on f^t, that $(x_q, \ldots, x_{r-1}, x_r)$ is the unique path in $G_{\sim}^{\overline{A}^t}$ connecting b^t and a^k. Therefore, $x_{q+1}, \ldots, x_{r-1}, x_r \notin \langle a^t, b^t | \mathcal{T}_t^A \rangle$. Then, the concatenated path $(z_1, \ldots, z_w = x_q, \ldots, x_{r-1}, x_r)$ in $G_{\sim}^A$ connects z and $a^k = x_r$, and includes x_{r-1}. Consequently, by the construction of $\mathcal{T}_s^A$, it is true that $z \in \underline{A}^s$, as required. This completes the verification of the claim.

Claim 3: Given $b^t = a^s$, we have $\langle a^t, b^t | \mathcal{T}_t^A \rangle \subseteq \underline{A}^s$.

Clearly, $x_q = b^t = a^s = x_r$ implies $q = r$. First, Observation 3(ii) on f^s implies that $(x_1, \ldots, x_p, \ldots, x_r)$ is the unique path in $G_{\sim}^A$ connecting a_1 and a^s. Similarly, according to f^t, Observation 3(iii) implies that $(x_q, \ldots, x_v)$ is the unique path in $G_{\sim}^A$ connecting b^t and a_m. Consequently, the line $\mathcal{L} = (x_1, \ldots, x_p, \ldots, x_r = x_q, \ldots, x_v)$ must be the unique path in $G_{\sim}^A$ connecting a_1 and a_m.

Let $\mathcal{N}_{\sim}^A(a^s) = \{z \in A : z \sim a^s\}$. We next show $\mathcal{N}_{\sim}^A(a^s) = \mathcal{N}_{\sim}^A(a^s) \cap \mathcal{L}$, in other words, there exists no $z \in A \backslash \mathcal{L}$ such that $z \sim a^s$. Suppose not, i.e., there exists $z \in A\backslash\mathcal{L}$ such that $z \sim a^s$. On the one hand, if $\underline{A}^s = \{a^s\}$, it is clear that $z \in A\backslash\underline{A}^s$; if $\underline{A}^s \neq \{a^s\}$, then Observation 3(ii) on f^s implies $a^s \in Ext(\mathcal{T}_s^{\underline{A}_s})$ which by $z \sim a^s$ and $z \in A\backslash\mathcal{L}$ further implies $z \in A\backslash\underline{A}^s$. Given $a_1 \in \underline{A}^s$ by Observation 3(ii) on f^s, by Observation 3(vi) on f^s, (a^s, b^s)-semi-hybridness on $\mathcal{T}_s^A$ implies $a^s \underline{P}_i z$. On the other hand, if $\overline{A}^t = \{b^t\}$, it is clear that $z \in A\backslash\overline{A}^t$; if $\overline{A}^t \neq \{b^t\}$, then Observation 3(iii) on f^t implies $b^t \in Ext(\mathcal{T}_t^{\overline{A}_t})$ which by $z \sim a^s = b^t$ and $z \in A\backslash\mathcal{L}$ further implies $z \in A\backslash\overline{A}^t$. Given $a_m \in \overline{A}^t$ by Observation 3(iii) on f^t, by Observation 3(vi) on f^t, (a^t, b^t)-semi-hybridness on $\mathcal{T}_t^A$ implies $b^t \overline{P}_i z$. Thus, $\underline{P}_i$ and $\overline{P}_i$ agree on the relative ranking of $a^s = b^t$ and z, which contradicts the hypothesis that $\underline{P}_i$ and $\overline{P}_i$ are complete reversals. Hence, there exists no $z \in A\backslash\mathcal{L}$ such that $z \sim a^s$, as required. Thus, we have $\mathcal{N}_\sim^A(a^s) = \mathcal{N}_\sim^A(a^s) \cap \mathcal{L} = \{x_{q-1}, x_{q+1}\}$ (if $q < v$) and $\mathcal{N}_\sim^A(a^s) = \mathcal{N}_\sim^A(a^s) \cap \mathcal{L} = \{x_{q-1}\}$ (if $q = v$). Furthermore, since Observation 3(iv) on f^t implies $x_{q-1} \in \langle a^t, b^t | \mathcal{T}_t^A\rangle$ and $x_{q+1} \notin \langle a^t, b^t | \mathcal{T}_t^A\rangle$, we have $\mathcal{N}_\sim^A(a^s) \cap \langle a^t, b^t | \mathcal{T}_t^A\rangle = \{x_{q-1}\}$.

Now, we prove the claim. Fixing an arbitrary $z \in \langle a^t, b^t | \mathcal{T}_t^A\rangle$ in $\mathcal{T}_t^A$, we show $z \in \underline{A}^s$. By Observation 3(iv) on f^t, there exists a path $(z_1, \dots, z_{w-1}, z_w)$ in $G_\sim^{\langle a^t, b^t | \mathcal{T}_t^A\rangle}$ connecting z and $b^t = a^s$. Since $z_{w-1} \sim z_w = a^s$ and $z_{w-1} \in \langle a^t, b^t | \mathcal{T}_t^A\rangle$, we know $z_{w-1} \in \mathcal{N}_\sim^A(a^s) \cap \langle a^t, b^t | \mathcal{T}_t^A\rangle$. Therefore, $z_{w-1} = x_{q-1} = x_{r-1}$. Consequently, we have a path $(z_1, \dots, z_{w-1}, z_w)$ in $G_\sim^A$ that connects z and $a^s = x_r$, and includes x_{r-1}. This by the construction of $\mathcal{T}_s^A$ implies $z \in \underline{A}_s$, as required. This completes the verification of the claim.

Henceforth, for notational convenience, let $B = \underline{A}^t \cup \langle a^t, b^t | \mathcal{T}_t^A\rangle$. Thus, $B \subseteq \underline{A}^s$.

Claim 4: The adjacency graph $G_\sim^A$ is a tree.

Since $B \subseteq \underline{A}^s$, it is clear that $G_\sim^B \subseteq G_\sim^{\underline{A}^s}$. According to f^t, Observation 3(v) implies that $G_\sim^B = G_\sim^{\underline{A}^t} \cup G_\sim^{\langle a^t, b^t | \mathcal{T}_t^A\rangle}$ is a connected graph. By Observation 3(ii) on f^s, we know that $G_\sim^{\underline{A}^s} = \mathcal{T}_s^{\underline{A}^s}$ is a tree. Therefore, $G_\sim^B \subseteq G_\sim^{\underline{A}^s}$ implies that $G_\sim^B$ must be a tree nested in $G_\sim^{\underline{A}^s} = \mathcal{T}_s^{\underline{A}^s}$. Last, according to f^t, since $G_\sim^{\overline{A}^t} = \mathcal{T}_t^{\overline{A}^t}$ is tree, $G_\sim^A = G_\sim^{\underline{A}^t} \cup G_\sim^{\langle a^t, b^t | \mathcal{T}_t^A\rangle} \cup G_\sim^{\overline{A}^t} = G_\sim^B \cup G_\sim^{\overline{A}^t}$ in Observation 3(v) implies that $G_\sim^A$ is a tree. This completes the verification of the claim.

Claim 5: According to the tree $G_\sim^A$, for all $z \in B$, we have $b^t \in \langle z, a^s | G_\sim^A\rangle$.

Given $z \in B$, we have the unique path $\langle z, b^t | G^A_\sim \rangle$ connecting z and b^t in $G^A_\sim$. Since $G^B_\sim$ is a tree nested in $G^A_\sim$ and $z, b^t \in B$, it is true that $\langle z, b^t | G^A_\sim \rangle \subseteq B$. Next, since $a^s \in \overline{A}^t$ by the contradictory hypothesis, we know, by Observation 3(iii) on f^t, that $(x_q, \ldots, x_r)$ is the unique path in $G^{\overline{A}^t}_\sim$ that connects b^t and a^s, and $\{x_q, \ldots, x_r\} \subseteq \overline{A}^t$. In particular, if $b^t = a^s$, then $q = r$ and hence $(x_q, \ldots, x_r)$ is a null path. Then, the concatenation of $\langle z, b^t | G^A_\sim \rangle$ and $(b^t = x_q, \ldots, x_r = a^s)$ forms a path in $G^A_\sim$ connecting z and a^s. Last, since $G^A_\sim$ is a tree, it is true that the concatenated path equals $\langle z, a^s | G^A_\sim \rangle$. Therefore, $b^t \in \langle z, a^s | G^A_\sim \rangle$. This completes the verification of the claim.

Claim 6: Domain $\mathbb{D}$ is semi-single-peaked on the tree $G^A_\sim$ w.r.t. b^t.

Fix an arbitrary preference $P_i \in \mathbb{D}$ and let $r_1(P_i) = x$. Note that b^t separates $G^A_\sim$ into two subtrees $G^{\overline{A}^t}_\sim$ and $G^B_\sim$. If $x = b^t$, P_i by definition is semi-single-peaked on $G^A_\sim$ w.r.t. b^t. Henceforth, we consider two cases: $x \in \overline{A}^t \backslash \{b^t\}$ and $z \in B \backslash \{b^t\}$.

First, assume $x \in \overline{A}^t \backslash \{b^t\}$. According to f^t, by Observation 3(vi), the definition of (a^t, b^t)-semi-hybridness on $\mathcal{T}^A_t$ implies that P_i is semi-single-peaked on $\mathcal{T}^{\overline{A}^t}_t$ w.r.t. b^t, where $\mathcal{T}^{\overline{A}^t}_t = G^{\overline{A}^t}_\sim$ by Observation 3(iii), $\max^{P_i}\left(\langle a^t, b^t | \mathcal{T}^A_t \rangle\right) = b^t$ and $\max^{P_i}(\underline{A}^t) = a^t$. Since $\underline{A}^t \cap \langle a^t, b^t | \mathcal{T}^A_t \rangle = \{a^t\}$, $b^t = \max^{P_i}\left(\langle a^t, b^t | \mathcal{T}^A_t \rangle\right)$ and $a^t = \max^{P_i}(\underline{A}^t)$ together imply $b^t = \max^{P_i}\left(\underline{A}^t \cup \langle a^t, b^t | \mathcal{T}^A_t \rangle\right) = \max^{P_i}(B)$. Therefore, P_i is semi-single-peaked on $G^A_\sim$ w.r.t. b^t.

Next, assume $z \in B \backslash \{b^t\}$. Since $B \subseteq \underline{A}^s$, we have $x \in \underline{A}^s$. Then, by Observation 3(vi) on f^s, (a^s, b^s)-semi-hybridness w.r.t. $\mathcal{T}^A_s$ implies that P_i is semi-single-peaked on $\mathcal{T}^{\underline{A}^s}_s$ w.r.t. a^s. Recall that $G^B_\sim$ is a tree nested in $G^{\underline{A}^s}_\sim$, and $G^{\underline{A}^s}_\sim = \mathcal{T}^{\underline{A}^s}_s$ by Observation 3(ii) on f^s. Then, Claim 5 implies that P_i is semi-single-peaked on $G^B_\sim$ w.r.t. b^t. Furthermore, since $x \in B \backslash \{b^t\} \subset \underline{A}^t \cup \langle a^t, b^t | \mathcal{T}^A_t \rangle$, by Observation 3(vi) on f^t, (a^t, b^t)-semi-hybridness w.r.t $\mathcal{T}^A_t$ implies $\max^{P_i}(\overline{A}^t) = b^t$. Therefore, P_i is semi-single-peaked on $G^A_\sim$ w.r.t. b^t. This completes the verification of the claim.

Thus, $\mathbb{D}$ is semi-single-peaked, and hence admits an invariant, tops-only and strategy-proof rule by the sufficiency part of Statement (i), which contradictions the hypothesis of Statement (ii). Therefore, the contradictory hypothesis $a^s \in \overline{A}^t$ cannot hold, and hence $a^s \notin \overline{A}^t$ holds, as required. This proves the lemma. ■

Now, let $M^* = \{x \in A : x \notin \underline{A}^s \cup \overline{A}^t\} \cup \{a^s, b^t\}$. Clearly, $M^* \neq \emptyset$. Then, we arrange all alternatives of M^* on a line $(a^s, \ldots, b^t)$ where a and b are the two extreme vertices. By the definition of M^* and Lemma 23, we know $\underline{A}^s \cap M^* = \{a^s\}$, $\overline{A}^t \cap M^* = \{b^t\}$ and $\underline{A}^s \cap \overline{A}^t = \emptyset$. Then, we combine the tree $\mathcal{T}_s^{\underline{A}^s}$, the line $(a^s, \ldots, b^t)$ and the tree $\mathcal{T}_t^{\overline{A}^t}$ to construct a tree $\mathcal{T}^A$, i.e., $\mathcal{T}^A = \mathcal{T}_s^{\underline{A}^s} \cup (a^s, \ldots, b^t) \cup \mathcal{T}_t^{\overline{A}^t}$. Clearly, a^s and b^t are thresholds in $\mathcal{T}^A$. Henceforth, for notational convenience, let $a = a^s$ and $b = b^t$. Thus, according to $\mathcal{T}^A$, we have $A^{a\rightharpoonup b} = \underline{A}^s$, $\langle a, b|\mathcal{T}^A\rangle = (a^s, \ldots, b^t)$ and $A^{b\rightharpoonup a} = \overline{A}^t$. Note that $\langle a, b|\mathcal{T}^A\rangle = M^* \subseteq [A\backslash\underline{A}^s] \cup \{a^s\} = A\backslash[\underline{A}^s\backslash\{a^s\}]$ and $\langle a, b|\mathcal{T}^A\rangle = M^* \subseteq [A\backslash\overline{A}^t] \cup \{b^t\} = A\backslash[\overline{A}^t\backslash\{b^t\}]$.

Lemma **24** *Domain $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$.*

Proof: The proof consists of 4 claims.

Claim 1: Domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$.

We fix an arbitrary preference $P_i \in \mathbb{D}$ and let $r_1(P_i) = x$.

First, let $x \in A^{a\rightharpoonup b}\backslash\{a\} = \underline{A}^s\backslash\{a^s\}$. By Observation 3(vi) on f^s, (a^s, b^s)-semi-hybridness on $\mathcal{T}_s^A$ implies that P_i is semi-single-peaked on $\mathcal{T}_s^{\underline{A}^s} = \mathcal{T}^{A^{a\rightharpoonup b}}$ w.r.t. $a^s = a$, and $\max^{P_i}\left(\langle a^s, b^s|\mathcal{T}_s^A\rangle \cup \overline{A}^t\right) = a^s = a$. Furthermore, since $\langle a, b|\mathcal{T}^A\rangle \subseteq \langle a^s, b^s|\mathcal{T}_s^A\rangle \cup \overline{A}^t$, it is evident that $\max^{P_i}\left(\langle a, b|\mathcal{T}^A\rangle\right) = a$. Moreover, since $x \in \underline{A}^s\backslash\{a^s\}$, we know $x \notin \overline{A}^t$ by Lemma 23. Then, according to f^t, by Observation 3(vi), (a^t, b^t)-semi-hybridness on $\mathcal{T}_t^A$ implies $\max^{P_i}(\overline{A}^t) = b^t$. Hence, we have $\max^{P_i}(A^{b\rightharpoonup a}) = b$, as required by Definition 4.

Symmetrically, if $x \in A^{b\rightharpoonup a}\backslash\{b\}$, we can show that P_i is semi-single-peaked on $\mathcal{T}^{A^{b\rightharpoonup a}}$ w.r.t. b, $\max^{P_i}\left(\langle a, b|\mathcal{T}^A\rangle\right) = b$ and $\max^{P_i}(A^{a\rightharpoonup b}) = a$, as required by Definition 4.

Last, let $x \in \langle a, b|\mathcal{T}^A\rangle$. Since $x \in \langle a, b|\mathcal{T}^A\rangle$, we know $x \notin A^{a\rightharpoonup b}\backslash\{a\} = \underline{A}^s\backslash\{a^s\}$ and $x \notin A^{b\rightharpoonup a}\backslash\{b\} = \overline{A}^t\backslash\{b^t\}$. Furthermore, according to $\mathcal{T}_s^A$, $x \notin \underline{A}^s\backslash\{a^s\}$ implies $x \in \langle a^s, b^s|\mathcal{T}_s^A\rangle \cup \overline{A}^s$. Then, according to f^s, by Observation 3(vi), (a^s, b^s)-semi-hybridness on $\mathcal{T}_s^A$ implies $\max^{P_i}(\underline{A}^s) = b^s$. Hence, we have $\max^{P_i}(A^{a\rightharpoonup b}) = a$, as required by Definition 4. Symmetrically, according to $\mathcal{T}_t^A$, $x \notin \overline{A}^t\backslash\{b^t\}$ implies $x \in \underline{A}^t \cup \langle a^t, b^t|\mathcal{T}_t^A\rangle$. Then, according to f^t, by Observation 3(vi), (a^t, b^t)-semi-hybridness on $\mathcal{T}_t^A$ implies $\max^{P_i}(\overline{A}^s) = b^t$. Hence, $\max^{P_i}(A^{b\rightharpoonup a}) = b$, as required by Definition 4. This confirms the third condition of Definition 4. Therefore, $P_i \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$. This completes the verification of the claim.

Claim 2: We have $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$.

Clearly $|\langle a, b\rangle| \geq 2$. Suppose $|\langle a, b\rangle| = 2$. Consequently, $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b) = \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, a) \cap \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, b)$. Therefore, $\mathbb{D}$ is semi-single-peaked on $\mathcal{T}^A$, which implies that $\mathbb{D}$ admits an invariant, tops-only and strategy-proof rule by the sufficiency part of Statement (i). This contradicts the hypothesis of Statement (ii). Therefore, $|\langle a, b\rangle| \geq 3$. This completes the verification of the claim.

Claim 3: Every tops-only and strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$.

First, by the Second Ramification Theorem, to prove the claim, it suffices to show that every rule of Ω behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Next, recall by Observation 3(vi) that for each $k = 1, \ldots, \eta$, f^k behaves like a dictatorship on $\langle a^k, b^k|\mathcal{T}_k^A\rangle$. Then, it suffices to show $\langle a, b|\mathcal{T}^A\rangle \subseteq \langle a^k, b^k|\mathcal{T}_k^A\rangle$ for all $k = 1, \ldots, \eta$. Given $1 \leq k \leq \eta$, suppose by contradiction that we have $x \in \langle a, b|\mathcal{T}^A\rangle \backslash \langle a^k, b^k|\mathcal{T}_k^A\rangle$. Thus, according to $\mathcal{T}_k^A$, it is true that either $x \in \underline{A}^k \backslash \{a^k\}$ or $x \in \overline{A}^k \backslash \{b^k\}$ holds. We assume w.l.o.g. that $x \in \underline{A}^k \backslash \{a^k\}$. On the one hand, recall $\langle a, b|\mathcal{T}^A\rangle \subseteq A\backslash[\underline{A}^s \backslash \{a^s\}]$. Therefore, $x \in A\backslash[\underline{A}^s \backslash \{a^s\}]$, and hence $x \notin \underline{A}^s \backslash \{a^s\}$. On the other hand, it is clear that either $a^k \neq a^s$ or $a^k = a^s$ holds. If $a^k \neq a^s$, the proof of Lemma 22 implies $\underline{A}^k \subseteq \underline{A}^s \backslash \{a^s\}$. Hence, $x \in \underline{A}^s \backslash \{a^s\}$. If $a^k = a^s$, we know $\underline{A}^k = \underline{A}^s$ by the construction of $\mathcal{T}_k^A$ and $\mathcal{T}_s^A$. Hence, $x \in \underline{A}^k \backslash \{a^k\} = \underline{A}^s \backslash \{a^s\}$. Overall, $x \in \underline{A}^s \backslash \{a^s\}$. Contradiction! Therefore, $\langle a, b|\mathcal{T}^A\rangle \subseteq \langle a^k, b^k|\mathcal{T}_k^A\rangle$, as required. This completes the verification of the claim.

