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Are tournaments optimal over piece rates under limited liability for the principal?

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ABSTRACT

A highly acclaimed result is that tournaments are superior to piece rates when the agents are risk averse and their production activities are subject to a relatively large common shock. The reason is that tournaments allow the principal to trade insurance for lower income to the agents. Our analysis shows that this celebrated result does not carry over to the case when a limited liability (bankruptcy) constraint limits the payments the principal can make, provided that the liquidation value of the firm is sufficiently small. This finding has important implications for the vast number of limited liability firms. Even though limited liability becomes an issue for different ranges of liquidation values under the two schemes, tournaments are still superior when the liquidation value of the firm is intermediate or large, even though the limited liability constraint is still binding for intermediate values. Surprisingly, uncertainty in the price of output strengthens the need for tournaments by expanding the range of liquidation values over which tournaments are dominant, because price uncertainty introduces additional bankruptcy risk. These findings provide insight into policy implications in the contracting out of services by state and local governments, in procurement, in rent-seeking contests and in tournaments used by HMOs.

1. Introduction

Following the seminal work of Lazear and Rosen (1981), Holmström (1982), Green and Stokey (1983) and Nalebuff and Stiglitz (1983),[2] a significant part of the current literature on relative performance evaluation has focused on two-part piece rate (cardinal) tournaments that include a base payment and a bonus or penalty based on an agent's performance relative to the group average, and contrasted these schemes with standard linear piece rates that include a base payment and a variable payment based on the agent's absolute performance.[3] The latter are sometimes expressed as "fixed performance standards" when an agent's performance is evaluated against a fixed standard instead of the average output obtained. Note that Lazear and Rosen focused on rank-order (ordinal) tournaments,

however, these tournaments are informationally wasteful by ignoring the agents' cardinal performance (see Holmström (1982)).[4] Cardinal tournaments are popular in several occupations or industries where cardinal data are available (e.g., contracts for salesmen, contracts for physicians contracting with HMOs, agricultural contracts, promotion tournaments and annual salary raises for faculty), partly because they are simple to design and easy to implement and enforce. To some extent, the non-linearity of the theoretically optimal contract is due to the fact that contracts accommodate all possible events. Holmström and Milgrom (1987), however, have argued that schemes that adjust compensation to account for rare events may not provide correct incentives in ordinary high probability circumstances. For the most part, this literature has overlooked the implications of limited liability for the firm (principal), an issue of importance for the vast number of limited liability firms. The focus of this paper is the implications of limited liability in contrasting tournaments (relative performance evaluation) to piece rates (absolute performance evaluation).

Absent limited liability for the principal, tournaments constitute a move closer to the First Best. This is because relative performance evaluation partially alleviates the agents' moral hazard problem by providing information about the value of common shocks. The principal filters away common shocks from the responsibility of agents and charges a premium for this insurance. The move from absolute

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[2] An associated literature is the one on contests, often based on contest success functions. See Tullock (1980) for rent-seeking contests, Pérez-Castrillo and Verdier (1992) for a more recent discussion of rent-seeking games, Skaperdas (1996) for an axiomatization of contest success functions, Wärneryd (2003) for uncertain prizes, Konrad and Kovenock (2009) for contests with multiple rounds, and Riis (2010) for an extension of the Lazear and Rosen efficiency result to heterogeneous agents.

[3] See Knoeber (1989), Knoeber and Thurman (1994, 1995), Tsoulouhas (1999), Tsoulouhas and Vukina (1999, 2001), Levy and Vukina (2004), Wu and Roe (2005, 2006), Tsoulouhas and Marinakis (2007), Vandegrift et al. (2007) and Marinakis and Tsoulouhas (2012).

[4] Moreover, Tsoulouhas (2012) shows that switching from ordinal to cardinal tournaments improves efficiency.

performance schemes to tournaments is Pareto improving because the principal's expected profit increases without hurting the agent.

The dominance of tournaments is less clear when the principal is subject to limited liability, and bankruptcy is an issue because the firm's liquidation value is small or because it is possible for the output state to be quite unfavorable. When switching from absolute performance schemes, such as piece rates, to tournaments, the risk premium the principal charges for insurance against common shocks reduces the base payment the agent receives. Further, the filtering of common uncertainty enables the principal to implement a higher-power incentive scheme. However, because higher effort by the agent reduces his utility, the base payment will need to adjust to ensure the participation of the agent. Thus, under a tournament, the agent will receive better insurance but he will have to exert more effort, and even though the bonus factor under tournament should increase, the direction in the adjustment of the base payment is not clear a priori. Because the total wage bill under tournament is related to the base payment, the direction of the change in the total wage bill under tournament is also ambiguous a priori.[5] Because it is not clear if total payments go up or down when moving from piece rate schemes to tournaments, tournaments may or may not be better than piece rates under limited liability. Section 2 develops the model we will use to investigate this question.

Our analysis of piece rate schemes in Section 3 and of tournaments in Section 4 shows that absent limited liability the base payment and, hence, the total wage bill increases under tournament (note that Section 5 briefly examines how the contractual parameters adjust to changes in the model parameters). This is so because the expected bonus payment in a tournament is zero, whereas with piece rate compensation it is positive. Therefore, agents expect to be compensated for effort through the base payment in a tournament.[6] Thus, in the presence of a limited liability constraint which limits payments in unfavorable states, tournaments may not be dominant over piece rates. The intuition is that contracts with risk neutrality and limited liability for the principal look very much like those that would have been obtained with risk aversion. In other words, if the principal is concerned about the allocation of profit across states, he may no longer offer insurance against common shocks via tournaments and may resort to piece rate schemes or fixed performance standards. One might expect that limited liability for the principal would reduce the cost of bankruptcy to him and, hence, it would distort his incentives by encouraging risk-taking, thereby increasing the risk that bankruptcy arises. A Coasian criticism of this view is that contracts would adapt accordingly. It is true that limited liability makes bankruptcy tempting. It is also true that it encourages the principal to promise high payments in low states that he cannot really deliver, because he can only deliver the assets available to the firm in loss states. Following the footsteps of Sappington (1983), Farmer (1985) and Kahn and Scheinkman (1985), the approach taken in this paper is that rational agents (workers or input suppliers), knowing that limited liability prevents them from pursuing the assets of the principal, cannot be suckered by the prospect of payments that the principal clearly cannot deliver. Thus, they will sign a contract only if it stipulates that losses cannot exceed the firm's liquidation value. In this sense, the principal cares about the allocation of profit across states in order to satisfy the bankruptcy constraint required by the workers.

Thus, our analysis provides answers to the following questions: How should the piece rate and tournament schemes be modified to ensure that the firm will not be better off pleading bankruptcy? Is it possible to implement these schemes under a bankruptcy constraint? When both schemes can be implemented, which is better and why?

Our comparison of piece rate schemes to tournaments in Section 6 shows that, surprisingly, when both schemes can be implemented, the liquidation value of the firm must be sufficiently small for tournaments to be inferior. In particular, we show that there exists a critical liquidation value above which tournaments are still superior even if the limited liability constraint is binding under tournaments. However, below the critical liquidation value, piece rate schemes are always superior. In a sense the limited liability constraint must be really tight for tournaments to be inferior. To the best of our knowledge, this result has never been obtained in the literature. Our finding is analogous to showing that if the principal were sufficiently risk-averse he would be unable to offer insurance against common shocks by using tournaments, and he would resort to piece rate schemes. By contrast, if the principal were less risk-averse than the agent, he would still provide insurance through tournaments.

At this point, a word of caution is in order. Given the form of the two compensation schemes the way they are used in practice, and the adjustment of the contractual parameters after adding the limited liability constraint, the range of liquidation values over which the constraint becomes binding differs for each scheme. The intuition is that the expected bonus under tournaments is zero, whereas that under piece rates is positive. Thus, the base payment is positive for tournaments and negative for piece rates, yielding no range of liquidation values over which the limited liability constraint is binding under both schemes. Moreover, given that by construction each contract reacts in an entirely different way to the addition of the constraint, to be precise, the piece rate under limited liability provides higher power incentives, while the tournament provides more insurance, our goal is to contrast the two schemes via the overall effects of limited liability. That is, we derive the reduction of expected profit for both contracts in the presence of limited liability but, more importantly, we aim at analyzing the range of liquidation values over which each contract is affected by the limited liability constraint. In our analysis, a scheme may be characterized as "superior" because it leads to higher profits or because, due to its structure, it is less vulnerable to limited liability, that is, limited liability is binding for a smaller range of liquidation values or for smaller liquidation values than in the other scheme. In all, we characterize the superior scheme at each liquidation value and the scheme which is superior because it is less vulnerable to limited liability. In this sense, the piece rate scheme is less vulnerable to limited liability (bankruptcy) because the bankruptcy constraint is only binding for negative liquidation values and because it is completely immune to bankruptcy when both schemes are implementable. Tournaments are not even implementable for negative liquidation values (meaning that the principal would not find it profitable to make an offer that the agent would accept).

The analysis also shows in Section 6.3 that the superiority of tournaments over piece rate schemes critically depends on the agent's risk aversion rate, as well as on the variance of common uncertainty. The more risk-averse the agent is or the higher the magnitude of common uncertainty, the more the agent is willing to pay for insurance or the more the principal can charge for insurance, which raises the principal's profit. As a result, the range of liquidation values over which tournaments are dominant increases with the agents' risk aversion and the magnitude of common uncertainty. The number of agents has a similar effect, in that a large number of agents is necessary to eliminate idiosyncratic noise from the average output obtained by the agents. Hence, more insurance is provided against common shocks when the number of agents is large.

In the main analysis, we take the number of agents to be fixed. Section 6.4 extends the analysis by letting the number of agents vary to make the point that the principal can affect the superiority of tournaments over piece rates, and therefore his profit, by adjusting the number of agents. In particular, even if the liquidation value of the firm is large, the principal may find it profitable to increase the number

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[5] Specifically, the analysis below shows that the total wage bill is the number of agents multiplied by the base payment.

[6] Under piece rate, the bonus also compensates the agents for their effort costs. In fact, the expected bonus exceeds the cost of effort and the base payment is negative.

of agents and maximize profit by using a piece rate contract. By contrast, if the liquidation value of the firm is small, the principal may find it profitable to decrease the number of agents and maximize profit by using a tournament. Interestingly, our analysis shows that, if the number of agents is not limited by exogenous or organizational factors, a tournament will never be superior because the principal will find it profitable to keep increasing the number of agents until piece rate contracts become superior.

