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Asset classes

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Abstract

This paper proposes a theory of endogenous differences in liquidity of assets based on the interaction between differences in their risk and differences in liquidity needs of agents. An equilibrium of the model displays a class structure, where agents sort themselves across different types of assets according to their types. High-liquidity-need agents hold onto safer portfolios than lower-liquidity-need agents whenever the variation in the value of liquidity across states raises the value of safe assets more than that of riskier assets, and vice versa. I also derive CAPM-like formulae for excess returns where a liquidity premium and absolute excess returns are added to the traditional risk premium, although the risk and liquidity premia are interdependent and specific to each type of agent.

Keywords: Liquidity; Asset Prices; Classes.

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1 Introduction

Assets differ greatly in their risk and liquidity characteristics. Although the early asset pricing literature focused exclusively on the role of risk,[1] it has since been recognized that liquidity can be an important factor in the pricing of an asset.[2] However, it is not clear whether we should expect any systematic relationship between risk and liquidity, and if yes, under what conditions and what that relationship would be. The existence of a systematic relationship would be important. First, it means that risk and liquidity premia are not independent, which has implications for asset pricing. Moreover, it also matters for the way data is used to infer the liquidity of assets and for the formulation of policy decisions.

In this paper I propose a theory of endogenous variations in liquidity across assets based on the interaction between the heterogeneity in their risk and the heterogeneity of consumers' liquidity needs, and I use the model to examine the role of individual and aggregate risks on the cross-sectional variations in asset returns, the velocity of trade, as well as the decomposition of asset returns into risk and liquidity premia. More specifically, I consider a simple three-period consumption-based Capital Asset Pricing Model (CAPM) augmented with features of Rocheteau and Wright's (2005) model of exchange to incorporate heterogeneous liquidity needs. On one side of the market there are claims that all mature in the last period and only intrinsically differ in the covariance of their payoffs with those of the market portfolio at maturity (the difference in risk). On the other side of the market there are consumers, who, like in Diamond and Dybvig (1983), may want to consume early, i.e., before assets mature, which creates a value for liquidity. The only *ex ante* difference among consumers is that some are more likely than others to want to consume early (the difference in the frequency of liquidity needs). As a result all consumers value risk at maturity the same way *ex ante*, while the differentiated frequency of liquidity needs means different types of consumers have different willingness to pay for the liquidity services rendered by the different types of assets.

The first main result of the paper is that whenever differences in liquidity needs are large enough the equilibrium displays a *class structure*, by which I mean that each type of consumer belongs to a given group, all consumers of that group *only* hold onto assets with risk in a certain range, and assets with risk in this range are held *only* by consumers in this group. The class structure arises because the difference across consumer types in willingness to pay for the value of liquidity services of different assets leads to different types of consumers choosing different portfolios in equilibrium. This leads to the formation of *asset classes* when consumers' liquidity needs are different enough: they wish to hold portfolios so different that a segmentation of the asset market appears.

A direct consequence of the existence of classes is that the asset pricing function is convex in risk, because safer (riskier) assets are held by consumers whose marginal valuation of assets is most (least) sensitive to risk. I also show that if instead one focuses on rates of returns, traditional CAPM formulae hold within each class and, as a result, the model admits a representation not unlike that of the traditional CAPM despite the existence of an endogenous class structure. There are a couple of notable differences, however. First, although the only factor in the model is risk, one can decompose the excess returns (beyond absolute excess

[1] See, for instance, the CAPM or the vast consumption-based asset pricing literature that followed Lucas (1978).

[2] A non-exhaustive list of theoretical contributions includes Glosten and Milgrom (1985), Kyle (1985), Amihud and Mendelson (1986), Constantidines (1986), Grossman and Miller (1988), Allen and Gale (1994), Shleifer and Vishny (1992, 1997), Geanakoplos (1997, 2003), Vayanos (1998, 1999), Brunnermeier and Pedersen (2009), Kiyotaki and Moore (2005), Duffie, et al. (2005), Weill (2008), Lagos and Rocheteau (2009), Lagos (2010), Geromichalos et al. (2007), Rocheteau (2011), Jacquet and Tan (2012). See also Amihud et al. (2012), Foucault et al. (2013), or Vayanos and Wang (2013) for extensive reviews of the empirical and theoretical literatures.

returns) of a risky asset relative to the risk-free asset into risk and liquidity premia: the risk premium is determined by the additional return an illiquid version of the risky asset must pay relative to an illiquid risk-free claim, while the liquidity premium is an adjustment that takes into account the covariance between returns at maturity and the marginal value of liquidity. But because the differences in liquidity services originate from differences in risk, the risk and liquidity premia of an asset must be determined jointly. And there is no market-wide premium for liquidity or for risk, for the split of excess returns into risk and liquidity premia varies across consumer types. Second, the convexity of the pricing function resulting from the existence of classes means the model is able to generate absolute excess returns for assets not belonging to the same class as riskless assets. This property of the equilibrium can be used to estimate the class structure of assets without using consumption data.

I then study sorting of consumers types across asset types when there are two types of consumers and the utility derived from consumption as an early and as a late consumer displays constant relative risk aversion (CRRA) with the same risk-aversion parameter. I show that sorting is driven by the variation across states of the marginal value of liquidity: if the variation across states in the marginal value of liquidity amplifies the effect of risk at maturity, it increases the value of claims whose payoffs have a low correlation with those of the market portfolio (the "safe" claims) *relative* to the value of claims whose payoffs are highly correlated with those of the market portfolio (the "risky" claims). And *vice versa* if the variation in the value of liquidity dampens the effect of risk at maturity. Since consumers with greater frequency of liquidity needs are disproportionately affected by the variation in the marginal value of liquidity, they hold safer portfolios in the former case and riskier portfolios in the latter. I also show that when sorting is observed in equilibrium, segmentation appears whenever the difference in frequency of liquidity needs across the two types of consumers is large enough. Otherwise both types of consumers belong to the same class.

Sorting implies that even though all assets are intrinsically equally liquid (they can be traded with the same ease), they differ in their velocity in equilibrium because of a *clientele effect*: some assets are traded more than others simply because they are held in greater proportion by agents who are more likely to need to consume early and therefore trade more often. This calls for caution when using some common liquidity measures that rely on velocity of trade to infer variations in assets' intrinsic liquidity properties.

In an extension I disentangle the roles of risk aversion and of the Intertemporal Elasticity of Substitution (IES) by replacing the CRRA utility function with a recursive utility function à la Epstein-Zin. The variation in the ratio of the marginal value of liquidity still plays a crucial role, but in this case the variation across states in the ratio only depends on the IES, while the coefficient of risk aversion determines how this variation affects sorting.

Related Literature - Among the literature that is interested in the role of market liquidity for asset pricing, this paper is particularly related to the papers that are interested in the role of liquidity differences and how these interact with risk in explaining the cross-sectional distribution of asset returns. In Vayanos (2004), assets differ exogenously in their transaction costs while investors are fund managers that are subject to withdrawals when fund performance falls below a certain threshold. This generates a preference for liquidity which is time varying and increasing in volatility, so that liquidity premia increase in volatile times. Acharya and Pedersen (2005) introduce differences in liquidity across assets, in the form of differentiated transaction costs, in an otherwise standard consumption-based CAPM framework. Their main focus is on the role of liquidity risk, i.e., the volatility of the transaction costs. Lagos (2010) studies a consumption-based asset pricing model à la Lucas, but augmented to allow assets to also facilitate trade. He considers two assets, a risk-free asset and a risky equity, and investigates theoretically and quantitatively the extent to which the risk-free rate and equity premium puzzles can be explained once differences in liquidity are taken into account.[3] He and Milbradt (2014) study the interaction between default and liquidity for corporate bonds that are traded in an over-the-counter market with search frictions in a model à la Duffie *et al.* (2005). In their paper the risk faced by a bond holder is endogenous, because it depends on the endogenous default decision by equity holders and the endogenous market liquidity, i.e., how easy it is to find a trading partner. Assuming that defaulted bonds are less liquid than non-defaulted bonds, the authors show that the closer a bond is to default, the more illiquid it is. There are two important differences between these papers and mine. First, they adopt a representative-agent framework and therefore cannot generate clientele effects. Second, with the exception of one version of Lagos' model, in these papers intrinsic differences in liquidity properties of assets, in the form of differences in either the cost or ease of trade, play an essential role in explaining differences in liquidity premia, whereas this paper generates differences in liquidity premia even when all assets are intrinsically equally liquid.

The paper is also related to the literature that is interested in the segmentation of markets with two-sided heterogeneity. Jacquet and Tan (2007) and Bidner *et al.* (2014) build on the work of Burdett and Coles (1997), Eeckhout (1999), and Smith (2006), and study how agents looking for a long-term partner can endogenously sort themselves into different markets when agents' search strategy implies the existence of a class structure. There also exists a literature that is interested in the segmentation of asset markets. Vayanos and Vila (2009) and Guibaud *et al.* (2013) study models of the yield curve where agents sort themselves into different maturities, based on either a preferred-habitat assumption (in the former) or on the part of an agent's life-cycle (in the latter). Vayanos and Wang (2007) and Vayanos and Weill (2007) develop models that explain why two assets with identical cash-flows can trade at different prices through endogenous differences in liquidity. In the latter work there exist long and short traders who meet according a matching function with increasing returns to scale, and the authors show there exists an asymmetric equilibrium where the concentration of short traders in one asset increases the supply of that asset increases the ease with which trading partners are met, i.e., its liquidity, and thus its price. In the former paper the difference is coming from a clientele effect where agents with short or long trading horizons specialize in one or the other asset.

Amihud and Mendelson (1986), Pagano (1989), Studart (2014), and Biais et al. (2018) also develop models of segmentation of asset markets with heterogeneous agents and heterogeneous assets. In the latter segmentation appears because agents with different degrees of aversion for risk trade on endogenously incomplete markets (due to imperfect pledgeability of assets). And in the former three papers sorting of agents across assets originates from exogenous and differentiated transaction costs: the clientele effect acts as an amplifier to the intrinsic liquidity differences that the differentiated transaction costs represent, which is in contrast to the present paper where assets are all intrinsically equally liquid. This difference has consequences regarding whether velocity or volume of trade and liquidity measures using these data are appropriate instruments for intrinsic liquidity properties of assets.

The paper is organized as follows. The next section introduces the setup. I then provide an illustrative example in section 3 before deriving the portfolio problem in section 4 and presenting the formation of asset classes and asset pricing implications, including CAPM-like formulae, in Section 5. Section 6 analyzes sorting in the case with two types of agents, while section 7 disentangles the roles of risk-aversion and of the

[3]See also Chapter 11.3 in Nosal and Rocheteau (2011).

Intertemporal Elasticity of Substitution for sorting. And section 8 concludes. All proofs not in the main text can be found, unless indicated otherwise, in the appendix at the end of this manuscript.

2 The Setup

2.1 The Model

The environment is a simple consumption-based CAPM in which I incorporate features of the competitive model of exchange in Rocheteau and Wright (2005) in order to introduce liquidity needs in a tractable way.

Time Periods and States - There are three time periods indexed by $t \in \{0, 1, 2\}$. The economy can be in either of two states, which is information revealed at $t = 1$ before trade takes place. The state of the world is indexed by $\omega \in \{l, h\}$, and the *ex ante* probability that the state of the economy is h is $\pi(h) = \pi$ and it is $\pi(l) = 1 - \pi$ for state l. There is one perfectly divisible and perishable good, which is traded in standard Walrasian markets at all dates.[4]

Lucas Trees and Securities - There is a mass E of trees and each tree is characterized by a fixed and known risk parameter ϵ. The risk parameter ϵ for the population of trees is distributed over $[\epsilon_{\min}, \epsilon_{\max}]$ according to some cumulative distribution function (*cdf*) G, with $\epsilon_{\min} \leq 0$ and $\epsilon_{\max} \in (0, (1-\pi)/\pi]$.[5] The quantities of goods delivered at date $t = 2$ by a type-ϵ tree in state h and l are respectively given by

$$d_\epsilon(h) = 1 + \epsilon \text{ and } d_\epsilon(l) = 1 - \frac{\pi}{1-\pi}\epsilon, \tag{1}$$

which ensures that in expected terms all trees yield one unit of the good at $t = 2$. I assume, for simplicity, that the distribution of risk is without mass points and denote by g the probability distribution function (*pdf*) of the distribution of risk associated with G.

There is also a mass E of securities, with each security corresponding to a claim to the payoffs of a given tree. Securities can at no point be shorted, stripped, or repackaged, but they can be freely traded in all periods, and in that sense all securities are intrinsically perfectly liquid. The risk of the market portfolio can be defined by $\mathcal{E} = \int_{\epsilon_{\min}}^{\epsilon_{\max}} \epsilon g(\epsilon) d\epsilon > 0$. Note that ϵ drives the covariance of its payoffs with those of the market portfolio at *maturity*, which is $(\pi/(1-\pi))\mathcal{E}\epsilon$. Low-$\epsilon$ claims are therefore lower risk claims than high-ϵ claims.[6]

Agents - The economy is populated by two types of three-period lived agents who do not discount the future. There is a mass one of each type of agents, called *consumers* and *producers* because of their respective roles in the middle period. All producers are alike and are indexed by 0, whereas the population of consumers is split into I types indexed by $i \in \{1, ..., I\}$, with μ^i denoting the mass of agents of type i. The type of an agent is fixed and known at $t = 0$ and is indicated as a superscript.

All agents can consume and produce the good at $t = 0$, and an agent who consumes and produces respectively c and h units of the good enjoys utility $c - h$ for the period.

[4] The results carry over if trade takes place in an over-the-counter (OTC) market at date $t = 1$ (see Jacquet, 2015).

[5] It is possible to assume that the lower bound is strictly positive without changing any of the results.

[6] If we take the model literally, ϵ also drives the variance of payoffs of a claim at maturity. However, a more general interpretation of the model is that there is some idiosyncratic risk associated with each claim, i.e., the payoff of a claim in state ω is given by $d_\epsilon(z|\omega)$ with z drawn from a distribution with cdf F, but that a law of large numbers applies so that the realized payoff of the subportfolio made of ϵ-type claims in an agent's portfolio is always $d_\epsilon(\omega) = \int d_\epsilon(z|\omega) dF(z)$ in state ω.

At $t = 1$ producers do not derive utility from consumption but they can produce the good. On the other hand, consumers cannot produce and for each type i a randomly drawn fraction σ^i want to consume, they are *early* consumers, while other consumers, the *late* consumers, do not obtain any utility out of consumption. In this middle period a producer producing h units of the good bears a utility cost $(A(\omega))^{-1}h$ when the state is ω, $A(\omega) > 0$, while an early consumers obtains utility $\theta(\omega)u(c)$ out of consuming c units of consumption. I assume, without loss of generality, that $\sigma^1 > \sigma^2 > ... > \sigma^I > 0$ and that consumers learn their preference shock at date $t = 1$ after the aggregate state ω is revealed.

No production takes place at date $t = 2$ and the utility of consuming c units is c for producers while it is $U(c)$ for early and late consumers.[7] I assume that u and U are both strictly increasing and twice continuously differentiable and strictly concave functions, with $\lim_{c\to 0^+} u'(c) = \lim_{c\to 0^+} U'(c) = +\infty$.

Each agent is endowed with the same portfolio $\mathbf{e}$ of securities which replicates the market portfolio and is such that $e_\epsilon = 0.5Eg(\epsilon)$, where a bold symbol indicates the vector of all values of the non-bold symbol defined on the range $[\epsilon_{\min}, \epsilon_{\max}]$, i.e., $\mathbf{e} \equiv (e_\epsilon)_{\epsilon\in[\epsilon_{\min},\epsilon_{\max}]}$ in this case.[8] I also assume that $U'(\mathbf{d}(h) \cdot 2\mathbf{e}) \geq 1$, where for any two vectors $\mathbf{p}$ and $\mathbf{m}$ defined on $[\epsilon_{\min}, \epsilon_{\max}]$ $\mathbf{p} \cdot \mathbf{m} \equiv \int p_\epsilon m_\epsilon d\epsilon$. This is a technical assumption which ensures that only consumers choose to hold onto claims when leaving the first period, because it implies that if claims are split evenly across all consumers the marginal value of each type of claims is strictly greater than 1 for all types, and therefore in equilibrium there is at least one type of consumers with a marginal value of all claims strictly greater than 1. Since, as I will show later, 1 is the marginal value of all claims for producers, they always leave the first period with empty portfolios.[9]

I further assume that: (i) agents cannot commit to honor contracts; (ii) contracts cannot be enforced; and (iii) agents are anonymous so that they cannot be punished for not abiding by a contract. These assumptions imply that exchange must be *quid pro quo*, and in particular agents need a medium of exchange in the form of either a means of payment or collateral at $t = 1$.[10]

2.2 Setting Up the Portfolio Problem and Equilibrium Definition

At this point we only need a couple of additional assumptions and a bit more notation in order to formally set up the problem of the different types of agents and define an equilibrium. First, I consider symmetric equilibria where all agents of the same type make the same decisions. Second, I restrict functions for initial portfolios and portfolio transfers at date 1 to being positive and integrable functions on $[\epsilon_{\min}, \epsilon_{\max}]$.

In terms of notation, denote respectively by c_0^i, h_0^i, and $\mathbf{x}^i$ the date-0 consumption, effort, and portfolio choice of a type-i agent, $i \in \{0, ..., I\}$. Also, denote by $c_1^i(\omega)$ the consumption at date 1 in state ω of an early consumer of type i, and let $\mathbf{y}^i(\omega)$ be the part of her portfolio she uses as payment. That is, $y_\epsilon^i(\omega) > 0$ indicates she is selling type-ϵ claims at date 1. Late consumers do not enjoy consuming early but could rebalance their portfolios in the middle period and $\mathbf{Y}^i(\omega)$ denotes the portfolio traded at date 1 by a type-i

[7]All the main results of the paper are essentially unchanged if one assumes that early consumers do not value consumption in the last period.

[8]The linearity of the date-0 utility function implies this assumption is without loss of generality.

[9]The proof can be found in the online appendix. If I do not make this assumption it can be that producers leave period 0 with some claims (at one end of the risk-spectrum), and when this happens the price of these claims must be 1.

[10]There is no difference in the current setup between selling an asset and obtaining credit by putting the asset down as collateral, as long as the entire asset can serve as collateral. I study the impact of frictions that restrict the fraction of a tree that can be used as collateral in section 3 of part 2 of the online appendix .

late consumer, where $Y^i_\epsilon(\omega) > 0$ indicates she is selling type-ϵ claims at date 1. Finally, $c^i_2(\omega)$ and $C^i_2(\omega)$ respectively denote the consumption at date 2 in state ω of an early and a late consumer of type i.

Let $\mathbf{y}^i \equiv (\mathbf{Y}^i(h), \mathbf{Y}^i(l), \mathbf{y}^i(h), \mathbf{y}^i(l))$, $\mathbf{c}^i_1 \equiv (c^i_1(h), c^i_1(l))$, and $\mathbf{c}^i_2 \equiv (C^i_2(h), C^i_2(l), c^i_2(h), c^i_2(l))$ for $i \in \{1, ..., I\}$, and let $\mathbf{q}$ and $\mathbf{Q}$ respectively denote the vectors of date-0 and date-1 prices for claims, with $\mathbf{Q} \equiv (\mathbf{Q}(h), \mathbf{Q}(l))$. The problem of a type-$i$ consumer then is to choose $\{c^i_0, h^i_0, \mathbf{x}^i, \mathbf{c}^i_1, \mathbf{y}^i, \mathbf{c}^i_2\}$ to maximize

$$c^i_0 - h^i_0 + \sum_{\omega \in \{h,l\}} \pi(\omega)\{(1-\sigma^i)\, U(C^i_2(\omega)) + \sigma^i \left[\theta(\omega) u(c^i_1(\omega)) + U(c^i_2(\omega))\right]\}.$$

subject to the following constraints:

- At date 0 - the budget constraint $c^i_0 + \mathbf{q} \cdot \mathbf{x}^i \leq h^i_0 + \mathbf{q} \cdot \mathbf{e}$, the feasibility constraints $c^i_0, h^i_0 \geq 0$ and no-short-selling constraints $x^i_\epsilon \geq 0$ for all ϵ;
- At date 1 in state $\omega \in \{h, l\}$ - either the consumer is an early consumer and faces the budget constraint $c^i_1(\omega) \leq \mathbf{Q}(\omega) \cdot \mathbf{y}^i(\omega)$ and the feasibility constraint $c^i_1(\omega) \geq 0$ and no-short-selling constraints $y^i_\epsilon(\omega) \leq x^i_\epsilon$ for all ϵ; or she is a late consumer in which case she faces the budget constraint $\mathbf{Q}(\omega) \cdot \mathbf{Y}^i(\omega) \geq 0$ and no-short-selling constraints $Y^i_\epsilon(\omega) \leq x^i_\epsilon$ for all ϵ.
- At date 2 in state $\omega \in \{h, l\}$ - the budget constraint $c^i_2(\omega) \leq \mathbf{d}(\omega) \cdot (\mathbf{x}^i - \mathbf{y}^i(\omega))$ for an early consumer and $C^i_2(\omega) \leq \mathbf{d}(\omega) \cdot (\mathbf{x}^i - \mathbf{Y}^i(\omega))$ for a late consumer.

As for producers, let how much they work at date 1 in state ω be $h^0_1(\omega)$, the portfolio they accept in exchange of their production be $\mathbf{y}^0(\omega)$, and their date-2 consumption in state ω be $c^0_2(\omega)$. If $\mathbf{h}^0_1 \equiv (h^0_1(h), h^0_1(l))$, $\mathbf{y}^0 \equiv (\mathbf{y}^0(h), \mathbf{y}^0(l))$, while $\mathbf{c}^0_2 \equiv (c^0_2(h), c^0_2(l))$, the problem of a producer is then to choose $\{c^0_0, h^0_0, \mathbf{x}^0, \mathbf{h}^0_1, \mathbf{y}^0, \mathbf{c}^0_2\}$ to maximize

$$c^0_0 - h^0_0 + \sum_{\omega \in \{h,l\}} \pi(\omega)[-(A(\omega))^{-1} h^0_1(\omega) + c^0_2(\omega)]$$

subject to the following constraints:

1. At date 0 - the budget constraint $c^0_0 + \mathbf{q} \cdot \mathbf{x}^0 \leq h^0_0 + \mathbf{q} \cdot \mathbf{e}$, the feasibility constraints $c^0_0, h^0_0 \geq 0$, and no-short-selling constraints $x^0_\epsilon \geq 0$ for all ϵ;
2. At date 1 in state $\omega \in \{h, l\}$ - the budget constraint $h^0_1(\omega) \geq \mathbf{Q}(\omega) \cdot \mathbf{y}^0(\omega)$, the feasibility constraints $h^0_1(\omega) \geq 0$ and no-short-selling constraints $y^0_\epsilon(\omega) \geq -x^0_\epsilon$ for all ϵ.
3. At date 2 in state $\omega \in \{h, l\}$ - the budget constraint $c^0_2(\omega) \leq \mathbf{d}(\omega) \cdot (\mathbf{x}^0 + \mathbf{y}^0(\omega))$.

We now have all the pieces needed to define a symmetric equilibrium.

Definition 1: *A symmetric equilibrium are prices for claims* $(\mathbf{q}, \mathbf{Q})$ *and allocations* $\{c^i_0, h^i_0, \mathbf{x}^i, \mathbf{c}^i_1, \mathbf{y}^i, \mathbf{c}^i_2\}^I_{i=1}$ *for consumers* and $\{c^0_0, h^0_0, \mathbf{x}^0, \mathbf{h}^0_1, \mathbf{y}^0, \mathbf{c}^0_2\}$ *for producers such that:*

1. *Given* $(\mathbf{q}, \mathbf{Q})$, $\{c^i_0, h^i_0, \mathbf{x}^i, \mathbf{c}^i_1, \mathbf{y}^i, \mathbf{c}^i_2\}$ *solves the problem of a type-i* consumer, *for each* $i \in \{1, ...I\}$, and $\{c^0_0, h^0_0, \mathbf{x}^0, \mathbf{h}^0_1, \mathbf{y}^0, \mathbf{c}^0_2\}$ *solves* the problem of a producer; and
2. *Markets clear:*

(a) *At date 0 -*

$$\textit{Goods}: \sum_{i=0}^{I} \mu^i c_0^i = \sum_{i=0}^{I} h_0^i, \textit{ and}$$

$$\textit{Claims}: \int_{\epsilon_{\min}}^{\epsilon} \sum_{i=0}^{I} \mu^i x_{\epsilon'}^i d\epsilon' = \int_{\epsilon_{\min}}^{\epsilon} 2e_{\epsilon'} d\epsilon' \textit{ for all } \epsilon;$$

(b) *At date 1 - for each* $\omega \in \{h, l\}$,

$$\textit{Goods}: \sum_{i=1}^{I} \mu^i \sigma^i c_1^i(\omega) = h_1^0(\omega) \textit{ and}$$

$$\textit{Claims}: \int_{\epsilon_{\min}}^{\epsilon} \sum_{i=1}^{I} \mu^i (\sigma^i y_{\epsilon'}^i(\omega) + (1 - \sigma^i) Y_{\epsilon'}^i(\omega)) d\epsilon' = \int_{\epsilon_{\min}}^{\epsilon} y_{\epsilon'}^0(\omega) d\epsilon' \textit{ for all } \epsilon.$$

2.3 Interpretation

The model is designed to capture the fact that most assets have a given maturity, which is infinity for some, whereas investors do not know with certainty the timing of their liquidity needs (for consumption or investment purposes), and some investors are more likely to face such liquidity needs. This means investors regularly need to sell assets in their portfolio before they mature, i.e., there is a mismatch between the maturity of assets in an agent's portfolio and the timing of her consumption. This in turn implies that agents care about more than just the risk of an asset at maturity.