Since by hypothesis there exists no invariant, tops-only and strategy-proof, Statement (i) of the Theorem implies that $\mathbb{D}$ is never semi-single-peaked. Therefore, there exists no tree $\widehat{\mathcal{T}}^A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widehat{\mathcal{T}}^A, a)$ or $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widehat{\mathcal{T}}^A, b)$. Now, by Definition 4, to prove that $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$, it suffices to show that there exist no tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle a, b|\mathcal{T}^A\rangle \subset \langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle$.

Claim 4: There exist no tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \subset \langle a, b | \mathcal{T}^A \rangle$.

Suppose that the claim is not correct. Thus, we have a tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \subset \langle a, b | \mathcal{T}^A \rangle$. Let $\hat{A}^{a \rightharpoonup b} = \big\{x \in A : a \in \langle x, b | \widehat{\mathcal{T}}^A \rangle\big\}$ and $\hat{A}^{b \rightharpoonup a} = \big\{x \in A : b \in \langle x, a | \widehat{\mathcal{T}}^A \rangle\big\}$. According to $\widehat{\mathcal{T}}^A$, we construct the following SCF:

$$f^*(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } r_1(P_1) \in \langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle, \\ \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right) & \text{if } r_1(P_1) \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}, \\ \text{Proj}\left(\hat{b}, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right) & \text{if } r_1(P_1) \in \hat{A}^{\hat{b} \rightharpoonup \hat{a}} \backslash \{\hat{b}\}. \end{cases}$$

It is clear that f^* is unanimous, and hence is a rule. Moreover, f^* satisfies the tops-only property. Next, we show that f^* is strategy-proof.

Given (P_1, P_2) and (P_1, P_2'), voter 2 has two possible manipulations:
(1) $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right)$ and $f^*(P_1, P_2') = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2') | \widehat{\mathcal{T}}^A \rangle\right)$, and
(2) $f^*(P_1, P_2) = \text{Proj}\left(\hat{b}, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right)$ and $f^*(P_1, P_2') = \text{Proj}\left(\hat{b}, \langle r_1(P_1), r_1(P_2') | \widehat{\mathcal{T}}^A \rangle\right)$.
The two possible manipulations are symmetry, and we hence focus on the first one. In the first possible manipulation, it is true that $r_1(P_1) \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$. Thus, we induce $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right) \in \langle r_1(P_1), \hat{a} | \widehat{\mathcal{T}}^A \rangle \subseteq \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$ and $f^*(P_1, P_2') = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2') | \widehat{\mathcal{T}}^A \rangle\right) \in \langle r_1(P_1), \hat{a} | \widehat{\mathcal{T}}^A \rangle \subseteq \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$. We consider two cases: $r_1(P_2) \in \langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \cup \hat{A}^{\hat{b} \rightharpoonup \hat{a}}$ and $r_1(P_2) \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$. In the first case, $f^*(P_1, P_2) = \hat{a}$. Since P_2 is $(\hat{a}, \hat{b})$-semi-hybrid on $\widehat{\mathcal{T}}^A$ and $r_1(P_2) \in \langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \cup \hat{A}^{\hat{b} \rightharpoonup \hat{a}}$, we have $\hat{a} = \max^{P_2}\left(\hat{A}^{\hat{a} \rightharpoonup \hat{b}}\right)$, which implies either $f(P_1, P_2) = f(P_1, P_2') = \hat{a}$ or $f(P_1, P_2) P_2 f(P_1, P_2')$. In the second case, P_2 is semi-single-peaked on $\widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$ w.r.t. $\hat{a}$, and $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^A \rangle\right) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2) | \widehat{\mathcal{T}}^{\hat{A}^{a \rightharpoonup b}} \rangle\right) \in \langle r_1(P_2), \hat{a} | \widehat{\mathcal{T}}^{\hat{A}^{a \rightharpoonup b}} \rangle$. If $r_1(P_2') \in \langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \cup \hat{A}^{\hat{b} \rightharpoonup \hat{a}}$, then $f(P_1, P_2') = \hat{a}$. Furthermore, since P_2 is semi-single-peaked on $\widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$ w.r.t. $\hat{a}$, we have $\hat{a} = \min^{P_2}\left(\langle r_1(P_2), \hat{a} | \widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}} \rangle\right)$. Hence, either $f^*(P_1, P_2) = f^*(P_1, P_2')$ or $f^*(P_1, P_2) P_2 f^*(P_1, P_2')$ holds. If $r_1(P_2') \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$, $f^*(P_1, P_2') = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2') | \widehat{\mathcal{T}}^A \rangle\right) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2') | \widehat{\mathcal{T}}^{\hat{A}^{a \rightharpoonup b}} \rangle\right)$. Then, either $f^*(P_1, P_2) = f^*(P_1, P_2')$ holds, or $f^*(P_1, P_2) \neq f^*(P_1, P_2')$ and semi-single-peakedness of P_2 on $\widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$ w.r.t. $\hat{a}$ implies $f^*(P_1, P_2) P_2 f^*(P_1, P_2')$. Therefore, voter 2 has no incentive to manipulate.

Given (P_1, P_2) and (P_1', P_2), voter 1 has six possible manipulations:
(1) $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$ and $f^*(P_1', P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1'), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$,
(2) $f^*(P_1, P_2) = \text{Proj}\left(\hat{b}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$ and $f^*(P_1', P_2) = \text{Proj}\left(\hat{b}, \langle r_1(P_1'), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$,
(3) $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$ and $f^*(P_1', P_2) = \text{Proj}\left(\hat{b}, \langle r_1(P_1'), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$,
(4) $f^*(P_1, P_2) = \text{Proj}\left(\hat{b}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$ and $f^*(P_1', P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1'), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$,
(5) $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$ and $f^*(P_1', P_2) = r_1(P_1')$, and
(6) $f^*(P_1, P_2) = \text{Proj}\left(\hat{b}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right)$ and $f^*(P_1', P_2) = r_1(P_1')$.

Similar to voter 2, voter 1's first two possible manipulations are not profitable. In the third case, we know $r_1(P_1) \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$, $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right) \in \langle r_1(P_1), \hat{a}|\widehat{\mathcal{T}}^A\rangle \subseteq \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$ and $f^*(P_1', P_2) = \text{Proj}\left(\hat{b}, \langle r_1(P_1'), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right) \in \hat{A}^{\hat{b} \rightharpoonup \hat{a}}$. Since P_1 is $(\hat{a}, \hat{b})$-semi-hybrid on $\widehat{\mathcal{T}}^A$ and $r_1(P_1) \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$, it is true that $\hat{a} = \min^{P_1}\left(\langle r_1(P_1), \hat{a}|\widehat{\mathcal{T}}^A\rangle\right)$ and $\hat{a} = \max^{P_1}\left(\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \cup \hat{A}^{\hat{b} \rightharpoonup \hat{a}}\right)$. Therefore, $f^*(P_1, P_2) P_1 f^*(P_1', P_2)$. Symmetrically, in the fourth case, we have $f^*(P_1, P_2) P_1 f^*(P_1', P_2)$. In the fifth case, we know $r_1(P_1) \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$, $f^*(P_1, P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right) \in \langle r_1(P_1), \hat{a}|\widehat{\mathcal{T}}^A\rangle$, $r_1(P_1') \in \langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle$ and $f^*(P_1', P_2) = r_1(P_1') \in \langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle$. Since P_1 is $(\hat{a}, \hat{b})$-semi-hybrid on $\widehat{\mathcal{T}}^A$, it is true that $\hat{a} = \min^{P_1}\left(\langle r_1(P_1), \hat{a}|\widehat{\mathcal{T}}^A\rangle\right)$ and $\hat{a} = \max^{P_1}\left(\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle\right)$. Therefore, either $f^*(P_1, P_2) = f^*(P_1', P_2) = \hat{a}$ or $f^*(P_1, P_2) P_1 f^*(P_1', P_2)$ holds. Symmetrically, in the last case, we have $f^*(P_1, P_2) = f^*(P_1', P_2) = \hat{b}$ or $f^*(P_1, P_2) P_1 f^*(P_1', P_2)$. Therefore, voter 1 has no incentive to manipulate. In conclusion, f^* is strategy-proof.

Now, we are ready to induce a contradiction from f^*. Since f^* is strategy-proof, we know $f^* \in \Omega$. Since $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$, we have $x \in \langle a, b|\mathcal{T}^A\rangle \backslash \langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle$. According to $\widehat{\mathcal{T}}^A$, it is clear that $x \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$ or $x \in \hat{A}^{\hat{b} \rightharpoonup \hat{a}} \backslash \{\hat{b}\}$. We assume w.l.o.g. that $x \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$. On the one hand, by construction, we have $f^*(x, \hat{a}) = \text{Proj}\left(\hat{a}, \langle x, \hat{a}|\widehat{\mathcal{T}}^A\rangle\right) = \hat{a}$. On the other hand, since $f^* \in \Omega$, Claim 3 implies that f^* behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Then, by the construction of f^*, it must be the case that voter 1 dictates on $\langle a, b|\mathcal{T}^A\rangle$. Consequently, given $x, \hat{a} \in \langle a, b|\mathcal{T}^A\rangle$, we have $f^*(x, \hat{a}) = x$. Contradiction! This completes the verification of the claim.

This complete the verification of the lemma, and hence proves the necessity part of Statement (ii) of the Theorem. ■

E Proof of Corollary 2

To prove statements (i) and (ii) of Corollary 2, we fix a rich non-dictatorial tops-only domain $\mathbb{D}$.

We first show statement (i). Clearly, the sufficiency part follows from the sufficiency part of Statement (i) of the Theorem. Next, given that $\mathbb{D}$ admits an invariant and strategy-proof rule, by the necessity part of Statement (i) of the Theorem, we know that $\mathbb{D}$ is semi-single-peaked on a tree $\mathcal{T}^A$ w.r.t. some $\bar{x} \in A$. Furthermore, since $\mathbb{D}$ is a tops-only domain, statement (i) of Proposition 1 implies that $\mathbb{D}$ must be single-peaked on $\mathcal{T}^A$. Last, since $\mathbb{D}$ includes the pair of completely reversed preferences $\underline{P}_i = (a_1 \cdots a_k\, a_{k+1} \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_{k+1}\, a_k \cdots a_1)$, it must be the case that $\mathcal{T}^A = \mathcal{L}^A$. Therefore, $\mathbb{D}$ is single-peaked on $\mathcal{L}^A$. This proves statement (i) of Corollary 2.

We next move to statement (ii). To verify the sufficiency part, we further let $\mathbb{D}$ be a non-trivial (a_p, a_q)-hybrid domain on $\mathcal{L}^A$, where $1 < q - p < m - 1$. Clearly, $\mathbb{D}$ is also a semi-hybrid domain, and hence the sufficiency part of Statement (ii) of the Theorem implies that there exists no invariant, tops-only and strategy-proof rule. Moreover, since $\mathbb{D}$ is a tops-only domain, there exists no invariant and strategy-proof rule. This proves the sufficiency part of statement (ii).

To prove the necessity part of statement (ii), let $\mathbb{D}$ admit no invariant and strategy-proof rule. We show that $\mathbb{D}$ is a non-trivial (a_p, a_q)-hybrid domain on $\mathcal{L}^A$, where $1 < q - p < m - 1$. Since $\mathbb{D}$ is a tops-only domain, the hypothesis also implies that $\mathbb{D}$ admits no invariant, tops-only and strategy-proof rule. Then, by applying Statement (ii) of the Theorem and its proof on $\mathbb{D}$, we know that (i) $\mathbb{D}$ is a semi-hybrid domain on a tree $\mathcal{T}^A$ w.r.t. some thresholds a and b, and (ii) every tops-only and strategy-proof rule behaves likes a dictatorship on $\langle a, b | \mathcal{T}^A \rangle$. Moreover, since $\mathbb{D}$ is a tops-only domain, it is natural that every strategy-proof rule behaves likes a dictatorship on $\langle a, b | \mathcal{T}^A \rangle$. Then, the necessity part of the Auxiliary Proposition implies that $\mathbb{D}$ is non-trivial on $\langle a, b | \mathcal{T}^A \rangle$. Furthermore, since $\mathbb{D}$ is a tops-only domain, statement (ii) of Proposition 1 refines $\mathbb{D}$ to be (a, b)-hybrid on $\mathcal{T}^A$. Therefore, $\mathbb{D}$ is a non-trivial (a, b)-hybrid domain on $\mathcal{T}^A$. We next show that $\mathbb{D}$ is non-degenerate. Otherwise, $\mathbb{D}$ is a non-trivial and degenerate (a,b)-hybrid domain on $\mathcal{T}^A$. Consequently, the sufficiency part of the Auxiliary Proposition implies that every strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle = A$, and hence $\mathbb{D}$ is a dictatorial domain which contradicts the non-dictatorial-domain hypothesis. Therefore, $\mathbb{D}$ is a non-trivial and non-degenerate (a,b)-hybrid domain on $\mathcal{T}^A$. Furthermore, since $\mathbb{D}$ includes the completely reversed preferences $\underline{P}_i = (a_1 \cdots a_k\, a_{k+1} \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_{k+1}\, a_k \cdots a_1)$, to be compatible with (a,b)-hybridness on $\mathcal{T}^A$, it must be the case that $\mathcal{T}^A$ is a line. Thus, $\mathcal{T}^{A^{a\rightharpoonup b}}$ is a line if $A^{a\rightharpoonup b} \neq \{a\}$, and $\mathcal{T}^{A^{b\rightharpoonup a}}$ is a line if $A^{b\rightharpoonup a} \neq \{b\}$. We last refine $\mathbb{D}$ to be an (a_p, a_q)-hybrid domain on the line $\mathcal{L}^A$, where $1 < q-p < m-1$.[70]

Lemma **25** *Domain $\mathbb{D}$ is an (a_p, a_q)-hybrid domain on $\mathcal{L}^A$, where $1 < q-p < m-1$.*

Proof: Recall Step 1 in the proof of Statement (ii) of the Theorem, where we elicit the line $\mathcal{L} = (a_1 = x_1, \dots, x_v = a_m)$ (when there exists a unique path in $G^A_{\sim}$ connecting a_1 and a_m) and construct the line $\mathcal{L} = (a_1 = x_1, \dots, x_v = a_m)$ (when there are multiple paths in $G^A_{\sim}$ connecting a_1 and a_m). Furthermore, by the construction of $\mathcal{T}^A$ right above Lemma 24 and the definition of (a,b)-hybridness on $\mathcal{T}^A$, we know that (i) $a = x_p$ and $b = x_t$ for some $1 \leq p < t \leq v$, (ii) $(x_1, \dots, x_p)$ is included in $\mathcal{T}^{A^{a\rightharpoonup b}}$, and (iii) $(x_t, \dots, x_v)$ is included in $\mathcal{T}^{A^{b\rightharpoonup a}}$.

We next show $\mathcal{T}^{A^{a\rightharpoonup b}} = (x_1, \dots, x_p)$. If $A^{a\rightharpoonup b} = \{a\}$, it is evident that $\mathcal{T}^{A^{a\rightharpoonup b}}$ is a graph of the singleton vertex $a = x_1$. Next, let $A^{a\rightharpoonup b} \neq \{a\}$. Then, the inclusion of two completely reversed preferences implies $x_p = a \in Ext(\mathcal{T}^{A^{a\rightharpoonup b}})$ (see Clarification 3 of Appendix G). Recall that $\mathcal{T}^{A^{a\rightharpoonup b}}$ is a line and contains $(x_1, \dots, x_p)$. Hence, to show $\mathcal{T}^{A^{a\rightharpoonup b}} = (x_1, \dots, x_p)$, it suffices to show $x_1 \in Ext(\mathcal{T}^{A^{a\rightharpoonup b}})$. Suppose not, i.e., we have $x \in A^{a\rightharpoonup b}$ such that $x \neq x_2$ and $(x, x_1) \in \mathcal{E}^{A^{a\rightharpoonup b}}$. Clearly, $(x, x_1) \in \mathcal{E}^A$. Note that $a_m = x_v \in A\backslash A^{a\rightharpoonup b}$. Thus, since $\mathcal{T}^A$ is a line, it is true that $x_1 \in \langle a_m, x|\mathcal{T}^A\rangle$. Consequently, (a,b)-hybridness on $\mathcal{T}^A$ implies $x_1 P_i x$ for all $P_i \in \mathbb{D}^{a_m}$. This contradicts the fact that $x_1 = a_1$ is bottom ranked in

[70]Note that so far the line $\mathcal{T}^A$ here is not necessarily the line $\mathcal{L}^A$.

the preference $\overline{P}_i$. Therefore, $x_1 \in Ext(\mathcal{T}^{A^{a\rightharpoonup b}})$, and hence $\mathcal{T}^{A^{a\rightharpoonup b}} = (x_1, \ldots, x_p)$. Symmetrically, we have $\mathcal{T}^{A^{b\rightharpoonup a}} = (x_t, \ldots, x_v)$.

Moreover, according to the labeling of alternatives in the preference $\underline{P}_i = (a_1 \cdots a_k\, a_{k+1} \cdots a_m)$ which is (a,b)-hybrid on $\mathcal{T}^A$, $\mathcal{T}^{A^{a\rightharpoonup b}} = (x_1, \ldots, x_p)$ and $\mathcal{T}^{A^{b\rightharpoonup a}} = (x_t, \ldots, x_v)$ respectively imply $x_k = a_k$ for all $k = 1, \ldots, p$, and $x_l = a_{m-v+l}$ for all $l = t, \ldots, v$. For notational convenience, let $q = m - v + t$. Therefore, $a = a_p$, $b = a_q$, $\mathcal{T}^{A^{a\rightharpoonup b}} = (a_1, \ldots, a_p)$ and $\mathcal{T}^{A^{b\rightharpoonup a}} = (a_q, \ldots, a_m)$. Since $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$ by the definition of (a,b)-hybridness on $\mathcal{T}^A$, we know $A^{a\rightharpoonup b} \cup A^{b\rightharpoonup a} = \{a_1, \ldots, a_p, a_q, \ldots, a_m\} \neq A$ and hence $q - p > 1$. Therefore, $\mathbb{D}$ is an (a_p, a_q)-hybrid domain on $\mathcal{L}^A$. Last, since $\mathbb{D}$ is non-degenerate, we know that either $p > 1$ or $q < m$ holds, which implies $q - p < m - 1$. This completes the verification of the lemma and hence proves the necessity part of statement (ii) of Corollary 2. ■

Next, we show that given a rich domain $\mathbb{D}$, it is a tops-only domain if and only if it is single-peaked on $\mathcal{L}^A$ or non-trivially hybrid on $\mathcal{L}^A$.

