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Liangjun Su, Xia Wang

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THE SCHOOL OF ECONOMICS, SMU

Corrigendum to "On Time-varying Factor Models: Estimation and Testing"[J. Econometrics 198 (2017) 84-101]*

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February 27, 2020

Abstract

We note that Su and Wang (2017, On Time-varying Factor Models: Estimation and Testing, Journal of Econometrics 198, 84-101) ignore the bias terms when estimating the time-varying factor models. In this note, we correct the theoretical results on the estimation of time-varying factor models. The asymptotic results for testing the correct specification of time invariant factor loadings are not affected.

JEL Classification: C12, C14, C33, C38.

Key Words: Approximation error; Bias; Correction; Factor Model; Time-varying

1 Introduction

Su and Wang (2017, SW hereafter) introduced a time-varying factor model where factor loadings are allowed to change smoothly over time and proposed a local version of the principal component analysis (PCA) to estimate the latent factors and time-varying factor loadings simultaneously. After the paper was published, we found that some bias terms have been ignored. As a result, the limiting distributions of the estimated factors and factor loadings should be revised. In this paper, we correct the main results in SW. The notations are the same as SW unless otherwise specified. All proofs are contained in the online supplementary appendix.

We summarize the main corrections as follows. First, the leading bias terms in the factor loading estimators are of $O\left(h^2\right)$ as in standard kernel regressions. Second, the estimator of the common factor remains asymptotically unbiased for a rotational version of the true factors. Third, the bias of the estimated factor loadings does not affect the distributions of our test statistic under the null hypothesis and the usual sequence of Pitman local alternatives.

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2 Correction on the limiting distributions of the estimated factors and factor loadings

The time-varying factor model in SW is given as follows:

$$X_{it} = \lambda_{it}' F_t + e_{it}, \tag{2.1}$$

where $\{X_{it}, i = 1, 2, \ldots, N;\ t = 1, 2, \ldots, T\}$ is an N-dimensional time series with T observations, F_t is an $R \times 1$ vector of common factors, e_{it} is the idiosyncratic error, and the time-varying factor loading λ_{it} is assumed to be a nonrandom function of t/T:

$$\lambda_{it} = \lambda_i(t/T). \tag{2.2}$$

Under the assumption that $\lambda_i : [0,1] \to \mathbb{R}$ is a smooth function, we can approximate $\lambda_i\left(\frac{t}{T}\right)$ by $\lambda_i\left(\frac{r}{T}\right)$ whenever t/T is close to r/T for any fixed $r \in \{1, 2, \cdots, T\}$. Let $d_i(t,r) = \lambda_i\left(\frac{t}{T}\right) - \lambda_i\left(\frac{r}{T}\right)$ denote the approximation error. Then

$$\lambda_{it} = \lambda_i\left(\frac{t}{T}\right) = \lambda_i\left(\frac{r}{T}\right) + \left[\lambda_i\left(\frac{t}{T}\right) - \lambda_i\left(\frac{r}{T}\right)\right] \equiv \lambda_i\left(\frac{r}{T}\right) + d_i(t,r) \tag{2.3}$$

and

$$X_{it} = \lambda_{ir}' F_t + d_i(t,r)' F_t + e_{it} \equiv \lambda_{ir}' F_t + \Delta_i(t,r) + e_{it}, \tag{2.4}$$

where $\Delta_i(t,r) = d_i(t,r)' F_t$. Compared with Eq. (3.1) in SW, we explicitly give out the additional term $\Delta_i(t,r)$, which represents the approximation error in the common component and will generate some bias terms in the estimation of the factor loadings.

Recall that $F_t^{(r)} = k_{h,tr}^{*1/2} F_t$, where $k_{h,tr}^* = h^{-1} K_r^*\left((t-r)/(Th)\right)$ and K_r^* is the boundary kernel used in SW and constructed from a univariate kernel K with compact support $[-1,1]$. As in SW, we can obtain the local PCA (LPCA) estimator $\hat{F}_t^{(r)}$ of $F_t^{(r)}$ and the associated LPCA estimator $\hat{\lambda}_{ir}$ of the the factor loadings λ_{ir}. SW's estimator of F_t is obtained by running the cross-sectional regression of X_{it} on $\hat{\lambda}_{it}$.

Let $\Lambda_r = (\lambda_{1r}, \ldots \lambda_{Nr})'$ and $\Lambda_r^{(c)} = (\lambda_1^{(c)}(r/T), \ldots, \lambda_N^{(c)}(r/T))'$ where $\lambda_i^{(c)}(\cdot)$ denote the cth order derivative of $\lambda_i(\cdot)$ for $c = 1$, 2. Let $\kappa_2 = \int_{-1}^{1} u^2 K(u) du$. To derive the asymptotic distribution of these estimators, we strengthen SW's conditions on the factor loadings and bandwidth.

Assumption A.1*: (i) $\lambda_i(\cdot)$ is third-order continuously differentiable with $\max_{i,t} \|\lambda_i^{(c)}(t/T)\| \le \bar{c}_\lambda$ for $c = 1, 2, 3$ and some finite constant $\bar{c}_\lambda$.

(ii) As $(N,T) \to \infty$, $Th^7 \to 0$, $Nh^6 \to 0$, $Th/N \to 0$, $Nh/T \to 0$, $Th^2 \to \infty$, $Nh^2 \to \infty$, and $Th/N^{1/2} \to \infty$.

Assumption A.1*(i) requires that $\lambda_i(\cdot)$ be third-order continuously differentiable, which will greatly simplify the derivation. Assumption A.1*(ii) strengthens the conditions in SW's Assumption A.3(ii).

Theorems 2.1-2.3 below provide the limiting distributions of $\hat{F}_t^{(r)}$, $\hat{\lambda}_{ir}$, and $\hat{F}_t$, which are parallel to Theorems 3.1-3.3 in SW.

Theorem 2.1 *Suppose that Assumptions A.1, A.2(i) and A.3(i) in SW and Assumption A.1* hold. Then, for each $t = 1, 2, \ldots, T$ and $r = 1, 2, \ldots, T$ such that $|r-t| \leq Th$, we have*

$$K_r^* \left(\frac{r-t}{Th}\right)^{-1/2} \sqrt{Nh} \left[\hat{F}_t^{(r)} - H^{(r)'} F_t^{(r)} - B_t^{(r)}\right] \xrightarrow{d} N\left(0, V_r^{-1} Q_r \Gamma_{rt} Q_r' V_r^{-1}\right),$$

where the bias term

$$B_t^{(r)} = \left[\bar{C}_1^{(r)} \left(\frac{t-r}{T}\right) + \bar{C}_2^{(r)} \kappa_2 h^2 + \bar{C}_3^{(r)} \left(\frac{t-r}{T}\right)^2\right] F_t^{(r)}$$

with $\bar{C}_1^{(r)} = V_{NT}^{(r)-1} H^{(r)\prime} \Sigma_F (\Lambda_r' \Lambda_r^{(1)}/N)$, $\bar{C}_2^{(r)} = V_{NT}^{(r)-1}[\frac{1}{2} H^{(r)\prime} \Sigma_F (\Lambda_r^{(2)\prime} \Lambda_r/N) + \bar{C}_1^{(r)} \Sigma_F (\Lambda_r' \Lambda_r^{(1)}/N)]$, $\bar{C}_3^{(r)} = \frac{1}{2} V_{NT}^{(r)-1} H^{(r)\prime} \Sigma_F (\Lambda_r' \Lambda_r^{(2)} N)$, $H^{(r)} = (N^{-1} \Lambda_r' \Lambda_r)(T^{-1} F^{(r)'} \hat{F}^{(r)}) V_{NT}^{(r)-1}$, $V_{NT}^{(r)}$ denotes the $R \times R$ diagonal matrix of the first R largest eigenvalues of $(NT)^{-1} X^{(r)} X^{(r)'}$, V_r is the diagonal matrix consisting of the eigenvalues of $\Sigma_{\Lambda_r}^{1/2} \Sigma_F \Sigma_{\Lambda_r}^{1/2}$ in descending order with Υ_r being the corresponding (normalized) eigenvector matrix ($\Upsilon_r' \Upsilon_r = \mathbb{I}_R$), and $Q_r = V_r^{1/2} \Upsilon_r^{-1} \Sigma_{\Lambda_r}^{-1/2}$.

When we treat $H^{(r)'} F_t^{(r)}$ as the pseudo-true factors, we note that the bias term $B_t^{(r)}$ consists of three parts. The first and third parts are related to $\frac{t-r}{T}$ and $\left(\frac{t-r}{T}\right)^2$ that are of respective orders $O_P(h)$ and $O_P(h^2)$ and generated from the third-order Taylor expansion of $d_i(t, r)$. In the eigenvalue analysis, there is no summation running over r or t so that terms associated with $\frac{t-r}{T}$ and $\left(\frac{t-r}{T}\right)^2$ cannot be smoothed out. The second part in $B_t^{(r)}$ contains two components, namely, $\frac{1}{2} V_{NT}^{(r)-1} H^{(r)\prime} \Sigma_F (\Lambda_r^{(2)\prime} \Lambda_r/N) F_t^{(r)} \kappa_2 h^2$ and $V_{NT}^{(r)-1} \bar{C}_1^{(r)} \Sigma_F (\Lambda_r' \Lambda_r^{(1)}/N) F_t^{(r)} \kappa_2 h^2$. The first component is derived from the usual local constant estimation of the common factors while the second one is generated from the summation over $\bar{C}_1^{(r)} \left(\frac{t-r}{T}\right)$ appearing in the derivation. Consequently, $B_t^{(r)}$ is $O_P(h)$, which is quite large but does not cause much trouble in the asymptotic analyses of $\hat{\lambda}_{ir}$ and $\hat{F}_t$.

Theorem 2.2 *Suppose that Assumptions A.1, A.2(ii) and A.3(i) in SW and Assumption A.1* hold. Then, for each $i = 1, 2, \ldots, N$ and $r = 1, 2, \ldots, T$, we have*

$$\sqrt{Th} \left[\hat{\lambda}_{ir} - H^{(r)-1} \lambda_{ir} - B_\Lambda(i, r)\right] \xrightarrow{d} N\left(0, (Q_r')^{-1} \Omega_{ir} Q_r^{-1}\right),$$

where $B_\Lambda(i,r) = \left[\bar{C}_1^{(r)} \Sigma_F \lambda_{ir}^{(1)} + \frac{1}{2} Q_r^{-1\prime} \Sigma_F \lambda_{ir}^{(2)} - (H^{(r)\prime} \Sigma_F \bar{C}_2^{(r)} + H^{(r)\prime} \Sigma_F \bar{C}_3^{(r)} + \bar{C}_1^{(r)} \Sigma_F \bar{C}_1^{(r)\prime}) H^{(r)-1} \lambda_{ir}\right] \kappa_2 h^2$.

When we treat $H^{(r)-1} \lambda_{ir}$ as the pseudo-true factor loadings, Theorem 2.2 indicates that the bias of $\hat{\lambda}_{ir}$ contains three terms that are associated with λ_{ir} and its first and second order derivatives, respectively. The term related to λ_{ir} is introduced by the bias terms in $\hat{F}_t^{(r)}$, the term related to $\lambda_{ir}^{(2)}$ is derived from the conventional nonparametric kernel estimation, and the term associated with $\lambda_{ir}^{(1)}$ is obtained from the interaction between the other two bias terms.

Theorem 2.3 *Suppose that Assumptions A.1, A.2(i) and A.3(i) in SW and Assumption A.1* hold. Then, for each $t = 1, 2, \ldots, T$, we have*

$$\sqrt{N} \left[\hat{F}_t - \bar{H}^{(t)'} F_t\right] \xrightarrow{d} N\left(0, (\Sigma_{\Lambda_t}^{-1} Q_t^{-1})' \Gamma_{tt} \Sigma_{\Lambda_t}^{-1} Q_t^{-1}\right),$$

where $\bar{H}^{(t)} = H^{(t)} - \breve{H}^{(t)} (Q_t \Sigma_{\Lambda_t} Q_t')^{-1}$ and $\breve{H}^{(t)\prime} = h^2 \kappa_2 H^{(t)-1} \{\frac{1}{2} \frac{\Lambda_t' \Lambda_t^{(2)}}{N} \Sigma_F H^{(t)} + \frac{\Lambda_t' \Lambda_t^{(1)}}{N} \Sigma_F \bar{C}_1^{(t)\prime} - \frac{\Lambda_t' \Lambda_t}{N} H^{(t)-1\prime}$ $\times [\bar{C}_1^{(t)} \Sigma_F \bar{C}_1^{(t)\prime} + (\bar{C}_2^{(t)} + \bar{C}_3^{(t)}) \Sigma_F H^{(t)}]\} H^{(t)\prime}$.

Theorem 2.3 suggests that $\hat{F}_t$ is asymptotically unbiased for $\bar{H}^{(t)'}F_t$. The difference between $\bar{H}^{(t)}$ and $H^{(t)}$ is given by $-\breve{H}^{(t)}(Q_t\Sigma_{\Lambda_t}Q_t')^{-1}$, an $O_P(h^2)$ term that arises from the approximation error $d_i\left(t,r\right)$.

Following Bai (2003) and SW, we can also derive the limit theory of the estimated common components from Theorems 2.2-2.3. Recall that $C_{it}^0 = \lambda_{it}'F_t$ and $\hat{C}_{it} = \hat{\lambda}_{it}'\hat{F}_t$. The following theorem studies the asymptotic distribution of $\hat{C}_{it}$.

Theorem 2.4 *Suppose that Assumptions A.1, A.2, A.3(1) in SW and Assumption A.1* hold. Then, for each $i = 1, 2, \ldots, N$ and $r = 1, 2, \ldots, T$, we have*

$$\left(\frac{1}{N}V_{1it} + \frac{1}{Th}V_{2it}\right)^{-1/2}\left(\hat{C}_{it} - C_{it}^0 - B_C(i,t)\right) \xrightarrow{d} N\left(0,1\right),$$

where $B_C(i,t) = \lambda_{it}'Q_t'B_F(t) + B_\Lambda(i,t)'\left(Q_t^{(-1)}\right)'F_t$, $B_F(t) = -(Q_t\Sigma_{\Lambda_t}Q_t')^{-1}\breve{H}^{(t)\prime}F_t$, $V_{1it} = \lambda_{it}'\Sigma_{\Lambda_t}^{-1}\Gamma_{tt}\Sigma_{\Lambda_t}^{-1}\lambda_{it}$, *and* $V_{2it} = F_t'\Sigma_F^{-1}\Omega_{i,t}\Sigma_F^{-1}F_t$.

In comparison with the result in Theorem 3.4 in SW, the estimator $\hat{C}_{it}$ also exhibits biases that are $O_P(h^2)$, carried over from estimates $\hat{\lambda}_{ir}$ and $\hat{F}_t$.

The following theorem is a correction of Theorem 3.5 in SW and it reflects the contribution of the approximation error.

Theorem 2.5 *Suppose that Assumptions A.1, A.3(i) and A.4 in SW and Assumption A.1* hold. Then*
(i) $\max_{i,t}\left\|\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}\right\| = O_P\left((Th/\ln T)^{-1/2} + h^2\right)$,
(ii) $\max_t\left\|\hat{F}_t - H^{(t)'}F_t\right\| = O_P\left((N/\ln T)^{-1/2} + h^2\right)$,
(iii) $\max_{i,t}\left|\hat{C}_{it} - C_{it}^0\right| = o_P\left((Th/\ln T)^{-1/2}T^{1/8} + h^2T^{1/8}\right)$.

3 Effects of the approximation error on the limiting distribution of the test statistic

SW propose a statistic to test the null hypothesis of constant factor loadings, namely, $\mathbb{H}_0 : \lambda_{it} = \lambda_{i0}$ for all (i,t). The test is based on the comparison between the conventional PCA estimates of the common components under the null (e.g., Bai and Ng (2002)) and the LPCA estimates under the alternative of time-varying factor loadings: $\hat{M} = \frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T(\hat{\lambda}_{it}'\hat{F}_t - \tilde{\lambda}_{i0}'\tilde{F}_t)^2$, where $\tilde{\lambda}_{i0}$ and $\tilde{F}_t$ are the PCA estimates of the factor loadings and factors under the null. SW study the asymptotic distribution of $\hat{M}$ under $\mathbb{H}_0$ and a sequence of Pitman local alternatives $\mathbb{H}_1\left(a_{NT}\right) : \lambda_{it} = \lambda_{i0} + a_{NT}g_i(\frac{t}{T})$ for each i and t, where $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$. Like Assumption A.1*(i), we now require $g_i\left(\cdot\right)$ to be third order continuously differentiable.

According to the results in Section 2, we note that

$$\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} = \frac{1}{T}H^{(t)\prime}\sum_{s=1}^T k_{h,st}^*F_se_{is} + B_\Lambda(i,t) + R_\Lambda(i,t) \text{ and}$$

$$\hat{F}_t - H^{(t)\prime}F_t = \hat{S}_{\lambda,t}^{-1}H^{(t)-1}\frac{1}{N}\sum_{i=1}^N\lambda_{it}e_{it} + B_F(t) + R_F(t),$$

where $R_\Lambda(i,t)$ and $R_F(t)$ represent the smaller order terms in the asymptotic expansions. Under $\mathbb{H}_0$, all derivatives of $\lambda_i(\cdot)$ are zero and one can readily verify that $B_\Lambda(i,t) = B_F(t) = 0$ so that the limiting null distribution of $\hat{M}$ in SW continues to hold. Under $\mathbb{H}_1(a_{NT})$, the first and second order derivatives of $\lambda_i(t/T)$ are given by $\lambda_{it}^{(c)} = a_{NT} g_i^{(c)}(t/T)$ with $c = 1, 2$. As a result, $B_\Lambda(i,t) = O_P(a_{NT}h^2)$ and $B_F(t) = O_P(a_{NT}h^2)$, both of which are sufficiently small so that Lemmas B.1-B.3 in SW still hold (see Appendices B and C in the online supplement). In short, the presence of approximation error in $\hat{\lambda}_{it}$ and $\hat{F}_t$ does not affect the limiting distributions of our test statistic under $\mathbb{H}_0$ and $\mathbb{H}_1(a_{NT})$. We do not repeat the results here but rather provide a sketched proof in the online supplement.

4 Conclusion

In this paper, we correct the errors in SW which are due to the ignorance of an approximation error. The approximation error causes some bias terms to show up in the limiting distributions of the estimated factor loadings and alters the rotational matrix in the factor estimates. But it does not affect the limiting distributions of our test statistic under either the null hypothesis or the local alternative hypothesis.

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(NOT for Publication)

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This online supplement is composed of three appendices. Appendix A contains the proof of the results in Section 2. Appendix B contains the proof of the results in Section 4 of Su and Wang (2017, SW hereafter). Appendix C contains the proofs of some technical lemmas in Appendices A and B.

We use $\sum_i$, $\sum_{i,j}$, $\sum_t$, and $\sum_{s,t}$ to denote $\sum_{i=1}^N$, $\sum_{i=1}^N\sum_{j=1}^N$, $\sum_{t=1}^T$, and $\sum_{t=1}^T\sum_{s=1}^T$, respectively. Let $C_{NT}=\min\{\sqrt{Th},\sqrt{N},h^{-2}\}$ and $C_{0NT}=\min\{\sqrt{T},\sqrt{N}\}$. Denote $B_{1t}^{(r)}=\bar{C}_1^{(r)}(\frac{t-r}{T})F_t^{(r)}+\bar{C}_3^{(r)}(\frac{t-r}{T})^2F_t^{(r)}$, $B_{21}^{(r)}=\frac{1}{2}V_{NT}^{(r)-1}H^{(r)\prime}\Sigma_F(\Lambda_r^{(2)\prime}\Lambda_r/N)$, and $B_{22}^{(r)}=V_{NT}^{(r)-1}\bar{C}_1^{(r)}\Sigma_F(\Lambda_r'\Lambda_r^{(1)}/N)$. Note that $B_{2t}^{(r)}=\kappa_2h^2\bar{C}_2^{(r)}F_t^{(r)}=\kappa_2h^2(B_{21}^{(r)}+B_{22}^{(r)})F_t^{(r)}\equiv B_{2t,1}^{(r)}+B_{2t,2}^{(r)}$. Let $\lambda_{ir}^{(c)}=\lambda_i^{(c)}(\frac{r}{T})$ for $c=1,2$.

A Proofs of Theorems in Section 2

We first state two lemmas that are used in proving the main results, which are parallel to Lemmas A.1 and A.2 in SW.