In the main analysis, we also assume that the price of output is known ex ante. In Appendix A, we incorporate price uncertainty. With price uncertainty the principal should be even more concerned about the allocation of profit across states, hence, one would expect that tournaments would be less likely to be superior. Our analysis shows that, surprisingly, if the lowest possible price exceeds the bonus factor, the form of the dominant scheme is completely unaffected by the presence of price uncertainty. By contrast, if the lowest possible price is smaller than the bonus factor, the increased bankruptcy risk in fact strengthens the need for tournaments by expanding the range of liquidation values over which tournaments are dominant. We trace this surprising result to the potential tension, from the principal's perspective, between providing insurance to the agent against common shocks and insuring himself against the variability in the total wage bill. In the presence of significant price uncertainty, the principal prefers to offer a tournament in order to eliminate the variability of the total wage bill and partially insure himself, even though the limited liability constraint is tighter under significant price uncertainty. To the best of our knowledge, this result as well has never been obtained in the literature.

Our analysis is related to several papers, some theoretical and some empirical. Lazear and Rosen (1981), Holmström (1982), Green and Stokey (1983), Nalebuff and Stiglitz (1983), Knoeber (1989) and Knoeber and Thurman (1994, 1995) provide the groundwork for comparing relative to absolute performance evaluation schemes. Without limited liability, Tsoulouhas (1999) analyzed tournaments and fixed performance standards when there is a two-sided moral hazard problem between the principal and the agents, to make the point that using tournaments to monitor the agents relaxes the principal's moral hazard problem when he takes a single action. One early attempt for analyzing the implications of limited liability for the principal was made by Tsoulouhas and Vukina (1999) who focused on explaining why tournaments are used in certain industries (e.g., the broiler industry) and not in other similar industries (e.g., the swine industry). It is worth noting that in their empirical analysis Tsoulouhas and Vukina assumed that price volatility should discourage the use of tournaments. Tsoulouhas and Vukina (2001) theoretically examined the welfare effects of tournaments versus fixed performance standards, when tournaments insulate agents from common shocks but they expose them to "group composition risk" emanating from the imperfect knowledge of the abilities of the agents they are competing against. Their analysis indicated that fixed performance standards decrease agent insurance without raising welfare, unless the magnitude of the piece rate is regulated, in which case fixed performance standards can increase both income insurance and welfare. In a related article, Levy and Vukina (2004) empirically investigated this issue. Their "league composition effect" takes the place of the "group composition risk" in Tsoulouhas and Vukina. In another related article, Wu and Roe (2005) contrasted the efficiency and welfare effects of tournaments versus piece rates. Their experiments indicated that agent welfare increases under fixed performance standards, unless the variance of common uncertainty crosses a threshold. Paradoxically, however, agents exert more effort under fixed performance standards than under tournaments, and effort under tournaments decreases with the variance of common shock. Wu and Roe (2006) showed that inequity-averse agents may perceive tournaments to be less fair and more risky than piece rates. Tsoulouhas and Marinakis (2007) showed that ex post agent heterogeneity compromises the insurance function

of tournaments, in particular, tournaments become less desirable when the variance of the distribution of ability types is large, so that absolute performance piece rates should be preferred when agents are very heterogeneous. Vandegrift et al. (2007) compared the efficiency properties of tournaments versus piece rates in an experimental study. They found that the "winner-take-all" tournament induces more effort than the "graduated" tournament in which the runner-ups also receive a reward, but the graduated tournament is better at sorting the most talented performer. Lastly, Marinakis and Tsoulouhas (2012) contrasted tournaments to piece rates when agents, instead of the principal, are subject to limited liability. Their analysis indicated that tournaments are still superior when agents are liquidity constrained. The rationale is that, by providing insurance against common shocks through a tournament, payments to the agents in unfavorable states increase and payments in favorable states decrease which enables the principal to satisfy tight liquidity constraints for the agents without paying any ex ante rents to them, while simultaneously providing higher-power incentives than under piece rates.

Our contribution to this literature is that the liquidation value of the firm is a far more definitive factor than price uncertainty in the determination of the contract the principal should offer. Thus, regardless of price uncertainty, if the liquidation value is really small the principal will prefer to offer a piece rate scheme, and if the liquidation value is really large the principal will prefer to offer a tournament. Note, however, that we do not characterize the overall optimal contract or tournament.[7] We rather focus on comparing relative performance evaluation schemes, such as tournaments, to absolute performance evaluation schemes, such as piece rates, the way they are used in practice at least in the case of contracts for salesmen, contracts for physicians contracting with HMOs, agricultural contracts and annual salary raises for faculty.

Section 7 provides our concluding remarks. Appendix B provides the proofs of our comparative statics results which are discussed in Section 6.3.

2. Model

A risk-neutral principal signs a contract with n homogeneous agents.[8] Each agent i produces output according to the production function $x_i = a + e_i + \eta + \varepsilon_i$, where a is the agent's known ability, e_i is the agent's effort, η is a common shock inflicted on all agents and ε_i is an idiosyncratic shock. Both shocks follow independent normal distributions with zero means and finite variances $var(\eta) = \sigma_\eta^2$ and $var(\varepsilon_i) = \sigma_\varepsilon^2$, $\forall i$. Each agent's effort and the subsequent realizations of the production shocks are private information to him, but the output obtained is publicly observed.[9] In the baseline model the price of output is normalized to 1 so that the output produced by the agents is revenue to the principal. The principal compensates agents for their effort based on their outputs by using a piece rate scheme or a tournament. Agent preferences are represented by a CARA utility function

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[7] Tsoulouhas (2010a) has shown that "hybrid" cardinal tournaments of the form $w_i = b + \beta x_i - \gamma \bar{x}$ are dominant over standard tournaments of the form $w_i = b + \beta(x_i - \bar{x})$, however, the former are not used in practice and they are more complicated to calculate. Further, in the limit, that is, for a sufficiently large number of workers, $\beta = \gamma$.

[8] Agent heterogeneity has been examined in a number of recent papers. Konrad and Kovenock (2010) examine discriminating contests with stochastic contestant abilities. Tsoulouhas et al. (2007) consider CEO contests that are open to heterogeneous outsider contestants. Tsoulouhas and Marinakis (2007) analyze ex post agent heterogeneity. Instead, Riis (2010) allows for agents who are heterogeneous ex ante. In a similar framework, Tsoulouhas (2011) shows that there is no scope for screening agents ex ante via a menu of offers when agents are not very heterogeneous and the principal utilizes a cardinal tournament. Also see Tsoulouhas (2010b) and the symposium papers introduced therein.

[9] Thus, in this model there is one-sided moral hazard. The link between two-sided moral hazard and the optimality of tournaments has been analyzed by Carmichael (1983) and Tsoulouhas (1999).

$u(w_i, e_i) = -\exp(-rw_i + \frac{1}{2}\frac{r}{a}e_i^2)$, where r is the agent's coefficient of absolute risk aversion and w_i is the compensation he receives from the principal. Note that the cost of effort decreases with agent ability and is measured in monetary units. This utility function has been widely used in the literature (for instance, see Meyer and Vickers (1997)).

One of the advantages of this model is that its results, absent limited liability, conform with those obtained by Lazear and Rosen (1981) who, even though they utilized more general utility functions, relied on first-order Taylor approximations of those functions.[10] The benefit of using a model which provides an analytical solution instead of an approximate solution, at least in the baseline cases, is that the results can be extrapolated in a wide range of parameter values without incurring approximation errors.

3. The piece rate scheme

3.1. The piece rate scheme without limited liability

The piece rate scheme (R) is the payment scheme in which the compensation to the ith agent takes the form $w_i = b_R + \beta_R x_i$, where (b_R and β_R) are the contractual parameters to be determined by the principal. The principal will determine these parameters by backward induction.

First, the principal calculates each agent's expected utility:

$$EU_R = -\exp\left\{-r\left[b_R + \beta_R(a + e_i) - \frac{e_i^2}{2a} - \frac{r\beta_R^2\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right)}{2}\right]\right\}, \qquad (1)$$

where the expression in square brackets is the certainty equivalent compensation of the agent.[11] Observe that expected utility rises with increases in the expected payment from the principal, reductions in the effort level implemented by the principal and reductions in the variance of the payments. To ensure the compatibility of the contract with agent incentives to perform, the principal calculates the effort level that maximizes Eq. (1). First order conditions yield

$$e_i^* = a\beta_R. \qquad (2)$$

To ensure the compatibility of the contract with agent incentives to participate, the principal selects the value of the base payment, b_R, that satisfies the agent's individual rationality constraint with equality so that the agent receives no rents but still accepts the contract. The agent receives no rents because the principal is endowed with the bargaining power. For ease of exposition, we normalize the agent's reservation utility to -1,[12] hence, given Eq. (1) the agent's individual rationality constraint implies

$$EU_R = -1 \Leftrightarrow$$
$$\Leftrightarrow b_R = \frac{r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right) - a}{2}\beta_R^2 - a\beta_R. \qquad (3)$$

Thus, by choosing the piece rate β_R, the principal can precisely determine the agent's effort because the agent will optimally set his effort according to Eq. (2). In addition, by setting b_R in accordance with Eq. (3) the principal can induce agent participation at least cost. That is, agent incentives to perform are only determined by the

bonus factor β_R, whereas agent incentives to participate are determined by the base payment b_R.

Given conditions (2) and (3), the principal maximizes his expected total profit

$$ET\prod_R = \sum_{i=1}^n [Ex_i - Ew_i] = n\left[a + a\beta_R - \frac{r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right) + a}{2}\beta_R^2\right]. \qquad (4)$$

Maximizing Eq. (4) with respect to β_R and then using condition (3) proves Proposition 1.