First, let $\mathbb{D}$ be a rich tops-only domain. We know that $\mathbb{D}$ is either a non-dictatorial domain or a dictatorial domain. If $\mathbb{D}$ is a non-dictatorial domain, statements (i) and (ii) of Corollary 2 imply that $\mathbb{D}$ is either single-peaked on $\mathcal{L}^A$, or non-trivially and non-degenerate hybrid on $\mathcal{L}^A$. Next, let $\mathbb{D}$ be a dictatorial domain. Clearly, $\mathbb{D} \subseteq \mathbb{P} = \mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_1, a_m)$. Thus, as a dictatorial domain, every strategy-proof rule behaves like a dictatorship on $A = \langle a_1, a_m|\mathcal{L}^A\rangle$. Then, by the necessity part of the Auxiliary Proposition and Remark 1, we know that $\mathbb{D}$ is a non-trivial (a_1, a_m)-semi-hybrid domain on $\langle a_1, a_m|\mathcal{L}^A\rangle$. Furthermore, $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_1, a_m) = \mathbb{D}_{\text{H}}(\mathcal{L}^A, a_1, a_m)$ implies that $\mathbb{D}$ is also a non-trivial and degenerate (a_1, a_m)-hybrid domain on $\mathcal{L}^A$. In conclusion, $\mathbb{D}$ is either single-peaked on $\mathcal{L}^A$ or non-trivially hybrid on $\mathcal{L}^A$.

Second, let a rich domain $\mathbb{D}$ be single-peaked on $\mathcal{L}^A$, or non-trivially hybrid on $\mathcal{L}^A$. We show that $\mathbb{D}$ is a tops-only domain. If $\mathbb{D}$ is single-peaked on $\mathcal{L}^A$, we can apply Theorem 3 of Achuthankutty and Roy (2018) to show that $\mathbb{D}$ is a tops-only domain.[71]

[71] Achuthankutty and Roy (2018) study a single-peaked domain on $\mathcal{L}^A$ satisfying the following

Last, let $\mathbb{D}$ be a non-trivial (a_p, a_q)-hybrid domain on $\mathcal{L}^A$ where $q-p>1$. We show that $\mathbb{D}$ is a tops-only domain. We will adopt an inductive argument for the proof. To simplify the proof, we follow the method of Chatterji and Sen (2011) by including the single-voter SCFs into consideration.

Clearly, unanimity implies the tops-only property in every single-voter SCF. We then provide an induction hypothesis.

Induction Hypothesis: Given $n \geq 2$, for all $1 \leq n' < n$, every strategy-proof rule $f' : \mathbb{D}^{n'} \to A$ satisfies the tops-only property.

We show that every n-voter strategy-proof rule satisfies the tops-only property. Henceforth, we **fix** an arbitrary n-voter strategy-proof rule $f : \mathbb{D}^n \to A$, and show that it satisfies the tops-only property. It suffices to show that for all $i \in N$ and $(P_i, P_{-i}), (P'_i, P_{-i}) \in \mathbb{D}^n$, $\left[r_1(P_i) = r_1(P'_i)\right] \Rightarrow \left[f(P_i, P_{-i}) = f(P'_i, P_{-i})\right]$.

First, by the sufficiency part of the Auxiliary Proposition, we know that f behaves like a dictatorship on $\langle a_p, a_q|\mathcal{L}^A\rangle$. Furthermore, we can identify $1 \leq s \leq p$ and $q \leq t \leq m$ such that (i) f behaves like a dictatorship on $\langle a_s, a_t|\mathcal{L}^A\rangle$, and (ii) for all $1 \leq s' \leq s$ and $t \leq t' \leq m$ with $t'-s' > t-s$, f does not behaves like a dictatorship on $\langle a_{s'}, a_{t'}|\mathcal{L}^A\rangle$. In particular, if condition (i) is satisfied at $s=1$ and $t=m$, then f is a dictatorship and hence satisfies the tops-only property. Henceforth, we further assume either $s>1$ or $t<m$. According to condition (i), we assume w.l.o.g. that voter 1 dictates on $\langle a_s, a_t|\mathcal{L}^A\rangle$ at f. Furthermore, if $n>2$, according to f, we induce a two-voter function: $h(P_1, P_2) = f(P_1, P_2, \cdots, P_2)$ for all $P_1, P_2 \in \mathbb{D}$. It is clear that h is a well defined SCF, inherits unanimity and strategy-proofness from f, and hence is a strategy-proof rule.

The claim below shows that when voter 1 reports a preference with the peak in $\langle a_s, a_t|\mathcal{L}^A\rangle$, the social outcome follows exactly from voter 1's peak regardless of the others' preferences.

Claim 1: Given a preference $P_1^* \in \mathbb{D}$ with $r_1(P_1^*) = a_k \in \langle a_s, a_t|\mathcal{L}^A\rangle$, we have $f(P_1^*, P_{-1}) = a_k$ for all $P_{-1} \in \mathbb{D}^{n-1}$.

richness assumption: $a_k \in \mathcal{S}(\mathbb{D}^{a_{k+1}})$ and $a_{k+1} \in \mathcal{S}(\mathbb{D}^{a_k})$ for all $1 \leq k \leq m-1$, which is weaker than the imposition of path-connectedness on a single-peaked domain on $\mathcal{L}^A$.

Suppose not, i.e., there exists $P_{-1} \in \mathbb{D}^{n-1}$ such that $f(P_1^*, P_{-1}) = a_r \neq a_k$. It is clear that strategy-proofness implies $f\big(P_1^*, (a_r \cdots)\big) = a_r$ (if $n = 2$) and $f\big(P_1^*, (a_r \cdots), \ldots, (a_r \cdots)\big) = a_r$ (if $n > 2$). There are three cases: (i) $a_r \in \langle a_s, a_t | \mathcal{L}^A \rangle$, (ii) $a_r \in \big\langle a_1, a_{s-1} | \mathcal{L}^A \big\rangle$ and (iii) $a_r \in \big\langle a_{t+1}, a_m | \mathcal{L}^A \big\rangle$. In each case, we induce a contradiction.

In case (i), since voter 1 dictates $\langle a_s, a_t | \mathcal{L}^A \rangle$, we have $f\big(P_1^*, (a_r \cdots)\big) = a_k$ (if $n = 2$) and $f\big(P_1^*, (a_r \cdots), \ldots, (a_r \cdots)\big) = a_k$ (if $n > 2$). Contradiction! In case (ii), we first identify a path $(x_1, \ldots, x_v)$ in $G_\sim^A$ that connects a_k and a_r. By the definition of (a_p, a_q)-hybridness on $\mathcal{L}^A$, according to $G_\sim^A$, there exist $1 \leq \eta < v$ such that $x_\eta = a_s$, the subpath $(x_1, \ldots, x_\eta)$ is contained in $G_\sim^{\langle a_s, a_t | \mathcal{L}^A \rangle}$, and the subpath $(x_\eta, \ldots, x_v) = \langle a_s, a_r | \mathcal{L}^A \rangle$ is the unique path in $G_\sim^A$ connecting a_s and a_r. If $\eta > 1$, $x_1, x_2 \in \langle a_s, a_t | \mathcal{L}^A \rangle$. If $\eta = 1$, we identify $x_0 \in \langle a_s, a_t | \mathcal{L}^A \rangle$ such that $x_0 \sim x_1$, and construct the path $(x_0, x_1, \ldots, x_v)$. Overall, we have a path $\pi = (x_1, x_2, \ldots, x_v)$, $v \geq 3$, in $G_\sim^A$ such that $x_1, x_2 \in \langle a_s, a_t | \mathcal{L}^A \rangle$, $a_k \in \{x_1, x_2\}$ and $a_r = x_v$.

We first consider the case $n = 2$. On the one hand, given a preference $P_2^* \in \mathbb{D}^{a_r}$ by path-connectedness, we have $f(P_1^*, P_2^*) = a_r$. On the other hand, since voter 1 dictates on $\langle a_s, a_t | \mathcal{L}^A \rangle$ at f, we have $f\big((x_1 \cdots), (x_2 \cdots)\big) = x_1$. Then, according to the path $\pi = (x_1, x_2, \ldots, x_v)$, statement (i) of Lemma 1 implies $f\big((x_1 \cdots), (x_v \cdots)\big) = x_1$ and $f\big((x_2 \cdots), (x_v \cdots)\big) = x_2$. Since $a_k \in \{x_1, x_2\}$, we have either $f(P_1^*, P_2^*) = x_1 \neq a_r$ or $f(P_1^*, P_2^*) = x_2 \neq a_r$. Contradiction! Next, assume $n > 2$. On the one hand, given a preference $P_2^* \in \mathbb{D}^{a_r}$ by path-connectedness, we have $h(P_1^*, P_2^*) = f(P_1^*, P_2^*, \ldots, P_2^*) = a_r$. On the other hand, given $\hat{P}_1 \in \mathbb{D}^{x_1}$ and $\hat{P}_2 \in \mathbb{D}^{x_2}$ by path-connectedness, by voter 1's dictatorship on $\langle a_s, a_t | \mathcal{L}^A \rangle$ at f, we have $h(\hat{P}_1, \hat{P}_2) = f(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_2) = x_1$. Then, according to the path $\pi = (x_1, x_2, \ldots, x_v)$, statement (i) of Lemma 1 implies $h\big((x_1 \cdots), (x_v \cdots)\big) = x_1$ and $h\big((x_2 \cdots), (x_v \cdots)\big) = x_2$. Consequently, since $a_k \in \{x_1, x_2\}$, we have either $f(P_1^*, P_2^*, \ldots, P_2^*) = h(P_1^*, P_2^*) = x_1 \neq a_r$ or $f(P_1^*, P_2^*, \ldots, P_2^*) = h(P_1^*, P_2^*) = x_2 \neq a_r$. Contradiction!

By a symmetric argument, a contradiction can induced in case (iii). This completes the verification of the claim.

The next claim shows that when voter 1 reports a preference with a peak $a_k \in \langle a_1, a_{s-1}|\mathcal{L}^A\rangle$ (respectively, $a_k \in \langle a_{t+1}, a_m|\mathcal{L}^A\rangle$), then social outcome falls into the interval $\langle a_k, a_s|\mathcal{L}^A\rangle$ (respectively, $\langle a_t, a_k|\mathcal{L}^A\rangle$).

Claim 2: Given a preference $P_1^* \in \mathbb{D}$ with $r_1(P_1^*) = a_k \notin \langle a_s, a_t|\mathcal{L}^A\rangle$, the following two statements hold:

(i) If $a_k \in \langle a_1, a_{s-1}|\mathcal{L}^A\rangle$, then $f(P_1^*, P_{-1}) \in \langle a_k, a_s|\mathcal{L}^A\rangle$ for all $P_{-1} \in \mathbb{D}^{n-1}$.

(ii) If $a_k \in \langle a_{t+1}, a_m|\mathcal{L}^A\rangle$, then $f(P_1^*, P_{-1}) \in \langle a_t, a_k|\mathcal{L}^A\rangle$ for all $P_{-1} \in \mathbb{D}^{n-1}$.

The two statements are symmetric, and we hence focus on showing statement (i). Given $P_{-1} \in \mathbb{D}^{n-1}$, let $f(P_1^*, P_{-1}) = a_r$. Clearly, Claim 1 implies $f\big((a_s \cdots), P_{-1}\big) = a_s$, and (a_p, a_q)-hybridness on $\mathcal{L}^A$ implies $a_s = \max^{P_1^*}\big(\langle a_s, a_m|\mathcal{L}^A\rangle\big)$. Suppose $a_r \notin \langle a_k, a_s|\mathcal{L}^A\rangle$. Clearly, either $a_r \in \langle a_{s+1}, a_m|\mathcal{L}^A\rangle$ or $a_r \in \langle a_1, a_{k-1}|\mathcal{L}^A\rangle$ holds. If $a_r \in \langle a_{s+1}, a_m|\mathcal{L}^A\rangle$, then voter 1 will manipulate at (P_1^*, P_{-1}) via some $P_1 \in \mathbb{D}^{a_s}$, i.e., $f(P_1, P_{-1}) = a_s P_1^* a_r = f(P_1^*, P_{-1})$. Next, assume $a_r \in \langle a_1, a_{k-1}|\mathcal{L}^A\rangle$. Consider the unique path $\pi = (a_r, a_{r+1}, \ldots, a_k, a_{k+1}, \ldots, a_s)$ in $G_\sim^A$ that connects a_r and a_s. Since $a_r \sim a_{r+1}$, we have $\hat{P}_1, \hat{P}_1' \in \mathbb{D}$ such that $r_1(\hat{P}_1) = r_2(\hat{P}_1') = a_r$, $r_1(\hat{P}_1') = r_2(\hat{P}_1) = a_{r+1}$ and $r_l(\hat{P}_1) = r_l(\hat{P}_1')$ for all $l = 3, \ldots, m$. Clearly, strategy-proofness first implies $f(\hat{P}_1, P_{-1}) = a_r = r_2(\hat{P}_1')$. Then, strategy-proofness implies $f(\hat{P}_1', P_{-1}) \in \big\{r_2(\hat{P}_1'), r_1(\hat{P}_1')\big\} = \{a_r, a_{r+1}\}$. Suppose $f(\hat{P}_1', P_{-1}) = a_r$. Then, strategy-proofness implies $f\big(\hat{P}_1', (a_r \cdots)\big) = a_r$ (if $n = 2$) and $f\big(\hat{P}_1', (a_r \cdots), \ldots, (a_r \cdots)\big) = a_r$ (if $n > 2$).

We first consider the case $n = 2$. Given a preference $P_2^* \in \mathbb{D}^{a_r}$, we have $f(\hat{P}_1', P_2^*) = a_r$. Then, according to the path $\pi = (a_r, a_{r+1}, \ldots, a_k, a_{k+1}, \ldots, a_s)$, statement (ii) of Lemma 1 implies $f\big((a_s \cdots), (a_r \cdots)\big) = a_r \neq a_s$. This contradicts Claim 1. Next, assume $n > 2$. Given a preference $P_2^* \in \mathbb{D}^{a_r}$, we have $h(\hat{P}_1', P_2^*) = f(\hat{P}_1', P_2^*, \ldots, P_2^*) = a_r$. Then, according to the path $\pi = (a_r, a_{r+1}, \ldots, a_k, a_{k+1}, \ldots, a_s)$, statement (ii) of Lemma 1 implies $h\big((a_s \cdots), (a_r \cdots)\big) = a_r$. Consequently, we have $f\big((a_s \cdots), P_2^*, \ldots, P_2^*\big) = h\big((a_s \cdots), P_2^*\big) = a_r \neq a_s$, which contradicts Claim 1. This means that the contradictory hypothesis $f(\hat{P}_1', P_{-1}) = a_r$ cannot hold. Therefore, $f(\hat{P}_1', P_{-1}) = a_{r+1} = r_1(\hat{P}_1')$ holds, and hence strategy-proofness implies $f\big((a_{r+1}\cdots), P_{-1}\big) = a_{r+1}$. Following the path π from a_{r+1} to a_k, by repeatedly applying the argument above step by step, we eventually induce $f\big((a_k\cdots), P_{-1}\big) = a_k$, which contradicts the hypothesis $f(P_1^*, P_{-1}) = a_r \notin \langle a_k, a_s|\mathcal{L}^A\rangle$. Therefore, $f(P_1^*, P_{-1}) \in \langle a_k, a_s|\mathcal{L}^A\rangle$. This completes the verification of the claim.

Claim 3: Given two distinct profiles $(P_1, P_{-1}), (P_1', P_{-1}) \in \mathbb{D}^n$ with $r_1(P_1) = r_1(P_1') = a_k \notin \langle a_s, a_t|\mathcal{L}^A\rangle$, we have $f(P_1, P_{-1}) = f(P_1', P_{-1})$.

We know that either $a_k \in \langle a_1, a_{s-1}|\mathcal{L}^A\rangle$ or $a_k \in \langle a_{t+1}, a_m|\mathcal{L}^A\rangle$ holds. The verification for these two cases are symmetric. We hence assume w.l.o.g. that $a_k \in \langle a_1, a_{s-1}|\mathcal{L}^A\rangle$. Then, by Claim 2, we can assume $f(P_1, P_{-1}) = a_r \in \langle a_k, a_s|\mathcal{L}^A\rangle$ and $f(P_1', P_{-1}) = a_{r'} \in \langle a_k, a_s|\mathcal{L}^A\rangle$. Suppose $a_r \neq a_{r'}$. We know that either $k \leq r < r' \leq s$ or $k \leq r' < r \leq s$ holds. If $k \leq r < r' \leq s$, (a_p, a_q)-hybridness on $\mathcal{L}^A$ implies $a_r P_1' a_{r'}$, and hence voter 1 will manipulate at (P_1', P_{-1}) via P_1, i.e., $f(P_1, P_{-1}) = a_r P_1' a_{r'} = f(P_1', P_{-1})$. If $k \leq r' < r \leq s$, (a_p, a_q)-hybridness on $\mathcal{L}^A$ implies $a_{r'} P_1 a_r$, and hence voter 1 will manipulate at (P_1, P_{-1}) via P_1', i.e., $f(P_1', P_{-1}) = a_{r'} P_1 a_r = f(P_1, P_{-1})$. Therefore, it must be the case that $f(P_1, P_{-1}) = f(P_1', P_{-1})$. This completes the verification of the claim.

Now, one would easily observe from Claims 1 and 3 that for all distinct preference profiles $(P_1, P_{-1}), (P_1', P_{-1}) \in \mathbb{D}^n$ with $r_1(P_1) = r_1(P_1')$, we have $f(P_1, P_{-1}) = f(P_1', P_{-1})$. Next, we move to voters other than 1.

Claim 4: Given $i \neq 1$ and two distinct profiles $(P_1, P_i, P_{-\{1,i\}}), (P_1, P_i', P_{-\{1,i\}}) \in \mathbb{D}^n$ with $r_1(P_i) = r_1(P_i')$, we have $f(P_1, P_i, P_{-\{1,i\}}) = f(P_1, P_i', P_{-\{1,i\}})$.[72]

For notational convenience, let $r_1(P_1) = a_k$ and $r_1(P_i) = r_1(P_i') = a_o$. Clearly, if $a_k \in \langle a_s, a_t|\mathcal{L}^A\rangle$, Claim 1 implies $f(P_1, P_i, P_{-\{1,i\}}) = f(P_1, P_i', P_{-\{1,i\}}) = a_k$. Next, we assume $a_k \notin \langle a_s, a_t|\mathcal{L}^A\rangle$. Thus, either $a_k \in \langle a_1, a_{s-1}|\mathcal{L}^A\rangle$ or $a_k \in \langle a_{t+1}, a_m|\mathcal{L}^A\rangle$ holds. The verification for these two cases are symmetric. We hence assume w.l.o.g. that $a_k \in \langle a_1, a_{s-1}|\mathcal{L}^A\rangle$. Suppose $f(P_1, P_i, P_{-\{1,i\}}) = a_r \neq a_{r'} = f(P_1, P_i', P_{-\{1,i\}})$. Clearly, strategy-proofness implies $a_r P_i a_{r'}$ and $a_{r'} P_i' a_r$, and

[72] If $n = 2$, then $i = 2$ and the notation $(P_1, P_i, P_{-\{1,i\}})$ represents the profile (P_1, P_2).