Lemma A.1 *Suppose that Assumptions A.1 and A.3(i) in SW and Assumption A.1* hold. Then (i)* $T^{-1}\hat{F}^{(r)\prime}[(NT)^{-1}X^{(r)}X^{(r)\prime}]\hat{F}^{(r)}=V_{NT}^{(r)}=V_r+O_P\left(C_{NT}^{-1}\right)$, *(ii)* $T^{-1}\hat{F}^{(r)\prime}F^{(r)}=Q_r+O_P\left(C_{NT}^{-1}\right)$, *(iii)* $H^{(r)}=Q_r^{-1}+O_P\left(C_{NT}^{-1}\right)$, *and (iv)* $H^{(r)}H^{(r)\prime}=\Sigma_F^{-1}+O_P(C_{NT}^{-1})$, *where* V_r *and* Q_r *are as defined in Theorem 2.1.*

Lemma A.2 *Suppose that Assumptions A.1 and A.3(i) in SW and Assumption A.1* hold. Denote* $B^{(r)}=\left(B_1^{(r)},\cdots,B_T^{(r)}\right)'$. *Then (i)* $\frac{1}{T}\left\|\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right\|^2=O_P(C_{NT}^{-2})$, *(ii)* $\frac{1}{T}\left\|(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'F^{(r)}H^{(r)}\right\|=O_P(C_{NT}^{-2})$, *(iii)* $\frac{1}{T}\left\|(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'\hat{F}^{(r)}\right\|=O_P(C_{NT}^{-2})$.

Proof of Theorem 2.1 Let $D(s,r)=(d_1(s,r),...,d_N(s,r))'$. Noting that $(NT)^{-1}X^{(r)}X^{(r)\prime}\hat{F}^{(r)}=\hat{F}^{(r)}V_{NT}^{(r)}$ and $X_s^{(r)}=\Lambda_rF_s^{(r)}+D(s,r)F_s^{(r)}+e_s^{(r)}$, we can decompose $\hat{F}_t^{(r)}-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}$ as follows:

$$
\begin{aligned}
&\hat{F}_t^{(r)}-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}\\
&=V_{NT}^{(r)-1}\frac{1}{NT}\sum_s\hat{F}_s^{(r)}X_s^{(r)\prime}X_t^{(r)}-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}\\
&=V_{NT}^{(r)-1}\frac{1}{NT}\sum_s\hat{F}_s^{(r)}\left[\Lambda_rF_s^{(r)}+D(s,r)F_s^{(r)}+e_s^{(r)}\right]'\left[\Lambda_rF_t^{(r)}+D(t,r)F_t^{(r)}+e_t^{(r)}\right]-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}\\
&=V_{NT}^{(r)-1}\left\{\frac{1}{T}\sum_s\hat{F}_s^{(r)}E(e_s^{(r)\prime}e_t^{(r)}/N)+\frac{1}{T}\sum_s\hat{F}_s^{(r)}\left[e_s^{(r)\prime}e_t^{(r)}/N-E(e_s^{(r)\prime}e_t^{(r)}/N)\right]\right.\\
&\quad+\frac{1}{T}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}\Lambda_r'e_t^{(r)}/N+\frac{1}{T}\sum_s\hat{F}_s^{(r)}F_t^{(r)\prime}\Lambda_r'e_s^{(r)}/N+\left[\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'\Lambda_rF_t^{(r)}-V_{NT}^{(r)}B_{2t}^{(r)}\right]\\
&\quad+\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'D(t,r)F_t^{(r)}+\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'e_t^{(r)}+\frac{1}{TN}\sum_s\hat{F}_s^{(r)}e_s^{(r)\prime}D(t,r)F_t^{(r)}\\
&\quad\left.+\left[\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}\Lambda_r'D(t,r)F_t^{(r)}-V_{NT}^{(r)}B_{1t}^{(r)}\right]\right\}\equiv\sum_{l=1}^9A_l(t,r),
\end{aligned}
\tag{A.1}
$$

where $A_1(t,r),...,A_4(t,r)$ are as defined in SW and $A_l(t,r)$, $l=5,...,9$, arise from the approximation error $D\left(t,r\right)$. Note that $V_{NT}^{(r)-1}$ is well defined by Lemma A.1(i) and Assumptions A1(ii)-(iii). By Lemma A.3 below, $\sqrt{Nh}\sum_{j=1,j\neq 3}^{9}A_j(t,r)=o_P(1)$. In addition, SW have shown that $K_r^*\left(\frac{t-r}{Th}\right)^{-1/2}\sqrt{Nh}A_3(t,r)\xrightarrow{d} N\left(0,V_r^{-1}Q_r\Gamma_{rt}Q_r'V_r^{-1}\right)$. This completes the proof of Theorem 2.1. ■

Lemma A.3 *Suppose that Assumptions A.1 and A.3(i) in SW and Assumption A.1* hold. Then (i)* $\sqrt{Nh}\left[A_1(t,r)+A_2(t,r)\right]=o_P(1)$*, and (ii)*$\sqrt{Nh}A_l(t,r)=o_P(1)$ *for* $l=4,...,9$.

Proof of Theorem 2.2. Let $\Delta^{(r)}=(\Delta_1^{(r)},...,\Delta_N^{(r)})$, where $\Delta_i^{(r)}=(\Delta_{i1}^{(r)},...,\Delta_{iT}^{(r)})'$ and $\Delta_{it}^{(r)}=d_i\left(t,r\right)'F_t^{(r)}$. Noting that $\hat{\Lambda}_r'=T^{-1}\hat{F}^{(r)\prime}X^{(r)}$, $T^{-1}\hat{F}^{(r)\prime}\hat{F}^{(r)}=\mathbb{I}_R$, and $X^{(r)}=F^{(r)}\Lambda_r'+\Delta^{(r)}+e^{(r)}$, we have

$$\begin{aligned}
\hat{\lambda}_{ir} &= \frac{1}{T}\hat{F}^{(r)\prime}X_i^{(r)}=\frac{1}{T}\hat{F}^{(r)\prime}F^{(r)}\lambda_{ir}+\frac{1}{T}\hat{F}^{(r)\prime}e_i^{(r)}+\frac{1}{T}\hat{F}^{(r)\prime}\Delta_i^{(r)}\\
&= H^{(r)-1}\lambda_{ir}+\frac{1}{T}H^{(r)\prime}F^{(r)\prime}e_i^{(r)}+\frac{1}{T}\left[\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right]'e_i^{(r)}\\
&\quad -\frac{1}{T}\hat{F}^{(r)\prime}\left[\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right]H^{(r)-1}\lambda_{ir}+\frac{1}{T}\left[\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right]'\Delta_i^{(r)}\\
&\quad +\frac{1}{T}H^{(r)\prime}F^{(r)\prime}\Delta_i^{(r)}+\frac{1}{T}B^{(r)\prime}e_i^{(r)}-\frac{1}{T}\hat{F}^{(r)\prime}B^{(r)}H^{(r)-1}\lambda_{ir}+\frac{1}{T}B^{(r)\prime}\Delta_i^{(r)}\\
&\equiv H^{(r)-1}\lambda_{ir}+\sum_{l=1}^{8}D_l(i,r). \qquad\qquad (A.2)
\end{aligned}$$

By Lemma A.4 below, $\sqrt{Th}\sum_{l=2}^{8}D_l(i,r)=\sqrt{Th}B_\Lambda(r)+o_P(1)$, where $B_\Lambda(r)=\{\bar{C}_1^{(r)}\Sigma_F\lambda_{ir}^{(1)}+\frac{1}{2}Q_r^{-1\prime}\Sigma_F\lambda_{ir}^{(2)}-(H^{(r)\prime}\Sigma_F\bar{C}_2^{(r)}+H^{(r)\prime}\Sigma_F\bar{C}_3^{(r)}+\bar{C}_1^{(r)}\Sigma_F\bar{C}_1^{(r)\prime})H^{(r)-1}\lambda_{ir}\}\kappa_2h^2$. As in SW, $\sqrt{Th}D_1(i,r)=\frac{1}{\sqrt{Th}}H^{(r)\prime}\sum_s K_r^*(\frac{s-r}{Th})F_se_{is}\xrightarrow{d}N(0,\left(Q_r^{-1}\right)'\Omega_{i,r}Q_r^{-1})$ by A.1(iii) and Assumption A.2(ii). This completes the proof. ■

Lemma A.4 *Suppose that Assumptions A.1 and A.3(i) in SW and Assumption A.1* hold. Then*

(i) $\sqrt{Th}D_l(i,r)=o_P(1)$ *for* $l=2,3,4,6$,

(ii) $\sqrt{Th}D_5(i,r)=\sqrt{Th}\frac{\kappa_2h^2}{2}Q_r^{-1\prime}\Sigma_F\lambda_{ir}^{(2)}+o_P(1)$,

(iii) $\sqrt{Th}D_7(i,r)=-\sqrt{Th}\kappa_2h^2[H^{(r)\prime}\Sigma_F\bar{C}_2^{(r)}+H^{(r)\prime}\Sigma_F\bar{C}_3^{(r)}+\bar{C}_1^{(r)}\Sigma_F\bar{C}_1^{(r)\prime}]H^{(r)-1}\lambda_{ir}+o_P(1)$,

(iv) $\sqrt{Th}D_8(i,r)=\sqrt{Th}\kappa_2h^2\bar{C}_1^{(r)}\Sigma_F\lambda_{ir}^{(1)}+o_P(1)$.

Lemma A.5 *Suppose that Assumptions A.1 and A.3(i) in SW and Assumption A.1* hold. Then for* $t=1,2,\ldots,T$,

(i) $\hat{S}_{\lambda,t}=\frac{1}{N}\sum_i\hat{\lambda}_{it}\hat{\lambda}_{it}'=Q_t\Sigma_{\Lambda_t}Q_t'+o_P(1)$,

(ii) $\frac{1}{\sqrt{N}}\sum_i(\hat{\lambda}_{it}-H^{(t)-1}\lambda_{it})e_{it}=o_P(1)$,

(iii) $\frac{1}{\sqrt{N}}\sum_i\hat{\lambda}_{it}(\hat{\lambda}_{it}-H^{(t)-1}\lambda_{it})'H^{(t)\prime}F_t=\sqrt{N}\breve{H}^{(t)\prime}F_t+o_P(1)$ *where* $\breve{H}^{(t)}$ *is defined in Theorem 2.3.*

Proof of Theorem 2.3. Noting that $X_{it}=\hat{\lambda}_{it}'H^{(t)\prime}F_t+e_{it}+(H^{(t)-1}\lambda_{it}-\hat{\lambda}_{it})'H^{(t)\prime}F_t$, we have

$$\begin{aligned}
\hat{F}_t-H^{(t)\prime}F_t &= \hat{S}_{\lambda,t}^{-1}\left(\frac{1}{N}\sum_i\hat{\lambda}_{it}X_{it}\right)-H^{(t)\prime}F_t\\
&= \hat{S}_{\lambda,t}^{-1}\left\{H^{(t)-1}\frac{1}{N}\sum_i\lambda_{it}e_{it}+\frac{1}{N}\sum_i(\hat{\lambda}_{it}-H^{(t)-1}\lambda_{it})e_{it}-\frac{1}{N}\sum_i\hat{\lambda}_{it}(\hat{\lambda}_{it}-H^{(t)-1}\lambda_{it})'H^{(t)\prime}F_t\right\}\\
&\equiv A_1(t)+A_2(t)+A_3(t), \qquad\qquad (A.3)
\end{aligned}$$

where $\hat{S}_{\lambda,t} = \frac{1}{N}\sum_i \hat{\lambda}_{it}\hat{\lambda}_{it}'$. By Lemmas A.5, $\sqrt{N}A_2(t) = o_P(1)$ and $\sqrt{N}A_3(t) = -\sqrt{N}(Q_t\Sigma_{\Lambda_t}Q_t')^{-1}\breve{H}^{(t)\prime}F_t + o_P(1)$. By Lemmas A.1(iii) and A.5(i), and Assumption A.2(i), we have

$$\sqrt{N}A_1(t) = \hat{S}_{\lambda,t}^{-1}H^{(t)-1}\frac{1}{\sqrt{N}}\sum_i \lambda_{it}e_{it} = \left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}Q_t\frac{1}{\sqrt{N}}\sum_i \lambda_{it}e_{it} + o_P(1)$$
$$\xrightarrow{d} N\left(0, \left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}Q_t\Gamma_{tt}Q_t'\left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}\right) = N\left(0, \left(\Sigma_{\Lambda_t}^{-1}Q_t^{-1}\right)'\Gamma_{tt}\Sigma_{\Lambda_t}^{-1}Q_t^{-1}\right).$$

It follows that $\sqrt{N}\left[\hat{F}_t - \bar{H}^{(t)\prime}F_t\right] \xrightarrow{d} N\left(0, \left(\Sigma_{\Lambda_t}^{-1}Q_t^{-1}\right)'\Gamma_{tt}\Sigma_{\Lambda_t}^{-1}Q_t^{-1}\right)$ where $\bar{H}^{(t)} = H^{(t)} - (Q_t\Sigma_{\Lambda_t}Q_t')^{-1}\breve{H}^{(t)}$. ■

Proof of Theorem 2.4. Recall that $B_F(t) = -(Q_t\Sigma_{\Lambda_t}Q_t')^{-1}\breve{H}^{(t)\prime}F_t$. Note that

$$\begin{aligned}
\hat{C}_{it} - C_{it}^0 &= \hat{\lambda}_{it}'\hat{F}_t - \lambda_{it}'F_t \\
&= \left(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)\right)'\left(\hat{F}_t - H^{(t)\prime}F_t - B_F(t)\right) + B_\Lambda(i,t)'B_F(t) \\
&+ \left(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)\right)'H^{(t)\prime}F_t + \left(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)\right)'B_F(t) \\
&+ \lambda_{it}'(H^{(t)-1})'\left(\hat{F}_t - H^{(t)\prime}F_t - B_F(t)\right) + \lambda_{it}'(H^{(t)-1})'B_F(t) \\
&+ B_\Lambda(i,t)'\left(\hat{F}_t - H^{(t)\prime}F_t - B_F(t)\right) + B_\Lambda(i,t)'H^{(t)\prime}F_t \equiv \sum_{l=1}^{8} C_{it}^{(l)},
\end{aligned} \tag{A.4}$$

By Theorems 2.2-2.3 and the fact that $B_\Lambda(i,t)$ and $B_F(t)$ are both $O_P\left(h^2\right)$, $C_{it}^{(1)} = O_P((NTh)^{-1/2})$, $C_{it}^{(2)} = O_P(h^4)$, $C_{it}^{(4)} = O_P((Th)^{-1/2}h^2)$, and $C_{it}^{(7)} = O_P(N^{-1/2}h^2)$. Noting that $H^{(t)} = Q_t^{-1} + O_P\left(C_{NT}^{-1}\right)$ by Lemma A.1(iii), $C_{it}^{(6)} + C_{it}^{(8)} = \lambda_{it}'(H^{(t)-1})'B_F(t) + B_\Lambda(i,t)'H^{(t)\prime}F_t = B_C(i,t) + o_P(\bar{C}_{NT}^{-1})$ where $\bar{C}_{NT} \equiv \min\{\sqrt{Th}, \sqrt{N}\}$. By the proofs of Theorems 2.2-2.3, $\sqrt{Th}(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it} - B_\Lambda(i,t)) = \frac{1}{\sqrt{Th}}H^{(t)\prime}\sum_s K_t^*(\frac{s-t}{Th})F_s e_{is}$ $+o_P(1)$ and $\sqrt{N}(\hat{F}_t - H^{(t)\prime}F_t - B_F(t)) = \left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}Q_t\frac{1}{\sqrt{N}}\sum_i \lambda_{it}e_{it} + o_P(1)$. These results, along with the fact that $H^{(t)}H^{(t)\prime} = \Sigma_F^{-1} + O_P(C_{NT}^{-1})$ by Lemmas A.1(iv), imply that

$$\begin{aligned}
\bar{C}_{NT}\left[\hat{C}_{it} - C_{it}^0 - B_C(i,t)\right] &= \bar{C}_{NT}[C_{it}^{(5)} + C_{it}^{(3)}] + o_P(1) = \frac{\bar{C}_{NT}}{\sqrt{N}}\lambda_{it}'Q_t'\left(Q_t\Sigma_{\Lambda_t}Q_t'\right)^{-1}Q_t\frac{1}{\sqrt{N}}\sum_i \lambda_{it}e_{it} \\
&\quad + \frac{\bar{C}_{NT}}{\sqrt{Th}}F_t'H^{(t)}H^{(t)\prime}\frac{1}{\sqrt{Th}}\sum_s K_t^*\left(\frac{s-t}{Th}\right)F_s e_{is} + o_P(1) \\
&= \frac{\bar{C}_{NT}}{\sqrt{N}}\xi_{1it} + \frac{\bar{C}_{NT}}{\sqrt{Th}}\xi_{2it} + o_P(1),
\end{aligned}$$

where $\xi_{1it} \equiv \lambda_{it}'\Sigma_{\Lambda_t}^{-1}\frac{1}{\sqrt{N}}\sum_i \lambda_{it}e_{it}$ and $\xi_{2it} \equiv F_t'\Sigma_F^{-1}\frac{1}{\sqrt{Th}}\sum_s K_t^*(\frac{s-t}{Th})F_s e_{is}$. By Assumption A.2, $\xi_{1it} \xrightarrow{d} N(0, V_{1it})$ and $\xi_{2it} \xrightarrow{d} N(0, V_{2it})$. It is easy to show that ξ_{1it} and ξ_{2it} are asymptotically independent. Consequently, we have $V_{it}^{-1/2}\bar{C}_{NT}[\hat{C}_{it} - C_{it}^0 - B_C(i,t)] \xrightarrow{d} N(0,1)$, where $V_{it} = \frac{\bar{C}_{NT}^2}{N}V_{1it} + \frac{\bar{C}_{NT}^2}{Th}V_{2it}$. ■

The next lemma is needed in the proof of the uniform convergence results in Theorem 2.5.

Lemma A.6 *Let $S_{\lambda,r} = Q_r\Sigma_{\Lambda_r}Q_r'$, $B^{(r)} = (B_1^{(r)}, ..., B_T^{(r)})'$, and $B_\Lambda^{(r)} = (B_\Lambda(1,r), ..., B_\Lambda(N,r))'$. Suppose that the conditions in Theorem 2.5 hold. Then*

(i) $\max_r \left\|V_{NT}^{(r)} - V_r\right\| = o_P(1)$,

(ii) $\max_r \left\|H^{(r)} - Q_r^{-1}\right\| = o_P(1)$, *and* $\max_r \left\|T^{-1}\hat{F}^{(r)\prime}F^{(r)} - Q_r\right\| = o_P(1)$,

(iii) $\max_r \frac{1}{T}\left\|\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right\|^2=O_P\left(T^{-1}h^{-1}+N^{-1}\ln T\right)+o_P(h^4)$,

(iv) $\max_r \left\|\frac{1}{T}(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'F^{(r)}H^{(r)}\right\|=O_P\left(T^{-1}h^{-1}+N^{-1}\ln T\right)+o_P(h^4)$,

(v) $\max_{i,r} \left\|\frac{1}{T}(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'e_i^{(r)}\right\|=O_P\left(T^{-1}h^{-1}+N^{-1}\ln T\right)+o_P(h^4)\,$,

(vi) $\max_{i,r} \left\|\frac{1}{T}(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'\Delta_i^{(r)}\right\|=O_P\left(T^{-1}h^{-1}+N^{-1}\ln T\right)+o_P(h^4)\,$,

(vii) $\max_{i,r} \left\|\frac{1}{T}H^{(r)\prime}F^{(r)\prime}\Delta_i^{(r)}\right\|=O_P((Th)^{-1}+(Th/\ln T)^{-1/2}h+h^2)$,

(viii) $\max_{i,r} \left\|\frac{1}{T}B^{(r)\prime}e_i^{(r)}\right\|=O_P((Th/\ln T)^{-1/2}h)$,

(ix) $\max_{i,r} \left\|\frac{1}{T}\hat{F}^{(r)\prime}B^{(r)}H^{(r)-1}\lambda_{ir}\right\|=O_P((Th)^{-1}+(Th/\ln T)^{-1/2}h+h^2+N^{-1}h\ln T)$,

(x) $\max_{i,r} \left\|\frac{1}{T}B^{(r)\prime}\Delta_i^{(r)}\right\|=O_P(h^2)$,

(xi) $\max_r \left\|\hat{S}_{\lambda,r}-S_{\lambda,r}\right\|=o_P(1)\,$,

(xii) $\max_r \frac{1}{N}\left\|\hat{\Lambda}_r-\Lambda_r H^{(r)-1\prime}-B_\Lambda^{(r)}\right\|^2=O_P\left(C_{NT}^{-2}\ln T\right)$,

(xiii) $\max_r \frac{1}{N}\left\|\left(\hat{\Lambda}_r-\Lambda_r H^{(r)-1\prime}-B_\Lambda^{(r)}\right)'\Lambda_r H^{(r)-1\prime}\right\|=O_P\left(C_{NT}^{-2}\ln T\right)$,

(xiv) $\max_r \left\|\frac{1}{N}\left(\hat{\Lambda}_r-\Lambda_r H^{(r)-1\prime}-B_\Lambda^{(r)}\right)'e_r\right\|=O_P\left(C_{NT}^{-2}\ln T\right)$.