The parameter ϵ drives the covariance of payoffs of an asset *at maturity* with those of the market portfolio. A T-bill, for instance, would be a $\epsilon = 0$ asset, for it always pays the same when it matures, whereas the stock of a blue chip company whose dividends do not fluctuate much would be a low $\epsilon > 0$ asset, and examples of $\epsilon = (1 - \pi)/\pi$ assets would be the stock of a highly-leveraged company that would file for bankruptcy in a recession or a call option that is in-the-money only in state h.

The parameter σ captures the *frequency* of liquidity needs, which is state independent but type specific, while θ measures the *intensity* of these needs and A impacts the terms of trade in the middle period, with both A and θ being state-dependent but independent of the type of the agent. These features of the model aim to capture the facts that (i) different types of investors face different liquidity needs through differences in the frequency of liquidity needs, and (ii) the extent of the liquidity needs can vary with market conditions through the variations in A and θ.

Consumers in the model can be interpreted as being investors - like pension funds, Money Market Mutual Funds (MMMF), Mutual Funds (MF), commercial and investment banks, life insurance companies, hedge funds or non-financial corporations, which have funds to invest but also face the possibility of having to liquidate part of their portfolio because of liquidity needs - unusually large withdrawal of funds by depositors or holders of shares, termination of insurance policies, or cash-flow needs. These different types of investors tend to have different liquidity needs: withdrawal of funds is typically much easier for funds invested in a MF or regular deposit accounts at a bank than it is for hedge funds or for the liquidation of a life insurance policy, where there are typically notice periods and/or penalties to be paid if withdrawal happens before a certain horizon. Hence, agents in the model with a greater likelihood of needing to consume in the middle period, σ, can be interpreted as investors who face a greater risk of having to sell a portion of their portfolio to meet larger than usual withdrawal of funds. The variation in the intensity of liquidity needs driven by the state-dependence of θ can be interpreted as capturing variations in the demand for liquidity from investors. Assuming $\theta(l) > \theta(h)$, for instance, amounts to assuming that funds and banks tend to face larger than average withdrawals in times of poor asset market performance, which translate into needing to sell a greater part of their portfolios early. The role of A on the other hand is to capture the variations in the supply of liquidity: assuming that $A(h) > A(l)$, for instance, can be interpreted as indicating that liquidity tends to be more plentiful when asset market valuations are high, which translates into lower costs of obtaining funds to purchase assets. This could be because credit constraints are looser or because there is greater ease of attracting investors to start or expand existing investment vehicles.

3 An Illustrative Example

Simplified Setup - Before solving and analyzing the full-scale model it is useful to consider a simple example highlighting the mechanism driving the main results of the paper. The timing of events and of the release of information is as in the general setup. The simplifications are as follows. First, I assume that there exist only two Lucas trees with expected value of payoffs 1 at date 2. The payoffs at maturity of the first tree are $d_b(h) = d_b(l) = 1$, and I label claims to this tree as "bonds." The second tree's payoffs are $d_s(h) = 1 + \epsilon > d_s(l) = 1 - \pi\epsilon/(1-\pi)$ for some $\epsilon > 0$, and I label claims to this risky tree as "stocks." Second, I assume that there are only two consumers, 1 and 2. Furthermore, in order to focus on the portfolio decisions of consumers at date $t = 0$, I ignore producers as well as decisions after $t = 0$ and assume that the value function of a consumer leaving the first period is linear in the quantity of each asset held. More specifically, denoting the bond and stock holdings of consumer i by x_b^i and x_s^i respectively, I assume that her value of holding onto portfolio (x_b^i, x_s^i) when leaving date 0 is

$$v^i(x_b^i, x_s^i) = \mathbb{E}\left[(1-\sigma^i) \times V_2 \times (x_b^i + d_s x_s^i) + \sigma^i \times V_1 \times (Q_b x_b^i + Q_z x_s^i)\right], \tag{2}$$

where the expectation operator is with respect to the state of the world ω, $Q_z(\omega)$ is the resale price at date 1 of type $z \in \{b, s\}$ claims in state ω, while $V_t(\omega)$ is the constant marginal utility of consumption at date t when the state is ω. I will now explain this expression. Consider agent i when she reaches date 1 not yet knowing whether she needs to consume early, but knowing that the state of the world is ω. She will be a late consumer with probability $1 - \sigma^i$, in which case the value of having portfolio (x_b^i, x_s^i) is assumed to be $V_2(\omega) \times (x_b^i + d_s(\omega)x_s^i)$, while she will be an early consumer with complementary probability σ^i, in which case the value of holding on to portfolio (x_b^i, x_s^i) is assumed to be $V_1(\omega) \times (Q_b(\omega)x_b^i + Q_s(\omega)x_s^i)$. One way to interpret the latter part is that early consumers always sell their entire portfolio at date $t = 1$ and only consume early. This is optimal if and only if $Q_z(\omega)V_1(\omega) \geq d_z(\omega)V_2(\omega)$ for both ω, and I assume this is true. The assumption of linearity of the value function simplifies the analysis in two ways. First, it eliminates the desire to smooth consumption, be it intertemporally or across states, although agents still care about risk. Second, it enables me to focus on the role of the risk of assets and ignore the impact that the size of the portfolio has in general.

Portfolio Problem, Risk, and Liquidity Services - The date-0 maximization problem of consumer i who is endowed at date $t = 0$ with a portfolio (e_b^i, e_s^i) of claims to the two trees is

$$\max_{(c,h,x_b^i,x_s^i)} c - h + v^i(x_b^i, x_s^i),$$

subject to the budget constraint $c + q_b x_b^i + q_s x_s^i \leq h + q_b e_b^i + q_s e_s^i$, feasibility constraints $c, h \geq 0$, and no-short-selling constraints $x_b^i, x_s^i \geq 0$, where q_z denotes the date-0 price of a unit claim to the fruits of tree $z \in \{b, s\}$ at maturity. Assuming the budget constraint holds with equality and substituting it into the objective function, we obtain that the optimal portfolio choice of consumer i is such that for $z \in \{b, s\}$,

$$-q_z + v_z^i \leq 0, = \text{ if } x_z^i > 0, \tag{3}$$

where, given (2), the marginal value for agent i of purchasing at date $t = 0$ a unit of claims to z is

$$v_z^i = \mathbb{E}\left(d_z V_2\right) + \sigma^i \mathbb{E}\left(Q_z V_1 - d_z V_2\right). \tag{4}$$

The marginal value of both types of claims is therefore the sum of two components. The first is the marginal value attached to an illiquid version of the asset, that is an asset with the same payout at maturity but which cannot be traded before it matures. For a unit of bond this marginal value is simply the expected marginal utility at maturity, $\mathbb{E}(V_2)$, whereas for a unit of stock it is $\mathbb{E}(d_s V_2) = \mathbb{E}(V_2) + cov(V_2, d_s)$ and therefore also includes an adjustment for the risk as a *late* consumer, $cov(V_2, d_s)$. This first component is common to both consumers and is what is obtained in traditional asset pricing models that abstract from liquidity considerations. The second component is the product of the frequency of liquidity needs σ and of $\mathbb{E}(Q_z V_1 - d_z V_2)$, the expected value of the liquidity services that the asset provides. Note that, since $\mathbb{E}(Q_z V_1 - d_z V_2) = \mathbb{E}(Q_z)\mathbb{E}(V_1) - \mathbb{E}(V_2) + cov(V_1, Q_z) - cov(V_2, d_z)$, the expected value of liquidity services also includes an adjustment for risk, but it is for the risk faced as an early consumer *relative* to that as a late consumer. This adjustment is $cov(V_1, Q_b)$ for a bond, which highlights that, although a bond is riskless if held until maturity, it is not necessarily totally riskless because consumers face some risk as early consumers whenever the resale price is not certain.

Pricing and Liquidity Premium - I now turn to the pricing of claims to the two trees and to the composition of portfolios of the two consumers. First, (3) implies that $q_z = \max_{i \in \{1,2\}} v_z^i$ for $z \in \{b, s\}$, which together with (4) imply

$$q_z = \mathbb{E}\left(d_z V_2\right) + \max_{i \in \{1,2\}} \sigma^i \mathbb{E}\left(Q_z V_1 - d_z V_2\right).$$

That is, the price of a claim is determined by the agent with the greatest willingness to pay, which is general feature of Walrasian consumption-based asset pricing models with endogenous funding liquidity constraints (e.g., Alvarez and Jermann, 2000; Chien and Lustig, 2010; or Geanakoplos and Zame, 2014) and of models where *ex ante* heterogeneous agents form different *clienteles* who hold different sets of assets. Examples include models where investors with different degrees of risk aversion face endogenous funding liquidity constraints (Biais et al., 2018), or where investors have heterogeneous beliefs (e.g., Miller, 1977; Harrison and Kreps, 1978; Scheinkman and Xiong, 2003; or Hong, Scheinkman and Xiong, 2006), as well as models where agents with heterogeneous trading needs trade assets that differ in their trading costs (e.g., Amihud and Mendelson, 1986; Pagano, 1989; and Studart, 2014), or where investors face differentiated tax liabilities (see, for instance, Miller and Modigliani, 1961, who hypothesized it and Long, 1977, and Dybvig and Ross, 1986, for formal treatments). Moreover, very much like in liquidity-based asset pricing models of Kiyotaki and Moore (2005) or Lagos (2010), among others, the price of a claim can be decomposed into the price of an illiquid claim, $\mathbb{E}(d_z V_2)$, and a liquidity premium, $\max_{i \in \{1,2\}} \sigma^i \mathbb{E}(Q_z V_1 - d_z V_2)$.

Since consumer 1 is more likely than consumer 2 to want to consume early, consumer 1 holds asset z if and only if $\mathbb{E}(Q_z V_1) \geq \mathbb{E}(d_z V_2)$, and she is the only one holding it if the inequality is strict. The factor determining which agent holds onto an asset is therefore simply the sign of the expected value of its liquidity services, which is intuitive: if the expected value of liquidity services is positive, then the consumer with the greater likelihood to benefit from these services is the one most willing to pay for them; and *vice versa* if the expected value of liquidity services is negative.

Sorting - To move forward I assume that the price at date 1 of asset $z \in \{b, s\}$ is $Q_z(\omega) = A(\omega)d_z(\omega)$ for some $A(\omega) > 0$ for both ω. This implies that the date-1 prices of both assets vary in the same proportion relative to their payoffs at maturity, which will be the case in the general model. This assumption implies that the expected value of liquidity services can be re-expressed as a function of terminal payoffs:

$$\mathbb{E}\left[d_b\left(AV_1 - V_2\right)\right] = \mathbb{E}(AV_1 - V_2), \text{ and} \tag{5}$$

$$\mathbb{E}\left[d_s\left(AV_1 - V_2\right)\right] = \mathbb{E}(AV_1 - V_2) + cov(d_s, AV_1 - V_2). \tag{6}$$

Let me define $A(\omega)V_1(\omega) - V_2(\omega)$, the utility gain generated by an early consumer over that generated as a late consumer by holding extra assets worth 1 unit of consumption at maturity in state ω,[11] as the **marginal value of liquidity** in state ω. We then have that the expected value of liquidity services of an asset is equal to the expected marginal value of liquidity plus an adjustment to take into account the correlation between the payoffs of the asset at maturity and the marginal value of liquidity.

There are five possible equilibria depending on which consumer holds which type of claims. First, it is obvious from (5) and (6) that both agents hold onto both types of claims if and only if $A(l)V_1(l) - V_2(l) = A(h)V_1(h) - V_2(h) = 0$. That is, agents do not sort themselves across claims if the marginal value of liquidity is the same in both states, with value zero in this example.

More generally, (5) implies that consumer 1 is willing to hold bonds if and only if

$$\mathbb{E}\left(AV_1 - V_2\right) \geq 0, \tag{7}$$

and she is the only one holding bonds when the inequality is strict. That is, the consumer with the greatest liquidity needs holds onto bonds if and only if it the expected marginal value of liquidity is positive. Similarly, (6) implies that consumer 2 is willing to hold stocks if and only if

$$cov(d_s, AV_1 - V_2) \leq -\mathbb{E}(AV_1 - V_2), \tag{8}$$

and she is the only one holding stocks if the inequality is strict. That is, stocks are held by the consumer with the lesser liquidity needs if and only if the covariance between its payoffs at maturity and the marginal value of liquidity is negative enough.

Focusing on the case where consumer 1 holds onto the entire all bonds and consumer 2 holds all stocks, the structure of payoffs at maturity implies that condition (8) with a strict inequality is equivalent to

$$\mathbb{E}(AV_1 - V_2) < \epsilon\pi\left[\left(A(l)V_1(l) - V_2(l)\right) - \left(A(h)V_1(h) - V_2(h)\right)\right],$$

which, given $\mathbb{E}(AV_1 - V_2) > 0$ and $\epsilon > 0$, is equivalent to

$$A(l)V_1(l) - V_2(l) > A(h)V_1(h) - V_2(h) \tag{9}$$

[11] Assets worth 1 unit of consumption at date 2 generate $V_2(\omega)$ units of utility at date 2, whereas they are worth $A(\omega)$ units of goods at date 1, which generates utility $A(\omega)V_1(\omega)$.

and

$$\epsilon > \epsilon^* \equiv \frac{\mathbb{E}(AV_1 - V_2)}{\pi\left[(A(l)V_1(l) - V_2(l)) - (A(h)V_1(h) - V_2(h))\right]}, \tag{10}$$

That is, (i) the marginal value of liquidity in state l, $A(l)V_1(l) - V_2(l)$, is greater than in state h, $A(h)V_1(h) - V_2(h)$, which reduces how attractive stocks are due to the variability of their payoffs (condition (9)); and (ii) the variability of stocks' payoffs is large enough[12] to make the covariance between its payoffs and the marginal value of liquidity negative enough to induce the high-liquidity-need consumer not to want to hold stocks (condition (10)). In this example with linear marginal utility of consumption, the expected value of the liquidity services of stocks is negative when conditions (9) and (10) are satisfied, which leads consumers to discount its value relative to what it would have been without liquidity needs. And naturally the consumer with less frequent liquidity needs is the one discounting the value of stocks the least and holds it.

Among the three conditions (7), (9), and (10) used to characterize sorting in the second case of this example, only condition (9) will be central in the general setup. This is because (7) and (10) are conditions that are needed in this example to guarantee that each agent hold onto claims to one of the trees, bonds for consumer 1 and stocks for consumer 2 in the case presented, whereas in the general setup the assumption of unbounded marginal utility for early consumers in the middle period ensures each consumer type always holds onto some claims.

4 Derivation of the Portfolio Problem and Asset Pricing Function

I now turn my attention to the general model as presented in section 2 and I derive the first-order conditions characterizing the optimal portfolio choice of agents at date 0 by working backwards.

Last Period ($t = 2$) - An agent consumes in the last period whatever she is entitled to given her portfolio, which given her budget constraint, means that a producer and a consumer respectively derive utility $\mathbf{d}(\omega) \cdot \mathbf{x}$ and $U(\mathbf{d}(\omega) \cdot \mathbf{x})$ of entering the last period with portfolio $\mathbf{x}$ in state ω.

Middle Period ($t = 1$) - Moving on to the middle period and defining by $V^i(\mathbf{x};\omega)$ the value for an early consumer of type $i \in \{1, ..., I\}$ of entering the middle period with portfolio $\mathbf{x}$ in state ω, we have that

$$V(\mathbf{x};\omega) = \max_{(c,\mathbf{y})} \theta(\omega)u(c) + U\left(\mathbf{d}(\omega) \cdot (\mathbf{x} - \mathbf{y})\right),$$

subject to the budget constraint $c \leq \mathbf{Q}(\omega) \cdot \mathbf{y}$, the feasibility constraint $c \geq 0$ and no-short-selling constraints $y_\epsilon \leq x_\epsilon$ for all ϵ. Since early consumers' marginal utility goes to infinity as their consumption in the middle period goes to zero, we can anticipate that consumers bring and sell non-empty portfolios into the middle period. We can thus ignore the non-negativity constraint on consumption, replace c in the objective function by its expression from the budget constraint, and obtain that the first-order condition with respect to y_ϵ is

$$Q_\epsilon(\omega)\theta(\omega)u'(\mathbf{Q}(\omega) \cdot \mathbf{y}) \geq d_\epsilon(\omega)U'(\mathbf{d}(\omega) \cdot (\mathbf{x} - \mathbf{y})),$$

with equality if $y_\epsilon < x_\epsilon$.

The problem of a late consumer is

$$\max_{\mathbf{Y}} U\left(\mathbf{d}(\omega) \cdot (\mathbf{x} - \mathbf{Y})\right),$$

[12]In fact, the variance of the variable part of the risky asset's payoffs is $Var(d_\epsilon - 1) = \epsilon^2 \times \pi/(1-\pi)$.

subject to the budget constraint $\mathbf{Q}(\omega) \cdot \mathbf{Y} \geq 0$ and no-shorting-selling constraints $Y_\epsilon \leq x_\epsilon$ for all ϵ. The first-order condition with respect to Y_ϵ is

$$d_\epsilon(\omega)U'\left(\mathbf{d}(\omega) \cdot (\mathbf{x} - \mathbf{Y})\right) - \lambda_\epsilon(\omega) = \Lambda(\omega)Q_\epsilon(\omega),$$

where Λ and λ_ϵ are respectively the Lagrange multipliers on the budget constraint and the no short-selling constraint for type-ϵ claims.

Finally, the problem of a producer entering the middle period with portfolio $\mathbf{x}$ when the state of the world is ω is

$$\max_{(h,\mathbf{y})} - (A(\omega))^{-1} h + \mathbf{d}(\omega) \cdot (\mathbf{x} + \mathbf{y}),$$

subject to the budget constraint $\mathbf{Q}(\omega) \cdot \mathbf{y} \leq h$, the feasibility constraint $h \geq 0$, and no-short-selling constraints $y_\epsilon \geq -x_\epsilon$ for all ϵ. Since consumers bring and sell non-empty portfolios in the middle period, it must be that production takes place then. The constraint on effort of producers must therefore be non-binding. The first-order condition with respect to the quantity of type-ϵ claims to accept in state ω, $y_\epsilon(\omega)$, is thus

$$- (A(\omega))^{-1} Q_\epsilon(\omega) + d_\epsilon(\omega) \leq 0, \ = \ \text{if } y_\epsilon(\omega) > -x_\epsilon.$$

We then have the following lemma.[13]

Lemma 1 *In a symmetric equilibrium the date-1 price of a type-ϵ claim in state $\omega \in \{1, 2\}$ is*

$$Q_\epsilon(\omega) = A(\omega)d_\epsilon(\omega), \tag{11}$$

and late consumers are indifferent between trading and not trading.
Furthermore, the marginal value at $t = 1$ of claims of type ϵ for an early consumer of type $i \in \{1, ..., I\}$ is

$$V_\epsilon^i(\mathbf{x};\omega) = Q_\epsilon(\omega)\theta^\omega u'(\mathbf{Q}(\omega) \cdot \mathbf{y}^i(\mathbf{x};\omega)),$$

where $\mathbf{y}^i(\mathbf{x};\omega)$ solves

$$Q_\epsilon(\omega)\theta(\omega)u'\left(\mathbf{Q}(\omega) \cdot \mathbf{y}^i(\mathbf{x};\omega)\right) = d_\epsilon(\omega)U'(\mathbf{d}(\omega) \cdot (\mathbf{x} - \mathbf{y}^i(\mathbf{x};\omega))). \tag{12}$$

The lemma establishes that the first-order conditions with respect to transfers of claims in the middle period for both early consumers and producers hold with equality, and late consumers do not gain from rebalancing their portfolios. This is because at that point all information has been revealed and therefore all claims are equally good means of payment in the middle period. We also obtain that the prices of claims in the middle period take the same form as in the example presented earlier. In particular, the price of $\epsilon = 0$ claims in the middle period varies with the state if A does, thereby implying that these claims are perfectly risk-free if and only if either A is constant or the claims are held until maturity. Note that the linearity of the producer's utility function and the implied price of claims in the middle period imply that the demand for claims and effort of producers adjust to clear the claims and goods markets.

Let $v^i(\mathbf{x})$ be the value for a type $i \in \{0, ..., I\}$ agent of entering the middle period with portfolio $\mathbf{x}$ before consumers know whether they want to consume and before the state of the world is revealed. The above lemma and the envelope condition imply that the marginal value of entering the middle period with portfolio $\mathbf{x}$ is

$$v_\epsilon^0(\mathbf{x}) = 1$$

[13] All proofs are, unless indicated otherwise, in the Appendix.

for producers, while for a consumer of type $i \in \{1, ..., I\}$ it is

$$v_\epsilon^i(\mathbf{x}) = v_\epsilon^{illiq}(\mathbf{x}) + \sigma^i v_\epsilon^{liq}(\mathbf{x}), \tag{13}$$

where

$$v_\epsilon^{illiq}(\mathbf{x}) \equiv \mathbb{E}\left[d_\epsilon U'(\mathbf{d} \cdot \mathbf{x})\right] \text{ and } v_\epsilon^{liq}(\mathbf{x}) \equiv \mathbb{E}[d_\epsilon L(\mathbf{x})], \tag{14}$$

for

$$L(\mathbf{x};\omega) \equiv A(\omega)\theta(\omega)u'(A(\omega)\mathbf{d}(\omega) \cdot \mathbf{y}(\mathbf{x};\omega))) - U'(\mathbf{d}(\omega) \cdot \mathbf{x}), \tag{15}$$

the marginal value of liquidity in state ω.

(13) implies that $v_\epsilon^i(\mathbf{x})$ can be re-expressed as the sum of $v_\epsilon^{illiq}(\mathbf{x})$, the marginal value of an illiquid version of a type-ϵ claim, and the product of the frequency of liquidity needs σ^i and of $v_\epsilon^{liq}(\mathbf{x})$, the expected marginal value of liquidity services provided by this type of claims. Note for future reference that although neither $v_\epsilon^{illiq}(\mathbf{x})$ nor $v_\epsilon^{liq}(\mathbf{x})$ directly depend on the agent's type, they do indirectly depend on it through her portfolio choice. And, as in the example of section 3, the two components can each be decomposed into two elements. The value that an agent attaches to an illiquid claim can be decomposed into the marginal value attached to an illiquid riskless asset, $\mathbb{E}\left[U'(\mathbf{d} \cdot \mathbf{x})\right]$, and a traditional risk premium added to the price of the illiquid riskless asset and which is $cov(d_\epsilon, U'(\mathbf{d} \cdot \mathbf{x}))$. As for the expected marginal value of liquidity services that a given type of claim provides, it can as well be decomposed into the expected marginal value of a unit of liquidity $\mathbb{E}\left[L(\mathbf{x})\right]$, and $cov(d_\epsilon, L(\mathbf{x}))$, an adjustment for the risk linked to the marginal value of liquidity.[14]

In order to simplify the presentation I assume that when early consumers purchase goods in the middle period they transfer the same fraction of each type of claim, so that $\mathbf{y}(\mathbf{x};\omega) = \alpha(\mathbf{x};\omega)\mathbf{x}$ for some $\alpha(\mathbf{x};\omega) \in [0, 1]$. Even though this assumption is restrictive for a given agent, it is much less so at the aggregate. In fact, this assumption is equivalent to assuming that agents randomize between the different types of claims, which they are happy to do since they are indifferent between the different types of claims as media of exchange, and that a law of large numbers exists and ensures that this is correct at the aggregate.

Note that since late consumers are indifferent between trading and not trading, and they have no other means to transfer resources between the middle and final periods, I assume they do not participate in trade. I show in section 2 of the online appendix that introducing a storage technology between the last two periods to enable late consumers to sell early their portfolio whenever the terms of trade are favorable essentially does not change the results.

