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Restricted Probabilistic Fixed Ballot Rules and Hybrid Domains*

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Abstract

We study Random Social Choice Functions (or RSCFs) in a standard ordinal mechanism design model. We introduce a new preference domain called a hybrid domain which includes as special cases as the complete domain and the single-peaked domain. We characterize the class of unanimous and strategy-proof RSCFs on these domains and refer to them as Restricted Probabilistic Fixed Ballot Rules (or RPFBRs). These RSCFs are not necessarily decomposable, i.e., cannot be written as a convex combination of their deterministic counterparts. We identify a necessary and sufficient condition under which decomposability holds for anonymous RPFBRs. Finally, we provide an axiomatic justification of hybrid domains and show that every connected domain satisfying some mild conditions is a hybrid domain where the RPFBR characterization still prevails.

Keywords: Strategy-proofness; hybrid domain; restricted probabilistic fixed ballot rule; decomposability; connectedness

JEL Classification: D71; H41

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1 Introduction

Two familiar preference domains in the literature on mechanism design in voting environments are the complete domain and the domain of single-peaked preferences. The complete domain arises naturally when there are no a priori restrictions on preferences. The classic results of Gibbard (1973), Satterthwaite (1975) and Gibbard (1977) apply here. According to them, requiring strategy-proofness forces the mechanism to be a dictatorship in the deterministic case and to be a random dictatorship in the probabilistic case. Single-peaked preferences on the other hand, require more structure on the set of alternatives. However, they arise naturally in a variety of situations such as preference aggregation (Black, 1948), strategic voting (Moulin, 1980), public facility allocation (Bochet and Gordon, 2012), fair division (Sprumont, 1991) and assignment (Bade, 2019). The single-peaked domain also admits well-behaved strategy-proof social choice functions. In this paper, we propose a flexible preference domain that admits both the complete domain and the single-peaked domain as special cases. We call them *hybrid domains* and completely characterize unanimous and strategy-proof *random social choice functions* (or RSCFs) over the hybrid domains. We refer to these random social choice functions as *Restricted Probabilistic Fixed Ballots Rules* (or RPFBRs) and analyze their salient properties. Finally, we provide an axiomatic justification of hybrid domains and show that all domains that satisfy some richness properties must be hybrid.

We briefly recall the definition of single-peaked preferences. The set of alternatives is a finite set $A = \{a_1, a_2, \dots, a_m\}$ which is endowed with the prior order $a_1 \prec a_2 \prec \cdots \prec a_m$. A preference ordering over A is *single-peaked* if there exists a unique top-ranked alternative, say a_k, such that preferences decline when alternatives move "farther away" from a_k. For instance, if "$a_r \prec a_s \prec a_k$ or $a_k \prec a_s \prec a_r$" , then a_s is strictly preferred to a_r. A preference is *hybrid* if there exist *threshold* alternatives $a_{\underline{k}}$ and $a_{\overline{k}}$ with $a_{\underline{k}} \prec a_{\overline{k}}$ such that preferences over the alternatives in the interval between $a_{\underline{k}}$ and $a_{\overline{k}}$ are "unrestricted" relative to each other, while preferences over other alternatives retain features of single-peakedness. Thus, the set A can be decomposed into three parts: left interval $L = \{a_1, \dots, a_{\underline{k}}\}$, right interval $R = \{a_{\overline{k}}, \dots, a_m\}$ and middle interval $M = \{a_{\underline{k}}, \dots, a_{\overline{k}}\}$. Formally, a preference is $(\underline{k}, \overline{k})$-*hybrid* if the following holds: (i) for a voter whose best alternative lies in L (respectively in R), preferences over alternatives in the set $L \cup R$ are conventionally single-peaked, while preferences over alternatives in M are arbitrary subject to the restriction that the best alternative in M is the left threshold $a_{\underline{k}}$ (respectively, right threshold $a_{\overline{k}}$), and (ii) for a voter whose peak lies in M, preferences restricted to $L \cup R$ are single-peaked but arbitrary over M. Observe that if $\underline{k} = 1$ and $\overline{k} = m$, then preferences are unrestricted, while the case where $\overline{k} - \underline{k} = 1$ coincides with the case of single-peaked preferences.

A $(\underline{k}, \overline{k})$-hybrid preference is a preference ordering which is single-peaked everywhere except over the alternatives in the middle interval. Consider the location of candidates in the forthcoming Democratic party primary elections in the USA, in the usual political left-right spectrum. It is clear that candidates such as Sanders and Warren belong to the left, while others such as Biden (perhaps) belong to the right. However, there are several candidates who cannot easily be ordered in this manner. The typical reason is that they are left on some issues and right on others. Hybrid preferences treat these candidates as ones belonging to the middle part, and the hybrid domain reflects the reversals in the relative rankings of these alternatives that arise from the underlying multidimensional issues. A more general way to model departures from single-peaked preferences would be to consider several intervals of alternatives where single-peakedness fails. However, as suggested by Theorem 3, this complicates the analysis significantly without adding substantial new insights.

We study unanimous and strategy-proof RSCFs on hybrid domains. A RSCF associates a lottery over alternatives to each profile of preferences. Randomization is a way to resolve conflicts of interest by ensuring a measure of ex-ante fairness in the collective decision process. More importantly, it has recently been shown that randomization significantly enlarges the scope of designing well-behaved mechanisms, e.g., the compromise RSCF of Chatterji et al. (2014) and the maximal-lottery mechanism of Brandl et al. (2016).

In order to define the notion of strategy-proofness, we follow the standard approach of Gibbard (1977). For every voter, truthfully revealing her preference ordering must yield a lottery that stochastically dominates the lottery arising from any unilateral misrepresentation of preferences according to the sincere preference. Unanimity is a weak efficiency requirement which says that the alternative that is unanimously best at a preference profile is selected with probability one.

The main theorem of the paper shows that a RSCF defined on the $(\underline{k}, \overline{k})$-hybrid domain is unanimous and strategy-proof if and only if it is a RPFBR (see Theorem 1). A RPFBR is a special case of a *Probabilistic Fixed Ballot Rule* (or PFBR) introduced by Ehlers et al. (2002). A PFBR is specified by a collection of probability distributions β_S, where S is a coalition of voters, over the set of alternatives. We formally call β_S a *probabilistic ballot*. If $\overline{k}-\underline{k}=1$, then a RPFBR reduces to a PFBR. However, if $\overline{k}-\underline{k}>1$, then a RPFBR requires an additional restriction on the probabilistic ballots: each voter i has a fixed probability weight ε_i such that the probability of the right interval R according to β_S is the total weight $\sum_{i\in S}\varepsilon_i$ of the voters in S and that of the left interval L is the total weight $\sum_{i\notin S}\varepsilon_i$ of the voters outside S.

We use our characterization result to investigate the the following classical decomposability question on these domains: Can every unanimous and strategy-proof RSCF be decomposed as a mixture of finitely many deterministic unanimous and strategy-proof social choice functions? Decomposability holds on several well-known domains, for instance the complete domain (Gibbard, 1977) and the single-peaked domains (Peters et al., 2014; Pycia and Ünver, 2015). Thus, decomposability holds for the cases when $\overline{k}-\underline{k}=1$ or $\overline{k}-\underline{k}=m-1$. Surprisingly, it does *not* hold for any intermediate values of $\overline{k}$ and $\underline{k}$. In other words, randomization non-trivially expands the scope for designing strategy-proof mechanisms. We identify a necessary and sufficient condition for decomposability under an additional assumption of anonymity, which requires the RSCF be non-sensitive to the identities of voters (see Theorem 2). We further observe that non-decomposable RPFBRs dominate almost all decomposable RPFBRs in recognizing social compromises.

Finally, we formally demonstrate the salience of hybrid domains. We consider *connected* domains, where connectedness is a property of a graph that is induced by the domain. Essentially, connectedness ensures the existence of a path from one preference to another by a sequence of specific preference switches. Connected domains have been used extensively in the literature on strategic social choice (e.g. Monjardet, 2009; Sato, 2013; Puppe, 2018). According to Theorem 3, every connected domain that satisfies the weak no-restoration property of Sato (2013) and includes two completely reversed preferences must be a hybrid domain over which the RPFBR characterization still holds. An important feature of this result is that the condition on the domain does not specify an underlying structure of single-peakedness or threshold alternatives. These are derived endogenously from our hypotheses.

The paper is organized as follows. Section 1.1 reviews the literature, while Section 2 sets out the model and definitions. Section 3 and 4 introduce hybrid preferences and RPFBRs, respectively. Section 5 presents the main characterization result as well as the result on decomposability. Section 6 provides an axiomatic justification for hybrid domains.

1.1 Relationship with the Literature

The analysis of strategy-proof deterministic social choice functions on single-peaked domains was initiated by Moulin (1980) and developed further by Barberà et al. (1993), Ching (1997) and Weymark (2011). In the deterministic setting, Nehring and Puppe (2007), Chatterji et al. (2013), Reffgen (2015), Chatterji and Massó (2018), Achuthankutty and Roy (2018) and Bonifacio and Massó (2019) analyze the structure of unanimous and strategy-proof social choice functions on domains closely related to single-peakedness.

The structure of unanimous and strategy-proof RSCFs on single-peaked domains was first studied by Ehlers et al. (2002). They considered the case where the set of alternatives is an interval in the real line and characterized the unanimous and strategy-proof RSCFs in terms of probabilistic fixed ballot rules. Recently, Roy and Sadhukhan (2018) strengthen the characterization result on a single-peaked domain which does not require maximal cardinality. Characterizations of unanimous and strategy-proof RSCFs as convex combinations of counterpart deterministic social choice functions were provided by Peters et al. (2014) and Pycia and Ünver (2015).

Recently, Peters et al. (2019) have considered the case where the set of alternatives is endowed with a graph structure. Single-peakedness is defined w.r.t. such graphs as in Demange (1982) and Chatterji et al. (2013). Peters et al. (2019) investigate the structure of unanimous and strategy-proof RSCFs. Their characterization result (Theorem 5.6 of Peters et al. (2019)) implies our Theorem 1 for a special graph structure. However, the extension of our result in Theorem 3 is more general than their result since we do not assume a prespecified graph over the set of alternatives. In particular, our result covers many domains that are excluded by theirs. Finally, we emphasize that the motivation, formulation, and proof techniques in the two papers are completely different.

2 Preliminaries

Let $A = \{a_1, a_2, \dots, a_m\}$ be a finite set of alternatives with $m \geq 3$. Let $N = \{1, 2, \dots, n\}$ be a finite set of voters with $n \geq 2$. Each voter i has a preference ordering P_i (i.e., a complete, transitive and antisymmetric binary relation) over the alternatives. We interpret $a_s P_i a_t$ as "a_s is strictly preferred to a_t according to P_i". For each $1 \leq k \leq m$, $r_k(P_i)$ denotes the kth ranked alternative in P_i. We use the following notational convention: $P_i = (a_k\, a_s\, a_t \cdots)$ refers to a preference ordering where a_k is first-ranked, a_s is second-ranked, and a_t is third-ranked, while the rest of the rankings in P_i are arbitrary.

We denote the set of all preference orderings by $\mathbb{P}$, which we call **the complete domain**. A domain $\mathbb{D}$ is a subset of $\mathbb{P}$. We say that two distinct preferences $P_i, P_i' \in \mathbb{D}$ are **adjacent**, denoted $P_i \sim P_i'$, if there exist $a_s, a_t \in A$ such that (i) $r_k(P_i) = r_{k+1}(P_i') = a_s$ and $r_k(P_i') = r_{k+1}(P_i) = a_t$ for some $1 \leq k \leq m-1$, and (ii) $r_l(P_i) = r_l(P_i')$ for all $l \notin \{k, k+1\}$. In other words, alternatives a_s and a_t are consecutively ranked in both P_i and P_i' and are swapped between the two preferences, while the ordering of all remaining alternatives is unchanged. In this case, we say alternatives a_s and a_t are *locally switched* between P_i and P_i'. Given distinct $P_i, P_i' \in \mathbb{D}$, a sequence of preferences $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ is called a **path** connecting P_i and P_i' if $P_i^1 = P_i$, $P_i^t = P_i'$ and $P_i^k \sim P_i^{k+1}$ for all $k = 1, \dots, t-1$. Two preferences P_i, P_i' are **completely reversed** if for all $a_s, a_t \in A$, we have $[a_s P_i a_t] \Leftrightarrow [a_t P_i' a_s]$.

A domain $\mathbb{D}$ is **minimally rich** if for each $a_k \in A$, there exists a preference $P_i \in \mathbb{D}$ such that $r_1(P_i) = a_k$. Throughout the paper, we assume the domain in question is minimally rich. A preference profile is an n-tuple of preferences, i.e., $P = (P_1, P_2, \dots, P_n) = (P_i, P_{-i}) \in \mathbb{D}^n$.

Let $\Delta(A)$ denote the space of all lotteries over A. An element $\lambda \in \Delta(A)$ is a lottery or a probability distribution over A, where $\lambda(a_k)$ denotes the probability received by alternative a_k. For notational convenience, we let $\boldsymbol{e}_{a_k}$ denote the degenerate lottery where alternative a_k receives probability one. A **Random Social Choice Function** (or RSCF) is a map $\varphi : \mathbb{D}^n \to \Delta(A)$ which associates each preference profile to a lottery. Let $\varphi_{a_k}(P)$ denote the probability assigned to a_k by φ at the preference profile P. If a RSCF selects a degenerate lottery at every preference profile, it is called a **Deterministic Social Choice Function** (or DSCF). More formally, a DSCF is a mapping $f : \mathbb{D}^n \to A$.

In this paper, we impose two basic axioms on RSCFs: unanimity and strategy-proofness. A RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ is **unanimous** if for all $P \in \mathbb{D}^n$ and $a_k \in A$, $[r_1(P_i) = a_k$ for all $i \in$

$N] \Rightarrow [\varphi(P) = e_{a_k}]$. We adopt the first-order stochastic dominance notion of strategy-proofness proposed by Gibbard (1977). This requires the lottery from truthtelling stochastically dominate the lottery obtained by any misrepresentation by any voter at any possible profile of other voters' preferences. Formally, a RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ is **strategy-proof** if for all $i \in N, P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, $\varphi(P_i, P_{-i})$ stochastically dominates $\varphi(P'_i, P_{-i})$ according to P_i, i.e., $\sum_{t=1}^{k} \varphi_{r_t(P_i)}(P_i, P_{-i}) \geq \sum_{t=1}^{k} \varphi_{r_t(P_i)}(P'_i, P_{-i})$ for all $k = 1, \ldots, m$. In addition, a RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ satisfies the **tops-only property** if for all $P, P' \in \mathbb{D}^n$, we have $[r_1(P_i) = r_1(P'_i)$ for all $i \in N] \Rightarrow [\varphi(P) = \varphi(P')]$. In other words, the tops-only property ensures that the social outcome at each preference profile depends only on the first-ranked alternatives at that preference profile.

An important class of unanimous and strategy-proof RSCFs is the class of random dictatorships. Formally, a RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ is a **random dictatorship** if there exists a "dictatorial coefficient" $\varepsilon_i \geq 0$ for each $i \in N$ with $\sum_{i \in N} \varepsilon_i = 1$ such that $\varphi(P) = \sum_{i \in N} \varepsilon_i \boldsymbol{e}_{r_1(P_i)}$ for all $P \in \mathbb{D}^n$. In particular, if $\varepsilon_i = 1$ for some $i \in N$, the random dictatorship degenerates to a *dictatorship*. It is evident that every random dictatorship is a *mixture* (equivalently, a convex combination) of dictatorships. Gibbard (1977) showed that every unanimous and strategy-proof RSCF on the complete domain $\mathbb{P}$ is a random dictatorship.

An important restricted domain is the domain of single-peaked preferences (Black, 1948; Moulin, 1980). A preference P_i is **single-peaked** w.r.t. a prior order $\prec$ over A if for all $a_s, a_t \in A$, we have $[a_s \prec a_t \prec r_1(P_i)$ or $r_1(P_i) \prec a_t \prec a_s] \Rightarrow [a_t P_i a_s]$. Let $\mathbb{D}_{\prec}$ denote **the single-peaked domain** which contains all single-peaked preferences w.r.t. $\prec$. Whenever we do not mention the prior order $\prec$, we assume that it is the *natural order*, $a_{k-1} \prec a_k$ for all $k = 2, \ldots, m$. For notational convenience, let $a_s \preceq a_t$ denote either $a_s \prec a_t$ or $a_s = a_t$, and $[a_s, a_t] = \{a_k \in A : a_s \preceq a_k \preceq a_t\}$ denote the set of alternatives between a_s and a_t on $\prec$, provided $a_s \preceq a_t$. Note that the single-peaked domain $\mathbb{D}_{\prec}$ contains a pair of completely reversed preferences $\underline{P}_i = (a_1 \cdots a_{k-1}\, a_k \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_k\, a_{k-1} \cdots a_1)$.[1]

3 Hybrid Domains

Hybrid domains are supersets of single-peaked domains where single-peakedness may be violated over a subset of alternatives that lie in the "middle" of the alternative set. We use the term "hybrid" to emphasize the coexistence of such violations, with other features of single-peakedness.

Consider the natural order $\prec$ over A. Fix two alternatives $a_{\underline{k}}$ and $a_{\overline{k}}$ with $a_{\underline{k}} \prec a_{\overline{k}}$, which we refer to as the *left threshold* and the *right threshold*, respectively. We define three subsets of A using these two thresholds: **Left Interval** $L = [a_1, a_{\underline{k}}]$, **Right Interval** $R = [a_{\overline{k}}, a_m]$

[1]The notation $\underline{P}_i = (a_1 \cdots a_{k-1}\, a_k \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_k\, a_{k-1} \cdots a_1)$ denote the preferences $\underline{P}_i$ and $\overline{P}_i$ where $a_{k-1}\underline{P}_i a_k$ and $a_k \overline{P}_i a_{k-1}$ for all $k = 2, \ldots, m$.

and **Middle Interval** $M = [a_{\underline{k}}, a_{\overline{k}}]$.[2] In what follows, we present the structure of preference orderings in a hybrid domain.

Consider a preference ordering whose peak belongs to M (see the first diagram of Figure 1). The ranking of the alternatives in M is completely arbitrary, while the ranking of the alternatives in L and R follows the conventional single-peakedness restriction w.r.t. $\prec$. In other words, the only restriction that the preference ordering satisfies is that preference declines as one moves from $a_{\underline{k}}$ towards a_1, or from $a_{\overline{k}}$ towards a_m. Note that this allows some alternatives in L or R be ranked above some alternatives in M.

Next, consider a preference ordering whose peak belongs to L (see the second diagram of Figure 1). The ranking of the alternatives in L and R follows single-peakedness w.r.t. $\prec$. In other words, preference declines as one moves from the peak towards a_1 or $a_{\underline{k}}$, or moves from $a_{\overline{k}}$ towards a_m. Furthermore, all alternatives in M are ranked below $a_{\underline{k}}$ in an arbitrary manner. Notice that an alternative in R may be ranked above some alternative in M, but can never be ranked above $a_{\underline{k}}$. For a preference ordering with the peak in R, the restriction is analogous.

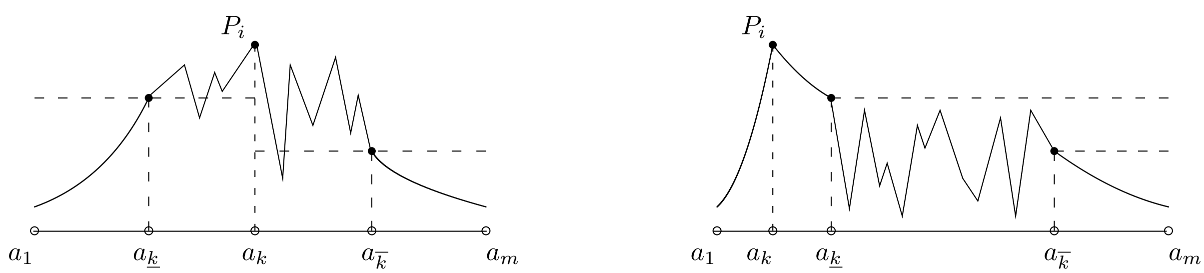

Figure 1: A graphic illustration of hybrid preference orderings

The formal definition of hybrid domains is given below.

DEFINITION 1 *Let $\prec$ be the natural order over A and let $1 \leq \underline{k} < \overline{k} \leq m$. A preference P_i is called $(\underline{k}, \overline{k})$-**hybrid** if the following two conditions are satisfied:*

(i) *For all $a_r, a_s \in L$ or $a_r, a_s \in R$, $[a_r \prec a_s \prec r_1(P_i)$ or $r_1(P_i) \prec a_s \prec a_r] \Rightarrow [a_s P_i a_r]$.*

(ii) *$[r_1(P_i) \in L] \Rightarrow [a_{\underline{k}} P_i a_r$ for all $a_r \in M$ with $a_r \neq a_{\underline{k}}]$ and*
$[r_1(P_i) \in R] \Rightarrow [a_{\overline{k}} P_i a_s$ for all $a_s \in M$ with $a_s \neq a_{\overline{k}}]$.

Let $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ denote **the** $(\underline{k}, \overline{k})$**-hybrid domain** which contains *all* $(\underline{k}, \overline{k})$-hybrid preference orderings. Note that $\mathbb{D}_{\prec} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for all $1 \leq \underline{k} < \overline{k} \leq m$, and $\mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}') \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for all $\underline{k} \leq \underline{k}' < \overline{k}' \leq \overline{k}$.

Now, we explain the relation of hybrid domains with five important preference domains studied in the literature.

[2]Note that $L \cap M = \{a_{\underline{k}}\}$, $R \cap M = \{a_{\overline{k}}\}$ and $L \cap R = \emptyset$.

The single-peaked domain: Consider a hybrid domain $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $\overline{k} - \underline{k} = 1$. This means $M = \{a_{\underline{k}}, a_{\overline{k}}\}$ and $L \cup R = A$. Then, conditions (i) and (ii) of Definition 1 boil down to the single-peakedness restriction (see the first diagram of Figure 2), and consequently, $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ coincides with the single-peaked domain $\mathbb{D}_{\prec}$.

The complete domain: Consider the hybrid domain $\mathbb{D}(\underline{k}, \overline{k})$ with $\overline{k} - \underline{k} = m - 1$ (equivalently, $\underline{k} = 1$ and $\overline{k} = m$). This means $L = \{a_{\underline{k}}\}$, $R = \{a_{\overline{k}}\}$, and $M = A$. Then, both the conditions of Definition 1 become vacuous. In other words, no restriction is imposed on the preference orderings (see the second diagram of Figure 2) in $\mathbb{D}_{\mathrm{H}}(1, m)$, and consequently, $\mathbb{D}_{\mathrm{H}}(1, m)$ becomes the complete domain $\mathbb{P}$.

Figure 2: Two hybrid preferences with $\overline{k} - \underline{k} = 1$ and $\overline{k} - \underline{k} = m - 1$

Multiple single-peaked domains: Hybrid domains generalize the notion of multiple single-peaked domains introduced by Reffgen (2015). Let $\Omega = \{\prec_r\}_{r=1}^{s}$, $s \geq 2$ be a collection of linear orders over A. For each order $\prec_r$ in Ω, let the single-peaked domain w.r.t. $\prec_r$ be denoted by $\mathbb{D}_{\prec_r}$. Then, the union $\mathbb{D}_{\Omega} = \cup_{r=1}^{s} \mathbb{D}_{\prec_r}$ is called *the multiple single-peaked domain w.r.t.* Ω.[3]

One can first identify the maximum common left part L_{Ω} of all orders $\{\prec_r\}_{r=1}^{s}$ over A, and relabel all alternatives of $L_{\Omega} = \{a_1, \ldots, a_{\underline{k}}\}$ (if $L_{\Omega} \neq \emptyset$), i.e., for all orders $\prec_r$ in Ω, after relabeling, either $a_1 \prec_r \cdots \prec_r a_{\underline{k}} \prec_r a_p$ for all $a_p \in A \backslash L_{\Omega}$, or $a_p \prec_r a_{\underline{k}} \prec_r \cdots \prec_r a_1$ for all $a_p \in A \backslash L_{\Omega}$ holds. Second, one can symmetrically identify and relabel the maximum common right part $R_{\Omega} = \{a_{\overline{k}}, \ldots, a_m\} \subseteq A \backslash L_{\Omega}$ of all orders $\{\prec_r\}_{r=1}^{s}$ over A (if $R_{\Omega} \neq \emptyset$) and finally arbitrarily relabel all remaining alternatives as $a_{\underline{k}+1}, \ldots, a_{\overline{k}+1}$. We correspondingly relabel all alternatives in the preferences of $\mathbb{D}_{\Omega}$. Then, after setting $a_{\underline{k}}$ and $a_{\overline{k}}$ as two thresholds, it is clear that each preference ordering in $\mathbb{D}_{\Omega}$ is $(\underline{k}, \overline{k})$-hybrid.[4] Usually, $\mathbb{D}_{\Omega}$ is "strictly" contained in $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. This will be illustrated in the following example.

Note that by definition, a multiple single-peaked domain cannot be a single-peaked domain, whereas a hybrid domain can be single-peaked for a suitable choice of thresholds (when $\overline{k} - \underline{k} = 1$).