Proof of Theorem 2.5. (i) By (A.2), $\hat{\lambda}_{ir}-H^{(r)-1}\lambda_{ir}=\sum_{l=1}^{8}D_l(i,r)$. SW have shown that $\max_{i,r}\|D_1(i,r)\|=O_P((Th/\ln T)^{-1/2})$. By Lemma A.6(v), $\max_{i,r}\|D_2(i,r)\|=O_P(T^{-1}h^{-1}+N^{-1}\ln T)+o_P(h^4)$. By Lemma A.6(iii)-(iv), $\max_{i,r}\|D_3(i,r)\|=O_P(T^{-1}h^{-1}+N^{-1}\ln T)+o_P(h^4)$. By Lemma A.4(vii)-(x), $\sum_{l=4}^{8}\max_{i,r}\|D_l(i,r)\|=O_P((Th)^{-1}+(Th/\ln T)^{-1/2}h+h^2+N^{-1}h\ln T)$. Then $\max_{i,r}\left\|\lambda_{ir}-H^{(r)-1}\lambda_{ir}\right\|=O_P((Th/\ln T)^{-1/2}+h^2)$.

(ii)-(iii) The proof follows from that of Theorem 3.5(ii)-(iii) in SW by using Lemma A.6(xi)-(xiv) with obvious modifications. ■

B Proofs of Theorems in Section 4 of SW

The proofs of Theorems 4.1 to 4.3 in SW rely on Lemmas B.1-B.3. We argue that these three lemmas all hold under $\mathbb{H}_1(a_{NT})$. Then the results in SW's Theorems 4.1 to 4.3 continue to hold.

Lemma B.1 *Suppose that Assumptions A.1, A.3(i), A.4, and A.7 in SW and Assumption A.1* hold. Then under $\mathbb{H}_1(a_{NT})$ with $a_{NT}=T^{-1/2}N^{-1/4}h^{-1/4}$,*

(i) $\max_r \left\|V_{NT}^{(r)}-V_0\right\|=O_P(C_{NT}^{-1}(\ln T)^{1/2})$,

(ii) $\max_r \left\|H^{(r)}-H_0\right\|=O_P(C_{NT}^{-1}(\ln T)^{1/2})$,

(iii) $\max_r \left\|\hat{S}_{\lambda,r}-S_{\lambda,0}\right\|=O_P(C_{NT}^{-1}(\ln T)^{1/2})$,

(iv) $\frac{1}{T}\sum_t\left\|\frac{1}{N}\sum_j(\hat{\lambda}_{jt}-H^{(t)-1}\lambda_{jt})e_{jt}\right\|^2=O_P\left(T^{-2}h^{-2}+N^{-2}\ln T\right)$,

(v) $\frac{1}{T}\sum_t\left\|\hat{F}_t-H^{(t)\prime}F_t\right\|^2=O_P\left(N^{-1}\right)$.

Lemma B.2 *Suppose that Assumptions A.1, A.3(i), A.4 and A.7 in SW and Assumption A.1* hold. Then under $\mathbb{H}_1(a_{NT})$ with $a_{NT}=T^{-1/2}N^{-1/4}h^{-1/4}$,*

(i) $\frac{1}{T}\left\|\tilde{F}-FH\right\|^2=O_P\left(C_{0NT}^{-2}\right)$,

(ii) $\frac{1}{T}(\tilde{F}-FH)'FH=O_P\left(C_{0NT}^{-2}\right)+o_P(a_{NT})$,

(iii) $\frac{1}{T}(\tilde{F}-FH)'\tilde{F}=O_P\left(C_{0NT}^{-2}\right)+o_P(a_{NT})$,

(iv) $\frac{1}{T}(\tilde{F}'\tilde{F}-H'F'FH)=O_P\left(C_{0NT}^{-2}\right)+o_P\left(a_{NT}\right)$,
(v) $V_{NT}=V_0+O_P\left(C_{0NT}^{-1}\right)$,
(vi) $H=Q_0^{-1}+O_P\left(C_{0NT}^{-1}\right)$,
(vii) $\frac{1}{N}\sum_i\left\|\tilde{\lambda}_{i0}-H^{-1}\lambda_{i0}\right\|^2=O_P\left(C_{0NT}^{-2}\right)$,
(viii) $\max_r\left\|\left(H^{(r)-1}\right)'\hat{S}_{\lambda,r}^{-1}H^{(r)-1}-\left(H^{-1}\right)'V_{NT}^{-1}(\frac{1}{T}\tilde{F}'F)\right\|=O_P(C_{NT}^{-1}(\ln T)^{1/2})$.

Lemma B.3 *Let $R_\Lambda(i,t)=\sum_{l=2}^{8}D_l(i,t)$ and $R_F(t)=A_2(t)+A_3(t)$. Let $R_\Lambda^0(i)$ and $R_F^0(t)$ be as defined in SW (Lemma B.3). Suppose that Assumptions A.1, A.3(i), A.4 and A.7 in SW and Assumption A.1* hold. Then under $\mathbb{H}_1(a_{NT})$ with $a_{NT}=T^{-1/2}N^{-1/4}h^{-1/4}$,*
(i) $\frac{1}{NT}\sum_i\sum_t\|R_\Lambda(i,t)\|^2=\left(T^{-2}h^{-2}+N^{-2}(\ln T)^2\right)$,
(ii) $\frac{1}{NT}\sum_i\sum_t\|R_F(t)\|^2=O_P(C_{NT}^{-4}(\ln T)^2)$,
(iii) $\frac{1}{N}\sum_i\left\|R_\Lambda^0(i)\right\|^2=O_P\left(C_{0NT}^{-4}\right)+o_P\left(a_{NT}^2\right)$,
(iv) $\frac{1}{T}\sum_t\left\|R_F^0(t)\right\|^2=O_P\left(C_{0NT}^{-4}\right)+o_P\left(a_{NT}^2\right)$.

Proofs of Theorems 4.1-4.3 in SW As Lemmas B.1-B.3 remain valid with little modifications, the proofs in SW continue to hold. ■

C Proof of the Technical Lemmas in Appendices A and B

Proof of Lemma A.1. (i) From the method of local PCA, we have

$$(NT)^{-1}X^{(r)}X^{(r)\prime}\hat{F}^{(r)}=\hat{F}^{(r)}V_{NT}^{(r)}. \tag{C.1}$$

Premultiplying both sides by $T^{-1}\hat{F}^{(r)\prime}$ and using the fact that $T^{-1}\hat{F}^{(r)\prime}\hat{F}^{(r)}=\mathbb{I}_R$, we obtain the first equality in (i). Premultiplying both sides of (C.1) by $\left(\frac{1}{N}\Lambda_r'\Lambda_r\right)^{1/2}\frac{1}{T}F^{(r)\prime}$ and plugging $X^{(r)}=F^{(r)}\Lambda_r'+\Delta^{(r)}+e^{(r)}$ with $\Delta^{(r)}$ being defined in the proof of Theorem 2.2 yield

$$\left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{1/2}\left(\frac{F^{(r)\prime}F^{(r)}}{T}\right)\frac{\Lambda_r'\Lambda_r}{N}\left(\frac{F^{(r)\prime}\hat{F}^{(r)}}{T}\right)+d_{NT}^{(r)}=\left(\frac{\Lambda_r'\Lambda_r}{N}\right)^{1/2}\left(\frac{F^{(r)\prime}\hat{F}^{(r)}}{T}\right)V_{NT}^{(r)}, \tag{C.2}$$

where $d_{NT}^{(r)}=d_{NT,1}^{(r)}+d_{NT,2}^{(r)}$, $d_{NT,1}^{(r)}=(\frac{\Lambda_r'\Lambda_r}{N})^{1/2}\{\frac{F^{(r)\prime}F^{(r)}}{T}\frac{\Lambda_r'e^{(r)\prime}\hat{F}^{(r)}}{NT}+\frac{F^{(r)\prime}e^{(r)}\Lambda_r}{NT}\frac{F^{(r)\prime}\hat{F}^{(r)}}{T}+\frac{F^{(r)\prime}e^{(r)}e^{(r)\prime}\hat{F}^{(r)}}{NT^2}\}$, and $d_{NT,2}^{(r)}=(\frac{\Lambda_r'\Lambda_r}{N})^{1/2}\{\frac{F^{(r)\prime}F^{(r)}}{T}\frac{\Lambda_r'\Delta^{(r)\prime}\hat{F}^{(r)}}{NT}+\frac{F^{(r)\prime}\Delta^{(r)}\Lambda_r}{NT}\frac{F^{(r)\prime}\hat{F}^{(r)}}{T}+\frac{F^{(r)\prime}\Delta^{(r)}\Delta^{(r)\prime}\hat{F}^{(r)}}{NT^2}+\frac{F^{(r)\prime}\Delta^{(r)}e^{(r)\prime}\hat{F}^{(r)}}{NT^2}+\frac{F^{(r)\prime}e^{(r)}\Delta^{(r)\prime}\hat{F}^{(r)}}{NT^2}\}$. SW show that $\left\|d_{NT,1}^{(r)}\right\|=O_P(T^{-1}h^{-1}+N^{-1/2})$. Now, we want to show $\left\|d_{NT,2}^{(r)}\right\|=O_P(T^{-1}h^{-1}+N^{-1/2}+h^2)$. Note that $N^{-1}\Lambda_r'\Lambda_r=\Sigma_{\Lambda_r}+O_P(N^{-1/2})$ and $\frac{1}{T}\sum_t F_t^{(r)}F_t^{(r)\prime}=\Sigma_F+O_P(T^{-1/2}h^{-1/2})$ as in SW. First, we study $\frac{1}{NT}\Lambda_r'\Delta^{(r)\prime}\hat{F}^{(r)}$. Note that

$$\frac{1}{NT}\Lambda_r'\Delta^{(r)\prime}\hat{F}^{(r)}=\frac{1}{NT}\sum_i\sum_t\lambda_{ir}\Delta_{it}^{(r)}\hat{F}_t^{(r)\prime}=\sum_{l=1}^{3}\frac{1}{NT}\sum_i\sum_t\lambda_{ir}d_i(t,r)'F_t^{(r)}\varphi_{l,tr}\equiv\sum_{l=1}^{3}\Theta_l,$$

where $\varphi_{1,tr}=\hat{F}_t^{(r)}-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}$, $\varphi_{2,tr}=H^{(r)\prime}F_t^{(r)}$, and $\varphi_{3,tr}=B_t^{(r)}$. By (A.1), $\Theta_1=\sum_{l=1}^{9}\Theta_{1l}$, where $\Theta_{1l}=\frac{1}{NT}\sum_i\sum_t\lambda_{ir}d_i(t,r)F_t^{(r)}A_l(t,r)'$. By tedious but straightforward analysis, we can show that $\sum_{l=1}^{9}\Theta_{1l}=O_P(h^2)$. In addition, we can show that $\Theta_2=\frac{1}{NT}\sum_i\sum_t k_{h,tr}^*\lambda_{ir}d_i(t,r)F_tF_t'H^{(r)}=O_P(h^2)$ and

$$\Theta_3=\frac{1}{NT}\sum_i\sum_t k_{h,tr}^*\lambda_{ir}d_i(t,r)F_tF_t'\left[\bar{C}_1^{(r)}\left(\frac{t-r}{T}\right)+\bar{C}_2^{(r)}\kappa_2h^2+\bar{C}_3^{(r)}\left(\frac{t-r}{T}\right)^2\right]'=O_P(h^2).$$

Then $\frac{1}{NT}\Lambda_r'\Delta^{(r)\prime}\hat{F}^{(r)} = O_P(h^2)$. Similarly, we have that $\frac{1}{NT}\Lambda_r'\Delta^{(r)\prime}F^{(r)} = O_P(h^2)$.

Second, we study $\frac{1}{NT^2}F^{(r)\prime}\Delta^{(r)}\Delta^{(r)\prime}\hat{F}^{(r)}$. Noting that $T^{-1/2}\left\|\hat{F}^{(r)}\right\|_{\rm sp} = 1$, we have $\frac{1}{NT^2}\left\|F^{(r)\prime}\Delta^{(r)}\Delta^{(r)\prime}\hat{F}^{(r)}\right\| \leq \frac{1}{NT^{3/2}}\left\|F^{(r)\prime}\Delta^{(r)}\Delta^{(r)\prime}\right\|$. In addition,

$$\begin{aligned}\frac{1}{N^2T^3}\left\|F^{(r)\prime}\Delta^{(r)}\Delta^{(r)\prime}\right\|^2 &= \frac{1}{N^2T^3}\sum_{i,j}\sum_{t,s}\sum_{l=1}^{T}k_{h,tr}^*k_{h,sr}^*k_{h,lr}^*{\rm tr}\left\{F_tF_t'd_i(t,r)d_i(s,r)'F_sF_s'd_j(s,r)d_j(l,r)'F_lF_l'\right\} \\ &\leq \frac{O_P\left(h^4\right)}{T}\sum_s k_{h,sr}^*\left(\frac{1}{T}\sum_t k_{h,tr}^*\left\|F_t\right\|^2\right)^2 = O_P(h^4).\end{aligned}$$

Then $\frac{1}{NT^2}F^{(r)\prime}\Delta^{(r)}\Delta^{(r)\prime}\hat{F}^{(r)} = O_P(h^2)$.

Third, we study $\frac{1}{NT^2}F^{(r)\prime}\Delta^{(r)}e^{(r)\prime}\hat{F}^{(r)}$. Note that $\frac{1}{NT^2}\left\|F^{(r)\prime}\Delta^{(r)}e^{(r)\prime}\hat{F}^{(r)}\right\| \leq \frac{1}{NT^{3/2}}\left\|F^{(r)\prime}\Delta^{(r)}e^{(r)\prime}\right\|$. In view of the fact that $\frac{1}{T}\sum_t k_{h,tr}^*F_tF_t'd_i(t,r) = O_P\left(h^2 + (Th/\ln T)^{-1/2}h\right)$ uniformly in i, we can readily show that

$$\begin{aligned}\frac{1}{N^2T^3}\left\|F^{(r)\prime}\Delta^{(r)}e^{(r)\prime}\right\|^2 &= \frac{1}{N^2T^3}\sum_{i,j}\sum_{t,s}\sum_{l=1}^{T}k_{h,tr}^*k_{h,sr}^*k_{h,lr}^*{\rm tr}\left\{F_tF_t'd_i(t,r)e_{is}e_{js}d_j(l,r)'F_lF_l'\right\} \\ &= \frac{1}{N^2}\sum_{i,j}{\rm tr}\left\{\left[\frac{1}{T}\sum_t k_{h,tr}^*F_tF_t'd_i(t,r)\right]\frac{1}{T}\sum_s k_{h,sr}^*e_{is}e_{js}\left[\frac{1}{T}\sum_{l=1}^{T}k_{h,lr}^*d_j(l,r)'F_lF_l'\right]\right\} \\ &\leq \frac{1}{N^2}\sum_{i,j}\left|\frac{1}{T}\sum_s k_{h,sr}^*e_{is}e_{js}\right|\max_i\left\|\frac{1}{T}\sum_t k_{h,tr}^*F_tF_t'd_i(t,r)\right\|^2 = O_P(h^4 + (Th)^{-1}).\end{aligned}$$

Then $\frac{1}{NT^2}F^{(r)\prime}\Delta^{(r)}e^{(r)\prime}\hat{F}^{(r)} = O_P((Th)^{-1/2} + h^2)$ and $\left\|d_{NT,2}^{(r)}\right\| = O_P\left((Th)^{-1/2} + N^{-1/2} + h^2\right)$. The rest of the proof then follows that of Lemma A.1(i) in SW.

(ii)-(iii) The proof is exactly the same as that of Lemma A.1(ii)-(iii) in SW. ■

Proof of Lemma A.2. (i) Noting that $\hat{F}_t^{(r)} - H^{(r)\prime}F_t^{(r)} - B_t^{(r)} = \sum_{l=1}^{9}A_l(t,r)$ by (A.1), we have $\frac{1}{T}\sum_t\left\|\hat{F}_t^{(r)} - H^{(r)\prime}F_t^{(r)} - B_t^{(r)}\right\|^2 \leq \left\|V_{NT}^{-(r)}\right\|^2\frac{9}{T}\sum_t\sum_{l=1}^{9}\left\|V_{NT}^{(r)}A_l(t,r)\right\|^2$ by the Cauchy-Schwarz (CS hereafter) inequality. By Lemma A.1(i), we can bound $\frac{1}{T}\sum_t\left\|A_l(t,r)\right\|^2$ by determining the probability order of $\frac{1}{T}\sum_t\left\|V_{NT}^{(r)}A_l(t,r)\right\|^2$. The terms associated with $A_1(t,r)$ to $A_4(t,r)$ have been analyzed by SW. Now, we consider the other terms in the above summation . First,

$$\frac{1}{T}\sum_t\left\|V_{NT}^{(r)}A_5(t,r)\right\|^2 \leq 3\sum_{l=1}^{3}\frac{1}{T}\sum_t\left\|\frac{1}{TN}\sum_s\varphi_{l,sr}F_s^{(r)\prime}D(s,r)'\Lambda_rF_t^{(r)} - \chi_{2l,t}^{(r)}\right\|^2 \equiv 3\sum_{l=1}^{3}A_{5l}(r).$$

where $\chi_{21,t}^{(r)} = 0$, $\chi_{22,t}^{(r)} = V_{NT}^{(r)}B_{2t,1}^{(r)}$ and $\chi_{23,t}^{(r)} = V_{NT}^{(r)}B_{2t,2}^{(r)}$. By (A.1) and the CS inequality, $A_{51}(r) \leq 9\sum_{l=1}^{9}A_{51l}(r)$, where $A_{51l}(r) = \frac{1}{T}\sum_t||\frac{1}{TN}\sum_s A_l(s,r)F_s^{(r)\prime}D(s,r)'\ \Lambda_rF_t^{(r)}||^2$. By analyzing $A_{51l}(r)$ one by one, we can show that $A_{51}(r) = O_P(C_{NT}^{-2})$. For $A_{52}(r)$ we have

$$A_{52}(r) = \frac{1}{T}\sum_t\left\|H^{(r)\prime}\left[\frac{1}{TN}\sum_s F_s^{(r)}F_s^{(r)\prime}D(s,r)'\Lambda_r - \frac{h^2\kappa_2}{2}\Sigma_F\frac{\Lambda_r^{(2)\prime}\Lambda_r}{N}\right]F_t^{(r)}\right\|^2 \leq \left\|H^{(r)}\right\|\frac{1}{T}\sum_t\left\|F_t^{(r)}\right\|^2\bar{A}_{52}(r)^2,$$

where $\bar{A}_{52}(r) = \left\|\frac{1}{TN}\sum_s F_s^{(r)}F_s^{(r)\prime}D(s,r)'\Lambda_r - \frac{h^2\kappa_2}{2}\Sigma_F\frac{\Lambda_r^{(2)\prime}\Lambda_r}{N}\right\|$. Note that $\bar{A}_{52}(r) \leq \left\|\frac{1}{T}\sum_s F_s^{(r)}F_s^{(r)\prime}\left(\frac{s-r}{T}\right)\frac{\Lambda_r^{(1)\prime}\Lambda_r}{N}\right\| +$

$O_P\left(h^2\right)$, where the leading term is bounded above by

$$\left\{\left\|\frac{1}{T}\sum_s k^*_{h,sr}(F_sF_s'-\Sigma_F)\frac{s-r}{T}\right\|+\left\|\frac{1}{T}\sum_s k^*_{h,sr}\Sigma_F\frac{s-r}{T}\right\|\right\}\left\|\frac{\Lambda_r^{(1)\prime}\Lambda_r}{N}\right\|=O_P((Th)^{-1/2}h)+O((Th)^{-1})$$

by the CS inequality and the uniform approximation of Riemann summation to a definite integral. Then $A_{52}(r)=O_P((Th)^{-1}h^2+(Th)^{-2}+h^4)=O_P(C_{NT}^{-2})$. Similarly,

$$\begin{aligned}A_{53}(r)&\leq\frac{O_P(1)}{T}\sum_t\left\|\frac{1}{TN}\sum_s\sum_{l=1}^{3}c_{l,sr}F_s^{(r)}F_s^{(r)\prime}D(s,r)'\Lambda_rF_t^{(r)}-V_{NT}^{(r)}B_{22}^{(r)}h^2\kappa_2F_t^{(r)}\right\|^2\\&\leq\frac{O_P(1)}{T}\sum_t\left\|\frac{1}{TN}\sum_s\bar{C}_1^{(r)}\left(\frac{s-r}{T}\right)F_s^{(r)}F_s^{(r)\prime}D(s,r)'\Lambda_rF_t^{(r)}-\bar{C}_1^{(r)}\Sigma_F\frac{\Lambda_r'\Lambda_r^{(1)}}{N}h^2\kappa_2F_t^{(r)}\right\|^2+O_P\left(h^6\right)\\&\leq\frac{O_P(1)}{T}\sum_t\left\|F_t^{(r)}\right\|^2\left\|\bar{C}_1^{(r)}\left[\frac{1}{TN}\sum_s k^*_{h,sr}\left(\frac{s-r}{T}\right)^2F_sF_s'-\Sigma_Fh^2\kappa_2\right]\frac{\Lambda_r'\Lambda_r^{(1)}}{N}\right\|^2+O_P\left(h^6\right)\\&=O_P((Th)^{-1}h^2+(Th)^{-2}+h^6)=o_P(C_{NT}^{-2}),\end{aligned}$$

where $c_{1,sr}=\bar{C}_1^{(r)}(\frac{s-r}{T})$, $c_{2,sr}=\bar{C}_2^{(r)}\kappa_2h^2$, and $c_{3,sr}=\bar{C}_3^{(r)}(\frac{s-r}{T})^2$. Then $\frac{1}{T}\sum_t\|A_5(t,r)\|^2=O_P(C_{NT}^{-2})$.