First Period ($t = 0$) - The maximization problem of an agent of type $i \in \{0, ..., I\}$ who is endowed at date $t = 0$ with a portfolio $\mathbf{e}$ of claims to trees is

$$\max_{(c,h,\mathbf{x})} c - h + v^i(\mathbf{x}),$$

subject to the budget constraint $c + \mathbf{q} \cdot \mathbf{x} \leq h + \mathbf{q} \cdot \mathbf{e}$, feasibility constraints $c, h \geq 0$, and the no-short-selling constraints $x_\epsilon \geq 0$ for all ϵ. Assuming again that the budget constraint holds with equality and substituting

[14] This decomposition is slightly different from the one provided in (4) in the example of section 3. I present this decomposition, which also holds in the example for $Q_\epsilon(\omega) = A(\omega)d_\epsilon(\omega)$, because it will be useful when I consider sorting later on. One can alternatively decompose the value just like in the example into $\mathbb{E}[Q_\epsilon]\mathbb{E}\left[\theta u'(\mathbf{Q} \cdot \mathbf{y}(\mathbf{x}^i))\right] - \mathbb{E}[U'(\mathbf{d} \cdot \mathbf{x}^i)]$ and $cov(Q_\epsilon, \theta u'(\mathbf{Q} \cdot \mathbf{y}(\mathbf{x}^i))) - cov(d_\epsilon, U'(\mathbf{d} \cdot \mathbf{x}^i)$. The former can be interpreted as the expected marginal value of the liquidity services provided by an asset which is truly riskless, with fixed prices $\mathbb{E}[Q_\epsilon]$ at $t = 1$ and 1 at $t = 2$, while the latter is a risk adjustment to capture the additional risk faced by an agent arising from the need to consume early.

it into the objective function, we obtain that the optimal portfolio choice of a consumer of type i is such that for all ϵ

$$-q(\epsilon) + v_\epsilon^i(\mathbf{x}) \leq 0, = \text{ if } x_\epsilon > 0. \tag{16}$$

An equilibrium can then be fully characterized by the vector of date-0 portfolios $(\mathbf{x}^i)_{i=0}^I$ and the vector $\mathbf{q}$ of prices for claims such that (i) (16) holds for all ϵ and all i, and (ii) the markets for all claims clear at date 0. In fact, the date-0 portfolio choice $\mathbf{x}^i$ for an agent of type i implies: $c_0^i - h_0^i$ from the date-0 budget constraint; $c_1^i(\mathbf{x}^i;\omega) = \mathbf{Q}(\omega) \cdot \mathbf{y}^i(\mathbf{x}^i;\omega)$, with $\mathbf{y}^i(\mathbf{x}^i;\omega)$ solving (12) and given $\mathbf{Q}$ from lemma 1, and $C_2^i(\mathbf{x}^i;\omega) = \mathbf{d}(\omega) \cdot \mathbf{x}^i$ while $c_2^i(\mathbf{x}^i;\omega) = \mathbf{d}(\omega) \cdot (\mathbf{x}^i - \mathbf{y}^i(\mathbf{x}^i;\omega))$ for $i \in \{1, ..I\}$; finally, $h_1^0(\omega)$ and $\mathbf{y}^0(\omega)$ ensure the date-1 goods market and markets for claims clear for each ω given $\{c_1^i(\mathbf{x}^i;\omega), \mathbf{y}^i(\mathbf{x}^i;\omega)\}_{i=1}^I$, and $c_2^0(\mathbf{x}^0;\omega) = \mathbf{d}(\omega) \cdot (\mathbf{x}^0 - \mathbf{y}^0(\omega))$.

Asset Pricing Equation - (16) implies that the price of a type-ϵ claim tree is given by

$$q_\epsilon = \max_i v_\epsilon^i(\mathbf{x}^i). \tag{17}$$

Hence, as noted in the example of section 3, the price of a claim is determined by the types of agents with the greatest willingness to pay. The decomposition of the marginal valuation of claims (13), together with (17) , naturally leads to a similar decomposition of the price of claims:

$$q_\epsilon = \max_i \{v_\epsilon^{illiq}(\mathbf{x}^i) + \sigma^i v_\epsilon^{liq}(\mathbf{x}^i)\}. \tag{18}$$

That is, the price of a claim can be decomposed into the price of an illiquid claim and a liquidity premium. However, contrary to the example of section 3, the two components are not independent of each other, for they are jointly determined by the type(s) of agents holding this type of claims and the portfolio choice of these agents. Moreover, the decomposition of the price of a claim is not unique whenever there is more than one type of agents valuing the claim the same way, for the types' marginal valuations of an illiquid claim and liquidity services differ. This means that there are as many risk and liquidity premia for a type of claims as there are types of consumers holding onto these claims. These two points will be reinforced when I derive CAPM-like formulae that decompose the rate of return of a claim into risk and liquidity premia.

5 Existence Asset Classes and Liquidity-Need Adjusted CAPM

5.1 Equilibrium Class Structure and First Implications for Asset Prices

Asset Classes - In the previous section I showed that the marginal value of a claim can be decomposed into the expected value of its payoffs at maturity and the expected marginal value of the liquidity services it provides. This decomposition is useful because it highlights the role of liquidity needs in the pricing of claims. However, the characterization of the equilibrium properties of the model involves establishing the composition of the portfolios of the different types of agents, and to this aim it is useful to instead decompose the marginal value of a claim into the marginal value of a riskless claim and an adjustment for risk.

As discussed above, (13) and (14) imply that $v_\epsilon^i(\mathbf{x}^i) = v_0^i(\mathbf{x}^i) + cov(d_\epsilon, U'(\mathbf{d} \cdot \mathbf{x}^i)) + \sigma^i cov(d_\epsilon, L(\mathbf{x}^i))$. However, the structure of payoffs and the fact that there exist two states together imply that covariances are linear functions of risk: $cov(d_\epsilon, U'(\mathbf{d} \cdot \mathbf{x}^i)) = -\epsilon\pi \left[U'(\mathbf{d}^l \cdot \mathbf{x}^i) - U'(\mathbf{d}^h \cdot \mathbf{x}^i)\right]$ and $cov(d_\epsilon, L(\mathbf{x}^i)) = -\epsilon\pi \left[L^l(\mathbf{x}^i) - L^h(\mathbf{x}^i)\right]$. We thus obtain the following lemma.

Lemma 2 *The marginal value that a type-i consumer attaches to a claim of type ϵ can be expressed as*

$$v^i_\epsilon(\mathbf{x}^i) = v^i_0(\mathbf{x}^i) - \epsilon\pi\Delta^i(\mathbf{x}^i), \tag{19}$$

where

$$\Delta^i(\mathbf{x}^i) \equiv U'(\mathbf{d}^l \cdot \mathbf{x}^i) - U'(\mathbf{d}^h \cdot \mathbf{x}^i) + \sigma^i \left[L^l(\mathbf{x}) - L^h(\mathbf{x})\right]. \tag{20}$$

The marginal value that a consumer attaches to a claim thus depends linearly on the risk of that claim, with gradient $-\pi\Delta^i(\mathbf{x}^i)$. This leads neatly to a class structure. Consider any two types i and j of consumers. If $\Delta^i(\mathbf{x}^i) = \Delta^j(\mathbf{x}^j)$, then it must be that $v^i_0(\mathbf{x}^i) = v^j_0(\mathbf{x}^j)$ so that the two types of consumers have the same marginal valuation for all claims. In fact, if it were otherwise one of the two types of consumers would have empty portfolios when leaving the first period and would therefore not consume later on, which cannot be true in equilibrium. If instead $\Delta^i(\mathbf{x}^i) \neq \Delta^j(\mathbf{x}^j)$, then, either one type has a strictly greater marginal valuation of all claims except maybe for $\epsilon_{\min}$- or $\epsilon_{\max}$-claims, or $v^i_\epsilon(\mathbf{x}^i)$ and $v^j_\epsilon(\mathbf{x}^j)$ intersect exactly once in $(\epsilon_{\min}, \epsilon_{\max})$. The former cannot be true, because it would imply that one type of consumers has infinite marginal utility of consumption in periods 1 and 2. Figure 1 presents a case where there are three types of consumers whose marginal valuation functions have different slopes.

Insert Figure 1 about here.

The fact that the marginal value that an agent attaches to a claim changes linearly with the risk of the claim leads to the following proposition.

Proposition 1 *An equilibrium is such that the sets of claims and agent types can each be partitioned into a sequence of $K \leq I$ groups $\{\Xi_k, \mathcal{I}_k\}_{k=1}^K$ such that:*
(i) $\Xi_k = [\epsilon^*_{k-1}, \epsilon^*_k]$, where $\epsilon^*_0 = \epsilon_{\min}$, $\epsilon^*_K = \epsilon_{\max}$, and $\epsilon^*_{k-1} < \epsilon^*_k$ for all k;
(ii) $\mathcal{I}_k \subseteq \{1, ..., I\}$, with $\mathcal{I}_k \cap \mathcal{I}_m = \{\varnothing\}$ for all $k \neq m$, and agents with type $i \in \mathcal{I}_k$ only hold onto claims with risk $\epsilon \in \Xi_k$.
(iii) $\Delta^i > \Delta^j$ for any two types $i \in \mathcal{I}_k$ and $j \in \mathcal{I}_m$ with $k < m$.

Parts (i) and (ii) are about the existence of a class structure. The set of types of agents can be partitioned in $K \leq I$ subsets, or classes, and all types belonging to a given class hold onto the same types of claims whose risk belong to a given interval. And the spectrum of risk for claims can itself be split into $K \leq I$ intervals that do not overlap, except at boundaries, these are the asset classes, such that claims in a certain asset class are held only by agents of a certain class. Part (iii) says that agents whose marginal valuation of claims is the most sensitive to risk, those with the largest Δ, hold the safest assets, and vice versa.

First Implications for Asset Pricing - The model has clean predictions for how asset prices depend on the risk of an asset. In fact, $\Delta^i(\mathbf{x}^i) = \Delta^j(\mathbf{x}^j) = \Delta$ for any two types i and j when there is a unique class, in which case the price of claims is a linear function of the risk index ϵ with slope $-\pi\Delta$. If, however, there are $K > 1$ classes, then one can see from figure 1 that the asset pricing function is the upper-envelope of a family of affine functions with increasing slopes, which is a convex function. More generally, we have that $\Delta^i(\mathbf{x}^i) > \Delta^j(\mathbf{x}^j) > ... > \Delta^n(\mathbf{x}^n)$ for $i \in \mathcal{I}_1$, $j \in \mathcal{I}_2$, and $n \in \mathcal{I}_K$ and the slope of the asset pricing functions for claims in class k is $-\pi\Delta^i(\mathbf{x}^i)$ for $i \in \mathcal{I}_k$. If $\Delta > 0$ for all types, then the price of safer assets is more sensitive to risk than the price of riskier assets. I will later on establish that it is possible to have $\Delta < 0$ for some types, thereby implying that the price of claims held by these agents is increasing in risk. This can happen if the terms of trade or the liquidity effects lead (some) consumers to want to consume more in state h than in state l, because then the marginal value of a claim increases with risk.

Since the asset pricing function is linear when all types of agents belong to the same class, it is possible in this case to use simple securities like Arrow securities to price the claims that agents trade in this paper. But this is no longer true when there is more than one class. Let $p(\omega)$ denote the price of an Arrow security that delivers one unit of the general good at $t = 2$ when the state of the world is ω. It is such that

$$p(\omega) = \max_i \pi(\omega) \left\{ U'(\mathbf{d}(\omega) \cdot \mathbf{x}^i) + \sigma^i L(\mathbf{x}^i; \omega) \right\}.$$

When there is a unique class, the cost of obtaining in the last period $1 + \epsilon$ units of the good in the h state and $1 - (\pi/(1-\pi))\epsilon$ in the l state is therefore

$$(1+\epsilon)\, p(h) + \left(1 - \frac{\pi}{1-\pi}\epsilon\right) p(l) = q_\epsilon. \tag{21}$$

When there is more than one class, however, (21) does not hold. In fact, if we use Arrow securities to build a risk-free asset, its price is

$$p(h) + p(l) = \sum_{\omega \in \{h,l\}} \max_i \pi(\omega) \left\{ U'(\mathbf{d}(\omega) \cdot \mathbf{x}^i) + \sigma^i L(\mathbf{x}^i; \omega) \right\},$$

whereas the price of a risk-free claim in this paper is

$$q_0 = \max_i \sum_{\omega \in \{h,l\}} \pi(\omega) \left\{ U'(\mathbf{d}(\omega) \cdot \mathbf{x}^i) + \sigma^i L(\mathbf{x}^i; \omega) \right\},$$

and thus $q_0 \leq p(h) + p(l)$, with $q_0 < p(h) + p(l)$ whenever there is more than one class. This is different from the asset pricing literature with limited commitment, .e.g., Alvarez and Jermann (2000) and Chien and Lustig (2010), where the price of a complex security is equal to the price of a set of simple securities replicating that complex security. In fact, agents can pledge the entire set of future cashflows, thereby implying there would otherwise be an arbitrage possibility by purchasing a complex security whose price is lower than the price of a set of simple securities replicating it, stripping it into Arrow securities, and reselling the Arrow securities. I neither allow for short-selling nor for stripping and repackaging of securities, thereby implying that agents cannot take advantage of such arbitrage possibilities. Note that Biais et al. (2018) obtain a similar deviation from the law of one price in a model with limited commitment, but where agents with heterogeneous degrees of risk aversion have limited ability to pledge future cash flows.

5.2 Liquidity-Need Adjusted CAPM

I have so far focused on the differentiated valuations of claims and the implications for pricing these claims. In this section I instead consider the rates of return and show that it is possible, very much like in Holmström and Tirole (2001) or Acharya and Pedersen (2005), to derive CAPM-like formulae where liquidity concerns are added. The proofs of the results in this section can be found in the online appendix.

5.2.1 Risk and Liquidity Premia and Absolute Excess Returns

I here present a formula for the excess returns of claims. Two adjustments are required in this setup relative to a traditional consumption-based CAPM with a representative agent. First, the existence of liquidity needs leads to the existence of a liquidity premium. This adjustment is simply the other side of the coin of the adjustment made when looking at prices. Second, absolute excess returns exist whenever there is more than one class. In fact, the existence of classes implies that one needs to adjust for the fact that agents belonging to different classes typically value type-0 claims differently. Let ϕ^{km} be the ratio of the marginal values of an $\epsilon = 0$ claim for any two types of agents $i_k \in \mathcal{I}_k$ and $i_m \in \mathcal{I}_m$, i.e.,

$$\phi^{km} \equiv \frac{v_0^{i_k}(\mathbf{x}^{i_k})}{v_0^{i_m}(\mathbf{x}^{i_m})}. \tag{22}$$

If we assume that risk-free claims belong to class k, i.e., $0 \in \Xi_k$, then $\phi^{km} \geq 1$. This is because $\epsilon = 0$ claims must be held by agents with the highest value for them. The inequality is weak because it could be that risk-free claims are at the boundary of two classes. It is important to note that ϕ^{km} is indexed by classes k and m that agents belong to, but not by the specific types of agents. In fact, all agents belonging to the same class have the same marginal valuation of risk-free claims, even though the decomposition between the values of an illiquid claim and the value of liquidity services is different across types in the same class.

Proposition 2 *Let $R_\epsilon(\omega)$ denote the gross rate of return on a type-ϵ claim in state ω, i.e., $R_\epsilon(\omega) \equiv d_\epsilon(\omega)/q_\epsilon$ and assume that risk-free claims belong to class k, i.e., $0 \in \Xi_k$.*
(i) Then the excess returns on claims of type $\epsilon \in \Xi_m$ can be expressed as

$$\begin{aligned}\mathbb{E}(R_\epsilon) - R_0 = &-\phi^{km} R_0 cov\left(R_\epsilon, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})\right) \\ &- \phi^{km} R_0 cov\left(R_\epsilon, \sigma^{i_m} L(\mathbf{x}^{i_m})\right) + (\phi^{km} - 1)R_0\end{aligned} \tag{23}$$

for $i_m \in \mathcal{I}_m$.
(ii) Furthermore, $\phi^{km} \geq 1$, with strict inequality whenever 0 is not at the boundary of two classes, and it is increasing in m whenever $0 \in \Xi_1$ for $K \geq 3$.

The model thus admits an additive representation, very much like the traditional CAPM, but with a couple of differences.[15] The first is that there is a liquidity premium, $-\phi^{km} R_0 cov\left(R_\epsilon, \sigma^{i_m} L(\mathbf{x}^{i_m})\right)$, which is an adjustment to the risk-free rate of a liquid claim to take into account the covariance between returns at maturity and the marginal value of liquidity. It also turns out that the risk premium, $-\phi^{km} R_0 cov\left(R_\epsilon, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})\right)$, features ϕ^{km}, an adjustment to the traditional formula. And because $\phi^{km} \geq 1$, with strict inequality whenever type-0 claims are not at the boundary of two classes, the difference in valuation of $\epsilon = 0$ claims amplifies the role of the covariances for the risk and liquidity premia. Finally, the model generates absolute excess returns, or "alphas," for claims that do not belong to the same class as risk-free claims. The latter two adjustments are necessary because the traditional CAPM formula, with no adjustment other than the introduction of a liquidity premium, holds within each class when one expresses excess returns of claims in a given class relative to the valuation of risk-free claims for agents in that class, but the formula above expresses excess returns relative to the *market* risk-free rate, which is lower than the marginal valuation of risk-free claims for agents in classes not holding these claims.

As has been mentioned earlier, even though the model admits an additive representation, the risk and liquidity premia have to be computed jointly, because $U'(\mathbf{d}\cdot\mathbf{x}^i)$ and $L(\mathbf{x}^i)$ are not independent of each other. Furthermore, the variations of the terms of trade and urgency of liquidity needs can exacerbate or dampen

[15] In the online appendix I also show that excess returns to a given type of claims obtained can be represented as a function of the value of aggregate risk before maturity and at maturity, and its exposure to these two aggregate risks.

the impact of risk, for, as I will soon establish, they can lead $L(\mathbf{x})$ to move across states in the same or opposite direction as $U'(\mathbf{d}\cdot\mathbf{x})$.

The decomposition presented in this section is useful for thinking about the forces driving the rates of return of claims, and in particular the roles of risk before and at maturity. But because it is specific to each type of agent, micro consumption data and a detailed analysis are required to estimate or calibrate the model's parameters. In fact, the variability in an agent's consumption predicted by the model depends on aggregate variables $(A(h), A(l), \theta(h), \theta(l), \pi, \gamma)$ and on both the exogenous frequency of liquidity needs σ^i and the endogenous choice of portfolios, in particular the average risk ε^i. Moreover, the resulting endogenous class structure determines the properties of aggregate consumption and therefore the mapping between the model and aggregate properties of the data. One would thus need to infer from the data both the class structure, i.e., the number of types I and $\{\mu^i\}_{i=1}^I$ as well as the split of these types into classes K, to be able to provide the decompositions of excess returns. But, as the next section shows, estimating the class structure for assets is less demanding.

5.2.2 Returns and the Market Portfolio

I now show how excess returns on a given type of claims can be expressed as a function of excess returns of any reference portfolio and the exposure of the claims to the risk of the portfolio of reference. An advantage of this formulation over the one of the previous section is that this relationship can be estimated using only market data. The starting point of the analysis is the price of the portfolio of reference considered. Denote by q^M the price of a portfolio M made up of all claims with risk in some interval $M=\left[\epsilon^M_{\min}, \epsilon^M_{\max}\right]\subseteq[\epsilon_{\min}, \epsilon_{\max}]$ for which there is a non-zero mass of claims. That is, $q^M\equiv\int_{\epsilon^M_{\min}}^{\epsilon^M_{\max}} q_\epsilon dG^M(\epsilon)$, where for all $\epsilon\in M$

$$G^M(\epsilon)\equiv\frac{G(\epsilon)-G(\epsilon_{\min})}{G(\epsilon^M_{\max})-G(\epsilon^M_{\min})}.$$

We then have that when M covers all claims in classes i to j for $i\le j$, i.e., $M=\left[\epsilon^*_{i-1}, \epsilon^*_j\right]$, then

$$q^M=\sum_{k=i}^{j}\int_{\epsilon^*_{k-1}}^{\epsilon^*_k}\mathbb{E}\left[d_\epsilon\left(U'(\mathbf{d}\cdot\mathbf{x}^{i_k})+\sigma^{i_k}L(\mathbf{x}^{i_k})\right)\right]dG^M(\epsilon),$$

where type i_k belongs to class k. If we define ε^M as the average risk for portfolio M, i.e., $\varepsilon^M\equiv\int_{\epsilon^M_{\min}}^{\epsilon^M_{\max}}\epsilon dG^M(\epsilon)$, the convexity of the asset pricing function implies that $q^M\ge q_{\varepsilon^M}$, with strict inequality if M contains two or more classes. Defining $R^M(\omega)$ as the gross return of portfolio M in state ω, we have the following proposition.

Proposition 3 *The excess returns for claims of type $\epsilon\in\Xi_n\subseteq M$ can be expressed as*

$$\mathbb{E}\left(R_\epsilon\right)-R_0=\alpha_{\epsilon,M}+\beta_{\epsilon,M}\left[\mathbb{E}\left(R^M\right)-R_0\right], \tag{24}$$

where $\alpha_{\epsilon,M}\neq 0$ unless M contains a unique class, and for $\varepsilon^M\in\Xi_m$, $i_n\in\mathcal{I}_n$ and $i_m\in\mathcal{I}_m$,

$$\beta_{\epsilon,M}\equiv\frac{cov\left(R_\epsilon, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right]}\times\frac{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{im}L(\mathbf{x}^{i_m})\right]}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{im}L(\mathbf{x}^{i_m})\right)}.$$

Furthermore, if M is the market portfolio, $m=k$, and the asset pricing function is strictly decreasing everywhere, then $\alpha_{\epsilon,M}>0$ for $\epsilon>0$.

In this formulation both the absolute excess return $\alpha_{\epsilon,M}$ and the exposure of the claim to the risk of the portfolio of reference $\beta_{\epsilon,M}$ depend on the risk of a claim, but it has the advantage of ensuring that absolute excess returns are zero when excess returns to a claim are regressed on excess returns of a portfolio of claims belonging to the same class. This feature of returns can be exploited to break down claims into different classes. One can, for instance, start with assets considered riskless when held until maturity, which are reasonably easily identified as they are limited to certain set of sovereign debt instruments. The next step is to find the set of assets which belong to the same class by looking for all claims which do not generate absolute excess returns when they are included in the proposed portfolio for the class. Once the first class has been constructed, one can continue with the surrounding one(s), and so on and so forth.

Finally, expression (24) is not unlike expression (7) giving the conditional expected net returns of a security in the model with random trading costs of Acharya and Pedersen (2005). In fact, they obtain an expression for an asset's excess returns where the "beta" takes into account the variability in trading costs in addition to the variability in gross returns, very much like (24) takes into account the variability of the liquidity services in addition to the variability of the marginal utility at maturity. A notable difference between the two approaches is that Acharya and Pedersen (2005) take the variability in trading costs as a primitive, whereas the variability in liquidity services is endogenous in my framework. This is going to be of importance when I discuss the prediction of the model for velocity or volume of trade and the implications regarding to use of such variables as proxies for cost of trade.

6 Equilibrium Sorting

6.1 Two Additional Assumptions

I have so far looked in detail into the implications of the existence of asset classes for the decomposition of asset prices and returns. In order to be able to go further and provide a more detailed characterization of the equilibrium, and in particular of the types of sorting that are possible, I make two additional assumptions. First, I assume that there are two types of agents. Although this assumption is not necessary for all the results that follow,[16] it simplifies the analysis because there can then be only three types of equilibria. In the first type, which I call Non-Segmented Equilibria (NSE), the two types of consumers belong to the same class. In the other two types of equilibria possible the two types of consumers belong to different classes, with the difference between these two types of equilibria being about which types holds the safer claims. The second assumption that I make is with regards to the utility of consuming early and late for consumers:

Assumption 1: $u(c) = U(c) = (c^{1-\gamma} - 1)/(1-\gamma)$, for $\gamma > 0$.

This assumption ensures that the intensity of liquidity needs, that is how large the marginal utility of early consumption of an early consumer is relative to that of a late consumer, is driven exclusively by the parameter θ.[17] I now consider the implications of these two assumptions in turn.

[16] In particular, all the results about Non-Segmented Equilibria hold with an arbitrary number of types.

[17] A previous version of the paper (Jacquet, 2015) considered a case where the intensity of liquidity needs is in-built, by assuming that (i) late consumers are risk-neutral with $U(c) = c$, and (ii) the utility function of early consumers $u(c)$ is either of the Constant Relative Risk Aversion or quadratic type. In that case the intensity of liquidity needs is in equilibrium always greater in state l than in state h.