[3]If two orders $\prec_1$ and $\prec_2$ are completely reversed, the two single-peaked domains $\mathbb{D}_{\prec_1}$ and $\mathbb{D}_{\prec_2}$ become identical. Therefore, we assume that there is no pair of orders in Ω that are completely reversed.

[4]As Ω contains at least two orders and no pair of orders are completely reversed, it must be the case that $\overline{k} - \underline{k} > 1$ when $L_{\Omega} \neq \emptyset$ and $R_{\Omega} \neq \emptyset$. If $L_{\Omega} = \emptyset$ and $R_{\Omega} \neq \emptyset$, then $\mathbb{D}_{\Omega}$ is $(1, \overline{k})$-hybrid, while if $L_{\Omega} \neq \emptyset$ and $R_{\Omega} = \emptyset$, then $\mathbb{D}_{\Omega}$ is $(\underline{k}, m)$-hybrid. If both L_{Ω} and R_{Ω} are empty sets, then $\mathbb{D}_{\Omega} \subseteq \mathbb{P} = \mathbb{D}_{\mathrm{H}}(1, m)$ and $\mathbb{D}_{\Omega} \not\subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for any other $\underline{k}$ and $\overline{k}$.

Multidimensional single-peaked domains in voting under constraints: We provide an example to show that hybrid preferences arise from a model of voting under constraints studied in Barberà et al. (1995).

Let $X = X_1 \times X_2$, $X_1 = \{1, 2, 3, 4, 5\}$ and $X_2 = \{1, 2, 3\}$, where both X_1 and X_2 are ordered according to the natural order, denoted by $<_1$ and $<_2$. A preference P_i, with $r_1(P_i) = x$, is *multidimensional single-peaked* over X w.r.t. $<_1$ and $<_2$ if for all $y, z \in X$, we have $[z_k \leq_k y_k \leq_k x_k$ or $x_k \leq_k y_k \leq_k z_k$ for both $k = 1, 2] \Rightarrow [yP_iz]$. Meanwhile, let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\} \subset X$ be the set of *feasible* alternatives, which are depicted by the black nodes in Figure 3 below.

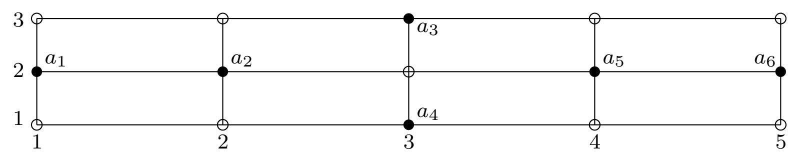

Figure 3: The Cartesian product of $<_1$ and $<_2$

Note that in a multidimensional single-peaked preference, (i) if a_1 is first-ranked, then a_2 must be second-ranked within A, and a_5 is preferred to a_6; if a_2 is first-ranked, then a_5 is preferred to a_6, and (ii) if a_3 is first-ranked, then a_2 is better than a_1, and a_5 is better than a_6. Analogous preference restrictions over the ranking of feasible alternatives are observed for multidimensional single-peaked preferences with peaks a_6, a_5 and a_4. These two observations coincide with the two preference restrictions in the definition of the $(2, 5)$-hybrid domain $\mathbb{D}_{\mathrm{H}}(2, 5)$ if we rearrange all feasible alternatives according to the natural order $\prec$. In conclusion, when we restrict attention to all multidimensional single-peaked preferences whose peaks are feasible, the domain of induced preferences over the feasible alternatives is identical to $\mathbb{D}_{\mathrm{H}}(2, 5)$.

We may alternatively extract the two linear orders $\prec_1 = (a_1a_2a_3a_4a_5a_6)$ and $\prec_2 = (a_1a_2a_4a_3a_5a_6)$ over feasible alternatives from Figure 3, and induce the multiple single-peaked domain $\mathbb{D}_{\prec_1} \cup \mathbb{D}_{\prec_2}$. Notice that $\mathbb{D}_{\prec_1} \cup \mathbb{D}_{\prec_2}$ is strictly contained in $\mathbb{D}_{\mathrm{H}}(2, 5)$. For instance, a_3 and a_4 are always ranked above a_5 and a_6 in every preference of $\mathbb{D}_{\prec_1} \cup \mathbb{D}_{\prec_2}$ that has peak a_1, whereas we can identify a particular multidimensional single-peaked preference with peak a_1 that induces the preference ordering over feasible alternatives as $(a_1a_2a_5a_6a_3a_4)$.

This illustrates the additional flexibility that a hybrid domain affords, and may be useful for formulations (for example, political economy or public goods location models) that seek to reduce a model where the underlying issues are multidimensional, to one where the preference restriction is generated via a one dimensional order over alternatives.

Semi-single-peaked domains: The notion of semi-single-peaked domains was introduced by Chatterji et al. (2013). Consider the natural order $\prec$ and fix *one* threshold alternative. The semi-single-peakedness restriction on a preference requires that (i) the usual single-peakedness restriction prevail in the interval between the peak and the threshold, and (ii) each alternative located beyond the threshold be ranked below the threshold.

One can extend the semi-single-peakedness notion by adding more thresholds and requiring preferences to be semi-single-peaked w.r.t. each threshold alternative. In particular, suppose that there are two distinct thresholds $a_{\underline{k}}$ and $a_{\overline{k}}$ with $a_{\underline{k}} \prec a_{\overline{k}}$. Consider a preference P_i with $a_{\underline{k}} \preceq r_1(P_i) \preceq a_{\overline{k}}$. If P_i is $(\underline{k}, \overline{k})$-hybrid, then the usual single-peakedness restriction prevails on the left and right intervals, and no restriction is imposed on the ranking of the alternatives in the middle interval (see the first diagram of Figure 4). On the contrary, if P_i is semi-single-peaked w.r.t. both $a_{\underline{k}}$ and $a_{\overline{k}}$, then the single-peakedness restriction prevails on the middle interval but fails on the left and right intervals (see the second diagram of Figure 4). Thus, the notions of hybrid preferences and semi-single-peaked preferences are not entirely compatible with each other.

Chatterji et al. (2013) show that under a mild domain richness condition, semi-single-peakedness is necessary and sufficient for the existence of a unanimous, anonymous, tops-only and strategy-proof DSCF.[5] This, in particular, implies that when $\overline{k} - \underline{k} > 1$, the $(\underline{k}, \overline{k})$-hybrid domain cannot admit such a well-behaved strategy-proof DSCF.

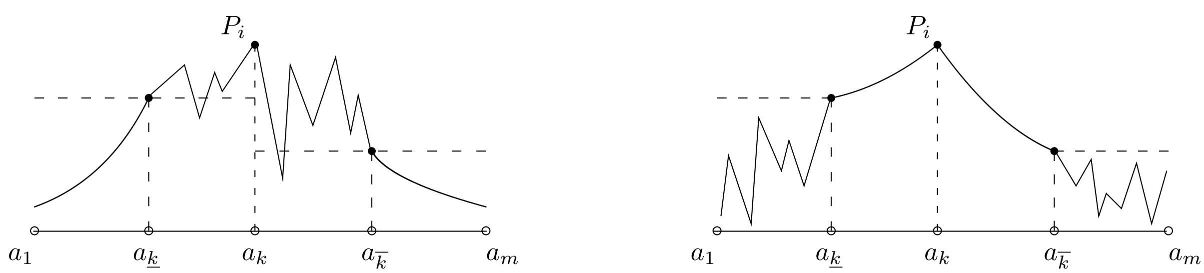

Figure 4: A hybrid preference *v.s.* a semi-single-peaked preference

4 Restricted Probabilistic Fixed Ballot Rules

In this section, we introduce the notion of Restricted Probabilistic Fixed Ballot Rules (or RPFBRs). Ehlers et al. (2002) introduce the notion of Probabilistic Fixed Ballot Rules (or PFBR); RPFBRs are special cases of these rules.

A PFBR φ is based on a collection of parameters $(\beta_S)_{S \subseteq N}$, called **probabilistic ballots**. Each probabilistic ballot β_S, which is associated to the coalition $S \subseteq N$, is a probability distribution on A satisfying the following two properties.

- **Ballot unanimity**: β_N assigns probability 1 to a_m, and $\beta_\emptyset$ assigns probability 1 to a_1.

[5] Recently, Chatterji and Massó (2018) introduce the semilattice single-peaked domain which significantly generalizes semi-single-peakedness, and Bonifacio and Massó (2019) characterize all unanimous, anonymous, tops-only and strategy-proof DSCFs on the semilattice single-peaked domain.

- **Monotonicity**: probabilities according to β_S move towards right as S gets bigger, i.e., $\beta_S([a_k, a_m]) \leq \beta_T([a_k, a_m])$ for all $S \subset T$ and all $a_k \in A$.[6]

For an example, suppose that there are two agents $\{1, 2\}$ and four alternatives $\{a_1, a_2, a_3, a_4\}$. Then, a choice of probabilistic ballots could be $\beta_\emptyset = (1, 0, 0, 0)$, $\beta_{\{1\}} = (0.5, 0.2, 0.1, 0.2)$, $\beta_{\{2\}} = (0.4, 0.3, 0.2, 0.1)$ and $\beta_N = (0, 0, 0, 1)$. Here, we denote by (x, y, w, z) a probability distribution where a_1, a_2, a_3 and a_4 receive probabilities x, y, w and z, respectively.

A PFBR φ w.r.t. a collection of probabilistic ballots $(\beta_S)_{S \subseteq N}$ works as follows. For each $1 \leq k \leq m$, let $S(k, P) = \{i \in N : a_k \preceq r_1(P_i)\}$ be the set of agents whose peaks are not to the left of a_k. Consider an arbitrary preference profile P and an arbitrary alternative a_k. We induce the probabilities $\beta_{S(k,P)}([a_k, a_m])$ and $\beta_{S(k+1,P)}([a_{k+1}, a_m])$. If $a_k = a_m$, then set $\beta_{S(m+1,P)}([a_{m+1}, a_m]) = 0$. The probability of the alternative a_k selected at the preference profile P is defined as the difference between these two probabilities, i.e., $\varphi_{a_k}(P) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m])$.[7] For an example, consider the PFBR φ w.r.t. the parameters presented in the predecessor paragraph. Consider a preference profile $P = (P_1, P_2)$ where $r_1(P_1) = a_2$ and $r_1(P_2) = a_4$. Then, we calculate

$$\varphi_{a_1}(P) = \beta_{S(1,P)}([a_1, a_4]) - \beta_{S(2,P)}([a_2, a_4]) = \beta_N([a_1, a_4]) - \beta_N([a_2, a_4]) = 0,$$
$$\varphi_{a_2}(P) = \beta_{S(2,P)}([a_2, a_4]) - \beta_{S(3,P)}([a_3, a_4]) = \beta_N([a_2, a_4]) - \beta_{\{2\}}([a_3, a_4]) = 1 - 0.3 = 0.7,$$
$$\varphi_{a_3}(P) = \beta_{S(3,P)}([a_3, a_4]) - \beta_{S(4,P)}([a_4, a_4]) = \beta_{\{2\}}([a_3, a_4]) - \beta_{\{2\}}([a_4, a_4]) = 0.3 - 0.1 = 0.2, \text{ and}$$
$$\varphi_{a_4}(P) = \beta_{S(4,P)}([a_4, a_4]) - 0 = \beta_{\{2\}}([a_4, a_4]) = 0.1.$$

Clearly, the PFBR satisfies the tops-only property.

It is worth mentioning that the probabilistic ballot β_S for a coalition $S \subseteq N$ represents the outcome of φ at the "boundary profile" where agents in S have the preference $\overline{P}_i = (a_m \cdots a_k\, a_{k-1} \cdots a_1)$, while the others have the preference $\underline{P}_i = (a_1 \cdots a_{k-1}\, a_k \cdots a_m)$. For ease of presentation, we call such a preference profile a *S-boundary profile*.[8] If a PFBR φ is unanimous, then it follows that $\beta_\emptyset$ assigns probability 1 to a_1 and β_N assigns probability 1 to a_m, which in turn implies ballot unanimity. In what follows, we argue that if φ is strategy-proof, then $(\beta_S)_{S \subseteq N}$ must be monotonic. Consider a proper subset $S \subset N$ and $i \in N \setminus S$. Let P and P' be the S-boundary and $S \cup \{i\}$-boundary profiles, respectively. In other words, only agent i changes her preference $\overline{P}_i$ in the $S \cup \{i\}$-boundary profile to $\underline{P}_i$. Strategy-proofness of φ implies that the probability of each upper contour set of $\overline{P}_i$ is weakly increased from $\varphi(P)$ to $\varphi(P')$. Since the interval $[a_k, a_m]$ coincides with the upper contour set

[6] For a subset B of A, we denote the probability of B according to β_S by $\beta_S(B)$.

[7] Since $S(k+1, P) \subseteq S(k, P)$ and $[a_{k+1}, a_m] \subset [a_k, a_m]$, monotonicity ensures $\varphi_{a_k}(P) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m]) \geq 0$. Moreover, note that $\sum_{k=1}^m \varphi_{a_k}(P) = \sum_{k=1}^m \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m]) = \beta_{S(1,P)}([a_1, a_m]) = 1$. Therefore, $\varphi(P) \in \Delta(A)$ and φ is a well defined RSCF.

[8] Note that for every $S \subseteq N$, there is a unique S-boundary profile.

of a_k at $\overline{P}_i$, it follows that $\beta_S([a_k, a_m]) \leq \beta_{S\cup\{i\}}([a_k, a_m])$. Monotonicity of $(\beta_S)_{S\subseteq N}$ follows from the repeated application of this argument.

Note that the outcome of a PFBR at any preference profile is uniquely determined by its outcomes at boundary profiles. It is shown in Ehlers et al. (2002) that every PFBR is unanimous and strategy-proof on the single-peaked domain. Thus, unanimity and strategy-proofness of a PFBR at every preference profile can be ensured by imposing those only on the boundary profiles.

The deterministic versions of PFBRs can be obtained by additionally requiring the probabilistic ballots be degenerate, i.e., $\beta_S(a_k) \in \{0, 1\}$ for all $S \subseteq N$ and $a_k \in A$. These DSCFs were introduced by Moulin (1980); we refer to these as *Fixed Ballot Rules* (or FBRs).[9] Moulin (1980) showed that a DSCF is unanimous, tops-only and strategy-proof on the single-peaked domain if and only if it is an FBR. It can be easily verified that an arbitrary mixture of FBRs is unanimous and strategy-proof on the single-peaked domain, and is a PFBR. Theorem 3 of Peters et al. (2014) and Theorem 5 of Pycia and Ünver (2015) prove that the converse is also true.

Below, we present the formal definition of PFBRs.

Definition 2 *A RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ is called a* ***Probabilistic Fixed Ballot Rule*** *(or* ***PFBR****) if there exists a collection of probabilistic ballots $(\beta_S)_{S\subseteq N}$ satisfying ballot unanimity and monotonicity such that for all $P \in \mathbb{D}^n$ and $a_k \in A$, we have*

$$\varphi_{a_k}(P) = \beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m]),$$

where $\beta_{S(m+1,P)}([a_{m+1}, a_m]) = 0$.

We are now ready to present the notion of RPFBRs. The structure of a $(\underline{k}, \overline{k})$-RPFBR depends on the values of $\underline{k}$ and $\overline{k}$. If $\overline{k} - \underline{k} = 1$, then the $(\underline{k}, \overline{k})$-RPFBR is the same as a PFBR. However, if $\overline{k} - \underline{k} > 1$, then the $(\underline{k}, \overline{k})$-RPFBR is a PFBR whose probabilistic ballots satisfy the following additional restriction: for each agent $i \in N$, there is a "conditional dictatorial coefficient" $\varepsilon_i \geq 0$ with $\sum_{i\in N} \varepsilon_i = 1$ such that for all $S \subseteq N$, $\beta_S([a_{\overline{k}}, a_m]) = \sum_{i\in S} \varepsilon_i$ and $\beta_S([a_1, a_{\underline{k}}]) = \sum_{i\in N\setminus S} \varepsilon_i$. Note that this, in particular, means that *no* β_S assigns positive probability to an alternative that lies (strictly) between $a_{\underline{k}}$ and $a_{\overline{k}}$, i.e., $\beta_S(a_k) = 0$ for all $S \subseteq N$ and $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$. In what follows, we present an example of a RPFBR.

Example 1 Let $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4, a_5\}$. Take $\underline{k} = 2$ and $\overline{k} = 4$, and consider the $(2, 4)$-hybrid domain $\mathbb{D}_{\mathrm{H}}(2, 4)$. Let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$. Consider the 8 probabilistic ballots in Table 1, where both ballot unanimity and monotonicity can be easily verified. Note

[9] Moulin (1980) called these Augmented Median Voter Rules, while Barberà et al. (1993) called these Generalized Median Voter Schemes. For an FBR φ, the subtraction form in Definition 2 can be simplified to a max-min form (see Definition 10.3 in Nisan et al., 2007). Moulin (1980) originally defined an augmented median voter rule in the min-max form which can be equivalently translated to a max-min form.

that they also satisfy the property that $\beta_S([a_4, a_5]) = \sum_{i\in S} \varepsilon_i$ and $\beta_S([a_1, a_2]) = \sum_{i\in N\setminus S} \varepsilon_i$ for all $S \subseteq N$. Therefore, the PFBR w.r.t. these probabilistic ballots is a $(2, 4)$-RPFBR. $\square$

	$\beta_\emptyset$	$\beta_{\{1\}}$	$\beta_{\{2\}}$	$\beta_{\{3\}}$	$\beta_{\{1,2\}}$	$\beta_{\{1,3\}}$	$\beta_{\{2,3\}}$	β_N
a_1	1	1/3	1/3	1/3	1/3	1/3	1/3	0
a_2	0	1/3	1/3	1/3	0	0	0	0
a_3	0	0	0	0	0	0	0	0
a_4	0	0	0	0	1/3	1/3	1/3	0
a_5	0	1/3	1/3	1/3	1/3	1/3	1/3	1

Table 1: The probabilistic ballots $(\beta_S)_{S\subseteq N}$

Below, we present a formal definition of RPFBRs.

Definition **3** *Let $1 \le \underline{k} < \overline{k} \le m$. A PFBR φ w.r.t. probabilistic ballots $(\beta_S)_{S\subseteq N}$ is called a $(\underline{k}, \overline{k})$-**Restricted Probabilistic Fixed Ballots Rule** (or $(\underline{k}, \overline{k})$-**RPFBR**) if $\overline{k} - \underline{k} > 1$ implies that for each $i \in N$, there exists $\varepsilon_i \ge 0$ with $\sum_{i\in N} \varepsilon_i = 1$ such that for all $S \subseteq N$, $\beta_S([a_{\overline{k}}, a_m]) = \sum_{i\in S} \varepsilon_i$ and $\beta_S([a_1, a_{\underline{k}}]) = \sum_{i\in N\setminus S} \varepsilon_i$.*

It is worth mentioning that when $\overline{k}-\underline{k} > 1$, at the preference profiles where all peaks are in the middle interval $M = [a_{\underline{k}}, a_{\overline{k}}]$, a $(\underline{k}, \overline{k})$-RPFBR behaves like a random dictatorship where each agent i's dictatorial coefficient is ε_i. More formally, if φ is a $(\underline{k}, \overline{k})$-RPFBR, then $\varphi(P) = \sum_{i\in N} \varepsilon_i \, \boldsymbol{e}_{r_1(P_i)}$ for all preference profile P such that $r_1(P_i) \in [a_{\underline{k}}, a_{\overline{k}}]$ for all $i \in N$. Therefore, in the extreme case where $\underline{k} = 1$ and $\overline{k} = m$, the $(1, m)$-RPFBR reduces to a random dictatorship. For ease of presentation, we call the condition satisfied by the probabilistic ballots $(\beta_S)_{S\subseteq N}$ in Definition 3 the **constrained random-dictatorship condition**.

5 A Characterization of Unanimous and Strategy-proof RSCFs on Hybrid Domains

In this section, we provide a characterization of unanimous and strategy-proof RSCFs on hybrid domains. Theorem 1 says that a RSCF φ is unanimous and strategy-proof on the $(\underline{k}, \overline{k})$-hybrid domain if and only if it is a $(\underline{k}, \overline{k})$-RPFBR. Ehlers et al. (2002) consider the case of continuum of alternatives (for instance, the interval $[0, 1]$) and show that a RSCF is unanimous and strategy-proof on the single-peaked domain if and only if it is a PFBR. Since when $\overline{k} - \underline{k} = 1$, the $(\underline{k}, \overline{k})$-hybrid domain boils down to the single-peaked domain and the $(\underline{k}, \overline{k})$-RPFBR becomes a PFBR, Theorem 1 implies their result in the case of finite alternatives.

Theorem **1** *Let $1 \le \underline{k} < \overline{k} \le m$. A RSCF $\varphi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is unanimous and strategy-proof if and only if it is a $(\underline{k}, \overline{k})$-RPFBR.*

We present a formal proof of Theorem 1 in Appendix A. Here, we provide an intuitive explanation. The "if part" of the theorem, i.e., the fact that every RPFBR on a hybrid domain is unanimous and strategy-proof, intuitively follows from the observations: (i) the $(\underline{k}, \overline{k})$-hybrid domain satisfies single-peakedness on the intervals $[a_1, a_{\underline{k}}]$ and $[a_{\overline{k}}, a_m]$, and (ii) the RPFBR behaves like a PFBR over these intervals. For the "only-if part", we first show how in a two-voter setting a PFBR fails to satisfy strategy-proofness on the $(\underline{k}, \overline{k})$-hybrid domain if any of its probabilistic ballots assigns a positive probability to some alternative in the interval $[a_{\underline{k}+1}, a_{\overline{k}-1}]$.

Consider the model with two agents. Suppose that some probabilistic ballot of φ, say $\beta_{\{2\}}$, assigns a strictly positive probability to some alternative $a_k \in [a_{\underline{k}+1}, a_{\overline{k}-1}]$. First, by the definition of the $(\underline{k}, \overline{k})$-hybrid domain, there is a preference where a_1 is the first-ranked alternative and $a_{\overline{k}}$ is preferred to a_k. Correspondingly, consider a preference profile where agent 1 has such a preference and the first-ranked alternative of agent 2 is $a_{\overline{k}}$. By the definition of PFBR, the probability of a_k at this profile equals $\beta_{\{2\}}(a_k)$, which is strictly positive by our assumption. However, using unanimity agent 1 can manipulate by misreporting a preference that has $a_{\overline{k}}$ as the first-ranked alternative.[10]

An important point to note is that the aforementioned argument only indicates that a PFBR which is strategy-proof on the $(\underline{k}, \overline{k})$-hybrid domain is a $(\underline{k}, \overline{k})$-RPFBR. In order to complete the verification of the "only-if part", a crucial step in the proof of Theorem 1 is to show that every unanimous and strategy-proof RSCF on the hybrid domain is some PFBR.

5.1 Decomposability of anonymous RPFBRs

In this section, we investigate the decomposability property of RSCFs. We say that a unanimous and strategy-proof RSCF is *decomposable* if it can be expressed as a mixture (equivalently, a convex combination) of finitely many unanimous and strategy-proof DSCFs. Formally, a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ is **decomposable** if there exist finitely many unanimous and strategy-proof DSCFs $f^k : \mathbb{D}^n \to A$, $k = 1, \ldots, q$ and weights $\alpha^1, \ldots, \alpha^q > 0$ with $\sum_{k=1}^{q} \alpha^k = 1$, such that $\varphi(P) = \sum_{k=1}^{q} \alpha^k \boldsymbol{e}_{f^k(P)}$ for all $P \in \mathbb{D}^n$.

Decomposability is an important property of RSCFs and has been widely investigated in a large class of domains (e.g., Gibbard, 1977; Peters et al., 2014; Pycia and Ünver, 2015; Gaurav et al., 2017). As mentioned earlier, when $\overline{k} - \underline{k} = 1$, the $(\underline{k}, \overline{k})$-hybrid domain coincides with the single-peaked domain, and the $(\underline{k}, \overline{k})$-RPFBR becomes a PRBR. It is shown in Peters et al. (2014) and Pycia and Ünver (2015) that every PFBR is a mixture of their deterministic counterparts. In the other extreme case where $\overline{k} - \underline{k} = m - 1$, every $(\underline{k}, \overline{k})$-RPFBR becomes

[10] Note that the strength of unanimity reduces considerably as the number of agents increases. So, the argument presented above does not extend straightforwardly to the case of arbitrary number of agents. We provide these details in our formal proof.

a random dictatorship, which is, by definition, a mixture of dictatorships. Thus, a $(\underline{k}, \overline{k})$-RPFBR is decomposable when $\overline{k}-\underline{k}=1$ or $\overline{k}-\underline{k}=m-1$. However, for the remaining cases $1<\overline{k}-\underline{k}<m-1$, we observe that decomposability fails in some RPFBRs (see Example 2 below). A complete characterization of decomposable RPFBRs in the general case, appears to be difficult.[11] In this section, we investigate the decomposition of *anonymous* RPFBRs for the remaining cases $1<\overline{k}-\underline{k}<m-1$.[12]

Formally, a RSCF $\varphi: \mathbb{D}^n \to \Delta(A)$ is **anonymous** if for all permutations $\sigma: N \to N$ and profile $(P_1, \dots, P_n) \in \mathbb{D}^n$, we have $\varphi(P_1, \dots, P_n) = \varphi(P_{\sigma(1)}, \dots, P_{\sigma(n)})$. More specifically, one can easily verify that a $(\underline{k}, \overline{k})$-RPFBR $\varphi: \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ is anonymous if and only if all probabilistic ballots are invariant to the size of coalitions, i.e., for all nonempty $S, S' \subseteq N$ with $|S| = |S'|$, we have $\beta_S = \beta_{S'}$. For instance, recall the probabilistic ballots in Table 1. The corresponding RPFBR is anonymous.