Next, noting that $\frac{1}{T}\sum_s\left\|\hat{F}_s^{(r)}\right\|^2=R$, we have

$$\begin{aligned}\frac{1}{T}\sum_t\left\|V_{NT}^{(r)}A_6(t,r)\right\|^2&=\frac{1}{T}\sum_t\left\|\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'D(t,r)F_t^{(r)}\right\|^2\\&\leq R\left[\frac{1}{NT}\sum_s\left\|F_s^{(r)\prime}D(s,r)'\right\|^2\right]\left[\frac{1}{NT}\sum_t\left\|D(t,r)F_t^{(r)}\right\|^2\right]=O_P(h^4),\text{ and}\\\frac{1}{T}\sum_t\left\|V_{NT}^{(r)}A_7(t,r)\right\|^2&=\frac{1}{T}\sum_t\left\|\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'e_t^{(r)}\right\|^2\\&\leq R\left[\frac{1}{TN^2}\sum_{s,t}\left\|F_s^{(r)\prime}D(s,r)'e_t^{(r)}\right\|^2\right]=O_P(N^{-1}h^2).\end{aligned}$$

Similarly, we can show that $\frac{1}{T}\sum_t\left\|V_{NT}^{(r)}A_8(t,r)\right\|^2=O_P(N^{-1}h^2)$. Finally, we shall show that

$$\begin{aligned}&\frac{1}{T}\sum_t\left\|V_{NT}^{(r)}A_9(t,r)\right\|^2=\frac{1}{T}\sum_t\left\|\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}\Lambda_r'D(t,r)F_t^{(r)}-V_{NT}^{(r)}B_{1t}^{(r)}\right\|^2\\&=\frac{1}{T}\sum_t\left\|\frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}\Lambda_r'D(t,r)F_t^{(r)}-H^{(r)\prime}\Sigma_F\left[\frac{\Lambda_r'\Lambda_r^{(1)}}{N}\frac{t-r}{T}+\frac{\Lambda_r'\Lambda_r^{(2)}}{2N}(\frac{t-r}{T})^2\right]F_t^{(r)}\right\|^2=O_P(h^4).\end{aligned}$$

To establish the last claim, we first obtain a rough bound for $\frac{1}{T}\sum_t\left\|V_{NT}^{(r)}A_9(t,r)\right\|^2=O_P(h^2)$. Then combining the above results, we obtain $\frac{1}{T}\sum_t\left\|\hat{F}_t^{(r)}-H^{(r)\prime}F_t^{(r)}-B_t^{(r)}\right\|^2=O_P(T^{-1}h^{-1}+N^{-1}+h^2)$. With this preliminary result, we can readily obtain a better control on $\frac{1}{T}\sum_t \left\| V_{NT}^{(r)} A_9(t,r) \right\|^2$:

$$\frac{1}{T}\sum_t \left\| V_{NT}^{(r)} A_9(t,r) \right\|^2 \leq 3\sum_{l=1}^{3} \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \varphi_{1,sr} F_s^{(r)\prime} \Lambda_r' D(t,r) F_t^{(r)} - \chi_{1l,t}^{(r)} \right\|^2$$
$$= O_P(T^{-1}h^{-1} + N^{-1} + h^2) O_P(h^2) + O_P((Th)^{-1} h^2 + (Th)^{-2} + h^4) + O_P(h^4)$$
$$= O_P(T^{-1}h + N^{-1}h^2 + h^4),$$

where $\chi_{11,t}^{(r)} = \chi_{13,t}^{(r)} = 0$ and $\chi_{12,t}^{(r)} = V_{NT}^{(r)} B_{1t}^{(r)}$. Combining the above results yields the desired result.

(ii) By (A.1), $\frac{1}{T}(\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)})' F^{(r)} H^{(r)} = V_{NT}^{(r)-1} \sum_{l=1}^{9} \bar{A}_l(r) H^{(r)}$, where $\bar{A}_l(r) = \frac{1}{T}\sum_t V_{NT}^{(r)} A_l(t,r) F_t^{(r)\prime}$. $\bar{A}_l(r)$, $l = 1, ..., 4$, have been analyzed by SW. For $\bar{A}_5(r)$, we have

$$\bar{A}_5(r) = \frac{1}{T}\sum_t \left[\frac{1}{NT}\sum_s \sum_i \hat{F}_s^{(r)} F_s^{(r)\prime} d_i(s,r) \lambda_{ir}' F_t^{(r)} - V_{NT}^{(r)} B_{2t}^{(r)} \right] F_t^{(r)\prime}$$
$$= \sum_{l=1}^{3} \frac{1}{T}\sum_t \left[\frac{1}{NT}\sum_s \sum_i \varphi_{l,sr} F_s^{(r)\prime} d_i(s,r) \lambda_{ir}' F_t^{(r)} - \chi_{2l,t}^{(r)} \right] F_t^{(r)\prime} \equiv \sum_{l=1}^{3} \bar{A}_{5l}(r).$$

First, $\left\| \bar{A}_{51}(r) \right\| \leq \check{A}_{51}(r) \frac{1}{T}\sum_t \left\| F_t^{(r)} F_t^{(r)\prime} \right\| = \check{A}_{51}(r) O_P(1)$, where $\check{A}_{51}(r) = || \frac{1}{NT}\sum_s \sum_i \varphi_{1,sr} F_s^{(r)\prime} d_i(s,r) \lambda_{ir}' ||$. By (A.1), $\check{A}_{51}(r) = \sum_{l=1}^{9} \check{A}_{51l}(r)$, where $\check{A}_{51l}(r) = \frac{1}{NT}\sum_s \sum_i A_l(s,r) F_s^{(r)\prime} d_i(s,r) \lambda_{ir}'$. By tedious calculations, we can show that $\sum_{l=1}^{9} \check{A}_{51l}(r) = O_P(C_{NT}^{-1} h^2) = O_P(C_{NT}^{-2})$. For, $\bar{A}_{52}(r)$, we can apply Lemma A.1(i) to obtain $\bar{A}_{52}(r) = H^{(r)\prime} \check{A}_{52}(r) \frac{1}{T}\sum_t F_t^{(r)} F_t^{(r)\prime} + O_P(C_{NT}^{-1} h^2)$, where $\check{A}_{52}(r) = \frac{1}{NT}\sum_s \sum_i F_s^{(r)} F_s^{(r)\prime} d_i(s,r) \lambda_{ir}' - \Sigma_F \frac{h^2 \kappa_2}{2} \frac{\Lambda_r^{(2)\prime} \Lambda_r}{N}$. In addition, it is easy to show that

$$\left\| \check{A}_{52}(r) \right\| = \left\| \frac{1}{NT}\sum_s \sum_i k_{h,sr}^* F_s F_s' d_i(s,r) \lambda_{ir}' - \Sigma_F \frac{h^2 \kappa_2}{2} \frac{\Lambda_r^{(2)\prime} \Lambda_r}{N} \right\| = O_P((Th)^{-1/2} h^2 + (Th)^{-1}).$$

Then $\bar{A}_{52}(r) = O_P(C_{NT}^{-2})$. Next, noting that $B_s^{(r)} = \left[\bar{C}_1^{(r)} \left(\frac{s-r}{T}\right) + \bar{C}_2^{(r)} \kappa_2 h^2 + \bar{C}_3^{(r)} \left(\frac{s-r}{T}\right)^2 \right] F_s^{(r)}$, we have

$$\bar{A}_{53}(r) = \frac{1}{T}\sum_t \left[\frac{1}{NT}\sum_s \sum_i B_s^{(r)} F_s^{(r)\prime} d_i(s,r) \lambda_{ir}' F_t^{(r)} - V_{NT}^{(r)} B_{2t,2}^{(r)} \right] F_t^{(r)\prime}$$
$$= \frac{1}{T}\sum_t \left[\frac{1}{NT}\sum_s \sum_i \bar{C}_1^{(r)} \left(\frac{s-r}{T}\right) F_s^{(r)} F_s^{(r)\prime} d_i(s,r) \lambda_{ir}' - h^2 \kappa_2 \bar{C}_1^{(r)} \Sigma_F \frac{\Lambda_r' \Lambda_r^{(1)}}{N} \right] F_t^{(r)} F_t^{(r)\prime}$$
$$+ \frac{1}{NT^2}\sum_t \sum_s \sum_i \left[\bar{C}_2^{(r)} \kappa_2 h^2 + \bar{C}_3^{(r)} \left(\frac{s-r}{T}\right)^2 \right] F_s^{(r)} F_s^{(r)\prime} d_i(s,r) \lambda_{ir}' F_t^{(r)} F_t^{(r)\prime} \equiv \bar{A}_{53,1}(r) + \bar{A}_{53,2}(r).$$

It is easy to show that

$$\bar{A}_{53,1}(r) = \frac{1}{T}\sum_t \bar{C}_1^{(r)} \left[\frac{1}{T}\sum_s F_s^{(r)} F_s^{(r)} \left(\frac{s-r}{T}\right)^2 - \Sigma_F \kappa_2 h^2 \right] \frac{\Lambda_r' \Lambda_r^{(1)}}{N} F_t^{(r)} F_t^{(r)\prime} = O_P((Th)^{-1/2} h^2 + (Th)^{-1})$$

and $\bar{A}_{53,2}(r) = O_P((Th)^{-1/2} h^2 + h^4)$. Then $\bar{A}_{53}(r) = O_P(C_{NT}^{-2})$ and $\bar{A}_5(r) = O_P(C_{NT}^{-2})$.

Next, note that $\bar{A}_6(r) = \frac{1}{T^2 N}\sum_{t,s} \hat{F}_s^{(r)} F_s^{(r)\prime} D(s,r)' D(t,r) F_t^{(r)} F_t^{(r)\prime} = \sum_{l=1}^{3} \frac{1}{T^2 N}\sum_{t,s} \varphi_{l,sr} F_s^{(r)\prime} D(s,r)'$ $D(t,r) F_t^{(r)} F_t^{(r)\prime} \equiv \sum_{l=1}^{3} \bar{A}_{6l}(r)$. For $\bar{A}_{61}(r)$, we have

$$\left\| \bar{A}_{61}(r) \right\| \leq \left\{ \frac{1}{T}\sum_s \left\| \varphi_{1,sr} \right\|^2 \right\}^{1/2} \left\{ \frac{1}{T^3 N^2}\sum_s \left\| \sum_t F_s^{(r)\prime} D(s,r)' D(t,r) F_t^{(r)} F_t^{(r)\prime} \right\|^2 \right\}^{1/2} = O_P(C_{NT}^{-1}) h^2.$$

Note that $\bar{A}_{62}(r) = H^{(r)\prime}\breve{A}_{62}(r)$, where $\breve{A}_{62}(r) = \frac{1}{T^2N}\sum_t\sum_s F_s^{(r)}F_s^{(r)\prime}D(s,r)'D(t,r)F_t^{(r)}F_t^{(r)\prime}$. Noting that $d_i(s,r) = \lambda_{ir}^{(1)}(\frac{s-r}{T}) + \frac{1}{2}\lambda_{ir}^{(2)}(\frac{s-r}{T})^2 + O_P((\frac{s-r}{T})^3)$, we can readily show that

$$\begin{aligned}\|\breve{A}_{62}(r)\| &= \left\|\frac{1}{NT^2}\sum_{t,s}\sum_i k^*_{h,sr}k^*_{h,tr}F_sF_s'd_i(s,r)d_i(t,r)'F_tF_t'\right\| \\ &\leq 3\left\|\frac{1}{NT^2}\sum_{t,s}\sum_i k^*_{h,sr}k^*_{h,tr}F_sF_s'\lambda_{ir}^{(1)}\frac{s-r}{T}\frac{t-r}{T}\lambda_{ir}^{(1)\prime}F_tF_t'\right\| + O_P\left(h^4\right) \\ &\leq 3\frac{1}{N}\sum_i\left\|\frac{1}{T}\sum_t k^*_{h,sr}F_sF_s'\lambda_{ir}^{(1)}\frac{s-r}{T}\right\|^2 + O_P\left(h^4\right) = O_P((Th)^{-1}h^2 + (Th)^{-2} + h^4) \text{ and} \\ \|\bar{A}_{63}(r)\| &= \frac{1}{T^2N}\sum_{t,s}\left\|B_s^{(r)}F_s^{(r)\prime}D(s,r)'D(t,r)F_t^{(r)}F_t^{(r)\prime}\right\| \\ &= \frac{1}{T^2N}\sum_{t,s}\left\|\bar{C}_1^{(r)}\frac{s-r}{T}F_s^{(r)}F_s^{(r)\prime}D(s,r)'D(t,r)F_t^{(r)}F_t^{(r)\prime}\right\| + O_P\left(h^4\right) = O_P((Th)^{-1/2}h^3 + h^4).\end{aligned}$$

Thus, we have $\bar{A}_6(r) = O_P(C_{NT}^{-2})$.

Next, $\bar{A}_7(r) = \frac{1}{T^2N}\sum_{t,s}\hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'e_t^{(r)}F_t^{(r)\prime} = \sum_{l=1}^3\frac{1}{T^2N}\sum_{t,s}\varphi_{l,sr}F_s^{(r)\prime}D(s,r)'e_t^{(r)}F_t^{(r)\prime} \equiv \sum_{l=1}^3\bar{A}_{7l}(r)$. For $\bar{A}_{71}(r)$, we have

$$\|\bar{A}_{71}(r)\| \leq \left\{\frac{1}{T}\sum_s\|\varphi_{1,sr}\|^2\right\}^{1/2}\left\{\frac{1}{T^3N^2}\sum_s\left\|\sum_t F_s^{(r)\prime}D(s,r)'e_t^{(r)}F_t^{(r)\prime}\right\|^2\right\}^{1/2} = O_P(C_{NT}^{-1})O_P((NT/h)^{-1/2}),$$

where we use the fact that

$$\begin{aligned}\frac{1}{T^3N^2}\sum_s\left\|\sum_t F_s^{(r)\prime}D(s,r)'e_t^{(r)}F_t^{(r)\prime}\right\|^2 &= \frac{1}{T^3N^2}\sum_s k^*_{h,sr}\left\|\sum_i\sum_t k^*_{h,tr}F_s'd_i(s,r)e_{it}F_t'\right\|^2 \\ &\leq \frac{1}{T}\sum_s k^*_{h,sr}\|F_s\|^2\left\|\frac{1}{NT}\sum_i\sum_t k^*_{h,tr}e_{it}d_i(s,r)F_t'\right\|^2 = O_P((NTh)^{-1}h^2).\end{aligned}$$

Note that $\bar{A}_{72}(r) = H^{(r)\prime}\breve{A}_{72}(r)$, where $\breve{A}_{72}(r) = \frac{1}{T^2N}\sum_t\sum_s F_s^{(r)}F_s^{(r)\prime}D(s,r)'e_t^{(r)}F_t^{(r)\prime}$. It is easy to show that $\breve{A}_{72}(r) = \frac{1}{NT}\sum_i\sum_s k^*_{sr}F_sF_s'd_i(s,r)\frac{1}{T}\sum_t k^*_{h,tr}e_{it}F_t' = O_P((Th)^{-1/2}h^2)$. Then $\bar{A}_{72}(r) = O_P(C_{NT}^{-2})$. Similarly, we can show that

$$\|\bar{A}_{73}(r)\| = \frac{1}{T^2N}\sum_t\sum_s\left\|\left[\bar{C}_1^{(r)}\frac{s-r}{T} + \bar{C}_2^{(r)}h^2\kappa_2 + \bar{C}_3^{(r)}\left(\frac{s-r}{T}\right)^3\right]F_s^{(r)}F_s^{(r)\prime}D(s,r)'e_t^{(r)}F_t^{(r)\prime}\right\| = O_P(C_{NT}^{-2}).$$

Hence, we have $\bar{A}_7(r) = O_P(C_{NT}^{-2})$.

Next, $\bar{A}_8(r) = \frac{1}{NT^2}\sum_{t,s}\hat{F}_s^{(r)}e_s^{(r)\prime}D(t,r)F_t^{(r)}F_t^{(r)\prime} = \sum_{l=1}^3\frac{1}{NT^2}\sum_{t,s}\varphi_{l,sr}e_s^{(r)\prime}D(t,r)F_t^{(r)}F_t^{(r)\prime} \equiv \sum_{l=1}^3\bar{A}_{8l}(r)$. We can show that $\bar{A}_8(r) = O_P(C_{NT}^{-2})$ by showing that $\bar{A}_{8l}(r) = O_P(C_{NT}^{-2})$ for $l = 1,2,3$. Similarly, $\bar{A}_9(r) = O_P(C_{NT}^{-2})$. Combining the above results and using Lemma A.1(i) yield the claim in part (ii) of the lemma.

(iii) This follows from (i) and (ii) and the triangle inequality. ■

Proof of Lemma A.3. (i) This has been shown in SW.

(ii) SW establish the result for $l = 4$. We now show that $\bar{A}_l(t,r) \equiv V_{NT}^{(r)}A_l(t,r) = o_P((Nh)^{-1/2})$ for $l = 5,...,9$. For $\bar{A}_5(t,r)$, we have

$$\bar{A}_5(t,r) = \frac{1}{TN}\sum_s\hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'\Lambda_rF_t^{(r)} - V_{NT}^{(r)}B_{2t}^{(r)} = \sum_{l=1}^3\left[\frac{1}{TN}\sum_s\varphi_{l,sr}F_s^{(r)\prime}D(s,r)'\Lambda_rF_t^{(r)} - \chi_{2l,t}^{(r)}\right] \equiv \sum_{l=1}^3\bar{A}_{5,l}(t,r).$$

By Lemma A.2(i),

$$\left\|\bar{A}_{5,1}(t,r)\right\| \leq \frac{1}{TN}\sum_s \left\|\varphi_{1,sr}F_s^{(r)\prime}D(s,r)'\Lambda_r\right\| \left\|F_t^{(r)}\right\|$$

$$\leq \left\{\frac{1}{T}\sum_s \left\|\varphi_{1,sr}\right\|^2\right\}^{1/2} \left\{\frac{1}{T}\sum_s \left\|F_s^{(r)\prime}\frac{D(s,r)'\Lambda_r}{N}\right\|^2\right\}^{1/2} O_P(h^{-1/2}) = O_P(C_{NT}^{-1})O_P(h)O_P(h^{-1/2}).$$

Then $\sqrt{Nh}\bar{A}_{5,1}(t,r) = O_P(C_{NT}^{-1}\sqrt{N}h) = O_P(T^{-1/2}N^{1/2}h^{1/2} + h + N^{1/2}h^3) = o_P(1)$ as $Nh/T \to 0$ and $Nh^6 \to 0$ under Assumption A.1*(ii). Next, note that $\bar{A}_{5,2}(t,r) = H^{(r)\prime}\dot{A}_{5,2}(t,r)F_t^{(r)}$, where $\dot{A}_{5,2}(t,r) = \frac{1}{TN}\sum_s k_{h,sr}^* F_s F_s'\, D\,(s,r)'\,\Lambda_r - \frac{h^2\kappa_2}{2}\Sigma_F(\Lambda_r^{(2)\prime}\Lambda_r/N)$. We can show that

$$\dot{A}_{5,2}(t,r) = \frac{1}{T}\sum_s k_{h,sr}^* F_s F_s'\left(\frac{s-r}{T}\right)\frac{\Lambda_r^{(1)}\Lambda_r}{N} + \frac{1}{2}\left[\frac{1}{T}\sum_s k_{h,sr}^* F_s F_s'\left(\frac{s-r}{T}\right)^2 - h^2\kappa_2\Sigma_F\right]\frac{\Lambda_r^{(2)\prime}\Lambda_r}{N} + O_P(h^3)$$

$$= O_P((Th)^{-1/2}\,h + (Th)^{-1}) + O_P((Th)^{-1/2}\,h^2 + (Th)^{-1} + h^3) + O_P(h^3)$$

$$= O_P((Th)^{-1/2}\,h + (Th)^{-1} + h^3),$$

where we use the fact that $\frac{1}{T}\sum_s k_{h,sr}^* F_s F_s'(\frac{s-r}{T}) = O_P((Th)^{-1/2}\,h) + O((Th)^{-1})$. Then $\sqrt{Nh}\bar{A}_{5,2}(t,r)$ $= \sqrt{Nh}O_P((Th)^{-1/2}\,h + h^3 + C_{NT}^{-1}h^2)O_P\left(h^{-1/2}\right) = O_P(N^{1/2}T^{-1/2}h^{1/2} + N^{1/2}h^{6/2} + h^2) = o_P(1)$. Noting that $\chi_{22,t}^{(r)} = V_{NT}^{(r)-1}B_{22,2}^{(r)} = h^2\kappa_2\bar{C}_1^{(r)}\Sigma_F(\Lambda_r'\Lambda_r^{(1)}/N)F_t^{(r)}$, we have

$$\bar{A}_{5,3}(t,r) = \left[\frac{1}{TN}\sum_s B_s^{(r)}F_s^{(r)\prime}D(s,r)'\Lambda_r - h^2\kappa_2\bar{C}_1^{(r)}\Sigma_F\frac{\Lambda_r'\Lambda_r^{(1)}}{N}\right]F_t^{(r)} \equiv \tilde{A}_{5,3}(t,r)F_t^{(r)}.$$

Using the definitions of $B_s^{(r)}$ and $D(s,r)$, we can show that

$$\tilde{A}_{5,3}(t,r) = \bar{C}_1^{(r)}\left[\frac{1}{T}\sum_s\left(\frac{s-r}{T}\right)^2 F_s^{(r)}F_s^{(r)\prime} - h^2\kappa_2\Sigma_F\right]\frac{\Lambda_r'\Lambda_r^{(1)}}{N} + O_P\left(h^3\right) = O_P((Th)^{-1/2}h^2 + (Th)^{-1} + h^3).$$

Then $\sqrt{Nh}\bar{A}_{5,3}(t,r) = \sqrt{Nh}O_P((Th)^{-1/2}h^2 + h^3 + (Th)^{-1})O_P(h^{-1/2}) = o_P(1)$. In sum, we have shown that $\sqrt{Nh}\bar{A}_5(t,r) = o_P(1)$.