6.1.1 Portfolios as Mutual Funds

By restricting both the number of consumer types and the states of the world to two, it is possible and useful to transform the portfolio problem of agents into the choice of a mutual fund characterized by its size and its average risk. Let X^i and $\mathcal{E}^i$ be the size and risk at maturity of the portfolio $\mathbf{x}^i$ of an agent of type i, i.e.,

$$X^i \equiv \int x^i_\epsilon d\epsilon, \text{ and } \mathcal{E}^i \equiv \frac{1}{X^i}\int \epsilon x^i_\epsilon d\epsilon. \tag{25}$$

The value of this type's portfolio in the last period when the state of the world is ω is then

$$\mathbf{d}(\omega)\cdot\mathbf{x}^i = (1+\mathcal{E}^i(\omega))X^i,$$

where

$$1+\mathcal{E}^i(h) \equiv 1+\mathcal{E}^i \text{ and } 1+\mathcal{E}^i(l) \equiv 1-\frac{\pi}{1-\pi}\mathcal{E}^i.$$

And since there are only two types of agents we have that $x^2(\epsilon) = \left(Eg(\epsilon) - \mu^1 x^1_\epsilon\right)/(1-\mu^1)$, and therefore the size and risk of the portfolio of type 2 agents can easily be expressed as functions of the characteristics of the portfolio of type 1 agents:

$$X^2 = \frac{E-\mu^1 X^1}{1-\mu^1} \text{ and } \mathcal{E}^2 = \frac{E\mathcal{E}-\mu^1 X^1\mathcal{E}^1}{1-\mu^1 X^1}. \tag{26}$$

Importantly, there is an admissible range of values for the average risk of the portfolio $\mathcal{E}^i$ for a given size of portfolio X^i (and for given distribution of risk G and size of the asset pool E). I denote respectively by $\mathcal{E}^i_{\min}(X^i)$ and $\mathcal{E}^i_{\max}(X^i)$ the minimum and maximum admissible values of the average risk for agents of type i given the size of the portfolio X^i.[18] These restrictions implicitly incorporate the adding-up constraint that the portfolio sizes must also satisfy, $\sum_{i=1}^2 \mu^i X^i = E$.

We can also define $\widetilde{v}^i_\epsilon(X^i,\mathcal{E}^i)$ as the marginal value of a type-ϵ claim for an agent of type i when consumers of type 1 hold onto portfolios of size X^i and average risk $\mathcal{E}^i$,

$$\widetilde{v}^i_\epsilon(X^i,\mathcal{E}^i) \equiv \mathbb{E}\{d_\epsilon[U'((1+\mathcal{E}^i)X^i)+\sigma^i\widetilde{L}(X^i,\mathcal{E}^i)]\} \tag{27}$$

where

$$\widetilde{L}(X^i,\mathcal{E}^i;\omega) \equiv A(\omega)\theta(\omega)u'[A(\omega)\widetilde{\alpha}(X^i,\mathcal{E}^i;\omega)(1+\mathcal{E}^i(\omega))X^i] - U'((1+\mathcal{E}^i(\omega))X^i).$$

Definition 1 *A pair of portfolios* $(\mathbf{x}^1,\mathbf{x}^2)$*is* feasible *if (i)* $x^i_\epsilon \in [0, 2e_\epsilon/\mu^i]$ *for all* ϵ *and for both* i *and (ii)* $\mu^1 x^1_\epsilon + \mu^2 x^2_\epsilon = 2e_\epsilon$.

By definition a pair of portfolios is feasible if and only if $\mathcal{E}^1 \in [\mathcal{E}^1_{\min}(X^1), \mathcal{E}^1_{\max}(X^1)]$. We then have the following lemma.

Lemma 3 *Suppose there exists* $\left(X^1,\mathcal{E}^1\right) \in \left\{(X,\mathcal{E}) : X \in \left[0, E/\mu^1\right] \text{ and } \mathcal{E} \in \left[\mathcal{E}^1_{\min}(X), \mathcal{E}^1_{\max}(X)\right]\right\}$ *such that either:*
(i) NSE:

$$\widetilde{v}^1_\epsilon(X^1,\mathcal{E}^1) = \widetilde{v}^2_\epsilon\left(X^2,\mathcal{E}^2\right) \text{ for all } \epsilon, \text{ or} \tag{28}$$

[18] The expressions and some properties of these boundaries can be found in the online appendix.

(ii) SE: either (a) $X^1 = \int_{\epsilon_{\min}}^{\epsilon^*} 2e_\epsilon d\epsilon$ *and* $\mathcal{E}^1 = \mathcal{E}^1_{\min}(X^1)$ *for some* $\epsilon^* \in (\mathcal{E}^1_{\min}(X^1), \mathcal{E}^1_{\max}(X^1))$ *such that*

$$\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1_{\min}(X^1)) > (=)[<]\widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2_{\max}(X^2)) \text{ for } \epsilon < (=)[>]\epsilon^*, \text{ or} \tag{29}$$

(b) $X^1 = \int_{\epsilon^*}^{\epsilon_{\max}} 2e_\epsilon d\epsilon$ *and* $\mathcal{E}^1 = \mathcal{E}^1_{\max}(X^1)$ *for some* $\epsilon^* \in (\mathcal{E}^1_{\min}(X^1), \mathcal{E}^1_{\max}(X^1))$ *such that*

$$\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1_{\max}(X^1)) < (=)[>]\widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2_{\min}(X^2)) \text{ for } \epsilon < (=)[>]\epsilon^*. \tag{30}$$

Then any feasible pair of portfolios $(\mathbf{x}^1, \mathbf{x}^2)$ *such that (25) holds and with* $v^i_\epsilon(\mathbf{x}^i) = \widetilde{v}^i_\epsilon(X^i, \mathcal{E}^i)$ *for all* ϵ *and for both* i *is an equilibrium pair of portfolios. Furthermore, the pricing function* $\mathbf{q}$ *is the same for all equilibrium portfolios associated with the same couple* $(X^1, \mathcal{E}^1)$ *and it is such that* $q_\epsilon = \max\{\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1); \widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2)\}$ *for all* ϵ.

This lemma enables us to work with portfolios as defined by their size and average risk: once we have found a couple $(X^1, \mathcal{E}^1)$ which is feasible and implies that the marginal valuation functions $\widetilde{v}^1_\epsilon$ and $\widetilde{v}^2_\epsilon$ satisfy one of the three sets of conditions (28)-(30), then an equilibrium exists and a pair of equilibrium portfolios can be backed out by using (25). And it is straightforward to obtain the asset pricing function since it is the upper-envelope of the family of marginal valuation functions. Note that equilibrium portfolios might not be unique, even if there is a unique couple $(X^1, \mathcal{E}^1)$ solving one of the three sets of conditions indicated in the lemma. In fact, there might be more than one portfolio leading to a given couple $(X^1, \mathcal{E}^1)$. If, for instance, we consider a distribution of risk which is uniformly distributed, then there are a continuum of ways to obtain a given $(X^1, \mathcal{E}^1)$.[19]

When the pair $(X^1, \mathcal{E}^1)$ associated with an equilibrium pair of portfolio is such that (29) holds, then consumers with the greater liquidity needs hold onto all assets with risk below a certain threshold, while the opposite is true when instead it is such that (30) holds. In this case we can use the following definition of sorting.

Definition 2 *An equilibrium displays* positive (negative) sorting with segmentation *if it is such that type-1 consumers hold onto all claims with risk below (above) a certain threshold* ϵ^*.

But what about equilibria where the pair $(X^1, \mathcal{E}^1)$ associated with an equilibrium pair of portfolio is such that (28) holds? In this case there is no segmentation but we can use the following definition.

Definition 3 *An equilibrium displays* positive (no) [negative] sorting *if it is such that* $\mathcal{E}^1 < (=)[>]\mathcal{E}^2$.

In this weaker definition of sorting an equilibrium with positive sorting is such that agents with greater liquidity needs hold portfolios with lower *average* risk. And *vice versa* in the case of negative sorting.

6.1.2 Liquidity Needs as Preference Shocks

Lemma 1 characterizes half of the terms of trade by establishing that the price at which a type-ϵ claim is sold in state ω is $Q_\epsilon(\omega) = A(\omega)d_\epsilon(\omega)$. The next lemma characterizes the fraction of her portfolio that an agent spends in the middle period.

[19] For instance, if agents of type i hold onto all claims with risk level in the intervals $(\mathcal{E}^i - \eta - 0.5X^i(\epsilon_{\max} - \epsilon_{\min}), \mathcal{E}^i - \eta)$ and $(\mathcal{E}^i + \eta, \mathcal{E}^i + \eta + 0.5X^i(\epsilon_{\max} - \epsilon_{\min}))$ for $\eta \geq 0$, then type-i agents have portfolios of size X^i and average risk $\mathcal{E}^i$, irrespective of the exact value for η.

Lemma 4 *Suppose Assumption 1 holds. Then the fraction of her portfolio that an agent spends in state $\omega \in \{h, l\}$ is given by*

$$\alpha(\omega) = \frac{1}{1 + (A(\omega))^{1-\frac{1}{\gamma}}(\theta(\omega))^{-\frac{1}{\gamma}}}. \tag{31}$$

We hence obtain that the fraction of an agent's portfolio that she spends in the middle period does not depend on her type or on the size and composition of her portfolio, which simplifies the analysis significantly. Moreover, $\alpha(\omega)$ is always increasing with the *intensity of the liquidity needs* as measured by θ. The impact of the price, which I call the *Terms of Trade effect* and originates from the variation in the disutility of production for producers A^{-1}, depends on the strength of the substitution effect. If the substitution effect is strong, i.e., $\gamma \in (0, 1)$, a decrease in the cost of production, which implies an increase in the selling price of claims and therefore a drop in the cost of consumption in the middle period relative to the last period, leads to a decrease in last period consumption. And as a result the fraction of the portfolio being sold for consumption in this middle period increases. If instead the substitution effect is weak because $\gamma > 1$, last period consumption increases following the decrease in the relative cost of middle period consumption, which means that the fraction of portfolio sold in the middle period decreases. And, as usual, the substitution and wealth effects cancel out when $\gamma = 1$, in which case the terms of trade do not matter for last period consumption, implying that the fraction of portfolio sold in the middle period is not state dependent.

A corollary to lemma 4 is that

$$\widetilde{L}(X^i, \mathcal{E}^i; \omega) = \{[1 + (A(\omega))^{\frac{1}{\gamma}-1}(\theta(\omega))^{\frac{1}{\gamma}}]^{\gamma} - 1\}U'((1 + \mathcal{E}^i(\omega))X^i). \tag{32}$$

and therefore the expression for the marginal valuation of a claim in period 1 (27) can be re-expressed as

$$\widetilde{v}^i_{\epsilon}(X^i, \mathcal{E}^i) = \overline{\sigma}^i \mathbb{E}\left[d_{\epsilon}(\widetilde{\sigma}^i/\overline{\sigma}^i)U'((1 + \mathcal{E}^i)X^i)\right], \tag{33}$$

where

$$\widetilde{\sigma}^i(\omega) \equiv 1 + \sigma^i\{[1 + (A(\omega))^{\frac{1}{\gamma}-1}(\theta(\omega))^{\frac{1}{\gamma}}]^{\gamma} - 1\}, \text{ and} \tag{34}$$

$$\overline{\sigma}^i \equiv \mathbb{E}(\widetilde{\sigma}^i). \tag{35}$$

(33) shows that the model is equivalent to a standard consumption-based CAPM where there is no middle period, and therefore no liquidity needs, but where agents differ in their preferences in two ways. First, they differ in their absolute desire to consume through differences in $\overline{\sigma}$. Second, the different types of agents face preference shocks $(\widetilde{\sigma}/\overline{\sigma})$ that are correlated across types, but with type-specific volatility. I will soon establish that: (i) $\overline{\sigma}$ matters for the size of agents' portfolios - agents with the largest absolute desire $\overline{\sigma}$ to consume choosing larger portfolios; and (ii) it is the variation in $(\widetilde{\sigma}^i/\overline{\sigma}^i)$ which matters for sorting, because it determines the covariance between d_{ϵ} and $(\widetilde{\sigma}^i/\overline{\sigma}^i)U'((1 + \mathcal{E}^i)X^i)$ - no sorting takes place when $\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h) = \widetilde{\sigma}^2(l)/\widetilde{\sigma}^2(h)$, while type-1 consumers hold onto safer (riskier) portfolios than type-2 consumers if $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h))/(\widetilde{\sigma}^2(l)/\widetilde{\sigma}^2(h))$ is greater (smaller) than 1. This is because the utility function is homogeneous (Assumption 1).

Before I move on to the complete characterization of equilibrium and the intuition, it is useful to present the following result, for it links the relative variation in $\widetilde{\sigma}$ to the variation in terms of trade and intensity of liquidity needs.

Lemma 5 $\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h) \geq \widetilde{\sigma}^2(l)/\widetilde{\sigma}^2(h)$ *if and only if* $(A(l))^{1-\gamma}\theta(l) \geq (A(h))^{1-\gamma}\theta(h)$, *with strict inequality if* $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$.

6.2 Equilibrium Portfolios and Sorting

I now present the second main result of the paper, after which I will go over the intuition. And I leave it to the interested reader to go to the appendix for further details.

Proposition 4 *Suppose Assumption 1 holds. (i) There exists a unique* $(X^1, \mathcal{E}^1)$ *solving one of (28)-(30). It is such that* $X^1 > X^2$ *always, and* $\mathcal{E}^1 < (=)[>]\mathcal{E}^2$ *if and only if* $(A(l))^{1-\gamma}\theta^l > (=)[<](A(h))^{1-\gamma}\theta^h$*.*
(ii) Furthermore, let $\sigma^1 = \sigma + \eta$ *and* $\sigma^2 = \sigma - \eta$*, for* $\eta \in (0, \sigma)$*. Then, whenever* $(A(l))^{1-\gamma}\theta^l \neq (A(h))^{1-\gamma}\theta^h$ *there exists* $\eta^* \in (0, \sigma)$ *such that there is no segmentation in equilibrium for* $\eta < \eta^*$*, whereas segmentation appears whenever* $\eta \geq \eta^*$*.*

A direct corollary to this proposition is that an equilibrium is essentially unique, in that payoffs are unique because all equilibrium variables other than portfolios are uniquely pinned. And therefore a SE exists if and only if a NSE does not. Let me provide the intuition as to why an equilibrium displays segmentation whenever the difference in frequency of liquidity needs of the two types of consumers is beyond a certain threshold (part (ii)). First note that there always exists $\left(X^1, \mathcal{E}^1\right)$ such that $\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1) = \widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2)$ for all ϵ.[20] If the two types of agents have very similar liquidity needs, because σ^1 is only slightly greater than σ^2, they intuitively should hold very similar portfolios, in that their sizes and average risks do not differ very much. If we assume, for the sake of exposition, that $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$, one can intuit from part (i) of proposition 4 that $\mathcal{E}^1$ is only slightly smaller than $\mathcal{E}$, which is itself slightly smaller than $\mathcal{E}^2$. Hence, unless the distribution of risk G is of an odd shape, the couple $\left(\mathcal{E}^1, X^1\right)$ for which $\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1) = \widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2)$ for all ϵ is such that $\mathcal{E}^1$ falls within $[\mathcal{E}^1_{\min}(X^1), \mathcal{E}^1_{\max}(X^1)]$. In this case a NSE exists, and one can show that a SE does not. But when σ^1 is much greater than σ^2, type-1 consumers want to hold onto a portfolio much larger and much safer than that of type-2 consumers, and it can therefore be that given the distribution of risk G the couple $\left(\mathcal{E}^1, X^1\right)$ for which $\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1) = \widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2)$ for all ϵ is such that $\mathcal{E}^1$ falls outside $[\mathcal{E}^1_{\min}(X^1), \mathcal{E}^1_{\max}(X^1)]$. In this case a NSE does not exist, but then a SE does: when a NSE does not exist one can find a threshold ϵ^* such that type 1 agents can hold onto a large and safe portfolio by holding onto all claims with a level of risk below ϵ^* and the marginal values of claims of the two types of consumers cross at ϵ^*. We therefore have that the equilibrium is always essentially unique, and a NSE is an interior equilibrium (where the feasibility constraint for portfolios is not binding) while a SE is a corner equilibrium (where the feasibility constraint for portfolios is binding).

I now discuss the forces that drive sorting and I assume, for the sake of clarity, that the equilibrium is a NSE. The intuition carries over to the case with segmentation, since such equilibria are corner solutions that exist only when the difference in liquidity needs is so large that the two types of consumers want to hold such different portfolios that they cannot hold the same claims. Since $\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1) = \widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2)$ for all ϵ

[20] In fact, (27) can, just like v^i_ϵ, be expressed as a linear function of risk:

$$\widetilde{v}^i_\epsilon(X^i, \mathcal{E}^i) \equiv \widetilde{v}^i_0(X^i, \mathcal{E}^i) - \epsilon\pi\widetilde{\Delta}^i(X^i, \mathcal{E}^i), \tag{36}$$

where

$$\widetilde{\Delta}^i(X^i, \mathcal{E}^i) \equiv \widetilde{\sigma}^i(l)U'((1+\mathcal{E}^i(l))X^i) - \widetilde{\sigma}^i(h)U'((1+\mathcal{E}^i(h))X^i).$$

However, the couple $\left(\mathcal{E}^1, X^1\right)$ such that $\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1) = \widetilde{v}^2_\epsilon(X^2, \mathcal{E}^2)$ for all ϵ solves the system

$$\widetilde{v}^1_0(X^1, \mathcal{E}^1) = \widetilde{v}^2_0(X^2, \mathcal{E}^2) \text{ and } \widetilde{\Delta}^1(X^1, \mathcal{E}^1) = \widetilde{\Delta}^2(X^2, \mathcal{E}^2). \tag{37}$$

in a NSE, we have that $(X^1, \mathcal{E}^1)$ is such that

$$\overline{\sigma}^1(X^1)^{-\gamma}\mathbb{E}\left[d_\epsilon(\widetilde{\sigma}^1/\overline{\sigma}^1)(1+\mathcal{E}^1)^{-\gamma}\right] = \overline{\sigma}^2(X^2)^{-\gamma}\mathbb{E}\left[d_\epsilon(\widetilde{\sigma}^2/\overline{\sigma}^2)(1+\mathcal{E}^2)^{-\gamma}\right] \text{ for all } \epsilon. \tag{38}$$

As mentioned earlier, the relative variations in $\widetilde{\sigma}/\overline{\sigma}$ for the two types is what drives sorting, and it is mathematically intuitive. If, in particular, $\widetilde{\sigma}^i(l) = \widetilde{\sigma}^i(h) = \overline{\sigma}^i$ for both i, which is true whenever $(A(l))^{1-\gamma}\theta(l) = (A(h))^{1-\gamma}\theta(h) = \overline{z}$, it is clear from (38) that a NSE exists and is such that

$$\overline{\sigma}^1(X^1)^{-\gamma} = \overline{\sigma}^2(X^2)^{-\gamma} \text{ and } \mathcal{E}^1 = \mathcal{E}^2.$$

As for the economic intuition, remember that $\widetilde{\sigma}^i(\omega)U'((1+\mathcal{E}^i(\omega))X^i)) = U'((1+\mathcal{E}^i(\omega))X^i)+\sigma^i\widetilde{L}(\omega)(X^i,\mathcal{E}^i)$, which is to say that liquidity needs raise the marginal value of consumption in each state for a consumer who does not yet know whether she will be an early consumer, while $\widetilde{\sigma}^i(\omega) = 1 + \sigma^i[\widetilde{L}(X^i,\mathcal{E}^i;\omega)/U'((1+\mathcal{E}^i(\omega))X^i)) - 1]$. But when $(A(l))^{1-\gamma}\theta(l) = (A(h))^{1-\gamma}\theta(h) = \overline{z}$, there is no sorting because in this case liquidity needs raise the marginal value of consumption *in the same proportion* for both states, and therefore liquidity needs do not change the relative preference for the different types of claims of the two types of consumers. In fact, in this case $\widetilde{\sigma}^i(l) = \widetilde{\sigma}^i(h) = \overline{\sigma}^i$ and therefore $\widetilde{L}(X^i,\mathcal{E}^i;\omega) = \overline{Z}U'((1+\mathcal{E}^i(\omega))X^i))$ for both the state, with $\overline{Z} \equiv (1+\overline{z}^{1/\gamma})^\gamma$.

If instead $(A(l))^{1-\gamma}\theta(l) > \overline{z} > (A(h))^{1-\gamma}\theta(h)$, then $\widetilde{\sigma}^i(l)/\overline{\sigma}^i > \widetilde{\sigma}^i(h)/\overline{\sigma}^i$ for both i, because in this case $\widetilde{L}(X^i,\mathcal{E}^i;l) > \overline{Z}U'((1+\mathcal{E}^i(l))X^i)$ while $\widetilde{L}(X^i,\mathcal{E}^i;h) < \overline{Z}U'((1+\mathcal{E}^i(h))X^i)$. That is, liquidity needs raise the marginal value of consumption relatively more for state l than for state h. This makes lower risk claims even more attractive than in the case $(A(l))^{1-\gamma}\theta(l) = (A(h))^{1-\gamma}\theta(h) = \overline{z}$ for both types of consumers. But $\sigma^1 > \sigma^2$ means type-1 consumers are disproportionately more affected (equivalently, $\widetilde{\sigma}^i/\overline{\sigma}^i$ varies more for type-1 consumers as established lemma 5), and as a result they hold safer portfolios than consumers of type 2 in equilibrium.[21] The case where $(A(l))^{1-\gamma}\theta(l) < \overline{z} < (A(h))^{1-\gamma}\theta(h)$ is symmetric.

Note that in the example of section 3 it was the variation across states in $AV_1 - V_2$, the absolute difference across states of the value of liquidity, that mattered for sorting. This difference with the general case exists because in the example the marginal values of consumption are constant, whereas in the general setup they depend on wealth.

Having worked out the role of the variation in the value of liquidity in sorting, it is easy to separate the effects of the intensity of liquidity needs, which are driven by θ, and of the terms of trade, which are driven by A. In the absence of terms of trade effects, which happens when $A(h) = A(l)$, sorting is positive whenever the intensity of liquidity needs is stronger in state l than it is in state h. This is intuitive: the marginal utility of consumption of an early consumer relative to that of a late consumer is greater in state l than in state h when $\theta(l) > \theta(h)$, and this raises the value of liquidity in state l relative to state h.[22]

If one instead abstracts from the effect of the intensity of liquidity needs and assume that $\theta(h) = \theta(l)$, one can see that the impact of the terms of trades depends on the strength of the substitution effect. If γ

[21] And because consumers of type 1 hold disproportionately more low-ϵ claims, which reduces (raises) the value of consumption in state l (h) for a portfolio of a given size, and vice versa for type-2 consumers, the two types of consumers can have the same marginal value of consumption in both states in equilibrium, which then means they share the same marginal valuation for all claims.

[22] I mentioned earlier that I obtained a similar result in a previous version of the paper with no terms-of-trade effects and where the intensity of liquidity needs is in-built, because $u(c)$ is of the CRRA or quadratic types and $U(c) = c$. We can now see that in these cases the ratio of marginal utilities is correlated with the risk of the portfolio held by an agent, which is why in this earlier version the intensity of the liquidity needs is in-built and always in one direction.

is below 1, so that the substitution effect is strong, the value of liquidity is greater in state l than in state h whenever $A(l)$ is greater than $A(h)$. This is because the terms of trade in the middle period then imply that the ratio of the price of claims in state l in the middle period, $A(l)d_\epsilon(l)$, relative to its price in the last period, $d_\epsilon(l)$, is higher than the ratio of these two prices in state h. The effects of the terms of trade are opposite if the substitution effect is weak because $\gamma > 1$, while if the income and substitution effect cancel out, i.e., $\gamma = 1$, then the terms of trade have no impact.

6.3 Asset Pricing Implications

The implications of sorting for asset prices are summarized in the following corollary to proposition 4.

Corollary 1 *Suppose Assumption 1 holds.*
(i) If $(A(l))^{1-\gamma}\theta(l) \geq (A(h))^{1-\gamma}\theta(h)$, then q_ϵ is a strictly decreasing function, which is linear with slope $-\pi\widetilde{\Delta}^1 = -\pi\widetilde{\Delta}^2$ if the equilibrium is a NSE, or piecewise linear with slope $-\pi\widetilde{\Delta}^1$ at first and then slope $-\pi\widetilde{\Delta}^2 > -\pi\widetilde{\Delta}^1$ if it is a SE.
(ii) If $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$ and the equilibrium is a NSE, then q_ϵ is a linear function which is decreasing, flat, or increasing depending on whether $\widetilde{\Delta}^1 = \widetilde{\Delta}^2$ is positive, zero, or negative respectively; if instead the equilibrium is a SE, then q_ϵ is a piecewise linear function which is strictly decreasing everywhere, decreasing and then increasing, or increasing everywhere depending on whether $\widetilde{\Delta}^2 > \widetilde{\Delta}^1 \geq 0$, $\widetilde{\Delta}^2 > 0 > \widetilde{\Delta}^1$, or $0 \geq \widetilde{\Delta}^2 > \widetilde{\Delta}^1$ respectively.