We next provide a necessary and sufficient condition, *per-capita monotonicity*, for the decomposition of all anonymous RPFBRs. Consider a $(\underline{k}, \overline{k})$-RPFBR φ w.r.t. the probabilistic ballots $(\beta_S)_{S \subseteq N}$. Recall the left interval $L = [a_1, a_{\underline{k}}]$ and the right interval $R = [a_{\overline{k}}, a_m]$. This condition imposes two restrictions that strengthen the monotonicity requirement between the probabilistic ballots of two nonempty coalitions $S, S' \subset N$ with $S \subset S'$. The first restriction says that the average probability, $\frac{\beta_{S'}}{|S'|}$, of any interval $[a_t, a_m]$ in R for the coalition S' is at least as much as the counterpart for the coalition S, i.e., for all $a_t \in R$, $\frac{\beta_{S'}([a_t,a_m])}{|S'|} \geq \frac{\beta_{S}([a_t,a_m])}{|S|}$. The second restriction is the analogue of the first one. Here, we consider any interval $[a_1, a_s]$ in L and the respective complements of S' and S. Recall from the constrained random-dictatorship condition that the probabilities $\beta_{N\setminus S'}([a_1, a_s])$ and $\beta_{N\setminus S}([a_1, a_s])$ are related to the conditional dictatorial coefficients of voters in S' and S respectively. We require here that the average probability $\frac{\beta_{N\setminus S'}([a_1,a_s])}{|S'|}$ be weakly higher than $\frac{\beta_{N\setminus S}([a_1,a_s])}{|S|}$.

Definition 4 *A RPFBR $\varphi: \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ satisfies **per-capita monotonicity** if, for all nonempty $S \subset S' \subset N$, $a_t \in R$ and $a_s \in L$, we have*

$$\frac{\beta_{S'}([a_t, a_m])}{|S'|} \geq \frac{\beta_{J}([a_t, a_m])}{|S|} \text{ and } \frac{\beta_{N\setminus S'}([a_1, a_s])}{|S'|} \geq \frac{\beta_{N\setminus S}([a_1, a_s])}{|S|}.$$

Our main theorem of this section says that per-capita monotonicity is both necessary and sufficient for the decomposability of anonymous RPFBRs. The proof of Theorem 2 is contained in Appendix B.

[11]In the general case, we show that every two-voter $(\underline{k}, \overline{k})$-RPFBR is unconditionally decomposable, and provide a necessary condition for the decomposition of a $(\underline{k}, \overline{k})$-RPFBR with more than two voters. These results are available in the Supplementary Material to this paper.

[12]It is important to mention that in the case $1<\overline{k}-\underline{k}<m-1$, Theorem 1 implies that there exists no anonymous, unanimous and strategy-proof DSCFs on the $(\underline{k}, \overline{k})$-hybrid domain. Therefore, the decomposition of an anonymous $(\underline{k}, \overline{k})$-RPFBR (if it exists) is a mixture of finitely many unanimous and strategy-proof DSCFs, all of which violate anonymity.

Theorem 2 *Let* $1 < \overline{k} - \underline{k} < m-1$. *Then, an anonymous* $(\underline{k}, \overline{k})$*-RPFBR* $\varphi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ *is decomposable if and only if it satisfies per-capita monotonicity.*

To conclude this section, we observe using an example that a non-decomposable RPFBR may dominate a decomposable one in terms of admitting "social compromises". This indicates that randomization enhances possibilities for economic design in a meaningful way, since the non-decomposable RPFBRs we characterize may allow for more flexibility in assigning probabilities to compromise alternatives.

Example 2 Let $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4, a_5\}$. Recall the $(2, 4)$-hybrid domain $\mathbb{D}_{\mathrm{H}}(2, 4)$ and the probabilistic ballots $(\beta_S)_{S \subseteq N}$ in Table 1. It is easy to verify that $(\beta_S)_{S \subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition when the conditional dictatorial coefficients are $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{3}$, and are invariant to the size of coalitions. Therefore, the PFBR $\varphi : \left[\mathbb{D}_{\mathrm{H}}(2, 4)\right]^3 \to \Delta(A)$ w.r.t. $(\beta_S)_{S \subseteq N}$ is an anonymous $(2, 4)$-RPFBR. Furthermore, it can be verified that φ is not decomposable as it fails to satisfy per-capita monotonicity, i.e., $\frac{\beta_{\{1,2\}}(a_5)}{|\{1,2\}|} = \frac{1}{6} < \frac{1}{3} = \frac{\beta_{\{1\}}(a_5)}{|\{1\}|}$.

Consider now a random dictatorship, $\phi(P) = \sum_{i \in N} \frac{1}{3} \boldsymbol{e}_{r_1(P_i)}$ for all $P \in [\mathbb{D}_{\mathrm{H}}(2, 4)]^3$. We show that φ dominates ϕ in admitting "social compromises". Formally, we recognize an alternative a_k as a social compromise alternative at a preference profile P if some voters disagree on the peaks, and all voters agree on a_k as the second best.

First, as a random dictatorship, ϕ at every preference profile assigns zero probability to any alternative that is not first-ranked in any voter's preference, and therefore admits no social compromise. However, we notice that for *all* profile $P \in [\mathbb{D}_{\mathrm{H}}(2, 4)]^3$, whenever a social compromise alternative a_k arises, the probability of a_k in φ is at least as much as that in ϕ, i.e., $\varphi_{a_k}(P) \geq \phi_{a_k}(P)$,[13] and at *some* profile $P \in [\mathbb{D}_{\mathrm{H}}(2, 4)]^3$ which has a social compromise alternative, φ assigns strictly higher probability to the social compromise alternative than ϕ. Indeed, consider a preference profile $P \in [\mathbb{D}_{\mathrm{H}}(2, 4)]^3$ such that $r_1(P_1) = a_3 \neq a_5 = r_1(P_2) = r_1(P_3)$ and $r_2(P_1) = r_2(P_2) = r_2(P_3) = a_4$; we have $\varphi_{a_4}(P) = \frac{1}{3} > 0 = \phi_{a_4}(P)$. Thus a non-decomposable anonymous RPFBR may dominate a decomposable one in terms of admitting social compromises.[14] □

[13] It is possible that both φ and ϕ assign zero probability to the social compromise alternative at the same preference profile. For instance, consider a preference profile $P \in [\mathbb{D}_{\mathrm{H}}(2, 4)]^3$ such that $r_1(P_1) = a_2 \neq a_4 = r_1(P_2) = r_1(P_3)$ and $r_2(P_1) = r_2(P_2) = r_2(P_3) = a_3$. Then, $\varphi_{a_3}(P) = \phi_{a_3}(P) = 0$.

[14] In the Supplementary Material to this paper, we provide a general analysis on social compromises which (i) characterizes all RPFBRs that are dominated in admitting social compromises, and (ii) identifies a condition under which an anonymous decomposable RPFBR is dominated in admitting social compromises by an anonymous non-decomposable RPFBR.

6 The Salience of Hybrid Domains and RPFBRs

Our purpose in this section is two-fold. We first propose an axiomatic justification of hybrid domains. Specifically, we show that any domain that satisfies certain "connectedness" and "richness" properties must be contained in a hybrid domain (say the $(\overline{k}, \underline{k})$-hybrid domain). Secondly, and more importantly, the set of unanimous and strategy-proof RSCFs on this domain is precisely the set of unanimous and strategy-proof RSCFs on the $(\overline{k}, \underline{k})$-hybrid domain, i.e., $(\overline{k}, \underline{k})$-RPFBRs. Thus, the set of unanimous and strategy-proof RSCFs on such a domain is the set of RPFBRs associated with the minimal hybrid domain in which it is embedded.

Recall the notions of adjacency and path introduced in the beginning of Section 2. A domain is said *connected* if every pair of two distinct preferences is connected by a path in the domain. We restrict attention to a class of connected domains which in addition satisfies the *weak no-restoration property* of Sato (2013).

Definition 5 *A domain $\mathbb{D}$ satisfies the* ***weak no-restoration property*** *if for all distinct $P_i, P_i' \in \mathbb{D}$ and $a_p, a_q \in A$, there exists a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i' such that we have*

$$[a_p P_i^{k^*} a_q \text{ and } a_q P_i^{k^*+1} a_p \text{ for some } 1 \leq k^* < t] \\ \Rightarrow [a_p P_i^k a_q \text{ for all } k = 1, \ldots, k^*, \text{ and } a_q P_i^l a_p \text{ for all } l = k^* + 1, \ldots, t].$$

Evidently, the weak no-restoration property implies connectedness, and suggests that according to each pair of alternatives a_p and a_q, one path can be constructed in the domain to reconcile the difference of P_i and P_i' shortly in the manner that the relative ranking of a_p and a_q is switched for *at most* once on the path. In particular, if a_p and a_q are identically ranked in P_i and P_i', then their relative ranking does not change along the path.

Proposition 3.2 of Sato (2013) shows that the weak no-restoration property is necessary for all DSCFs which only forbid misrepresentations of preferences that are adjacent to the sincere one, to retain strategy-proofness. The weak no-restoration property is satisfied by many important voting domains in the literature, e.g., the complete domain, the single-peaked domain and some multiple single-peaked domains, and also covers our hybrid domains (see the proof of Fact 1 in Appendix D).

Our last result establishes two features of domains that satisfy the weak no-restoration property and include two completely reversed preferences. The first is that every such domain is a subset of some hybrid domain. The second is that every unanimous and strategy-proof RSCF on such a domain is a RPFBR. The proof Theorem 3 is available in Appendix C.

Theorem 3 *Let domain $\mathbb{D}$ satisfy the weak no-restoration property and contain two completely reversed preferences. Then, there exist $1 \leq \underline{k} < \overline{k} \leq m$ such that $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ and $\mathbb{D} \nsubseteq \mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$ where $\underline{k}' > \underline{k}$ or $\overline{k}' < \overline{k}$. Moreover, a RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ is unanimous and strategy-proof if and only if it is a $(\underline{k}, \overline{k})$-RPFBR, where $\underline{k}$ and $\overline{k}$ are as described above.*

References

Achuthankutty, G. and S. Roy (2018): "Strategy-proof rules on partially single-peaked domains," *working paper, Indian Statistical Institute (Kolkata).*

Bade, S. (2019): "Matching with single-peaked preferences," *Journal of Economic Theory*, 180, 81–99.

Barberà, S., F. Gul, and E. Stacchetti (1993): "Generalized median voter schemes and committees," *Journal of Economic Theory*, 61, 262–289.

Barberà, S., J. Massó, and A. Neme (1995): "Voting under constraints," *Journal of Economic Theory*, 76, 298–321.

Black, D. (1948): "On the rationale of group decision-making," *Journal of Political Economy*, 56, 23.

Bochet, O. and S. Gordon (2012): "Priorities in the location of multiple public facilities," *Games and Economic Behavior*, 74, 52–67.

Bonifacio, A. G. and J. Massó (2019): "On strategy-proofness and semilattice single-peakedness," *working paper, Universitat Autònoma de Barcelona.*

Brandl, F., F. Brandt, and G. Seedig, H. (2016): "Consistent probabilistic social choice," *Econometrica*, 84, 1839–1880.

Chatterji, S. and J. Massó (2018): "On strategy-proofness and the salience of single-peakedness," *International Economic Review*, 59, 163–189.

Chatterji, S., R. Sanver, and A. Sen (2013): "On domains that admit well-behaved strategy-proof social choice functions," *Journal of Economic Theory*, 148, 1050–1073.

Chatterji, S., A. Sen, and H. Zeng (2014): "Random dictatorship domains," *Games and Economic Behavior*, 86, 212–236.

Chatterji, S. and H. Zeng (2018): "On random social choice functions with the tops-only property," *Games and Economic Behavior*, 109, 413–435.

Ching, S. (2018): "Strategy-proofness and "Median Voters"," *International Journal of Game Theory*, 26(4), 473–490.

Cho, W. (2016): "Incentive properties for ordinal mechanisms," *Games and Economic Behavior*, 95, 168–177.

Demange, G. (1982): "Single-peaked orders on a tree," *Mathematical Social Sciences*, 3(4), 389–396.

Ehlers, L., H. Peters, and T. Storcken (2002): "Strategy-proof probabilistic decision schemes for one-dimensional single-peaked preferences," *Journal of Economic Theory*, 105, 408–434.

Gaurav, A., J. Picot, and A. Sen (2017): "The decomposition of strategy-proof random social choice functions on dichotomous domains," *Mathematical Social Sciences*, 90, 28–34.

Gibbard, A. (1973): "Manipulation of voting schemes: a general result," *Econometrica*, 587–601.

——— (1977): "Manipulation of schemes that mix voting with chance," *Econometrica*, 665–681.

Monjardet, B. (2009): "Acyclic domains of linear orders: a survey," in *The Mathematics of Preference, Choice and Order*, Springer, 139–160.

Moulin, H. (1980): "On strategy-proofness and single peakedness," *Public Choice*, 35, 437–455.

Nehring, K. and C. Puppe (2007): "The structure of strategy-proof social choice: Part I: general characterization and possibility results on median spaces," *Journal of Economic Theory*, 135, 269–305.

Nisan, N., T. Roughgarden, E. Tardos, and V. Vazirani (2007): *Algorithmic Game Theory*, vol. 1, Cambridge University Press Cambridge.

Peters, H., S. Roy, and S. Sadhukhan (2019): "Unanimous and strategy-proof probabilistic rules for single-peaked preference profiles on graphs," *mimeo*.

Peters, H., S. Roy, A. Sen, and T. Storcken (2014): "Probabilistic strategy-proof rules over single-peaked domains," *Journal of Mathematical Economics*, 52, 123–127.

Puppe, C. (2018): "The single-peaked domain revisited: a simple global characterization," *Journal of Economic Theory*, 176, 55–80.

Pycia, M. and U. Ünver (2015): "Decomposing random mechanisms," *Journal of Mathematical Economics*, 61, 21–33.

Reffgen, A. (2015): "Strategy-proof social choice on multiple and multi-dimensional single-peaked domains," *Journal of Economic Theory*, 157, 349–383.

Roy, S. and S. Sadhukhan (2018): "A characterization of random min–max domains and its applications," *Economic Theory*, 1–20.

Sato, S. (2013): "A sufficient condition for the equivalence of strategy-proofness and nonmanipulability by preferences adjacent to the sincere one," *Journal of Economic Theory*, 148, 259–278.

Satterthwaite, M. A. (1975): "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory*, 10, 187–217.

Sprumont, Y. (1991): "The division problem with single-peaked preferences: a characterization of the uniform allocation rule," *Econometrica*, 509–519.

Weymark, J. (2011): "A unified approach to strategy-proofness for single-peaked preferences," *SERIEs: Journal of the Spanish Economic Association, Special Issue in Honor of Salvador Barberà, Part I: On Strategy-Proofness*, 2(4), 529–550.

Appendix

A Proof of Theorem 1

When $\overline{k} - \underline{k} = 1$, $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k}) = \mathbb{D}_{\prec}$, and then Theorem 1 follows from Theorem 4.1 and Proposition 5.2 of Ehlers et al. (2002). Henceforth, we assume $\overline{k} - \underline{k} > 1$.

(**Sufficiency part**) Let $\varphi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$ be a $(\underline{k}, \overline{k})$-RPFBR. First, ballot unanimity implies that φ is unanimous. We next show strategy-proofness of φ in two steps. In the first step, we introduce a notion weaker than strategy-proofness, *local strategy-proofness*, which only requires a RSCF be immune to the misrepresentation of preferences that are adjacent to the sincere one.[15] Fact 1 below shows that every locally strategy-proof RSCF on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ is strategy-proof. In the second step, we show that φ is locally strategy-proof.

Fact 1 *Every locally strategy-proof RSCF on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ is strategy-proof.*

By Theorem 1 of Cho (2018), to prove Fact 1, it suffices to show that $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ satisfies the *no-restoration property* of Sato (2013). Therefore, the verification of Fact 1 is independent of RPFBR φ, and for ease of presentation, is delegated to Appendix D.

Now, to complete the verification, we show that φ is locally strategy-proof. Fixing $i \in N$, $P_i, P_i' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $P_i \sim P_i'$ and $P_{-i} \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^{n-1}$, we show that $\varphi(P_i, P_{-i})$ stochastically dominates $\varphi(P_i', P_{-i})$ according to P_i. Let $r_1(P_i) = a_s$ and $r_1(P_i') = a_t$. Evidently, if $a_s = a_t$, the tops-only property implies $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$. Next, assume $a_s \neq a_t$. Then, $P_i \sim P_i'$ implies $r_1(P_i) = r_2(P_i') = a_s$, $r_1(P_i') = r_2(P_i) = a_t$ and $r_k(P_i) = r_k(P_i')$ for all $k \notin \{1, 2\}$. Thus, to show local strategy-proofness, it suffices to show the following condition:

$$\begin{aligned}&\varphi_{a_s}(P_i, P_{-i}) \geq \varphi_{a_s}(P_i', P_{-i}) \text{ or } \varphi_{a_t}(P_i, P_{-i}) \leq \varphi_{a_t}(P_i', P_{-i}), \text{ and}\\ &\varphi_{a_k}(P_i, P_{-i}) = \varphi_{a_k}(P_i', P_{-i}) \text{ for all } a_k \notin \{a_s, a_t\}.\end{aligned} \tag{\#}$$

By the definition of $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$, $P_i \sim P_i'$ implies one of the following three cases: (i) $a_s, a_t \in L$ and $a_t \in \{a_{s-1}, a_{s+1}\}$, (ii) $a_s, a_t \in R$ and $a_t \in \{a_{s-1}, a_{s+1}\}$, and (iii) $a_s, a_t \in M$. Note that the first two cases are symmetric. Therefore, we focus on cases (i) and (iii).

Claim 1: In case (i), condition (#) holds.

If $a_t = a_{s-1}$, then we know $S(s, (P_i, P_{-i})) \supset S(s, (P_i', P_{-i}))$ and $S(k, (P_i, P_{-i})) = S(k, (P_i', P_{-i}))$ for all $a_k \in A \backslash \{a_s\}$, and derive

$$\begin{aligned}\varphi_{a_s}(P_i, P_{-i}) &= \beta_{S(s,(P_i,P_{-i}))}([a_s, a_m]) - \beta_{S(s+1,(P_i,P_{-i}))}([a_{s+1}, a_m])\\ &\geq \beta_{S(s,(P_i',P_{-i}))}([a_s, a_m]) - \beta_{S(s+1,(P_i',P_{-i}))}([a_{s+1}, a_m]) \quad \text{by monotonicity}\\ &= \varphi_{a_s}(P_i', P_{-i}),\end{aligned}$$

and for all $a_k \notin \{a_{s-1}, a_s\}$,

$$\begin{aligned}\varphi_{a_k}(P_i, P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1}, a_m])\\ &= \beta_{S(k,(P_i',P_{-i}))}([a_k, a_m]) - \beta_{S(k+1,(P_i',P_{-i}))}([a_{k+1}, a_m]) = \varphi_{a_k}(P_i', P_{-i}).\end{aligned}$$

[15] Formally, a RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ is **locally strategy-proof** if for all $i \in N$, $P_i, P_i' \in \mathbb{D}$ with $P_i \sim P_i'$ and $P_{-i} \in \mathbb{D}^{n-1}$, $\varphi(P_i, P_{-i})$ stochastically dominates $\varphi(P_i', P_{-i})$ according to P_i.

If $a_t = a_{s+1}$, then we know $S(s+1,(P_i,P_{-i})) \subset S(s+1,(P'_i,P_{-i}))$ and $S(k,(P_i,P_{-i})) = S(k,(P'_i,P_{-i}))$ for all $a_k \in A \backslash \{a_{s+1}\}$, and derive

$$\begin{aligned}\varphi_{a_{s+1}}(P_i,P_{-i}) &= \beta_{S(s+1,(P_i,P_{-i}))9}([a_{s+1},a_m]) - \beta_{S(s+2,(P_i,P_{-i}))}([a_{s+2},a_m]) \\ &\leq \beta_{S(s+1,(P'_i,P_{-i}))}([a_{s+1},a_m]) - \beta_{S(s+2,(P'_i,P_{-i}))}([a_{s+2},a_m]) \quad \text{by monotonicity} \\ &= \varphi_{a_{s+1}}(P'_i,P_{-i}).\end{aligned}$$

and for all $a_k \notin \{a_s, a_{s+1}\}$,

$$\begin{aligned}\varphi_{a_k}(P_i,P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k,a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1},a_m]) \\ &= \beta_{S(k,(P'_i,P_{-i}))}([a_k,a_m]) - \beta_{S(k+1,(P'_i,P_{-i}))}([a_{k+1},a_m]) = \varphi_{a_k}(P'_i,P_{-i}).\end{aligned}$$

This completes the verification of the claim.

Claim 2: In case (iii), condition (#) holds.

We assume $a_t \prec a_s$. The verification related to the situation $a_s \prec a_t$ is symmetric, and we hence omit it. First, note that $S(a_k,(P_i,P_{-i})) = S(a_k,(P'_i,P_{-i}))$ for all $a_k \in A$ with $a_k \preceq a_t$ or $a_s \prec a_k$. Then, for each $a_k \in A$ with $a_k \prec a_t$ or $a_s \prec a_k$, we have

$$\begin{aligned}\varphi_{a_k}(P_i,P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k,a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1},a_m]) \\ &= \beta_{S(k,(P'_i,P_{-i}))}([a_k,a_m]) - \beta_{S(k+1,(P'_i,P_{-i}))}([a_{k+1},a_m]) = \varphi_{a_k}(P'_i,P_{-i}).\end{aligned}$$

Next, given $a_t \prec a_k \prec a_s$, we know $a_{\underline{k}} \prec a_k \prec a_{\overline{k}}$ and $a_{\underline{k}} \prec a_{k+1} \preceq a_{\overline{k}}$. Then, Definition 3 implies that for all $S \subseteq N$, $\beta_S([a_k,a_m]) = \sum_{j\in S}\varepsilon_j = \beta_S([a_{k+1},a_m])$. Moreover, note that $S(k,(P_i,P_{-i}))\backslash S(k+1,(P_i,P_{-i})) = \{j \in N\backslash\{i\} : r_1(P_j) = a_k\} = S(k,(P'_i,P_{-i}))\backslash S(k+1,(P'_i,P_{-i}))$. Therefore, we have

$$\begin{aligned}\varphi_{a_k}(P_i,P_{-i}) &= \beta_{S(k,(P_i,P_{-i}))}([a_k,a_m]) - \beta_{S(k+1,(P_i,P_{-i}))}([a_{k+1},a_m]) \\ &= \sum\nolimits_{j\in S(k,(P_i,P_{-i}))\backslash S(k+1,(P_i,P_{-i}))} \varepsilon_j \\ &= \sum\nolimits_{j\in S(k,(P'_i,P_{-i}))\backslash S(k+1,(P'_i,P_{-i}))} \varepsilon_j \\ &= \beta_{S(k,(P'_i,P_{-i}))}([a_k,a_m]) - \beta_{S(k+1,(P'_i,P_{-i}))}([a_{k+1},a_m]) = \varphi_{a_k}(P'_i,P_{-i}).\end{aligned}$$

Overall, we have $\varphi_{a_k}(P_i,P_{-i}) = \varphi_{a_k}(P'_i,P_{-i})$ for all $a_k \notin \{a_s,a_t\}$. Last, since $a_t \prec a_s$ implies $S(s,(P_i,P_{-i})) \supset S(s,(P'_i,P_{-i}))$ and $S(a_{s+1},(P_i,P_{-i})) = S(a_{s+1},(P'_i,P_{-i}))$, we have

$$\begin{aligned}\varphi_{a_s}(P_i,P_{-i}) &= \beta_{S(s,(P_i,P_{-i}))}([a_s,a_m]) - \beta_{S(s+1,(P_i,P_{-i}))}([a_{s+1},a_m]) \\ &\geq \beta_{S(s,(P'_i,P_{-i}))}([a_s,a_m]) - \beta_{S(s+1,(P'_i,P_{-i}))}([a_{s+1},a_m]) \quad \text{by monotonicity} \\ &= \varphi_{a_s}(P'_i,P_{-i}).\end{aligned}$$

This completes the verification of the claim.

Therefore, φ is locally strategy-proof, as required. This hence completes the verification of the sufficiency part of Theorem 1.