Next, $\bar{A}_6(t,r) = \frac{1}{TN}\sum_s \hat{F}_s^{(r)}F_s^{(r)\prime}D(s,r)'D(t,r)F_t^{(r)} = \sum_{l=1}^3 \frac{1}{TN}\sum_s \varphi_{l,sr}F_s^{(r)\prime}D(s,r)'D(t,r)F_t^{(r)} \equiv \sum_{l=1}^3 \bar{A}_{6,l}(t,r)$. For $\bar{A}_{6,1}(t,r)$, we have

$$\begin{aligned}\left\|\bar{A}_{6,1}(t,r)\right\| &\leq \left\{\frac{1}{T}\sum_s \left\|\varphi_{1,sr}\right\|^2\right\}^{1/2}\left\{\frac{1}{TN^2}\sum_s\left\|F_s^{(r)\prime}D(s,r)'D(t,r)\right\|^2\right\}^{1/2}\left\|F_t^{(r)}\right\| \\ &= O_P(C_{NT}^{-1})O_P(h^2)O_P(h^{-1/2}) = o_P((Nh)^{-1/2}),\end{aligned}$$

where we use the fact that $\frac{1}{TN^2}\sum_s\left\|F_s^{(r)\prime}D(s,r)'D(t,r)\right\|^2 = O_P(h^4)$ for $|t-r| \leq Th$. For $\bar{A}_{6,2}(t,r)$, we have

$$\bar{A}_{62}(t,r) = \frac{1}{T}H^{(r)\prime}\sum_s k_{h,sr}^* F_s F_s'\left(\frac{s-r}{T}\right)\frac{\Lambda_r^{(1)\prime}\Lambda_r^{(1)}}{N}\left(\frac{t-r}{T}\right)F_t^{(r)} + O_P\left(h^{5/2}\right)$$

$$= O_P((Th)^{-1/2}\,h^2 + (Th)^{-1}\,h + h^{5/2}) = o_P((Nh)^{-1/2}).$$

In addition, we can readily show that $\bar{A}_{6,3}(t,r) = \frac{1}{TN}\sum_s B_s^{(r)} F_s^{(r)\prime} D(s,r)' D(t,r) F_t^{(r)} = O_P(h^{5/2}) = o_P((Nh)^{-1/2})$. Consequently, $\sqrt{Nh}\bar{A}_6(t,r) = o_P(1)$. Next, $\bar{A}_7(t,r) = \frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} D(s,r)' e_t^{(r)} = \sum_{l=1}^3 \frac{1}{TN}\sum_s \varphi_{l,sr} F_s^{(r)\prime} D(s,r)' e_t^{(r)} \equiv \sum_{l=1}^3 \bar{A}_{7,l}(t,r)$. Note that

$$\left\|\bar{A}_{7,1}(t,r)\right\| \le \left\{\frac{1}{T}\sum_s \left\|\varphi_{1,sr}\right\|^2\right\}^{1/2} \left\{\frac{1}{TN^2}\sum_s \left\|F_s^{(r)\prime} D(s,r)' e_t^{(r)}\right\|^2\right\}^{1/2} = O_P(C_{NT}^{-1}) O_P(N^{-1/2}h^{1/2} + h^{5/2}),$$

where we use the fact that

$$\begin{aligned}
\frac{1}{TN^2}\sum_s \left\|F_s^{(r)\prime} D(s,r)' e_t^{(r)}\right\|^2 &\le \frac{1}{TN^2}\sum_s \left\|\sum_i F_s^{(r)\prime}\left[\lambda_{ir}^{(1)}\frac{s-r}{T} + \frac{1}{2}\lambda_{ir}^{(2)}(\frac{s-r}{T})^2\right] e_{it}\right\|^2 k_{h,tr}^* + O_P(h^5) \\
&\le \frac{1}{NT}\sum_s k_{h,sr}^*(\frac{s-r}{T})^2 \left\|F_s\right\|^2 \left\|\frac{1}{\sqrt{N}}\sum_i [\lambda_{ir}^{(1)} + \frac{1}{2}\lambda_{ir}^{(2)}\frac{s-r}{T}] e_{it}\right\|^2 k_{h,tr}^* + O_P(h^5) \\
&= O_P(N^{-1}h + h^5).
\end{aligned}$$

In addition,

$$\begin{aligned}
\bar{A}_{7,2}(t,r) &= \frac{1}{TN} H^{(r)\prime}\sum_s\sum_i F_s^{(r)} F_s^{(r)\prime}\left[\lambda_{ir}^{(1)}\frac{s-r}{T} + \frac{1}{2}\lambda_{ir}^{(2)}(\frac{s-r}{T})^2\right] e_{it} k_{h,tr}^{*1/2} + O_P(h^{5/2}) \\
&= H^{(r)\prime}\frac{1}{Th}\sum_s k_{h,sr}^* F_s F_s' \frac{s-r}{T}\left(\frac{1}{N}\sum_i [\lambda_{ir}^{(1)} + \lambda_{ir}^{(2)}\frac{s-r}{T}] e_{it}\right) k_{h,tr}^{*1/2} + O_P(h^{5/2}) = O_P(N^{-1/2}h^{1/2} + h^{5/2})
\end{aligned}$$

and $\bar{A}_{7,3}(t,r) = \frac{1}{TN}\sum_s [\bar{C}_1^{(r)}(\frac{s-r}{T}) + \bar{C}_2^{(r)}\kappa_2 h^2 + \bar{C}_3^{(r)}(\frac{s-r}{T})^2] F_s^{(r)} F_s^{(r)\prime} D(s,r)' e_t k_{h,tr}^{*1/2} = O_P(N^{-1/2}h^{3/2} + h^{5/2})$. Therefore, we have $\sqrt{Nh}\bar{A}_7(t,r) = o_P(1)$. Next, $\bar{A}_8(t,r) = \sum_{l=1}^3 \frac{1}{TN}\sum_s \varphi_{l,sr} F_t^{(r)\prime} D(t,r)' e_s^{(r)} \equiv \sum_{l=1}^3 \bar{A}_{8,l}(t,r)$. Following the analysis of $\bar{A}_7(t,r)$, we can also show that $\sqrt{Nh}\bar{A}_8(t,r) = o_P(1)$.

Lastly, $\bar{A}_9(t,r) = \sum_{l=1}^3 \left(\frac{1}{TN}\sum_s \varphi_{l,sr} F_s^{(r)\prime}\Lambda_r' D(t,r) F_t^{(r)} - \chi_{1l,t}^{(r)}\right) \equiv \sum_{l=1}^3 \bar{A}_{9,l}(t,r)$. Note that

$$\left\|\bar{A}_{9,1}(t,r)\right\| \le \left\{\frac{1}{T}\sum_s \left\|\varphi_{1,sr}\right\|^2\right\}^{1/2} \left\{\frac{1}{TN^2}\sum_s \left\|F_s^{(r)\prime}\Lambda_r' D(t,r) F_t^{(r)}\right\|^2\right\}^{1/2} = O_P(C_{NT}^{-1}) O_P(h^{1/2}) = o_P((Nh)^{-1/2}).$$

In view of $B_{1t}^{(r)} = \left[\bar{C}_1^{(r)}(\frac{t-r}{T}) + \bar{C}_3^{(r)}(\frac{t-r}{T})^2\right] F_t^{(r)} = V_{NT}^{(r)-1} H^{(r)\prime}\Sigma_F\left[\frac{\Lambda_r'\Lambda_r^{(1)}}{N}(\frac{t-r}{T}) + \frac{1}{2}\frac{\Lambda_r'\Lambda_r^{(2)}}{N}(\frac{t-r}{T})^2\right] F_t^{(r)}$ and Lemma A.1(i), we can readily show that

$$\bar{A}_{9,2}(t,r) = \left[\frac{1}{TN}\sum_s H^{(r)\prime} F_s^{(r)} F_s^{(r)\prime}\Lambda_r' D(t,r) F_t^{(r)} - V_{NT}^{(r)} B_{1t}^{(r)}\right] = H^{(r)\prime}\tilde{A}_{9,2}(t,r) F_t^{(r)},$$

where $\tilde{A}_{9,2}(t,r) = \frac{1}{TN}\sum_s F_s^{(r)} F_s^{(r)\prime}\Lambda_r' D(t,r) - \Sigma_F\left[\frac{\Lambda_r'\Lambda_r^{(1)}}{N}(\frac{t-r}{T}) + \frac{1}{2}\frac{\Lambda_r'\Lambda_r^{(2)}}{N}(\frac{t-r}{T})^2\right] = O_P((Th)^{-1/2}h + (Th)^{-1}) = o_P((Nh)^{-1/2})$. Similarly,

$$\begin{aligned}
\left\|\bar{A}_{9,3}(t,r)\right\| &\le \frac{1}{TN}\sum_s \left\|\left[\bar{C}_1^{(r)}(\frac{s-r}{T}) + \bar{C}_2^{(r)}\kappa_2 h^2 + \bar{C}_3^{(r)}(\frac{s-r}{T})^2\right] F_s^{(r)} F_s^{(r)\prime}\Lambda_r' D(t,r) F_t^{(r)}\right\| \\
&\le \frac{1}{TN}\sum_s \left\|\bar{C}_1^{(r)}(\frac{s-r}{T}) F_s^{(r)} F_s^{(r)\prime}\Lambda_r' D(t,r)\right\| \left\|F_t^{(r)}\right\| + O_P(h^{5/2}) \\
&= O_P((Th)^{-1/2}h^{3/2} + (Th)^{-1}h^{1/2} + h^{5/2}) = o_P((Nh)^{-1/2}).
\end{aligned}$$

Then $\bar{A}_9(t,r) = o_P((Nh)^{-1/2})$. ■

Proof of Lemma A.4. (i) SW (Lemma A.4 (i)-(ii)) have shown $\sqrt{Th}D_l\left(i,r\right) = o_P\left(1\right)$ for $l = 1, 2$. For $D_4(i,r)$, we have

$$\|D_4(i,r)\| = \left\|\frac{1}{T}\sum_t \varphi_{1,tr}F_t^{(r)\prime}d_i(t,r)\right\| \leq \left\{\frac{1}{T}\sum_t \left\|\varphi_{1,tr}\right\|^2\right\}^{1/2}\left\{\frac{1}{T}\sum_t \left\|F_t^{(r)\prime}d_i(t,r)\right\|^2\right\}^{1/2} = O_P(C_{NT}^{-1})O_P(h)$$

by Lemma A.2(i) and the fact that $\frac{1}{T}\sum_t \left\|F_t^{(r)\prime}d_i(t,r)\right\|^2 = O_P(h^2)$. Then $\sqrt{Th}D_4(i,r) = O_P(T^{1/2}h^{1/2}C_{NT}^{-1}h) = O_P(h + T^{1/2}h^{3/2}N^{-1/2} + T^{1/2}h^{7/2}) = o_P(1)$. For $D_6(i,r)$, we have

$$D_6(i,r) = \frac{1}{T}\sum_t B_t^{(r)}e_{it}^{(r)} = \frac{1}{T}\sum_t k_{h,tr}^*\left[\bar{C}_1^{(r)}(\frac{t-r}{T}) + \bar{C}_2^{(r)}\kappa_2 h^2 + \bar{C}_3^{(r)}(\frac{t-r}{T})^2\right]F_t e_{it} = O_P((Th)^{-1/2}h).$$

That is, $\sqrt{Th}D_6(i,r) = O_P(h) = o_P(1)$.

(ii) Note that

$$\begin{aligned}
D_5(i,r) &= \frac{1}{T}H^{(r)\prime}F^{(r)\prime}\Delta_i^{(r)} = \frac{1}{2T}H^{(r)\prime}\sum_t k_{h,tr}^*F_tF_t'\lambda_{ir}^{(2)}(\frac{t-r}{T})^2 + \frac{1}{T}H^{(r)\prime}\sum_t k_{h,tr}^*F_tF_t'\lambda_{ir}^{(1)}(\frac{t-r}{T}) + O_P\left(h^3\right)\\
&= \frac{h^2\kappa_2}{2}Q_r^{-1\prime}\Sigma_F\lambda_{ir}^{(2)} + O_P((Th)^{-1/2}h + (Th)^{-1}) + O_P\left(h^3\right).
\end{aligned}$$

Then $\sqrt{Th}D_5(i,r) = \sqrt{Th}\frac{h^2\kappa_2}{2}Q_r^{-1\prime}\Sigma_F\lambda_{ir}^{(2)} + o_P(1)$.

(iii) Note that $-D_7(i,r) = \frac{1}{T}\sum_t \hat{F}_t^{(r)}B_t^{(r)\prime}H^{(r)-1}\lambda_{ir} = \sum_{l=1}^3 \frac{1}{T}\sum_t \varphi_{l,tr}B_t^{(r)\prime}H^{(r)-1}\lambda_{ir} \equiv \sum_{l=1}^3 D_{7,l}(i,r)$. For $D_{7,1}(i,r)$, we have $\|D_{7,1}(i,r)\| \leq \left\{\frac{1}{T}\sum_t \left\|\varphi_{1,tr}\right\|^2\right\}^{1/2}\left\{\frac{1}{T}\sum_t \left\|B_t^{(r)}\right\|^2\right\}^{1/2}\left\|H^{(r)-1}\right\|\left\|\lambda_{ir}\right\| = O_P(C_{NT}^{-1})O_P(h)$ $= o_P((Th)^{-1/2})$. For $D_{7,2}(i,r)$ and $D_{7,3}(i,r)$, we have

$$\begin{aligned}
D_{7,2}(i,r) &= \frac{1}{T}\sum_t H^{(r)\prime}F_t^{(r)}B_t^{(r)\prime}H^{(r)-1}\lambda_{ir}\\
&= \frac{1}{T}\sum_t H^{(r)\prime}F_t^{(r)}F_t^{(r)\prime}\left[\bar{C}_1^{(r)}(\frac{t-r}{T}) + \bar{C}_2^{(r)}h^2\kappa_2 + \bar{C}_3^{(r)}(\frac{t-r}{T})^2\right]H^{(r)-1}\lambda_{ir}\\
&= h^2\kappa_2 H^{(r)\prime}\Sigma_F\left[\bar{C}_2^{(r)} + \bar{C}_3^{(r)}\right]H^{(r)\prime-1}\lambda_{ir} + O_P((Th)^{-1/2}h + (Th)^{-1}) \text{ and}\\
D_{7,3}(i,r) &= \frac{1}{T}\sum_t B_t^{(r)}B_t^{(r)\prime}H^{(r)-1}\lambda_{ir} = \frac{1}{T}\sum_t \bar{C}_1^{(r)}(\frac{t-r}{T})^2F_t^{(r)}F_t^{(r)\prime}\bar{C}_1^{(r)\prime} + O_P\left(h^3\right)\\
&= h^2\kappa_2\bar{C}_1^{(r)}\Sigma_F\bar{C}_1^{(r)\prime}H^{(r)-1}\lambda_{ir} + O_P\left(h^3\right)
\end{aligned}$$

Thus, we have $\sqrt{Th}D_7(i,r) = -\sqrt{Th}h^2\kappa_2\left[H^{(r)\prime}\Sigma_F\bar{C}_2^{(r)} + H^{(r)\prime}\Sigma_F\bar{C}_3^{(r)} + \bar{C}_1^{(r)}\Sigma_F\bar{C}_1^{(r)\prime}\right]H^{(r)-1}\lambda_{ir} + o_P(1)$.

(iv) For $D_8(i,r)$, we have

$$\begin{aligned}
D_8(i,r) &= \frac{1}{T}\sum_t B_t^{(r)}F_t^{(r)\prime}d_i(t,r) = \frac{1}{T}\sum_t\left[\bar{C}_1^{(r)}\frac{t-r}{T} + \bar{C}_2^{(r)}h^2\kappa_2 + \bar{C}_3^{(r)}(\frac{t-r}{T})^2\right]F_t^{(r)}F_t^{(r)\prime}\lambda_{ir}^{(1)}\frac{t-r}{T} + O_P(h^3)\\
&= \frac{1}{T}\sum_t \bar{C}_1^{(r)}(\frac{t-r}{T})^2F_t^{(r)}F_t^{(r)\prime}\lambda_{ir}^{(1)} + O_P\left(h^3\right) = h^2\kappa_2\bar{C}_1^{(r)}\Sigma_F\lambda_{ir}^{(1)} + O_P((Th)^{-1/2}h^2 + (Th)^{-1}) + O_P(h^3).
\end{aligned}$$

Then $\sqrt{Th}D_8(i,r) = h^2\kappa_2\bar{C}_1^{(r)}\Sigma_F\lambda_{ir}^{(1)} + o_P(1)$. ■

Proof Lemma A.5. (i) By (A.2),

$$\frac{1}{N}\sum_i \hat{\lambda}_{it}\hat{\lambda}_{it}' = \frac{1}{N}\sum_i \left[H^{(t)-1}\lambda_{it} + \sum_{l=1}^{8} D_l(i,t)\right]\left[H^{(t)-1}\lambda_{it} + \sum_{l=1}^{8} D_l(i,t)\right]' = J_1 + J_2 + J_3 + J_4,$$

where $J_1 = H^{(t)-1}[\frac{1}{N}\sum_i \lambda_{it}\lambda_{it}'][H^{(t)^{-1}}]'$, $J_2 = \frac{1}{N}\sum_i H^{(t)-1}\lambda_{it}[\sum_{l=1}^{8} D_l(i,t)]'$, $J_3 = J_2'$, and $J_4 = \frac{1}{N}\sum_i [\sum_{l=1}^{8} D_l(i,t)][\sum_{l=1}^{8} D_l(i,t)]'$. SW show that $J_1 \xrightarrow{p} Q_t\Sigma_{\Lambda_t}Q_t'$ by Assumption A.1(iii) and Lemma A.1(iii). Despite the presence of some additional terms in $\sum_{l=1}^{8} D_l(i,t)$ due to the approximation bias, we can easily follow SW and show that $J_2 = o_P(1)$ and $J_4 = o_P(1)$. Then $\frac{1}{N}\sum_i \hat{\lambda}_{it}\hat{\lambda}_{it}' = Q_t\Sigma_{\Lambda_t}Q_t' + o_P(1)$.