Claim 2 implies $a_r, a_{r'} \in \langle a_k, a_s | \mathcal{L}^A \rangle$. We can assume w.l.o.g. that $k \leq r < r' \leq s$. The verification for the case $k \leq r' < r \leq s$ is symmetric and hence is omitted.

We first show $a_o \in \langle a_{r+1}, a_{r'-1} | \mathcal{L}^A \rangle$. If $a_o \in \langle a_1, a_r | \mathcal{L}^A \rangle$, (a_p, a_q)-hybridness on $\mathcal{L}^A$ implies $a_r P_i' a_{r'}$. Contradiction! If $a_o \in \langle a_{r'}, a_m | \mathcal{L}^A \rangle$, (a_p, a_q)-hybridness on $\mathcal{L}^A$ implies $a_{r'} P_i a_r$. Contradiction! Therefore, $a_o \in \langle a_{r+1}, a_{r'-1} | \mathcal{L}^A \rangle$.

We next induce a manipulation in the case $n = 2$. Thus, $i = 2$, and unanimity implies $f(P_2', P_2') = a_o$. Since $r_1(P_1) = a_k \in \langle a_1, a_{s-1} | \mathcal{L}^A \rangle$, $a_r, a_{r'} \in \langle a_k, a_s | \mathcal{L}^A \rangle$ and $a_o \in \langle a_{r+1}, a_{r'-1} | \mathcal{L}^A \rangle$, (a_p, a_q)-hybridness on $\mathcal{L}^A$ implies $a_o P_1 a_{r'}$. Consequently, voter 1 will manipulate at (P_1, P_2') via P_2', i.e., $f(P_2', P_2') = a_o P_1 a_{r'} = f(P_1, P_2')$. Therefore, it must be the case that $f(P_1, P_2) = f(P_1, P_2')$.

Last, we induce a manipulation in the case $n > 2$. We combine voters 1 and i as one, and induce an $(n-1)$-voter function: $g(\hat{P}_1, \hat{P}_{-\{1,i\}}) = f(\hat{P}_1, \hat{P}_1, \hat{P}_{-\{1,i\}})$ for all $\hat{P}_1 \in \mathbb{D}$ and $\hat{P}_{-\{1,i\}} \in \mathbb{D}^{n-2}$. Clearly, g is a well defined SCF, inherits unanimity and strategy-proofness from f, and hence is an $(n-1)$-voter strategy-proof rule. Therefore, the induction hypothesis implies that g satisfies the tops-only property. Hence, we have $f(P_i, P_i, P_{-\{1,i\}}) = g(P_i, P_{-\{1,i\}}) = g(P_i', P_{-\{1,i\}}) = f(P_i', P_i', P_{-\{1,i\}}) \equiv a_w$. Furthermore, according to the profile $(P_i, P_i, P_{-\{1,i\}})$, Claim 2 implies $a_w \in \langle a_o, a_s | \mathcal{L}^A \rangle$. Thus, since $a_o \in \langle a_{r+1}, a_{r'-1} | \mathcal{L}^A \rangle$, either $a_w \in \langle a_o, a_{r'-1} | \mathcal{L}^A \rangle$ or $a_w \in \langle a_{r'}, a_s | \mathcal{L}^A \rangle$ holds. If $a_w \in \langle a_o, a_{r'-1} | \mathcal{L}^A \rangle$, (a_p, a_q)-hybridness on $\mathcal{L}^A$ implies $a_w P_1 a_{r'}$, and then voter 1 will manipulate at $(P_1, P_i', P_{-\{1,i\}})$ via P_i', i.e., $f(P_i', P_i', P_{-\{1,i\}}) = a_w P_1 a_{r'} = f(P_1, P_i', P_{-\{1,i\}})$. Therefore, it must be the case that $a_w \in \langle a_{r'}, a_s | \mathcal{L}^A \rangle$. Recall $a_r P_i a_{r'}$ by strategy-proofness of f, and $a_{r'} = \max^{P_i}\big(\langle a_{r'}, a_s | \mathcal{L}^A \rangle\big)$ by the definition of (a_p, a_q)-hybridness on $\mathcal{L}^A$. Therefore, $a_r P_i a_w$. Consequently, voter 1 will manipulate at $(P_i, P_i, P_{-\{1,i\}})$ via P_1, i.e., $f(P_1, P_i, P_{-\{1,i\}}) = a_r P_1 a_w = f(P_i, P_i, P_{-\{1,i\}})$. Therefore, it must be the case that $f(P_1, P_i, P_{-\{1,i\}}) = f(P_1, P_i', P_{-\{1,i\}})$. This completes the verification of the claim.

Now, by Claims 1, 3 and 4, we know that for all $i \in N$ and $(P_i, P_{-i}), (P_i', P_{-i}) \in \mathbb{D}^n$, $\big[r_1(P_i) = r_1(P_i')\big] \Rightarrow \big[f(P_i, P_{-i}) = f(P_i', P_{-i})\big]$, as required. Therefore, f satisfies the tops-only property. This completes the verification of the induction hypothesis, and hence shows that $\mathbb{D}$ is a tops-only domain.

F Proof of Proposition 3

Fixing a tree $\mathcal{T}^A$ and thresholds $a, b \in A$, let $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$ and the semi-hybrid domain $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ contain a pair of completely reversed preferences. We fix a tops-only and strategy-proof rule $f : \big[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\big]^n \to A$. We henceforth assume w.l.o.g. that $A^{a\rightharpoonup b} \neq \{a\}$ and $A^{b\rightharpoonup a} \neq \{b\}$, and show that f is a hybrid rule w.r.t. a and b.[73] Since $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ includes the completely reversed preferences $\underline{P}_i$ and $\overline{P}_i$, we infer that $a \in Ext(\mathcal{T}^{A^{a\rightharpoonup b}})$ and $b \in Ext(\mathcal{T}^{A^{b\rightharpoonup a}})$. Henceforth, let $\bar{a}$ be the unique neighbor of a in $\mathcal{T}^{A^{a\rightharpoonup b}}$, and $\bar{b}$ be the unique neighbor of b in $\mathcal{T}^{A^{b\rightharpoonup a}}$. Furthermore, note that every two alternatives of $\langle a, b|\mathcal{T}^A\rangle$ are adjacent, and hence $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \emptyset$ which implies that $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ is non-trivial on $\langle a, b|\mathcal{T}^A\rangle$. Then, the sufficiency part of the Auxiliary Proposition implies that f behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Henceforth, we assume w.l.o.g. that voter 1 dictates on $\langle a, b|\mathcal{T}^A\rangle$, i.e., for all $P \in \big[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\big]^n$, $\big[r_1(P_i) \in \langle a, b|\mathcal{T}^A\rangle \text{ for all } i \in N\big] \Rightarrow \big[f(P) = r_1(P_1)\big]$.

Claim 1: For all $P \in \big[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\big]^n$, if $r_1(P_1) \in \langle a, b|\mathcal{T}^A\rangle$, then $f(P) = r_1(P_1)$.

We can employ the verification of Claim 1 in the proof of Corollary 2 to prove this claim.

Given arbitrary $x \in Ext(\mathcal{T}^A) \cap A^{a\rightharpoonup b}$ and $y \in Ext(\mathcal{T}^A) \cap A^{b\rightharpoonup a}$, we can label the path $\langle x, y|\mathcal{T}^A\rangle = (x_1, \dots, x_{s-1}, x_s, \dots, x_t, x_{t+1}, \dots, x_v)$ where $x_1 = x$, $x_{s-1} = \bar{a}$, $x_s = a$, $x_t = b$, $x_{t+1} = \bar{b}$ and $x_v = y$. Clearly, $(x_1, \dots, x_{s-1}, x_s) = \langle x, a|\mathcal{T}^A\rangle = \langle x, a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle = \langle x, a|G_{\sim}^{A^{a\rightharpoonup b}}\rangle$ and $(x_t, x_{t+1}, \dots, x_v) = \langle b, y|\mathcal{T}^A\rangle = \langle b, y|\mathcal{T}^{A^{b\rightharpoonup a}}\rangle = \langle b, y|G_{\sim}^{A^{b\rightharpoonup a}}\rangle$. Furthermore, since every two alternatives of $\langle a, b|\mathcal{T}^A\rangle$ form an edge in $G_{\sim}^A$, it is true that the line $\langle a, b|\mathcal{T}^A\rangle$ is contained in the adjacency subgraph $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$. Therefore, $(x_s, \dots, x_t)$ is a path in $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ that connects a and b, and includes all alternatives of $\langle a, b|\mathcal{T}^A\rangle$. Last, we construct a linear order $\prec^{x,y}$ over all alternatives of $\langle x, y|\mathcal{T}^A\rangle$ such that $x_k \prec^{x,y} x_{k+1}$ for all $k = 1, \dots, v-1$.[74]

Since $\mathbb{D}_{\text{SP}}(\mathcal{T}^A) \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, we can extract an SCF from f: $\bar{f}(P) = f(P)$ for all $P \in \big[\mathbb{D}_{\text{SP}}(\mathcal{T}^A)\big]^n$. Clearly, $\bar{f}$ inherits unanimity, the tops-only property and

[73] When $A^{a\rightharpoonup b} = \{a\}$ or $A^{b\rightharpoonup a} = \{b\}$ holds, the proof is relative simpler.

[74] For notational convenience, given $c, d \in \langle x, y|\mathcal{T}^A\rangle$, let $c \preceq^{x,y} d$ denote $c = d$ or $c \prec^{x,y} d$.

strategy-proofness from f. Then, by Theorem 1 of Schummer and Vohra (2002), we know that

(1) for all $x \in Ext(\mathcal{T}^A) \cap A^{a \rightharpoonup b}$ and $y \in Ext(\mathcal{T}^A) \cap A^{b \rightharpoonup a}$, $\bar{f}$ *behaves like a generalized median voter rule* on $\langle x, y|\mathcal{T}^A\rangle$, i.e., there exists a fixed ballot $\boldsymbol{b}_J^{x,y} = \bar{f}\big(\frac{y}{J}, \frac{x}{N\backslash J}\big) \in \langle x, y|\mathcal{T}^A\rangle$ for each coalition $J \subseteq N$, which satisfies ballot unanimity, $\boldsymbol{b}_N^{x,y} = y$ and $\boldsymbol{b}_{\emptyset}^{x,y} = x$, and monotonicity, $[J \subset J'] \Rightarrow \big[\boldsymbol{b}_J^{x,y} \preceq^{x,y} \boldsymbol{b}_{J'}^{x,y}\big]$, such that for all $P \in \big[\mathbb{D}_{\text{SP}}(\mathcal{T}^A)\big]^n$, we have

$$\big[r_1(P_i) \in \langle x, y|\mathcal{T}^A\rangle \text{ for all } i \in N\big] \Rightarrow \left[\bar{f}(P) = \max_{J \subseteq N}{}^{\prec^{x,y}} \left(\min_{j \in J}{}^{\prec^{x,y}} \big(r_1(P_j), \boldsymbol{b}_J^{x,y}\big)\right)\right].$$

(2) for all $x, x' \in Ext(\mathcal{T}^A) \cap A^{a \rightharpoonup b}$ and $y, y' \in Ext(\mathcal{T}^A) \cap A^{b \rightharpoonup a}$, given $x \neq x'$ or $y \neq y'$, $\bar{f}$ behaves *consistently* on $\langle x, y|\mathcal{T}^A\rangle$ and $\langle x', y'|\mathcal{T}^A\rangle$, i.e., given the two families of fixed ballots $(\boldsymbol{b}_J^{x,y})_{J \subseteq N}$ and $(\boldsymbol{b}_J^{x',y'})_{J \subseteq N}$, for all $J \subseteq N$, we have

$$\Big[\big|\langle x, y|\mathcal{T}^A\rangle \cap \langle x', y'|\mathcal{T}^A\rangle \cap \langle y, y'|\mathcal{T}^A\rangle\big| \leq 1\Big] \Rightarrow \Big[\big|\langle x, y|\mathcal{T}^A\rangle \cap \langle x', y'|\mathcal{T}^A\rangle \cap \langle \boldsymbol{b}_J^{x,y}, \boldsymbol{b}_J^{x',y'}|\mathcal{T}^A\rangle\big| \leq 1\Big] \text{ and}$$
$$\Big[\big|\langle x, y|\mathcal{T}^A\rangle \cap \langle x', y'|\mathcal{T}^A\rangle \cap \langle x, x'|\mathcal{T}^A\rangle\big| \leq 1\Big] \Rightarrow \Big[\big|\langle x, y|\mathcal{T}^A\rangle \cap \langle x', y'|\mathcal{T}^A\rangle \cap \langle \boldsymbol{b}_J^{x,y}, \boldsymbol{b}_J^{x',y'}|\mathcal{T}^A\rangle\big| \leq 1\Big].$$

Since both $\mathbb{D}_{\text{SP}}(\mathcal{T}^A)$ and $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ are minimally rich, by the tops-only property of f and the construction of $\bar{f}$, condition (1) implies that for all $x \in Ext(\mathcal{T}^A) \cap A^{a \rightharpoonup b}$ and $y \in Ext(\mathcal{T}^A) \cap A^{b \rightharpoonup a}$, f also behaves like a generalized median voter rule on $\langle x, y|\mathcal{T}^A\rangle$ w.r.t. the fixed ballots $(\boldsymbol{b}_J^{x,y})_{J \subseteq N}$. Hence, $\boldsymbol{b}_J^{x,y} = f\big(\frac{y}{J}, \frac{x}{N\backslash J}\big)$ for all $J \subseteq N$.

The claim below shows that by voter 1's dictatorship on $\langle x, y|\mathcal{T}^A\rangle$ and the definition of $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, additional restrictions are embedded in the fixed ballots.

Claim 2: Given $x \in Ext(\mathcal{T}^A) \cap A^{a \rightharpoonup b}$ and $y \in Ext(\mathcal{T}^A) \cap A^{b \rightharpoonup a}$, we have $[1 \in J] \Rightarrow \big[\boldsymbol{b}_J^{x,y} \in \{b, y\}\big]$ and $[1 \notin J] \Rightarrow \big[\boldsymbol{b}_J^{x,y} \in \{x, a\}\big]$.

We focus on showing $[1 \in J] \Rightarrow \big[\boldsymbol{b}_J^{x,y} \in \{b, y\}\big]$. The verification for $[1 \notin J] \Rightarrow \big[\boldsymbol{b}_J^{x,y} \in \{x, a\}\big]$ is symmetric.

Fixing a coalition $J \subseteq N$, let $1 \in J$. Clearly, J is non-empty. If $J = N$, $\boldsymbol{b}_J^{x,y} = y$ by definition. We next assume $J \subset N$. Consider the profile $\big(b, \frac{y}{J\backslash\{1\}}, \frac{x}{N\backslash J}\big)$. Claim 1 first implies $f\big(b, \frac{y}{J\backslash\{1\}}, \frac{x}{N\backslash J}\big) = b$. Next, along the path $\langle b, y|\mathcal{T}^A\rangle =$

$\langle b, y|G^{A^{b\rightharpoonup a}}_{\sim}\rangle$, similar to statement (iii) of Lemma 9, $f\big(b, \frac{y}{J\backslash\{1\}}, \frac{x}{N\backslash J}\big) = b$ implies $f\big(y, \frac{y}{J\backslash\{1\}}, \frac{x}{N\backslash J}\big) \in \langle b, y|\mathcal{T}^A\rangle$. Hence, $\boldsymbol{b}_J^{x,y} \in \langle b, y|\mathcal{T}^A\rangle$.

Suppose $\boldsymbol{b}_J^{x,y} = z \in \langle b, y|\mathcal{T}^A\rangle\backslash\{b, y\}$. We induce a two-voter SCF: $\hat{f}(P_i, P_j) = f\big(\frac{P_i}{J}, \frac{P_j}{N\backslash J}\big)$ for all $P_i, P_j \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$. Clearly, $\hat{f}$ is well defined, and inherits its unanimity, tops-onlyness and strategy-proofness from f. Thus, $\hat{f}(y, x) = f\big(\frac{y}{J}, \frac{x}{N\backslash J}\big) = \boldsymbol{b}_J^{x,y} = z$. By the definition of $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, we have a preference $P_j \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ such that $r_1(P_j) = x$ and yP_jz. Consequently, agent j will manipulate at $\hat{f}$, i.e., $\hat{f}(y, y) = yP_jz = \hat{f}(y, P_j)$. Therefore, $\boldsymbol{b}_J^{x,y} \in \{b, y\}$. This completes the verification of the claim.

Claim 3: Given $x \in Ext(\mathcal{T}^A) \cap A^{a\rightharpoonup b}$ and $y \in Ext(\mathcal{T}^A) \cap A^{b\rightharpoonup a}$, there exist $W^y \subseteq N$ and $W^x \subseteq N$ with $1 \in W^y \cap W^x$ such that the following two statements hold: for all $J \subseteq N$,

(i) given $1 \in J$, $[W^y \subseteq J] \Rightarrow [\boldsymbol{b}_J^{x,y} = y]$ and $[W^y \nsubseteq J] \Rightarrow [\boldsymbol{b}_J^{x,y} = b]$, and

(ii) given $1 \notin J$, $[W^x \subseteq J \cup \{1\}] \Rightarrow [\boldsymbol{b}_J^{x,y} = x]$ and $[W^y \nsubseteq J \cup \{1\}] \Rightarrow [\boldsymbol{b}_J^{x,y} = a]$.

The two statements are symmetric, and we hence focus on showing the first one. Clearly, $\boldsymbol{b}_{\{1\}}^{x,y} \in \{b, y\}$ by Claim 2. If $\boldsymbol{b}_{\{1\}}^{x,y} = y$, monotonicity implies $\boldsymbol{b}_J^{x,y} = y$ for all $J \subseteq N$ with $1 \in J$. Then, we set $W^y = \{1\}$ to meet statement (i). Next, we assume $\boldsymbol{b}_{\{1\}}^{x,y} = b$. Since $\boldsymbol{b}_N^{x,y} = y$ by definition, there exists $W^y \subseteq N$ with $1 \in W^y$ such that $\boldsymbol{b}_{W^y}^{x,y} = y$ and $[1 \in J \text{ and } |J| < |W^y|] \Rightarrow [\boldsymbol{b}_J^{x,y} = b]$. By monotonicity, we know that for all $J \subseteq N$, $[W^y \subseteq J] \Rightarrow [\boldsymbol{b}_J^{x,y} = y]$. In the rest of verification, we consider an arbitrary coalition $J \subseteq N$ such that $1 \in J$ and $W^y \nsubseteq J$.