Proposition 1. *The piece rate scheme without limited liability for the principal,* (b_R *and* β_R), *satisfies*:

$$b_R = -\frac{a^2}{2}\frac{r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right) + 3a}{\left[r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right) + a\right]^2}, \qquad (5)[13]$$

$$\beta_R = \frac{a}{a + r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right)}. \qquad (6)$$

Given Proposition 1, the principal's expected profit per agent under the piece rate scheme and absent limited liability is

$$E\prod_R = a + \frac{a^2}{2}\frac{1}{a + r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right)}. \qquad (7)$$

3.2. The piece rate scheme with limited liability

Suppose now that the principal is subject to limited liability, that is, he cannot be required to pay the agents more than the revenue available to him plus the liquidation value, A, of the firm. This is particularly important if the state turns out to be unfavorable (that is, if production shocks turn out to be unfavorable). The principal could promise payments in these states that are higher than the sum of the revenue available to him and A, in order to reduce payments in high states. But then the principal could renege on his promise by pleading bankruptcy or by threatening to plead bankruptcy, and reduce the actual payments in low states in accord with the assets available to him. Rational agents, however, cannot be suckered by the prospect of a payment that the principal clearly cannot make. The agents take this constraint into account in deciding whether to participate and which effort to exert (see Innes, 1990, 1993a, 1993b; Tsoulouhas, 1996). They will sign a contract with the principal only if it stipulates that potential losses cannot exceed the firm's liquidation value (equivalently, the wage bill does not exceed revenue plus the liquidation value, because this is effectively what can be recouped in the case of bankruptcy). Thus, implicit in the literature above, albeit never clearly stated, is a principle for bankruptcy analogous to the celebrated Revelation Principle.[14] In particular, payments actually made when bankruptcy is a possibility can be characterized directly with a limited liability or bankruptcy constraint limiting the payments the principal makes in low states so that bankruptcy is prevented. To conclude, in order to

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[10] By contrast, Tsoulouhas (1999) and Tsoulouhas and Vukina (1999) considered first-order Taylor approximations of the optimal non-linear contract in order to approximate it by a linear tournament.

[11] This is so because $E\left[\exp(-rw_i + \frac{1}{2}\frac{r}{a}e_i^2)\right] = \exp\left[\mu + \frac{\sigma^2}{2}\right]$, when $-rw_i + \frac{1}{2}\frac{r}{a}e_i^2 \sim N(\mu, \sigma^2)$, which allows us to obtain a closed form solution for the expected utility.

[12] Note that the analysis is directly applicable to any (negative) normalization other than -1.

[13] Note that, given our assumption for the reservation utility, the base payment b_R is negative. This corresponds to cases where agents need to pay a fee in order to work, for instance, franchisees, growers of chickens who need to build chicken houses, people who buy telemarketing products in order to benefit from the real estate market etc.

[14] According to the Revelation Principle, any equilibrium allocation of any mechanism can be achieved by a truthful direct revelation mechanism.

provide correct incentives, the principal incorporates a limited liability or bankruptcy constraint (LLP) in determining the contract to offer. The LLP constraint is

$$T\prod_R(\eta, \varepsilon_1, ..., \varepsilon_n) + n\alpha \geq 0, \forall(\eta, \varepsilon_1, ..., \varepsilon_n), \tag{8}$$

where $\alpha = A/n$ is the liquidation value per agent. For generality, we allow the liquidation value to be negative, that is, we allow the company to be in debt from prior operations, or to have no collateral and be in need to borrow funds exogenously to get started.

It is easy to show that total profit $T\prod_R(\eta, \varepsilon_1, ..., \varepsilon_n)$ is increasing in the state (to be precise, it is increasing in $n\eta + \Sigma\varepsilon_i$). Therefore, if Eq. (8) is satisfied in the lowest possible state (i.e., in the lowest possible realization of production shocks) then it is satisfied in all states.[15] We assume for simplicity that the lowest possible realization of the state is the one that yields no output, but we also assume that the output distribution has a sufficiently high mean (in other words, the agent's known ability a is sufficiently high). When the agent's ability is sufficiently high, then, the output distribution area over non-positive values is negligible.[16] Therefore the LLP constraint is satisfied at all states if

$$b_R \leq \alpha, \tag{9}$$

meaning that the base payment cannot exceed the liquidation value per agent when agents exert effort but obtain no output because of unfavorable shocks. This constraint is binding if

$$\alpha_R \equiv -\frac{a^2}{2} \frac{r(\sigma_\eta^2 + \sigma_\varepsilon^2) + 3a}{\left[r(\sigma_\eta^2 + \sigma_\varepsilon^2) + a\right]^2} > \alpha, \tag{10}$$

that is, if the solution without the LLP constraint (i.e., condition (5)), which was obtained in the previous section, violates the LLP constraint. In other words, the LLP constraint is binding if the liquidation value of the firm is sufficiently small (and negative). When the LLP constraint is binding, the contractual parameters $\left(\hat{b}_R \text{ and } \hat{\beta}_R\right)$ must satisfy the non-linear system consisting of the LLP constraint (9) with equality and the individual rationality constraint (3). Therefore, the piece rate scheme with limited liability, when the liquidation value of the firm is sufficiently small satisfies

$$\hat{b}_R = \alpha, \tag{11}$$

$$\frac{r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a}{2}\hat{\beta}_R^2 - a\hat{\beta}_R - \alpha = 0, \tag{12}$$

where Eq. (12) derives from Eq. (3). Compared to the case without limited liability, Eqs. (11) and (10) indicate that the base payment b_R is reduced to satisfy the limited liability constraint, that is,

$$\hat{b}_R < b_R. \tag{13}$$

Given that $\alpha < 0$ for the LLP constraint to be binding, assuming that $a^2 > -2[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a]\alpha$ implies that there are two candidate real solutions for the piece rate in Eq. (12):

$$\hat{\beta}_R = \frac{a \pm \sqrt{a^2 + 2\left[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a\right]\alpha}}{r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a}. \tag{14}$$

Given condition (10) and $\beta_R \in [0,1]$, it can easily be shown that the higher root yields a lower profit than the lower root.[17] Therefore,

$$\hat{\beta}_R = \frac{a - \sqrt{a^2 + 2\left[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a\right]\alpha}}{r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a}. \tag{15}$$

Note that if $a^2 = -2[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a]\alpha$, then there is only one real solution

$$\hat{\beta}_R = \frac{a}{r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a}, \tag{16}$$

however, in the remaining analysis we assume that $a^2 > -2[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a]\alpha$.[18] Given $\alpha < 0$, it can easily be shown that this condition is satisfied if $a > \alpha + \sqrt{\alpha[\alpha - 2r(\sigma_\eta^2 + \sigma_\varepsilon^2)]}$, that is, if ability is sufficiently high. Observe that the piece rate is not defined if

$$\alpha < \alpha_0 \equiv -\frac{a^2}{2\left[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a\right]}, \tag{17}$$

because in that case the solution is not a real number. Compared to the case without limited liability (i.e., condition (6)), condition (15) implies

$$\hat{\beta}_R > \beta_R. \tag{18}$$

The rationale behind condition (18) is that because the LLP constraint limits the base payment, the piece rate β_R must increase to satisfy the agent's individual rationality constraint. Proposition 2 below summarizes the findings under limited liability.

Proposition 2. *Assuming that* $a > \alpha + \sqrt{\alpha[\alpha - 2r(\sigma_\eta^2 + \sigma_\varepsilon^2)]}$, *the piece rate scheme with limited liability for the principal,* $\left(\hat{b}_R \text{ and } \hat{\beta}_R\right)$, *when the liquidation value is sufficiently small and negative so that Eq. (10) holds and the LLP constraint (9) is binding, satisfies:*

$$\hat{b}_R = \alpha, \tag{19}$$

$$\hat{\beta}_R = \frac{a - \sqrt{a^2 + 2\left[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a\right]\alpha}}{r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a}. \tag{20}$$

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[15] Note that obtaining the lowest possible production level is a low probability event. Again, according to Holmström and Milgrom (1987) schemes that adjust compensation to account for rare events may not provide correct incentives in ordinary high probability circumstances. An alternative way to incorporate limited liability is to assume that the principal offers a contract such that bankruptcy is avoided $(1 - \omega)$ 100% of the time, where ω is a fraction close to 1. In this case, the lowest possible state is the lower bound of a one-sided confidence interval of production at a significance level ω. Then the contractual parameters can only be determined by computational techniques.

[16] In this case, the untruncated distribution we use in the paper is a close proxy of the truncated distribution. Appendix C, available from the authors, shows that with a truncated distribution neither a closed form solution exists nor any robust computational results can be obtained. Further, as shown below, in contrasting piece rates to tournaments, we focus on positive liquidation values α, because the tournament is not defined for negative liquidation values. However, for positive liquidation values, the limited liability constraint is non-binding under piece rates. Therefore, the core analysis of the piece rates against tournaments effectively only requires the benchmark case for piece rates which was analyzed in Section 3.1.

[17] Note that whereas the lower root is inside the interval [0,1] when $r(\sigma_\varepsilon^2 + \sigma_\eta^2) \neq a$, the higher root may or may not be inside the interval. However, when both are in the interval, the lower root is optimal as argued above. Obviously, neither root is defined when $r(\sigma_\varepsilon^2 + \sigma_\eta^2) = a$.

[18] Note that in principle, given $\alpha < 0$, assumption $a^2 > -2[r(\sigma_\eta^2 + \sigma_\varepsilon^2) - a]\alpha$ is consistent with $r(\sigma_\eta^2 + \sigma_\varepsilon^2) < a$ or $r(\sigma_\eta^2 + \sigma_\varepsilon^2) > a$. Thanks to one of the referees for pointing this out.

Proposition 2 is saying that when bankruptcy is an issue, to stay away from it, the base payment must be reduced but the piece rate must be increased to ensure agent participation. As a result, agent effort rises. Bankruptcy is an issue only when the liquidation value of the firm is sufficiently negative. Given Proposition 2, the expected profit per agent under the piece rate scheme when limited liability is binding is

$$E\widehat{\prod}_R = a\left(\hat{\beta}_R - \hat{\beta}_R^2\right) - \alpha. \tag{21}$$

By contrast, if the liquidation value is sufficiently large, that is, if condition (10) is violated, the scheme characterized in Proposition 1, instead, is implementable because the LLP constraint is non-binding.

4. The tournament

4.1. The tournament without limited liability

The (two-part piece rate) tournament (T) is the payment scheme in which the compensation to each agent is determined by a relative performance evaluation. Specifically the payment scheme is

$$w_i = b_T + \beta_T(x_i - \bar{x}) = b_T + \beta_T\left(\frac{n-1}{n}x_i - \frac{1}{n}\sum_{j \neq i} x_j\right), \tag{22}$$

where $\bar{x}$ is the average output obtained by all agents, and (b_T and β_T) are the contractual parameters to be determined by the principal. Note that under tournament the total wage bill is proportional to the base payment b_T, in particular, $\Sigma w_i = nb_T$.[19] Thus, in contrast to the piece rate scheme, the principal's total payment to the agents and, hence, the expected payment per agent are independent of output. This observation is very useful throughout the remaining analysis.