(ii) By (A.2), we have $\frac{1}{\sqrt{N}}\sum_i(\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it})e_{it} = \sum_{l=1}^{8}\bar{D}_l(t)$, where $\bar{D}_l(t) = \frac{1}{\sqrt{N}}\sum_i D_l(i,t)e_{it}$ for $l = 1, ..., 8$. Following SW (Lemma A.6(ii)) with some minor changes (e.g., $\bar{D}_2(t)$ should be decomposed into nine terms according to (A.1)), we can show that $\bar{D}_l(t) = o_P(1)$ for $l = 1, 2, 3$. Now, we consider the terms $\bar{D}_4(t)$ to $\bar{D}_8(t)$. For $\bar{D}_4(t)$, using $d_i(s,t) = \lambda_{it}^{(1)}\frac{s-t}{T} + \frac{1}{2}\lambda_{it}^{(2)}(\frac{s-t}{T})^2 + O_P((\frac{s-t}{T})^3)$ and Lemma A.2(i), we have

$$\begin{aligned}
\left\|\bar{D}_4(t)\right\| &= \left\|\frac{1}{\sqrt{N}}\sum_i \frac{1}{T}\sum_s \varphi_{1,st}F_s^{(t)\prime}d_i(s,t)e_{it}\right\| = \left\|\frac{1}{T}\sum_s \varphi_{1,st}F_s^{(t)\prime}\frac{s-t}{T}\frac{1}{\sqrt{N}}\sum_i \lambda_{it}^{(1)}e_{it}\right\| + \sqrt{N}O_P(C_{NT}^{-1}h^2) \\
&\le \left\{\frac{1}{T}\sum_s \left\|\varphi_{1,st}\right\|^2\right\}^{1/2}\left\{\frac{1}{T}\sum_s \left\|F_s^{(t)}\frac{s-t}{T}\right\|\right\}^{1/2}\left\|\frac{1}{\sqrt{N}}\sum_i \lambda_{it}^{(1)}e_{it}\right\| + o_P(1) \\
&= O_P(C_{NT}^{-1})O_P(h)O_P(1) + o_P(1) = o_P(1),
\end{aligned}$$

where we also use the fact that $\frac{1}{T}\sum_s \left\|F_s^{(t)}(\frac{s-t}{T})\right\| = O_P(h)$ and that $N^{-1/2}\sum_i \lambda_{it}^{(1)}e_{it} = O_P(1)$. For $\bar{D}_5(t)$,

$$\begin{aligned}
\left\|\bar{D}_5(t)\right\| &= \frac{1}{\sqrt{N}}\left\|\sum_i \frac{1}{T}\sum_s H^{(t)}F_s^{(t)}F_s^{(t)\prime}d_i(s,t)e_{it}\right\| \\
&= \frac{1}{\sqrt{N}}\left\|\sum_i \frac{1}{T}\sum_s k_{h,st}^* H^{(t)}F_sF_s'\sum_{l=1}^{2}\frac{1}{l}\lambda_{it}^{(l)}\left(\frac{s-t}{T}\right)^l e_{it}\right\| + \sqrt{N}O_P(h^3) \\
&\le \left\|H^{(t)}\right\|\sum_{l=1}^{2}\left\|\frac{1}{lT}\sum_s k_{h,st}^* F_sF_s'\left(\frac{s-t}{T}\right)^l\right\|\left\|\frac{1}{\sqrt{N}}\sum_i \lambda_{it}^{(l)}e_{it}\right\| + o_P(1) \le O_P(h) + o_P(1) = o_P(1).
\end{aligned}$$

Next, noting that $B_s^{(t)} = \sum_{l=1}^{3} c_{l,st}F_s^{(t)}$,

$$\begin{aligned}
\left\|\bar{D}_6(t)\right\| &= \left\|\frac{1}{\sqrt{N}}\sum_i \frac{1}{T}\sum_s B_s^{(t)}e_{is}^{(t)}e_{it}\right\| = \left\|\frac{1}{T}\sum_s k_{h,st}^*\sum_{l=1}^{3} c_{l,st}F_s\frac{1}{\sqrt{N}}\sum_i e_{is}e_{it}\right\| \\
&\le \left\{\frac{1}{T}\sum_s \left\|k_{h,st}^*\left[\bar{C}_1^{(t)}\frac{s-t}{T} + \bar{C}_2^{(t)}\kappa_2h^2 + \bar{C}_3^{(t)}\left(\frac{s-t}{T}\right)^2\right]\right\|^2\right\}^{1/2}\left\{\frac{N}{T}\sum_s \left\|\frac{1}{N}\sum_i F_se_{is}e_{it}\right\|^2\right\}^{1/2} \\
&= O_P(h)O_P(N^{1/2}T^{-1/2} + 1) = o_P(1),
\end{aligned}$$

where we use the fact that $\sum_s \left\|\frac{1}{N}\sum_i F_se_{is}e_{it}\right\|^2 \le \sum_s \left\|\frac{1}{N}\sum_i E(F_se_{is}e_{it})\right\|^2 + \sum_s \left\|\frac{1}{N}\sum_i [F_se_{is}e_{it} - E(F_se_{is}e_{it})]\right\|^2 = \sum_s \left\|\gamma_{N,F}(s,t)\right\|^2 + O_P(T/N) = O(1) + O_P(T/N)$ with $\gamma_{N,F}(s,t) = \frac{1}{N}\sum_i E(F_se_{is}e_{it})$.

For $\bar{D}_7(t)$, it suffices to consider a rough bound:

$$\left\|\bar{D}_7(t)\right\| = \left\|\frac{1}{\sqrt{N}}\sum_i \frac{1}{T}\sum_s \hat{F}_s^{(t)} B_s^{(t)\prime} H^{(t)-1}\lambda_{it}e_{it}\right\| = \left\|\frac{1}{\sqrt{N}}\sum_i \frac{1}{T}\sum_s \hat{F}_s^{(t)} B_s^{(t)\prime} H^{(t)-1}\lambda_{it}e_{it}\right\|$$
$$\leq \left\|\frac{1}{T}\sum_s \hat{F}_s^{(t)} B_s^{(t)\prime}\right\| \left\|H^{(t)-1}\right\| \left\|\frac{1}{\sqrt{N}}\sum_i \lambda_{it}e_{it}\right\| = O_P(h)O_P(1)O_P(1) = o_P(1).$$

Finally,

$$\left\|\bar{D}_8(t)\right\| = \left\|\frac{1}{\sqrt{N}}\sum_i \frac{1}{T}\sum_s B_s^{(t)} F_s^{(t)\prime} d_i(s,t)e_{it}\right\| = \frac{1}{\sqrt{N}}\sum_i \frac{1}{T}\sum_s B_s^{(t)} F_s^{(t)\prime}\lambda_{it}^{(1)}\frac{s-t}{T}e_{it} + O_P(\sqrt{N}h^3)$$
$$\leq \left\|\frac{1}{T}\sum_s \left[\bar{C}_1^{(t)}\frac{s-t}{T} + \bar{C}_2^{(t)}\kappa_2 h^2 + \bar{C}_3^{(t)}\left(\frac{s-t}{T}\right)^2\right] F_s^{(t)} F_s^{(t)\prime}\frac{s-t}{T}\right\| \left\|\frac{1}{\sqrt{N}}\sum_i \lambda_{it}^{(1)}e_{it}\right\| + o_P(1)$$
$$= O_P(h^2)O_P(1) + o_P(1) = o_P(1).$$

This proves the desired result in (ii).

(iii) We make the following decomposition

$$\begin{aligned}\frac{1}{\sqrt{N}}\sum_i \hat{\lambda}_{it}[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]' H^{(t)\prime}F_t &= \frac{1}{\sqrt{N}}\sum_i [\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}][\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]' H^{(t)\prime}F_t \\ &\quad + \frac{1}{\sqrt{N}}\sum_i H^{(t)-1}\lambda_{it}[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]' H^{(t)\prime}F_t \equiv L_1(t) + L_2(t).\end{aligned}$$

By Theorem 2.2, $\left\|\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}\right\| = O_P(h^2 + (Th)^{-1/2}) = O_P(C_{NT}^{-1})$. Following the proof of Theorem 2.2, we can readily show that

$$\left\|L_1(t)\right\| \leq \frac{1}{\sqrt{N}}\sum_i \left\|\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}\right\|^2 \left\|H^{(t)\prime}F_t\right\| = O_P(N^{1/2}C_{NT}^{-2}) = o_P(1).$$

For $L_2(t)$, we have $L_2(t) = \sum_{l=1}^8 \frac{1}{\sqrt{N}}\sum_i H^{(t)-1}\lambda_{it}D_l(i,t)' H^{(t)\prime}F_t \equiv \sum_{l=1}^8 L_{2,l}(t)$ by (A.2). Let $\bar{L}_{2,1}(t) = \frac{1}{T\sqrt{N}}\sum_i \lambda_{it}e_i^{(t)\prime}F^{(t)}$. Then $L_{2,1}(t) = H_{2,1}^{(t)-1}\bar{L}_{2,1}(t)H^{(t)}H^{(t)\prime}F_t$. In view of the fact that

$$\bar{L}_{2,1}(t) = \frac{1}{T\sqrt{N}}\sum_i\sum_s \lambda_{it}e_{is}^{(t)}F_s^{(t)\prime} = \frac{1}{\sqrt{Th}}\left\{\frac{h^{1/2}}{(NT)^{1/2}}\sum_i\sum_s k_{h,st}^*\lambda_{it}e_{is}F_s'\right\} = O_P((Th)^{-1/2})$$

by Assumption A.1(vii), we have $L_{2,1}(t) = o_P(1)$. For $L_{22}(t)$ and $L_{2,3}(t)$, we have by Lemma A.2

$$\begin{aligned}\|L_{2,2}(t)\| &= \left\| \frac{1}{\sqrt{N}} \sum_i H^{(t)-1} \lambda_{it} \left[\frac{1}{T} e_i^{(t)\prime} \left(\hat{F}^{(t)} - F^{(t)} H^{(t)} - B^{(t)} \right) \right] H^{(t)\prime} F_t \right\| \\ &= \left\| H^{(t)-1} \frac{1}{T} \sum_s \left(\frac{1}{\sqrt{N}} \sum_i \lambda_{it} e_{is}^{(t)} \right) \varphi_{1,st}' H^{(t)\prime} F_t \right\| \\ &\le O_P(1) \left\{ \frac{1}{T} \sum_s \left\| \frac{1}{\sqrt{N}} \sum_i \lambda_{it} e_{is}^{(t)} \right\|^2 \right\}^{1/2} \left\{ \frac{1}{T} \sum_s \|\varphi_{1,st}\|^2 \right\}^{1/2} = O_P(1) O_P(C_{NT}^{-1}) \text{ and} \\ \|L_{2,3}(t)\| &= \left\| H^{(t)-1} \frac{1}{T\sqrt{N}} \sum_i \lambda_{it} \lambda_{it}' H^{(t)-1\prime} \left[\hat{F}^{(t)} - F^{(t)} H^{(t)} - B^{(t)} \right]' \hat{F}^{(t)} H^{(t)\prime} F_t \right\| \\ &\le \frac{O_P(1)}{T} \left\| \left[\hat{F}^{(t)} - F^{(t)} H^{(t)} - B^{(t)} \right]' \hat{F}^{(t)} H^{(t)\prime} \right\| \frac{1}{\sqrt{N}} \sum_i \|\lambda_{it}\|^2 = O_P(C_{NT}^{-2}) O_P(N^{1/2}) = o_P(1).\end{aligned}$$

For $L_{2,4}(t)$, we have

$$\begin{aligned}L_{2,4}(t) &= \frac{1}{\sqrt{N}} \sum_i H^{(t)-1} \lambda_{it} \frac{1}{T} \sum_s d_i(s,t)' F_s^{(t)} \varphi_{1,st}' H^{(t)\prime} F_t \\ &= H^{(t)-1} \frac{1}{\sqrt{N}} \sum_i \lambda_{it} \lambda_{it}^{(1)\prime} \left[\frac{1}{T} \sum_s \frac{s-t}{T} F_s^{(t)} \varphi_{1,st}' \right] H^{(t)\prime} F_t + O_P(N^{1/2} h^2 C_{NT}^{-1}) \\ &= O_P(N^{1/2}) O_P(C_{NT}^{-2}) + o_P(1) = o_P(1)\end{aligned}$$

where we use the fact that $\frac{1}{T} \sum_s \frac{s-t}{T} F_s^{(t)} \varphi_{1,st}' = O_P(C_{NT}^{-2})$ by similar analysis as used in the proof of Lemma A.2(ii).

For $L_{2,5}(t)$, using $\Delta_{is}^{(t)} = d_i(s,t)' F_s^{(t)} = [\lambda_{it}^{(1)} \frac{s-t}{T} + \frac{1}{2} \lambda_{it}^{(2)} (\frac{s-t}{T})^2 + O_P((\frac{s-t}{T})^3)]' F_s^{(t)}$, we can readily show that

$$L_{2,5}(t) = \frac{1}{\sqrt{N}} \sum_i H^{(t)-1} \lambda_{it} \left[\frac{1}{T} H^{(t)\prime} F^{(t)\prime} \Delta_i^{(t)} \right]' H^{(t)\prime} F_t = H^{(t)-1} \sum_{l=1}^2 L_{2,5,l}(t) H^{(t)} H^{(t)\prime} F_t + O_P(N^{1/2} h^3),$$

where $L_{2,5,l}(t) = \frac{1}{l\sqrt{N}} \sum_i \lambda_{it} \lambda_{it}^{(l)\prime} \frac{1}{T} \sum_s k_{h,st}^* F_s F_s' (\frac{s-t}{T})^l$ for $l = 1, 2$. Note that $\|L_{2,5,1}(t)\| \le \left\| \frac{1}{\sqrt{N}} \sum_i \lambda_{it} \lambda_{it}^{(1)\prime} \right\| \times \left\| \frac{1}{T} \sum_s k_{h,st}^* F_s F_s' \frac{s-t}{T} \right\| = O_P(N^{1/2}) O_P((Th)^{-1/2} h + (Th)^{-1}) = o_P(1)$ and

$$L_{2,5,2}(t) = \frac{\sqrt{N}}{2} \frac{\Lambda_t' \Lambda_t^{(2)}}{N} \frac{1}{T} \sum_s k_{h,st}^* F_s F_s' \left(\frac{s-t}{T} \right)^2 = \frac{\sqrt{N} h^2 \kappa_2}{2} \frac{\Lambda_t' \Lambda_t^{(2)}}{N} \Sigma_F + o_P(1).$$

Then $L_{2,5}(t) = \frac{\sqrt{N} h^2 \kappa_2}{2} H^{(t)-1} \frac{\Lambda_t' \Lambda_t^{(2)}}{N} \Sigma_F H^{(t)} H^{(t)\prime} F_t + o_P(1) = o_P(1)$.

Let $\bar{L}_{2,6}(t) = \frac{1}{\sqrt{N}} \sum_i \lambda_{it} \left[\frac{1}{T} \sum_s B_s^{(t)} e_{is}^{(t)} \right]'$. Then $L_{2,6}(t) = H^{(t)-1} \bar{L}_{2,6}(t) H^{(t)\prime} F_t$. For $\bar{L}_{2,6}(t)$, we have $\bar{L}_{2,6}(t) = \frac{1}{\sqrt{N}} \sum_i \lambda_{it} \frac{1}{T} \sum_s B_s^{(t)\prime} e_{is}^{(t)} = \sum_{l=1}^3 \frac{1}{\sqrt{N}} \sum_i \lambda_{it} \frac{1}{T} \sum_s F_s^{(t)\prime} c_{l,st}' e_{is}^{(t)} \equiv \sum_{l=1}^3 \bar{L}_{2,6,l}(t)$. Note that

$$\begin{aligned}\|\bar{L}_{2,6,1}(t)\| &= \left\| \frac{1}{T} \sum_s k_{h,st}^* F_s' \frac{s-t}{T} \frac{1}{\sqrt{N}} \sum_i \lambda_{it} e_{is} \bar{C}_1^{(t)\prime} \right\| \\ &\le \left\{ \frac{1}{T} \sum_s \left\| k_{h,st}^* F_s \frac{s-t}{T} \right\|^2 \right\}^{1/2} \left\{ \frac{1}{T} \sum_s \left\| \frac{1}{\sqrt{N}} \sum_i \lambda_{it} e_{is} \right\|^2 \right\}^{1/2} \left\| \bar{C}_1^{(t)} \right\| = O_P(h^{1/2}) O_P(1) = o_P(1).\end{aligned}$$

Similarly, we can show that $\bar{L}_{2,6,l}(t) = o_P(1)$ for $l = 2, 3$. Then $L_{2,6}(t) = o_P(1)$. For $L_{2,7}(t)$, we make the following decomposition:

$$-L_{2,7}(t) = \frac{1}{\sqrt{N}T}\sum_i\sum_s H^{(t)-1}\lambda_{it}\lambda_{it}' H^{(t)-1\prime} B_s^{(t)} \hat{F}_s^{(t)\prime} H^{(t)\prime} F_t = H^{(t)-1}\frac{\Lambda_t'\Lambda_t}{N} H^{(t)-1\prime}\left(\sum_{l=1}^{3} L_{2,7,l}(t)\right) H^{(t)\prime} F_t,$$

where $L_{2,7,l}(t) = \frac{\sqrt{N}}{T}\sum_s B_s^{(t)}\varphi_{l,st}'$. Note that

$$\|L_{2,7,1}(t)\| \leq N^{1/2}\left\{\frac{1}{T}\sum_s \left\|\varphi_{1,st}\right\|^2\right\}^2 \left\{\frac{1}{T}\sum_s \left\|B_s^{(t)}\right\|^2\right\}^{1/2} = N^{1/2} O_P(C_{NT}^{-1}) O_P(h) = o_P(1).$$

For $L_{2,7,2}(t)$, we have

$$\begin{aligned} L_{2,7,2}(t) &= \frac{\sqrt{N}}{T}\sum_s B_s^{(t)} F_s^{(t)\prime} H^{(t)} = \frac{\sqrt{N}}{T}\sum_s k_{h,ts}^*\left[\bar{C}_1^{(t)}\frac{s-t}{T} + \bar{C}_2^{(t)}\kappa_2 h^2 + \bar{C}_3^{(t)}\left(\frac{s-t}{T}\right)^2\right] F_s F_s' H^{(t)} \\ &= \sqrt{N} h^2 \kappa_2\left[\bar{C}_2^{(t)} + \bar{C}_3^{(t)}\right]\Sigma_F H^{(t)} + o_P(1), \end{aligned}$$

where we use the fact that $\frac{\sqrt{N}}{T}\sum_s k_{h,ts}^*(\frac{s-t}{T}) F_s F_s' = \sqrt{N} O_P((Th)^{-1/2} h + (Th)^{-1}) = o_P(1)$. For $L_{2,7,3}(t)$, we use $B_s^{(t)} = \bar{C}_1^{(t)}\frac{s-t}{T} + \bar{C}_2^{(t)} h^2\kappa_2 + \bar{C}_3^{(r)}(\frac{s-t}{T})^2$ to obtain

$$L_{2,7,3}(t) = \frac{\sqrt{N}}{T}\sum_s B_s^{(t)} B_s^{(t)\prime} = \sqrt{N}\bar{C}_1^{(t)}\left[\frac{1}{T}\sum_s (\frac{s-t}{T})^2 F_s^{(t)} F_s^{(t)\prime}\right]\bar{C}_1^{(t)\prime} + o_P(1) = \sqrt{N} h^2\kappa_2 \bar{C}_1^{(t)}\Sigma_F \bar{C}_1^{(t)\prime} + o_P(1).$$

Then by Lemma A.1(iii),

$$L_{2,7}(t) = -\sqrt{N} h^2\kappa_2 H^{(t)-1}\frac{\Lambda_t'\Lambda_t}{N} H^{(t)-1\prime}\left(\bar{C}_1^{(t)}\Sigma_F\bar{C}_1^{(t)\prime} + \left[\bar{C}_2^{(t)} + \bar{C}_3^{(t)}\right]\Sigma_F H^{(t)}\right) H^{(t)\prime} F_t + o_P(1).$$

Next,

$$\begin{aligned} L_{2,8}(t) &= \frac{1}{\sqrt{N}}\sum_i H^{(t)-1}\lambda_{it} D_8(i,t)' H^{(t)\prime} F_t = H^{(t)-1}\frac{1}{\sqrt{N}}\sum_i \lambda_{it}\frac{1}{T}\sum_s B_s^{(t)\prime} F_s^{(t)\prime}\lambda_{it}^{(1)}\frac{s-t}{T} H^{(t)\prime} F_t + O_P(N^{1/2}h^3) \\ &= H^{(t)-1}\frac{1}{\sqrt{N}}\sum_i \lambda_{it}\lambda_{it}^{(1)\prime}\left[\frac{1}{T}\sum_s F_s^{(t)} F_s^{(t)\prime}(\frac{s-t}{T})^2\right]\bar{C}_1^{(t)\prime} H^{(t)\prime} F_t + o_P(1) \\ &= \sqrt{N} h^2\kappa_2 H^{(t)-1}\frac{\Lambda_t'\Lambda_t^{(1)}}{N}\Sigma_F \bar{C}_1^{(t)\prime} H^{(t)\prime} F_t + o_P(1). \end{aligned}$$

Combine the above results yields $\frac{1}{\sqrt{N}}\sum_i \hat{\lambda}_{it}[\hat{\lambda}_{it} - H^{(t)-1}\lambda_{it}]' H^{(t)\prime} F_t = \sqrt{N}\breve{H}^{(t)\prime} F_t + o_P(1)$. ■

Proof of Lemma A.6. (i)-(ii) The proof follows from that of Lemma A.7(i)-(ii) of SW and the proof of Lemma A.1 in this paper.