(**Necessity part**) Let $\varphi : [\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})]^n \to \Delta(A)$ be a unanimous and strategy-proof RSCF. Since $\mathbb{D}_{\prec} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})$, we can elicit a unanimous and strategy-proof RSCF $\phi : [\mathbb{D}_{\prec}]^n \to \Delta(A)$ such that $\phi(P) = \varphi(P)$ for all $P \in [\mathbb{D}_{\prec}]^n$. First, Theorem 3 of Peters et al. (2014) or Theorem 5 of Pycia and Ünver (2015) and Proposition 3 of Moulin (1980) together imply that ϕ is a mixture of finitely many FBRs. Then, it follows immediately that ϕ is a PFBR. Let $(\beta_S)_{S\subseteq N}$ be the probabilistic ballots of ϕ. Evidently, $(\beta_S)_{S\subseteq N}$ satisfies ballot unanimity and monotonicity. Next, by the proof of Fact 1 and Proposition 1 of Chatterji and Zeng (2018), we know that φ satisfies the tops-only property. Last, since both $\mathbb{D}_{\prec}$ and $\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})$ are minimally rich, the tops-only property of φ implies that φ is also a PFBR and inherits ϕ's probabilistic ballots $(\beta_S)_{S\subseteq N}$. Therefore, for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\right]^n$ and $a_k \in A$, we have $\varphi_{a_k}(P) = \beta_{S(k,P)}([a_k,a_m]) - \beta_{S(k+1,P)}([a_{k+1},a_m])$, where $\beta_{S(m+1,P)}([a_{m+1},a_m]) = 0$. To complete the proof, we show that φ is a $(\underline{k},\overline{k})$-RPFBR.

Let $\overline{\mathbb{D}} = \left\{P_i \in \mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k}) : r_1(P_i) \in M\right\}$ denote the subdomain of hybrid preferences whose peaks are in M. Since $|M| \geq 3$ and $\overline{\mathbb{D}}$ has no restriction on the ranking of alternatives in M, according to the random dictatorship characterization theorem of Gibbard (1977), we easily infer that there exists a "conditional dictatorial coefficient" $\varepsilon_i \geq 0$ for each $i \in N$ with $\sum_{i\in N}\varepsilon_i = 1$ such that $\varphi(P) = \sum_{i\in N}\varepsilon_i\, \boldsymbol{e}_{r_1(P_i)}$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\right]^n$ with $r_1(P_i) \in M$ for all $i \in N$.

Fix an arbitrary coalition $S \subseteq N$. We first show $\beta_S([a_{\overline{k}},a_m]) = \sum_{j\in S}\varepsilon_j$. We construct a profile $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\right]^n$ where every voter of S has the preference with the peak $a_{\overline{k}}$ and all other voters have the preference with the peak $a_{\underline{k}}$. Thus, $S = S(\overline{k},P)$ and $\varphi(P) = \sum_{j\in S}\varepsilon_j\, \boldsymbol{e}_{a_{\overline{k}}} + \sum_{j\in N\backslash S}\varepsilon_j\, \boldsymbol{e}_{a_{\underline{k}}}$. We then have $\beta_S([a_{\overline{k}},a_m]) = \beta_{S(\overline{k},P)}([a_{\overline{k}},a_m]) = \sum_{k=\overline{k}}^{m}\left[\beta_{S(k,P)}([a_k,a_m]{-}\beta_{S(k+1,P)}([a_{k+1},a_m])\right] = \sum_{k=\overline{k}}^{m}\varphi_{a_k}(P) = \varphi_{a_{\overline{k}}}(P^*) = \sum_{j\in S}\varepsilon_j$.

Last, we show $\beta_S([a_1,a_{\underline{k}}]) = \sum_{j\in N\backslash S}\varepsilon_j$. Since $\beta_S([a_1,a_{\underline{k}}]) = 1 - \beta_S([a_{\overline{k}},a_m]) - \beta_S([a_{\underline{k}+1},a_{\overline{k}-1}]) = \sum_{j\in N\backslash S}\varepsilon_j - \beta_S([a_{\underline{k}+1},a_{\overline{k}-1}])$, it suffices to show $\beta_S(a_k) = 0$ for all $a_k \in [a_{\underline{k}+1},a_{\overline{k}-1}]$. Given $a_{\underline{k}} \prec a_k \prec a_{\overline{k}}$, since $S(k,P) = S = S(k+1,P)$, we have $\beta_S(a_k) = \beta_S([a_k,a_m]) - \beta_S([a_{k+1},a_m]) = \beta_{S(k,P)}([a_k,a_m]) - \beta_{S(k+1,P)}([a_{k+1},a_m]) = \varphi_{a_k}(P) = 0$, as required. This completes the verification of the necessity part of Theorem 1.

B Proof of Theorem 2

We first show the sufficiency part of Theorem 2, and then turn to proving the necessity part. Before proceeding the proof, we formally introduce the deterministic version of a $(\underline{k},\overline{k})$-RPFBR, which we call a $(\underline{k},\overline{k})$-*Restricted Fixed Ballot Rule* (or $(\underline{k},\overline{k})$-*RFBR*).

Definition 6 *A DSCF* $f : \left[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\right]^n \to \Delta(A)$ *is called a* $(\underline{k},\overline{k})$-***Restricted Fixed Ballot Rule*** *(or* $(\underline{k},\overline{k})$-***RFBR****) if it is an FBR, i.e., there exists a collection of deterministic ballots* $(b_S)_{S\subseteq N}$ *satisfying ballot unanimity, i.e.,* $b_N = a_m$ *and* $b_\emptyset = a_1$*, and monotonicity, i.e.,* $[S \subset T \subseteq N] \Rightarrow [b_S \preceq b_T]$*, such that for all* $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\right]^n$*, we have* $f(P) = \max^{\prec}_{S\subseteq N}\left(\min^{\prec}_{j\in S}\left(r_1(P_j), b_S\right)\right)$*, and in addition,* $(b_S)_{S\subseteq N}$ *satisfy the* ***constrained dictatorship condition****, i.e.,* $\overline{k} - \underline{k} > 1$ *implies that there exists* $i \in N$ *such that* $[i \in S] \Rightarrow [b_S \in R]$ *and* $[i \notin S] \Rightarrow [b_S \in L]$*.*

(**Sufficiency part**) Fixing an anonymous $(\underline{k},\overline{k})$-RPFBR $\varphi : \left[\mathbb{D}_{\mathrm{H}}(\underline{k},\overline{k})\right]^n \to \Delta(A)$, assume that φ satisfy per-capita monotonicity. Let $(\beta_S)_{S\subseteq N}$ be the corresponding probabilistic ballots. By anonymity and the constrained random-dictatorship condition, $\beta_S = \beta_{S'}$ for all nonempty $S, S' \subseteq N$ with $|S| = |S'|$, and each voter has the conditional dictatorial coefficient $\frac{1}{n}$. We are going to decompose φ as a mixture of finitely many $(\underline{k},\overline{k})$-RFBRs.

We provide some new notation which will be repeatedly used in the proof. Given $S \subseteq N$, let $supp(\beta_S) = \{a_k \in A : \beta_S(a_k) > 0\}$ denote the support of β_S. Given $S \subseteq N$ with $S \neq \emptyset$ and $N\backslash S \neq \emptyset$, the constrained random-dictatorship condition implies $supp(\beta_S) \cap R \neq \emptyset$ and $supp(\beta_S) \cap L \neq \emptyset$. Hence, we define

$$\hat{b}_S^R = \min^{\prec}\left(supp(\beta_S) \cap R\right) \text{ and } \hat{b}_S^L = \max^{\prec}\left(supp(\beta_S) \cap L\right).$$

Evidently, $\hat{b}_S^L \prec \hat{b}_S^R$. Moreover, let $\hat{b}_N^R = a_m$ and let $\hat{b}_\emptyset^L = a_1$. It is evident that (i) $\beta_N(\hat{b}_N^R) = 1$ and $\beta_\emptyset(\hat{b}_\emptyset^L) = 1$, and (ii) for all nonempty $S \subset N$, $\beta_S(\hat{b}_S^R) > 0$, $\beta_S(\hat{b}_S^L) > 0$ and $\beta_S(a_k) = 0$ for all $a_k \in A$ with $\hat{b}_S^L \prec a_k \prec \hat{b}_S^R$. Note that anonymity of φ implies $\hat{b}_S^R = \hat{b}_{S'}^R$ and $\hat{b}_S^L = \hat{b}_{S'}^L$ for all nonempty $S, S' \subseteq N$ with $|S| = |S'|$.

Lemma 1 *For all nonempty $S \subset S' \subseteq N$, we have $\hat{b}_S^R \preceq \hat{b}_{S'}^R$.*

Proof: If $S' = N$, it is evident that $\hat{b}_S^R \preceq a_m = \hat{b}_{S'}^R$. Next, let $S' \subset N$. Suppose $\hat{b}_S^R \succ \hat{b}_{S'}^R$. We then have $\frac{\beta_{S'}([\hat{b}_S^R, a_m])}{|S'|} \leq \frac{\beta_{S'}([a_{\overline{k}}, a_m]) - \beta_{S'}(\hat{b}_{S'}^R)}{|S'|} < \frac{|S'|/n}{|S'|} = \frac{1}{n} = \frac{\beta_S([a_{\overline{k}}, a_m])}{|S|} = \frac{\beta_S([\hat{b}_S^R, a_m])}{|S|}$, which contradicts per-capita monotonicity. □

Lemma 2 *For all $S \subset S' \subset N$, we have $\hat{b}_S^L \preceq \hat{b}_{S'}^L$.*

Proof: If $S = \emptyset$, it is evident that $\hat{b}_S^L = a_1 \preceq \hat{b}_{S'}^L$. Next, let $S \neq \emptyset$. Suppose $\hat{b}_S^L \succ \hat{b}_{S'}^L$. For notational convenience, let $\hat{S} = N \backslash S$ and $\hat{S}' = N \backslash S'$. Thus, $\hat{S} \neq \emptyset$, $\hat{S}' \neq \emptyset$, $\hat{S} \supset \hat{S}'$ and $\hat{b}_{N\backslash\hat{S}}^L = \hat{b}_S^L \succ \hat{b}_{S'}^L = \hat{b}_{N\backslash\hat{S}'}^L$. We then have $\frac{\beta_{N\backslash\hat{S}}([a_1, \hat{b}_{N\backslash\hat{S}'}^L])}{|\hat{S}|} \leq \frac{\beta_{N\backslash\hat{S}}([a_1, a_{\underline{k}}]) - \beta_{N\backslash\hat{S}}(\hat{b}_{N\backslash\hat{S}}^L)}{|\hat{S}|} < \frac{|\hat{S}|/n}{|\hat{S}|} = \frac{1}{n} = \frac{\beta_{N\backslash\hat{S}'}([a_1, a_{\underline{k}}])}{|\hat{S}'|} = \frac{\beta_{N\backslash\hat{S}'}([a_1, \hat{b}_{N\backslash\hat{S}'}^L])}{|\hat{S}'|}$, which contradicts per-capita monotonicity. □

Given an arbitrary $i \in N$, we construct deterministic ballots $(b_S^i)_{S \subseteq N}$:

$$b_S^i = \hat{b}_S^R \text{ and } b_{N\backslash S}^i = \hat{b}_{N\backslash S}^L \text{ for all } S \subseteq N \text{ with } i \in S.$$

Since $b_N^i = \hat{b}_N^R = a_m$ and $b_\emptyset^i = \hat{b}_{N\backslash N}^L = \hat{b}_\emptyset^L = a_1$, ballot unanimity is satisfied. Next, we show monotonicity is satisfied. Fix $S \subset S' \subset N$. If $i \in S$, then $i \in S'$, and Lemma 1 implies $b_S^i = \hat{b}_S^R \preceq \hat{b}_{S'}^R = b_{S'}^i$. If $i \notin S'$, then $i \notin S$, and Lemma 2 implies $b_S^i = b_{N\backslash[N\backslash S]}^i = \hat{b}_{N\backslash[N\backslash S]}^L = \hat{b}_S^L \preceq \hat{b}_{S'}^L = \hat{b}_{N\backslash[N\backslash S']}^L = b_{N\backslash[N\backslash S']}^i = b_{S'}^i$. If $i \in S'\backslash S$, then $b_S^i \in L$ and $b_{S'}^i \in R$, and hence $b_S^i \prec b_{S'}^i$. Overall, $b_S^i \preceq b_{S'}^i$, as required. Correspondingly, let f^i be the FBR w.r.t. the deterministic ballots $(b_S^i)_{S \subseteq N}$. Moreover, given $S \subseteq N$, we have $[i \in S] \Rightarrow [b_S^i = \hat{b}_S^R \in R]$, and $[i \in N\backslash S] \Rightarrow \big[b_S^i = b_{N\backslash[N\backslash S]}^i = \hat{b}_{N\backslash[N\backslash S]}^L \in L\big]$ which meet the constrained dictatorship condition. Therefore, f^i is a $(\underline{k}, \overline{k})$-RFBR which is strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ by Theorem 1.

Next, we mix all $(\underline{k}, \overline{k})$-RFBRs $(f^i)_{i \in N}$ with the equal weight $\frac{1}{n}$, and construct the $(\underline{k}, \overline{k})$-RPFBR:

$$\phi(P) = \sum_{i \in N} \frac{1}{n} \boldsymbol{e}_{f^i(P)} \text{ for all } P \in \big[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\big]^n.$$

Let $(\gamma_S)_{S \subseteq N}$ denote the corresponding probabilistic ballots, which obviously satisfies ballot unanimity, monotonicity and the constrained random-dictatorship condition. We make two observations on $(\gamma_S)_{S \subseteq N}$: (1) $\gamma_S = \sum_{i \in N} \frac{1}{n} \boldsymbol{e}_{b_S^i} = \frac{1}{n} \sum_{i \in S} \boldsymbol{e}_{\hat{b}_S^R} + \frac{1}{n} \sum_{i \in N\backslash S} \boldsymbol{e}_{\hat{b}_S^L} = \frac{|S|}{n} \boldsymbol{e}_{\hat{b}_S^R} + \frac{n-|S|}{n} \boldsymbol{e}_{\hat{b}_S^L}$ for all $S \subseteq N$, and (2) ϕ is anonymous. Given distinct $S, S' \subseteq N$ with $|S| = |S'|$, anonymity of φ implies $e_{\hat{b}_S^R} = e_{\hat{b}_{S'}^R}$ and $e_{\hat{b}_S^L} = e_{\hat{b}_{S'}^L}$. We then have $\gamma_S = \frac{1}{n} \boldsymbol{e}_{\hat{b}_S^R} + \frac{n-|S|}{n} e_{\hat{b}_S^L} = \frac{1}{n} \boldsymbol{e}_{\hat{b}_{S'}^R} + \frac{n-|S'|}{n} e_{\hat{b}_{S'}^L} = \gamma_{S'}$, as required.

Furthermore, we identify the real number:

$$\alpha = \min_{S \subset N: S \neq \emptyset} \Big(\min \Big(\frac{\beta_S(\hat{b}_S^R)}{|S|}, \frac{\beta_S(\hat{b}_S^L)}{n - |S|} \Big) \Big).$$

Evidently, $0 < \alpha \leq \frac{\beta_S(\hat{b}_S^R)}{|S|}$ for all nonempty $S \subset N$. Moreover, given a nonempty $S \subset N$, the constrained random-dictatorship condition implies $\alpha \leq \frac{\beta_S(\hat{b}_S^R)}{|S|} \leq \frac{\sum_{j \in S} \frac{1}{n}}{|S|} = \frac{1}{n}$.

Lemma **3** *We have* $\alpha = \frac{1}{n}$ *if and only if* $|supp(\beta_S)| = 2$ *for all nonempty* $S \subset N$*. Moreover, if* $\alpha = \frac{1}{n}$*, then* $\varphi(P) = \phi(P)$ *for all* $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$*, and hence* φ *is decomposable.*

Proof: First, assume $|supp(\beta_S)| = 2$ for all nonempty $S \subset N$. Thus, for all nonempty $S \subset N$, we know $supp(\beta_S) = \{\hat{b}_S^R, \hat{b}_S^L\}$, $\beta_S(\hat{b}_S^R) = \frac{|S|}{n}$ and $\beta_S(\hat{b}_S^L) = \frac{n-|S|}{n}$ by the constrained random-dictatorship condition. Consequently, $\alpha = \frac{1}{n}$ by definition.

Next, assume $\alpha = \frac{1}{n}$. Fix an arbitrary nonempty $S \subset N$. By definition, $\frac{\beta_S(\hat{b}_S^R)}{|S|} \geq \alpha = \frac{1}{n}$ and $\frac{\beta_S(\hat{b}_S^L)}{n-|S|} \geq \alpha = \frac{1}{n}$. Meanwhile, the constrained random-dictatorship condition implies $\beta_S(\hat{b}_S^R) \leq \frac{|S|}{n}$ and $\beta_S(\hat{b}_S^L) \leq \frac{n-|S|}{n}$. Therefore, $\beta_S(\hat{b}_S^R) = \frac{|S|}{n}$ and $\beta_S(\hat{b}_S^L) = \frac{n-|S|}{n}$, and hence $|supp(\beta_S)| = 2$.

Furthermore, note that (i) $\beta_N = \boldsymbol{e}_{a_m} = \gamma_N$ and $\beta_\emptyset = \boldsymbol{e}_{a_m} = \gamma_\emptyset$, and (ii) for all nonempty $S \subset N$, $\beta_S = \frac{|S|}{n}\boldsymbol{e}_{\hat{b}_S^R} + \frac{n-|S|}{n}\boldsymbol{e}_{\hat{b}_S^L} = \sum_{i \in N} \frac{1}{n}\boldsymbol{e}_{b_S^i} = \gamma_S$. Therefore, $\varphi(P) = \phi(P)$ for all $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$, and hence, φ is decomposable. □

Henceforth, we assume $0 < \alpha < \frac{1}{n}$, and define the following

$$\hat{\beta}_S = \frac{\beta_S - \alpha n \gamma_S}{1 - \alpha n} = \frac{\beta_S - \alpha|S|\boldsymbol{e}_{\hat{b}_S^R} - \alpha(n - |S|)\boldsymbol{e}_{\hat{b}_S^L}}{1 - \alpha n} \text{ for all } S \subseteq N, \text{ and}$$

$$\psi(P) = \frac{\varphi(P) - \alpha n \phi(P)}{1 - \alpha n} \text{ for all } P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n.$$

It is easy to show that $\hat{\beta}_S \in \Delta(A)$ for each $S \subseteq N$. Hence, $(\hat{\beta}_S)_{S \subseteq N}$ are probabilistic ballots. It is evident that $(\hat{\beta}_S)_{S \subseteq N}$ satisfy ballot unanimity. Since both φ and ϕ are anonymous, ψ is also anonymous by construction. Next, let each voter have the conditional dictatorial coefficient $\frac{1}{n}$. We show that $(\hat{\beta}_S)_{S \subseteq N}$ satisfy the constrained random-dictatorship condition. Given nonempty $S \subset N$, we have $\hat{\beta}_S([a_{\overline{k}}, a_m]) = \frac{\beta_S([a_{\overline{k}}, a_m]) - \alpha|S|}{1 - \alpha n} = \frac{\frac{|S|}{n} - \alpha|S|}{1 - \alpha n} = \frac{|S|}{n}$ and $\hat{\beta}_S([a_1, a_{\underline{k}}]) = \frac{\beta_S([a_1, a_{\underline{k}}]) - \alpha(n - |S|)}{1 - \alpha n} = \frac{\frac{n-|S|}{n} - \alpha(n-|S|)}{1 - \alpha n} = \frac{n-|S|}{n}$, as required. Next, we show that ψ is a PFBR w.r.t. $(\hat{\beta}_S)_{S \subseteq N}$. Given $P \in \left[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\right]^n$ and $a_k \in A$, we have $\psi_{a_k}(P) = \frac{\varphi_{a_k}(P) - \alpha n \phi_{a_k}(P)}{1 - \alpha n} = \frac{\left(\beta_{S(k,P)}([a_k, a_m]) - \beta_{S(k+1,P)}([a_{k+1}, a_m])\right) - \alpha n\left(\gamma_{S(k,P)}([a_k, a_m]) - \gamma_{S(k+1,P)}([a_{k+1}, a_m])\right)}{1 - \alpha n} = \frac{\beta_{S(k,P)}([a_k, a_m]) - \alpha n \gamma_{S(k,P)}([a_k, a_m])}{1 - \alpha n} - \frac{\beta_{S(k+1,P)}([a_{k+1}, a_m]) - \alpha n \gamma_{S(k+1,P)}([a_{k+1}, a_m])}{1 - \alpha n} = \hat{\beta}_{S(k,P)}([a_k, a_m]) - \hat{\beta}_{S(k+1,P)}([a_{k+1}, a_m])$, as required.

The next two lemmas show that $(\hat{\beta}_S)_{S \subseteq N}$ satisfy monotonicity and ψ satisfies per-capita monotonicity respectively. Hence, we conclude that ψ is an anonymous $(\underline{k}, \overline{k})$-RPFBR and satisfies per-capita monotonicity.

Lemma **4** *Probabilistic ballots* $(\hat{\beta}_S)_{S \subseteq N}$ *satisfy monotonicity.*

Proof: Given $S \subset S' \subseteq N$, if $S = \emptyset$ or $S' = N$, monotonicity holds evidently. We hence assume $S \neq \emptyset$ and $S' \neq N$. We first identify $\hat{b}_S^L \preceq \hat{b}_{S'}^L \preceq a_{\underline{k}} \prec a_{\overline{k}} \preceq \hat{b}_S^R \preceq \hat{b}_{S'}^R$ by Lemmas 1 and 2. We assume w.l.o.g. that $|S'| = |S| + 1$. Given $a_t \in A$, we have five cases: (1) $\hat{b}_{S'}^R \prec a_t$, (2) $\hat{b}_S^R \prec a_t \preceq \hat{b}_{S'}^R$, (3) $\hat{b}_{S'}^L \prec a_t \preceq \hat{b}_S^R$, (4) $\hat{b}_S^L \prec a_t \preceq \hat{b}_{S'}^L$, and (5) $a_t \preceq \hat{b}_S^L$. We show $\hat{\beta}_{S'}([a_t, a_m]) \geq \hat{\beta}_S([a_t, a_m])$ in each case.

First, in either case (1) or case (5), $\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t, a_m]) - \beta_S([a_t, a_m])}{1 - \alpha n} \geq 0$.

In case (2), $\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t, a_m]) - \alpha|S'| - \beta_S([a_t, a_m])}{1 - \alpha n} \geq \frac{\frac{|S'|}{n} - \alpha(|S'|) - \left[\beta_S([\hat{b}_S^R, a_m]) - \beta_S(\hat{b}_S^R)\right]}{1 - \alpha n} = \frac{\frac{|S|+1}{n} - \alpha(|S|+1) - \frac{|S|}{N} + \beta_S(\hat{b}_S^R)}{1 - \alpha n} = \frac{(\frac{1}{n} - \alpha) + |S|\left(\frac{\beta_S(\hat{b}_S^R)}{|S|} - \alpha\right)}{1 - \alpha n} > 0$, where the first inequality follows from $\hat{b}_S^L \prec a_t \preceq \hat{b}_{S'}^L$ and the constrained random dictatorship condition of φ, and the last inequality follows from the hypothesis $\alpha < \frac{1}{n}$ and the definition of α.

In case (3), $\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t, a_m]) - \alpha|S'| - [\beta_S([a_t, a_m]) - \alpha|S|]}{1 - \alpha n} = \frac{\frac{|S'|}{n} - \alpha|S'| - (\frac{|S|}{n} - \alpha|S|)}{1 - \alpha n} = \frac{\frac{1}{n} - \alpha}{1 - \alpha n} > 0$.

Last, in case (4), we have $\hat{\beta}_{S'}([a_t, a_m]) - \hat{\beta}_S([a_t, a_m]) = \frac{\beta_{S'}([a_t,a_m])-\alpha|S'|-\alpha(n-|S'|)-\left[\beta_S([a_t,a_m])-\alpha|S|\right]}{1-\alpha n} = \frac{\frac{|S'|}{n}+\beta_{S'}([a_t,a_{\underline{k}}])-\alpha(n-|S|)-\left[\frac{|S|}{n}+\beta_S([a_t,a_{\underline{k}}])\right]}{1-\alpha n} \geq \frac{\frac{1}{n}+\beta_{S'}(\hat{b}^L_{S'})-\alpha(n-|S'|+1)}{1-\alpha n} = \frac{(\frac{1}{n}-\alpha)+(n-|S'|)\left(\frac{\beta_{S'}(\hat{b}^L_{S'})}{n-|S'|}-\alpha\right)}{1-\alpha n} > 0$, where the first inequality follows from $\hat{b}^R_S \prec a_t \preceq \hat{b}^R_{S'}$ and the constrained random dictatorship condition of φ, and the last inequality follows from the hypothesis $\alpha < \frac{1}{n}$ and the definition of α.

In conclusion, $\hat{\beta}_{S'}([a_t, a_m]) \geq \hat{\beta}_S([a_t, a_m])$ for all $a_t \in A$. □

Lemma **5** *RPFBR ψ satisfies per-capita monotonicity.*

Proof: Fixing $S \subset S' \subseteq N$, we have $\hat{b}^R_S \preceq \hat{b}^R_{S'}$ and $\hat{b}^L_{N\setminus S'} \preceq \hat{b}^L_{N\setminus S}$ by Lemmas 1 and 2. If $S = \emptyset$ or $S' = N$, per-capita monotonicity holds evidently. We hence assume $S \neq \emptyset$ and $S' \neq N$.

Given $a_t \in R$, either one of the three cases occurs: (1) $\hat{b}^R_{S'} \prec a_t$, (2) $\hat{b}^R_S \prec a_t \preceq \hat{b}^R_{S'}$, and (3) $a_t \preceq \hat{b}^R_S$.

In case (1), $\frac{\hat{\beta}_{S'}([a_t,a_m])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{S'}([a_t,a_m])}{|S'|} \geq \frac{1}{1-\alpha n}\frac{\beta_S([a_t,a_m])}{|S|} = \frac{\hat{\beta}_S([a_t,a_m])}{|S|}$, where the inequality follows from per-capita monotonicity of φ.