Under a tournament the agent's expected utility is

$$EU_T = -\exp\left\{-r\left(b_T + \beta_T\frac{n-1}{n}(a + e_i) - \beta_T\frac{1}{n}\sum_{j \neq i}\left(a + e_j\right) - \frac{e_i^2}{2a} - \frac{n-1}{n}\frac{r\beta_T^2\sigma_\varepsilon^2}{2}\right)\right\}. \tag{23}$$

The effort level that maximizes Eq. (23) satisfies

$$e_i^{**} = \frac{n-1}{n}a\beta_T. \tag{24}$$

Further, the individual rationality constraint implies

$$EU_T = -1 \Leftrightarrow$$
$$\Leftrightarrow b_T = \frac{1}{2}\frac{n-1}{n}\left(\frac{n-1}{n}a + r\sigma_\varepsilon^2\right)\beta_T^2. \tag{25}$$

Then, given conditions (24) and (25), the principal maximizes expected total profit

$$ET\prod_T = n\left[a + \frac{n-1}{n}a\beta_T - \frac{1}{2}\frac{n-1}{n}\left(\frac{n-1}{n}a + r\sigma_\varepsilon^2\right)\beta_T^2\right]. \tag{26}$$

Maximizing Eq. (26) with respect to β_T and then using condition (25) completes the proof of Proposition 3.

Proposition 3. *The tournament without limited liability for the principal, (b_T and β_T), satisfies:*

$$b_T = \frac{1}{2}\frac{a^2}{a + \frac{n}{n-1}r\sigma_\varepsilon^2}, \tag{27}$$

$$\beta_T = \frac{a}{\frac{n-1}{n}a + r\sigma_\varepsilon^2}. \tag{28}$$

Compared to condition (5), condition (27) implies that

$$b_T > b_R. \tag{29}$$

As argued above, a priori it was not clear whether the base payment would go up or down under tournament. This is because it can be reduced by the risk premium for the insurance against common uncertainty, but at the same time it would need to be increased to ensure agent participation. It turns out that the second effect is dominant.[20] Further, it is straightforward to show that

$$\beta_T > \beta_R, \tag{30}$$

where β_R was characterized in condition (6), that is, the principal implements higher-power incentives when common uncertainty is removed from the responsibility of the agent under tournament. The principal's expected profit per agent under tournament and absent limited liability is

$$E\prod_T = a + \frac{a^2}{2}\frac{1}{a + \frac{n}{n-1}r\sigma_\varepsilon^2}. \tag{31}$$

4.2. The tournament with limited liability

Similar to the case where the principal uses a piece rate and is subject to limited liability, the limited liability constraint under a tournament is satisfied at all states if

$$b_T \leq \alpha, \tag{32}$$

and it is binding if the liquidation value of the firm is sufficiently small in the sense that

$$\alpha_T \equiv \frac{1}{2}\frac{a^2}{a + \frac{n}{n-1}r\sigma_\varepsilon^2} > \alpha. \tag{33}$$

When the LLP constraint is binding, the contractual parameters $\left(\hat{b}_T \text{ and } \hat{\beta}_T\right)$ must satisfy the non-linear system consisting of Eq. (32), with equality, and Eq. (25). Proposition 4 characterizes the tournament in this case.

Proposition 4. *The tournament with limited liability for the principal, $\left(\hat{b}_T \text{ and } \hat{\beta}_T\right)$, when the liquidation value is sufficiently small and positive so that Eq. (33) holds, satisfies:*

$$\hat{b}_T = \alpha, \tag{34}$$

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[19] Also see footnote 7, and observe that under hybrid tournaments the total wage bill (prize) is not fixed.

[20] We conjecture that this result holds generally and not just for our model specification. As mentioned above, this is so because the expected bonus payment in a tournament is zero, whereas with piece rate compensation it is positive. Therefore, agents expect to be compensated for effort through the base payment in a tournament.

$$\hat{\beta}_T = \sqrt{\frac{2\alpha}{\frac{n-1}{n}\left(\frac{n-1}{n}a + r\sigma_\varepsilon^2\right)}}. \tag{35}$$

Observe, first, that Eq. (35) derives from the agent's individual rationality constraint (25) (simply replace b_T with $\hat{b}_T = \alpha$, and β_T with $\hat{\beta}_T$). Second, in the case with limited liability the tournament is never defined when the liquidation value per agent, α, is negative (because $\hat{\beta}_T$ is not a real number). Thus, Proposition 4 is saying that tournaments are implementable over a smaller range of liquidation values than piece rates, as they are not implementable for any negative liquidation values, whereas piece rates are implementable for a range of negative liquidation values, as well as over positive liquidation values. In addition, unlike tournaments, bankruptcy under piece rates is never an issue under positive liquidation values. Conditions (27) and (34) imply

$$\hat{b}_T < b_T, \tag{36}$$

that is, the base payment is smaller when the LLP constraint is binding. Conditions (35) and (28) imply

$$\hat{\beta}_T < \beta_T, \tag{37}$$

that is, the bonus factor decreases under tournament with limited liability. This is in sharp contrast to the result obtained above for piece rates. That is, the two schemes deal very differently with how to avoid bankruptcy. Under tournament the base payment still needs to be reduced to satisfy the LLP constraint, which reduces the agent's expected utility, but the bonus factor now needs to be reduced to satisfy the agent's individual rationality constraint. Providing the agent with lower-power incentives in this case increases his expected utility because his expected wage remains constant but his cost of effort is reduced. This is also consistent with the principal's objectives because if the principal offered higher-power incentives, given that the total wage bill is independent of output, the LLP constraint would be violated in unfavorable states. Finally, the principal's expected profit per agent under the tournament when limited liability is binding is

$$E\,\widehat{\prod}_T = a + a\sqrt{\frac{2\alpha}{a + \frac{n}{n-1}r\sigma_\varepsilon^2}} - \alpha. \tag{38}$$

Note that if the liquidation value is sufficiently large, that is, if condition (33) is violated, the scheme characterized in Proposition 3, instead, is implementable because the LLP constraint is non-binding.

5. The impact of changes in r, σ_η^2 and σ_ε^2

Based on the analysis above, we can obtain some useful insights on how the contractual parameters in the two compensation schemes respond to changes in the model parameters. Fig. 1 illustrates the impact of changes in r, σ_η^2 and σ_ε^2 on the contractual parameters b and β under both schemes, and on expected profit per agent, by using the values: $n = 30$, $a = 10$, $r = \sigma_\eta^2 = \sigma_\varepsilon^2 = 1$. As the graphs show, when the variance of common uncertainty or the variance of idiosyncratic uncertainty increase ceteris paribus, then, b_R increases, β_R decreases and $E\Pi_R$ decreases. This is so because the agent needs to be compensated more on average in order to accept a contract when there is more total uncertainty. Further, output will depend more on total uncertainty rather than on effort; hence, the principal provides lower powered incentives through a reduced β_R. Clearly, the same should be true with an increase in the agent's risk aversion rate, r, because an increase in uncertainty given the risk aversion rate, or an increase in the risk aversion rate given uncertainty, should lead to the same results. The graphs indicate that the results are similar indeed.

With respect to tournaments, the variance of common uncertainty does not affect b_T, β_T or $E\Pi_T$ because tournaments filter away common uncertainty from the responsibility of the agent. By contrast, when idiosyncratic uncertainty increases, ceteris paribus, output variability is less dependent on effort choice. Then there is less need for providing incentives to exert effort and, hence, the agent will be compensated less on average. Thus both b_T and β_T decline. Because the agent is expected to exert less effort, expected profit per agent also declines. When the risk aversion rate increases ceteris paribus, then the principal charges more for the insurance against common uncertainty that he provides; hence, b_T declines. The bonus factor β_T is also reduced when the agent is more risk-averse because, then, he is more concerned about the idiosyncratic uncertainty against which he is not insured. In this case, the principal benefits by providing lower-power incentives which implement a lower effort for the agent, than by providing higher power incentives at a substantial monetary cost in order to induce high effort by an agent who is not very motivated. The inability to motivate the agent at a reasonable cost also reduces the expected profit per agent.

6. The dominant scheme

The principal's decision about which payment scheme to offer depends entirely on the expected profit each scheme will yield. As Lazear and Rosen (1981) and Green and Stokey (1983) pointed out, tournaments are superior provided that common uncertainty is sufficient to warrant insurance provision against common shocks. In the model at hand, it is feasible to calculate the exact magnitude of common uncertainty relative to that of idiosyncratic required for a tournament to be superior over a piece rate scheme. Our analysis shows that tournaments are superior provided that the common uncertainty is larger than only a specific fraction of idiosyncratic uncertainty.[21] The implication of this finding is that in practice tournaments should normally be favored over piece rate schemes absent limited liability or when bankruptcy is not an issue. This is because empirical research (for instance, Knoeber and Thurman, 1995) has found that the magnitude of common uncertainty is approximately equal to that of idiosyncratic uncertainty.

Because tournaments offer better insurance, they lead to lower payments in favorable output states but to higher payments in unfavorable states. When limited liability is introduced, because the LLP constraint limits the payments the principal can make in low states, tournaments may cease to be superior. In other words, when the allocation of profit across states is important in satisfying the limited liability constraint, the principal may not be able to offer insurance against common shocks by using tournaments. Surprisingly, the LLP constraint must be really tight, in the sense that the liquidation value of the firm must be sufficiently small for tournaments to be inferior. Otherwise, we show that tournaments will still be superior when the liquidation value of the firm is not really small, but it is sufficient for the LLP constraint to be binding (i.e., when the liquidation value is intermediate). Thus, the allocation of profit across states must be of significant concern in order for piece rate schemes to dominate tournaments.