The asset pricing function is therefore unambiguously decreasing whenever sorting is positive or absent, whereas the relationship between the price of claims and their risk is not fixed and can be non-monotonic when sorting is negative. However, because the asset pricing function is always convex, if we consider three types of claims with risk $\epsilon > \epsilon' > \epsilon''$, it is always true that $q_\epsilon - q_{\epsilon'} \geq q_{\epsilon'} - q_{\epsilon''}$, with strict inequality unless all three types of claims belong to the same class.

I will now provide some intuition for the shape of the asset pricing function. A useful starting point is the fact that the risk adjustment to the marginal valuation of a consumer of type i, $cov(d_\epsilon, U'((1+\mathcal{E}^i)X^i) + \sigma^i\widetilde{L}(X^i, \mathcal{E}^i))$, can be decomposed into the adjustment for the risk at maturity,

$$cov(d_\epsilon, U'((1+\mathcal{E}^i)X^i)) = -\epsilon\pi[(1+\mathcal{E}^i(l))^{-\gamma} - (1+\mathcal{E}^i(h))^{-\gamma}](X^i)^{-\gamma},$$

and σ^i times the risk adjustment for liquidity services

$$cov(d_\epsilon, \widetilde{L}(X^i, \mathcal{E}^i)) = -\epsilon\pi[Z(l)(1+\mathcal{E}^i(l))^{-\gamma} - Z(h)(1+\mathcal{E}^i(h))^{-\gamma}](X^i)^{-\gamma},$$

where $Z(\omega) = (1 + (A(\omega))^{1/\gamma-1}(\theta(\omega))^{1/\gamma})^\gamma$. Whenever $(A(l))^{1-\gamma}\theta(l) \geq (A(h))^{1-\gamma}\theta(h)$, then $Z(l) \geq Z(h)$, with strict inequality when $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$. That is, liquidity needs in this case make consumers like low-ϵ claims even more than without. In equilibrium both consumers choose portfolios whose average risk is positively correlated with the risk of the market portfolio, i.e., $\mathcal{E}^i > 0$ for both i, and as a result one gets that $cov(d_\epsilon, \widetilde{L}(X^i, \mathcal{E}^i)) < cov(d_\epsilon, U'((1+\mathcal{E}^i)X^i)) < 0$ for all $\epsilon > 0$ and both i. That is, the risk adjustment for liquidity services raises the adjustment for risk of marginal valuations, and therefore the asset pricing function decreases more with risk.

When $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$ instead, then liquidity needs no longer always imply an adjustment for risk which amplifies the adjustment for risk at maturity. If the variation in $(A)^{1-\gamma}\theta$ is not too strong, it is still the case that $cov(d_\epsilon, \widetilde{L}(X^i, \mathcal{E}^i)) < 0$ for all $\epsilon > 0$ and both i. In this case liquidity needs still amplify the adjustment for risk at maturity, and therefore the marginal valuations of the two types and the asset pricing function are still decreasing in risk everywhere. But if $Z(h)$ is large enough relative to $Z(l)$ (because the variation in $(A)^{1-\gamma}\theta$ is large enough), then we have that $cov(d_\epsilon, \widetilde{L}(X^i, \mathcal{E}^i)) > 0$ for all $\epsilon > 0$, which means the adjustments for risk at maturity and for liquidity are of opposite signs: consumers prefer low-ϵ claims for late consumption, but they prefer high-ϵ claims for early consumption. As a result $cov(d_\epsilon, U'((1+\mathcal{E}^i)X^i)+\sigma^i\widetilde{L}(X^i, \mathcal{E}^i))$ is negative or positive for $\epsilon > 0$ depending on which effect dominates, and this depends on the frequency of the liquidity needs and is more likely to be positive for type-1 consumers. It can therefore be that $cov(d_\epsilon, U'((1+\mathcal{E}^1)X^1)+\sigma^1\widetilde{L}(X^1, \mathcal{E}^1)) > 0$ and $cov(d_\epsilon, U'((1+\mathcal{E}^2)X^2)+\sigma^2\widetilde{L}(X^2, \mathcal{E}^2)) < 0$, in which case the asset pricing function is decreasing in risk at first, on the range of risk of claims held by type-2 consumers, while it is increasing for the riskier claims, on the range of claims held by type-1 consumers. And if the variation in $(A)^{1-\gamma}\theta$ is larger still, then $cov(d_\epsilon, U'((1+\mathcal{E}^i)X^i)+\sigma^i\widetilde{L}(X^i, \mathcal{E}^i))$ is positive for $\epsilon > 0$ for both types of consumers, in which case the asset pricing function is upward sloping everywhere.

In other words, if the intensity of liquidity needs and/or the terms of trade in the middle period make consuming early in state h more desirable than consuming early in state l, then liquidity needs make high ϵ claims less undesirable. And if the preference for consuming early in state h rather than in state l is strong enough, then liquidity needs can even make high ϵ claims more desirable than lower ϵ claims for at least one type of consumers, explaining why the asset pricing function can be upward sloping either in parts (type-1 consumers like risky claims) or everywhere (both types of consumers like risky claims).

6.4 Velocity and Clientele Effect

I now investigate the relationships between risk, price, and velocity of claims, where the velocity of claims with risk in a given interval is defined as being the volume of trade for these claims over the stock of such claims. Since all late consumers hold on to all of their asset maturity, differences in the velocity of trade across claims are driven exclusively by differences in the frequency of liquidity needs of consumers and velocity of trade in state ω for claims with risk in an interval (ϵ', ϵ'') can be formally defined as

$$VELO(\epsilon', \epsilon''|\omega) = \alpha(\omega)\frac{\int_{\epsilon'}^{\epsilon''}\sum_{i=1,2}\sigma^i\mu^i x_\epsilon^i d\epsilon}{2\int_{\epsilon'}^{\epsilon''} e_\epsilon d\epsilon} \in [\alpha(\omega)\sigma^2, \alpha(\omega)\sigma^1],$$

where I continue to assume that all early consumers sell a fraction $\alpha(\omega)$ of their claims in state ω in the middle period. Clearly, velocity increases as the fraction of claims with risk in (ϵ', ϵ'') held by type-1 agents increases, and when there is segmentation it is equal to $\alpha(\omega)\sigma^1$ for claims held by type-1 consumers and $\alpha(\omega)\sigma^2$ for those held by type-2 consumers.[23]

Sorting thus creates a correlation between risk and velocity when considering the entire set of claims. In fact, by definition high-liquidity-need consumers hold onto relatively safer portfolios when sorting is positive, i.e., $\varepsilon^1 < \varepsilon^2$, while the opposite holds with negative sorting. In the former case the model generates a negative correlation between risk and velocity, whereas the correlation is positive with negative sorting. And if no sorting takes place, no correlation is observed because both types of consumers hold onto portfolios with the same average risk.

[23]This is because in this case $x_\epsilon^i = 2e_\epsilon/\mu^i$ for claims held by type i consumers.

Regarding the correlation between the price and the velocity of trade of claims, it also depends on the type of sorting observed in equilibrium. Since the asset pricing function is strictly decreasing in risk when there is positive or no sorting, prices and velocity are positively correlated in the former case, while there is no relationship in the latter. And because the asset pricing function can be decreasing, or decreasing and then increasing, or even increasing everywhere when sorting is negative, the correlation between asset prices and velocity can be anything in this case.[24]

Clientele effect - The differences in measured liquidity arise from a selection effect, which is known in the literature as a clientele effect. The clientele effect appears in papers like Amihud and Mendelson (198), Pagano (1989), and Studart (2014), because, similar to this framework, some agents have shorter trading horizon than others, but, and this is in contrast to this paper, assets differ in their cost of trade, which is meant to capture costs arising from transactions, like broker fees and bid-ask spread, or from search and delays of trades. However, in this paper differences in velocity arise from risk differences across assets and these do not have any direct relationship with the ease of trade or any other determinant of intrinsic liquidity.

The distinction between the two sources of differences in velocity matters and highlights that the difference between measured and intrinsic liquidity is subtle, but important. In fact, if differences of velocity across assets arise from differences in risk, then differences in measured liquidity need not imply differences in intrinsic liquidity. Hence, if one considers some assets with a low velocity of trade and the low velocity is purely the consequence of a clientele effect originating in risk differences, then it would not be possible to increase the measured liquidity of this asset market without changing the intrinsic risk characteristics of this type of assets, which might not be feasible or even desirable since it means reducing the risk of these assets. Furthermore, as I discuss in greater detail in the online appendix, changes in velocity observed for some assets can, through changes in the types of agents who hold them, be the natural consequence of changes in the perceived risk of these assets. In particular, sudden drops in measured liquidity observed in times of so-called market freezes need not have anything to do with disruptions to the functioning of a market.

The fact that intrinsic differences in liquidity need not be the sole driver of differences in velocity or volume of trade means that it can be problematic to use liquidity measures that rely on velocity or trading volume of an asset as a proxy for its intrinsic liquidity properties. For instance, consider the following measure of illiquidity for some asset i in month t proposed by Amihud (2002) and used by Acharya and Pedersen (2005),

$$ILLIQ_{iy} \equiv \frac{1}{D_{iy}} \sum_{t=1}^{D_{iy}} \frac{|R_{iyd}|}{VOLD_{ivyd}},$$

where R_{iyd} and $VOLD_{ivyd}$ are respectively the return and dollar volume for asset i on day d of year y, and D_{iy} is the number of days for which data are available for asset i in year y. Assume the model predicts there are two classes, because σ^1 is significantly larger than σ^2. Then, claims in the first class have a much greater velocity that assets in the second class. Consider two assets with similar risk parameters $\epsilon = \epsilon^* - \eta$ and $\epsilon' = \epsilon^* + \eta$, for η small and where ϵ^* is the cutoff between the two classes. These two types of claims must have similar average daily returns, for the asset pricing function is continuous. But since they have very different velocities of trade, we would have that $ILLIQ_{\epsilon}$ is significantly lower than $ILLIQ_{\epsilon'}$, which would be interpreted as implying that type-ϵ are a lot more liquid than the ones of type-ϵ' even though they are

[24]These results hold on *average* over the entire range $[\epsilon_{\min}, \epsilon_{\max}]$ *and* if we assume that consumers pay in the middle period with portfolios that are a fraction of their initial portfolios. I refer the interested reader to the discussion in part 1 of the online appendix for more details.

intrinsically equally liquid. Even if we introduce shocks (please see the online appendix) that lead to changes in the cutoff ϵ^* and to shifts in the asset pricing function, because the asset pricing function is continuous in risk while velocity is not, claims with not too different risk levels still have not too different average daily returns while they can have quite different average velocity.

Note also that another difference between the risk-based source of differences in liquidity put forward in this paper and the cost-based explanation found in Amihud and Mendelson (1986) is that the risk-based explanation requires heterogeneity of agents' liquidity needs to generate differences in the velocity of trade of assets, whereas the cost-based explanation does not. In fact, if we assume that all agents are alike, live for infinitely-many periods, and face common liquidity needs, in the form of consumption needs, that arise randomly and with a varying intensity over time, agents would trade low-cost assets more often that high-cost ones to save on the cost of trade: if consumption needs are low agents sell only assets with the lowest cost of trade, while when consumption needs are larger agents also sell assets with larger cost of trade. In this sense the differences in velocity obtained in the cost-based explanation are actually driven by differences in assets' characteristics, whereas the risk-based explanation put forward in this paper also requires heterogeneity in agents' liquidity needs.

7 Risk Aversion and Intertemporal Substitution

The discussion in section 6.2 explained that the impact of the terms of trade effect on sorting depends on the strength of the substitution effect. However, because the degree of risk aversion and the elasticity of substitution are controlled by the same parameter in the CRRA utility function used so far, it was not possible to distinguish between the static and intertemporal substitution effects. In this section I address this shortcoming and characterize sorting in a slightly modified model where consumers are endowed with a recursive utility function à la Epstein-Zin. I restrict my attention to the case where there is no segmentation in equilibrium. And barring any ambiguity I drop the superscript "i" referring to an agent's type in order to save on notation. All proofs and derivations not in the main text can be found in the online appendix.

Let $\widehat{v}(\mathbf{x})$ be the value at date 0 of choosing portfolio $\mathbf{x}$ and denote by $\widehat{V}^s(\mathbf{x};\omega)$ the value of entering date 1 with portfolio $\mathbf{x}$ when the state of the world is ω and s is an indicator function equal to 1 if the consumer is an early consumer and 0 otherwise. I assume that

$$\widehat{v}(\mathbf{x}) = \left\{ C_0^{1-\rho}(\mathbf{x}) + \left(\mathbb{E}\left[\widehat{V}^{1-\gamma}(\mathbf{x}) \right] \right)^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}, \tag{39}$$

where $C_0(\mathbf{x}) = H + \mathbf{q} \cdot (\mathbf{k} - \mathbf{x})$ for some endowment of goods $H > 0$ large enough to avoid corner solutions, the expectation is with respect to the pair (s, ω), and

$$\widehat{V}^s(\mathbf{x};\omega) = \begin{cases} \mathbf{d}^\omega \cdot \mathbf{x}, \text{ if } s = 0; \\ \max_{\mathbf{y}(\mathbf{x};\omega): y_\epsilon \in [0, x_\epsilon]} \left\{ \theta(\omega) \left(A(\omega) \mathbf{d}(\omega) \cdot \mathbf{y}(\omega)(\mathbf{x}) \right)^{1-\rho} + \left(\mathbf{d}(\omega) \cdot (\mathbf{x} - \mathbf{y}(\omega)(\mathbf{x})) \right)^{1-\rho} \right\}^{\frac{1}{1-\rho}}, \text{ if } s = 1, \end{cases} \tag{40}$$

where I have already taken advantage of the fact that we know the terms of trade at date 1. As usual with this family of utility functions, the constant coefficient of relative risk aversion (CCRRA) is γ, while the intertemporal elasticity of substitution (IES) is $1/\rho$. I once again derive the first-order conditions for the date-0 portfolio problem by working backwards.

Lemma 6 *In equilibrium,* $\widehat{V}^1(\mathbf{x};\omega) = \widehat{Z}(\omega)\left(\mathbf{d}(\omega)\cdot\mathbf{x}\right)$*, where*

$$\widehat{Z}(\omega) \equiv \left[1 + (\theta(\omega))^{1/\rho}\,(A(\omega))^{(1-\rho)/\rho}\right]^{\frac{\rho}{1-\rho}}. \tag{41}$$

All uncertainty has been resolved in period 1 and therefore the value of entering period 1 with a given portfolio only depends on the parameter controlling the intertemporal elasticity of substitution. Two remarks are in order. First, although I use the same IES for the desire to substitute consumption intertemporally between dates 0 and 1 and 1 and 2, the IES coefficient appearing in (41) is the IES between dates 1 and 2. Note also for future reference that the lemma implies that

$$\widehat{Z}(\omega) = \frac{\widehat{V}^1(\mathbf{x};\omega)}{\widehat{V}^0(\mathbf{x};\omega)} = \frac{\widehat{V}^1_\epsilon(\mathbf{x};\omega)}{\widehat{V}^0_\epsilon(\mathbf{x};\omega)} \text{ for all } \epsilon,$$

where $\widehat{V}^1_\epsilon(\mathbf{x};\omega)$ is the marginal value of type ϵ claims. The second equality follows from the linearity of $\widehat{V}_1$ with respect to x_ϵ. That is, $\widehat{Z}(\omega)$ is the ratio in state ω of the marginal values of a unit of claims at date 1 to that of date 2, and the marginal value of liquidity in that state is therefore $(\widehat{Z}(\omega)-1)\left(\mathbf{d}(\omega)\cdot\mathbf{x}\right)$.

Consider the role of the IES in the impact of the variations in the terms of trade and intensity of liquidity needs on $\widehat{Z}$. If $\theta(h)=\theta(l)$, then $\widehat{Z}(l) > \widehat{Z}(h)$ if and only if $A(l) > A(h)$. This is intuitive as we should expect the date-1 marginal values of claims for early consumers, relative to that of late consumers, to be greater in the state where terms of trade in the middle period are relatively more favorable, irrespective of the value of the IES. If instead one assumes that $A(h) = A(l)$, then $\widehat{Z}(l) > \widehat{Z}(h)$ if and only if $\theta(l) > \theta(h)$ when $\rho \in (0,1)$ or $\theta(l) < \theta(h)$ when $\rho > 1$. The IES thus matters for role of the variation in the intensity of the liquidity needs. If the intertemporal substitution effect between dates 1 and 2 is large, then early consumers are happy to allocate a greater share of resources to early consumption in the state with the greatest intensity of liquidity needs, say state l. And because this does not affect much the marginal utility of consumption in either state (controlling for θ), the marginal values of claims for early consumers relative to that of late consumers is greater in state l than in h. And vice versa if instead the willingness to intertemporally substitute consumption is low.

The date-0 first-order condition with respect to the choice of type-ϵ claims at date 0 is

$$\widehat{v}_\epsilon(\mathbf{x}^i) \le 0, \;=\; \text{if } x_\epsilon > 0,$$

However, I here focus on NSE and in these type of equilibria it must be that $\widehat{v}_\epsilon(\mathbf{x}^i) = 0$ for all ϵ *and* for all consumer types. Let this time

$$\widehat{\sigma}^i(\omega) \equiv 1 + \sigma^i[(\widehat{Z}(\omega))^{1-\gamma} - 1].$$

We then have the following lemma.

Lemma 7 *The date-0 price of a type-*ϵ *claim in a NSE is given by*

$$\widehat{q}_\epsilon = (C_0^i)^\rho \left\{\widehat{\Gamma}^i(\mathbf{x}^i) - \epsilon\pi\widehat{\Delta}^i(\mathbf{x}^i)\right\},\ \textit{all } i,$$

where

$$\widehat{\Gamma}^i(\mathbf{x}^i) \equiv \left(\mathbb{E}\left[\widehat{\sigma}^i\left(\mathbf{d}\cdot\mathbf{x}^i\right)^{1-\gamma}\right]\right)^{\frac{\gamma-\rho}{1-\gamma}} \mathbb{E}\left[\widehat{\sigma}^i\left(\mathbf{d}\cdot\mathbf{x}^i\right)^{-\gamma}\right],\ \textit{and}$$

$$\widehat{\Delta}^i(\mathbf{x}^i) \equiv \left(\mathbb{E}\left[\widehat{\sigma}^i\left(\mathbf{d}\cdot\mathbf{x}^i\right)^{1-\gamma}\right]\right)^{\frac{\gamma-\rho}{1-\gamma}} \left[\widehat{\sigma}^i(l)\left(\mathbf{d}^l\cdot\mathbf{x}^i\right)^{-\gamma} - \widehat{\sigma}^i(h)\left(\mathbf{d}^h\cdot\mathbf{x}^i\right)^{-\gamma}\right].$$

We thus obtain that, although the expression is more complicated than with the CRRA utility function, the asset pricing function is still linear in the risk parameter ϵ. Note in particular that if we set $\rho = \gamma$, then $\widehat{\sigma}^i(\omega) = \widetilde{\sigma}^i(\omega)$ and the first term in $\widehat{\Gamma}^i(\mathbf{x}^i)$ and in $\widehat{\Delta}^i(\mathbf{x}^i)$ are equal 1, so that $\widehat{q}_\epsilon = (C_0^i)^\gamma q_\epsilon$. We do not recover exactly the previous expression for asset prices because in this section the marginal utility out of date-0 consumption is not fixed and depends on γ, whereas it is constant and equal to 1 in the benchmark case.

Since all types of agents must have the same date-0 marginal valuation of all types of claims in a NSE, it is still possible to provide conditions for sorting, as the following proposition establishes.

Proposition 5 *Suppose that all is as in the original setup except that consumers are endowed with $H > 0$ units of goods instead of time at date 0 and that their utility function is recursive and given by (39) and (40). Then a NSE displays positive (no) [negative] sorting if and only if*

$$(\widehat{Z}(l))^{(1-\gamma)} > (=)[<](\widehat{Z}(h))^{(1-\gamma)}. \tag{42}$$

We thus have that no sorting ever takes place if $\gamma = 1$. If, however, the substitution effect is strong, i.e., $\gamma \in (0, 1)$, then positive sorting is obtained if and only if $\widehat{Z}(l) > \widehat{Z}(h)$. In this case liquidity needs raise the marginal value of consumption more in state l than in state h, which combined with the high willingness to substitute consumption across states means consumers prefer low-ϵ claims even more than without liquidity needs. And since this preference is relatively stronger for high-liquidity need agents, they hold relatively more safe assets. When the substitution effect is weak instead, i.e., $\gamma > 1$, then positive sorting is obtained if and only if $\widehat{Z}(h) > \widehat{Z}(l)$. In this case liquidity needs raise the marginal value of consumption more in state h than in state l when $\widehat{Z}(h) > \widehat{Z}(l)$, which combined with the low willingness to substitute consumption across states also means consumers prefer low-ϵ claims even more than without liquidity needs. And once again this preference is relatively stronger for high-liquidity need agents, so they hold relatively more safe assets.

Considering lemma 6 and proposition 5 together we obtain that if $\theta^h = \theta^l$, then positive sorting is observed in equilibrium if and only if $A(l) > A(h)$ whenever $\gamma \in (0, 1)$ and if and only if $A(l) < A(h)$ if instead $\gamma > 1$. If instead I focus on the role of the intensity of liquidity needs and assume that $A(l) = A(h)$, (42) implies that positive sorting is obtained in equilibrium if and only if $\theta(l) > \theta(h)$ whenever $(1-\gamma)/(1-\rho) > 0$ and if and only if $\theta(l) < \theta(h)$ whenever $(1-\gamma)/(1-\rho) < 0$.[25] That is, if $A(l) = A(h)$ then positive sorting is observed in equilibrium if and only if

$$\theta(l) \begin{matrix} > \\ < \end{matrix} \theta(h) \text{ whenever } \begin{matrix} \text{either } (\gamma,\rho) \in (0,1)^2 \text{ or } (\gamma,\rho) \in (1,+\infty)^2 \\ \text{either } (\gamma,\rho) \in (0,1) \times (1,+\infty) \text{ or } (\gamma,\rho) \in (1,+\infty) \times (0,1). \end{matrix}$$

In conclusion, we have from lemma 6 and proposition 5 that both the CCRRA and the IES matter for sorting in equilibrium, but that their roles are different. Lemma 6 established that the IES $1/\rho$ determines how the variations in the intensity of liquidity needs and the terms of trade affect the variation across states in $\widehat{Z}$ and of the the value of liquidity, while a corollary to proposition 5 is that the impact on sorting of the variation across state of the value of liquidity, through its impact on $\widehat{Z}$, is driven by the CCRRA γ.

[25] Since $\gamma = \rho$ with CRRA utility functions, $(1-\gamma)/(1-\rho) = 1 > 0$ always, and therefore the results obtained here are consistent with those obtained earlier.

8 Concluding Remarks

In this paper I have presented a theory of endogenous liquidity differences across assets based on the interaction between differences in the risk of assets and differences in liquidity needs of investors. I have shown that the equilibrium displays a class structure, and characterized the type of sorting observed in equilibrium, as well as the implications for asset prices and velocity, a measure of liquidity. The model's predictions on sorting, and therefore on the relationship between risk, price, and velocity, depend crucially on the variability of the terms of trade and the intensity of liquidity needs, which can vary over time. Hence, establishing whether the data support the predictions of the model would require a detailed empirical analysis, which is beyond the scope of the paper. The existing literature is also of limited help in providing support for the theory in this paper, for most of the literature interested in the role of liquidity in the pricing of assets uses measures of liquidity other than velocity. However, there are exceptions, like Datar *et al.* (1998), Hu (1997), and Spiegel and Wang (2005), and their results are consistent with the findings of this paper in the case of positive sorting. The two former papers find a negative correlation between velocity and rates of return, while Spiegel and Wang (2005) find that idiosyncratic risk and liquidity are highly correlated for stocks,[26] and that after controlling for idiosyncratic risk the explanatory power of the liquidity measures they consider for stock returns, which include trading volume, mostly disappears.