In case (2), $\frac{\hat{\beta}_{S'}([a_t,a_m])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{S'}([a_t,a_m])-\alpha|S'|}{|S'|} = \frac{1}{1-\alpha n}\frac{\frac{|S'|}{n}-\alpha|S'|}{|S'|} = \frac{1}{1-\alpha n}(\frac{1}{n} - \alpha) \geq \frac{1}{1-\alpha n}\left(\frac{1}{n} - \frac{\beta_S(\hat{b}^R_S)}{|S|}\right) = \frac{1}{1-\alpha n}\frac{\frac{|S|}{n}-\beta_S(\hat{b}^R_S)}{|S|} = \frac{1}{1-\alpha n}\frac{\beta_S([a_{\overline{k}},a_m])-\beta_S(\hat{b}^R_S)}{|S|} \geq \frac{1}{1-\alpha n}\frac{\beta_S([a_t,a_m])}{|S|} = \frac{\hat{\beta}_S([a_t,a_m])}{|S|}$, where the first inequality follows from the definition of α and the second inequality follows from $\hat{b}^R_S \prec a_t$.

Last, in case (3), $\frac{\hat{\beta}_{S'}([a_t,a_m])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{S'}([a_t,a_m])-\alpha|S'|}{|S'|} = \frac{1}{(1-\alpha n)}\left[\frac{\beta_{S'}([a_t,a_m])}{|S'|} - \alpha\right] \geq \frac{1}{(1-\alpha n)}\left[\frac{\beta_S([a_t,a_m])}{|S|} - \alpha\right] = \frac{1}{1-\alpha n}\frac{\beta_S([a_{\overline{k}},a_t])-\alpha|S|}{|S|} = \frac{\hat{\beta}_S([a_t,a_m])}{|S|}$, where the inequality follows from per-capita monotonicity of φ.

Symmetrically, given $a_s \in L$, either one of the three cases occurs: (i) $a_s \prec \hat{b}^L_{N\setminus S'}$, (ii) $\hat{b}^L_{N\setminus S'} \preceq a_s \prec \hat{b}^L_{N\setminus S}$, and (iii) $\hat{b}^L_{N\setminus S} \preceq a_s$.

In case (i), $\frac{\hat{\beta}_{N\setminus S'}([a_1,a_s])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S'}([a_1,a_s])}{|S'|} \geq \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_s])}{|S|} = \frac{\hat{\beta}_{N\setminus S}([a_1,a_s])}{|S|}$, where the inequality follows from per-capita monotonicity of φ.

In case (ii), $\frac{\hat{\beta}_{N\setminus S'}([a_1,a_s])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S'}([a_1,a_s])-\alpha[n-(n-|S'|)]}{|S'|} = \frac{1}{1-\alpha n}(\frac{1}{n} - \alpha) \geq \frac{1}{1-\alpha n}\left(\frac{1}{n} - \frac{\beta_{N\setminus S}(\hat{b}^L_{N\setminus S})}{n-(n-|S|)}\right) = \frac{1}{1-\alpha n}\frac{\frac{|S|}{n}-\beta_{N\setminus S}(\hat{b}^L_{N\setminus S})}{|S|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_{\underline{k}}])-\beta_{N\setminus S}(\hat{b}^L_{N\setminus S})}{|S|} \geq \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_s])}{|S|} = \frac{\hat{\beta}_{N\setminus S}([a_1,a_s])}{|S|}$, where the first inequality follows from the definition of α and the second inequality follows from $a_s \prec \hat{b}^L_{N\setminus S}$.

Last, in case (iii), $\frac{\hat{\beta}_{N\setminus S'}([a_1,a_s])}{|S'|} = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S'}([a_1,a_s])-\alpha[n-(n-|S'|)]}{|S'|} = \frac{1}{1-\alpha n}\left[\frac{\beta_{N\setminus S'}([a_1,a_s])}{|S'|} - \alpha\right] \geq \frac{1}{1-\alpha n}\left[\frac{\beta_{N\setminus S}([a_1,a_s])}{|S|} - \alpha\right] = \frac{1}{1-\alpha n}\frac{\beta_{N\setminus S}([a_1,a_s])-\alpha[n-(n-|S|)]}{|S|} = \frac{\hat{\beta}_{N\setminus S}([a_1,a_s])}{|S|}$, where the inequality follows from per-capita monotonicity of φ.

In conclusion, ψ satisfies per-capita monotonicity. □

The next lemma shows that the support of every φ's probabilistic ballot is refined by that of ψ, and the support of some φ's probabilistic ballot is strictly refined.

Lemma **6** *For all nonempty $S \subset N$, supp$(\hat{\beta}_S) \subseteq$ supp(β_S), and for some nonempty $S^* \subset N$, supp$(\hat{\beta}_{S^*}) \subset$ supp(β_{S^*}).*

Proof: Given nonempty $S \subset N$, since $\hat{\beta}_S = \frac{\beta_S-\alpha|S|\boldsymbol{e}_{\hat{b}^R_S}-\alpha(n-|S|)\boldsymbol{e}_{\hat{b}^L_S}}{1-\alpha n}$, it is true that *supp*$(\hat{\beta}_S) \subseteq$ *supp*(β_S). Next, by the definition of α, there exists a nonempty $S^* \subset N$ such that $\alpha = \frac{\beta_{S^*}(\hat{b}^R_{S^*})}{|S^*|}$ or $\alpha = \frac{\beta_{S^*}(\hat{b}^L_{S^*})}{n-|S^*|}$. Hence, either $\hat{\beta}_{S^*}(\hat{b}^R_{S^*}) = 0$ or $\hat{\beta}_{S^*}(\hat{b}^L_{S^*}) = 0$ holds. Therefore, *supp*$(\hat{\beta}_{S^*}) \subset$ *supp*(β_{S^*}). □

By spirit of Lemma 6, we call ψ the *refined* $(\underline{k}, \overline{k})$-RPFBR of φ. Now, we have $(\underline{k}, \overline{k})$-RFBRs $(f^i)_{i\in N}$ and an anonymous $(\underline{k}, \overline{k})$-RPFBR ψ which satisfies per-capita monotonicity. More importantly, the original $(\underline{k}, \overline{k})$-RPFBR φ can be specified as a mixture of $(f^i)_{i\in N}$ and ψ, i.e., $\varphi(P) = \alpha n\phi(P) + (1-\alpha n)\psi(P) = \alpha \sum_{i\in N} \boldsymbol{e}_{f^i(P)} + (1-\alpha n)\psi(P)$ for all $P \in \left[\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})\right]^n$.

Note that if we repeat the procedure above on the anonymous $(\underline{k}, \overline{k})$-RPFBR ψ, we can further decompose φ. Therefore, by repeatedly applying the procedure, we eventually can decompose φ as a mixture of finitely many $(\underline{k}, \overline{k})$-RFBRs, *provided that the procedure can terminate in finite steps*. In each step of the procedure, Lemma 6 implies that the support of the refined $(\underline{k}, \overline{k})$-RPFBR's probabilistic ballots strictly shrinks. Since the alternative set A is finite, it must be the case that after finite steps, the support of the refined $(\underline{k}, \overline{k})$-RPFBR's every probabilistic ballot becomes a binary set. Furthermore, by Lemma 3, the refined $(\underline{k}, \overline{k})$-RPFBR becomes a mixture of n $(\underline{k}, \overline{k})$-RFBRs. Hence, the procedure terminates, and we finish the decomposition of φ. This completes the verification of the sufficiency part of Theorem 2.

(**Necessity part**) Fix an anonymous decomposable $(\underline{k}, \overline{k})$-RPFBR $\varphi : \left[\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$. Let $(\beta_S)_{S\subseteq N}$ be the corresponding probabilistic ballots. By Theorem 1, we know that $(\beta_S)_{S\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition. Moreover, anonymity of φ implies that every voter has the conditional dictatorial coefficient $\frac{1}{n}$, and $\beta_S = \beta_{S'}$ for all $S, S' \subseteq N$ with $|S| = |S'|$. By decomposability and Theorem 1, we have finitely many $(\underline{k}, \overline{k})$-RFBRs $f^k : \left[\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})\right]^n \to \Delta(A)$, $k = 1\ldots, q$, and weights $\alpha^1, \ldots, \alpha^q > 0$ with $\sum_{k=1}^q \alpha^k = 1$ such that $\varphi(P) = \sum_{k=1}^q \alpha^k e_{f^k(P)}$ for all $P \in \left[\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})\right]^n$. For each $1 \le k \le q$, let $(b_S^k)_{S\subseteq N}$ denote the deterministic ballots of f^k. Evidently, for each $1 \le k \le q$, $(b_S^k)_{S\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained-dictatorship condition. For ease of presentation, we call the voter specified in the constrained dictatorship condition of f^k the *constrained dictator*, denoted by i^k. Moreover, let $I_i = \left\{k \in \{1,\ldots,q\} : i^k = i\right\}$ collect the indexes of RFBRs where i is the constrained dictator. Last, by monotonicity of both $(\beta_S)_{S\subseteq N}$ and $(b_S^k)_{S\subseteq N}$, $k = 1, \ldots, q$, it is true that $\beta_S = \sum_{k=1}^q \alpha^k \boldsymbol{e}_{b_S^k}$ for all $S \subseteq N$.

Lemma **7** *For all $i \in N$, $\sum_{k\in I_i} \alpha^k = \frac{1}{n}$.*

Proof: Suppose that it is not true. Then, there exist $i, j \in N$ such that $\sum_{k\in I_i} \alpha^k \neq \sum_{k\in I_j} \alpha^k$. Then, by the constrained random dictatorship condition, we have $\beta_{\{i\}}([a_{\overline{k}}, a_m]) = \sum_{k=1}^q \alpha^k \mathbf{1}\big(b_{\{i\}}^k \in R\big) = \sum_{k\in I_i} \alpha^k \neq \sum_{k\in I_j} \alpha^k = \sum_{k=1}^q \alpha^k \mathbf{1}\big(b_{\{j\}}^k \in R\big) = \beta_{\{j\}}([a_{\overline{k}}, a_m])$, which contradicts the fact $\beta_{\{i\}} = \beta_{\{j\}}$.[16] □

For each $i \in N$, let $\varphi^i(P) = \sum_{k\in I_i} \alpha^k n \boldsymbol{e}_{f^k(P)}$ for all $P \in \left[\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})\right]^n$. By Lemma 7, φ^i is a mixture of RFBRs $(f^k)_{k\in I_i}$ according to the weights $(\alpha^k n)_{k\in I_i}$, and hence is a $(\underline{k}, \overline{k})$-RPFBR. Let $(\beta_S^i)_{S\subseteq N}$ denote the corresponding probabilistic ballots. Evidently, $(\beta_S^i)_{S\subseteq N}$ satisfy ballot unanimity and monotonicity, and φ^i satisfies the constrained random-dictatorship condition. Note that voter i has the conditional dictatorial coefficient 1 in φ^i.

Lemma **8** *For all $S \subseteq N$, $\beta_S = \sum_{i\in N} \frac{1}{n}\beta_S^i$.*

Proof: By the definition RPFBRs $(\varphi^i)_{i\in N}$, we can rewrite φ as follows: $\varphi(P) = \sum_{k=1}^q \alpha^k \boldsymbol{e}_{f^k(P)} = \sum_{i\in N}\sum_{k\in I_i} \alpha^k \boldsymbol{e}_{f^k(P)} = \sum_{i\in N} \frac{1}{n}\Big(\sum_{k\in I_i} \alpha^k n \boldsymbol{e}_{f^k(P)}\Big) = \sum_{i\in N} \frac{1}{n}\varphi^i(P)$ for all $P \in \left[\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})\right]^n$. Therefore, $\beta_S = \sum_{i\in N} \frac{1}{n}\beta_S^i$ for all $S \subseteq N$. □

Now, for each $i \in N$, we construct another collection of probabilistic ballots $(\bar{\beta}_S^i)_{S\subseteq N}$ by equally mixing probabilistic ballots $\{(\beta_S^j)_{S\subseteq N} : j \in N\}$ in a particular way. Specifically, given $S \subseteq N$, say $|S| = k$, we

[16] The notation $\mathbf{1}(\cdot)$ denotes an indicator function.

construct $\bar{\beta}^i_S$ in two steps. In the first step, we refer to each coalition $S' \subseteq N$ that has the same size as S, the k corresponding probabilistic ballots $(\beta^j_{S'})_{j\in S'}$ and the $n-k$ corresponding probabilistic ballots $(\beta^j_{S'})_{j\in N\backslash S'}$. We then make two equal mixtures $\sum_{j\in S'} \frac{1}{k}\beta^j_{S'}$ and $\sum_{j\in N\backslash S'} \frac{1}{n-k}\beta^j_{S'}$. In the second step, we check whether i is included in S or not. If $i \in S$, we refer to $\sum_{j\in S'} \frac{1}{k}\beta^j_{S'}$ for all $C^k_n = \frac{n!}{k!(n-k)!}$ subsets S' of N that have the same size as S, and make their equal mixture as $\bar{\beta}^i_S$, i.e.,

$$\bar{\beta}^i_S = \sum_{S'\subseteq N:|S'|=k} \frac{1}{C^k_n}\Big(\sum_{j\in S'} \frac{1}{k}\beta^j_{S'}\Big) = \frac{1}{C^k_n}\frac{1}{k} \sum_{S'\subseteq N:|S'|=k}\sum_{j\in S'} \beta^j_{S'};$$

otherwise we refer to $\sum_{j\in N\backslash S'} \frac{1}{n-k}\beta^j_{S'}$ for all $C^k_n = \frac{n!}{k!(n-k)!}$ subsets S' of N that have the same size as S, and make their equal mixture as $\bar{\beta}^i_S$, i.e.,

$$\bar{\beta}^i_S = \sum_{S'\subseteq N:|S'|=k} \frac{1}{C^k_n}\Big(\sum_{j\in N\backslash S'} \frac{1}{n-k}\beta^j_{S'}\Big) = \frac{1}{C^k_n}\frac{1}{n-k} \sum_{S'\subseteq N:|S'|=k}\sum_{j\in N\backslash S'} \beta^j_{S'}.$$

We are going to show that $(\bar{\beta}^i_S)_{S\subseteq N}$ satisfy ballot unanimity, monotonicity and the constrained random-dictatorship condition. First, it is easy to verify the following four statements:

(i) $\bar{\beta}^i_S \in \Delta(A)$ for all $S \subseteq N$ and $i \in N$.

(ii) $(\bar{\beta}^i_S)_{S\subseteq N}$ satisfy ballot unanimity, i.e., $\bar{\beta}^i_\emptyset = \frac{1}{n}\sum_{S'\subseteq N:|S'|=0}\sum_{j\notin S'} \beta^j_{S'} = \frac{1}{n}\sum_{j\in N}\beta^j_\emptyset = \boldsymbol{e}_{a_1}$ and $\bar{\beta}^i_N = \frac{1}{n}\sum_{S'\subseteq N:|S'|=n}\sum_{j\in S'}\beta^j_{S'} = \frac{1}{n}\sum_{j\in N}\beta^j_N = \boldsymbol{e}_{a_m}$.

(iii) $(\bar{\beta}^i_S)_{S\subseteq N}$ satisfy the constrained random dictatorship condition, i.e., given $S \subset N$, say $|S| = k$, if $i \in S$, we have $\bar{\beta}^i_S([a_{\overline{k}}, a_m]) = \sum_{S'\subseteq N:|S'|=k} \frac{1}{C^k_n}\big(\sum_{j\in S'} \frac{1}{k}\beta^j_{S'}([a_{\overline{k}}, a_m])\big) = 1$; otherwise, we have $\bar{\beta}^i_S([a_1, a_{\underline{k}}]) = \sum_{S'\subseteq N:|S'|=k} \frac{1}{C^k_n}\big(\sum_{j\in N\backslash S'} \frac{1}{n-k}\beta^j_{S'}([a_1, a_{\underline{k}}])\big) = 1$.

(iv) For all nonempty $S \subset N$ and distinct $i, j \in S$ or $i, j \notin S$, we have $\bar{\beta}^i_S = \bar{\beta}^j_S$.

Next, we focus on showing monotonicity of $(\bar{\beta}^i_S)_{S\subseteq N}$.

Lemma 9 *Given nonempty* $S \subset N$, $\beta_S = \sum_{i\in N} \frac{1}{n}\bar{\beta}^i_S$.

Proof: Let $|S| = k$. Thus, $0 < k < n$. We then have

$$\begin{aligned}
\beta_S =& \frac{1}{C^k_n}\sum_{S'\subseteq N:|S'|=k}\beta_{S'} \quad \text{(by anonymity)}\\
=& \frac{1}{C^k_n}\sum_{S'\subseteq N:|S'|=k}\sum_{i\in N}\frac{1}{n}\beta^i_{S'} \quad \text{(by Lemma 8)}\\
=& \frac{1}{C^k_n}\frac{1}{n}\sum_{S'\subseteq N:|S'|=k}\Big(\sum_{i\in S'}\beta^i_{S'} + \sum_{i\in N\backslash S'}\beta^i_{S'}\Big)\\
=& \frac{k}{n}\Big(\frac{1}{C^k_n}\frac{1}{k}\sum_{S'\subseteq N:|S'|=k}\sum_{i\in S'}\beta^i_{S'}\Big) + \frac{n-k}{n}\Big(\frac{1}{C^k_n}\frac{1}{n-k}\sum_{S'\subseteq N:|S'|=k}\sum_{i\in N\backslash S'}\beta^i_{S'}\Big)\\
=& \frac{k}{n}\bar{\beta}^i_S + \frac{n-k}{n}\bar{\beta}^j_S \quad \text{for some } i \in S \text{ and some } j \in N\backslash S \quad \text{(by the definition of } \bar{\beta}^i_S \text{ and } \bar{\beta}^j_S)\\
=& \sum_{i\in S}\frac{1}{n}\bar{\beta}^i_S + \sum_{j\in N\backslash S}\frac{1}{n}\bar{\beta}^j_S \quad \text{(by statement (iv) above)}\\
=& \sum_{i\in N}\frac{1}{n}\bar{\beta}^i_S.
\end{aligned}$$

This completes the verification of the lemma. □

Lemma **10** *Probabilistic ballots* $\left(\bar{\beta}_S^i\right)_{S\subseteq N}$ *satisfy monotonicity.*

Proof: Fix $S \subset S' \subseteq N$. If $S = \emptyset$ or $S' = N$, the condition of monotonicity holds evidently. Henceforth, let $S \neq \emptyset$ and $S' \neq N$. We assume w.l.o.g. that $|S| = k$ and $|S'| = k+1$. If $S'\backslash S = \{i\}$, we have $\bar{\beta}_{S'}^i([a_{\overline{k}}, a_m]) = 1$ and $\bar{\beta}_S^i[a_1, a_{\underline{k}}] = 1$ by the constrained random-dictatorship condition, which immediately imply the condition of monotonicity.

Next, assume $i \in S$. Then, $i \in S'$. Now, given $a_t \in A$, we have

$$\begin{aligned}
\bar{\beta}_{S'}^i([a_t, a_m]) - \bar{\beta}_S^i([a_t, a_m]) =& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \sum_{\bar{S}\subseteq N:|\bar{S}|=k+1} \sum_{j\in\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) - \frac{1}{C_n^k}\frac{1}{k} \sum_{\bar{S}\subseteq N:|\bar{S}|=k} \sum_{j\in\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \\
=& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \frac{1}{k} \left[\sum_{\bar{S}\subseteq N:|\bar{S}|=k+1} \left(k \sum_{j\in\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) - \sum_{\bar{S}\subseteq N:|\bar{S}|=k} \left((n-k) \sum_{j\in\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) \right] \\
=& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \frac{1}{k} \left[\sum_{\bar{S}\subseteq N:|\bar{S}|=k} \left(\sum_{\nu\in N\backslash\bar{S}} \sum_{j\in\bar{S}} \beta_{\bar{S}\cup\{\nu\}}^j([a_t, a_m]) \right) - \sum_{\bar{S}\subseteq N:|\bar{S}|=k} \left((n-k) \sum_{j\in\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) \right] \\
=& \frac{1}{C_n^{k+1}} \frac{1}{k+1} \frac{1}{k} \sum_{\bar{S}\subseteq N:|\bar{S}|=k} \sum_{\nu\in N\backslash\bar{S}} \sum_{j\in\bar{S}} \left(\beta_{\bar{S}\cup\{\nu\}}^j([a_t, a_m]) - \beta_{\bar{S}}^j([a_t, a_m]) \right) \\
\geq& 0. \quad \text{(by monotonicity of } (\beta_J^j)_{J\subseteq N}, j \in \bar{S})
\end{aligned}$$

Last, assume $i \notin S'$. Then, $i \notin S$. Now, given $a_t \in A$, we have

$$\begin{aligned}
&\bar{\beta}_{S'}^i([a_t, a_m]) - \bar{\beta}_S^i([a_t, a_m]) \\
=& \frac{1}{C_n^{k+1}} \frac{1}{n-(k+1)} \sum_{\bar{S}\subseteq N:|\bar{S}|=k+1} \sum_{j\in N\backslash\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) - \frac{1}{C_n^k} \frac{1}{n-k} \sum_{\bar{S}\subseteq N:|\bar{S}|=k} \sum_{j\in N\backslash\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \\
=& \frac{1}{C_n^k} \frac{1}{n-k} \frac{1}{n-(k+1)} \left[\sum_{\bar{S}\subseteq N:|\bar{S}|=k+1} \left((k+1) \sum_{j\in N\backslash\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) - \sum_{\bar{S}\subseteq N:|\bar{S}|=k} \left([n-(k+1)] \sum_{j\in N\backslash\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) \right] \\
=& \frac{1}{C_n^k} \frac{1}{n-k} \frac{1}{n-(k+1)} \left[\sum_{\bar{S}\subseteq N:|\bar{S}|=k+1} \left((k+1) \sum_{j\in N\backslash\bar{S}} \beta_{\bar{S}}^j([a_t, a_m]) \right) - \sum_{\bar{S}\subseteq N:|\bar{S}|=k+1} \left(\sum_{\nu\in\bar{S}} \sum_{j\in N\backslash\bar{S}} \beta_{\bar{S}\backslash\{\nu\}}^j([a_t, a_m]) \right) \right] \\
=& \frac{1}{C_n^k} \frac{1}{n-k} \frac{1}{n-(k+1)} \sum_{\bar{S}\subseteq N:|\bar{S}|=k+1} \sum_{\nu\in\bar{S}} \sum_{j\in N\backslash\bar{S}} \left[\beta_{\bar{S}}^j([a_t, a_m]) - \beta_{\bar{S}\backslash\{\nu\}}^j([a_t, a_m]) \right] \\
\geq& 0. \quad \text{(by monotonicity of } (\beta_J^j)_{J\subseteq N}, j \in N\backslash\bar{S})
\end{aligned}$$

This completes the verification of the lemma. □

Now, we are ready to show per-capita monotonicity of φ. Given nonempty $S \subset S' \subset N$, $a_t \in R$ and $a_s \in L$, we have

$$\begin{aligned}
\frac{\beta_{S'}([a_t, a_m])}{|S'|} - \frac{\beta_S([a_t, a_m])}{|S|} =& \frac{\sum_{i\in N} \frac{1}{n}\bar{\beta}_{S'}^i([a_t, a_m])}{|S'|} - \frac{\sum_{i\in N} \frac{1}{n}\bar{\beta}_S^i([a_t, a_m])}{|S|} \quad \text{(by Lemma 9)} \\
=& \frac{\sum_{i\in S'} \frac{1}{n}\bar{\beta}_{S'}^i([a_t, a_m])}{|S'|} - \frac{\sum_{i\in S} \frac{1}{n}\bar{\beta}_S^i([a_t, a_m])}{|S|} \quad \text{(by statement (iii))} \\
=& \frac{\bar{\beta}_{S'}^i([a_t, a_m]) - \bar{\beta}_S^i([a_t, a_m])}{n} \quad \text{(select } i \in S \text{ and apply statement (iv))} \\
\geq& 0 \quad \text{(by Lemma 10), and}
\end{aligned}$$

$$\begin{aligned}
\frac{\beta_{N\backslash S'}([a_1,a_s])}{|S'|}-\frac{\beta_{N\backslash S}([a_1,a_s])}{|S|}=&\frac{\sum_{i\in N}\frac{1}{n}\bar{\beta}^i_{N\backslash S'}([a_1,a_s])}{|S'|}-\frac{\sum_{i\in N}\frac{1}{n}\bar{\beta}^i_{N\backslash S}([a_1,a_s])}{|S|} \quad \text{(by Lemma 9)}\\
=&\frac{\sum_{i\in S'}\frac{1}{n}\bar{\beta}^i_{N\backslash S'}([a_1,a_s])}{|S'|}-\frac{\sum_{i\in S}\frac{1}{n}\bar{\beta}^i_{N\backslash S}([a_1,a_s])}{|S|} \quad \text{(by statement (iii))}\\
=&\frac{\bar{\beta}^i_{N\backslash S'}([a_1,a_s])-\bar{\beta}^i_{N\backslash S}([a_1,a_s])}{n} \quad \text{(select } i\in J \text{ and apply statement (iv))}\\
=&\frac{\bar{\beta}^i_{N\backslash S}([a_{s+1},a_m])-\bar{\beta}^i_{N\backslash S'}([a_{s+1},a_m])}{n}\\
\geq&0. \quad \text{(by Lemma 10)}
\end{aligned}$$

This completes the verification of the necessity part of Theorem 2.