It is important to note that our analysis of the superiority of a scheme does not amount to a comparison of expected profits for each scheme under a limited liability constraint which is binding for both schemes at the same time. Instead, we investigate how the limited liability constraint affects each scheme at different liquidation values and overall. In this sense, the terms "superior", "dominant" or "optimal" mean that, at a given liquidation value, a scheme is affected less or not at all by the limited liability constraint and its performance

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[21] Lazear and Rosen (1981) show that when there is no common uncertainty at all, workers are risk-averse and have a CARA utility function, then piece rates dominate tournaments (see p. 854).
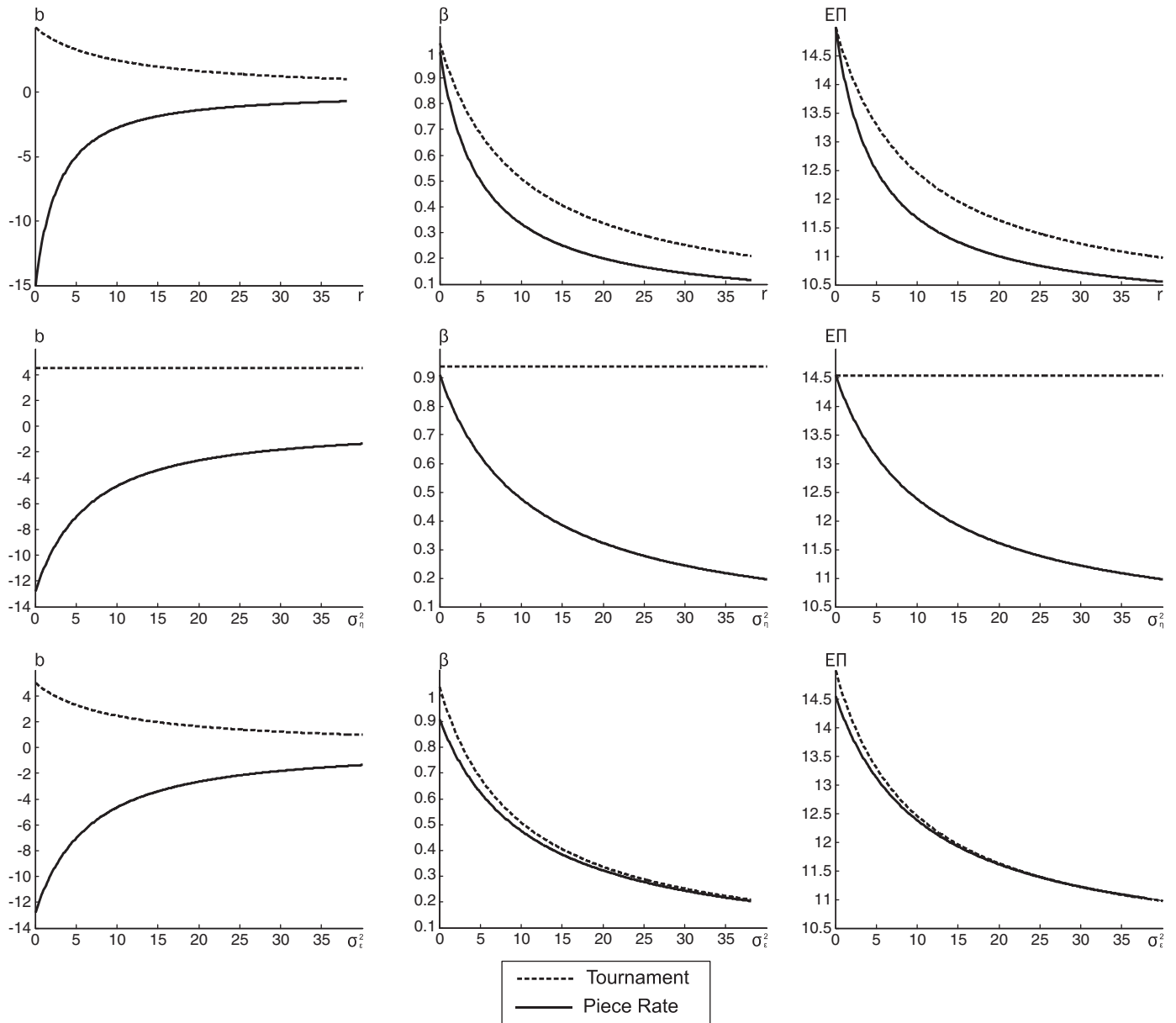
Fig. 1. The contractual parameters of piece rate contracts and tournaments for different values of r, σ_η^2, and σ_ε^2.

remains higher than that of the other scheme, or a scheme is less vulnerable to limited liability because the constraint is binding over a smaller range of liquidation values.

6.1. The dominant scheme without limited liability

Absent limited liability, recall that expected profit per agent under the piece rate scheme is shown by Eq. (7) and under the tournament it is shown by Eq. (31). Given these conditions, and Propositions 1 and 3, Proposition 5 characterizes the dominant scheme without limited liability.

Proposition 5. *Without limited liability for the principal, the tournament scheme (b_T and β_T) is more profitable for the principal than the piece rate scheme (b_R and β_R), that is, $E\Pi_T > E\Pi_R$, if*

$$\frac{1}{n-1}\sigma_\varepsilon^2 < \sigma_\eta^2. \tag{39}$$

Proposition 5 indicates that tournaments are superior provided that the variance of the common shock is larger than only a fraction of the variance of the idiosyncratic shock, where the fraction decreases when the number of agents increases. A large number of agents strengthens the dominance of tournaments over piece rates because idiosyncratic shocks cancel out, which enables the principal to offer better insurance by filtering away common shocks from the responsibility of the agents through the average output obtained by them.[22] Therefore, piece rate schemes are superior when the variance of the common shock is sufficiently small or the number of agents is sufficiently small.

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[22] Note that under tournament $Var(w_i) = \frac{n-1}{n}\beta_T^2\sigma_\varepsilon^2$ that is, the variance of agent compensation is increasing in n. Thus, from the perspective of the agent the tournament eliminates the common uncertainty from the responsibility of the agent, but on the other hand it introduces idiosyncratic uncertainty from the activities of other agents, which gets larger with more agents.
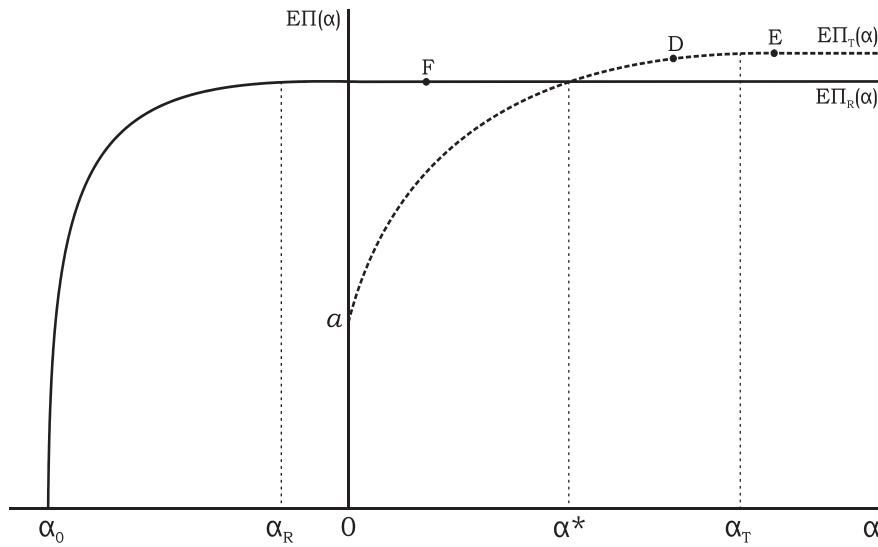
Fig. 2. Expected profit per agent under piece rate contracts and tournaments for different liquidation values.

6.2. The dominant scheme with limited liability

In order to contrast the superiority of piece rate schemes against tournaments with limited liability we compare the expected profit per agent under the two schemes. Fig. 2 depicts expected profit per agent for various liquidation values per agent, α, under both schemes. The LLP constraint for the piece rate scheme is binding for α values that are smaller than α_R and the LLP constraint for the tournament is binding for α values that are smaller than α_T. Recall that $\alpha_R < 0$ and $\alpha_T > 0$ were defined in conditions (10) and (33). That is, the LLP constraint under the piece rate scheme is binding when the liquidation value per capita is sufficiently negative (and given $a > \alpha + \sqrt{\alpha\left[\alpha - 2r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right)\right]}$ as in Proposition 2, that is, given a sufficiently high ability for the agent), while the constraint under the tournament is binding when the liquidation value is positive and sufficiently small.[23] Note that there is no discontinuity at α_T or α_R because, when α_T or α_R is crossed, the base payment b and the bonus factor β adjust in a continuous manner through the agent's individual rationality constraint which is always binding. Therefore, expected profit $E\Pi$ also changes continuously.

If the LLP constraint is non-binding under both the piece rate scheme and the tournament, that is, if $\alpha \geq \alpha_T$, then the analysis is identical to that without limited liability. In this case expected profit under tournament is higher independently of the liquidation value per agent, α, provided that condition (39) is satisfied.[24] Therefore, the focus is on the case when the LLP constraint is binding under at least one of the contractual forms.

If $\alpha < \alpha_0 < 0$, neither the piece rate scheme nor the tournament is defined as shown in Sections 3.2 and 4.2. The piece rate scheme is not defined because b_R must be reduced to $b_R = \alpha$ to satisfy the LLP constraint. But this violates the agent's individual rationality constraint because the bonus factor β_R would need to increase sufficiently in order to satisfy it. However, β_R cannot increase sufficiently without making production undertaking unprofitable. In other words, the individual rationality and the LLP constraints cannot be satisfied simultaneously.

If $\alpha_0 \leq \alpha < 0$, then, the LLP constraint for the piece rate scheme is binding if $\alpha < \alpha_R$ and non-binding if $\alpha \geq \alpha_R$. In either case, because the tournament is not defined for a negative α, the piece rate scheme is superior by default. That is a principal who is in debt but makes a contract offer, will propose a piece rate scheme. The tournament is not feasible for negative α values because the LLP constraint implies that the base payment $b_T = \alpha < 0$. Under a tournament the wage payment each agent expects to receive, Ew_i, is equal to the base payment. Therefore, each agent would expect a negative payment α which would violate his individual rationality constraint. In contrast to the tournament, a piece rate scheme is feasible for a negative α because the payment the agent expects to receive is larger than b_R, specifically it equals $b_R + a\beta_R + a\beta_R^2$.[25] Therefore the principal can satisfy the agent's individual rationality constraint by adjusting β_R while simultaneously satisfying LLP through a reduced b_R.

If $0 \leq \alpha < \alpha_T$, the LLP constraint is binding only under tournament. As stated in Proposition 6 below, conditions (7) and (38) then imply that there exists a critical value below which the piece rate contract is dominant and above which the tournament is dominant. As argued above, either the piece rate or the tournament or both schemes are not defined if $\alpha < 0$.