In the interest of space, a number of extensions and an application were left out of the main body of this article and can be found in part 2 of the online appendix. Let me highlight here that I study the impact of a fall in the stock of trees and of an increase in risk, either in the form of an increase in the risk of the market portfolio or of an increase in the volatility of market conditions. I show that the model is also able to match salient features of financial crises, like the 2007-8 subprime financial crisis or the European sovereign debt crisis that followed, when one introduces an increase in risk, but not when the stock of trees falls. The model in particular predicts that lower-risk claims are doubly safe, because their price increases when aggregate risk increases, while the price and velocity of riskier assets falls, potentially sharply. The framework can therefore be useful to think about what constitutes a safe asset and complements the recent work of Geromichalos *et al.* (2018) and Loberto (2019). And the paper therefore also contributes to the literature that investigates how shocks can impact asset prices through changes in liquidity (see, for instance, Li *et al.*, 2012, Guerrieri and Shimer, 2014, Chang, 2018, Shi, 2005, Cui and Radde, forthcoming, Del Negro *et al.*, 2017, and Vayanos, 2004).[27]

Finally, the theory also appears to provide a useful framework to think about policy issues, like the role of active liquidity management in times of crises. In particular, the response of the asset pricing function to

[26] In the model a change in the level of portfolio risk triggers a rotation of the asset pricing function that, if we were to assume a stable asset pricing function, would lead to interpret the data generated by the model as indicating that low (high) ϵ claims are also assets with high (low) idiosyncratic risk.

[27] The mechanisms in the first three papers are based on asymmetry of information, which is absent here, and these papers and the present work are therefore complementary. Shi (2015) studies the role of the saleability of assets and model liquidity shocks as a reduction in the stock of assets, which, like in this paper, leads, *ceteris paribus*, to an increase in asset prices. Cui and Radde (forthcoming) and Del Negro et al. (2017) each identify a fix to this anomaly: the introduction of nominal rigidities and a zero-lower bound on nominal interest rates in Del Negro et al. and, like in this paper, a concomitant reduction in the ease of trade in Cui and Radde. Finally, Vayanos (2004) also considers the interaction between changes in risk and changes in liquidity. In his model changes in volatility triggers changes in liquidity premia because fund managers, who are subject to the threat of withdrawal of funds if they do not perform, worry more about having to bear transaction costs in times of high volatility. However, differences in the liquidity of assets in his model originate from their exogenous differences in transaction costs.

a portfolio risk shock or to changes in market conditions provides some rationale for outright asset purchase programs like TARP, Quantitative Easing, or the European Stability Mechanism (ESM): an institution which is not facing short-term funding constrains can take advantage of a dynamic arbitrage possibility by purchasing risky (or selling safe) assets in times of high aggregate risk, and reselling these high risk (or purchasing back these low risk) assets in times of low aggregate risk. And by doing so the institution reduces the level of aggregate risk among assets in the market in times of high aggregate risk. A possible application of the framework would be to investigate further the role that outright asset purchases by governments can play in mitigating the effects of shocks, especially when also incorporating other considerations, like asymmetry of information.

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Appendix

Proof of Lemma 1: The first-order conditions to the problem of a producer are

$$-\left(A(\omega)\right)^{-1} + \lambda(\omega) + \mu_h(\omega) = 0, \text{ and}$$
$$d_\epsilon(\omega) - \lambda(\omega)Q_\epsilon(\omega) + \widetilde{\mu}_\epsilon(\omega) = 0,$$

where λ^ω, $\mu_h(\omega)$, and $\widetilde{\mu}_\epsilon(\omega)$ are respectively the Lagrange multipliers on the budget constraint, non-negativity constraint for h in state ω, and the no short-selling constraint for claim ϵ in state ω. Since the marginal utility of consumption of consumers in this period is infinity at zero consumption, they bring and sell claims and consume, implying that production takes place. It must therefore be that $\mu_h(\omega) = 0$ for $\omega = 1, 2$, implying that $\lambda(\omega) = (A(\omega))^{-1}$. It thus follows that for all ϵ

$$Q_\epsilon(\omega) = A(\omega)[d_\epsilon(\omega) + \widetilde{\mu}_\epsilon(\omega)]. \tag{A1}$$

Since $\lim_{c\to 0} u'(c) = +\infty$, we can ignore the non-negativity constraint for consumption in a consumer's problem and obtain that the optimal choice of transfer in state ω for a consumer with portfolio $\mathbf{x}$ who wants to consume, $y(\mathbf{x};\omega)$, is such that for all ϵ

$$Q_\epsilon(\omega)\theta(\omega)u'\left(\mathbf{Q}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega)\right) = d_\epsilon(\omega)U'(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{y}(\mathbf{x};\omega))) + \mu_\epsilon(\mathbf{x};\omega), \tag{A2}$$

where $\mu_\epsilon(\omega)$ is the Lagrange multiplier on the constraint $y_\epsilon(\omega) \le x_\epsilon$. From (A1) we have that for all ϵ

$$A(\omega)[d_\epsilon(\omega) + \widetilde{\mu}_\epsilon(\omega)] = d_\epsilon(\omega)\frac{U'(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{y}(\mathbf{x};\omega)))}{\theta(\omega)u'\left(\mathbf{Q}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega)\right)} + \frac{\mu_\epsilon(\mathbf{x};\omega)}{\theta(\omega)u'\left(\mathbf{Q}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega)\right)}.$$

However, if $\widetilde{\mu}_\epsilon(\omega) > 0$ for some ϵ, then there are consumers purchasing these claims, implying $\mu_\epsilon(\mathbf{x};\omega) = 0$ for these consumers. The previous equation for these consumers thus imply that

$$A(\omega) < \frac{U'(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{y}(\mathbf{x};\omega)))}{\theta(\omega)u'\left(\mathbf{Q}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega)\right)},$$

and therefore that $\widetilde{\mu}_\epsilon(\omega) > 0$ for all ϵ. That is, producers do not receive claims in the middle period, which cannot be true since it would imply they do not produce and thus that early consumers do not consume in the middle period. It must therefore be that $\widetilde{\mu}_\epsilon(\omega) = 0$ for all ϵ and

$$Q_\epsilon(\omega) = A(\omega)d_\epsilon(\omega). \tag{A3}$$

The first-order condition for the type-ϵ claim choice for a late consumer is therefore

$$d_\epsilon(\omega)U'\left(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{Y})\right) - \lambda_\epsilon(\omega) = \Lambda(\omega)A(\omega)d_\epsilon(\omega), \text{ or}$$
$$U'\left(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{Y})\right) - \widetilde{\lambda}_\epsilon(\omega) = \Lambda(\omega)A(\omega),$$

for $\widetilde{\lambda}_\epsilon(\omega) \equiv \lambda_\epsilon(\omega)/d_\epsilon$. It must therefore be that $\widetilde{\lambda}_\epsilon(\omega) = \widetilde{\lambda}(\omega) \ge 0$ for all ϵ. And since late consumers do not enjoy consumption in this middle period, they cannot optimally choose to sell all of their claims then. It must thus be that $\widetilde{\lambda}(\omega) = 0$, implying that late consumers are not constrained in the composition of the portfolios they trade. This means they are indifferent between trading one type of claims for another. In other words, late consumers are indifferent between trading and not trading in this middle period.

(A2) and (A3) together imply that for an early consumer

$$A(\omega)\theta(\omega)u'\left(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega)\right) = U'(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{y}(\mathbf{x};\omega))) + \mu_\epsilon(\omega)/d_\epsilon(\omega).$$

Hence, if $\mu_\epsilon(\omega) > 0$ for some ϵ, then $\mu_\epsilon(\omega) > 0$ for all ϵ, while if $\mu_\epsilon(\omega) = 0$ for some ϵ, then $\mu_\epsilon(\omega) = 0$ for all ϵ. That is, an agent is short of some types of claims in the middle period if and only if it is short of all other types of claims, which also means all claims are equally good media of exchanges.

Since

$$V_\epsilon^i(\mathbf{x};\omega) = d_\epsilon(\omega) \times A(\omega)\theta(\omega)u'(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega))\frac{\partial y_\epsilon(\omega)}{\partial x_\epsilon} + U'(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{y}(\mathbf{x};\omega))) \times \left(1 - \frac{\partial y_\epsilon(\omega)}{\partial x_\epsilon}\right),$$

it follows that if $\mu_\epsilon(\omega) = 0$, then $A(\omega)\theta(\omega)u'\left(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega)\right) = U'(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{y}(\mathbf{x};\omega))$, in which case

$$V_\epsilon^i(\mathbf{x};\omega) = d_\epsilon(\omega)\times A(\omega)\theta(\omega)u'(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega))\frac{\partial y_\epsilon(\omega)}{\partial x_\epsilon} + A(\omega)\theta(\omega)u'\left(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega)\right)\times\left(1 - \frac{\partial y_\epsilon(\omega)}{\partial x_\epsilon}\right),$$

so that $V_\epsilon^i(\mathbf{x};\omega) = d_\epsilon(\omega) \times A(\omega)\theta(\omega)u'(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega))$. If instead $\mu_\epsilon(\omega) > 0$, then $\partial y_\epsilon(\omega)/\partial x_\epsilon = 1$, in which case $V_\epsilon^i(\mathbf{x};\omega) = d_\epsilon(\omega) \times A(\omega)\theta(\omega)u'(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega))$.■

Proof of Proposition 1:

(i) and (ii): The linearity in risk of agents' valuation of claims implies that for any two types i and j of agents, either (a) one has a higher valuation of all claims; or (b) they have the same valuation for all claims; or (c) their marginal valuation functions cross once. Since it is assumed that the marginal utility of consumption in the middle period is infinity at zero, case (a) cannot be true in equilibrium.

I will now prove the existence of a class structure by construction. If all types of agents have the same valuation of all claims - case (a), then there exists a unique class of agents and of claims, so that $\mathcal{I}_1 = \{1, ..., I\}$, $\Xi_1 = [\epsilon_{\min}, \epsilon_{\max}]$ and $K = 1$.

If not all types of agents have the same valuation of all claims - case (c), then pick the type(s) of agents who have the highest (common) valuation of claims with risk level $\epsilon_0^* = \epsilon_{\min}$, and denote by $\mathcal{I}_1 \subset \{1, ..., I\}$ the set of all types of agents with that same valuation for these claims. The linearity of the marginal value of claims implies that if for any $(i, j) \in \mathcal{I}_1 \times \mathcal{I}_1$, $v_{\epsilon_{\min}}^i = v_{\epsilon_{\min}}^j$ and $v_\epsilon^i > v_\epsilon^j$ for some $\epsilon > \epsilon_{\min}$, then $v_\epsilon^i > v_\epsilon^j$ for all $\epsilon > \epsilon_{\min}$. This would imply that consumers of type j do not hold any claims other than type $\epsilon_{\min}$ claims, which given the absence of mass points in the distribution of claims means their marginal utility of consumption at dates 1 and 2 is arbitrarily large. That clearly cannot be true in equilibrium. It must therefore be that $v_\epsilon^i = v_\epsilon^j$ for all ϵ and any $(i, j) \in \mathcal{I}_1 \times \mathcal{I}_1$.

Consider next any two types of consumers i and j such that $v_{\epsilon_{\min}}^i > v_{\epsilon_{\min}}^j$. We know that type-$j$ consumers must consume at dates 1 and 2 and therefore we know there exists ϵ^{ij} the smallest ϵ such that consumers of type j value claims with $\epsilon \geq \epsilon^{ij}$ at least as much as consumers of type i . That is, ϵ^{ij} is such that $v_\epsilon^i > (=)[<]v_\epsilon^j$ for all $\epsilon < (=)[>]\epsilon^{ij}$ for any two types of consumers i and j such that $v_{\epsilon_{\min}}^i > v_{\epsilon_{\min}}^j$. Then let $\epsilon_1^* \equiv \min_j \epsilon^{ij}$ for $i \in \mathcal{I}_1$ and $j \notin \mathcal{I}_1$. That is, ϵ_1^* is the smallest value of ϵ for which some consumers whose type is not in $\mathcal{I}_1$ value these claims at least as much as consumers whose type is in $\mathcal{I}_1$. We thus have that agents of type in $\mathcal{I}_1$ hold onto claims with $\epsilon \in \Xi_1 = [\epsilon_0^*, \epsilon_1^*]$, and no other types hold onto claims with $\epsilon \in [\epsilon_0^*, \epsilon_1^*)$.

Let $\mathcal{I}_2 \equiv \{j \in \{1, ..., I\} \setminus \{\mathcal{I}_1\} : \epsilon^{ij} = \epsilon_1^* \text{ for } i \in \mathcal{I}_1\}$. From our analysis above it is clear that $v_\epsilon^i = v_\epsilon^j$ for all ϵ for all $(i, j) \in \mathcal{I}_2 \times \mathcal{I}_2$. If $\mathcal{I}_1 \cup \mathcal{I}_2 = \{1, ..., I\}$, then the class structure contains two classes of agents and claims with agents from the second class holding onto all claims with risk $\epsilon \in \Xi_2 = [\epsilon_1^*, \epsilon_{\max}]$. If instead $\mathcal{I}_1 \cup \mathcal{I}_2 \subset \{1, ..., I\}$, then we can reproduce the reasoning applied for the construction of the first class: there exists $\epsilon_2^* \equiv \min_j \epsilon^{ij}$ for $i \in \mathcal{I}_2$ and $j \notin (\mathcal{I}_1 \cup \mathcal{I}_2)$ such that $v_\epsilon^i > (=)[<] v_\epsilon^j$ for all $\epsilon < (=)[>] \epsilon_2^*$ for $i \in \mathcal{I}_2$ and all $j \notin (\mathcal{I}_1 \cup \mathcal{I}_2)$. In this case we have that agents of types in $\mathcal{I}_2$ hold onto claims with $\epsilon \in \Xi_2 = [\epsilon_1^*, \epsilon_2^*]$, and no other types of agents hold onto claims with $\epsilon \in (\epsilon_1^*, \epsilon_2^*)$.

And so on and so forth until class K such that $\mathcal{I}_1 \cup \mathcal{I}_2 \cup ... \cup \mathcal{I}_K = \{1, ..., I\}$ and $\Xi_K = [\epsilon_{K-1}^*, \epsilon_{\max}]$ is reached.

(iii) Consider any two types $i \in \mathcal{I}_k$ and $j \in \mathcal{I}_m$ with $k < m$. We then have that $v_0^i - \epsilon\pi\Delta^i > (=)[<] v_0^j - \epsilon\pi\Delta^j$ for all $\epsilon < (=)[>] \epsilon^{ij}$. If $\epsilon^{ij} \geq 0$, then $v_0^i - v_0^j = \epsilon^{ij}\pi(\Delta^i - \Delta^j) \geq 0$ implies that $\Delta^i \geq \Delta^j$. And because i and j belong to different classes, the inequality must be strict. Now assume that $\epsilon^{ij} < 0$ instead. Then $v_0^i - v_0^j = \epsilon^{ij}\pi(\Delta^i - \Delta^j) < 0$ also implies that $\Delta^i > \Delta^j$. That is, $\Delta^i > \Delta^j$ for any two types $i \in \mathcal{I}_k$ and $j \in \mathcal{I}_m$ with $k < m$.■

Proof of Lemma 3: Whenever there exists $\left(X^1, \mathcal{E}^1\right) \in \left\{(X, \mathcal{E}) : X \in \left[0, E/\mu^1\right] \text{ and} \mathcal{E} \in \left[\mathcal{E}_{\min}^1(X), \mathcal{E}_{\max}^1(X)\right]\right\}$ one can, by definition of the set of a feasible pair $\left(X^1, \mathcal{E}^1\right)$, build a pair $(\mathbf{x}^1, \mathbf{x}^2)$ of feasible portfolios such that (25) holds. Also, by definition of the set of feasible pairs $\left(X^1, \mathcal{E}^1\right)$ and the definition of $\left(X^2, \mathcal{E}^2\right)$ as functions of $\left(X^1, \mathcal{E}^1\right)$, the markets for all types of claims clear for any pair $(\mathbf{x}^1, \mathbf{x}^2)$ of feasible portfolios such that (25). Finally, since (25) holds, it follows that $v_\epsilon^i(\mathbf{x}^i) = \widetilde{v}_\epsilon^i(X^i, \mathcal{E}^i)$ for all ϵ and for both i.

We thus have that:

- Case 1 - If $\widetilde{v}_\epsilon^1(X^1, \mathcal{E}^1) = \widetilde{v}_\epsilon^2\left(X^2, \mathcal{E}^2\right)$ for all ϵ, then a pair $(\mathbf{x}^1, \mathbf{x}^2)$ of feasible portfolios such that (25) holds and the asset pricing function $\mathbf{q}$ such that $q_\epsilon = \widetilde{v}_\epsilon^1(X^1, \mathcal{E}^1) = \widetilde{v}_\epsilon^2(X^2, \mathcal{E}^2)$ for all ϵ satisfy (16) and deliver that the date-0 markets for all types of claims clear;

- Case 2: if instead $\widetilde{v}_\epsilon^1(X^1, \mathcal{E}^1) > (=)[<] \widetilde{v}_\epsilon^2\left(X^2, \mathcal{E}^2\right)$ for $\epsilon < (=)[>] \epsilon^*$ for $\epsilon^* \in (\mathcal{E}_{\min}^1\left(X^1\right), \mathcal{E}_{\max}^1\left(X^1\right))$ such that $X^1 = \int_{\epsilon_{\min}}^{\epsilon^*} 2e_\epsilon dG(\epsilon)$, then a pair $(\mathbf{x}^1, \mathbf{x}^2)$ of feasible portfolios such that (25) holds and the asset pricing function $\mathbf{q}$ such that $q_\epsilon = \widetilde{v}_\epsilon^1(X^1, \mathcal{E}^1)$ for $\epsilon \leq \epsilon^*$, with strict inequality for $\epsilon < \epsilon^*$, and $q_\epsilon = \widetilde{v}_\epsilon^2(X^2, \mathcal{E}^2)$ for $\epsilon > \epsilon^*$ satisfy (16) and deliver that the date-0 markets for all types of claims clear: type-1 consumers hold onto all claims $\epsilon < \epsilon^*$ and type-2 consumers hold onto all claims $\epsilon > \epsilon^*$ and the split is indeterminate for ϵ^*;

- Case 3 - finally, if $\widetilde{v}_\epsilon^1(X^1, \mathcal{E}^1) < (=)[>] \widetilde{v}_\epsilon^2\left(X^2, \mathcal{E}^2\right)$ for $\epsilon < (=)[>] \epsilon^*$ for $\epsilon^* \in (\mathcal{E}_{\min}^1\left(X^1\right), \mathcal{E}_{\max}^1\left(X^1\right))$ such that $X^1 = \int_{\epsilon^*}^{\epsilon_{\max}} 2e_\epsilon dG(\epsilon)$, then a pair $(\mathbf{x}^1, \mathbf{x}^2)$ of feasible portfolios such that (25) holds and the asset pricing function $\mathbf{q}$ such that $q_\epsilon = \widetilde{v}_\epsilon^1(X^1, \mathcal{E}^1)$ for $\epsilon \geq \epsilon^*$, with strict inequality for $\epsilon > \epsilon^*$, and $q_\epsilon = \widetilde{v}_\epsilon^2(X^2, \mathcal{E}^2)$ for $\epsilon < \epsilon^*$ satisfy (16) and deliver that the date-0 markets for all types of claims clear: type-1 consumers hold onto all claims $\epsilon > \epsilon^*$ and type-2 consumers hold onto all claims $\epsilon < \epsilon^*$ and the split is indeterminate for ϵ^*.

We have seen earlier that an equilibrium is fully characterized by the vector of date-0 portfolios $(\mathbf{x}^i)_{i=0}^I$ and the vector $\mathbf{q}$ of prices for claims such that (i) (16) holds for all ϵ and all i, and (ii) the markets for all claims clear at date 0. The proposed feasible pairs of portfolios and asset pricing function are therefore equilibrium portfolios and asset pricing functions.■

Proof of Lemma 4: Assuming that Assumption 1 holds, we can ignore corner solutions. The first-order conditions characterizing an agent's trading choice in the middle period are therefore

$$d_\epsilon(\omega) \times A(\omega)\theta(\omega)u'[A(\omega)\alpha(\omega)(1+\mathcal{E}^i(\omega))X^i] = d_\epsilon(\omega) \times U'[(1-\alpha(\omega))(1+\mathcal{E}^i(\omega))X^i], \text{ for all } \epsilon.$$

We thus have that $A(\omega)\theta(\omega)[A(\omega)\alpha(\omega)(1+\mathcal{E}^i(\omega))X^i]^{-\gamma} = [(1-\alpha(\omega))(1+\mathcal{E}^i(\omega))X^i]^{-\gamma}$, and the result follows.■

Proof of Proposition 4: (i) The proof of this first part of the proposition is in three parts. I first show in part A that if a pair $(X^1, \mathcal{E}^1)$ solving (28) exists, then it is unique and satisfies the characteristics in (i). I then show in part B that if a pair $(X^1, \mathcal{E}^1)$ solving (29) or (30) exists, then it also unique and satisfies the characteristics in (i). Lastly, I show in part C that there exists a unique pair $(X^1, \mathcal{E}^1)$ solving one of (28)-(30).

Part A

• *Uniqueness*: The two expressions in (28) are equivalent to

$$\frac{1+\mathcal{E}^1}{1-\frac{\pi}{1-\pi}\mathcal{E}^1} = \frac{\Sigma(h)}{\Sigma(l)} \times \frac{1-\mu^1+\mu^1\Sigma(l)}{1-\mu^1+\mu^1\Sigma(h)} \times \frac{1+\mathcal{E}}{1-\frac{\pi}{1-\pi}\mathcal{E}}, \text{ and} \tag{A4}$$

$$X^1 = \frac{\Sigma(\omega)}{1-\mu^1+\mu^1\Sigma(\omega)} \times \frac{1+\mathcal{E}(\omega)}{1+\mathcal{E}^1(\omega)}K \text{ for some } \omega, \tag{A5}$$

where $\mathcal{E}^k(h) \equiv \mathcal{E}^k$, $\mathcal{E}^k(l) \equiv -(\pi/(1-\pi))\mathcal{E}^k$, $k \in \{\varnothing, 1, 2\}$, and $\Sigma(\omega) \equiv \widetilde{\sigma}^1(\omega)/\widetilde{\sigma}^2(\omega) > 1$ for all ω. There clearly exists a unique $\mathcal{E}^1$ solving (A4), and given this unique $\mathcal{E}^1$ there exists a unique X^1 given by (A5) for a chosen ω. This proves that if a pair $(X^1, \mathcal{E}^1)$ solving (28) exists, then it is unique.

• *Sorting*: It is straightforward to show that $\Sigma(l) \geq \Sigma(h)$ if and only if $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) \geq 1$, with strict inequality if $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) > 1$. We then have directly from (A4) that $\mathcal{E}^1 \leq \mathcal{E}^2$ if and only if $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) \geq 1$, with strict inequality if $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) > 1$. However, $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) \geq 1$ if and only if $(A(l))^{1-\gamma}\theta(l) \geq (A(h))^{1-\gamma}\theta(h)$, with strict inequality if $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$.

• *Size*: To prove that $X^1 > X^2$ it is useful to use the fact that the following two equations are equivalent to the two (28):

$$\frac{1+\mathcal{E}^1(l)}{1+\mathcal{E}^1(h)} \times \frac{1+\mathcal{E}^2(h)}{1+\mathcal{E}^2(l)} = \left(\frac{\widetilde{\sigma}^1(l)}{\widetilde{\sigma}^1(h)} \times \frac{\widetilde{\sigma}^2(h)}{\widetilde{\sigma}^2(l)}\right)^{1/\gamma}, \text{ and} \tag{A6}$$

$$\frac{X^1}{X^2} = \left(\frac{\widetilde{\sigma}^1(\omega)}{\widetilde{\sigma}^2(\omega)}\right)^{1/\gamma} \times \frac{1+\mathcal{E}^2(\omega)}{1+\mathcal{E}^1(\omega)} \text{ for one } \omega. \tag{A7}$$

We have from (A7) for state h that $X^1 > X^2$ whenever $\mathcal{E}^1 \leq \mathcal{E}^2$, and from (A7) for state l we obtain that $X^1 > X^2$ whenever $\mathcal{E}^1 \geq \mathcal{E}^2$. Hence, if a pair $(X^1, \mathcal{E}^1)$ solving (28) exists, then it is such that $X^1 > X^2$.

Part B - I first prove that if a pair $(X^1, \mathcal{E}^1)$ solving (29) or (30) exists, then it is such that (i) $\mathcal{E}^1 < [>]\mathcal{E}^2$ if and only if $(A(l))^{1-\gamma}\theta(l) > [<](A(h))^{1-\gamma}\theta(h)$, and (ii) $X^1 > X^2$. I then prove that there is a unique pair solving either (29) or (30).