C Proof of Theorem 3

Let domain $\mathbb{D}$ satisfy the weak no-restoration property and contain two completely reversed preferences. Thus, $\mathbb{D}$ is connected. Note that $\mathbb{D}$ is minimally richness. We first show that $\mathbb{D}$ is $(\underline{k},\overline{k})$-hybrid for some unique $\underline{k}$ and $\overline{k}$. The proof consists of Lemmas 11 - 17.

We first introduce an important new notion. A pair of distinct alternatives $a_s, a_t \in A$ is said **adjacent** in $\mathbb{D}$, denoted $a_s \sim a_t$, if there exist $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) = a_s$ and $r_1(P_i') = a_t$ such that $P_i \sim P_i'$. Then, we induce a graph, denoted by $G_{\mathbb{D}}$, such that the set of vertex is A, and in the set of edges, every pair of alternatives forms an edge if and only if they are adjacent in $\mathbb{D}$. An **alternative-path**, denoted by $\mathcal{P}$, connecting a_s and a_t is a sequence of (non-repeated) vertices $\{x_k\}_{k=1}^l \subseteq A$ such that $x_1 = a_s$, $x_l = a_t$ and $x_k \sim x_{k+1}$ for all $k = 1,\dots,l-1$. For notational convenience, let $\Pi(a_s,a_t)$ denote the set of *all* alternative-paths connecting a_s and a_t,[17] and $\langle a_s,a_t\rangle$ denote *one* alternative-path connecting a_s and a_t.

Lemma **11** *Every pair of distinct alternatives $a_s,a_t \in A$ is connected via an alternative-path, i.e., $\Pi(a_s,a_t) \neq \emptyset$.*

Proof: Given $P_i \in \mathbb{D}$ with $r_1(P_i) = a_s$ and $P_i' \in \mathbb{D}$ with $r_1(P_i') = a_t$ by minimal richness, since $\mathbb{D}$ is connected, we have a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i'. We partition $\{P_i^k\}_{k=1}^t$ according to the peaks of preferences (without rearranging preferences in the path), and elicit all preference peaks:

$$\left\{\frac{P_i^1,\dots,P_i^{k_1}}{\text{the same peak } x_1},\frac{P_i^{k_1+1},\dots,P_i^{k_2}}{\text{the same peak } x_2},\dots,\frac{P_i^{k_{q-1}+1},\dots,P_i^{t}}{\text{the same peak } x_q}\right\}\xrightarrow{\text{Elicit peaks}}\{x_1,x_2,\dots,x_q\},$$

where $x_k \neq x_{k+1}$ and $x_k \sim x_{k+1}$ for all $k = 1,\dots,q-1$. Note that $\{x_1,x_2,\dots,x_q\}$ may contain repetitions. Whenever a repetition appears, we remove all alternatives strictly between the repetition and one alternative of the repetition. For instance, if $x_k = x_l$ where $1 \leq k < l \leq q$, we remove $x_k, x_{k+1},\dots,x_{l-1}$, and refine the sequence to $\{x_1,\dots,x_{k-1},x_l,\dots,x_q\}$. By repeatedly eliminating repetitions, we finally elicit an alternative-path $\{x_k\}_{k=1}^p$ connecting a_s and a_t. □

Let $\underline{P}_i$ and $\overline{P}_i$ be the pair of completely reversed preferences contained in $\mathbb{D}$. Assume w.l.o.g. that $\underline{P}_i = (a_1\cdots a_{k-1}a_k\cdots a_m)$ and $\overline{P}_i = (a_m\cdots a_k a_{k-1}\cdots a_1)$. Note that the way we specify $\underline{P}_i$ and $\overline{P}_i$ determines the labeling of all alternatives.

Lemma **12** *Given distinct $a_p,a_s,a_t \in A$, let a_t be included in every alternative-path of $\Pi(a_p,a_s)$. Given $P_i \in \mathbb{D}$, we have $[r_1(P_i) = a_p] \Rightarrow [a_t P_i a_s]$ and $[r_1(P_i) = a_s] \Rightarrow [a_t P_i a_p]$.*

[17] In particular, if $a_s = a_t$, then $\Pi(a_s,a_t) = \big\{\{a_s\}\big\}$ is a singleton set of a null alternative-path.

Proof: Suppose that $r_1(P_i) = a_p$ and $a_s P_i a_t$. Pick an arbitrary preference $P'_i \in \mathbb{D}$ with $r_1(P'_i) = a_s$ by minimal richness. By the weak no-restoration property, there exists a path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P'_i such that $a_s P_i^k a_t$ for all $k = 1, \ldots, l$. Thus, $r_1(P_i^k) \neq a_t$ for all $k = 1, \ldots, l$. According to path $\{P_i^k\}_{k=1}^l$, we elicit an alternative-path $\langle a_p, a_s \rangle$ which excludes a_t. This contradicts the hypothesis of the lemma. Therefore, $a_t P_i a_s$. Symmetrically, if $r_1(P_i) = a_s$, then $a_t P_i a_p$. □

LEMMA **13** *Given $a_s, a_t \in A \backslash \{a_1, a_m\}$ with $a_s \sim a_t$, If one alternative-path of $\Pi(a_1, a_m)$ includes a_t, there exists an alternative-path of $\Pi(a_1, a_m)$ including a_s.*

Proof: Let $\{x_k\}_{k=1}^p \in A$ and $a_t = x_\eta$ for some $1 < \eta < p$. If $a_s \in \{x_k\}_{k=1}^p$, the lemma holds evidently. Henceforth, assume $a_s \notin \{x_k\}_{k=1}^p$. Note the alternative-path $\{a_1 = x_1, x_2, \ldots, x_\eta = a_t, a_s\} \in \Pi(a_1, a_s)$, and the alternative-path $\{a_s, a_t = x_\eta, \ldots, x_{p-1}, x_p = a_m\} \in \Pi(a_s, a_m)$.

Since $\underline{P}$ and $\overline{P}_i$ are completely reversed, either $a_s \underline{P}_i a_t$ or $a_s \overline{P}_i a_t$ holds. Assume w.l.o.g. that $a_s \underline{P}_i a_t$. The verification related to $a_s \overline{P}_i a_t$ is symmetric and we hence omit it. Pick an arbitrary preference $P_i \in \mathbb{D}$ with $r_1(P_i) = a_s$ by minimal richness. By the weak no-restoration property, we have a path $\{P_i^k\}_{k=1}^\nu \subseteq \mathbb{D}$ connecting $\underline{P}_i$ and P_i such that $a_s P_i^k a_t$ for all $k = 1, \ldots, \nu$. Thus, $r_1(P_i^k) \neq a_t$ for all $k = 1, \ldots, \nu$. According to $\{P_i^k\}_{k=1}^\nu$, we elicit an alternative-path $\{y_k\}_{k=1}^q \in \Pi(a_1, a_s)$ such that $a_t \notin \{y_k\}_{k=1}^q$.

Evidently, $\{y_k\}_{k=1}^q \cap \{x_k\}_{k=1}^p \supseteq \{a_1\}$. If $\{y_k\}_{k=1}^q \cap \{x_k\}_{k=1}^p = \{a_1\}$, then the concatenated alternative-path $\{a_1 = y_1, \ldots, y_q = a_s; a_t = x_\eta, \ldots, x_p = a_m\} \in \Pi(a_1, a_m)$ includes a_s. Next, we assume $\{y_k\}_{k=1}^q \cap \{x_k\}_{k=1}^p \supset \{a_1\}$. We identify the alternative in $\{y_k\}_{k=1}^q$ that has the maximum index and is also included in $\{x_k\}_{k=1}^p$, i.e., $y_{\hat{k}} = x_{k^*}$ for some $1 < \hat{k} < q$ and $1 < k^* \leq p$ and $\{y_{\hat{k}+1}, \ldots, y_q\} \cap \{x_k\}_{k=1}^p = \emptyset$. Note that $a_t = x_\eta$, $1 < \eta < p$ and $a_t \neq y_{\hat{k}}$. Therefore, either $1 < k^* < \eta$ or $\eta < k^* \leq p$ must hold. If $1 < k^* < \eta$, the concatenated alternative-path $\{a_1 = x_1, \ldots, x_{k^*} = y_{\hat{k}}; y_{\hat{k}+1}, \ldots, y_q = a_s; a_t = x_\eta, \ldots, x_p = a_m\} \in \Pi(a_1, a_m)$ includes a_s. If $\eta < k^* \leq p$, the concatenated alternative-path $\{a_1 = x_1, \ldots, x_\eta = a_t; a_s = y_q, \ldots, y_{\hat{k}+1}; y_{\hat{k}} = x_{k^*}, \ldots, x_p = a_m\} \in \Pi(a_1, a_m)$ includes a_s. □

LEMMA **14** *Given $a_s \in A \backslash \{a_1, a_m\}$, there exists an alternative-path of $\Pi(a_1, a_m)$ including a_s.*

Proof: Pick an arbitrary preference $P_i \in \mathbb{D}$ with $r_1(P_i) = a_s$ by minimal richness. Note that $a_s \underline{P}_i a_m$ and $a_s P_i a_m$. By the weak no-restoration property, we have a path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting $\underline{P}_i$ and P_i such that $a_s P_i^k a_m$ for all $k = 1, \ldots, l$. Thus, $r_1(P_i^k) \neq a_m$ for all $k = 1, \ldots, l$. According to $\{P_i^k\}_{k=1}^l$, we elicit an alternative-path $\{x_k\}_{k=1}^p \in \Pi(a_1, a_s)$ that excludes a_m. Symmetrically, we have an alternative-path $\{y_k\}_{k=1}^q \in \Pi(a_s, a_m)$ that excludes a_1. Thus, $\{x_k\}_{k=1}^p \cap \{y_k\}_{k=1}^q \supseteq \{a_s\}$. If $\{x_k\}_{k=1}^p \cap \{y_k\}_{k=1}^q = \{a_s\}$, then the concatenated alternative-path $\{a_1 = x_1, \ldots, x_p = a_s = y_1, \ldots, y_q = a_m\} \in \Pi(a_1, a_m)$ includes a_s. If $\{x_k\}_{k=1}^p \cap \{y_k\}_{k=1}^q \supset \{a_s\}$, we identify the alternative a_t included in both $\{x_k\}_{k=1}^p$ and $\{y_k\}_{k=1}^q$ with the maximum index in $\{x_k\}_{k=1}^p$ and the minimum index in $\{y_k\}_{k=1}^q$, i.e., $a_t = x_{\hat{k}} = y_{k^*}$ for some $1 < \hat{k} < p$ and $1 < k^* < q$ such that $\{x_1, \ldots, x_{\hat{k}-1}\} \cap \{y_{k^*+1}, \ldots, y_q\} = \emptyset$. Thus, the concatenated alternative-path $\{x_1, \ldots, x_{\hat{k}-1}, x_{\hat{k}} = a_t = y_{k^*}, y_{k^*+1}, \ldots, y_q\} \in \Pi(a_1, a_m)$ includes a_t, and excludes a_s. Furthermore, we refer to the sub-alternative-path $\{a_t = x_{\hat{k}}, \ldots, x_p = a_s\}$, by repeatedly applying Lemma 13 step by step from a_t to a_s along the sub-alternative-path, we eventually find an alternative-path of $\Pi(a_1, a_m)$ that includes a_s. □

Note that $\Pi(a_1, a_m)$ is a finite nonempty set. Hence, we label $\Pi(a_1, a_m) = \{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$, and make sure that each alternative-path of $\Pi(a_1, a_m)$ starts from a_1 and ends at a_m. Given $\mathcal{P}_l \in \Pi(a_1, a_m)$ and $a_s, a_t \in \mathcal{P}_l$, let $\langle a_s, a_t \rangle^{\mathcal{P}_l}$ denote the interval between a_s and a_t on $\mathcal{P}_l$.

LEMMA **15** *If $\Pi(a_1, a_m)$ is a singleton set, $\mathbb{D}$ is $(\underline{k}, \overline{k})$-hybrid for all $1 \leq \underline{k} < \overline{k} \leq m$ with $\overline{k} - \underline{k} = 1$.*

Proof: Since $\Pi(a_1, a_m)$ is a singleton set, Lemma 14 implies that all alternatives must be included in a unique alternative-path. Thus, $G_{\mathbb{D}}$ must be a line and include all alternatives. More importantly, Lemma 12 implies that all preferences of $\mathbb{D}$ must be single-peaked w.r.t. $G_{\mathbb{D}}$. Since $\underline{P}_i$ and $\overline{P}_i$ are single-peaked w.r.t. $G_{\mathbb{D}}$, it must be the case that $G_{\mathbb{D}}$ is a line of $\{a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_m\}$ which coincides to the natural order $\prec$. Hence, $\mathbb{D} \subseteq \mathbb{D}_{\prec} = \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for all $1 \leq \underline{k} < \overline{k} \leq m$ with $\overline{k} - \underline{k} = 1$. Evidently, as $\mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$, where $\underline{k}' > \underline{k}$ or $\overline{k}' < \overline{k}$, is not well defined, $\mathbb{D} \nsubseteq \mathbb{D}_{\mathrm{H}}(\underline{k}', \overline{k}')$. □

Henceforth, we assume that $\Pi(a_1, a_m)$ is not a singleton set. Since all alternative-paths of $\Pi(a_1, a_m)$ start from a_1 and end at a_m, we can identify the left maximum common part and the right maximum common part of all alternative-paths of $\Pi(a_1, a_m)$, i.e., there exist two alternatives $a_{\underline{k}}, a_{\overline{k}} \in A$ (either $\underline{k} \leq \overline{k}$ or $\underline{k} \geq \overline{k}$ so far) such that the following three conditions are satisfied:

(i) $a_{\underline{k}}, a_{\overline{k}} \in \mathcal{P}_l$ for all $\mathcal{P}_l \in \Pi(a_1, a_m)$,

(ii) $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} = \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_\nu}$, and $\langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} = \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_\nu}$ for all $\mathcal{P}_l, \mathcal{P}_\nu \in \Pi(a_1, a_m)$, and

(iii) there exist *no* $a_{\underline{k}'}, a_{\overline{k}'} \in A$ such that $a_{\underline{k}'}, a_{\overline{k}'} \in \mathcal{P}_l$ for all $\mathcal{P}_l \in \Pi(a_1, a_m)$, and $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \subset \langle a_1, a_{\underline{k}'} \rangle^{\mathcal{P}_l}$ or $\langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} \subset \langle a_{\overline{k}'}, a_m \rangle^{\mathcal{P}_l}$ for all $\mathcal{P}_l \in \Pi(a_1, a_m)$.

We claim that $a_{\underline{k}} \neq a_{\overline{k}}$. Otherwise, $\Pi(a_1, a_m)$ degenerates to a singleton set. Note that condition (iii) implies that $a_{\underline{k}}$ and $a_{\overline{k}}$ are unique. Fix an arbitrary $\mathcal{P}_l \in \Pi(a_1, a_m)$. We first claim $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cap \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} = \emptyset$. Suppose not, i.e., there exists $a_s \in \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cap \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l}$ such that $\langle a_1, a_s \rangle^{\mathcal{P}_l} \cap \langle a_s, a_m \rangle^{\mathcal{P}_l} = \{a_s\}$. Since $a_{\underline{k}} \neq a_{\overline{k}}$, we know either $a_s \neq a_{\underline{k}}$ or $a_s \neq a_{\overline{k}}$. Consequently, the concatenated alternative-path $\{\langle a_1, a_s \rangle^{\mathcal{P}_l}, \langle a_s, a_m \rangle^{\mathcal{P}_l}\} \in \Pi(a_1, a_m)$ excludes either $a_{\underline{k}}$ or $a_{\overline{k}}$, which contradicts condition (i). Therefore, $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cap \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} = \emptyset$. Next, we claim that $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l} \cup \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_l} \neq A$. Otherwise, condition (ii) implies $\langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_\nu} \cup \langle a_{\overline{k}}, a_m \rangle^{\mathcal{P}_\nu} = A$ for all $\mathcal{P}_\nu \in \Pi(a_1, a_m)$, and consequently, $\Pi(a_1, a_m)$ degenerates to a singleton set.

Lemma **16** *The following two statements hold:*

(i) $\Pi(a_1, a_{\underline{k}})$ *is a singleton set of the unique alternative-path* $\{a_1, \dots, a_k, a_{k+1}, \dots, a_{\underline{k}}\}$.

(ii) $\Pi(a_{\overline{k}}, a_m)$ *is a singleton set of the unique alternative-path* $\{a_{\overline{k}}, \dots, a_k, a_{k+1}, \dots, a_m\}$.

Proof: By symmetry, we show the first statement, and omit the verification of the second statement.

First, let $\Pi(a_1, a_{\underline{k}})$ be a singleton set. We show that $\Pi(a_1, a_{\underline{k}}) = \big\{\{a_1, \dots, a_k, a_{k+1}, \dots, a_{\underline{k}}\}\big\}$, which coincides to the nature order $\prec$ from a_1 to $a_{\underline{k}}$. Since $\Pi(a_1, a_{\underline{k}})$ is a singleton set, Lemma 12 implies that all preferences of $\mathbb{D}$ must be single-peaked w.r.t. the unique alternative-path of $\Pi(a_1, a_{\underline{k}})$. Moreover, since the completely reversed preferences $\underline{P}_i = (a_1 \cdots a_k a_{k+1} \cdots a_{\underline{k}} \cdots a_{\overline{k}} \cdots a_m)$ and $\overline{P}_i = (a_m \cdots a_{\overline{k}} \cdots a_{\underline{k}} \cdots a_{k+1} a_k \cdots a_1)$ are contained in $\mathbb{D}$, this implies that the unique alternative-path of $\Pi(a_1, a_{\underline{k}})$ must be $\{a_1, \dots, a_k, a_{k+1}, \dots, a_{\underline{k}}\}$.

Next, we show that $\Pi(a_1, a_{\underline{k}})$ is a singleton set. If $a_1 = a_{\underline{k}}$, statement (i) holds by the definition of $\Pi(a_1, a_{\underline{k}})$. We next assume $a_1 \neq a_{\underline{k}}$. Pick an arbitrary alternative-path $\mathcal{P}_l = \{a_1 = x_1, \dots, x_v = a_{\underline{k}}, \dots, x_t = a_m\} \in \Pi(a_1, a_m)$. Given an arbitrary alternative-path $\langle a_1, a_{\underline{k}} \rangle = \{a_1 = y_1, \dots, y_u = a_{\underline{k}}\}$, we show $\langle a_1, a_{\underline{k}} \rangle = \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l}$. Since $a_{\underline{k}} = x_v = y_u$, we can identify the alternative $y_{\hat{k}} = x_{k^*}$ for some $1 < \hat{k} \leq u$ and $v \leq k^* \leq t$ such that $\{y_1, \dots, y_{\hat{k}-1}\} \cap \{x_{k^*+1}, \dots, x_t\} = \emptyset$. Then, we have a concatenated alternative-path $\mathcal{P}_\nu = \{y_1, \dots, y_{\hat{k}-1}, y_{\hat{k}} = x_{k^*}, x_{k^*+1}, \dots, x_t\} \in \Pi(a_1, a_m)$. By condition (i) above, we know $a_{\underline{k}} \in \mathcal{P}_\nu$. Since $a_{\underline{k}} \notin \{y_1, \dots, y_{\hat{k}-1}\}$ and $a_{\underline{k}} \notin \{x_{k^*+1}, \dots, x_t\}$, it must be the case $y_{\hat{k}} = a_{\underline{k}}$ and $x_{k^*} = a_{\underline{k}}$. Hence, $\langle a_1, a_{\underline{k}} \rangle = \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_\nu}$. Last, by condition (ii) above, we have $\langle a_1, a_{\underline{k}} \rangle = \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_\nu} = \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l}$. Since both $\mathcal{P}_l$ and $\langle a_1, a_{\underline{k}} \rangle$ are arbitrarily selected, $\langle a_1, a_{\underline{k}} \rangle = \langle a_1, a_{\underline{k}} \rangle^{\mathcal{P}_l}$ implies that $\Pi(a_1, a_{\underline{k}})$ is a singleton set. □

Henceforth, let $L = \{a_1, \dots, a_k, a_{k+1}, \dots, a_{\underline{k}}\}$, $R = \{a_{\overline{k}}, \dots, a_k, a_{k+1}, \dots, a_m\}$ and $M = \{a_{\underline{k}}, \dots, a_k, a_{k+1}, \dots, a_{\overline{k}}\}$. As mentioned before, we know $\overline{k} - \underline{k} > 1$.

Lemma **17** *Domain* $\mathbb{D} \subseteq \mathbb{D}_{\rm H}(\underline{k}, \overline{k})$, *and* $\mathbb{D} \nsubseteq \mathbb{D}_{\rm H}(\underline{k}', \overline{k}')$ *where* $\underline{k}' > \underline{k}$ *or* $\overline{k}' < \overline{k}$.

Proof: By Lemma 12, we know that all preferences of $\mathbb{D}$ are single-peaked w.r.t. the natural order $\prec$ on both L and R. Therefore, the first restriction of Definition 1 is satisfied. We focus on showing the second restriction of Definition 1.

Fix $P_i \in \mathbb{D}$ with $r_1(P_i) = a_p \in L$ and $a_r \in M\backslash\{a_{\underline{k}}\}$. If $a_p = a_{\underline{k}}$, $a_{\underline{k}} P_i a_r$ holds evidently. We next assume $a_p \neq a_{\underline{k}}$. By Lemma 12, to prove $a_{\underline{k}} P_i a_r$, it suffices to show that $a_{\underline{k}}$ is included in every alternative-path of $\Pi(a_p, a_r)$. Suppose not, i.e., there exists an alternative-path $\langle a_p, a_r\rangle$ such that $a_{\underline{k}} \notin \langle a_p, a_r\rangle$. Since $a_p \neq a_{\underline{k}}$, we have the alternative-path $\langle a_1, a_p\rangle = \{a_1, \ldots, a_k, a_{k+1}, \ldots, a_p\}$ which excludes $a_{\underline{k}}$. Next, if $a_r = a_{\overline{k}}$, we have the alternative-path $\langle a_r, a_m\rangle = \{a_{\overline{k}}, \ldots, a_m\}$ which excludes $a_{\underline{k}}$. If $a_r \in M\backslash\{a_{\underline{k}}, a_{\overline{k}}\}$, by Lemma 14, we have an alternative-path $\mathcal{P}_l \in \Pi(a_1, a_m)$ that includes a_r. Moreover, by condition (i) above and Lemma 16, we write $\mathcal{P}_l = \{a_1, \ldots, a_{\underline{k}}, x_1, \ldots, x_t, a_{\overline{k}}, \ldots, a_m\}$ where $a_r = x_v \in \{x_1, \ldots, x_t\} \subseteq M\backslash\{a_{\underline{k}}, a_{\overline{k}}\}$ for some $1 \leq v \leq t$. Then, we have an alternative-path $\{a_r = x_v, \ldots, x_t, a_{\overline{k}}, \ldots, a_m\}$ which excludes $a_{\underline{k}}$. Overall, we have an alternative-path $\langle a_r, a_m\rangle$ that excludes $a_{\underline{k}}$. Now, we have three alternative-paths $\langle a_1, a_p\rangle$, $\langle a_p, a_r\rangle$ and $\langle a_r, a_m\rangle$ which all exclude $a_{\underline{k}}$. By combining them and removing repeated alternatives, we can construct an alternative-path of $\Pi(a_1, a_m)$ that excludes $a_{\underline{k}}$. This contradicts condition (i) above. Therefore, $a_{\underline{k}}$ is included in every alternative-path of $\Pi(a_p, a_r)$, as required. Symmetrically, given $P_i \in \mathbb{D}$ with $r_1(P_i) \in R$ and $a_s \in M\backslash\{a_{\overline{k}}\}$, we have $a_{\overline{k}} P_i a_s$.

Last, recall condition (iii) above. Since $a_{\underline{k}}$ and $a_{\overline{k}}$ are uniquely identified, $\mathbb{D} \nsubseteq \mathbb{D}_{\rm H}(\underline{k}', \overline{k}')$ where $\underline{k}' > \underline{k}$ or $\overline{k}' < \overline{k}$. This completes the verification of the lemma, and hence proves the first part of Theorem 3. □

Now, we turn to the second part of Theorem 3. By the first part of Theorem 3, we know that $\mathbb{D} \subseteq \mathbb{D}_{\rm H}(\underline{k}, \overline{k})$ for some $1 \leq \underline{k} < \overline{k} \leq m$ and $\mathbb{D} \nsubseteq \mathbb{D}_{\rm H}(\underline{k}', \overline{k}')$ where $\underline{k}' > \underline{k}$ and $\overline{k}' < \overline{k}$. By the sufficiency part of Theorem 1, it is evident that every $(\underline{k}, \overline{k})$-RPFBR is unanimous and strategy-proof on $\mathbb{D}$. Therefore, we focus on showing that every unanimous and strategy-proof on $\mathbb{D}$ is a $(\underline{k}, \overline{k})$-RPFBR. We provides four independent lemmas which show some important properties on all unanimous and strategy-proof RSCFs defined on $\mathbb{D}$. Then, these four lemmas together enable us to complete the characterization of $(\underline{k}, \overline{k})$-RPFBRs.