Proposition 6. *With limited liability for the principal, assuming that $\alpha \geq 0$ and condition (39) is satisfied, there exists a critical value $\alpha^* \in (0, \alpha_T)$ satisfying*

$$\alpha^* = a^2\left[\frac{1}{a + \frac{n}{n-1}r\sigma_\varepsilon^2}\left(1 - \sqrt{\frac{r\left(\sigma_\eta^2 - \frac{1}{n-1}\sigma_\varepsilon^2\right)}{a + r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right)}}\right) - \frac{1}{2}\frac{1}{a + r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right)}\right] \quad (40)$$

such that: (i) the tournament scheme $\left(\hat{b}_T \text{ and } \hat{\beta}_T\right)$ is more profitable for the principal than the piece rate scheme (b_R and β_R), that is, $E\widehat{\Pi}_T(\alpha) > E\Pi_R$, iff $\alpha > \alpha^*$, $\forall \alpha \in [0, \alpha_T)$; (ii) the tournament scheme (b_T and β_T) is more profitable for the principal than the piece rate scheme (b_R and β_R), that is, $E\prod_T > E\prod_R, \forall \alpha \geq \alpha_T$.

In a nutshell, Proposition 6 demonstrates that while tournaments are, in many instances, superior to piece rate schemes, this ceases to be true if payments are constrained to satisfy the bankruptcy constraint

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[23] Recall that the tournament is not defined when the liquidation value of the firm is negative because β_T is not a real number in this case. For similar reasons, the piece rate contract is not defined for a sufficiently negative liquidation value (see condition (17)).

[24] If condition (39) is not satisfied, then, tournaments are dominated by piece rate contracts regardless of whether the LLP constraint is binding or not. That is, $E\prod_R(\alpha) > E\prod_T(\alpha)$, $\forall \alpha$.

[25] To see this, note that the expected payment is equal to $b_R + \beta_R(a + e_i)$, where e_i satisfies condition (2).

(i.e., avoid bankruptcy). The rationale is that when both schemes are implementable, the possibility of facing bankruptcy only arises under the tournament scheme. Along the liquidation value dimension, on the other hand, Proposition 6 demonstrates that when the liquidation value per agent is positive but smaller than the critical value α^*, the piece rate scheme dominates the tournament. That is, a principal with a small positive liquidation value will find it profitable to refrain from insuring the agent against common uncertainty. The intuition is that when the liquidation value of the firm is sufficiently small, the principal is concerned about the allocation of profit across states because he has to satisfy a tight limited liability constraint. In some sense, the suboptimality of offering insurance when the principal is risk-neutral but subject to a tight limited liability constraint is analogous to that had the principal been sufficiently risk-averse without being constrained by limited liability. However, if the liquidation value is larger than the critical value α^*, tournaments will dominate piece rate schemes because the principal will benefit by providing insurance against common uncertainty. Similar to the intuition above, a principal who is not very concerned about the allocation of profit across states will still provide insurance. Therefore, surprisingly, tournaments can still be dominant even if the LLP constraint is binding under tournament. Lastly, Proposition 6 indicates that for a sufficiently large liquidation value (i.e., for $\alpha \geq \alpha_T$), the LLP constraint is non-binding under either scheme, and the contracts that were characterized in the case without limited liability are implementable.

When comparing two alternative schemes, the best case scenario would be to consider their performance under identical states of the limited liability constraint. That is, the ideal would be a comparison of performance for each scheme under the assumption that the limited liability constraint is binding for both schemes over the same range of liquidation values. However, the literature comparing relative to absolute performance evaluation schemes (see Footnote 3) has established that these schemes are structurally different. The findings of our analysis are no exception. The response of the two schemes to the addition of a limited liability constraint is entirely different. Under the new constraint both schemes have to compensate for the decrease in expected profit. The piece rate compensates by increasing the power of incentives through an increase in the bonus factor. The tournament, on the other hand, compensates by decreasing the bonus factor, that is, by providing more insurance to the agents so that they will accept a lower payment. Given the different characteristics and behavior of the two schemes under limited liability, one should expect that, even though we start with the same formula for the limited liability constraint for both schemes, each scheme will be vulnerable to the constraint over different ranges of liquidation values.

Independently of any modeling considerations, the ranges of liquidation values over which the constraint is binding for each scheme cannot coincide. This is so because the expected bonus under piece rates is positive, whereas in a tournament it is zero, implying that agents must be compensated for effort through the base payment in a tournament. Therefore, the optimal base payments under piece rates and tournaments should be distinctly different (with that for tournaments being the larger one). From relationships (9) and (32) it is obvious that the base payment for each scheme marks the point from which the limited liability constraint becomes binding. Therefore, we can conclude that the limited liability constraint, independently of the utility function used and of the agent's reservation utility, will always have bite at higher liquidation values for the tournament than for the piece rate scheme.

In our analysis, the optimality of a compensation scheme arises from the fact that limited liability becomes binding for different ranges of liquidation values under each scheme. As shown above, binding limited liability is not a state that we can exogenously impose to the model, because each scheme will react in an entirely different way to the addition of the constraint. Thus, coordinating the model so that the critical liquidation value, α^*, would be within the range of liquidation values where the limited liability constraint is binding

for both schemes, would be a matter of intentional selection of parameters. Instead, we allow the limited liability constraint to enter the principal's profit maximization problem and freely yield the range of liquidation values over which limited liability affects the contractual parameters for each compensation scheme. As a result of the difference in the nature of the two schemes, we observe that the range of liquidation values over which the constraints are binding is indeed different. Based on our analysis, the principal will select to use the contract that yields the highest expected profit either because the limited liability constraint is not binding under either scheme (e.g., point E in Fig. 2), or because the binding constraint has not become a serious issue yet (e.g., point D in Fig. 2) or because the previously optimal scheme is now underperforming due to severely binding limited liability (e.g., point F in Fig. 2).

In our model, the intuition behind the different ranges of liquidation values over which the limited liability constraint is binding under each scheme becomes apparent. The expected bonus under tournaments is zero, whereas that under piece rates is positive. Thus, the base payment is positive for tournaments and negative for piece rates, yielding no range of liquidation values over which the limited liability constraint is binding under both schemes. Moreover, the analysis shows that tournaments are not defined for negative liquidation values and piece rates are not defined for sufficiently negative liquidation values. The rationale is that in these cases the principal will not find it profitable to offer a contract that the agents will accept. Given that it is impossible to characterize the superior contract when the limited liability constraint is binding under both schemes, the only possibility is to characterize the superior contract for different liquidation values. Thus, for reasonably negative liquidation values the principal, who is in debt, can only propose a piece rate contract, under which the limited liability constraint is binding for sufficiently small liquidation values. This contract is the only scheme that is implementable. For positive liquidation values, the principal can choose between piece rates and tournaments, but the liquidation value will never be binding under piece rates and it will be binding for small or intermediate liquidation values under tournaments. For small positive liquidation values, the superior scheme is the piece rate scheme. For intermediate liquidation values the superior scheme is the tournament scheme even though the limited liability constraint is binding. Lastly, for large liquidation values the superior scheme is the tournament scheme again, but the limited liability constraint is non-binding.

6.3. Comparative statics

In this section, we analyze the impact of changes in the coefficient of absolute risk aversion of the agent, r, the variance of the common shock, σ_η^2, and the number of agents, n, ceteris paribus. As shown in Appendix A, the following relationships hold:

$$\frac{\partial \alpha^*}{\partial r} < 0, \tag{41}$$

$$\frac{\partial \alpha^*}{\partial \sigma_\eta^2} < 0, \tag{42}$$

$$\frac{\partial \alpha^*}{\partial n} < 0. \tag{43}$$

The comparative statics results in relationships (41), (42) and (43) indicate that the critical liquidation value per agent α^*, defined in Eq. (40) above, decreases when the coefficient of risk aversion, the variance of common shock or the number of agents increase. This means that tournaments dominate piece rate schemes over a wider range of α values when these parameters increase. The intuition is that the
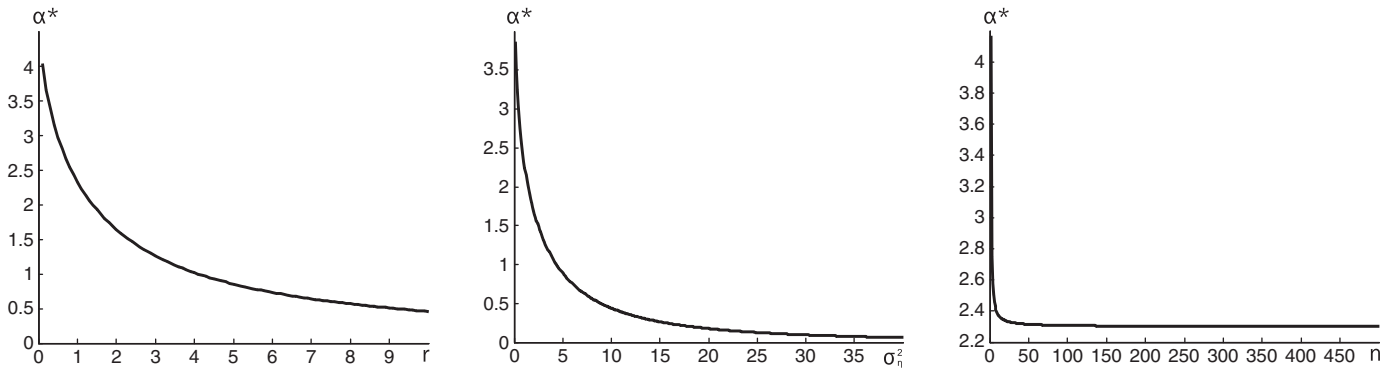
Fig. 3. The response of α^* in changes of r, σ_η^2 and n.

more risk-averse the agent is, the more he is willing to pay for insurance, or the higher the risk premium the principal can charge for insurance. The higher the variability in the common shock the more insurance is provided through the tournament and, as before, the higher the risk premium the principal can charge. Clearly in both cases tournaments become more profitable to the principal. An increase in the number of agents also makes tournaments dominant over a wider range of α values. A large number of agents is necessary to eliminate idiosyncratic noise from the average output obtained by the agents. Thus, the more agents, the more insurance is provided against common shocks. This leads to a result similar to that obtained above for increases in the variance of common shock.

A numerical example will illustrate the impact of the parameters above. For this example, we use the following values: $n = 30$, $a = 10$, $r = \sigma_\eta^2 = \sigma_\varepsilon^2 = 1$. As shown in Fig. 3, the impact of r, σ_η^2 and n on α^* is similar. However, the number of agents has a stronger negative impact initially. Once a sufficient number of agents is reached so that the impact of idiosyncratic noise on relative performance becomes negligible (i.e., once the idiosyncratic noise is virtually eliminated from the variance of $\bar{x}$), then additional agents do not have a significant impact on α^*. The impact of all three parameters eventually fades off because α^* cannot drop below zero.