(iii) By (A.1), $\frac{1}{T}\sum_t ||\hat{F}_t^{(r)} - H^{(r)\prime} F_t^{(r)} - B^{(r)}||^2 \leq 9||V_{NT}^{(r)-1}||\sum_{l=1}^{9} A_l(r)$, where $A_l(r) = \frac{1}{T}\sum_t ||V_{NT}^{(r)} A_l(t,r)||^2$. Noting that $\max_r \left\|V_{NT}^{(r)-1}\right\| \leq \max_r \left\|V_{NT}^{(r)-1} - V_r^{-1}\right\| + \left\|V_r^{-1}\right\| = O_P(1)$ by part (i), we can prove (iii) by finding the uniform probability bound for $A_l(r)$, $l = 1, \cdots, 9$. SW (Lemma A.7(iii)) have studied the terms $A_1(r)$ to $A_4(r)$ to obtain $\max_r \sum_{l=1}^{4} A_l(r) = O_P((Th)^{-1} + N\ln T)$. For the remaining terms, noting that $A_7(r)$ and $A_8(r)$ are higher order term than $A_3(r)$ and $A_4(r)$, respectively, it suffices to bound $A_5(r)$, $A_6(r)$

and $A_9(r)$. For $A_5(r)$, we have

$$A_5(r) = \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r F_t^{(r)} - V_{NT}^{(r)} B_{2t}^{(r)} \right\|^2$$
$$\leq 3\sum_{l=1}^{3} \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \varphi_{l,sr} F_s^{(r)\prime} D(s,r)'\Lambda_r F_t^{(r)} - \chi_{2l,t}^{(r)} \right\|^2 \equiv 3\sum_{l=1}^{3} A_{5,l}(r).$$

We will obtain rough probability bound for $\max_r A_{5,l}(r)$, $l = 1,2,3$. Note that

$$\max_r A_{5,1}(r) \leq \max_r \frac{1}{T}\sum_t \left\| F_t^{(r)} \right\|^2 \max_r \frac{1}{TN^2}\sum_s \left\| F_s^{(r)\prime} D(s,r)'\Lambda_r \right\|^2 \max_r \frac{1}{T}\sum_s \left\| \varphi_{1,sr} \right\|^2.$$

It is trivial to show that $\max_r \frac{1}{T}\sum_t \left\| F_t^{(r)} \right\|^2 = O_P(1)$ and $\max_r \frac{1}{TN^2}\sum_s \left\| F_s^{(r)\prime} D(s,r)'\Lambda_r \right\|^2 = O_P(h^2)$. In addition, $\max_r \frac{1}{T}\sum_s \left\| \varphi_{1,sr} \right\|^2 \leq O_P(1)\sum_{l=1}^{9} \max_r A_l(r)$. With arguments that are much simpler than those used in the analyses of $\max_r A_l(r)$ for $l = 5,...,9$ below, we can easily obtain a rough bound: $\max_r \sum_{l=5}^{9} A_l(r) = O_P((Th)^{-1}) + o_P(h^2)$. Then we obtain a rough probability order: $\max_r A_{5,1}(r) = O_P(((Th)^{-1} + N^{-1}\ln T)h^2) + o_P(h^4)$. For $A_{5,2}(r)$,

$$\max_r A_{5,2}(r) = \max_r \frac{1}{T}\sum_t \left\| H^{(r)\prime} \left[\frac{1}{TN}\sum_s F_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r F_t^{(r)} - \frac{h^2\kappa_2}{2}\Sigma_F \frac{\Lambda_r^{(2)\prime}\Lambda_r}{N} \right] F_t^{(r)} \right\|^2$$
$$\leq O_P(1) \max_r \left\| \frac{1}{T}\sum_s F_s^{(r)} F_s^{(r)\prime} \frac{s-r}{T} \frac{\Lambda_r^{(1)\prime}\Lambda_r}{N} \right\|^2 \frac{1}{T}\sum_t \left\| F_t^{(r)} \right\|^2$$
$$+ O_P(1) \max_r \left\| \left[\frac{1}{T}\sum_s F_s^{(r)} F_s^{(r)\prime} \left(\frac{s-r}{T}\right)^2 - \frac{h^2\kappa_2}{2}\Sigma_F \right] \frac{\Lambda_r^{(2)\prime}\Lambda_r}{N} \right\|^2 \frac{1}{T}\sum_t \left\| F_t^{(r)} \right\|^2 + O_P(h^6)$$
$$= O_P(h^2(Th/\ln T)^{-1} + (Th)^{-2} + h^6)$$

by Assumption A.4(ii) in SW and Lemma A.6(i). Similarly,

$$\max_r A_{5,3}(r) = \max_r \frac{1}{T}\sum_t \left\| \left[\frac{1}{TN}\sum_s B_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r - \bar{C}_1^{(r)}\Sigma_F \frac{\Lambda_r'\Lambda_r^{(1)}}{N}\kappa_2 h^2 \right] F_t^{(r)} \right\|^2$$
$$\leq \max_r \left\| \frac{1}{TN}\sum_s B_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r - \bar{C}_1^{(r)}\Sigma_F \frac{\Lambda_r'\Lambda_r^{(1)}}{N}\kappa_2 h^2 \right\|^2 \frac{1}{T}\sum_t \left\| F_t^{(r)} \right\|^2$$
$$= O_P(h^4(Th/\ln T)^{-1} + (Th)^{-2}).$$

Then $\max_r A_5(r) = O_P((T/h)^{-1}\ln T + N^{-1}h^2\ln T) + o_P(h^4) = o_P((Th)^{-1} + N\ln T + h^4)$. Next, $A_6(r) = \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} D(s,r)' D(t,r) F_t^{(r)} \right\|^2 \leq 3\sum_{l=1}^{3} \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \varphi_{l,sr} F_s^{(r)\prime} D(s,r)' D(t,r) F_t^{(r)} \right\|^2 \equiv 3\sum_{l=1}^{3} A_{6,l}(r)$. It is easy to show that $\max_r A_{6,3}(r) = O_P(h^6)$ and $\max_r A_{6,1}(r) \leq \max_r \left\{ \frac{1}{NT}\sum_t \left\| D(t,r) F_t^{(r)} \right\|^2 \right\}^2$

$\times \frac{1}{T}\sum_s \left\|\varphi_{1,sr}\right\|^2 = h^4[O_P((Th)^{-1} + N\ln T) + o_P(h^2)]$. For $A_{6,2}(r)$, we have

$$\begin{aligned}
\max_r A_{6,2}(r) &\leq O_P(1)\max_r \left\| \frac{1}{T}\sum_t \frac{1}{TN}\sum_s F_s^{(r)} F_s^{(r)\prime} D(s,r)^{\prime} D(t,r) F_t^{(r)} \right\|^2 \\
&= O_P(1)\max_r \left\| \frac{1}{T}\sum_s F_s^{(r)} F_s^{(r)\prime} \left(\frac{s-r}{T}\right) \frac{\Lambda_r^{(1)\prime}\Lambda_r^{(1)}}{N} \frac{1}{T}\sum_t F_t^{(r)} \left(\frac{t-r}{T}\right) \right\|^2 + O_P(h^6) \\
&\leq O_P(1)\max_r \left\| \frac{1}{T}\sum_s F_s^{(r)} F_s^{(r)\prime} \left(\frac{s-r}{T}\right) \right\|^2 \left\| \frac{\Lambda_r^{(1)\prime}\Lambda_r^{(1)}}{N} \right\|^2 \left\| \frac{1}{T}\sum_t F_t^{(r)} \left(\frac{t-r}{T}\right) \right\|^2 + O_P(h^6) \\
&= O_P\left((Th)^{-1}h^2\ln T + (Th)^{-2}\right) O_P\left((Th)^{-1}h^2\ln T + (Th)^{-2}\right) + O_P(h^6).
\end{aligned}$$

In sum, we have $\max_r A_6(r) = o_P((Th)^{-1} + N^{-1}\ln T + h^4)$.

For $A_7(r)$, it suffices to consider the rough probability bound

$$\begin{aligned}
\max_r A_7(r) &= \max_r \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} D(s,r)^{\prime} e_t^{(r)} \right\|^2 \\
&\leq \max_r \frac{1}{T}\sum_t \left\|\hat{F}_s^{(r)}\right\|^2 \max_r \frac{1}{T^2N^2}\sum_t\sum_s \left\| F_s^{(r)\prime} D(s,r)^{\prime} e_t^{(r)} \right\|^2 = O_P((N/\ln T)^{-1}h^2 + h^6)
\end{aligned}$$

where we use the fact that $\max_r \frac{1}{T^2N^2}\sum_{t,s} \left\| F_s^{(r)\prime} D(s,r)^{\prime} e_t^{(r)} \right\|^2 \leq \max_r \sum_{l=1}^2 \frac{1}{T^2} \left\| F_s^{(r)\prime}(\frac{s-r}{T})^l \frac{1}{lN}\sum_i \lambda_{ir}^{(l)} e_{it} \right\|^2$ $+ O_P\left(h^6\right) = O_P((N/\ln T)^{-1}h^2 + h^6)$. Similarly, $\max_r A_8(r) = O_P((N/\ln T)^{-1}h^2 + h^6)$. For $A_9(r)$, we have

$$\begin{aligned}
A_9(r) &= \frac{1}{T}\sum_t \left\| V_{NT}^{(r)} A_9(t,r) - V_{NT}^{(r)} B_{1t}^{(r)} \right\|^2 = \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} \Lambda_r^{\prime} D(t,r) F_t^{(r)} - V_{NT}^{(r)} B_{1t}^{(r)} \right\|^2 \\
&\leq 3\sum_{l=1}^3 \frac{1}{T}\sum_t \left\| \frac{1}{TN}\sum_s \varphi_{1,sr} F_s^{(r)\prime} \Lambda_r^{\prime} D(t,r) F_t - \chi_{1l,t}^{(r)} \right\|^2 \equiv \sum_{l=1}^3 A_{9,l}(r),
\end{aligned}$$

where $\chi_{11,t}^{(r)} = \chi_{13,t}^{(r)} = 0$ and $\chi_{12,t}^{(r)} = V_{NT}^{(r)} B_{1t}^{(r)}$. Following the analysis of $A_5(r)$, we can show that $\max_r A_{9,1}(r)$ $= O_P(((Th)^{-1} + N^{-1}\ln T)h^2) + o_P(h^4)$. It is easy to show that $\max_r A_{9,2}(r) = O_P\left((Th/\ln T)^{-1}h^2 + (Th)^{-2}\right)$ and $\max_r A_{9,1,3}(r) = O_P((Th/\ln T)^{-1}h^4 + h^6)$. Then $\max_r A_9(r) = o_P((Th)^{-1} + N^{-1}\ln T + h^4)$.

In sum, we have shown that $\frac{1}{T}\sum_t \left\| \hat{F}_t^{(r)} - H^{(r)\prime} F_t^{(r)} - B^{(r)} \right\|^2 = O_P\left(T^{-1}h^{-1} + N^{-1}\ln T\right) + o_P(h^4)$.

(iv) Following the proof of SW's Lemma A.7(iv) and the steps in the proof of Lemmas A.2(ii) and Lemma A.6(iii), we can show that $\max_r ||\frac{1}{T}(\hat{F}^{(r)} - F^{(r)}H^{(r)})^{\prime} F^{(r)} H^{(r)}|| = O_P\left(T^{-1}h^{-1} + N^{-1}\ln T\right) + o_P(h^4)$.

(v) By (A.1), we have $\frac{1}{T}(\hat{F}^{(r)} - F^{(r)}H^{(r)} - B^{(r)})^{\prime} e_i^{(r)} = V_{NT}^{-(r)} \sum_{l=1}^9 AE_l(i,r)$, where $AE_l(i,r) = \frac{1}{T}\sum_t V_{NT}^{(r)} A_l(t,r) e_{it}^{(r)}$. By SW (Lemma A.7(v)), $\max_{i,r} \left\| \sum_{l=1}^4 AE_l(i,r) \right\| = O_P\left(T^{-1}h^{-1} + N^{-1}\ln T\right)$. For the remaining terms, since $AE_7(i,r)$ and $AE_8(i,r)$ are higher order terms than $AE_3(i,r)$ and $AE_4(i,r)$, respectively, we only consider $AE_5(i,r)$, $AE_6(i,r)$ and $AE_9(i,r)$. For $AE_5(i,r)$, we have

$$\begin{aligned}
\|AE_5(i,r)\| &= \left\| \frac{1}{T}\sum_t \left[\frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} D(s,r)^{\prime} \Lambda_r F_t^{(r)} - V_{NT}^{(r)} B_{2t}^{(r)} \right] e_{it}^{(r)} \right\| \\
&\leq \sum_{l=1}^3 \left\| \frac{1}{T}\sum_t \left[\frac{1}{TN}\sum_s \varphi_{l,sr} F_s^{(r)\prime} D(s,r)^{\prime} \Lambda_r F_t^{(r)} - \chi_{2l,t}^{(r)} \right] e_{it}^{(r)} \right\| \equiv \sum_{l=1}^3 AE_{5,l}(i,r).
\end{aligned}$$

For the first two terms, we have

$$\max_{i,r} AE_{5,1}(i,r) \le \max_{i,r} \left\| \frac{1}{T}\sum_t F_t^{(r)} e_{it}^{(r)} \right\| \left\{ \frac{1}{T}\sum_s \|\varphi_{1,sr}\|^2 \right\}^{1/2} \left\{ \frac{1}{TN^2}\sum_s \left\| F_s^{(r)\prime} D(s,r)'\Lambda_r \right\|^2 \right\}^{1/2}$$
$$= O_P((Th/\ln T)^{-1/2})[O_P((Th)^{-1/2} + (N/\ln T)^{-1/2}) + o_P(h^2)]h = o_P((Th)^{-1} + N^{-1}\ln T + h^4)$$

and

$$\max_{i,r} AE_{5,2}(i,r) = \max_{i,r} \left\| \frac{1}{T}\sum_t H^{(r)\prime} \left[\frac{1}{TN}\sum_{s=1}^T F_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r - \frac{h^2\kappa_2}{2}\Sigma_F \frac{\Lambda_r^{(2)}\Lambda_r}{N} \right] F_t^{(r)} e_{it}^{(r)} \right\|$$
$$\le \max_{i,r} \left\| H^{(r)} \right\| \left\| \frac{1}{T}\sum_t F_t^{(r)} e_{it}^{(r)} \right\| \left\| \frac{1}{TN}\sum_s F_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r - \frac{h^2\kappa_2}{2}\Sigma_F \frac{\Lambda_r^{(2)}\Lambda_r}{N} \right\|$$
$$= O_P((Th/\ln T)^{-1/2}) O_P((Th/\ln T)^{-1/2}h + (Th)^{-1}) = o_P((Th)^{-1}).$$

For $AE_{5,3}(i,r)$, noting that $B_{2t,2}^{(r)} = \kappa_2 h^2 B_{22}^{(r)} F_t^{(r)} = \kappa_2 h^2 V_{NT}^{(r)-1} \bar{C}_1^{(r)} \Sigma_F (\Lambda_r' \Lambda_r^{(1)}/N) F_t^{(r)}$,

$$\max_{i,r} AE_{5,3}(i,r) = \max_{i,r} \left\| \frac{1}{T}\sum_t \left(\frac{1}{TN}\sum_s B_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r - \kappa_2 h^2 \bar{C}_1^{(r)} \Sigma_F \frac{\Lambda_r' \Lambda_r^{(1)}}{N} \right) F_t^{(r)} e_{it}^{(r)} \right\|$$
$$\le \max_{i,r} \left\| \frac{1}{T}\sum_t F_t^{(r)} e_{it}^{(r)} \right\| \max_{i,r} \left\| \frac{1}{TN}\sum_s B_s^{(r)} F_s^{(r)\prime} D(s,r)'\Lambda_r - \kappa_2 h^2 \bar{C}_1^{(r)} \Sigma_F \frac{\Lambda_r' \Lambda_r^{(1)}}{N} \right\|$$
$$= O_P((Th/\ln T)^{-1/2}) O_P((Th/\ln T)^{-1/2}h^2 + (Th)^{-1}) = o_P(T^{-1}h^{-1}).$$

Therefore, $\max_{i,r} \|AE_5(i,r)\| = o_P((Th)^{-1} + N^{-1}\ln T + h^4)$. For $AE_6(i,r)$, we have

$$\|AE_6(i,r)\| = \left\| \frac{1}{T}\sum_t \frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} D(s,r)' D(t,r) F_t^{(r)} e_{it}^{(r)} \right\|$$
$$\le \sum_{l=1}^3 \left\| \frac{1}{T}\sum_t \frac{1}{TN}\sum_s \varphi_{l,sr} F_s^{(r)\prime} D(s,r)' D(t,r) F_t^{(r)} e_{it}^{(r)} \right\| \equiv \sum_{l=1}^3 AE_{6,l}(i,r).$$

It is easy to show that

$$\max_{i,r} AE_{6,1}(i,r) \le \max_{i,r} \left\| \frac{1}{T\sqrt{N}}\sum_t D(t,r) F_t^{(r)} e_{it}^{(r)} \right\| \left\{ \frac{1}{T}\sum_s \|\varphi_{1,sr}\|^2 \right\}^{1/2} \left\{ \frac{1}{TN}\sum_s \left\| F_s^{(r)\prime} D(s,r)' \right\|^2 \right\}^{1/2}$$
$$= O_P((Th/\ln T)^{-1/2}h)[O_P((Th)^{-1/2} + (N/\ln T)^{1/2}) + o_P(h^2)]h = o_P((Th)^{-1} + N^{-1}\ln T + h^4),$$
$$\max_{i,r} AE_{6,2}(i,r \le \max_{i,r} \left\| H^{(r)} \right\| \left\| \frac{1}{TN^{1/2}}\sum_t D(t,r) F_t^{(r)} e_{it}^{(r)} \right\| \left\| \frac{1}{TN^{1/2}}\sum_s F_s^{(r)} F_s^{(r)\prime} D(s,r)' \right\|$$
$$= O_P((Th/\ln T)^{-1/2}h) O_P((Th/\ln T)^{-1/2}h) = o_P((Th)^{-1}),$$

and similarly $\max_{i,r} AE_{6,3}(i,r) = O_P((Th/\ln T)^{-1/2}h^2 + (Th)^{-1} + h^3) O_P((Th/\ln T)^{-1/2}h) = o_P((Th)^{-1} + h^4)$. Thus $\max_{i,r} \|AE_6(i,r)\| = o_P((Th)^{-1} + N^{-1}\ln T + h^4)$. Analogously, we can also show that

$$\max_{i,r} \|AE_9(i,r)\| = \max_{i,r} \left\| \frac{1}{T}\sum_t \frac{1}{TN}\sum_s \hat{F}_s^{(r)} F_s^{(r)\prime} \Lambda_r' D(t,r) F_t^{(r)} e_{it}^{(r)} - V_{NT}^{(r)} B_{1t}^{(r)} e_{it}^{(r)} \right\| = o_P((Th)^{-1} + N^{-1}\ln T + h^4).$$

Then $\max_{i,r}\left\|\frac{1}{T}(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'e_i^{(r)}\right\|=O_P\left(T^{-1}h^{-1}+N^{-1}\ln T\right)+o_P(h^4)$.

(vi) The proof is analogous to that of (v) and thus omitted.

(vii) Note that $\max_{i,r}\left\|\frac{1}{T}H^{(r)\prime}F^{(r)\prime}\Delta_i^{(r)}\right\|\leq\max_r\left\|H^{(r)}\right\|\Xi_1=O_P(1)\,\Xi_1$, where $\Xi_1=\max_{i,r}||\frac{1}{T}\sum_t F_t^{(r)}F_t^{(r)\prime}\times d_i(t,r)||$.

$$\begin{aligned}\Xi_1 &= \max_{i,r}\left\|\frac{1}{T}\sum_t k^*_{h,tr}F_tF_t'd_i(t,r)\right\| \\ &\leq \max_{i,r}\left\|\frac{1}{T}\sum_t k^*_{h,tr}\Sigma_F\left(\frac{t-r}{T}\right)\lambda_{ir}^{(1)}\right\|+\max_{i,r}\left\|\frac{1}{T}\sum_t k^*_{h,tr}\left(F_tF_t'-\Sigma_F\right)\left(\frac{t-r}{T}\right)\lambda_{ir}^{(1)}\right\|+O_P\left(h^2\right) \\ &= O((Th)^{-1})+O_P((Th/\ln T)^{-1/2}\,h)+O_P\left(h^2\right).\end{aligned}$$

Then $\max_{i,r}\left\|\frac{1}{T}H^{(r)\prime}F^{(r)\prime}\Delta_i^{(r)}\right\|=O_P((Th)^{-1}+(Th/\ln T)^{-1/2}\,h+h^2)$.