Lemma **18** *Every unanimous and strategy-proof RSCF* $\varphi : \mathbb{D}^n \to \Delta(A)$ *satisfies the tops-only property.*

Proof: Fix a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$. To prove the tops-only property, it suffices to show that for all $i \in N$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, $[r_1(P_i) = r_1(P_i')] \Rightarrow [\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})]$.

We prove this in two steps. In the first step, by the proof of Theorem 1 of Chatterji and Zeng (2018), we know that φ satisfies the following property: for all $i \in N$, $P_i, P_i' \in \mathbb{D}$ with $P_i \sim P_i'$ and $P_{-i} \in \mathbb{D}^{n-1}$, $[r_1(P_i) = r_1(P_i')] \Rightarrow [\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})]$.[18] In the second step, we consider $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) = r_1(P_i') \equiv a_s$, but P_i is not adjacent to P_i'.

First, strategy-proofness implies $\varphi_{a_s}(P_i, P_{-i}) = \varphi_{a_s}(P_i', P_{-i})$. Next, pick an arbitrary $a_t \in A\backslash\{a_s\}$, we show $\varphi_{a_t}(P_i, P_{-i}) = \varphi_{a_t}(P_i', P_{-i})$. By the weak no-restoration property, there exists a path $\{P_i^k\}_{k=1}^q \subseteq \mathbb{D}$ connecting P_i and P_i' such that $a_s P_i^k a_t$ for all $k = 1, \ldots, q$. Start from P_i^2. If $r_1(P_i^2) = r_1(P_i^1)$, the result in the first step implies $\varphi_{a_t}(P_i^1, P_{-i}) = \varphi_{a_t}(P_i^2, P_{-i})$. If $r_1(P_i^2) = a_r \neq a_s = r_1(P_i^1)$, then $P_i^1 \sim P_i^2$ implies $r_1(P_i^1) = r_2(P_i^2) = a_s$, $r_1(P_i^2) = r_2(P_i^1) = a_r$ and $r_l(P_i^1) = r_l(P_i^2)$ for all $l = 3, \ldots, m$. Hence, it must be the case that $a_t = r_l(P_i^1) = r_l(P_i^2)$ for some $3 \leq l \leq m$, and then strategy-proofness implies $\varphi_{a_t}(P_i^1, P_{-i}) =$

[18] Chatterji and Zeng (2018) introduce the interior and exterior properties on a domain and show that they together are sufficient for endogenizing the tops-only property on all unanimous and strategy-proof RSCFs. The weak no-restoration property implies the exterior property, but may not be compatible with the interior property. However, the proof of their Theorem 1 can be directly applied to show the first-step result here.

$\varphi_{a_t}(P_i^2, P_{-i})$. Overall, we have $\varphi_{a_t}(P_i^1, P_{-i}) = \varphi_{a_t}(P_i^2, P_{-i})$. By repeatedly applying this argument along the path from P_i^2 to P_i^q, we eventually have $\varphi_{a_t}(P_i^k, P_{-i}) = \varphi_{a_t}(P_i^{k+1}, P_{-i})$ for all $k = 1, \ldots, q-1$. Hence, $\varphi_{a_t}(P_i, P_{-i}) = \varphi_{a_t}(P_i', P_{-i})$. Therefore, $\varphi(P_i, P_{-i}) = \varphi(P_i', P_{-i})$, as required. □

Since $\mathbb{D}$ is minimally rich, the tops-only property implies that every unanimous and strategy-proof $\phi : \mathbb{D}^n \to \Delta(A)$ degenerates to a *random voting scheme* $\phi : A^n \to \Delta(A)$. Given an arbitrary random voting scheme $\phi : A^n \to \Delta(A)$, we say that (i) ϕ is *unanimous* on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ if for all $(P_1, \ldots, P_N) \in \big[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\big]^n$, $[r_1(P_1) = \cdots = r_1(P_n) = a_k] \Rightarrow [\phi(a_k, \ldots, a_k) = \boldsymbol{e}_{a_k}]$, and (ii) ϕ is *strategy-proof* (respectively, *locally strategy-proof*) on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ if for all $i \in N$, $P_i, P_i' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ (respectively, $P_i \sim P_i'$) and $P_{-i} \in \big[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\big]^{n-1}$, $\phi\big(r_1(P_i), r_1(P_{-i})\big)$ stochastically dominates $\phi\big(r_1(P_i'), r_1(P_{-i})\big)$ according to P_i, where $r_1(P_{-i}) = \big(r_1(P_1), \ldots, r_1(P_{i-1}), r_1(P_{i+1}), \ldots, r_1(P_n)\big)$.

To show a unanimous and strategy-proof $\varphi : \mathbb{D}^n \to \Delta(A)$ is a $(\underline{k}, \overline{k})$-RPFBR, by Lemma 18, Fact 1 and the necessity part of Theorem 1, it suffices to show that the corresponding random voting scheme $\varphi : A^n \to \Delta(A)$ is unanimous and locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. Note that both $\mathbb{D}$ and $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ are minimally rich. Consequently, since RSCF φ is unanimous and satisfies the tops-only property, it follows immediately that the random voting scheme $\varphi : A^n \to \Delta(A)$ is unanimous on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. In the rest of the proof, we show that every random voting scheme, which is induced from a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$, is locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

For notational convenience, with a little notational abuse, we write (a_s, a_t) as a two-voter preference profile where the first voter presents a preference with peak a_s while the second reports a preference with peak a_t. We also write (a_s, P_{-i}) as an n-voter preference profile where voter i presents a preference with peak a_s and $P_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$.

Lemma **19 (The uncompromising property)** *Let $\varphi : \mathbb{D}^n \to \Delta(A)$ be a unanimous and strategy-proof RSCF. Given an alternative-path $\{x_k\}_{k=1}^t$, $i \in I$ and $P_{-i} \in \mathbb{D}^{n-1}$, we have $\varphi_{a_s}(x_1, P_{-i}) = \varphi_{a_s}(x_t, P_{-i})$ for all $a_s \notin \{x_k\}_{k=1}^t$, and hence $\sum_{k=1}^t \varphi_{x_k}(x_1, P_{-i}) = \sum_{k=1}^t \varphi_{x_k}(x_t, P_{-i})$.*

Proof: We start with $\varphi(x_1, P_{-i})$ and $\varphi(x_2, P_{-i})$. Since $x_1 \sim x_2$, we have $P_i \in \mathbb{D}^{x_1}$ and $P_i' \in \mathbb{D}^{x_2}$ such that $P_i \sim P_i'$. Then, the tops-only property and strategy-proofness imply $\varphi_{a_s}(x_1, P_{-i}) = \varphi_{a_s}(P_i, P_{-i}) = \varphi_{a_s}(P_i', P_{-i}) = \varphi_{a_s}(x_2, P_{-i})$ for all $a_s \notin \{x_1, x_2\}$.

We next introduce an induction hypothesis: Given $2 < k \leq t$, for all $2 \leq k' < k$, $\varphi_{a_s}(x_1, P_{-i}) = \varphi_{a_s}(x_{k'}, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k'}$. We show $\varphi_{a_s}(x_1, P_{-i}) = \varphi_{a_s}(x_k, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k}$. Since $x_k \sim x_{k-1}$, we have $P_i \in \mathbb{D}^{x_k}$ and $P_i' \in \mathbb{D}^{x_{k-1}}$ such that $P_i \sim P_i'$. Then, the tops-only property and strategy-proofness imply $\varphi_{a_s}(x_k, P_{-i}) = \varphi_{a_s}(P_i, P_{-i}) = \varphi_{a_s}(P_i', P_{-i}) = \varphi_{a_s}(x_{k-1}, P_{-i})$ for all $a_s \notin \{x_{k-1}, x_k\}$. Moreover, since $\varphi_{a_s}(x_1, P_{-i}) = \varphi_{a_s}(x_{k-1}, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k-1}$ by the induction hypothesis, it is true that $\varphi_{a_s}(x_1, P_{-i}) = \varphi_{a_s}(x_k, P_{-i})$ for all $a_s \notin \{x_l\}_{l=1}^{k}$. This completes the verification of the induction hypothesis. Therefore, $\varphi_{a_s}(x_1, P_{-i}) = \varphi_{a_s}(x_t, P_{-i})$ for all $a_s \notin \{x_k\}_{k=1}^t$. Then, we have $\sum_{k=1}^t \varphi_{x_k}(x_1, P_{-i}) = 1 - \sum_{a_s \notin \{x_k\}_{k=1}^t} \varphi_{a_s}(x_1, P_{-i}) = 1 - \sum_{a_s \notin \{x_k\}_{k=1}^t} \varphi_{a_s}(x_t, P_{-i}) = \sum_{k=1}^t \varphi_{x_k}(x_t, P_{-i})$. □

Now, we can show that if $\overline{k} - \underline{k} = 1$, every unanimous and strategy-proof $\varphi : \mathbb{D}^n \to \Delta(A)$ is a PFBR. Recall that $\overline{k} - \underline{k} = 1$ implies $\mathbb{D} \subseteq \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k}) = \mathbb{D}_{\prec}$. Correspondingly, Lemma 19 degenerates to the uncompromising property of Ehlers et al. (2002), and the random voting scheme $\varphi : A^n \to \Delta(A)$ satisfies the uncompromising property on $\mathbb{D}_{\prec}$. Furthermore, Lemma 3.2 of Ehlers et al. (2002) implies that the random voting scheme φ is strategy-proof on $\mathbb{D}_{\prec}$, as required. This completes the verification of the second part of Theorem 3 in the case $\overline{k} - \underline{k} = 1$. Henceforth, we assume $\overline{k} - \underline{k} > 1$. We first make two observations on graph $G_{\mathbb{D}}$, which will be repeatedly used in the following-up proof.

Observation **1** Given $a_s \in M \backslash \{a_{\underline{k}}, a_{\overline{k}}\}$, there exists an alternative-path $\langle a_{\underline{k}}, a_{\overline{k}} \rangle \subseteq M$ that includes a_s. □

Observation **2** There exists a cycle $\mathcal{C}_1 = \{x_k\}_{k=1}^p \subseteq M$, $p \geq 3$, i.e., $x_k \sim x_{k+1}$ for all $k = 1, \ldots, p$ where $x_{p+1} = x_1$, such that $a_{\underline{k}} \in \mathcal{C}_1$.[19] There exists a cycle $\mathcal{C}_2 = \{y_k\}_{k=1}^q \subseteq M$, $q \geq 3$, i.e., $y_k \sim y_{k+1}$ for all $k = 1, \ldots, p-1$ where $y_{q+1} = y_1$, such that $a_{\overline{k}} \in \mathcal{C}_2$. □

Lemma **20** *Every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}} = \{P_i \in \mathbb{D} : r_1(P_i) \in M\}$, i.e., there exists a conditional dictatorial coefficient $\varepsilon_i \geq 0$ for each $i \in N$ with $\sum_{i \in N} \varepsilon_i = 1$ such that $\varphi(P) = \sum_{i \in N} \varepsilon_i \boldsymbol{e}_{r_1(P_i)}$ for all $P \in \overline{\mathbb{D}}^n$.*

Proof: We verify this lemma in two steps. In the first step, we restrict attention to the case $n = 2$, i.e., $N = \{1, 2\}$, and show by Claims 1 - 4 below that every two-voter unanimous and strategy-proof RSCF on $\mathbb{D}$ behaves like a random dictatorship on subdomain $\overline{\mathbb{D}}$. In the second step, we extend the result to the case $n > 2$ by adopting the Ramification Theorem of Chatterji et al. (2014).

Fix a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^2 \to \Delta(A)$. By Lemma 18, φ satisfies the tops-only property.

Claim 1: The following two statements hold:

(i) Given an alternative-path $\{z_k\}_{k=1}^l$, we have $\sum_{k=1}^l \varphi_{z_k}(z_1, z_l) = 1$.

(ii) Given a circle $\{z_k\}_{k=1}^l$, we have $\varphi_{z_s}(z_s, z_t) + \varphi_{z_t}(z_s, z_t) = 1$ for all $s \neq t$.

The first statement follows immediately from unanimity and the uncompromising property. Next, consider the circle $\{z_k\}_{k=1}^l$. Fixing z_s and z_t, assume w.l.o.g. that $s < t$. There are two alternative-paths connecting z_s and z_t: the clockwise alternative-path $\mathcal{P} = \{z_s, z_{s+1}, \ldots, z_t\}$ and the counter clockwise alternative-path $\mathcal{P}' = \{z_s, z_{s-1}, \ldots, z_1, z_l, z_{l-1}, \ldots, z_t\}$. It follows immediately from statement (i) that $\sum_{z \in \mathcal{P}} \varphi_z(z_s, z_t) = 1$ and $\sum_{z \in \mathcal{P}'} \varphi_z(z_s, z_t) = 1$. Last, since $\mathcal{P} \cap \mathcal{P}' = \{z_s, z_t\}$, it is true that $\varphi_{z_s}(z_s, z_t) + \varphi_{z_t}(z_s, z_t) = 1$. This completes the verification of the claim.

Claim 2: According to the cycle $\mathcal{C}_1 = \{x_k\}_{k=1}^p$ of Observation 2, φ behaves like a random dictatorship on the subdomain $\mathbb{D}^{\mathcal{C}_1} = \{P_i \in \mathbb{D} : r_1(P_i) \in \mathcal{C}_1\}$, i.e., there exists $0 \leq \varepsilon \leq 1$ such that $\varphi(x_k, x_{k'}) = \varepsilon \boldsymbol{e}_{x_k} + (1-\varepsilon)\boldsymbol{e}_{x_{k'}}$ for all $x_k, x_{k'} \in \mathcal{C}_1$.

Claim 1(ii) first implies $\varphi_{x_1}(x_1, x_2) + \varphi_{x_2}(x_1, x_2) = 1$. Let $\varepsilon = \varphi_{x_1}(x_1, x_2)$ and $1 - \varepsilon = \varphi_{x_2}(x_1, x_2)$. Fix another profile $(x_k, x_{k'})$. If $x_k = x_{k'}$, unanimity implies $\varphi(x_k, x_{k'}) = \varepsilon e_{x_k} + (1-\varepsilon) e_{x_{k'}}$. We next assume $x_k \neq x_{k'}$. There are four possible cases: (i) $x_1 \neq x_k$ and $x_2 = x_{k'}$, (ii) $x_1 = x_k$ and $x_2 \neq x_{k'}$, (iii) $x_1 \neq x_k$, $x_2 \neq x_{k'}$ and $(x_k, x_{k'}) \neq (x_2, x_1)$, and (iv) $(x_k, x_{k'}) = (x_2, x_1)$.

Since cases (i) and (ii) are symmetric, we focus on the verification of case (i), and omit the consideration of case (ii). We first have $\varphi_{x_k}(x_k, x_2) + \varphi_{x_2}(x_k, x_2) = 1$ by Claim 1(ii). We next show $\varphi_{x_2}(x_k, x_2) = 1 - \varepsilon$. Note that there exists an alternative-path in $\mathcal{C}_1$ that connects x_1 and x_k, and excludes x_2. Then, according to this alternative-path, the uncompromising property implies $\varphi_{x_2}(x_k, x_2) = \varphi_{x_2}(x_1, x_2) = 1 - \varepsilon$, as required.

In case (iii), we first know either $x_k \notin \{x_1, x_2\}$ or $x_{k'} \notin \{x_1, x_2\}$. Assume w.l.o.g. that $x_k \notin \{x_1, x_2\}$. Then, by the verification of cases (i), from (x_1, x_2) to (x_k, x_2), we have $\varphi(x_k, x_2) = \varepsilon \boldsymbol{e}_{x_k} + (1-\varepsilon)\boldsymbol{e}_{x_2}$. Furthermore, by case (ii), from (x_k, x_2) to $(x_k, x_{k'})$, we eventually have $\varphi(x_k, x_{k'}) = \varepsilon \boldsymbol{e}_{x_k} + (1-\varepsilon)\boldsymbol{e}_{x_{k'}}$.

Last, in case (iv), since the cycle $\mathcal{C}_1$ contains at least three alternatives, we first consider the profile (x_3, x_2) and have $\varphi(x_3, x_2) = \varepsilon \boldsymbol{e}_{x_3} + (1-\varepsilon)\boldsymbol{e}_{x_2}$ by the verification of case (i). Next, according to the verification of case (iii), from (x_3, x_2) to (x_2, x_1), we induce $\varphi(x_2, x_1) = \varepsilon \boldsymbol{e}_{x_2} + (1-\varepsilon)\boldsymbol{e}_{x_1}$. This completes the verification of the claim.

[19] By the identification of $a_{\underline{k}}$, we know that there exist at least two distinct alternatives of M that are adjacent to $a_{\underline{k}}$ in $\mathbb{D}$. Then, we can identify two distinct alternative-paths in M which connect $a_{\underline{k}}$ and $a_{\overline{k}}$. From these two alternative-paths, we can elicit a cycle in M that includes $a_{\underline{k}}$.

Symmetrically, according to the circle $\mathcal{C}_2$ of Observation 2, φ also mimics a random dictatorship on the subdomain $\mathbb{D}^{\mathcal{C}_2} = \{P_i \in \mathbb{D} : r_1(P_i) \in \mathcal{C}_2\}$, i.e., there exists $0 \leq \varepsilon' \leq 1$ such that $\varphi(y_k, y_{k'}) = \varepsilon' \boldsymbol{e}_{y_k} + (1-\varepsilon')\boldsymbol{e}_{y_{k'}}$ for all $y_k, y_{k'} \in \mathcal{C}_2$.

Claim 3: We have (i) $\varepsilon = \varepsilon'$, (ii) $\varphi(a_{\underline{k}}, a_{\overline{k}}) = \varepsilon\, \boldsymbol{e}_{a_{\underline{k}}} + (1-\varepsilon)\boldsymbol{e}_{a_{\overline{k}}}$, and (iii) $\varphi(a_{\overline{k}}, a_{\underline{k}}) = \varepsilon\, \boldsymbol{e}_{a_{\overline{k}}} + (1-\varepsilon)\boldsymbol{e}_{a_{\underline{k}}}$.

According to the graph $G_{\mathbb{D}}$ and the two cycles $\mathcal{C}_1$ and $\mathcal{C}_2$, we can construct an alternative-path $\mathcal{P} = \{z_1, z_2, \ldots, z_{l-1}, z_l\} \subseteq M$ such that (i) $l \geq 3$, (ii) $z_1, z_2 \in \mathcal{C}_1$ and $a_{\underline{k}} \in \{z_1, z_2\}$, and (iii) $z_{l-1}, z_l \in \mathcal{C}_2$ and $a_{\overline{k}} \in \{z_{l-1}, z_l\}$. First, Claim 2 and the uncompromising property imply $\varepsilon = \varphi_{z_1}(z_1, z_2) = \varphi_{z_1}(z_1, z_l)$ and $1 - \varepsilon = \varphi_{z_1}(z_2, z_1) = \varphi_{z_1}(z_l, z_1)$. Symmetrically, we have $1 - \varepsilon' = \varphi_{z_l}(z_{l-1}, z_l) = \varphi_{z_l}(z_1, z_l)$ and $\varepsilon' = \varphi_{z_l}(z_l, z_{l-1}) = \varphi_{z_l}(z_l, z_1)$. Thus, $\varepsilon + 1 - \varepsilon' = \varphi_{z_1}(z_1, z_l) + \varphi_{z_l}(z_1, z_l) \leq 1$ which implies $\varepsilon \leq \varepsilon'$, and $1 - \varepsilon + \varepsilon' = \varphi_{z_1}(z_l, z_1) + \varphi_{z_l}(z_l, z_1) \leq 1$ which implies $\varepsilon \geq \varepsilon'$. Therefore, $\varepsilon = \varepsilon'$. This completes the verification of statement (i).

Since statements (ii) and (iii) are symmetric, we focus on showing statement (ii) and omit the consideration of statement (iii). First, by the verification of statement (i), we have $\varphi(z_1, z_l) = \varepsilon\, \boldsymbol{e}_{z_1} + (1-\varepsilon)\boldsymbol{e}_{z_l}$. Second, according to $\mathcal{P}$, the uncompromising property implies $\varphi_{z_l}(z_2, z_l) = \varphi_{z_l}(z_1, z_l) = 1 - \varepsilon$ and $\varphi_{z_k}(z_2, z_l) = \varphi_{z_k}(z_1, z_l) = 0$ for all $2 < k < l$. Moreover, since $\sum_{k=2}^{l} \varphi_{z_k}(z_2, z_l) = 1$ by Claim 1(i), we have $\varphi_{z_2}(z_2, z_l) = 1 - \varphi_{z_l}(z_2, z_l) = \varepsilon$, and hence $\varphi(z_2, z_l) = \varepsilon\, \boldsymbol{e}_{z_2} + (1-\varepsilon)\boldsymbol{e}_{z_l}$. Symmetrically, we also have $\varphi(z_1, z_{l-1}) = \varepsilon\, \boldsymbol{e}_{z_1} + (1-\varepsilon)\boldsymbol{e}_{z_{l-1}}$. Recall that $a_{\underline{k}} \in \{z_1, z_2\}$ and $a_{\overline{k}} \in \{z_{l-1}, z_l\}$. We hence conclude that when $a_{\underline{k}} = z_1$ or $a_{\overline{k}} = z_l$, $\varphi(a_{\underline{k}}, a_{\overline{k}}) = \varepsilon\, \boldsymbol{e}_{a_{\underline{k}}} + (1-\varepsilon)\boldsymbol{e}_{a_{\overline{k}}}$. Last, we show that when $a_{\underline{k}} = z_2$ and $a_{\overline{k}} = z_{l-1}$, $\varphi(a_{\underline{k}}, a_{\overline{k}}) = \varepsilon\, \boldsymbol{e}_{a_{\underline{k}}} + (1-\varepsilon)\boldsymbol{e}_{a_{\overline{k}}}$. According to $\mathcal{P}$, the uncompromising property implies $\varphi_{a_{\underline{k}}}(a_{\underline{k}}, a_{\overline{k}}) = \varphi_{z_2}(z_2, z_{l-1}) = \varphi_{z_2}(z_2, z_l) = \varepsilon$ and $\varphi_{a_{\overline{k}}}(a_{\underline{k}}, a_{\overline{k}}) = \varphi_{z_{l-1}}(z_2, z_{l-1}) = \varphi_{z_{l-1}}(z_1, z_{l-1}) = 1 - \varepsilon$, as required. This completes the verification of statement (ii), and hence proves the claim.

Claim 4: Given distinct $a_s, a_t \in M$, $\varphi(a_s, a_t) = \varepsilon\, \boldsymbol{e}_{a_s} + (1-\varepsilon)\boldsymbol{e}_{a_t}$.

First, consider the situation that there exists $\mathcal{P}_l \in \Pi(a_1, a_m)$ such that $a_s, a_t \in \mathcal{P}_l$. Since $a_s, a_t \in M$, the interval $[a_{\underline{k}}, a_{\overline{k}}]^{\mathcal{P}_l} \equiv \{x_k\}_{k=1}^{l} \subseteq M$ must include a_s and a_t. By Claim 3, we have $\varphi(x_1, x_l) = \varepsilon\, e_{x_1} + (1-\varepsilon)e_{x_l}$ and $\varphi(x_l, x_1) = \varepsilon\, e_{x_l} + (1-\varepsilon)e_{x_1}$. Then, according to the alternative-path $\{x_k\}_{k=1}^{l}$, by repeatedly applying Claim 1(i) and the uncompromising property, we have $\varphi(x_k, x_{k'}) = \varepsilon\, \boldsymbol{e}_{x_k} + (1-\varepsilon)\boldsymbol{e}_{x_{k'}}$ for all distinct $1 \leq k, k' \leq l$. Hence, $\varphi(a_s, a_t) = \varepsilon\, \boldsymbol{e}_{a_s} + (1-\varepsilon)\boldsymbol{e}_{a_t}$.

Next, consider the situation that there exists no $\mathcal{P}_l \in \Pi(a_1, a_m)$ that includes both a_s and a_t. According to Observation 1, it must be the case that $a_s \notin \{a_{\underline{k}}, a_{\overline{k}}\}$ and $a_t \notin \{a_{\underline{k}}, a_{\overline{k}}\}$. Moreover, by Observation 1, let $\{b_k\}_{k=1}^{l} \subseteq M$ be an alternative-path that connects $a_{\underline{k}}$ and $a_{\overline{k}}$, and includes a_s, and let $\{c_k\}_{k=1}^{u} \subseteq M$ be an alternative-path that connects $a_{\underline{k}}$ and $a_{\overline{k}}$, and includes a_t. Evidently, $a_s \notin \{c_k\}_{k=1}^{n}$ and $a_t \notin \{b_k\}_{k=1}^{l}$. Let $a_s = b_p$ and $a_t = c_q$ for some $1 < p < l$ and $1 < q < u$. According to the sub-alternative-paths $\{b_1, b_2, \ldots, b_p\}$ and $\{c_1, c_2, \ldots, c_q\}$, since $b_1 = c_1 = a_{\underline{k}}$, $b_p \notin \{c_k\}_{k=1}^{u}$ and $c_q \notin \{b_k\}_{k=1}^{l}$, we identify $1 \leq \eta < p$ and $1 \leq \nu < q$ such that $b_\eta = c_\nu$ and $\{b_{\eta+1}, \ldots, b_p\} \cap \{c_{\nu+1}, \ldots, c_q\} = \emptyset$. Then, we have the concatenated alternative-path $\mathcal{P} = \{a_s = b_p, \ldots, b_\eta = c_\nu, \ldots, c_q = a_t\} \subseteq M$ which connects a_s and a_t. By the verification in the first situation, we have $\varphi_{b_p}(b_p, b_\eta) = \varepsilon$ and $\varphi_{c_q}(c_\nu, c_q) = 1 - \varepsilon$. Furthermore, according to $\mathcal{P}$, the uncompromising property implies $\varphi_{a_s}(a_s, a_t) = \varphi_{b_p}(b_p, c_q) = \varphi_{b_p}(b_p, c_\nu) = \varphi_{b_p}(b_p, b_\eta) = \varepsilon$ and $\varphi_{a_t}(a_s, a_t) = \varphi_{c_q}(b_p, c_q) = \varphi_{c_q}(b_\eta, c_q) = \varphi_{c_q}(c_\nu, c_q) = 1 - \varepsilon$. Therefore, $\varphi(a_s, a_t) = \varepsilon\, \boldsymbol{e}_{a_s} + (1-\varepsilon)\boldsymbol{e}_{a_t}$. This completes the verification of the claim.