6.4. The dominant scheme and the number of agents

In the preceding analysis, the number of agents was exogenous. A natural extension is the relationship between the form of the dominant contract and the number of agents, given the liquidation value of the principal's enterprise. A thorough examination of this issue is beyond the scope of this paper and is the subject of future work. However, for completeness, we briefly extend the analysis to allow for endogeneity in the number of agents.

Absent limited liability, and absent managerial and other inefficiencies or exogenous factors constraining the number of agents, the principal will find it profitable to keep increasing the number of agents because the expected profit per agent is positive (see condition (31)). Thus absent limited liability, as shown by condition (39), the principal will keep offering a tournament provided that the variance of the common shock is not negligible. By contrast, with limited liability, interestingly a tournament will never be superior provided that the number of agents is not limited by exogenous or organizational factors. This is because, starting from any exogenously given liquidation value such that the tournament yields higher expected profit per agent than the piece rate, the principal can increase the number of agents sufficiently so that the liquidation value per agent is less than the critical value α^* which was defined in Eq. (40). Then the principal will offer a piece rate contract which, even though it yields a lower expected profit per agent, it yields a higher total profit

when the number of agents is increased sufficiently.[26] An example of this is shown in Fig. 2, when the principal finds it profitable to move from a point such as D or E to a point such as F by increasing the number of agents sufficiently. A prerequisite for this result is that α^* never converges to zero when the number of agents keeps increasing; clearly this prerequisite is satisfied by condition (40).[27] Had the number of agents been limited, this result would only hold if the liquidation value A were sufficiently small. If the liquidation value of the firm were not sufficiently small, and the availability of agents were limited, increasing the number of agents might never reduce the liquidation value per agent α below α^*. Because in practice the number of agents is constrained by availability, as well as by managerial and other inefficiencies, firms with large liquidation values will be unable to increase the number of agents sufficiently to profit from switching to piece rate schemes. Any empirical investigations of the findings in this paper should also take into account the fact that companies with small liquidation values may employ a small number of agents, in which case they may not use tournaments because of the idiosyncratic noise that average agent performance will contain.

7. Conclusions and policy implications

Starting with Lazear and Rosen (1981) and Green and Stokey (1983), contract theory has long argued that tournaments dominate piece rate schemes in the presence of relatively large common shocks that affect agent performance, when agents are risk averse. Relative performance evaluation via tournaments constitutes a move closer to the First Best because the principal becomes better informed. By removing common uncertainty from the responsibility of the agents, and by charging a premium for this insurance, the principal increases his profit without hurting the agents.

This paper shows that this celebrated result, which, absent limited liability, holds if common uncertainty is larger than only a fraction of the idiosyncratic uncertainty, does not hold at all when the principal is subject to limited liability and the liquidation value of the firm is sufficiently small. Under a limited liability constraint, tournaments are dominated by piece rate schemes when the liquidation value of the firm is sufficiently small, because the principal is concerned about the allocation of profit across states. Piece rates allow the principal to decrease the total wage bill if the state of nature is unfavorable and output turns out to be small. By contrast the total wage bill under tournaments is constant and independent of output. Surprisingly, if the

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[26] Note that if both the number of agents and the liquidation value of the firm were endogenous, the piece rate contract would be dominant if the liquidation value per agent were sufficiently small.

[27] Condition (40) implies that $\lim_{n\to\infty} a^* = a^2 \left[\frac{1}{a+r\sigma_\varepsilon^2} \left(1 - \sqrt{\frac{r\sigma_\eta^2}{a+r(\sigma_\eta^2+\sigma_\varepsilon^2)}} \right) - \frac{1}{2} \frac{1}{a+r(\sigma_\eta^2+\sigma_\varepsilon^2)} \right] > 0$.

liquidation value is not sufficiently small tournaments dominate piece rates even when the limited liability constraint is binding, which it is for intermediate values. Interestingly, if the number of agents is not limited by exogenous or organizational factors, the principal will not prefer to offer a tournament over a piece rate because the principal will find it profitable to keep increasing the number of agents until piece rate contracts become superior. Surprisingly again, when price uncertainty is introduced (see Appendix A), if the lowest possible output price is not very small the dominant contract form is completely unaffected by the presence of price uncertainty. By contrast, if the lowest possible price is sufficiently small, specifically if it is smaller than the piece rate, the increased bankruptcy risk strengthens the need for tournaments by expanding the range of liquidation values over which tournaments are superior. The rationale is that from the principal's perspective there is tension between providing insurance to the agent against common shocks and insuring himself against the variability in the total wage bill. The principal prefers to offer a tournament in order to eliminate the variability of total wages, even though the limited liability constraint is tighter with significant price uncertainty.

To conclude, our analysis shows that the liquidation value of the firm is much more important than the magnitude of price uncertainty in determining the form of the contract the principal should offer. Thus, regardless of price uncertainty, if the liquidation value is sufficiently small the principal will prefer to offer a piece rate scheme, and if the liquidation value is sufficiently large the principal will prefer to offer a tournament. In our analysis, limited liability does not have to be binding over the same range of liquidation values for both compensation schemes to be contrasted against. Instead, we allow the constraint to affect the maximization problem of the principal freely and, given the different nature of the two schemes, we observe that the range of liquidation values over which the constraints are binding is different. In this sense, the principal will use the scheme for which expected profit is higher because limited liability is less of an issue.

Our analysis may be useful in a number of circumstances. Financially constrained firms should use absolute performance standards and refrain from using tournaments because the latter would increase the firm's costs in unfavorable states, and second, that firms facing significant price volatility for their products should favor tournaments, unless the firms are in real financial distress. Thus, for instance, financially constrained HMOs should refrain from using tournaments among physicians. Fiscal stress plays an important role in explaining the decision to contract out the provision of services (for instance, family services, food services, prison health care, solid waste disposal, public relations and social services) by states and municipalities (see Brudney et al., 2005 and the references therein). Our analysis adds to this discussion by highlighting the importance of uncertainty in revenue (in this case, tax revenue). The policy implication is in favor of contracting out to independent companies and of rewards that are based on individual outputs (piece rates) in states under serious financial stress, where keeping costs down is essential. By contrast, states with tax revenue volatility should not rely on contracting out, even if they are under mild financial stress. Keeping the provision of services in house helps stabilize the variability in the costs of these services.

There is an analogy between our framework of tournaments among agents when the principal is subject to limited liability and procurement under limited liability for the bidders (say, contractors). Parlane (2003) considers a winner-takes-all auction to make the argument that limited liability of bidders distorts their attitudes toward risk, enhancing the bidding competition and, therefore, the likelihood of bankruptcy. She then argues that a first price auction gives a lower probability of bankruptcy than a second price auction. Our analysis is similar, in that competition through tournaments increases the likelihood of bankruptcy. However, we add to the discussion the finding that how the winning bidder compensates his employees should also be controlled by contractual provisions limiting the use of tournaments and favoring the use of piece rates, in order to avoid bankruptcy.

Our analysis also applies to rent-seeking contests. You can think of such contests as competitions in which the competitors spend resources to win rents. When financial distress or bankruptcy is an issue for the entity which is providing the rents, rent-seeking invites such distress. For instance, lobbying activities to secure and increase rents also increase the possibility of financial distress for a government. This is analogous to tournaments increasing the total wage bill for the principal. However, because a contest, similar to a tournament, provides higher power incentives to the contestants (for instance, they exert more effort in designing a better defense system for the Department of Defense in order to win the contract), our analysis implies that a contest should be avoided only if financial distress is a serious concern.

Appendix A. Price uncertainty and limited liability

In the main analysis, the price of the output was given, known ex ante and normalized to 1. We now extend the analysis to allow for the case when the price, p, is unknown ex ante. It is interesting to investigate whether our results carry over to the case when there is price uncertainty in addition to production uncertainty. In examining this issue we assume that the principal is a price taker. The additional uncertainty the principal bears can, in principle, have serious consequences on the dominant contractual form. Price uncertainty can make the limited liability constraint tighter. Therefore, price uncertainty coupled with limited liability may limit the principal's ability to provide insurance against production uncertainty because the principal is more concerned about the allocation of profit across states (in this sense, it is as if the principal were more risk-averse). On the other hand, as demonstrated in the preceding analysis, the total wage bill under tournaments is invariant while under piece rate schemes it is not. This is more important to the principal under price uncertainty because, if total output turns out to be high and the price happens to be low, limiting costs is of primary significance. Therefore, under tournaments, there is an interesting trade-off for the principal between providing insurance to the agents and providing insurance to himself against variation in his cost.

In formulating the limited liability constraint under price uncertainty, note that there are two candidates for the lowest state. Either output per agent is zero regardless of output price or price is the lowest possible and output is high. For tractability we assume that output per agent is in the interval $[0, x_h]$ with non trivial probabilities, while probability that output is greater than x_h is negligible. Further, output price is normalized to be in the interval $[p_l, p_h]$ with an expected price equal to 1 in order to simplify the comparison to the preceding analysis. Thus, the limited liability constraint under the piece rate scheme is:

$$\begin{cases} (p_l - \beta_R)x_h - b_R + \alpha \geq 0 \\ (p_h - \beta_R)0 - b_R + \alpha \geq 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha \geq b_R - (p_l - \beta_R)x_h \\ \alpha \geq b_R \end{cases}. \tag{A1}$$

Clearly if $p_l \geq \beta_R$, then when the second LLP condition is satisfied the first condition is also satisfied. Hence, if the marginal revenue from an additional unit of output, p_l, exceeds the marginal cost, β_R, the characterization of the piece rate scheme is identical to that above in Section 3.2. By contrast if $p_l < \beta_R$, then when the first condition is binding the second one is non-binding. Thus, when price uncertainty is relatively large, the LLP constraint is tighter than absent price uncertainty; that is, the constraint becomes binding over a wider range of α values. In this case, we denote the contractual parameters by $\left(\tilde{b}_R \text{ and } \tilde{\beta}_R\right)$ and by using a method analogous to that in Section 3.2, $\left(\tilde{b}_R \text{ and } \tilde{\beta}_R\right)$ must simultaneously satisfy

$$\begin{cases} \tilde{b}_R = \alpha + \left(p_l - \tilde{\beta}_R\right)x_h \\ \tilde{b}_R = \dfrac{r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right) - a}{2}\tilde{\beta}_R^2 - a\tilde{\beta}_R. \end{cases} \tag{A2}$$

It follows that

$$\tilde{\beta}_R = \frac{a - x_h + \sqrt{(x_h - a)^2 + 2\left[r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right) - a\right](\alpha + p_l x_h)}}{r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right) - a}, \tag{A3}$$

and

$$\tilde{b}_R = \alpha - \tilde{\beta}_R x_h. \tag{A4}$$

Thus,

$$E\tilde{\prod}_R = a\left(\tilde{\beta}_R - \tilde{\beta}_R^2\right) - \alpha - \tilde{\beta}_R x_h. \tag{A5}$$

Next, we turn to the tournament. Under tournament, the LLP constraint is stated as:

$$\begin{cases} p_l x_h - b_T + \alpha \geq 0 \\ p_h 0 - b_T + \alpha \geq 0 \end{cases} \Longleftrightarrow \begin{cases} \alpha \geq b_T - p_l x_h \\ \alpha \geq b_T \end{cases}. \tag{A6}$$

Clearly, when the second condition is binding the first one is non-binding. Thus, price uncertainty has no impact on the contractual parameters under tournament. As in Section 4.2, b_T, β_T and $E\prod_T$ satisfy conditions (34), (35) and (38) respectively. The intuition is that the invariability in the total wage bill under tournament dominates the principal's concern about providing insurance to the agent when total uncertainty is increased. Proposition 7 summarizes our findings for the case with price uncertainty.