• *Sorting and Size*: (a) Consider sorting first. Assume first that a pair $(X^1, \mathcal{E}^1)$ solving (29) exists, so it is such that $\mathcal{E}^1 < \mathcal{E}^2$. Since then type-1 agents hold onto the safer claims and type-2 agents hold onto the riskier claims, it must be that type-1 (2) consumers have a higher marginal value of consumption in state l (h) - for otherwise either type-1 or type-2 agents would be holding all claims, which cannot be true in equilibrium. That is, $\widetilde{\sigma}^1(l)[(1+\mathcal{E}^1(l))X^1]^{-\gamma} > \widetilde{\sigma}^2(l)[(1+\mathcal{E}^2(l))X^2]^{-\gamma}$ and $\widetilde{\sigma}^1(h)[(1+\mathcal{E}^1(h))X^1]^{-\gamma} < \widetilde{\sigma}^2(h)[(1+\mathcal{E}^2(h))X^2]^{-\gamma}$. This then implies that

$$\frac{1+\mathcal{E}^1(l)}{1+\mathcal{E}^1(h)} \times \frac{1+\mathcal{E}^2(h)}{1+\mathcal{E}^2(l)} < \left(\frac{\widetilde{\sigma}^1(l)}{\widetilde{\sigma}^1(h)} \times \frac{\widetilde{\sigma}^2(h)}{\widetilde{\sigma}^2(l)}\right)^{1/\gamma}. \tag{43}$$

Since $\mathcal{E}^1 < \mathcal{E}^2$, we have that the left-hand side is strictly greater than 1, thereby implying that the right-hand side is also strictly greater than 1, which in turn implies that $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$. If we instead assume that a pair $(X^1, \mathcal{E}^1)$ solving (30) exists, so it is such that $\mathcal{E}^1 > \mathcal{E}^2$. Since type-1 agents hold onto the riskier claims and type-2 agents hold onto the safer claims, it must be that type-1 (2) consumers have a higher marginal value of consumption in state h (l) - for otherwise either type-1 or type-2 agents would be holding all claims, which cannot be true in equilibrium. That is, it must be that $\widetilde{\sigma}^1(l)[(1+\mathcal{E}^1(l))X^1]^{-\gamma} < \widetilde{\sigma}^2(l)[(1+\mathcal{E}^2(l))X^2]^{-\gamma}$ and $\widetilde{\sigma}^1(h)[(1+\mathcal{E}^1(h))X^1]^{-\gamma} > \widetilde{\sigma}^2(h)(1+\mathcal{E}^2(h))^{-\gamma}$. This then implies that

$$\frac{1+\mathcal{E}^1(l)}{1+\mathcal{E}^1(h)} \times \frac{1+\mathcal{E}^2(h)}{1+\mathcal{E}^2(l)} > \left(\frac{\widetilde{\sigma}^1(l)}{\widetilde{\sigma}^1(h)} \times \frac{\widetilde{\sigma}^2(h)}{\widetilde{\sigma}^2(l)}\right)^{1/\gamma}. \tag{44}$$

Since $\mathcal{E}^1 > \mathcal{E}^2$ we have that the left-hand side is strictly less than 1, thereby implying that the right-hand side is also strictly less than 1, which in turn implies that $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$.

Now assume that $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$. This implies that $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) > 1$. But if negative sorting is observed, then (44) holds and it must therefore also be that $\mathcal{E}^1 < \mathcal{E}^2$. This contradicts the fact that sorting is negative. Symmetrically, if one assumes that $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$, then $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) \leq 1$. But if positive sorting is observed, then (43) holds and it must therefore be that $\mathcal{E}^1 > \mathcal{E}^2$. This contradicts the fact that sorting is positive. That is, if $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$, then sorting must be positive, while if $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$, then sorting must be negative.

Finally, if $(A(l))^{1-\gamma}\theta(l) = (A(h))^{1-\gamma}\theta(h)$, then $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) = 1$, and none of the conditions for segmentation can be satisfied.

(b) I now establish the properties for the sizes of portfolios. Since $\widetilde{\sigma}^1(h)[(1+\mathcal{E}^1(h))X^1]^{-\gamma} < \widetilde{\sigma}^2(h)[(1+\mathcal{E}^2(h))X^2]^{-\gamma}$ if a pair $(X^1, \mathcal{E}^1)$ solves (29) while $\widetilde{\sigma}^1(l)[(1+\mathcal{E}^1(l))X^1]^{-\gamma} < \widetilde{\sigma}^2(l)[(1+\mathcal{E}^2(l))X^2]^{-\gamma}$ if a pair $(X^1, \mathcal{E}^1)$ solves (30), we have that in the former case $[\widetilde{\sigma}^1(h)/\widetilde{\sigma}^2(h)]^{1/\gamma} < \left[(1+\mathcal{E}^1(h))X^1\right] / \left[(1+\mathcal{E}^2(h))X^2\right]$, while in the latter $[\widetilde{\sigma}^1(l)/\widetilde{\sigma}^2(l)]^{1/\gamma} < [(1+\mathcal{E}^1(l))X^1/(1+\mathcal{E}^2(l))X^2]$. Since $\widetilde{\sigma}^1(\omega)/\widetilde{\sigma}^2(\omega) > 1$ for both ω and in the former case $\mathcal{E}^1 < \mathcal{E}^2$ while in the latter $\mathcal{E}^1 > \mathcal{E}^2$, it must be that $X^1 > X^2$ in both cases.

• *Uniqueness*: I now prove that there is a unique pair $(X^1, \mathcal{E}^1)$ solving either (29) or (30) (we know that the set of parameters for which either case is feasible do not intersect, so only one of the two cases can exists for a given set of parameters). I prove the result by a series of contradictions. Assume first that there exists a pair $(X^1, \mathcal{E}^1)$ solving (29), in which case it must be that there exists $\epsilon^*(X^1, \mathcal{E}^1)$ such that

$$\widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1) \begin{cases} > \\ = \\ < \end{cases} \widetilde{v}^2_\epsilon\left(\frac{K-\mu^1 X^1}{1-\mu^1}, \frac{\mathcal{E}K-\mu^1\mathcal{E}^1X^1}{K-\mu^1X^1}\right) \text{ for } \epsilon \begin{cases} < \\ = \\ > \end{cases} \epsilon^*.$$

If another pair $(\widehat{X}^1, \widehat{\mathcal{E}}^1)$ solving (29) exists and is such that the cutoff is $\epsilon^*(\widehat{X}^1, \widehat{\mathcal{E}}^1) < \epsilon^*(X^1, \mathcal{E}^1)$, then type-1 (2) agents must consume less (more) in both states so that $\widetilde{v}^1_\epsilon(\widehat{X}^1, \widehat{\mathcal{E}}^1) > \widetilde{v}^1_\epsilon(X^1, \mathcal{E}^1)$ for all ϵ while

$\widetilde{v}^2_\epsilon(\widehat{X}^2,\widehat{\mathcal{E}}^2) < \widetilde{v}^2_\epsilon(X^2,\mathcal{E}^2)$ for all ϵ. But this implies that $\widetilde{v}^1_\epsilon(\widehat{X}^1,\widehat{\mathcal{E}}^1) = \widetilde{v}^2_\epsilon(\widehat{X}^2,\widehat{\mathcal{E}}^2)$ for $\epsilon > \epsilon^*(X^1,\mathcal{E}^1)$, a contradiction. And if we assume instead that another pair $(\widehat{X}^1,\widehat{\mathcal{E}}^1)$ solving (29) exists and is such that the cutoff is $\epsilon^*(\widehat{X}^1,\widehat{\mathcal{E}}^1) > \epsilon^*(X^1,\mathcal{E}^1)$, then type-1 (2) agents must consume more (less) in both states so that $\widetilde{v}^1_\epsilon(\widehat{X}^1,\widehat{\mathcal{E}}^1) < \widetilde{v}^1_\epsilon(X^1,\mathcal{E}^1)$ for all ϵ while $\widetilde{v}^2_\epsilon(\widehat{X}^2,\widehat{\mathcal{E}}^2) > \widetilde{v}^2_\epsilon(X^2,\mathcal{E}^2)$ for all ϵ. But this implies that $\widetilde{v}^1_\epsilon(\widehat{X}^1,\widehat{\mathcal{E}}^1) = \widetilde{v}^2_\epsilon(\widehat{X}^2,\widehat{\mathcal{E}}^2)$ for $\epsilon < \epsilon^*(X^1,\mathcal{E}^1)$, again a contradiction.

If we instead assume that there exists a pair $(X^1,\mathcal{E}^1)$ solving (30), the proof is symmetric (we can redo all the steps above, but switching the indices for consumers of type 1 and 2). This conclude the proof that if a SE exists, then it is unique.

Part C

I start this part with establising some results that will be useful in proving the existence and uniqueness of a pair $(X^1,\mathcal{E}^1)$. The system of two equations (A6) and (A7) is equivalent to

$$\Phi(X^1,\mathcal{E}^1) = 0, \text{ and} \tag{A8}$$

$$\Psi(X^1,\mathcal{E}^1;\omega) = 0 \text{ for some } \omega, \tag{A9}$$

where

$$\Phi(X^1,\mathcal{E}^1) \equiv \frac{1+\mathcal{E}^1(l)}{1+\mathcal{E}^1(h)} \times \frac{1+\frac{E\mathcal{E}-\mu^1X^1\mathcal{E}^1}{K-\mu^1X^1}}{1-\frac{\pi}{1-\pi}\frac{E\mathcal{E}-\mu^1X^1\mathcal{E}^1}{E-\mu^1X^1}} - \left(\frac{\widetilde{\sigma}^1(l)}{\widetilde{\sigma}^1(h)} \times \frac{\widetilde{\sigma}^2(h)}{\widetilde{\sigma}^2(l)}\right)^{1/\gamma},$$

and

$$\Psi(X^1,\mathcal{E}^1;\omega) \equiv \frac{(1-\mu^1)X^1}{E-\mu^1X^1}\frac{1+\mathcal{E}^1(\omega)}{1+\mathcal{E}^2(\omega)} - \left(\frac{\widetilde{\sigma}^1(\omega)}{\widetilde{\sigma}^2(\omega)}\right)^{1/\gamma}.$$

Note that there always exists a unique solution $(X^1,\mathcal{E}^1)$ to the system of two equations (A8)-(A9), which would be the unique pair if we were to ignore the constraint imposed by the distribution of risk for claims G. One can re-write the above expressions as

$$\Phi(X^1,\mathcal{E}^1) = \frac{1+\mathcal{E}^1(l)}{1+\mathcal{E}^1(h)} \times \frac{E(1+\mathcal{E}(h))-\mu^1X^1(1+\mathcal{E}^1(h))}{E(1+\mathcal{E}(l))-\mu^1X^1(1+\mathcal{E}^1(l))} - \left(\frac{\widetilde{\sigma}^1(l)}{\widetilde{\sigma}^1(h)} \times \frac{\widetilde{\sigma}^2(h)}{\widetilde{\sigma}^2(l)}\right)^{1/\gamma},$$

and

$$\Psi(X^1,\mathcal{E}^1;\omega) = \frac{(1-\mu^1)X^1(1+\mathcal{E}^1(\omega))}{E(1+\mathcal{E}(\omega))-\mu^1X^1(1+\mathcal{E}^1(\omega))} - \left(\frac{\widetilde{\sigma}^1(\omega)}{\widetilde{\sigma}^2(\omega)}\right)^{1/\gamma}.$$

If $\Phi(X^1,\mathcal{E}^1) = 0$ and $\Psi(X^1,\mathcal{E}^1;\omega) = 0$ for some ω, then $\Psi^{'}(X^1,\mathcal{E}^1;\omega') = 0$ for $\omega' \neq \omega$. Also, $\Phi(X^1,\mathcal{E}^1)$ is strictly decreasing in $\mathcal{E}^1$ and it is strictly increasing (constant) [decreasing] with X^1 for $\mathcal{E}^1 < \mathcal{E}$ ($\mathcal{E}^1 = \mathcal{E}$) [$\mathcal{E}^1 > \mathcal{E}$]. Furthermore, $\Psi(X^1,\mathcal{E}^1;\omega)$ is strictly increasing in X^1 for each ω, while $\Psi(X^1,\mathcal{E}^1;l)$ is strictly decreasing in $\mathcal{E}^1$ and $\Psi(X^1,\mathcal{E}^1;h)$ is strictly increasing in $\mathcal{E}^1$.

I prove uniqueness and existence at the same time by showing in part that (a) if a pair $(X^1,\mathcal{E}^1)$ solving (28) exists, then there does not exist another pair $(\widetilde{X}^1,\widetilde{\mathcal{E}}^1)$ solving either (29) or (30), while in part (b) I show that if there does not exist a pair $(X^1,\mathcal{E}^1)$ solving (28), then there exists a pair $(\widetilde{X}^1,\widetilde{\mathcal{E}}^1)$ solving either (29) or (30).

(a) Suppose a pair $(X^1,\mathcal{E}^1)$ solving (28) exists. Assume first that $(A(l))^{1-\gamma}\theta(l) \geq (A(h))^{1-\gamma}\theta(h)$. Keep X^1 (and therefore X^2) constant and set $\widehat{\mathcal{E}}^1 = \mathcal{E}^1_{\min}(X^1) < \mathcal{E}^1$, so that $\widehat{\mathcal{E}}^2 = \mathcal{E}^2_{\max}(X^1) > \mathcal{E}^2$. That is, type-1 consumers hold onto all claims below a certain risk level $\widehat{\epsilon}^1$ such that the average risk of their portfolio is $\mathcal{E}^1_{\min}(X^1)$. We then have that

$$\widetilde{\sigma}^1(l)U'((1+\widehat{\mathcal{E}}^1(l))X^1) < \widetilde{\sigma}^1(l)U'((1+\mathcal{E}^1(l))X^1) = \widetilde{\sigma}^2(l)U'((1+\mathcal{E}^2(l))X^2) < \widetilde{\sigma}^2(l)U'((1+\widehat{\mathcal{E}}^2(l))X^2),$$

and

$$\widetilde{\sigma}^1(h)U'((1+\widehat{\mathcal{E}}^1)X^1) > \widetilde{\sigma}^1(h)U'((1+\mathcal{E}^1)X^1) = \widetilde{\sigma}^2(h)U'((1+\mathcal{E}^2)X^2) > \widetilde{\sigma}^2(h)U'((1+\widehat{\mathcal{E}}^2)X^2),$$

which implies that there exists $\epsilon^*(X^1, \widehat{\mathcal{E}}^1)$ such that $\widetilde{v}^1_\epsilon(X^1, \widehat{\mathcal{E}}^1) < (=)[>]\widetilde{v}^2_\epsilon(X^2, \widehat{\mathcal{E}}^2)$ for $\epsilon < (=)[>]\epsilon^*(X^1, \widehat{\mathcal{E}}^1)$. This cannot hold in equilibrium, for it implies that type-2 consumers hold onto the safest claims. Then, decrease X^1 to $\widetilde{X}^1$ by lowering the maximum risk level of the claims held by type-1 consumers, which mechanically increases the size of the portfolio of type-2 consumers by decreasing the minimum risk level of the claims they hold, and let $\widetilde{\mathcal{E}}^1 \equiv \mathcal{E}^1_{\min}(\widetilde{X}^1)$. As a result the marginal valuation of all claims increases for type-1 consumers and decreases for type-2 consumers, thereby implying that the cutoff $\epsilon^*(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ such that $\widetilde{v}^1_\epsilon(\widetilde{X}^1, \widetilde{\mathcal{E}}^1) < (=)[>]\widetilde{v}^2_\epsilon(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ for $\epsilon < (=)[>]\epsilon^*(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ is strictly less than $\epsilon^*(X^1, \widehat{\mathcal{E}}^1)$. And type-1 consumers are still the ones holding the riskier assets, which cannot be true in an equilibrium with no or positive sorting. And if one instead increases X^1, we symmetrically obtain that the marginal valuation of all claims decreases for type-1 consumers and increases for type-2 consumers, thereby implying that the cutoff $\epsilon^*(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ such that $\widetilde{v}^1_\epsilon(\widetilde{X}^1, \widetilde{\mathcal{E}}^1) < (=)[>]\widetilde{v}^2_\epsilon(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ for $\epsilon < (=)[>]\epsilon^*(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ is strictly greater than $\epsilon^*(X^1, \widehat{\mathcal{E}}^1)$. And type-1 consumers are again still the ones holding the riskier assets, which again cannot be true in an equilibrium with no or positive sorting.

A symmetric reasoning holds when $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$. It thus follows that if a pair $(X^1, \mathcal{E}^1)$ with no segmentation exists, then a pair $(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ with segmentation does not exist.

(b) Assume now there exists a solution $(X^1, \mathcal{E}^1)$ to (A8)-(A9) but that it is such that $\mathcal{E}^1 < \mathcal{E}^1_{\min}(X^1)$. And assume again first that $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$. Then keep X^1 constant but set $\widehat{\mathcal{E}}^1 = \mathcal{E}^1_{\min}(X^1)$. We then have that

$$\widetilde{\sigma}^1(l)U'((1+\widehat{\mathcal{E}}^1(l))X^1) > \widetilde{\sigma}^1(l)U'((1+\mathcal{E}^1(l))X^1) = \widetilde{\sigma}^2(l)U'((1+\mathcal{E}^2(l))X^2) > \widetilde{\sigma}^2(l)U'((1+\widehat{\mathcal{E}}^2(l))X^2),$$

and

$$\widetilde{\sigma}^1(h)U'((1+\widehat{\mathcal{E}}^1)X^1) < \widetilde{\sigma}^1(h)U'((1+\mathcal{E}^1)X^1) = \widetilde{\sigma}^2(h)U'((1+\mathcal{E}^2)X^2) < \widetilde{\sigma}^2(h)U'((1+\widehat{\mathcal{E}}^2)X^2),$$

thereby implying that $\Delta^1 > \Delta^2$, i.e., the marginal valuation of claims for type-1 consumers is now more sensitive to risk than that of type-2 consumers. If the cutoff $\epsilon^*(X^1, \widehat{\mathcal{E}}^1) = \epsilon^1_{\min}(X^1)$, where $\epsilon^1_{\min}(X^1)$ solves

$$\int_{\epsilon_{\min}}^{\epsilon^1_{\min}(X^1)} \frac{\epsilon E dG(\epsilon)}{\mu^1} = \mathcal{E}^1_{\min}(X^1),$$

then we have found a pair $(\widetilde{X}^1, \widetilde{\mathcal{E}}^1) = (X^1, \mathcal{E}^1_{\min}(X^1))$ with segmentation. If instead $\epsilon^*(X^1, \widehat{\mathcal{E}}^1) < \epsilon^1_{\min}(X^1)$, then continuously decrease X^1 so that $\mathcal{E}^1_{\min}(X^1)$ and $\epsilon^1_{\min}(X^1)$ continuously decrease, thereby implying that v^1_ϵ shifts up and v^2_ϵ shift down for all ϵ, which implies that ϵ^* such that $\widetilde{v}^1_{\epsilon^*} = \widetilde{v}^2_{\epsilon^*}$ increases. And do so until reaching $\widehat{X}^1$ such that $\widetilde{v}^1_\epsilon = \widetilde{v}^2_\epsilon$ for $\epsilon = \epsilon^1_{\min}(\widehat{X}^1)$. If instead $\widehat{\epsilon} > \epsilon^1_{\min}(X^1)$, then do the opposite and continuously increase X^1 until reaching $\widehat{X}^1$ such that $\widetilde{v}^1_\epsilon = \widetilde{v}^2_\epsilon$ for $\epsilon = \epsilon^1_{\min}(\widehat{X}^1)$. That is, one can always find a pair $(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ for which segmentation is observed.

A symmetric reasoning applies if $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$ when $\mathcal{E}^1 > \mathcal{E}^1_{\max}(X^1)$.

(ii) I now prove the second part of the proposition. Assume that $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$. To prove the result I need three intermediary results. First, $\widetilde{\sigma}^1(\omega)/\widetilde{\sigma}^2(\omega)$ increases for both ω as η increases.

Second, since $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$, we also have that

$$\frac{d(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^2(l))}{d\eta} > \frac{d(\widetilde{\sigma}^1(h)/\widetilde{\sigma}^2(h))}{d\eta}. \tag{A10}$$

Finally,

$$\frac{d\Psi(X^1,\mathcal{E}^1;l)}{dX^1} > \frac{d\Psi(X^1,\mathcal{E}^1;h)}{dX^1}. \tag{A11}$$

In fact,

$$\frac{d\Psi(X^1,\mathcal{E}^1;l)}{dX^1} = \frac{(1-\mu^1)(1+\mathcal{E}^1)(1+\mathcal{E})E}{[E(1+\mathcal{E})-\mu^1 X^1(1+\mathcal{E}^1)]^2}, \text{ while}$$
$$\frac{d\Psi(X^1,\mathcal{E}^1;h)}{dX^1} = \frac{(1-\mu^1)(1-\frac{\pi}{1-\pi}\mathcal{E}^1)(1+\mathcal{E})E}{[E(1-\frac{\pi}{1-\pi}\mathcal{E})-\mu^1 X^1(1-\frac{\pi}{1-\pi}\mathcal{E}^1)]^2},$$

implying that

$$\frac{d\Psi(X^1,\mathcal{E}^1;l)}{dX^1} > \frac{d\Psi(X^1,\mathcal{E}^1;h)}{dX^1}$$

if and only if

$$E(1+\mathcal{E}) \times E(1-\frac{\pi}{1-\pi}\mathcal{E}) > \mu^1 X^1(1+\mathcal{E}^1) \times \mu^1 X^1(1-\frac{\pi}{1-\pi}\mathcal{E}^1).$$

This inequality must hold since type-2 agents must choose portfolios that ensure they consume non-zero amounts in each state and therefore $\mu^2 X^2(1+\mathcal{E}^2) = E(1+\mathcal{E}) - \mu^1 X^1(1+\mathcal{E}^1) > 0$, and $\mu^2 X^2(1-\frac{\pi}{1-\pi}\mathcal{E}^2) = E(1-\frac{\pi}{1-\pi}\mathcal{E}) - \mu^1 X^1(1-\frac{\pi}{1-\pi}\mathcal{E}^1) > 0$.

I can now turn my attention to the proof itself. First, if $\eta = 0$, then $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l)) = 1$, in which case $\Phi(X^1,\mathcal{E}^1) = 0$ implies that $\mathcal{E}^1 = \mathcal{E} > \mathcal{E}_{\min}(X^1)$. There is thus no sorting and no segmentation. As η increases, $\widetilde{\sigma}^1(l)/\widetilde{\sigma}^2(l)$ and $\widetilde{\sigma}^1(h)/\widetilde{\sigma}^2(h)$ increase, which with (A10) implies that $0 > \Psi(X^1,\mathcal{E}^1;h) > \Psi(X^1,\mathcal{E}^1;l)$ *ceteris paribus*. And since $\Psi(X^1,\mathcal{E}^1;\omega)$ is strictly increasing in X^1 for each ω, while $\Psi(X^1,\mathcal{E}^1;l)$ is strictly decreasing in $\mathcal{E}^1$ and $\Psi(X^1,\mathcal{E}^1;h)$ is strictly increasing in $\mathcal{E}^1$, it thus must be that X^1 increases when η increases. Finally, X^1 converges to E/μ^1 as η converges to σ, because $\widetilde{\sigma}^1(\omega)/\widetilde{\sigma}^2(\omega)$ converges to infinity as η converges to σ for each $\omega \in \{h,l\}$. Hence, $\mathcal{E}_{\min}(X^1)$ increases as X^1 increases, and it converges to $\mathcal{E}$ as η converges to σ because then X^1 converges to E/μ^1.

However, (A11), together with the fact that X^1 increases with η, implies that $\mathcal{E}^1$ decreases as η increases. There therefore exists $\eta^* \in (0,\sigma)$ such that $\mathcal{E}^1(\eta) \leq \mathcal{E}_{\min}(X^1(\eta))$ for $\eta \geq \eta^*$, with strict inequality for $\eta > \eta^*$, which implies that no segmentation is no longer possible when $\eta \geq \eta^*$.