For notational convenience, let $W^y = \{1, \ldots, l, l+1, \ldots, k\}$, $W^y\backslash J = \{l+1, \ldots, k\}$ and $J\backslash W^y = \{k+1, \ldots, r\}$. Clearly, $W^y\backslash J \neq \emptyset$, $J = \{1, \ldots, l, k+1, \ldots, r\}$ and $N\backslash J = \{l+1, \ldots, k, r+1, \ldots, n\} \neq \emptyset$. Note that if $J\backslash W^y = \emptyset$, then $|J| < |W^y|$, and hence the definition of W^y implies $\boldsymbol{b}_J^{x,y} = b$. Henceforth, we assume $J\backslash W^y \neq \emptyset$. Then, according to f, we induce an SCF: $\hat{f} : \big[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\big]^3 \to A$ such that $\hat{f}(P_i, P_j, P_\nu) = f\big(\frac{P_i}{\{1,\ldots,l\}}, \frac{P_j}{\{k+1,\ldots,r\}}, \frac{P_\nu}{N\backslash J}\big)$ for all $P_i, P_j, P_\nu \in \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$. Clearly, $\hat{f}$ is well defined and inherits unanimity, tops-onlyness and strategy-proofness from f.

We show $\boldsymbol{b}_J^{x,y} = b$ by contradiction. Suppose $\boldsymbol{b}_J^{x,y} \neq b$. Thus, $\boldsymbol{b}_J^{x,y} = y$ by Claim 2. First, since $\{1,\ldots,l\} \subset W^y$, we have $\hat{f}(y,x,x) = f\big(\frac{y}{\{1,\ldots,l\}}, \frac{x}{\{k+1,\ldots,r\}}, \frac{x}{N\backslash J}\big) = \boldsymbol{b}_{\{1,\ldots,l\}}^{x,y} = b$. Then, strategy-proofness implies $\hat{f}(y,b,x) = b$. Furthermore, since $\bar{b} \sim b$, strategy-proofness implies $\hat{f}(y,\bar{b},x) \in \{b,\bar{b}\}$.

Second, we have $\hat{f}(y,y,x) = f\big(\frac{y}{\{1,\ldots,l\}}, \frac{y}{\{k+1,\ldots,r\}}, \frac{x}{N\backslash J}\big) = f\big(\frac{y}{J}, \frac{x}{N\backslash J}\big) = \boldsymbol{b}_J^{x,y} = y$. According to the path $\langle y,\bar{b}|\mathcal{T}^A\rangle = \langle y,\bar{b}|G_{\sim}^{A^{b\rightharpoonup a}}\rangle$, similar to statement (iii) of Lemma 9, $\hat{f}(y,y,x) = y$ implies $\hat{f}(y,\bar{b},x) \in \langle y,\bar{b}|\mathcal{T}^A\rangle$. Therefore, we have $\hat{f}(y,\bar{b},x) \in \{b,\bar{b}\} \cap \langle y,\bar{b}|\mathcal{T}^A\rangle = \{\bar{b}\}$ and hence $\hat{f}(y,\bar{b},x) = \bar{b}$.

Third, since $W^y \subseteq \{1,\ldots,l,l+1,\ldots k,r+1,\ldots,n\} = \{1,\ldots,l\} \cup [N\backslash J]$, $\boldsymbol{b}_{W^y}^{x,y} = y$ implies $\boldsymbol{b}_{\{1,\ldots,l\}\cup[N\backslash J]}^{x,y} = y$ by monotonicity. Then, we have $\hat{f}(y,x,y) = f\left(\frac{y}{\{1,\ldots,l\}}, \frac{x}{\{k+1,\ldots,r\}}, \frac{y}{N\backslash J}\right) = \boldsymbol{b}_{\{1,\ldots,l\}\cup[N\backslash J]}^{x,y} = y$. Moreover, according to $G_{\sim}^A$, we have a path π that connects x and $\bar{b}$, and excludes y. Then, along the path π, similar to statement (ii) of Lemma 9, $\hat{f}(y,x,y) = y$ implies $\hat{f}(y,\bar{b},y) = y$.

Last, by the definition of $\mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A,a,b)$, we have $P_\nu \in \mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A,a,b)$ such that $r_1(P_v) = x$ and $yP_\nu\bar{b}$. Then, voter ν will manipulate at $\hat{f}$, i.e., $\hat{f}(y,\bar{b},y) = yP_\nu\bar{b} = \hat{f}(y,\bar{b},P_\nu)$. Therefore, the hypothesis $\boldsymbol{b}_J^{x,y} = y$ cannot hold, and hence we have $\boldsymbol{b}_J^{x,y} = b$. This completes the verification of the claim.

Claim 4: Given $x,x' \in Ext(\mathcal{T}^A) \cap A^{a\rightharpoonup b}$ and $y,y' \in Ext(\mathcal{T}^A) \cap A^{b\rightharpoonup a}$, the following two statements hold:

(i) Given $y \neq y'$, we have $[\boldsymbol{b}_J^{x,y} = b] \Leftrightarrow [\boldsymbol{b}_J^{x',y'} = b]$ and $[\boldsymbol{b}_J^{x,y} = y] \Leftrightarrow [\boldsymbol{b}_J^{x',y'} = y']$ for all $J \subseteq N$ with $1 \in J$.

(ii) Given $x \neq x'$, we have $[\boldsymbol{b}_J^{x,y} = a] \Leftrightarrow [\boldsymbol{b}_J^{x',y'} = a]$ and $[\boldsymbol{b}_J^{x,y} = x] \Leftrightarrow [\boldsymbol{b}_J^{x',y'} = x']$ for all $J \subseteq N$ with $1 \notin J$.

The two statements are symmetric, and we hence focus on showing the first one. Let $y \neq y'$. Given $J \subseteq N$ with $1 \in J$, we have $\boldsymbol{b}_J^{x,y} \in \{b,y\}$ and $\boldsymbol{b}_J^{x',y'} \in \{b,y'\}$ by Claim 2. Suppose by contradiction that $\boldsymbol{b}_J^{x,y} = b$ and $\boldsymbol{b}_J^{x',y'} = y'$.[75] Let $y'' = \mathrm{Proj}\big(b, \langle y,y'|\mathcal{T}^A\rangle\big)$. Clearly, $\langle x,y|\mathcal{T}^A\rangle \cap \langle x',y'|\mathcal{T}^A\rangle \cap \langle y,y'|\mathcal{T}^A\rangle = \{y''\}$. Then, condition (2) above implies $\big|\langle x,y|\mathcal{T}^A\rangle \cap \langle x',y'|\mathcal{T}^A\rangle \cap \langle b,y'|\mathcal{T}^A\rangle\big| = \big|\langle x,y|\mathcal{T}^A\rangle \cap$

[75] The verification for the case $\boldsymbol{b}_J^{x,y} = y$ and $\boldsymbol{b}_J^{x',y'} = b$ is symmetric, and we hence omit it.

$\langle x', y'|\mathcal{T}^A\rangle \cap \langle \boldsymbol{b}_J^{x,y}, \boldsymbol{b}_J^{x',y'}|\mathcal{T}^A\rangle\big| \leq 1$, which contradicts the fact $b, \bar{b} \in \langle x, y|\mathcal{T}^A\rangle \cap \langle x', y'|\mathcal{T}^A\rangle \cap \langle b, y'|\mathcal{T}^A\rangle$. Therefore, we have $[\boldsymbol{b}_J^{x,y} = b] \Leftrightarrow [\boldsymbol{b}_J^{x',y'} = b]$ and $[\boldsymbol{b}_J^{x,y} = y] \Leftrightarrow [\boldsymbol{b}_J^{x',y'} = y']$ for all $J \subseteq N$. This completes the verification of the claim.

Now, by combining Claims 2, 3 and 4, we induce two coalitions $\mathcal{W}^{b\rightharpoonup a} \subseteq N$ and $\mathcal{W}^{a\rightharpoonup b} \subseteq N$ with $1 \in \mathcal{W}^{b\rightharpoonup a} \cap \mathcal{W}^{a\rightharpoonup b}$ such that for all $x \in Ext(\mathcal{T}^A) \cap A^{a\rightharpoonup b}$, $y \in Ext(\mathcal{T}^A) \cap A^{b\rightharpoonup a}$ and $J \subseteq N$, we have that

- given $1 \in J$, $[\mathcal{W}^{b\rightharpoonup a} \subseteq J] \Rightarrow [\boldsymbol{b}_J^{x,y} = y]$ and $[\mathcal{W}^{b\rightharpoonup a} \nsubseteq J] \Rightarrow [\boldsymbol{b}_J^{x,y} = b]$, and
- given $1 \notin J$, $[\mathcal{W}^{a\rightharpoonup b} \subseteq J \cup \{1\}] \Rightarrow [\boldsymbol{b}_J^{x,y} = x]$ and $[\mathcal{W}^{a\rightharpoonup b} \nsubseteq J \cup \{1\}] \Rightarrow [\boldsymbol{b}_J^{x,y} = a]$.

We call $\mathcal{W}^{b\rightharpoonup a}$ and $\mathcal{W}^{a\rightharpoonup b}$ winning coalitions.

For the next three claims, we fix an arbitrary preference profile $(P_1, P_{-1}) \in \big[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\big]^n$ with $r_1(P_1) \in A^{b\rightharpoonup a}\backslash\{b\}$ and let $f(P_1, P_{-1}) = z$. Clearly, strategy-proofness implies $f(z, P_{-1}) = z$. For notational convenience, let $z_i = r_1(P_i)$ for all $i \in N$ and $\mathcal{W}^{b\rightharpoonup a} = \{1, \ldots, k\}$. We fix arbitrary $x \in Ext(\mathcal{T}^A) \cap A^{a\rightharpoonup b}$ and $y \in Ext(\mathcal{T}^A) \cap A^{b\rightharpoonup a}$ such that $z_1 \in \langle b, y|\mathcal{T}^A\rangle$. Then, we have the path $\langle x, y|\mathcal{T}^A\rangle$. Furthermore, let $y_i = \text{Proj}(z_i, \langle x, y|\mathcal{T}^A\rangle)$ for all $i \in N$. Clearly, $z_1 = y_1$, and $\langle z_i, y_i|\mathcal{T}^A\rangle$ is a path in $G_\sim^A$ connecting z_i and y_i for all $i \in N\backslash\{1\}$, which may be a null path when $z_i = y_i$. We use the following diagram to illustrate.

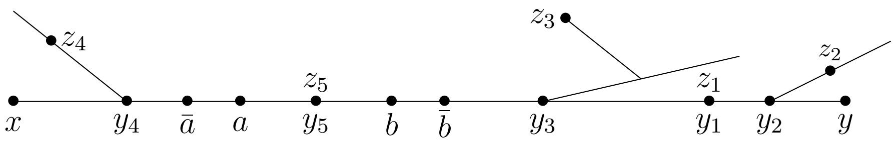


Figure 14: Given $\mathcal{W}^{b\rightharpoonup a} = \{1, 2, 3, 4, 5\}$, the peaks z_1, z_2, z_3, z_4, z_5 and their projections y_1, y_2, y_3, y_4, y_5 on $\langle x, y|\mathcal{T}^A\rangle$

By Claim 1, we know $f(b, P_{-1}) = b$. Then, according to the path $\langle b, z_1|\mathcal{T}^A\rangle = \langle b, z_1|G_\sim^{A^{b\rightharpoonup a}}\rangle$, similar to statement (iii) of Lemma 9, $f(b, P_{-1}) = b$ implies $z = f(z_1, P_{-1}) \in \langle b, z_1|\mathcal{T}^A\rangle$. Note that $\bar{b} \in \langle b, z_1|\mathcal{T}^A\rangle$ and $\bar{b}$ is the unique neighbor of b in $\mathcal{T}^{A^{b\rightharpoonup a}} = G_\sim^{A^{b\rightharpoonup a}}$. Then, we have two cases: $z \in \langle \bar{b}, z_1|\mathcal{T}^A\rangle$ and $z = b$.

Claim 5: We have $f(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) = z$.

Since $y_1 = z_1$, $f(y_1, P_{-1}) = f(z_1, P_{-1}) = z$. We next show $f(y_1, y_2, P_{-\{1,2\}}) = z$. Evidently, either $z = y_2$ or $z \neq y_2$ holds. If $z = y_2$, strategy-proofness implies $f(y_1, y_2, P_{-\{1,2\}}) = z$. If $z \neq y_2$, given $z \in \langle b, z_1|\mathcal{T}^A\rangle \subset \langle x, y|\mathcal{T}^A\rangle$, $y_2 = \text{Proj}(z_2, \langle x, y|\mathcal{T}^A\rangle)$ implies $z \notin \langle z_2, y_2|\mathcal{T}^A\rangle$. Then, along the path $\langle z_2, y_2|\mathcal{T}^A\rangle$, similar to statement (ii) of Lemma 9, $f(y_1, z_2, P_{-\{1,2\}}) = f(y_1, P_2, P_{-\{1,2\}}) = z$ implies $f(y_1, y_2, P_{-\{1,2\}}) = z$. By repeatedly applying this argument from voter 3 to n, we eventually have $f(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) = z$. This completes the verification of the claim.

Claim 6: Given $z \in \langle \bar{b}, z_1|\mathcal{T}^A\rangle$, we have $z = \text{Proj}\left(b, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b\rightharpoonup a}}\right)}\right)$.

We first show that $y_i \in \langle z, y|\mathcal{T}^A\rangle$ for all $i \in \mathcal{W}^{b\rightharpoonup a}$. Suppose not, i.e., there exists $i \in \mathcal{W}^{b\rightharpoonup a}$ such that $y_i \in \langle x, z|\mathcal{T}^A\rangle\backslash\{z\}$. Thus, $y_i \prec^{x,y} z$ which further implies that for all $J \subseteq N$ with $i \in J$, $\min^{\prec^{x,y}}_{j\in J}\left(y_j, \boldsymbol{b}^{x,y}_J\right) \preceq^{x,y} y_i \prec^{x,y} z$. Moreover, for all $J \subseteq N$ with $i \notin J$, we know either $1 \notin J$ which implies $\min^{\prec^{x,y}}_{j\in J}\left(y_j, \boldsymbol{b}^{x,y}_J\right) \preceq^{x,y} \boldsymbol{b}^{x,y}_J \preceq^{x,y} a \prec^{x,y} z$, or $1 \in J$ and $\mathcal{W}^{b\rightharpoonup a} \nsubseteq J$ which imply $\min^{\prec^{x,y}}_{j\in J}\left(y_j, \boldsymbol{b}^{x,y}_J\right) \preceq^{x,y} \boldsymbol{b}^{x,y}_J = b \prec^{x,y} z$. Overall, we have $\min^{\prec^{x,y}}_{j\in J}\left(y_j, \boldsymbol{b}^{x,y}_J\right) \prec^{x,y} z$ for all $J \subseteq N$. Consequently, we induce the following contradiction:

$$z = f(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) = \max^{\prec^{x,y}}_{J\subseteq N}\left(\min^{\prec^{x,y}}_{j\in J}\left(y_j, \boldsymbol{b}^{x,y}_J\right)\right) \prec^{x,y} z.$$

Therefore, $y_i \in \langle z, y|\mathcal{T}^A\rangle$ for all $i \in \mathcal{W}^{b\rightharpoonup a}$.

We next show that there exists $i \in \mathcal{W}^{b\rightharpoonup a}$ such that $y_i = z$. Suppose not, i.e., $y_i \neq z$ for all $i \in \mathcal{W}^{b\rightharpoonup a}$. Thus, $y_i \in \langle z, y|\mathcal{T}^A\rangle\backslash\{z\}$ for all $i \in \mathcal{W}^{b\rightharpoonup a}$. Consequently, we have $z \prec^{x,y} \min^{\prec^{x,y}}_{j\in\mathcal{W}^{b\rightharpoonup a}}\left(y_j, \boldsymbol{b}^{x,y}_{\mathcal{W}^{b\rightharpoonup a}}\right)$, which further induces the following contradiction:

$$\begin{aligned} z \prec^{x,y} \min^{\prec^{x,y}}_{j\in\mathcal{W}^{b\rightharpoonup a}}\left(y_j, \boldsymbol{b}^{x,y}_{\mathcal{W}^{b\rightharpoonup a}}\right) \preceq^{x,y} \max^{\prec^{x,y}}_{J\subseteq N}\left(\min^{\prec^{x,y}}_{j\in J}\left(y_j, \boldsymbol{b}^{x,y}_J\right)\right) \\ = \quad f(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) = z. \end{aligned}$$

Therefore, there exists $i \in \mathcal{W}^{b\rightharpoonup a}$ such that $y_i = z$.

Now, we have $\cap_{i\in\mathcal{W}^{b\rightharpoonup a}}\langle b, z_i|\mathcal{T}^A\rangle = \cap_{i\in\mathcal{W}^{b\rightharpoonup a}}\left[\langle b, y_i|\mathcal{T}^A\rangle \cup \left[\langle y_i, z_i|\mathcal{T}^A\rangle\backslash\{y_i\}\right]\right] = \left[\cap_{i\in\mathcal{W}^{b\rightharpoonup a}}\langle b, y_i|\mathcal{T}^A\rangle\right] \cup \left[\cap_{i\in\mathcal{W}^{b\rightharpoonup a}}\left[\langle y_i, z_i|\mathcal{T}^A\rangle\backslash\{y_i\}\right]\right] = \langle b, z|\mathcal{T}^A\rangle$,[76] which further

[76]First, note that $\cap_{i\in\mathcal{W}^{b\rightharpoonup a}}\langle b, y_i|\mathcal{T}^A\rangle = \langle b, z|\mathcal{T}^A\rangle$. Second, since $1 \in \mathcal{W}^{b\rightharpoonup a}$ and $z_1 = y_1$, we have $\langle y_1, z_1|\mathcal{T}^A\rangle\backslash\{y_1\} = \emptyset$.

implies $\text{Proj}\left(b, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b\rightharpoonup a}}\right)}\right) = z$. This completes the verification of the claim.

Claim 7: Given $z = b$, we have $z = \text{Proj}\left(b, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b\rightharpoonup a}}\right)}\right)$.

Clearly, $y_1 = z_1 \in A^{b\rightharpoonup a}\backslash\{b\}$ implies $b \prec^{x,y} y_1$. We first show $\mathcal{W}^{b\rightharpoonup a} \neq \{1\}$. Suppose by contradiction that $\mathcal{W}^{b\rightharpoonup a} = \{1\}$. Thus, we have $\boldsymbol{b}^{x,y}_{\{1\}} = y$ and induce the following contradiction:

$$\begin{aligned} y_1 = \min^{\prec^{x,y}}\left(y_1, \boldsymbol{b}^{x,y}_{\{1\}}\right) &\preceq^{x,y} \max_{J\subseteq N}{}^{\prec^{x,y}}\left(\min_{j\in J}{}^{\prec^{x,y}}\left(y_j, \boldsymbol{b}^{x,y}_J\right)\right) \\ &= \quad f(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) = z = b. \end{aligned}$$

Therefore, $\mathcal{W}^{b\rightharpoonup a} \neq \{1\}$.