Proposition 7. *With limited liability for the principal, assume that output per agent x_i is in the interval $[0, x_h]$ and the price of output p is uncertain with $p \in [p_l, p_h]$. If $p_l \geq \beta_R$, then the piece rate scheme takes the form $\left(\hat{b}_R \text{ and } \hat{\beta}_R\right)$, characterized in conditions (19) and (20). If $p_l < \beta_R$, then the piece rate scheme takes the form $\left(\tilde{b}_R \text{ and } \tilde{\beta}_R\right)$, characterized in conditions (A4) and (A3). Regardless of price, the tournament takes the form $\left(\hat{b}_T \text{ and } \hat{\beta}_T\right)$, characterized in conditions (34) and (35).*

Recall that $\left(\tilde{b}_R \text{ and } \tilde{\beta}_R\right)$ was characterized in Proposition 2 and $\left(\hat{b}_T \text{ and } \hat{\beta}_T\right)$ was characterized in Proposition 4. Similar to the analysis without price uncertainty, conditions (A5) and (38) imply that there exists a critical value α^{**} below which the piece rate is dominant and above which the tournament is dominant. Fig. 4 presents three examples depending on the value of the parameters. In the left graph where the parameters are set to $n = 30$, $a = 10$, $r = \sigma_\eta^2 = \sigma_\varepsilon^2 = 1$, $x_h = 5a$ and $p_l = 0.01$, the piece rate is never superior over the tournament. In the middle graph where the parameters are set to $n = 30$, $a = 10$, $r = \sigma_\eta^2 = \sigma_\varepsilon^2 = 1$, $x_h = 5a$, and $p_l = 0.4$, the range over which the piece rate is superior is smaller than in the case without price uncertainty. Finally, in the right graph where $n = 30$, $a = 10$, $r = \sigma_\eta^2 = \sigma_\varepsilon^2 = 1$, $x_h = 5a$, and $p_l = 0.6$, the range over which the piece rate is superior is unaffected.

To summarize our findings, if the lowest possible price p_l exceeds the piece rate β_R the form of the dominant scheme for different liquidation values per agent is completely unaffected by the presence of price uncertainty. By contrast, if the lowest possible price is smaller than the piece rate, for instance, if it is possible that the principal cannot find any buyers for the output produced so that $p_l = 0$, then the range of liquidation values over which piece rate schemes are superior is weakly reduced. The intuition why tournaments dominate piece rate schemes over a wider range of liquidation values is, again, that the principal benefits from the removal of the cost variability under tournament. In all, our analysis demonstrates that the liquidation value of the firm is the definitive factor in the determination of the scheme the principal should offer. Thus, unless the lowest possible price is extremely small so that the range of liquidation values over which piece rates are dominant is eliminated, if the liquidation value of the firm is sufficiently small, the principal will prefer to offer a piece rate scheme. Empirical evidence from the broiler, turkey and swine industries in Tsoulouhas and Vukina (1999) suggests that smaller companies do rely more heavily on piece rates rather than on tournaments, as opposed to larger companies. Specifically, the frequency of observing tournaments diminishes as we move from the broiler industry where the firms are largest, to the turkey industry where the firms are medium in size and to the swine industry where the companies are smallest.
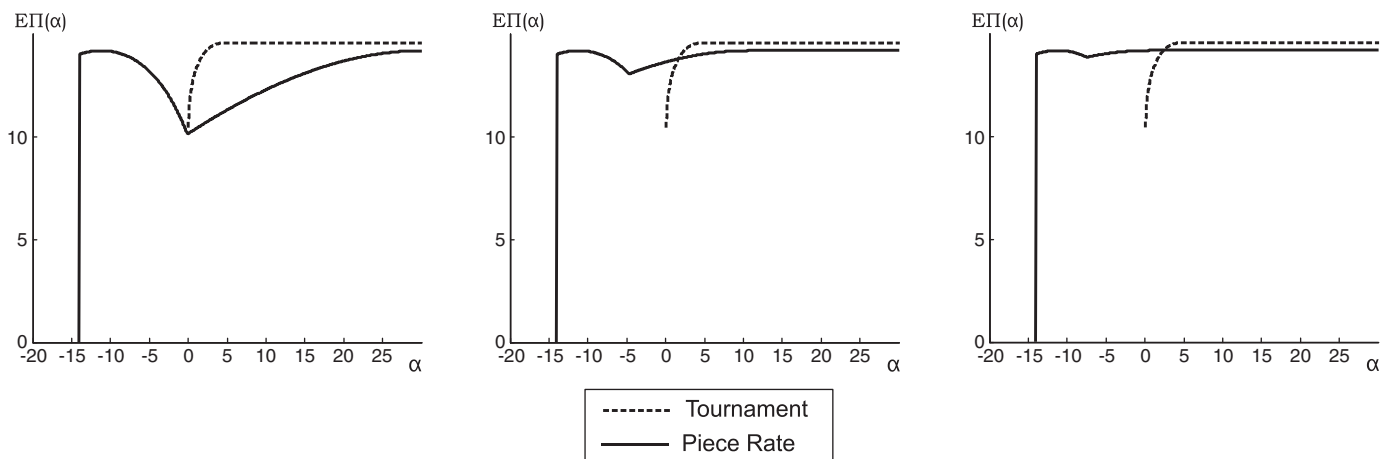

Fig. 4. The effect of price uncertainty on α^*.

Appendix B. Proofs

Proof of condition (42).

$$\frac{\partial \alpha^*}{\partial \sigma_\eta^2} < 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{ra^2}{2\left(a + r\sigma_\varepsilon^2 + r\sigma_\eta^2\right)^2} \left[1 - \frac{1}{\sqrt{\dfrac{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2}}} \right] < 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\sqrt{\dfrac{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2}}} > 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2} < 1 \Leftrightarrow$$

$$\Leftrightarrow -\frac{1}{n-1}r\sigma_\varepsilon^2 - \sigma_\varepsilon^2 < a.$$

(B1)

Condition (B1) is satisfied because the LHS is negative and the RHS is positive.   Q.E.D.

Proof of condition (43).

$$\frac{\partial \alpha^*}{\partial n} < 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{a^2}{\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)^2} \frac{r\sigma_\varepsilon^2}{(n-1)^2} \left[\left(1 - \sqrt{\frac{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2}} \right) - \frac{n}{2\sqrt{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}} \frac{a + \dfrac{n}{n-1}r\sigma_\varepsilon^2}{\sqrt{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2}} \right] < 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\sqrt{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}}{\sqrt{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2}} + \frac{\dfrac{n\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)}{2\sqrt{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}}}{\sqrt{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2}} > 1 \Leftrightarrow$$

(B2)

$$\Leftrightarrow \sqrt{r\sigma_\eta^2 - \frac{1}{n-1}r\sigma_\varepsilon^2} + \frac{n\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)}{2\sqrt{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}} > \sqrt{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2} \Leftrightarrow$$

$$\Leftrightarrow r\sigma_\eta^2 - \frac{1}{n-1}r\sigma_\varepsilon^2 + \frac{n^2\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)^2}{2\left(r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2\right)} + 2\sqrt{r\sigma_\eta^2 - \frac{1}{n-1}r\sigma_\varepsilon^2} \frac{n\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)}{2\sqrt{r\sigma_\eta^2 - \dfrac{1}{n-1}r\sigma_\varepsilon^2}} > a + r\sigma_\varepsilon^2 + r\sigma_\eta^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{n^2\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)^2}{2r\left(\sigma_\eta^2 - \dfrac{1}{n-1}\sigma_\varepsilon^2\right)} + (n-1)\left(a + \frac{n}{n-1}r\sigma_\varepsilon^2\right) > 0.$$

Given condition (39), condition (B2) holds.   Q.E.D.
Regarding condition (41), one can show that:

$$\frac{\partial \alpha^*}{\partial r} = a^2 \left[\frac{\sigma_\eta^2 + \sigma_\varepsilon^2 - \dfrac{a\left(\sigma_\eta^2 - \dfrac{1}{n-1}\sigma_\varepsilon^2\right)}{\sqrt{\dfrac{r\left(\sigma_\eta^2 - \dfrac{1}{n-1}\sigma_\varepsilon^2\right)}{a + r\sigma_\varepsilon^2 + r\sigma_\eta^2}}\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)}}{2\left(a + r\sigma_\varepsilon^2 + r\sigma_\eta^2\right)^2} - \frac{\left(1 - \sqrt{\dfrac{r\left(\sigma_\eta^2 - \dfrac{1}{n-1}\sigma_\varepsilon^2\right)}{a + r\left(\sigma_\eta^2 + \sigma_\varepsilon^2\right)}}\right)\dfrac{n}{n-1}\sigma_\varepsilon^2}{\left(a + \dfrac{n}{n-1}r\sigma_\varepsilon^2\right)^2} \right].$$

(B3)

This expression cannot be simplified in any meaningful way that would enable us to sign the derivative, however, the statement was easily verified by computational techniques.

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