In conclusion, every two-voter unanimous and strategy-proof RSCF behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}}$. For the general case $n > 2$, we adopt an induction argument.

Induction Hypothesis: Given $n \geq 3$, for all $2 \leq n' < n$, every unanimous and strategy-proof $\psi : \mathbb{D}^{n'} \to \Delta(A)$ behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}}$.

Given a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$, $n > 2$, we show that it behaves like a random dictatorship on the subdomain $\overline{\mathbb{D}}$. If $n \geq 4$, the verification follows exactly from Propositions 5 and 6 of Chatterji et al. (2014). Therefore, we focus on the case $n = 3$, i.e., $N = \{1, 2, 3\}$. Analogous to Propositions 4 and 6 of Chatterji et al. (2014), we split the verification into the following two parts:

1. There exists $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ such that for all $P \in \overline{\mathbb{D}}^3$, we have $[P_i = P_j$ for some distinct $i, j \in N] \Rightarrow \big[\varphi(P) = \varepsilon_1\, e_{r_1(P_1)} + \varepsilon_2\, e_{r_1(P_2)} + \varepsilon_3\, e_{r_1(P_3)}\big]$.
2. For all $P \in \overline{\mathbb{D}}^3$, we have $\varphi(P) = \varepsilon_1\, e_{r_1(P_1)} + \varepsilon_2\, e_{r_1(P_2)} + \varepsilon_3\, e_{r_1(P_3)}$.

The second part follows exactly from Proposition 6 of Chatterji et al. (2014). Therefore, we focus on showing the first part.[20]

According to φ, we first induce three two-voter RSCFs by merging two voters respectively: For all $P_1, P_2, P_3 \in \mathbb{D}$, let $\psi^1(P_1, P_2) = \varphi(P_1, P_2, P_2)$, $\psi^2(P_1, P_2) = \varphi(P_1, P_2, P_1)$ and $\psi^3(P_1, P_3) = \varphi(P_1, P_1, P_3)$. It is easy to verify that all ψ^1, ψ^2 and ψ^3 are unanimous and strategy-proof on $\mathbb{D}$. Therefore, the induction hypothesis implies that there exist $0 \leq \varepsilon_1, \varepsilon_2, \varepsilon_3 \leq 1$ such that for all $P_1, P_2, P_3 \in \overline{\mathbb{D}}$, $\psi^1(P_1, P_2) = \varepsilon_1\, \boldsymbol{e}_{r_1(P_1)} + (1 - \varepsilon_1)\boldsymbol{e}_{r_1(P_2)}$, $\psi^2(P_1, P_2) = (1 - \varepsilon_2)\boldsymbol{e}_{r_1(P_1)} + \varepsilon_2\, \boldsymbol{e}_{r_1(P_2)}$ and $\psi^3(P_1, P_3) = (1 - \varepsilon_3)\boldsymbol{e}_{r_1(P_1)} + \varepsilon_3\, \boldsymbol{e}_{r_1(P_3)}$. Note that to show the first part holds, it suffices to prove $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$.

Recall the cycle $\mathcal{C}_1 = \{x_k\}_{k=1}^p \subseteq M$ in Observation 2. First, according to the three alternative-paths $\{x_2, x_3\}$, $\{x_1, x_2\}$ and $\{x_1, x_p, \ldots, x_4, x_3\}$ in $\mathcal{C}_1$, the uncompromising property implies respectively that (i) $\varphi_{x_1}(x_1, x_2, x_3) = \varphi_{x_1}(x_1, x_2, x_2) = \psi^1_{x_1}(x_1, x_2) = \varepsilon_1$ and $\varphi_{a_s}(x_1, x_2, x_3) = \varphi_{a_s}(x_1, x_2, x_2) = \psi^1_{a_s}(x_1, x_2) = 0$ for all $a_s \notin \{x_1, x_2, x_3\}$, (ii) $\varphi_{x_3}(x_1, x_2, x_3) = \varphi_{x_3}(x_2, x_2, x_3) = \psi^3_{x_3}(x_2, x_3) = \varepsilon_3$, and (iii) $\varphi_{x_2}(x_1, x_2, x_3) = \varphi_{x_2}(x_3, x_2, x_3) = \psi^2_{x_2}(x_3, x_2) = \varepsilon_2$. Then, we have $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varphi_{x_1}(x_1, x_2, x_3) + \varphi_{x_2}(x_1, x_2, x_3) + \varphi_{x_3}(x_1, x_2, x_3) + \sum_{a_s \notin \{x_1, x_2, x_3\}} \varphi_{a_s}(x_1, x_2, x_3) = \sum_{a_s \in A} \varphi_{a_s}(x_1, x_2, x_3) = 1$, as required. This completes the verification of the induction hypothesis, and hence proves Lemma 20. □

LEMMA **21** *Let $\varphi : \mathbb{D}^n \to \Delta(A)$ be a unanimous and strategy-proof RSCF. Given distinct $a_s, a_t \in M$ and $P_{-i} \in \mathbb{D}^{n-1}$, we have $\varphi_{a_k}(a_s, P_{-i}) = \varphi_{a_k}(a_t, P_{-i})$ for all $a_k \notin \{a_s, a_t\}$.*

Proof: First, Lemma 18 implies that φ satisfies the tops-only property, and Lemma 20 implies that φ mimics a random dictatorship on the subdomain $\overline{\mathbb{D}} = \{P_i \in \mathbb{D} : r_1(P_i) \in M\}$.

CLAIM 1: The two statements hold: (i) $[a_{\underline{k}} \notin \{a_s, a_t\}] \Rightarrow [\varphi_{a_{\underline{k}}}(a_s, P_{-i}) = \varphi_{a_{\underline{k}}}(a_t, P_{-i})]$, and (ii) $[a_{\overline{k}} \notin \{a_s, a_t\}] \Rightarrow \big[\varphi_{a_{\overline{k}}}(a_s, P_{-i}) = \varphi_{a_{\overline{k}}}(a_t, P_{-i})\big]$.

By symmetry, we focus on showing statement (i) and omit the consideration of statement (ii). Note that if there exists an alternative-path that connects a_s and a_t and excludes $a_{\underline{k}}$, then the uncompromising property implies $\varphi_{a_{\underline{k}}}(a_s, P_{-i}) = \varphi_{a_{\underline{k}}}(a_t, P_{-i})$. Therefore, to complete the verification, we will construct such an alternative-path.

If $a_s \neq a_{\overline{k}}$, we pick an alternative-path $\langle a_{\underline{k}}, a_{\overline{k}} \rangle$ that includes a_s by Observation 1, and elicit the sub-alternative-path $\langle a_s, a_{\overline{k}} \rangle$. If $a_s = a_{\overline{k}}$, we refer to $\langle a_s, a_{\overline{k}} \rangle = \{a_s\}$. Thus, $a_{\underline{k}} \notin \langle a_s, a_{\overline{k}} \rangle$. Similarly, we have an alternative-path $\langle a_{\overline{k}}, a_t \rangle$ which excludes $a_{\underline{k}}$. According to $\langle a_s, a_{\overline{k}} \rangle$ and $\langle a_{\overline{k}}, a_t \rangle$, we construct an alternative-path which connects a_s and a_t, and excludes $a_{\underline{k}}$, as required. This completes the verification of the claim.

Since $a_s, a_t \in M$, by the verification of Claim 4 in the proof of Lemma 20, there exists an alternative-path $\{x_k\}_{k=1}^p \subseteq M$ connecting a_s and a_t. The uncompromising property first implies $\varphi_{a_k}(a_s, P_{-i}) = \varphi_{a_k}(a_t, P_{-i})$ for all $a_k \notin \{x_k\}_{k=1}^p$. Therefore, to complete the proof of the lemma, it suffices to show that $\varphi_{x_k}(a_s, P_{-i}) =$

[20] Proposition 4 of Chatterji et al. (2014) is not applicable for the verification of the first part since they impose an additional domain condition (see their Definition 18) which cannot be confirmed on domain $\mathbb{D}$.

$\varphi_{x_k}(a_t, P_{-i})$ for all $k = 2, \ldots, p-1$. If $x_k \in \{a_{\underline{k}}, a_{\overline{k}}\}$, it follows immediately from Claim 1 that $\varphi_{x_k}(a_s, P_{-i}) = \varphi_{x_k}(a_t, P_{-i})$. Hence, we let $\Theta = \{x_2, \ldots, x_{p-1}\} \backslash \{a_{\underline{k}}, a_{\overline{k}}\}$ and show $\varphi_z(a_s, P_{-i}) = \varphi_z(a_t, P_{-i})$ for all $z \in \Theta$.

For notational convenience, let $i = n$. We partition $\{1, \ldots, n-1\}$ into three parts: $\underline{I} = \{1, \ldots, j\}$, $\overline{I} = \{j+1, \ldots, l\}$ and $\hat{I} = \{l+1, \ldots, n-1\}$, and assume w.l.o.g that $r_1(P_1), \ldots, r_1(P_j) \in L \backslash \{a_{\underline{k}}\}$, $r_1(P_{j+1}), \ldots, r_1(P_l) \in R \backslash \{a_{\overline{k}}\}$ and $r_1(P_{l+1}), \ldots, r_1(P_{n-1}) \in M$. Note that if $l = 0$, Lemma 20 implies $\varphi_z(a_s, P_{-n}) = \varphi_z(a_t, P_{-n})$ for all $z \in \Theta$. Next, assume $l > 0$. We construct the following preference profiles: $P^{(\eta)} = \big(P_1, \ldots, P_\eta, \frac{a_{\underline{k}}}{\{\eta+1, \ldots, j\}}, \frac{a_{\overline{k}}}{\overline{I}}, P_{\hat{I}}, a_s\big)$, $\eta = 0, 1, \ldots, j$, and $P^{(\nu)} = \big(P_{\underline{I}}, P_{j+1}, \ldots, P_\nu, \frac{a_{\overline{k}}}{\{\nu+1, \ldots, l\}}, P_{\hat{I}}, a_s\big)$, $\nu = j+1, \ldots, l$. Note that $P^{(0)} = \big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\overline{I}}, P_{\hat{I}}, a_s\big)$ and $P^{(l)} = (a_s, P_{-n})$.

Given an arbitrary $0 \leq \eta < j$, consider $P^{(\eta)}$ and $P^{(\eta+1)}$. Note that voter $\eta+1$ has the preference peak $a_{\underline{k}}$ at $P^{(\eta)}$, and has the preference peak $r_1(P_{\eta+1}) = a_k \prec a_{\underline{k}}$ at $P^{(\eta+1)}$. By Lemma 16, $\{a_k, a_{k+1}, \ldots, a_{\underline{k}}\} \subseteq L$ is the unique alternative-path that connects a_k and $a_{\underline{k}}$, and hence excludes all alternatives of Θ. Then, the uncompromising property implies $\varphi_z(P^{(\eta)}) = \varphi_z(P^{(\eta+1)})$ for all $z \in \Theta$. Therefore, we have $\varphi_z(P^{(0)}) = \cdots = \varphi_z(P^{(j)})$ for all $z \in \Theta$. Next, given an arbitrary $j \leq \nu < l$, consider $P^{(\nu)}$ and $P^{(\nu+1)}$. Note that voter $\nu+1$ has the preference peak $a_{\overline{k}}$ at $P^{(\nu)}$, and has the preference peak $r_1(P_{\nu+1}) = a_k \succ a_{\overline{k}}$ at $P^{(\nu+1)}$. By Lemma 16, $\{a_{\underline{k}}, \ldots, a_{k-1}, a_k\} \subseteq R$ is the unique alternative-path that connects $a_{\overline{k}}$ and a_k, and hence excludes all alternatives of Θ. Then, the uncompromising property implies $\varphi_z(P^{(\nu)}) = \varphi_z(P^{(\nu+1)})$ for all $z \in \Theta$. Therefore, we have $\varphi_z(P^{(j)}) = \cdots = \varphi_z(P^{(l)})$ for all $z \in \Theta$. In conclusion, $\varphi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\overline{I}}, P_{\hat{I}}, a_s\big) = \varphi_z(P^{(0)}) = \cdots = \varphi_z(P^{(l)}) = \varphi_z(a_s, P_{-n})$ for all $z \in \Theta$.

Symmetrically, we also derive $\varphi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\overline{I}}, P_{\hat{I}}, a_t\big) = \varphi_z(a_t, P_{-n})$ for all $z \in \Theta$. Last, since Lemma 20 implies $\varphi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\overline{I}}, P_{\hat{I}}, a_s\big) = \varphi_z\big(\frac{a_{\underline{k}}}{\underline{I}}, \frac{a_{\overline{k}}}{\overline{I}}, P_{\hat{I}}, a_t\big)$ for all $z \in \Theta$, we have $\varphi_z(a_s, P_{-n}) = \varphi_z(a_t, P_{-n})$ for all $z \in \Theta$, as required. ☐

Now, fixing a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$, we are ready to show that the corresponding random voting scheme $\varphi : A^n \to \Delta(A)$ is locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$.

Fix $i \in N$, $P_i, P_i' \in \mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ with $P_i \sim P_i'$, and $P_{-i} \in \big[\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})\big]^{n-1}$. For notational convenience, let $r_1(P_i) = a_s$, $r_1(P_i') = a_t$ and $r_1(P_j) = x_j$ for all $j \neq i$. Let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. We show that $\varphi(a_s, x_{-i})$ stochastically dominates $\varphi(a_t, x_{-i})$ according to P_i. If $a_s = a_t$, $\varphi(a_s, x_{-i}) = \varphi(a_t, x_{-i})$, as required. Next, assume $a_s \neq a_t$. Then, $P_i \sim P_i'$ implies $r_1(P_i) = r_2(P_i') = a_s$, $r_1(P_i') = r_2(P_i) = a_t$ and $r_k(P_i) = r_k(P_i')$ for all $k = 3, \ldots, m$. To complete the verification, it suffices to show $\varphi_{a_s}(a_s, x_{-i}) \geq \varphi_{a_s}(a_t, x_{-i})$ and $\varphi_{a_k}(a_s, x_{-i}) = \varphi_{a_k}(a_t, x_{-i})$ for all $a_k \notin \{a_s, a_t\}$. Since $r_1(P_i) = a_s$, $r_1(P_i') = a_t$ and $P_i \sim P_i'$, we know $a_s \sim a_t$ in $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. Then, there are three possible cases: (i) $a_s, a_t \in L$ and $|s-t| = 1$, (ii) $a_s, a_t \in R$ and $|s-t| = 1$, and (iii) $a_s, a_t \in M$. The first two cases are symmetric, and hence we focus on the verification of the first case and omit the consideration of the second case. In the first case, since $|s-t| = 1$, it is also true that $a_s \sim a_t$ in $\mathbb{D}$. Hence, we have $\bar{P}_i, \bar{P}_i' \in \mathbb{D}$ such that $r_1(\bar{P}_i) = a_s$, $r_1(\bar{P}_i') = a_t$ and $\bar{P}_i \sim \bar{P}_i'$. Then, the tops-only property and strategy-proofness of φ on $\mathbb{D}$ imply $\varphi_{a_s}(a_s, x_{-i}) = \varphi_{a_s}(\bar{P}_i, x_{-i}) \geq \varphi_{a_s}(\bar{P}_i', x_{-i}) = \varphi_{a_s}(a_t, x_{-i})$, and $\varphi_{a_k}(a_s, x_{-i}) = \varphi_{a_k}(\bar{P}_i, x_{-i}) = \varphi_{a_k}(\bar{P}_i', x_{-i}) = \varphi_{a_k}(a_t, x_{-i})$ for all $a_k \notin \{a_s, a_t\}$, as required. Last, assume $a_s, a_t \in M$. Fixing $\bar{P}_i, \bar{P}_i' \in \mathbb{D}$ with $r_1(\bar{P}_i) = a_s$ and $r_1(\bar{P}_i') = a_t$ by minimal richness, we have $\varphi_{a_s}(a_s, x_{-i}) = \varphi_{a_s}(\bar{P}_i, x_{-i}) \geq \varphi_{a_s}(\bar{P}_i', x_{-i}) = \varphi_{a_s}(a_t, x_{-i})$ by the tops-only property and strategy-proofness of φ on $\mathbb{D}$, and $\varphi_{a_k}(a_s, x_{-i}) = \varphi_{a_k}(a_t, x_{-i})$ for all $a_k \notin \{a_s, a_t\}$ by Lemma 21, as required. Therefore, φ is locally strategy-proof on $\mathbb{D}_{\mathrm{H}}(\underline{k}, \overline{k})$. This completes the verification of the second part of Theorem 3 in the case $\overline{k} - \underline{k} > 1$, and hence completely proves Theorem 3.

D Proof of Fact 1

We first introduce some new notation and the formal definition of the no-restoration property of Sato (2013). Let $aP_i!b$ denote that a is *contiguously* preferred to b in P_i, i.e., aP_ib and there exists no $c \in A$ such that

aP_ic and cP_ib. Recall the notions of adjacency and path in the beginning of Section 2. A domain $\mathbb{D}$ satisfies the **no-restoration property** if for all distinct $P_i, P_i' \in \mathbb{D}$, there exists a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i' such that for all $a_p, a_q \in A$, we have

$$[a_pP_i^{k^*}a_q \text{ and } a_qP_i^{k^*+1}a_p \text{ for some } 1 \leq k^* < t] \Rightarrow [a_pP_i^ka_q \text{ for all } k = 1, \dots, k^*, \text{ and } a_qP_i^la_p \text{ for all } l = k^*+1, \dots, t].$$

By Theorem 1 of Cho (2018), to prove Fact 1, it suffices to show that $\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$ satisfies the no-restoration property. Before proceeding the proof, we introduce an important observation on $\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$.

Observation **3** Given $P_i \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$, let $r_1(P_i) = a_s$ and $a_pP_i!a_q$ (it is possible that $a_s = a_p$). Let P_i'' be a preference such that $P_i \sim P_i''$ and $a_qP_i''!a_p$. If one of the three conditions is satisfied: (i) $r_1(P_i) = r_1(P_i'')$, and $a_p \prec a_s \prec a_q$ or $a_q \prec a_s \prec a_p$, (ii) $r_1(P_i) = r_1(P_i'') \in M$ and neither both $a_p, a_q \in L$ nor both $a_p, a_q \in R$, and (iii) $r_1(P_i) \neq r_1(P_i'')$, and either $a_p, a_q \in L$ and $|p - q| = 1$, or $a_p, a_q \in R$ and $|p - q| = 1$, or $a_p, a_q \in M$, then $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. □

To show that $\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$ satisfies the no-restoration property, it suffices to show that for every pair of distinct preferences $P_i, P_i' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$, there exist $a_p, a_q \in A$ and $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$ such that $P_i \sim P_i''$, $a_pP_i!a_q$, $a_qP_i''!a_p$ and $a_qP_i'a_p$. Henceforth, we fix distinct $P_i, P_i' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$, and let $r_1(P_i) = a_s$ and $r_1(P_i') = a_t$.

We first assume $a_s = a_t$. We identify $1 < k \leq m$ such that $r_l(P_i) = r_l(P_i')$ for all $l = 1, \dots, k-1$, and $r_k(P_i) \neq r_k(P_i')$. Let $r_k(P_i') = a_q$ and $a_q = r_\nu(P_i)$ for some $k < \nu \leq m$. Meanwhile, let $r_{\nu-1}(P_i) = a_p$. We generate a preference P_i'' by locally switching a_p and a_q in P_i. Thus, $P_i \sim P_i''$, $a_pP_i!a_q$, $a_qP_i''!a_p$ and $a_qP_i'a_p$. Note that $r_1(P_i) = r_1(P_i'') = r_1(P_i')$. We next show $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. Suppose not, i.e., $P_i'' \notin \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. On the one hand, since P_i and P_i'' share the same peak and differ exactly on the relative rankings of a_p and a_q, $P_i \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$ and $P_i'' \notin \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$ imply that $a_qP_i''a_p$ must violate Definition 1. On the other hand, since P_i'' and P_i' share the same peak and the same relative ranking of a_p and a_q, $P_i' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$ implies that $a_qP_i''a_p$ does not violate Definition 1. Contradiction! Therefore, $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$.

Next, we assume $a_s \prec a_t$. The verification related to the situation $a_t \prec a_s$ is symmetric, and we hence omit it. We consider the four possible cases: (1) $a_s \prec a_{\underline{k}}$, (2) $a_{\overline{k}} \preceq a_s$, (3) $a_{\underline{k}} \preceq a_s \prec a_{\overline{k}} \preceq a_t$ and (4) $a_{\underline{k}} \preceq a_s \prec a_t \prec a_{\overline{k}}$.

In case (1), we notice $a_s \prec a_{s+1} \preceq a_{\underline{k}}$ and $a_s \prec a_{s+1} \preceq a_t$. Let $a_{s+1} = r_k(P_i)$ for some $1 < k \leq m$ and $r_{k-1}(P_i) = a_p$. Thus, $a_pP_i!a_{s+1}$. Since $r_1(P_i) = a_s \in L$, $a_pP_ia_{s+1}$ implies $a_p \preceq a_s$ by Definition 1. Hence, we know $a_p \preceq a_s \prec a_{s+1} \preceq a_{\underline{k}}$ and $a_p \preceq a_s \prec a_{s+1} \prec a_t$, which imply $a_{s+1}P_i'a_p$ by Definition 1. By locally switching a_p and a_{s+1} in P_i, we generate a preference P_i''. Thus, $P_i \sim P_i''$, $a_pP_i!a_{s+1}$, $a_{s+1}P_i''!a_p$ and $a_{s+1}P_i'a_p$. We last show $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') = r_1(P_i) = a_s$, it is true that $a_p \prec a_s \prec a_{s+1}$, and Observation 3(i) then implies $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') \neq r_1(P_i)$, it is true that $r_1(P_i) = a_s = a_p$ and $r_1(P_i'') = a_{s+1}$, and Observation 3(iii) then implies $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$.

The verification of case (2) is similar to that of case (1), and we hence omit it.

In case (3), let $a_{\overline{k}} = r_k(P_i)$ for some $1 < k \leq m$ and $r_{k-1}(P_i) = a_p$. Thus, $a_pP_i!a_{\overline{k}}$. Since $a_{\underline{k}} \preceq a_s \prec a_{\overline{k}}$, $a_pP_ia_{\overline{k}}$ implies $a_p \prec a_{\overline{k}}$ by Definition 1. Thus, we know either $a_p \prec a_{\underline{k}} \prec a_{\overline{k}} \preceq a_t$ which implies $a_{\overline{k}}P_i'a_{\underline{k}}$ and $a_{\underline{k}}P_i'a_p$ by Definition 1, or $a_{\underline{k}} \preceq a_p \prec a_{\overline{k}} \preceq a_t$ which implies $a_{\overline{k}}P_i'a_p$ by Definition 1. Overall, $a_{\overline{k}}P_i'a_p$. By locally switching a_p and $a_{\overline{k}}$ in P_i, we generate a preference P_i''. Thus, $P_i \sim P_i''$, $a_pP_i!a_{\overline{k}}$, $a_{\overline{k}}P_i''!a_p$ and $a_{\overline{k}}P_i'a_p$. We last show $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') = r_1(P_i) = a_s$, Observation 3(ii) implies $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') \neq r_1(P_i)$, it is true that $r_1(P_i) = a_s = a_p$ and $r_1(P_i'') = a_{\overline{k}}$, and Observation 3(iii) then implies $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$.

In case (4), let $a_t = r_k(P_i)$ for some $1 < k \leq m$ and $r_{k-1}(P_i) = a_p$. By locally switching a_p and a_t in P_i, we generate a preference P_i''. Thus, $P_i \sim P_i''$, $a_pP_i!a_t$, $a_tP_i''!a_p$ and $a_tP_i'a_p$ (recall $r_1(P_i') = a_t$). We last show $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') = r_1(P_i) = a_s$, Observation 3(ii) implies $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$. If $r_1(P_i'') \neq r_1(P_i)$, it is true that $r_1(P_i) = a_s = a_p$ and $r_1(P_i'') = a_t$, and Observation 3(iii) implies $P_i'' \in \mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$.

In conclusion, domain $\mathbb{D}_{\text{H}}(\underline{k}, \overline{k})$ satisfies the no-restoration condition of Sato (2013), as required. □