Clearly, we have $\text{Proj}\left(b, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b\rightharpoonup a}}\right)}\right) = b$ if and only if b is included in the tree $\mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b\rightharpoonup a}}\right)}$. Furthermore, given $z_1 \in A^{b\rightharpoonup a}\backslash\{b\}$, we know that b is included in the tree $\mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b\rightharpoonup a}}\right)}$ if and only if there exists $i \in \mathcal{W}^{b\rightharpoonup a}\backslash\{1\}$ such that $z_i \in A^{a\rightharpoonup b} \cup \langle a, b|\mathcal{T}^A\rangle$. Hence, in the rest of verification, we show that there exists $i \in \mathcal{W}^{b\rightharpoonup a}\backslash\{1\}$ such that $z_i \in A^{a\rightharpoonup b} \cup \langle a, b|\mathcal{T}^A\rangle$. Suppose not, i.e., $z_i \in A^{b\rightharpoonup a}\backslash\{b\}$ for all $i \in \mathcal{W}^{b\rightharpoonup a}\backslash\{1\}$. This implies $y_i \in \langle \bar{b}, y|\mathcal{T}^A\rangle$ for all $i \in \mathcal{W}^{b\rightharpoonup a}$. Consequently, we induce

$$\begin{aligned} \bar{b} \preceq^{x,y} \min_{j\in\mathcal{W}^{b\rightharpoonup a}}{}^{\prec^{x,y}}\left(y_j, \boldsymbol{b}^{x,y}_{\mathcal{W}^{b\rightharpoonup a}}\right) &\preceq^{x,y} \max_{J\subseteq N}{}^{\prec^{x,y}}\left(\min_{j\in J}{}^{\prec^{x,y}}\left(y_j, \boldsymbol{b}^{x,y}_J\right)\right) \\ &= \quad f(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) = z = b, \end{aligned}$$

which contradicts the fact $b \prec^{x,y} \bar{b}$. Therefore, there exists $i \in \mathcal{W}^{b\rightharpoonup a}\backslash\{1\}$ such that $z_i \in A^{a\rightharpoonup b} \cup \langle a, b|\mathcal{T}^A\rangle$, as required. This completes the verification of the claim.

Overall, Claims 6 and 7 imply that for $P \in \left[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\right]^n$ with $r_1(P_1) \in A^{b\rightharpoonup a}\backslash\{b\}$, $f(P) = \text{Proj}\left(b, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{b\rightharpoonup a}}\right)}\right)$. Symmetrically, we can show that for $P \in \left[\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)\right]^n$ with $r_1(P_1) \in A^{a\rightharpoonup b}\backslash\{a\}$, $f(P) = \text{Proj}\left(a, \mathcal{T}^{\Gamma\left(P_{\mathcal{W}^{a\rightharpoonup b}}\right)}\right)$. This proves Proposition 3.

G Other Clarifications

G.1 Clarification 1

Given the semi-single-peaked domain $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$, we show that $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ includes a pair of completely reversed preferences if and only if $|\mathcal{N}^A(\bar{x})| \leq 2$.

(**Sufficiency**) Since $|\mathcal{N}^A(\bar{x})| \leq 2$, we have a path $(x_1, \ldots, x_v)$ in $\mathcal{T}^A$ such that (i) $\bar{x} = x_{k^*}$ for some $1 \leq k^* \leq v$, (ii) $x_1, x_v \in Ext(\mathcal{T}^A)$ and (iii) $[1 < k^* < v] \Rightarrow [\mathcal{N}^A(\bar{x}) = \{x_{k^*-1}, x_{k^*+1}\}]$.

Given $k^* = 1$, since $\bar{x} = x_1 \in Ext(\mathcal{T}^A)$, we by definition have a preference $P_i \in \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, x_1)$ such that $r_1(P_i) = x_v$ and $r_m(P_i) = x_1$. Next, we construct a linear order P'_i that is a complete reversal of P_i. Since $r_1(P'_i) = x_1$, it is true that $P'_i \in \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, x_1)$. Symmetrically, if $k^* = v$, we also have two completely reversed preferences in $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$. Last, let $1 < k^* < v$. Since $\mathcal{N}^A(\bar{x}) = \{x_{k^*-1}, x_{k^*+1}\}$, it is natural that x_{k^*-1} and x_{k^*} are two thresholds in $\mathcal{T}^A$, and x_{k^*} and x_{k^*+1} are two thresholds in $\mathcal{T}^A$. Then, we identify the two subsets $A^{x_{k^*-1} \rightharpoonup x_{k^*}}$ and $A^{x_{k^*+1} \rightharpoonup x_{k^*}}$. Note that $x_{k^*} \notin A^{x_{k^*-1} \rightharpoonup x_{k^*}}$ and $x_{k^*} \notin A^{x_{k^*+1} \rightharpoonup x_{k^*}}$. Moreover, since $\mathcal{N}^A(\bar{x}) = \{x_{k^*-1}, x_{k^*+1}\}$, it is true that $A^{x_{k^*-1} \rightharpoonup x_{k^*}} \cap A^{x_{k^*+1} \rightharpoonup x_{k^*}} = \emptyset$ and $A^{x_{k^*-1} \rightharpoonup x_{k^*}} \cup \{x_{k^*}\} \cup A^{x_{k^*+1} \rightharpoonup x_{k^*}} = A$. Now, pick two arbitrary preferences $P_i, P'_i \in \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ such that $r_1(P_i) = x_1$ and $r_1(P'_i) = x_v$. Then, we construct two linear orders: $\hat{P}_i$ and $\hat{P}'_i$ over A such that (i) for all $x \in A^{x_{k^*-1} \rightharpoonup x_{k^*}}$, $y \in A^{x_{k^*+1} \rightharpoonup x_{k^*}}$, $x\hat{P}_i x_{k^*}$ and $x_{k^*}\hat{P}_i y$; and $y\hat{P}'_i x_{k^*}$ and $x_{k^*}\hat{P}'_i x$, (ii) $\hat{P}_i$ and P_i agree on the relative rankings over $A^{x_{k^*-1} \rightharpoonup x_{k^*}}$, i.e., for all $x, y \in A^{x_{k^*-1} \rightharpoonup x_{k^*}}$, $[x\hat{P}_i y] \Leftrightarrow [xP_i y]$, and $\hat{P}_i$ and P'_i completely disagree on the relative rankings over $A^{x_{k^*+1} \rightharpoonup x_{k^*}}$, i.e., for all $x, y \in A^{x_{k^*+1} \rightharpoonup x_{k^*}}$, $[x\hat{P}_i y] \Leftrightarrow [yP'_i x]$, and (iii) $\hat{P}'_i$ and P'_i agree on the relative rankings over $A^{x_{k^*+1} \rightharpoonup x_{k^*}}$, i.e., for all $x, y \in A^{x_{k^*+1} \rightharpoonup x_{k^*}}$, $[x\hat{P}'_i y] \Leftrightarrow [xP'_i y]$, and $\hat{P}'_i$ and P_i completely disagree on the relative rankings over $A^{x_{k^*-1} \rightharpoonup x_{k^*}}$, i.e., for all $x, y \in A^{x_{k^*-1} \rightharpoonup x_{k^*}}$, $[x\hat{P}'_i y] \Leftrightarrow [yP_i x]$. It is easy to show that $\hat{P}_i$ and $\hat{P}'_i$ are complete reversals and both are semi-single-peaked on $\mathcal{T}^A$ w.r.t. $\bar{x}$, i.e., $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$.

(**Necessity**) Let $P_i, P'_i \in \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ be two completely reversed preferences. We assume w.l.o.g. that $r_1(P_i) = a_1$ and $r_1(P'_i) = a_m$. We first show that

$\bar{x}$ is included in the path between a_1 and a_m in $\mathcal{T}^A$. Suppose not, i.e., $\bar{x} \notin \langle a_1, a_m | \mathcal{T}^A \rangle$. We identify $\hat{x} = \text{Proj}(\bar{x}, \langle a_1, a_m | \mathcal{T}^A \rangle)$. Clearly, $\hat{x} \in \langle a_1, \bar{x} | \mathcal{T}^A \rangle$ and $\hat{x} \in \langle a_m, \bar{x} | \mathcal{T}^A \rangle$. Consequently, the definition of $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ implies $\hat{x} P_i \bar{x}$ and $\hat{x} P_i' \bar{x}$, which contradict the hypothesis that P_i and P_i' are complete reversals. Therefore, $\bar{x} \in \langle a_1, a_m | \mathcal{T}^A \rangle$.

Now, we show $|\mathcal{N}^A(\bar{x})| \leq 2$. Suppose by contradiction $|\mathcal{N}^A(\bar{x})| > 2$. Then, there exists $x \in A$ such that $x \in \mathcal{N}^A(\bar{x})$ and $x \notin \langle a_1, a_m | \mathcal{T}^A \rangle$. Thus, $\bar{x} \in \langle a_1, x | \mathcal{T}^A \rangle$ and $\bar{x} \in \langle a_m, x | \mathcal{T}^A \rangle$. Consequently, the definition of $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ implies $\bar{x} P_i x$ and $\bar{x} P_i' x$, which contradict the hypothesis that P_i and P_i' are complete reversals. Therefore, $|\mathcal{N}^A(\bar{x})| \leq 2$.

G.2 Clarification 2

Given a path-connected domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$, we show that $\mathbb{D}$ satisfies extreme-vertex symmetry if and only if either $\bar{x} \notin Ext(\mathcal{T}^A)$, or $\bar{x} \in Ext(\mathcal{T}^A)$ and $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) \cap \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, x)$ where $\mathcal{N}^A(\bar{x}) = \{x\}$.

(**Sufficiency**) First, let $\bar{x} \notin Ext(\mathcal{T}^A)$. Then, semi-single-peakedness on $\mathcal{T}^A$ w.r.t. $\bar{x}$ implies $|\mathcal{S}(\mathbb{D}^z)| = 1$ for all $z \in Ext(\mathcal{T}^A)$. Next, let $\bar{x} \in Ext(\mathcal{T}^A)$ and $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) \cap \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, x)$ where $\mathcal{N}^A(\bar{x}) = \{x\}$. Clearly, $x \notin Ext(\mathcal{T}^A)$. Then, semi-single-peakedness on $\mathcal{T}^A$ w.r.t. x implies $|\mathcal{S}(\mathbb{D}^z)| = 1$ for all $z \in Ext(\mathcal{T}^A)$. Overall, $|\mathcal{S}(\mathbb{D}^z)| = 1$ for all $z \in Ext(\mathcal{T}^A)$. Last, since $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ is path-connected, it is true that $G^A_\sim = \mathcal{T}^A$. Then, we have $|\mathcal{S}(\mathbb{D}^z)| = 1$ for all $z \in Ext(\mathcal{T}^A) = Ext(G^A_\sim)$. Therefore, extreme-vertex symmetry is vacuously satisfied.

(**Necessity**) Let the path-connected domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ satisfy extreme-vertex symmetry. Clearly, $G^A_\sim = \mathcal{T}^A$. Note that either $\bar{x} \notin Ext(\mathcal{T}^A)$ or $\bar{x} \in Ext(\mathcal{T}^A)$ holds. If $\bar{x} \notin Ext(\mathcal{T}^A)$, the verification is completed. Henceforth, let $\bar{x} \in Ext(\mathcal{T}^A)$ and $\mathcal{N}^A(\bar{x}) = \{x\}$. Clearly, $G^A_\sim = \mathcal{T}^A$ implies $\bar{x} \in Ext(G^A_\sim)$ and $x \in \mathcal{S}(\mathbb{D}^{\bar{x}})$. Given the hypothesis $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$, to prove $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) \cap \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, x)$, it suffices to show $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, x)$.

Fix an arbitrary preference $P_i \in \mathbb{D}$. Either $r_1(P_i) \neq \bar{x}$ or $r_1(P_i) = \bar{x}$ holds. First, let $r_1(P_i) \neq \bar{x}$. Then, $\bar{x} \in Ext(\mathcal{T}^A)$ and $\mathcal{N}^A(\bar{x}) = \{x\}$ imply $x \in \langle r_1(P_i), \bar{x} | \mathcal{T}^A \rangle$.

Hence, semi-single-peakedness on $\mathcal{T}^A$ w.r.t. $\bar{x}$ implies that P_i is also semi-single-peaked on $\mathcal{T}^A$ w.r.t. x. Second, let $r_1(P_i) = \bar{x}$. According to $\bar{x} \in Ext(\mathcal{T}^A)$ and $\mathcal{N}^A(\bar{x}) = \{x\}$, to show that P_i is semi-single-peaked on $\mathcal{T}^A$ w.r.t. x, it suffices to show $r_2(P_i) = x$. Suppose not, i.e., $r_2(P_i) \neq x$. Thus, $|\mathcal{S}(\mathbb{D}^{\bar{x}})| > 1$, and then extreme-vertex symmetry implies that there exists $z \in \mathcal{S}(\mathbb{D}^{\bar{x}})$ such that $\bar{x} \in \mathcal{S}(\mathbb{D}^z)$ and $z \neq x$. Clearly, $z \neq \bar{x}$ and $x \in \langle z, \bar{x} | \mathcal{T}^A \rangle$. Then, by semi-single-peakedness on $\mathcal{T}^A$ w.r.t. $\bar{x}$, we have $x P_i' \bar{x}$ for all $P_i' \in \mathbb{D}^z$, which implies $\bar{x} \notin \mathcal{S}(\mathbb{D}^z)$. Contradiction! Therefore, we have $r_2(P_i) = x$, as required.

G.3 Clarification 3

Given the semi-hybrid domain $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, where $|\langle a, b | \mathcal{T}^A \rangle| \geq 3$, we show that $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ includes a pair of completely reversed preferences if and only if we have $[A^{a \rightharpoonup b} \neq \{a\}] \Rightarrow [a \in Ext(\mathcal{T}^{A^{a \rightharpoonup b}})]$ and $[A^{b \rightharpoonup a} \neq \{b\}] \Rightarrow [b \in Ext(\mathcal{T}^{A^{b \rightharpoonup a}})]$.

(**Sufficiency**) If $A^{a \rightharpoonup b} = \{a\}$ and $A^{b \rightharpoonup a} = \{b\}$, then $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b) = \mathbb{P}$ includes a pair of completely reversed preference. If $A^{a \rightharpoonup b} = \{a\}$ and $A^{b \rightharpoonup a} \neq \{b\}$, we know that b has a unique neighbor in the line $\langle a, b | \mathcal{T}^A \rangle$, and $[A^{b \rightharpoonup a} \neq \{b\}] \Rightarrow [b \in Ext(\mathcal{T}^{A^{b \rightharpoonup a}})]$ implies that b has a unique neighbor in the subtree $\mathcal{T}^{A^{b \rightharpoonup a}}$. Since $\mathcal{T}^A$ is a union of the line $\langle a, b | \mathcal{T}^A \rangle$ and the subtree $\mathcal{T}^{A^{b \rightharpoonup a}}$, it is true that $|\mathcal{N}^A(b)| = 2$. Moreover, since $A^{a \rightharpoonup b} = \{a\}$, it is true that $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, b) \subset \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$. Then, the sufficiency part of Clarification 1 implies that $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, b)$ contains a pair of completely reversed preferences. Therefore, $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ includes a pair of completely reversed preferences. Symmetrically, if $A^{a \rightharpoonup b} \neq \{a\}$ and $A^{b \rightharpoonup a} = \{b\}$, $\mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ includes a pair of completely reversed preferences.

Last, we consider the situation that $A^{a \rightharpoonup b} \neq \{a\}$ and $A^{b \rightharpoonup a} \neq \{b\}$. Thus, we have $a \in Ext(\mathcal{T}^{A^{a \rightharpoonup b}})$ and $b \in Ext(\mathcal{T}^{A^{b \rightharpoonup a}})$. Let $\bar{a}$ be the unique neighbor of a in the subtree $\mathcal{T}^{A^{a \rightharpoonup b}}$ and $\bar{b}$ be the unique neighbor of b in the subtree $\mathcal{T}^{A^{b \rightharpoonup a}}$. Note that $\mathbb{D}_{\text{SP}}(\mathcal{T}^A) \subset \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$. We fix two arbitrary alternatives $x, y \in Ext(\mathcal{T}^A)$ such that $x \in A^{a \rightharpoonup b}$ and $y \in A^{b \rightharpoonup a}$. Clearly, $x \notin \{a, b\}$ and $y \notin \{a, b\}$. According to $\mathbb{D}_{\text{SP}}(\mathcal{T}^A)$, we fix two single-peaked preferences P_i and P_i' such that $r_1(P_i) = x$ and $r_1(P_i') = y$. Clearly, P_i and P_i' completely disagree on the relative rankings over $\langle a,b|\mathcal{T}^A\rangle$, i.e., for all $z,z'\in\langle a,b|\mathcal{T}^A\rangle$, $[zP_iz']\Leftrightarrow[z'P_i'z]$. Now, we construct two linear orders $\hat{P}_i$ and $\hat{P}_i'$ over A such that the following three conditions are satisfied: (i) for all $z\in A^{a\rightharpoonup b}\backslash\{a\}$, $z'\in\langle a,b|\mathcal{T}^A\rangle$ and $z''\in A^{b\rightharpoonup a}\backslash\{b\}$, $z\hat{P}_iz'$, $z'\hat{P}_iz''$, $z''\hat{P}_i'z'$ and $z'\hat{P}_i'z$, (ii) $\hat{P}_i$ and P_i agree on the relative rankings over $A^{a\rightharpoonup b}\cup\langle a,b|\mathcal{T}^A\rangle$, and $\hat{P}_i$ and P_i' completely disagree on the relative rankings over $A^{b\rightharpoonup a}\backslash\{b\}$, and (iii) $\hat{P}_i'$ and P_i' agree on the relative rankings over $\langle a,b|\mathcal{T}^A\rangle\cup A^{b\rightharpoonup a}$, and $\hat{P}_i'$ and P_i completely disagree on the relative rankings over $A^{a\rightharpoonup b}\backslash\{a\}$. By construction, it is easy to show that $\hat{P}_i$ and $\hat{P}_i'$ are (a,b)-semi-hybrid on $\mathcal{T}^A$ and complete reversals.

(**Necessity**) Let $P_i,P_i'\in\mathbb{D}_{\text{SH}}(\mathcal{T}^A,a,b)$ be two completely reversed preferences. We assume w.l.o.g. that $r_1(P_i)=a_1$ and $r_1(P_i')=a_m$. Let $A^{a\rightharpoonup b}\neq\{a\}$. Thus, either $a_1\in A^{a\rightharpoonup b}\backslash\{a\}$ or $a_m\in A^{a\rightharpoonup b}\backslash\{a\}$ holds. We assume w.l.o.g. that $a_1\in A^{a\rightharpoonup b}\backslash\{a\}$. Then, it must be the case that $a_m\in A\backslash A^{a\rightharpoonup b}$. Suppose by contradiction that $a\notin Ext(\mathcal{T}^{A^{a\rightharpoonup b}})$. Then, we have distinct $x,y\in A^{a\rightharpoonup b}$ such that $(x,a)\in\mathcal{E}^{A^{a\rightharpoonup b}}$ and $(y,a)\in\mathcal{E}^{A^{a\rightharpoonup b}}$. Immediately, the definition of $\mathbb{D}_{\text{SH}}(\mathcal{T}^A,a,b)$ implies $aP_i'x$ and $aP_i'y$. Clearly, either $x\notin\langle a_1,a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle$ or $y\notin\langle a_1,a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle$ holds. We assume w.l.o.g. that $x\notin\langle a_1,a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle$. Then, $(x,a)\in\mathcal{E}^{A^{a\rightharpoonup b}}$ implies $\text{Proj}(x,\langle a_1,a|\mathcal{T}^{A^{a\rightharpoonup b}}\rangle)=a$, and hence the definition of $\mathbb{D}_{\text{SH}}(\mathcal{T}^A,a,b)$ implies aP_ix, which contradicts the hypothesis that P_i and P_i' are complete reversals. Therefore, $a\in Ext(\mathcal{T}^{A^{a\rightharpoonup b}})$. Symmetrically, we have $[A^{b\rightharpoonup a}\neq\{b\}]\Rightarrow[b\in Ext(\mathcal{T}^{A^{b\rightharpoonup a}})]$.