(viii) Note that $\frac{1}{T}B^{(r)\prime}e_i^{(r)}=\frac{1}{T}\sum_t B_t^{(r)}e_{it}^{(r)}\leq\frac{1}{T}\sum_t[\bar{C}_1^{(r)}\frac{t-r}{T}+\bar{C}_2^{(r)}\kappa_2h^2+\bar{C}_3^{(r)}(\frac{t-r}{T})^2]F_t^{(r)}e_{it}^{(r)}\equiv\sum_{l=1}^3\Xi_{2,l,ir}$. It is easy to show that $\max_{i,r}\Xi_{2,1,ir}\leq\max_r\left\|\bar{C}_1^{(r)}\right\|\max_{i,r}\left\|\frac{1}{T}\sum_t k^*_{h,tr}\frac{t-r}{T}F_te_{it}\right\|=O_P((Th/\ln T)^{-1/2}\,h)$ and similarly, $\max_{i,r}\Xi_{2,l,ir}=O_P((Th/\ln T)^{-1/2}\,h^2)$ for $l=2,3$. It follows that $\max_{i,r}\left\|\frac{1}{T}B^{(r)\prime}e_i^{(r)}\right\|=O_P((Th/\ln T)^{-1/2}\,h)$.

(ix) $\left\|\frac{1}{T}\hat{F}^{(r)\prime}B^{(r)}H^{(r)-1}\lambda_{ir}\right\|\leq\left\|\frac{1}{T}(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)})'B^{(r)}H^{(r)-1}\lambda_{ir}\right\|+\left\|\frac{1}{T}H^{(r)\prime}F^{(r)\prime}B^{(r)}H^{(r)-1}\lambda_{ir}\right\|+\left\|\frac{1}{T}B^{(r)\prime}B^{(r)}H^{(r)-1}\lambda_{ir}\right\|\equiv\sum_{l=1}^3\Xi_{3,l,ir}$. Following the proof of (iv), we can also show that

$$\max_{i,r}\Xi_{3,1,ir}\leq O_p(1)\max_{i,r}\left\|\frac{1}{T}\left(\hat{F}^{(r)}-F^{(r)}H^{(r)}-B^{(r)}\right)'B^{(r)}\right\|=O_P([(Th)^{-1}+N^{-1}\ln T]h)+o_P(h^4).$$

As in the proof of (vii), we can readily show that $\max_{i,r}\Xi_{3,2,ir}=O_P((Th)^{-1}+(Th'\ln T)^{-1/2}\,h+h^2)$ and $\max_{i,r}\Xi_{3,3,ir}=\max_{i,r}\Xi_{3,2,ir}=O_P(h^2)$. Then $\max_{i,r}\left\|\frac{1}{T}\hat{F}^{(r)\prime}B^{(r)}H^{(r)-1}\lambda_{ir}\right\|=O_P((Th)^{-1}+(Th/\ln T)^{-1/2}\,h+h^2+N^{-1}h\ln T)$.

(x) Noting that $B_t^{(r)}=[\bar{C}_1^{(r)}\frac{t-r}{T}+\bar{C}_2^{(r)}\kappa_2h^2+\bar{C}_3^{(r)}(\frac{t-r}{T})^2]F_t^{(r)}$, it is trivial to show that $\max_{i,r}\left\|\frac{1}{T}B^{(r)\prime}\Delta_i^{(r)}\right\|=\max_{i,r}\left\|\frac{1}{T}\sum_t B_t^{(r)}F_t^{(r)\prime}d_i(t,r)\right\|=O_P\left(h^2\right)$.

(xi) The proof is analogous to that of Lemmas A.5(i) in this paper and the Lemma A.6 (vi) in SW, and thus omitted here.

(xii) By (A.2), we have $\frac{1}{N}\left\|\hat{\Lambda}_r-\Lambda_rH^{(r)-1\prime}\right\|^2=\frac{1}{N}\sum_i\left\|\hat{\lambda}_{ir}-H^{(r)-1}\lambda_{ir}\right\|^2\leq 8\sum_{l=1}^8\frac{1}{N}\sum_i\|D_l(i,r)\|^2$. Following the proof of Lemma A.4 and using the previous results in this lemma, we can show that $\frac{1}{N}\left\|\hat{\Lambda}_r-\Lambda_rH^{(r)-1\prime}\right\|^2=O_P(C_{NT}^{-2}\ln T)$.

(xiii)-(xiv) The proof follows from that of Lemma A.7(viii)-(ix) with obvious modifications and thus omitted here. ■

Proof of Lemma B.1. (i)-(iii) The proof is essentially the same as that of Lemma B.1(i)-(iii) in SW and hence omitted here.

(iv) By (A.2) and the CS inequality, $\frac{1}{T}\sum_t\left\|\frac{1}{N}\sum_i(\hat{\lambda}_{it}-H^{(t)-1}\lambda_{it})e_{it}\right\|^2\leq 8\sum_{l=1}^8 D_l$, where $D_l=\frac{1}{T}\sum_t\left\|\frac{1}{N}\sum_i D_l(i,t)\,e_{it}\right\|^2$. By Lemma B.1(iv) in SW, $\sum_{l=1}^3 D_l=O_P\left(T^{-2}h^{-2}+N^{-2}\ln T\right)$. For D_4, we have by the CS inequality and Lemma A.6(iii)

$$D_4=\frac{1}{T}\sum_t\left\|\frac{1}{T}\sum_s\varphi_{1,st}\frac{1}{N}\sum_i F_s^{(t)\prime}d_i(s,t)e_{it}\right\|^2\leq\max_r\frac{1}{T}\sum_s\left\|\varphi_{1,st}\right\|^2\bar{D}_4=[O_P\left((Th)^{-1}+N^{-1}\ln T\right)+o_P(h^4)]\bar{D}_4,$$

where $\bar{D}_4 = \frac{1}{T^2}\sum_{t,s}\left\|\frac{1}{N}\sum_i F_s^{(t)\prime} d_i(s,t)e_{it}\right\|^2$. Let $\bar{D}_{4,l} = \frac{1}{T^2}\sum_{t,s}\left\|F_s^{(t)\prime}(\frac{s-t}{T})^l \frac{1}{lN}\sum_i \lambda_{it}^{(l)} e_{it}\right\|^2$ for $l = 1,2$. Note that

$$\bar{D}_4 = \frac{1}{T^2}\sum_{t,s}\left\|\frac{1}{N}\sum_i F_s^{(t)\prime} d_i(s,t)e_{it}\right\|^2 \leq 3(\bar{D}_{4,1}+\bar{D}_{4,2}) + O_P\left(a_{NT}^2 h^6\right) = O_P(a_{NT}^2 N^{-1}h^2 + a_{NT}^2 h^6),$$

where we use the fact that $\bar{D}_{4,l} \leq \frac{1}{T}\sum_t \left\|\frac{1}{lN}\sum_i \lambda_{it}^{(l)} e_{it}\right\|^2 \max_t \frac{1}{T}\sum_s \left\|F_s^{(t)\prime}(\frac{s-t}{T})^l\right\|^2 = O_P(a_{NT}^2 N^{-1} h^{2l})$ for $l = 1,2$. It follows that $D_4 = [O_P\left((Th)^{-1} + N^{-1}\ln T\right) + o_P(h^4)]O_P(a_{NT}^2 N^{-1}h^2 + a_{NT}^2 h^6) = o_P(T^{-2}h^{-2} + N^{-2}\ln T)$. For D_5, we have

$$\begin{aligned} D_5 &= \frac{1}{T}\sum_t \left\|\frac{1}{N}\sum_i D_5(i,t)\, e_{it}\right\|^2 = \frac{1}{T}\sum_t \left\|\frac{1}{N}\sum_i \frac{1}{T}\sum_s H^{(t)} F_s^{(t)} F_s^{(t)\prime} d_i(s,t) e_{it}\right\|^2 \\ &\leq 3\sum_{l=1}^{2}\frac{1}{T}\sum_t \left\|H^{(t)}\frac{1}{T}\sum_s F_s^{(t)}F_s^{(t)\prime}\left(\frac{s-t}{T}\right)^l \frac{1}{lN}\sum_i \lambda_{it}^{(l)} e_{it}\right\|^2 + O_P(a_{NT}^2 h^6) \equiv 3\sum_{l=1}^{2} D_{5,l} + O_P(a_{NT}^2 h^6). \end{aligned}$$

Noting that

$$\begin{aligned} D_{5,1} &\leq \max_r \left\|\frac{1}{N}\sum_i \lambda_{ir}^{(1)} e_{ir}\right\|^2 \left\|H^{(r)}\right\|^2 \frac{1}{T}\sum_t \left\{\left\|\frac{1}{T}\sum_s k_{h,st}^*\left(F_sF_s' - \Sigma_F\right)\frac{s-t}{T}\right\|^2 + \left\|\frac{1}{T}\sum_s k_{h,st}^*\frac{s-t}{T}\Sigma_F\right\|^2\right\} \\ &= O_P(a_{NT}^2 (N/\ln N)^{-1}) O_P((Th)^{-1}h + (Th)^{-2}) \end{aligned}$$

and similarly $D_{5,2} = O_P(a_{NT}^2(N/\ln N)^{-1})O_P\left(h^4\right)$, we have $D_5 = O_P\{a_{NT}^2(N/\ln N)^{-1}[(Th)^{-1}h + (Th)^{-2}] + a_{NT}^2(N/\ln N)^{-1})h^4 + a_{NT}^2 h^6\} = o_P\left(T^{-2}h^{-2} + N^{-2}\ln T\right)$. For D_6, we have

$$D_6 = \frac{1}{T}\sum_t \left\|\frac{1}{N}\sum_i \frac{1}{T}\sum_s B_s^{(t)} e_{is}^{(t)} e_{it}\right\|^2 \leq 3\sum_{l=1}^{3}\frac{1}{T}\sum_t \left\|\frac{1}{NT}\sum_i\sum_s k_{h,ts}^* c_{l,st} F_s e_{is} e_{it}\right\|^2 \equiv \sum_{l=1}^{3} D_{6,l}.$$

where $c_{1,st} = \bar{C}_1^{(t)}(\frac{s-t}{T})$, $c_{2,st} = \bar{C}_2^{(t)}\kappa_2 h^2$, and $c_{3,st} = \bar{C}_3^{(t)}(\frac{s-t}{T})^2$. Note that

$$\begin{aligned} D_{6,1} &\leq \max_r \left\|\bar{C}_1^{(r)}\right\|^2 \frac{1}{T}\sum_t \left\|\frac{1}{T}\sum_s k_{h,ts}^* \frac{t-s}{T}\frac{F_s e_s' e_t}{N}\right\|^2 \\ &\leq \max_r \left\|\bar{C}_1^{(r)}\right\|^2 \left\{\frac{1}{T}\sum_t \left\|\frac{1}{T}\sum_s k_{h,ts}^*\frac{t-s}{T}\gamma_{N,F}(s,t)\right\|^2 + \frac{1}{T}\sum_t \left\|\frac{1}{T}\sum_s k_{h,ts}^*\frac{t-s}{T}\bar{\gamma}_{N,F}(s,t)\right\|^2\right\} \\ &= O_P\left(a_{NT}^2\right) O_P(T^{-2} + (NT/h)^{-1}h^2) \end{aligned}$$

where $\bar{\gamma}_{N,F}(s,t) = \frac{F_s e_s' e_t}{N} - \gamma_{N,F}(s,t)$ and $\gamma_{N,F}(s,t) = \frac{1}{N}E\left[F_s e_s' e_t\right]$. Similarly $D_{6,l} = O_P\left(a_{NT}^2\right)O_P(T^{-2} + (NTh)^{-1}h^4)$ for $l = 2,3$. Then have $D_6 = O_P(a_{NT}^2(T^{-2} + (NTh)^{-1})) = o_P\left(T^{-2}h^{-2} + N^{-2}\ln T\right)$. For D_7, we have

$$D_7 = \frac{1}{T}\sum_t \left\|\frac{1}{NT}\sum_i\sum_s \hat{F}_s^{(t)} B_s^{(t)} H^{(t)-1}\lambda_{it} e_{it}\right\|^2 \leq 3\sum_{l=1}^{3}\frac{1}{T}\sum_t \left\|\frac{1}{T}\sum_s \varphi_{l,ts} B_s^{(t)\prime} H^{(t)-1}\frac{1}{N}\sum_i \lambda_{it} e_{it}\right\|^2 \equiv 3\sum_{l=1}^{3} D_{7,l},$$

where recall that $\varphi_{1,st}=\hat{F}_s^{(t)}-H^{(t)}F_s^{(t)}-B_s^{(t)}$, $\varphi_{2,st}=H^{(t)}F_s^{(t)}$, and $\varphi_{3,st}=B_s^{(t)}$. We can show

$$\begin{aligned}
D_{7,1} &\leq \frac{1}{T}\sum_t \left\| \frac{1}{T}\sum_s \varphi_{1,st}B_s^{(t)\prime}H^{(t)-1}\right\|^2 \max_t \left\| \frac{1}{N}\sum_i \lambda_{it}e_{it}\right\|^2 \\
&= h^2\left[O_P\left((Th)^{-1}+N^{-1}\ln T\right)+o_P\left(h^4\right)\right]O_P\left(N^{-1}\ln N\right)=o_P\left(T^{-2}h^{-2}+N^{-2}\ln T\right), \\
D_{7,2} &= \frac{1}{T}\sum_t \left\| \frac{1}{T}\sum_s H^{(t)}F_s^{(t)}B_s^{(t)\prime}H^{(t)-1}\frac{1}{N}\sum_i \lambda_{it}e_{it}\right\|^2 \\
&\leq O_P(1)\sum_{l=1}^3 \frac{1}{T}\sum_t \left\| \frac{1}{T}\sum_s F_s^{(t)}F_s^{(t)\prime}c_{l,ts}^{\prime}\right\|^2 \max_t \left\| \frac{1}{N}\sum_i \lambda_{it}e_{it}\right\|^2 \\
&= O_P\left((Th)^{-1}h^2+(Th)^{-2}+h^4\right)O_P\left(N^{-1}\ln N\right)=o_P(T^{-2}h^{-2}+N^{-2}\ln T),
\end{aligned}$$

and similarly, $D_{7,3}=\frac{1}{T}\sum_t\left\|\frac{1}{T}\sum_s B_s^{(t)}B_s^{(t)\prime}H^{(t)-1}\frac{1}{N}\sum_i\lambda_{it}e_{it}\right\|^2=O_P\left(a_{NT}^4h^4N^{-1}\ln N\right)=o_P(T^{-2}h^{-2}$ $+N^{-2}\ln T)$. It follows that $D_7=o_P(T^{-2}h^{-2}+N^{-2}\ln T)$. For D_8, it is easy to obtain a rough probability bound:

$$D_8\leq \sum_{l=1}^3\frac{1}{T}\sum_t\left\|\frac{1}{T}\sum_s c_{l,st}F_s^{(t)}F_s^{(t)\prime}\frac{1}{N}\sum_i d_i(s,t)e_{it}\right\|^2=O_P\left(a_{NT}^2(h^4N^{-1}\ln N+h^8)\right)=o_P\left(T^{-2}h^{-2}+N^{-2}\ln T\right).$$

Combining the above results concludes the proof of (iv).

(v) The proof is essentially the same as that of Lemma B.1 in SW and thus omitted. ■

Proof of Lemma B.2. (i)-(vii) are only involved with the conventional PCA estimate so the proof follows from that of Lemma B.2(i)-(vii) in SW. The proof of (viii) follows from that of Lemma B.2(viii) in SW with some modifications. ■

Proof of Lemma B.3. (i) Noting that $R_\Lambda(i,t)=\sum_{l=2}^8 D_l(i,t)$ by (A.2), we have $\frac{1}{NT}\sum_i\sum_t\|R_\Lambda(i,t)\|^2\leq 7\sum_{l=2}^8 R_l$, where $R_l=\frac{1}{NT}\sum_i\sum_t\|D_l(i,t)\|^2$. SW (Lemma B.3) show that $\sum_{l=2}^3 R_l=O_P\left((Th)^{-2}+(N/\ln T)^{-2})\right)$. Now, we consider the remaining terms. For R_4, we have

$$\begin{aligned}
R_4 &= \frac{1}{NT}\sum_i\sum_t\left\|\frac{1}{T}\sum_s\varphi_{1,st}F_s^{(t)}d_i(s,t)\right\|^2\leq \max_t\frac{1}{T}\sum_s\|\varphi_{1,st}\|^2\max_{i,t}\frac{1}{T}\sum_s\left\|F_s^{(t)}d_i(s,t)\right\|^2 \\
&= O_P(T^{-1}h^{-1}+N^{-1}\ln T+h^4)O_P(h^2a_{NT}^2)=o_P(T^{-2}h^{-2}+(N/\ln T)^{-2}).
\end{aligned}$$

For R_5, we have

$$\begin{aligned}
R_5 &= \frac{1}{NT}\sum_i\sum_t\left\|\frac{1}{T}H^{(r)\prime}\sum_s k_{h,st}^*F_sF_s^{\prime}d_i(s,t)\right\|^2 \\
&\leq \frac{2}{NT}\sum_i\sum_t\left\|\frac{1}{T}H^{(r)\prime}\sum_s k_{h,st}^*F_sF_s^{\prime}\frac{s-t}{T}\lambda_{it}^{(1)}\right\|^2+O_P(h^4a_{NT}^2) \\
&= O_P(a_{NT}^2[(Th)^{-1}h^2+(Th)^{-2}])+O_P(h^4a_{NT}^2)=o_P(T^{-2}h^{-2}).
\end{aligned}$$

Next, $R_6=\frac{1}{NT}\sum_i\sum_t\left\|\frac{1}{T}\sum_s B_s^{(t)}e_{is}^{(t)}\right\|^2\leq 3\sum_{l=1}^3\frac{1}{NT}\sum_i\sum_t\left\|\frac{1}{T}\sum_s k_{h,st}^*c_{l,st}F_se_{is}\right\|^2\equiv 3\sum_{l=1}^3 R_{6,l}$. Note that

$$R_{6,1}\leq \max_r\left\|\bar{C}_1^{(r)}\right\|^2\frac{1}{NT}\sum_i\sum_t\left\|\frac{1}{T}\sum_s k_{h,st}^*\frac{s-t}{T}F_se_{is}\right\|^2=O_P(a_{NT}^2)O_P((Th)^{-1}h^2)=o_P(T^{-2}h^{-2})$$

and similarly $R_{6,l} = O_P(a_{NT}^2)O_P((Th)^{-1}h^4)$ for $l = 2, 3$. Thus $R_6 = o_P(T^{-2}h^{-2})$. For R_7, we have

$$R_7 = \frac{1}{NT}\sum_i\sum_t \left\| \frac{1}{T}\sum_s \hat{F}_s^{(t)} B_s^{(t)\prime} H^{(r)-1}\lambda_{ir} \right\|^2 \leq 3\sum_{l=1}^{3} \frac{3}{NT}\sum_i\sum_t \left\| \frac{1}{T}\sum_s \varphi_{l,ts} B_s^{(t)\prime} H^{(r)-1}\lambda_{ir} \right\|^2 \equiv 3\sum_{l=1}^{3} R_{7,l}.$$

Following the analysis of R_4 and R_5, we can show that $R_{7,1} = o_P(T^{-2}h^{-2} + N^{-2}(\ln T)^2)$ and $R_{7,2} = o_P(T^{-2}h^{-2})$. In addition, $R_{7,3} = O_P(h^4 a_{NT}^4) = o_P(T^{-2}h^{-2})$. Then $R_7 = o_P(T^{-2}h^{-2} + N^{-2}(\ln T)^2)$. Lastly, it is easy to show that

$$R_8 = \frac{1}{NT}\sum_i\sum_t \left\| \frac{1}{T}\sum_s B_s^{(t)} F_s^{(t)\prime} d_i(s,t) \right\|^2 = O_P(h^4 a_{NT}^2) = o_P(T^{-2}h^{-2}).$$

Combining the above results yields $\frac{1}{NT}\sum_i\sum_t \|R_\Lambda(i,t)\|^2 = O_P\left(T^{-2}h^{-2} + N^{-2}(\ln T)^2\right)$.

(ii) The proof is essentially the same as that of Lemma B.3(ii) in SW. This is due to the fact that $B_F(t)$ is $O_P(a_{NT}h^2)$, which is $o_P(\min(T^{-1}h^{-1}, N^{-1}))$.

(iii)-(iv) This follows Lemma B.3(iii)-(iv) in SW as only the conventional PCA is involved. ■