G.4 Clarification 4

Suppose by contradiction that there exist a tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a},\hat{b}\in A$ such that $\mathbb{D}\subseteq\mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A,\hat{a},\hat{b})$ and $\langle\hat{a},\hat{b}|\widehat{\mathcal{T}}^A\rangle\subset\langle a_2,a_6|\mathcal{L}^A\rangle=\{a_2,a_3,a_4,a_5,a_6\}$. Thus, $\hat{a},\hat{b}\in\{a_2,a_3,a_4,a_5,a_6\}$. Let $\hat{A}^{\hat{a}\rightharpoonup\hat{b}}=\{x\in A:\hat{a}\in\langle x,\hat{b}|\widehat{\mathcal{T}}^A\rangle\}$ and $\hat{A}^{\hat{b}\rightharpoonup\hat{a}}=\{x\in A:\hat{b}\in\langle x,\hat{a}|\widehat{\mathcal{T}}^A\rangle\}$. Note that either $a_1\in\hat{A}^{\hat{a}\rightharpoonup\hat{b}}\backslash\{\hat{a}\}$ or $a_1\in\hat{A}^{\hat{b}\rightharpoonup\hat{a}}\backslash\{\hat{b}\}$ holds. We assume w.l.o.g. that $a_1\in\hat{A}^{\hat{a}\rightharpoonup\hat{b}}\backslash\{\hat{a}\}$.

Then, according to path-connectedness and $(\hat{a},\hat{b})$-semi-hybridness on $\widehat{\mathcal{T}}^A$, we know that $G_\sim^A$ is a union of the subtree $G_\sim^{\hat{A}^{\hat{a}\rightharpoonup\hat{b}}}=\widehat{\mathcal{T}}^{\hat{A}^{\hat{a}\rightharpoonup\hat{b}}}$, the connected graph $G_\sim^{\langle\hat{a},\hat{b}|\widehat{\mathcal{T}}^A\rangle}$ and the subtree $G_\sim^{\hat{A}^{\hat{b}\rightharpoonup\hat{a}}}=\widehat{\mathcal{T}}^{\hat{A}^{\hat{b}\rightharpoonup\hat{a}}}$.

First, since $\hat{A}^{\hat{a}\rightharpoonup\hat{b}}\neq\{\hat{a}\}$ and $|\langle\hat{a},\hat{b}|\widehat{\mathcal{T}}^A\rangle|\geq 2$, it is true that $\hat{a}$ has at least two neighbors in $G^A_\sim$. Therefore, $\hat{a} \neq a_3$ and hence $\hat{a} \in \{a_2, a_4, a_5, a_6\}$. Next, we show $\hat{a} = a_2$. Suppose $\hat{a} = a_6$. Since $a_1 \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$, the path $\langle a_1, a_6 | G^A_\sim \rangle = (a_1, a_2, a_4, a_5, a_6)$ must be included in $G^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}_\sim = \widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$. Moreover, since $a_3 \sim a_4$, a_3 must be contained in $\hat{A}^{\hat{a} \rightharpoonup \hat{b}}$ as well. Hence, $\{a_1, a_2, a_3, a_4, a_5, a_6\} \subseteq \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$ and the subgraph $G^{\{a_1, a_2, a_3, a_4, a_5, a_6\}}_\sim$ of $G^A_\sim$ in Figure 4 is included in $G^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}_\sim = \widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$. Thus, it must be the case that $\text{Proj}(a_3, \langle a_1, a_6 | \widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}} \rangle) = \text{Proj}(a_3, \langle a_1, a_6 | G^A_\sim \rangle) = a_4$. Consequently, $(\hat{a}, \hat{b})$-semi-hybridness on $\widehat{\mathcal{T}}^A$ implies that a_4 ranks above a_3 in every preference of $\mathbb{D}^{a_1}$, which contradicts preference P_1 of Table 1. Suppose $\hat{a} = a_5$. Since $a_1 \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$, the path $\langle a_1, a_5 | G^A_\sim \rangle = (a_1, a_2, a_4, a_5)$ must be included in $G^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}_\sim = \widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$. Moreover, since $a_3 \sim a_4$, a_3 must be contained in $\hat{A}^{\hat{a} \rightharpoonup \hat{b}}$ as well. Hence, $\{a_1, a_2, a_3, a_4, a_5\} \subseteq \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$ and the subgraph $G^{\{a_1, a_2, a_3, a_4, a_5\}}_\sim$ of $G^A_\sim$ in Figure 4 is included in $G^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}_\sim = \widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$. Thus, it must be the case that $\text{Proj}(a_3, \langle a_1, a_5 | \widehat{\mathcal{T}}^A \rangle) = \text{Proj}(a_3, \langle a_1, a_5 | G^A_\sim \rangle) = a_4$. Consequently, $(\hat{a}, \hat{b})$-semi-hybridness on $\widehat{\mathcal{T}}^A$ implies that a_4 ranks above a_3 in every preference of $\mathbb{D}^{a_1}$, which contradicts preference P_1 of Table 1. Last, suppose $\hat{a} = a_4$. Recall that $\mathbb{D}$ includes two completely reversed preference. Then, $a_1 \in \hat{A}^{\hat{a} \rightharpoonup \hat{b}} \backslash \{\hat{a}\}$ implies that a_4 has a unique neighbor in $\widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}} = G^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}_\sim$. Hence, $\widehat{\mathcal{T}}^{\hat{A}^{\hat{a} \rightharpoonup \hat{b}}}$ must be identical to the line $\langle a_1, a_4 | G^A_\sim \rangle = (a_1, a_2, a_4)$. Thus, $a_3 \notin \hat{A}^{\hat{a} \rightharpoonup \hat{b}}$, and $(\hat{a}, \hat{b})$-semi-hybridness on $\widehat{\mathcal{T}}^A$ implies that a_4 ranks above a_3 in every preference of $\mathbb{D}^{a_1}$, which contradicts preference P_1 of Table 1. Therefore, it must be the case that $\hat{a} = a_2$.

Now, according to the adjacency graph $G^A_\sim$ of Figure 4, we can infer $\hat{A}^{\hat{a} \rightharpoonup \hat{b}} = \{a_1, a_2\}$. Recall $\langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \subset \{a_2, a_3, a_4, a_5, a_6\}$. Then, it must be the case that $\hat{b} \neq a_2$ and $a_7 \in \hat{A}^{\hat{b} \rightharpoonup \hat{a}} \backslash \{\hat{b}\}$. Consequently, symmetric to $\hat{a}$, $\hat{b}$ also has two neighbors in $G^A_\sim$, which further implies $\hat{b} \neq a_3$. Therefore, $\hat{b} \in \{a_4, a_5, a_6\}$. We will induce a contradiction in each case. Suppose $\hat{b} = a_6$. We first infer according to the adjacency graph $G^A_\sim$ of Figure 4 that $\hat{A}^{\hat{b} \rightharpoonup \hat{a}} = \{a_6, a_7\}$. Consequently, $\hat{A}^{\hat{a} \rightharpoonup \hat{b}} = \{a_1, a_2\} = A^{a_2 \rightharpoonup a_6}$ and $\hat{A}^{\hat{b} \rightharpoonup \hat{a}} = \{a_6, a_7\} = A^{a_6 \rightharpoonup a_2}$ imply $\langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle = \langle a_2, a_6 | \mathcal{L}^A \rangle$. Contradiction! Suppose $\hat{b} = a_5$. Since $a_7 \in \hat{A}^{\hat{b} \rightharpoonup \hat{a}}$, the path $\langle a_5, a_7 | G^A_\sim \rangle = (a_5, a_6, a_7)$ must be included in $G^{\hat{A}^{\hat{b} \rightharpoonup \hat{a}}}_\sim = \widehat{\mathcal{T}}^{\hat{A}^{\hat{b} \rightharpoonup \hat{a}}}$. Then, $(\hat{a}, \hat{b})$-semi-hybridness on $\widehat{\mathcal{T}}^A$ implies that a_5 ranks above a_6 in every preference of

$\mathbb{D}^{a_2}$, which contradicts preference P_5 of Table 1. Suppose $\hat{b} = a_4$. Since $a_7 \in \hat{A}^{\hat{b}\rightharpoonup\hat{a}}$, the path $\langle a_4, a_7 | G_{\sim}^A \rangle = (a_4, a_5, a_6, a_7)$ in $G_{\sim}^A$ must be included in $G_{\sim}^{\hat{A}^{\hat{b}\rightharpoonup\hat{a}}} = \widehat{\mathcal{T}}^{\hat{A}^{\hat{b}\rightharpoonup\hat{a}}}$. Then, $(\hat{a}, \hat{b})$-semi-hybridness on $\widehat{\mathcal{T}}^A$ implies that a_4 ranks above a_5 in every preference of $\mathbb{D}^{a_1}$, which contradicts preference P_2 of Table 1.

In conclusion, there exist no tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}$ and $\hat{b}$ such that $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b} | \widehat{\mathcal{T}}^A \rangle \subset \langle a_2, a_6 | \mathcal{L}^A \rangle$.

G.5 Clarification 5

Let domain $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$ be path-connected and satisfy extreme-vertex symmetry. Moreover, let $\mathbb{D}$ be non-trivial on $\langle a, b | \mathcal{T}^A \rangle$. We show that $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$.

Since $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SH}}(\mathcal{T}^A, a, b)$ is path-connected, we know that $G_{\sim}^A$ is a union of the subtree $\mathcal{T}^{A^{a\rightharpoonup b}}$, the connected graph $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ and the subtree $\mathcal{T}^{A^{b\rightharpoonup a}}$.

First, we show that $\mathbb{D}$ satisfies condition (1) of Definition 4. Suppose by contradiction that $|\langle a, b | \mathcal{T}^A \rangle| = 2$. Thus, $\langle a, b | \mathcal{T}^A \rangle = \{a, b\}$. Since $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ is a connected graph, it is true that $\mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle} = \{(a, b), (b, a)\}$. Hence, $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) = \{a, b\}$. Consequently, we have $\max^{P_i}(\langle a, b | \mathcal{T}^A \rangle \backslash \{a\}) = \max^{P_i}(\{b\}) = b$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{a\rightharpoonup b}$ and $\max^{P_i'}(\langle a, b | \mathcal{T}^A \rangle \backslash \{b\}) = \max^{P_i'}(\{a\}) = a$ for all $P_i' \in \mathbb{D}$ with $r_1(P_i') \in A^{b\rightharpoonup a}$, which respectively violate conditions (ii) and (iii) of Definition 5. Therefore, $|\langle a, b | \mathcal{T}^A \rangle| \geq 3$.

Second, we show that $\mathbb{D}$ satisfies condition (3) of Definition 4. Suppose by contradiction that there exists a tree $\widehat{\mathcal{T}}^A$ such that either $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(\widehat{\mathcal{T}}^A, a)$ or $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(\widehat{\mathcal{T}}^A, b)$ holds. We assume w.l.o.g. that $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{SSP}}(\widehat{\mathcal{T}}^A, a)$. Since $\mathbb{D}$ is path-connected, it is true that $G_{\sim}^A = \widehat{\mathcal{T}}^A$. Recall that $G_{\sim}^A$ is a union of the subtree $G_{\sim}^{A^{a\rightharpoonup b}} = \mathcal{T}^{A^{a\rightharpoonup b}}$, the connected graph $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ and the subtree $G_{\sim}^{A^{b\rightharpoonup a}} = \mathcal{T}^{A^{b\rightharpoonup a}}$. Then, $G_{\sim}^A = \widehat{\mathcal{T}}^A$ implies $\widehat{\mathcal{T}}^{A^{a\rightharpoonup b}} = \mathcal{T}^{A^{a\rightharpoonup b}} = G_{\sim}^{A^{a\rightharpoonup b}}$, $\widehat{\mathcal{T}}^{A^{b\rightharpoonup a}} = \mathcal{T}^{A^{b\rightharpoonup a}} = G_{\sim}^{A^{b\rightharpoonup a}}$ and $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle} = \widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle}$ is a subtree nested in $\widehat{\mathcal{T}}^A$. Thus, $\widehat{\mathcal{T}}^A = G_{\sim}^A$ is a union of $G_{\sim}^{A^{a\rightharpoonup b}} = \widehat{\mathcal{T}}^{A^{a\rightharpoonup b}} = \mathcal{T}^{A^{a\rightharpoonup b}}$, $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle} = \widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle}$ and $G_{\sim}^{A^{b\rightharpoonup a}} = \widehat{\mathcal{T}}^{A^{b\rightharpoonup a}} = \mathcal{T}^{A^{b\rightharpoonup a}}$.

We consider two cases: (i) $\widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle}$ has an extreme vertex x which is neither a nor b, and (ii) $\widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle}$ has exactly two extreme vertices which are a

and b. In the first case, $x \in Ext(\widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle})\backslash\{a,b\} = Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})\backslash\{a,b\}$ implies $x \in Ext(G_{\sim}^A) = Ext(\widehat{\mathcal{T}}^A)$. On the one hand, since $x \in Ext(\widehat{\mathcal{T}}^A)$ and $x \neq a$, semi-single-peakedness on $\widehat{\mathcal{T}}^A$ w.r.t. a implies $|\mathcal{S}(\mathbb{D}^x)| = 1$. On the other hand, since $x \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})\backslash\{a,b\}$, condition (i) of Definition 5 implies $|\mathcal{S}(\mathbb{D}^x)| > 1$. Contradiction! In the second case, we know that $G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle} = \widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle}$ is a line which contains all alternatives of $\langle a,b|\mathcal{T}^A\rangle$ and has the extreme vertices a and b. For notational convenience, let $\widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle} = (z_1,\ldots,z_{\eta-1},z_\eta)$ where $z_1 = a$ and $z_\eta = b$. Since $\widehat{\mathcal{T}}^A$ is a union of the subtree $\mathcal{T}^{A^{a\rightharpoonup b}}$, the line $\widehat{\mathcal{T}}^{\langle a,b|\mathcal{T}^A\rangle} = (z_1,\ldots,z_{\eta-1},z_\eta)$ and the subtree $\mathcal{T}^{A^{b\rightharpoonup a}}$, we know that for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{b\rightharpoonup a}$ and $1 \leq k < \eta-1$, it is true that $z_k, z_{\eta-1} \in \langle r_1(P_i), a|\widehat{\mathcal{T}}^A\rangle$ and $z_{\eta-1} \in \langle r_1(P_i), z_k|\widehat{\mathcal{T}}^A\rangle$. Consequently, semi-single-peakedness on $\widehat{\mathcal{T}}^A$ w.r.t. a implies $z_{\eta-1} = \max^{P_i}(\{z_1,\ldots,z_{\eta-1}\}) = \max^{P_i}(\langle a,b|\mathcal{T}^A\rangle\backslash\{b\})$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{b\rightharpoonup a}$, which contradicts condition (iii) of Definition 5. Therefore, there exists no tree $\widehat{\mathcal{T}}^A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widehat{\mathcal{T}}^A, a)$ or $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widehat{\mathcal{T}}^A, b)$.

Last, we show that $\mathbb{D}$ satisfies condition (2) of Definition 4. Suppose by contradiction that there exists a tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a,b|\mathcal{T}^A\rangle$. Let $\hat{A}^{\hat{a}\rightharpoonup\hat{b}} = \{x \in A : \hat{a} \in \langle x, \hat{b}|\widehat{\mathcal{T}}^A\rangle\}$ and $\hat{A}^{\hat{b}\rightharpoonup\hat{a}} = \{x \in A : \hat{b} \in \langle x, \hat{a}|\widehat{\mathcal{T}}^A\rangle\}$. Since $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a,b|\mathcal{T}^A\rangle$, we have $x \in \langle a,b|\mathcal{T}^A\rangle\backslash\langle \hat{a},\hat{b}|\widehat{\mathcal{T}}^A\rangle$. We assume w.l.o.g. that $x \in \hat{A}^{\hat{a}\rightharpoonup\hat{b}}$. On the one hand, by the sufficiency part of the Auxilliary Proposition, we know that every strategy-proof rule defined on $\mathbb{D}$ behaves like a dictatorship on $\langle a,b|\mathcal{T}^A\rangle$. On the other hand, according to the contradictory hypothesis $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$, by the verification of Claim 3 in the proof of Lemma 24, we know that the following SCF:

$$f(P_1,P_2) = \begin{cases} r_1(P_1) & \text{if } r_1(P_1) \in \langle \hat{a},\hat{b}|\widehat{\mathcal{T}}^A\rangle, \\ \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right) & \text{if } r_1(P_1) \in \hat{A}^{\hat{a}\rightharpoonup\hat{b}}\backslash\{\hat{a}\}, \\ \text{Proj}\left(\hat{b}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right) & \text{if } r_1(P_1) \in \hat{A}^{\hat{b}\rightharpoonup\hat{a}}\backslash\{\hat{b}\}. \end{cases}$$

is a strategy-proof rule on $\mathbb{D}$. Clearly, voter 1 by construction dictates on $\langle \hat{a},\hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a,b|\mathcal{T}^A\rangle$. Recall $x \in \langle a,b|\mathcal{T}^A\rangle$ and $\hat{a} \in \langle a,b|\mathcal{T}^A\rangle$. According to $x \in \hat{A}^{\hat{a}\rightharpoonup\hat{b}}$ and $\hat{a} \in \langle \hat{a},\hat{b}|\widehat{\mathcal{T}}^A\rangle$, given $P_1 \in \mathbb{D}^x$ and $P_2 \in \mathbb{D}^{\hat{a}}$, the construction of f implies $f(P_1,P_2) = \text{Proj}\left(\hat{a}, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle\right) = \hat{a} \neq r_1(P_1)$. This indicates that voter

1 does not dictate on $\{x, \hat{a}\}$, and hence does not dictate on $\langle a, b|\mathcal{T}^A\rangle$. Contradiction! Therefore, there exist no tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$.

In conclusion, $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$.

G.6 Clarification 6

Fixing a tree $\mathcal{T}^A$ and two thresholds $a, b \in A$ with $A^{a \rightharpoonup b} = \{a\}$ and $A^{b \rightharpoonup a} = \{b\}$, let domain $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ be path-connected. Thus, $\langle a, b|\mathcal{T}^A\rangle = A$. We show that $\mathbb{D}$ satisfies the unique seconds property if and only if it violates the non-trivialness condition on $\langle a, b|\mathcal{T}^A\rangle$.

(**Sufficiency**) Let the path-connected domain $\mathbb{D}$ violate the non-trivialness condition on $\langle a, b|\mathcal{T}^A\rangle$. Thus, it must be the case that $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) \neq \emptyset$ and there exist $x \in Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ and $y \in \langle a, b|\mathcal{T}^A\rangle$ with $(x, y) \in \mathcal{E}_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}$ such that one of the following three conditions is satisfied:

(i) $x \notin \{a, b\}$ and $r_2(P_i) = y$ for all $P_i \in \mathbb{D}^x$,

(ii) $x = a$ and $r_2(P_i) = \max^{P_i}(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}) = y$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{a \rightharpoonup b} = \{a\}$, and

(iii) $x = b$ and $r_2(P_i) = \max^{P_i}(\langle a, b|\mathcal{T}^A\rangle \backslash \{b\}) = y$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{b \rightharpoonup a} = \{b\}$.

Each of these three conditions implies $\mathcal{S}(\mathbb{D}^x) = \{y\}$ and hence $|\mathcal{S}(\mathbb{D}^x)| = 1$. Therefore, domain $\mathbb{D}$ satisfies the unique seconds property.

(**Necessity**) Let the path-connected domain $\mathbb{D}$ satisfy the unique second property. Thus, we have some $x \in A$ such that $|\mathcal{S}(\mathbb{D}^x)| = 1$. We assume $\mathcal{S}(\mathbb{D}^x) = \{y\}$. Since $\mathbb{D}$ is path-connected, $\mathcal{S}(\mathbb{D}^x) = \{y\}$ implies that y is the unique neighbor of x in $G_{\sim}^A$. Thus, $x \in Ext(G_{\sim}^A) = Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle})$ and hence $Ext(G_{\sim}^{\langle a,b|\mathcal{T}^A\rangle}) \neq \emptyset$. Clearly, either $x \notin \{a, b\}$, or $x = a$, or $x = b$ holds. If $x \notin \{a, b\}$, then $\mathcal{S}(\mathbb{D}^x) = \{y\}$ violates condition (i) of Definition 5. If $x = a$, then $\mathcal{S}(\mathbb{D}^a) = \{y\}$ and $A^{a \rightharpoonup b} = \{a\}$ imply $\max^{P_i}(\langle a, b|\mathcal{T}^A\rangle \backslash \{a\}) = r_2(P_i) = y$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{a \rightharpoonup b}$, which clearly violates condition (ii) of Definition 5. Symmetrically, if $x = b$, then $\mathcal{S}(\mathbb{D}^b) = \{y\}$ and $A^{b\rightharpoonup a} = \{b\}$ imply $\max^{P_i}(\langle a, b|\mathcal{T}^A\rangle\backslash\{b\}) = r_2(P_i) = y$ for all $P_i \in \mathbb{D}$ with $r_1(P_i) \in A^{b\rightharpoonup a}$, which clearly violates condition (iii) of Definition 5. Therefore, domain $\mathbb{D}$ violates the non-trivialness condition on $\langle a, b|\mathcal{T}^A\rangle$.

Moreover, we provide an example of a degenerate semi-hybrid domain that satisfies the unique seconds property.

Example **5** Let $A = \{a_1, a_2, a_3, a_4\}$. All 9 preferences of the domain $\mathbb{D}$ and the adjacency graph $G_\sim^A$ are respectively specified in Table 4 and Figure 15.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a_1	a_1	a_1	a_2	a_2	a_3	a_3	a_3	a_4
a_2	a_2	a_3	a_1	a_3	a_1	a_2	a_4	a_3
a_3	a_4	a_2	a_3	a_1	a_3	a_1	a_2	a_2
a_4	a_3	a_4	a_4	a_4	a_4	a_4	a_1	a_1

Table 4: Domain $\mathbb{D}$

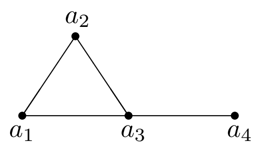


Figure 15: The adjacency graph $G_\sim^A$

First, $\mathbb{D}$ is path-connected according to the adjacency graph of Figure 15, satisfies extreme-vertex property vacuously, i.e., $Ext(G_\sim^A) = \{a_4\}$ and $|\mathcal{S}(\mathbb{D}^{a_4})| = 1$, and includes two completely reversed preferences P_1 and P_4. Therefore, $\mathbb{D}$ is a rich domain. Second, since $|\mathcal{S}(\mathbb{D}^{a_4})| = 1$, $\mathbb{D}$ satisfies the unique seconds property. Last, we observe that the adjacency graph $G_\sim^A$ contains a cycle, coincides with the counterpart adjacency graph of the semi-hybrid domain $\mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_1, a_3)$, and is strictly included in the counterpart adjacency graph of $\mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_1, a_4)$. Moreover, since $\mathbb{D} \subset \mathbb{P} = \mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_1, a_4)$ and P_2 is excluded by $\mathbb{D}_{\text{SH}}(\mathcal{L}^A, a_1, a_3)$, we infer that $\mathbb{D}$ is an (a_1, a_4)-semi-hybrid domain on $\mathcal{L}^A$, and hence a degenerate semi-hybrid domain. □

G.7 Clarification 7

Fixing a tree $\mathcal{T}^A$ and two thresholds $a, b \in A$, let $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$ be a rich domain. We show that every tops-only and strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$ if and only if $\mathbb{D}$ is an (a,b)-semi-hybrid domain on $\mathcal{T}^A$.

(**Sufficiency**) Let $\mathbb{D}$ be a rich (a,b)-semi-hybrid domain on $\mathcal{T}^A$. First, since $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b)$, by the verification of Claim 4 in the proof of Lemma 24, we know that the following SCF:

$$f(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } r_1(P_1) \in \langle a, b|\mathcal{T}^A\rangle, \\ \text{Proj}\left(a, \langle r_1(P_1), r_1(P_2)|\mathcal{T}^A\rangle\right) & \text{if } r_1(P_1) \in A^{a \rightharpoonup b}\backslash\{a\}, \\ \text{Proj}\left(b, \langle r_1(P_1), r_1(P_2)|\mathcal{T}^A\rangle\right) & \text{if } r_1(P_1) \in A^{b \rightharpoonup a}\backslash\{b\}, \end{cases}$$

is a tops-only and strategy-proof rule. Note that f by construction behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$, and does not behave like a dictatorship on any non-empty set that is not included in $\langle a, b|\mathcal{T}^A\rangle$.

Second, by the sufficiency part of Statement (ii) of the Theorem, since $\mathbb{D}$ is a rich (a,b)-semi-hybrid domain on $\mathcal{T}^A$, it never admits an invariant, tops-only and strategy-proof rule. Then, as a rich domain, the proof of the necessity part of the Theorem implies that $\mathbb{D}$ is a $(\hat{a}, \hat{b})$-semi-hybrid domain on some tree $\widehat{\mathcal{T}}^A$, and every tops-only and strategy-proof rule behaves like a dictatorship on $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle$. Therefore, rule f above must behave like a dictatorship on $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle$, which implies $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subseteq \langle a, b|\mathcal{T}^A\rangle$. Last, since $\mathbb{D}$ is an (a,b)-semi-hybrid domain on $\mathcal{T}^A$, condition (2) of Definition 4 implies $\langle a, b|\mathcal{T}^A\rangle = \langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle$. Therefore, every tops-only and strategy-proof rule behaves like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$.

(**Necessity**) Let every tops-only and strategy-proof rule defined on $\mathbb{D}$ behave like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$.

First, suppose $|\langle a, b|\mathcal{T}^A\rangle| = 2$. Then, $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\mathcal{T}^A, a, b) = \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, a) \cap \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, b)$. Consequently, $\mathbb{D}$ is semi-single-peaked on $\mathcal{T}^A$ w.r.t. a, and hence by the sufficiency part of Statement (i) of the Theorem admits the following invariant, tops-only and strategy-proof rule: $f(P_1, P_2) = \text{Proj}(a, \langle r_1(P_1), r_1(P_2)|\mathcal{T}^A\rangle)$ for all $P_1, P_2 \in \mathbb{D}$. Clearly, rule f does not behave like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$.

Contradiction! Therefore, $|\langle a, b|\mathcal{T}^A\rangle| \geq 3$, and hence $\mathbb{D}$ satisfies condition (1) of Definition 4.

Second, suppose that there exist a tree $\widehat{\mathcal{T}}^A$ and thresholds $\hat{a}, \hat{b} \in A$ such that $\mathbb{D} \subseteq \mathbb{D}_{\text{SH}}(\widehat{\mathcal{T}}^A, \hat{a}, \hat{b})$ and $\langle \hat{a}, \hat{b}|\widehat{\mathcal{T}}^A\rangle \subset \langle a, b|\mathcal{T}^A\rangle$. Consequently, the verification of Claim 4 in the proof of Lemma 24 indicates that $\mathbb{D}$ admits a tops-only and strategy-proof rule that does not behave like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Contradiction! Therefore, $\mathbb{D}$ satisfies condition (2) of Definition 4.

Last, suppose that there exists a tree $\widehat{\mathcal{T}}^A$ such that either $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widehat{\mathcal{T}}^A, a)$ or $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widehat{\mathcal{T}}^A, b)$ holds. We assume w.l.o.g. that $\mathbb{D} \subseteq \mathbb{D}_{\text{SSP}}(\widehat{\mathcal{T}}^A, a)$. Then, by the sufficiency part of Statement (i) of the Theorem, $\mathbb{D}$ admits the following invariant, tops-only and strategy-proof rule: $f(P_1, P_2) = \text{Proj}(a, \langle r_1(P_1), r_1(P_2)|\widehat{\mathcal{T}}^A\rangle)$ for all $P_1, P_2 \in \mathbb{D}$. Clearly, rule f does not behave like a dictatorship on $\langle a, b|\mathcal{T}^A\rangle$. Contradiction! Therefore, $\mathbb{D}$ satisfies condition (3) of Definition 4.

In conclusion, $\mathbb{D}$ is an (a, b)-semi-hybrid domain on $\mathcal{T}^A$.

G.8 Clarification 8

We explain in detail how Corollary 1 of Bonifacio and Massó (2020) is applied to establish Proposition 2.

Fix a tree $\mathcal{T}^A$ and an alternative $\bar{x} \in A$ such that $\bar{x} \notin \mathcal{N}^A(x)$ for any $x \in Ext(\mathcal{T}^A)$. We fix an anonymous, tops-only and strategy-proof rule $f : \left[\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})\right]^n \rightarrow A$.

First, we induce a binary relation $\succeq$ over A such that for all $x, y \in A$, $x \succeq y$ if and only if $x \in \langle \bar{x}, y|\mathcal{T}^A\rangle$. It is easy to show that $\succeq$ is a semilattice, i.e., $\sup_{\succeq}(x, y)$ uniquely exists for all $x, y \in A$. Clearly, for all $x \in A\backslash\{\bar{x}\}$, $\bar{x} \succ x$ i.e., $\bar{x} \succeq x$ and $x \not\succeq \bar{x}$. Moreover, it turns out that for all non-empty subset $B \subseteq A$, $\sup_{\succeq}(B) = \text{Proj}(\bar{x}, \mathcal{T}^{\Gamma(B)})$.

According to the semilattice $\succeq$, we construct the semilattice single-peaked domain $\mathcal{SSP}(\succeq)$ of Bonifacio and Massó (2020), which contains every linear order P_i over A satisfying the following condition: given $r_1(P_i) = x$, for all $y, z \in A$, $\left[\sup_{\succeq}(x, y) \neq \sup_{\succeq}(z, y)\right] \Rightarrow [\sup_{\succeq}(x, y) P_i \sup_{\succeq}(z, y)]$. We next show

$\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) = \mathcal{SSP}(\succeq)$.

Fixing $P_i \in \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$, we show $P_i \in \mathcal{SSP}(\succeq)$. Let $r_1(P_i) = x$. Given $y, z \in A$, we know $\sup_{\succeq}(x, y) = \text{Proj}(\bar{x}, \langle x, y|\mathcal{T}^A\rangle) = \text{Proj}(y, \langle x, \bar{x}|\mathcal{T}^A\rangle) \equiv \bar{y}$ and $\sup_{\succeq}(z, y) = \text{Proj}(\bar{x}, \langle z, y|\mathcal{T}^A\rangle) = \text{Proj}(z, \langle y, \bar{x}|\mathcal{T}^A\rangle) \equiv \bar{z}$. Assume $\bar{y} \neq \bar{z}$, and we show $\bar{y} P_i \bar{z}$. We consider two cases: $y \in \langle x, \bar{x}|\mathcal{T}^A\rangle$ and $y \notin \langle x, \bar{x}|\mathcal{T}^A\rangle$. If $y \in \langle x, \bar{x}|\mathcal{T}^A\rangle$, then $\bar{y} = y$. Since $\bar{z} = \text{Proj}(z, \langle y, \bar{x}|\mathcal{T}^A\rangle) \in \langle y, \bar{x}|\mathcal{T}^A\rangle = \langle \bar{y}, \bar{x}|\mathcal{T}^A\rangle$, we know $\bar{y}, \bar{z} \in \langle x, \bar{x}|\mathcal{T}^A\rangle$ and $\bar{y} \in \langle x, \bar{z}|\mathcal{T}^A\rangle$. Consequently, semi-single-peakedness on $\mathcal{T}^A$ w.r.t. $\bar{x}$ implies $\bar{y} P_i \bar{z}$, as required. Next, let $y \notin \langle x, \bar{x}|\mathcal{T}^A\rangle$. Note that $\langle y, \bar{x}|\mathcal{T}^A\rangle$ is the concatenation of $\langle y, \bar{y}|\mathcal{T}^A\rangle$ and $\langle \bar{y}, \bar{x}|\mathcal{T}^A\rangle$. Thus, $\bar{z} = \text{Proj}(z, \langle y, \bar{x}|\mathcal{T}^A\rangle) \in \langle y, \bar{x}|\mathcal{T}^A\rangle$ and $\bar{z} \neq \bar{y}$ imply either $\bar{z} \in \langle y, \bar{y}|\mathcal{T}^A\rangle \backslash \{\bar{y}\}$, or $\bar{z} \in \langle \bar{y}, \bar{x}|\mathcal{T}^A\rangle \backslash \{\bar{y}\}$. If $\bar{z} \in \langle y, \bar{y}|\mathcal{T}^A\rangle \backslash \{\bar{y}\}$, it is true that $\bar{z} \notin \langle x, \bar{x}|\mathcal{T}^A\rangle$ and $\text{Proj}(\bar{z}, \langle x, \bar{x}|\mathcal{T}^A\rangle) = \bar{y}$. Then, semi-single-peakedness on $\mathcal{T}^A$ w.r.t. $\bar{x}$ implies $\bar{y} P_i \bar{z}$, as required. If $\bar{z} \in \langle \bar{y}, \bar{x}|\mathcal{T}^A\rangle \backslash \{\bar{y}\}$, it is true that $\bar{z} \in \langle x, \bar{x}|\mathcal{T}^A\rangle$ and $\bar{y} \in \langle x, \bar{z}|\mathcal{T}^A\rangle$. Then, semi-single-peakedness on $\mathcal{T}^A$ w.r.t. $\bar{x}$ implies $\bar{y} P_i \bar{z}$, as required. Overall, we have $\bar{y} P_i \bar{z}$. Therefore, $P_i \in \mathcal{SSP}(\succeq)$ and hence $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) \subseteq \mathcal{SSP}(\succeq)$.

Conversely, fixing $P_i \in \mathcal{SSP}(\succeq)$, we show $P_i \in \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$. Let $r_1(P_i) = x$. First, given distinct $y, z \in \langle x, \bar{x}|\mathcal{T}^A\rangle$ such that $y \in \langle x, z|\mathcal{T}^A\rangle$, we show $y P_i z$. Clearly, $y \in \langle x, \bar{x}|\mathcal{T}^A\rangle$ implies $\sup_{\succeq}(x, y) = y$. Furthermore, $y, z \in \langle x, \bar{x}|\mathcal{T}^A\rangle$ and $y \in \langle x, z|\mathcal{T}^A\rangle$ imply $z \in \langle y, \bar{x}|\mathcal{T}^A\rangle$, which further implies $\sup_{\succeq}(z, y) = z$. Then, $\sup_{\succeq}(x, y) P_i \sup_{\succeq}(z, y)$ induces $y P_i z$, as required. Next, given $y \notin \langle x, \bar{x}|\mathcal{T}^A\rangle$ and $\bar{y} = \text{Proj}(y, \langle x, \bar{x}|\mathcal{T}^A\rangle)$, we show $\bar{y} P_i y$. Clearly, $\bar{y} = \text{Proj}(y, \langle x, \bar{x}|\mathcal{T}^A\rangle) = \text{Proj}(\bar{x}, \langle x, y|\mathcal{T}^A\rangle)$ implies $\sup_{\succeq}(x, y) = \bar{y}$. Then, $P_i \in \mathcal{SSP}(\succeq)$ implies $\bar{y} = \sup_{\succeq}(x, y) P_i \sup_{\succeq}(y, y) = y$, as required. Therefore, $P_i \in \mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})$ and hence $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) \supseteq \mathcal{SSP}(\succeq)$. In conclusion, $\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x}) = \mathcal{SSP}(\succeq)$. Thus, the anonymous, tops-only and strategy-proof rule $f : \left[\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})\right]^n \to A$ can be equivalently transferred to an anonymous, tops-only and strategy-proof rule $f : \mathcal{SSP}(\succeq)^n \to A$.

Furthermore, let $A^*(\succeq) = \big\{x \in A : \text{for each } y \in A\backslash\{\bar{x}\},\ [y \neq x] \Rightarrow [x \not\succeq y \text{ and } y \not\succeq x]\big\}$. Recall that $\bar{x} \notin \mathcal{N}^A(x)$ for any $x \in Ext(\mathcal{T}^A)$. We show $A^*(\succeq) = \emptyset$. Suppose by contradiction that $A^*(\succeq) \neq \emptyset$. Then, we have $x \in A^*(\succeq)$. Note that

$\bar{x} \succ x$, $x \not\succeq y$ and $y \not\succeq x$ for all $y \in A \backslash \{\bar{x}, x\}$. Accordingly, by the definition of $\succeq$, we know that for all $y \in A \backslash \{\bar{x}, x\}$, $x \notin \langle y, \bar{x} | \mathcal{T}^A \rangle$ and $y \notin \langle x, \bar{x} | \mathcal{T}^A \rangle$, which imply $x \in Ext(\mathcal{T}^A)$ and $\bar{x} \in \mathcal{N}^A(x)$. This contradicts the hypothesis that $\bar{x} \notin \mathcal{N}^A(z)$ for any $z \in Ext(\mathcal{T}^A)$. Therefore, $A^*(\succeq) = \emptyset$. Now, we can apply Corollary 1 of Bonifacio and Massó (2020), which implies that the anonymous, tops-only and strategy-proof rule $f : \mathcal{SSP}(\succeq)^n \to A$ has a supremum functional form, i.e., for all $P \in \mathcal{SSP}(\succeq)^n$, $f(P) = \sup_{\succeq} \big(r_1(P_1), \ldots, r_1(P_n)\big)$. Therefore, for all $P \in \big[\mathbb{D}_{\text{SSP}}(\mathcal{T}^A, \bar{x})\big]^n = \mathcal{SSP}(\succeq)^n$, we have $f(P) = \sup_{\succeq} \big(r_1(P_1), \ldots, r_1(P_n)\big) = \text{Proj}(\bar{x}, \mathcal{T}^{\Gamma(P)})$.