The proof is symmetric if one instead assumes that $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$.■

Proof of Corollary 1: The only element that does not directly follow from the assumptions is that the pricing function is always strictly increasing when $(A(l))^{1-\gamma}\theta(l) > (A(h))^{1-\gamma}\theta(h)$. In this case we know that $\mathcal{E}^1 < \mathcal{E}^2$, and therefore that $\widetilde{\Delta}^1 \geq \widetilde{\Delta}^2$. I will now prove that $\widetilde{\Delta}^2$ cannot be negative, which implies that $\widetilde{\Delta}^1$ cannot be negative either. Suppose that $\widetilde{\Delta}^2 < 0$. Since $\widetilde{\sigma}^i(l) > \widetilde{\sigma}^i(h)$ for both i, $\widetilde{\Delta}^2 < 0$ only if $\mathcal{E}^2 < 0$. This would in turn imply that $\mathcal{E}^1 < 0$, which cannot be true since $\mathcal{E} > 0$. We therefore obtain a contradiction.■

Figure

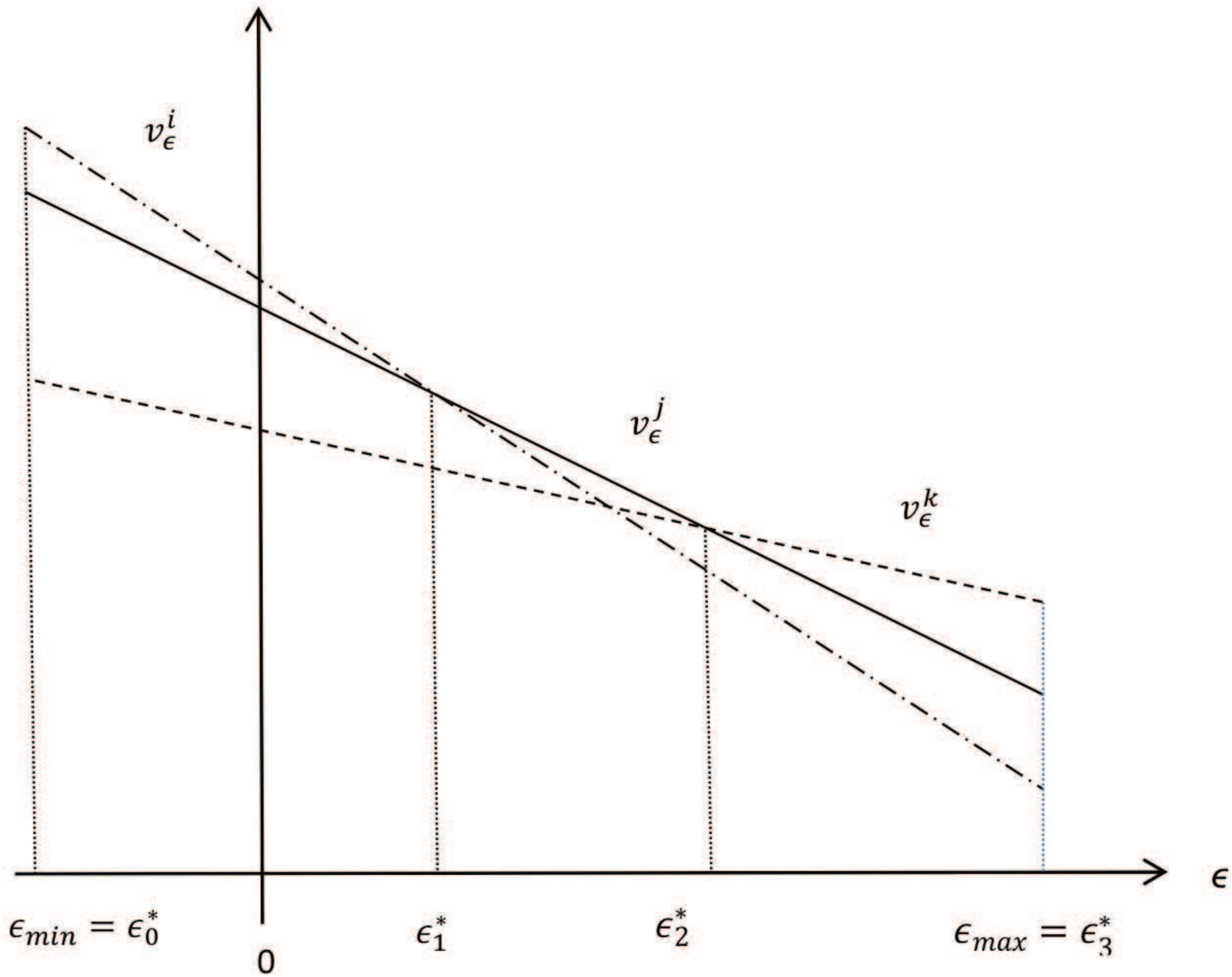

Figure 1: Example of a class structure with 3 classes, where agents of type i, j, and k respectively belong to class 1, 2, and 3.

Online Appendix

Part 1

This part of the online appendix contains proofs of results in the main text.

Proof that producers hold empty portfolios when leaving the first period if $U'(\mathbf{d}(h)\cdot 2\mathbf{e}) \geq 1$ - First, since $\mathcal{E}>0$, we have that $U'(\mathbf{d}(h)\cdot 2\mathbf{e}) \geq 1$ implies that $U'(\mathbf{d}(l)\cdot 2\mathbf{e}) > 1$. Second, $v^i_\epsilon(\mathbf{x}^i) = \mathbb{E}\left[d_\epsilon U'(\mathbf{d}\cdot\mathbf{x}^i)\right] + \sigma^i\mathbb{E}[d_\epsilon L(\mathbf{d}\cdot\mathbf{x}^i)] > \mathbb{E}[d_\epsilon U'(\mathbf{d}\cdot\mathbf{x}^i)]$. In fact, $L(\mathbf{x};\omega) \equiv A(\omega)\theta(\omega)u'(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{y}(\mathbf{x};\omega))) - U'(\mathbf{d}(\omega)\cdot\mathbf{x})$ and lemma 1 shows that $A(\omega)\theta(\omega)u'\left(\mathbf{Q}(\omega)\cdot\mathbf{y}^i(\mathbf{x};\omega)\right) = U'(\mathbf{d}(\omega)\cdot(\mathbf{x}-\mathbf{y}^i(\mathbf{x};\omega)))$ and therefore $A(\omega)\theta(\omega)u'\left(\mathbf{Q}(\omega)\cdot\mathbf{y}^i(\mathbf{x};\omega)\right) > U'(\mathbf{d}(\omega)\cdot(\mathbf{x}))$. Third, $2\mathbf{e} \geq \sum_{i=1}^I \mu^i\mathbf{x}^i$ and therefore $U'(\mathbf{d}(\omega)\cdot 2\mathbf{e}) \leq U'(\mathbf{d}(\omega)\cdot \sum_{i=1}^I \mu^i\mathbf{x}^i)$ for both ω. By Jensen's inequality we then have that $U'(\mathbf{d}(\omega)\cdot\sum_{i=1}^I\mu^i\mathbf{x}^i) \leq \sum_{i=1}^I\mu^i U'(\mathbf{d}(\omega)\cdot\mathbf{x}^i)$ for both ω (with strict inequality if $U'''(c)>0$). It thus follows that $\sum_{i=1}^I \mu^i\mathbb{E}[d_\epsilon U'(\mathbf{d}\cdot\mathbf{x}^i)] \geq \mathbb{E}[d_\epsilon U'(\mathbf{d}\cdot 2\mathbf{e})]$ for all ϵ, and therefore $\sum_{i=1}^I\mu^i\mathbb{E}[d_\epsilon U'(\mathbf{d}\cdot\mathbf{x}^i)] \geq 1$ for all ϵ, with strict inequality for all $\epsilon < (1-\pi)/\pi$. This implies that there exists at least one $j \in \{1,...,I\}$ such that $\mathbb{E}[d_\epsilon U'(\mathbf{d}\cdot\mathbf{x}^j)] \geq 1$ for all ϵ, with strict inequality for all $\epsilon < (1-\pi)/\pi$, and therefore $v^j_\epsilon(\mathbf{x}^j) > 1$ for all ϵ for at least one consumer type j. Producers therefore never hold any claims.■

The $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ functions - The expression for $\mathcal{E}^i_{\min}(X)$ and $\mathcal{E}^i_{\max}(X)$, respectively the minimum and maximum values admissible for the average risk of a portfolio of size X (given A and the aggregate distribution of risk G) for agents of type i, are respectively given by

$$\begin{aligned}\mathcal{E}^i_{\min}(X) &= \int_{\epsilon_{\min}}^{\underline{\epsilon}^i(X)} \epsilon\left(\frac{1}{X}\right)dG(\epsilon)\text{, and}\\ \mathcal{E}^i_{\max}(X) &= \int_{\overline{\epsilon}^i(X)}^{\epsilon_{\max}} \epsilon\left(\frac{1}{X}\right)dG(\epsilon)\text{,}\end{aligned}$$

where $\underline{\epsilon}^i(X)$ and $\overline{\epsilon}^i(X)$ for agents of type i are such that

$$\int_{\epsilon_{\min}}^{\underline{\epsilon}^i(X)} EdG(\epsilon) = \int_{\overline{\epsilon}^i(X)}^{\epsilon_{\max}} EdG(\epsilon) = \mu_i X.$$

These boundaries correspond respectively to the cases where the type i of agents hold onto all of the safest or all of the riskiest asset. We have that $\mathcal{E}^i_{\min}(X)$ increases while $\mathcal{E}^i_{\max}(X)$ decreases with X, and that

$$\begin{aligned}\mathcal{E}^i_{\min}(E) &< \mathcal{E} < \mathcal{E}^i_{\max}(E)\text{, and}\\ \mathcal{E}^i_{\min}(E/\mu_i) &= \mathcal{E}^i_{\max}(E/\mu_i) = \mathcal{E}.\end{aligned}$$

Further Discussion on Velocity - The results about the correlation between risk and velocity and price and velocity presented in the main text hold on *average* over the entire range $[\epsilon_{\min}, \epsilon_{\max}]$ *and* if we assume that consumers pay in the middle period with portfolios that are a fraction of their initial portfolios. In fact, even if I assume all early consumers sell a fraction $\alpha(\omega)$ of their claims in state ω in the middle period, it can perfectly well be that sorting is positive, i.e., $\varepsilon^1 < \varepsilon^2$, but there are claims $\epsilon^{'}$ and $\epsilon'' > \epsilon'$ such that $VELO(\epsilon'-\eta, \epsilon+\eta|\omega) < VELO(\epsilon''-\eta,\epsilon''+\eta|\omega)$ for some $\eta > 0$. This is possible in a NSE because there are many ways to obtain a portfolio with parameters (X^1, ε^1) and therefore the holding of a specific type of claims by a specific type of agent is not pinned down.

If, moreover, portfolios used as payments in the middle period are not a fixed fractions of initial portfolios, then correlations between risk and velocity and price and velocity can in principle be anything. Suppose, for instance, that the equilibrium is a NSE with positive sorting, but type-1 consumers always pay in the middle period with the highest possible ϵ claims in their portfolios (for a mass $\alpha(\omega)X^1$ in state ω), while type-2 consumers do the opposite and always pay with the lowest possible ϵ claims in their portfolios (for a mass $\alpha(\omega)X^2$ in state ω). If we denote by $\bar{\epsilon}(\alpha(\omega)X^2;\omega)$ the highest type of claims used by type-2 consumers in state ω, while $\underline{\epsilon}(\alpha(\omega)X^1;\omega)$ denotes the lowest type of claims used by type-1 consumers in state ω, it can be that $\bar{\epsilon}(\alpha(h)X^2;h)$ is greater than $\underline{\epsilon}(\alpha(h)X^1;h)$ while $\bar{\epsilon}(\alpha(l)X^2;l)$ is smaller than $\underline{\epsilon}(\alpha(l)X^1;l)$ and that

$$\begin{aligned} VELO(\bar{\epsilon}(\alpha(h)X^2;h),\epsilon_{\max}|h) &> VELO(\epsilon_{\min},\underline{\epsilon}(\alpha(h)X^1;h)|h), \text{ and} \\ VELO(\epsilon(\alpha(l)X^1;l),\epsilon_{\max}|l) &> VELO(\epsilon_{\min},\bar{\epsilon}(\alpha(l)X^2;l)|l), \end{aligned}$$

where

$$\begin{aligned} VELO(\bar{\epsilon}(\alpha(\omega)X^2;\omega),\epsilon_{\max}|\omega) &= \alpha^{\omega}\frac{\int_{\bar{\epsilon}(\alpha(\omega)X^2;\omega)}^{\epsilon_{\max}}\sigma^1\mu^1x^1_{\epsilon}d\epsilon}{\int_{\bar{\epsilon}(\alpha(\omega)X^2;\omega)}^{\epsilon_{\max}}2e_{\epsilon}d\epsilon}, \text{ and} \\ VELO(\epsilon_{\min},\underline{\epsilon}(\alpha(\omega)X^1;\omega)|\omega) &= \alpha^{\omega}\frac{\int_{\epsilon_{\min}}^{\underline{\epsilon}(\alpha(\omega)X^1;\omega)}\sigma^2\mu^2x^2_{\epsilon}d\epsilon}{\int_{\epsilon_{\min}}^{\underline{\epsilon}(\alpha(\omega)X^1;\omega)}2e_{\epsilon}d\epsilon}. \end{aligned}$$

It is then possible for the correlation between risk and velocity of claims to be negative, especially if the mass of claims around the cutoffs $\bar{\epsilon}(\alpha(l)X^1;l)$, $\bar{\epsilon}(\alpha(h)X^1;h)$, $\bar{\epsilon}(\alpha(l)X^2;l)$, and $\bar{\epsilon}(\alpha(h)X^2;h)$ is small.

Proof of Proposition 2:

(i) Consider $\epsilon\in\Xi_m$ claims. We have from (4) in the main text that

$$1=\mathbb{E}\left[R_{\epsilon}\right]\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right]+cov\left(R_{\epsilon},U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right).$$

And when $0\in\Xi_k$ we also have that

$$1=R_0\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_k})+\sigma^{i_k}L(\mathbf{x}^{i_k})\right].$$

It thus follows that

$$\mathbb{E}\left[R_{\epsilon}\right]\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right]+cov\left(R_{\epsilon},U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)=R_0\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_k})+\sigma^{i_k}L(\mathbf{x}^{i_k})\right],$$

and therefore

$$\begin{aligned} \mathbb{E}\left[R_{\epsilon}\right]-R_0 &= R_0\left\{\frac{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_k})+\sigma^{i_k}L(\mathbf{x}^{i_k})\right]}{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right]}-1\right\} \\ &\quad -R_0\frac{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_k})+\sigma^{i_k}L(\mathbf{x}^{i_k})\right]}{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right]}cov\left(R_{\epsilon},U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right). \end{aligned}$$

(ii) By definition, agents holding the $\epsilon=0$ type of claims are the ones with the highest value of such claims. That is,

$$\frac{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_k})+\sigma^{i_k}L(\mathbf{x}^{i_k})\right]}{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right]}>1,$$

unless type-0 claims are at the boundary of two classes in which case the ratio is 1.

Now assume that $K \geq 3$ and $0 \in \Xi_1$ and consider any two types $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_{m+1}$. If $m = 1$, then $\phi^{12} \geq 1$, with strict inequality unless $\epsilon_1^* = 0$. If instead we consider $m > 1$, then since $0 \in \Xi_1$, we have that $\epsilon^{ij} > 0$. And this, together with $v_0^i(\mathbf{x}^i) - \epsilon^{ij}\pi\Delta^i(\mathbf{x}^i) = v_0^j(\mathbf{x}^j) - \epsilon^{ij}\pi\Delta^j(\mathbf{x}^j)$ and $\Delta^i(\mathbf{x}^i) > \Delta^j(\mathbf{x}^j)$, implies that $v_0^i(\mathbf{x}^i) > v_0^j(\mathbf{x}^j)$. That is, $\phi^{ij} > 1$, and therefore $\phi^{1k}/\phi^{1(k+1)} = (v_0^1/v_0^k)/(v_0^1/v_0^{k+1}) = v_0^{k+1}/v_0^k < 1$, which proves that ϕ^{1k} is increasing in k.■

Decomposition of Excess Risk Premium into Risks Loadings - The excess returns to a given type of claims obtained in the main text can also be represented into its exposure to the two types of aggregate risks, at and before maturity, as the next corollary to the proposition 3.

Corollary 1 *Excess returns can be re-expressed as*

$$\begin{aligned}\mathbb{E}\left(R_\epsilon\right) - R_0 \;\; = \;\; & (\phi^{km} - 1)R_0 \\ & - \frac{(1-\sigma^{i_m})\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})\right]}{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m}) + \sigma^{i_m}L(\mathbf{x}^{i_m})\right]} \times \frac{cov(R_\epsilon, U'(\mathbf{d}\cdot\mathbf{x}^{i_m}))}{Var(U'(\mathbf{d}\cdot\mathbf{x}^{i_m}))} \times \frac{Var(U'(\mathbf{d}\cdot\mathbf{x}^{i_m}))}{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m})\right]} \\ & - \frac{\sigma^{i_m}\mathbb{E}\left[\theta u'(\mathbf{Q}\cdot\mathbf{y}^{i_m})\right]}{\mathbb{E}\left[U'(\mathbf{d}\cdot\mathbf{x}^{i_m}) + \sigma^{i_m}L(\mathbf{x}^{i_m})\right]} \times \frac{cov(R_\epsilon, A\theta u'(\mathbf{Q}\cdot\mathbf{y}^{i_m}))}{Var(\theta u'(\mathbf{Q}\cdot\mathbf{y}^{i_m}))} \times \frac{Var(\theta u'(\mathbf{Q}\cdot\mathbf{y}^{i_m}))}{\mathbb{E}\left[\theta u'(\mathbf{Q}\cdot\mathbf{y}^{i_m})\right]}.\end{aligned}$$

The last terms of lines 2 and 3 are the values of aggregate risk for consumers of type i_m in the last and middle period respectively. These are the counterparts to the market price for risk in traditional consumption-based CAPM model, although here there is no such thing as a market price because the same type of claims can be held by different types of agents who value the two risks differently. The second terms of the second and third lines are the risk loadings of a type-ϵ claim for the last and middle period respectively, while the first terms in each of the last two lines are the weights being put on each risk. Once again all these components are type specific, including when there is a unique class.

Note that, because the expression characterizes the decomposition of the excess returns until maturity, the risk loading before maturity is expressed as depending on the covariance between the rate of return until maturity and the marginal utility of consumption in the middle period, adjusted for the variation in the market conditions, rather than the covariance between the rate of return until the middle period and the marginal utility of consumption then.

Proof of Proposition 3: We have from Proposition 2 that for $0 \in \Xi_k$, $\epsilon \in \Xi_n$ and $\varepsilon^M \in \Xi_m$ and $i_n \in \mathcal{I}_n$ and $i_m \in \mathcal{I}_m$,

$$\begin{aligned}\mathbb{E}\left(R_\epsilon\right) - R_0 &= R_0\left(\phi^{kn} - 1\right) - R_0\phi^{kn} \times cov\left(R_\epsilon, U'(\mathbf{d}\cdot\mathbf{x}^{i_n}) + \sigma^{i_n}L(\mathbf{x}^{i_n})\right), \text{ and} \\ \mathbb{E}\left(R_{\varepsilon^M}\right) - R_0 &= R_0\left(\phi^{km} - 1\right) - R_0\phi^{km} \times cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m}) + \sigma^{i_m}L(\mathbf{x}^{i_m})\right).\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left(R_\epsilon\right) - (v_0^{i_n})^{-1} &= -(v_0^{i_n})^{-1} \times cov\left(R_\epsilon, U'(\mathbf{d}\cdot\mathbf{x}^{i_n}) + \sigma^{i_n}L(\mathbf{x}^{i_n})\right), \text{ and} \\ \mathbb{E}\left(R_{\varepsilon^M}\right) - (v_0^{i_m})^{-1} &= -(v_0^{i_m})^{-1} \times cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m}) + \sigma^{i_m}L(\mathbf{x}^{i_m})\right).\end{aligned}$$

$$\mathbb{E}\left(R_{\epsilon}\right)-(v_0^{i_n})^{-1} = \frac{v_0^{i_m}}{v_0^{i_n}} \times \frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)}\left[\mathbb{E}\left(R_{\varepsilon^M}\right)-(v_0^{i_m})^{-1}\right]$$

$$\mathbb{E}\left(R_{\epsilon}\right)-(v_0^{i_n})^{-1} = \underbrace{\frac{v_0^{i_m}}{v_0^{i_n}} \times \frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)}}_{\beta_{\epsilon,M}}\left[\mathbb{E}\left(R^M\right)-(v_0^{i_m})^{-1}\right]$$

$$+\underbrace{\frac{v_0^{i_m}}{v_0^{i_n}} \times \frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)}\xi_{\epsilon,M}}_{\alpha_{\epsilon,M}}$$

Hence,

$$\mathbb{E}\left(R_{\epsilon}\right)-R_0 = \frac{\phi^{kn}}{\phi^{km}}\frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)}\left[\mathbb{E}\left(R_{\varepsilon^M}\right)-R_0\right] \quad \text{(AA2)}$$

$$+\left[\left(\phi^{kn}-1\right)-\left(\phi^{km}-1\right)\frac{\phi^{kn}}{\phi^{km}}\frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}l(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}l(\mathbf{x}^{i_m})\right)}\right]R_0.$$

Since the asset pricing function is convex, we also have that $q^M \geq q_{\varepsilon^M}$, with strict inequality whenever there is more than one class, and therefore there exists exists $\xi \geq 0$, strictly positive if there is more than one class, such that

$$\mathbb{E}\left(R_{\varepsilon^M}\right)=\mathbb{E}\left(R^M\right)+\xi. \quad \text{(AA3)}$$

Combining (AA2) and (AA3) we have that

$$\mathbb{E}\left(R_{\epsilon}\right)-R_0=\beta_{\epsilon,M}\left[\mathbb{E}\left(R^M\right)-R_0\right]+\alpha_{\epsilon,M},$$

where

$$\beta_{\epsilon,M} \equiv \phi^{mn}\frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)}, \text{ and}$$

$$\alpha_{\epsilon,M} \equiv \left\{\left(\phi^{kn}-1\right)-\left(\phi^{km}-1\right)\phi^{mn}\frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)}\right\}R_0+\beta_{\epsilon}\xi.$$

And if $k=m$, then the absolute excess return, or "alpha," is

$$\left(\phi^{kn}-1\right)R_0+\frac{cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)}{cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)}\phi^{kn}\xi. \quad (1)$$

If $M=[\underline{\epsilon},\bar{\epsilon}]$, $m=k$, and the asset pricing function is strictly decreasing everywhere, then $cov\left(R_{\epsilon}, U'(\mathbf{d}\cdot\mathbf{x}^{i_n})+\sigma^{i_n}L(\mathbf{x}^{i_n})\right)=-\epsilon\pi\Delta(\mathbf{x}^{i_n})$, which is strictly negative for $\epsilon>0$, while $cov\left(R_{\varepsilon^M}, U'(\mathbf{d}\cdot\mathbf{x}^{i_m})+\sigma^{i_m}L(\mathbf{x}^{i_m})\right)=-\varepsilon\pi\Delta(\mathbf{x}^{i_m})<0$ since $\varepsilon>0$. The second term in (1) is therefore positive for $\epsilon>0$. And since $\phi^{kn}\geq 1$, with strict inequality for $n>k$ and with equality if $n=k$, we have that the absolute excess returns are therefore strictly positive for all claims with risk $\epsilon>0$.■

Proof of Lemma 6: The first-order conditions with respect to the type-ϵ claims sold by an early consumer is[1]

$$\frac{1}{1-\rho}\left\{(1-\rho)\,\theta(\omega)A(\omega)d_{\epsilon}(\omega)\left(A(\omega)\mathbf{d}(\omega).\mathbf{y}(\mathbf{x};\omega)\right)^{-\rho}-(1-\rho)\,d_{\epsilon}(\omega)\left(\mathbf{d}(\omega).(\mathbf{x}-\mathbf{y}(\mathbf{x};\boldsymbol{\omega}))\right)^{-\rho}\right\}\left[\widehat{V}_1^1(\omega)\right]^{\rho}=0,$$

[1] I can ignore the constraints because solution must be interior.

which is equivalent to

$$\begin{aligned}\theta(\omega)(A(\omega))^{1-\rho}\left(\mathbf{d}(\omega).\mathbf{y}(\mathbf{x};\omega)\right)^{-\rho} &= \left(\mathbf{d}(\omega).(\mathbf{x}-\mathbf{y}(\mathbf{x};\omega))\right)^{-\rho}, \text{ or}\\ \mathbf{d}(\omega).\mathbf{y}(\mathbf{x};\omega) &= \frac{(\theta(\omega))^{1/\rho}(A(\omega))^{1/\rho-1}}{1+(\theta(\omega))^{1/\rho}(A(\omega))^{1/\rho-1}}\left(\mathbf{d}(\omega).\mathbf{x}\right).\end{aligned}$$

It thus follows that

$$\begin{aligned}\widehat{V}_1^1(\omega) &= \left\{\theta(\omega)\left(A(\omega)\right)^{1-\rho}\left(\frac{(\theta(\omega))^{1/\rho}(A(\omega))^{1/\rho-1}}{1+(\theta(\omega))^{1/\rho}(A(\omega))^{1/\rho-1}}\mathbf{d}(\omega).\mathbf{x}\right)^{1-\rho}+\left(\frac{1}{1+(\theta(\omega))^{1/\rho}(A(\omega))^{1/\rho-1}}\mathbf{d}(\omega).\mathbf{x}\right)^{1-\rho}\right\}^{\frac{1}{1-\rho}}, \text{ or}\\ \widehat{V}_1^1(\omega) &= \left[1+(\theta(\omega))^{1/\rho}(A(\omega))^{1/\rho-1}\right]^{\frac{\rho}{1-\rho}}\left(\mathbf{d}(\omega).\mathbf{x}\right). \blacksquare\end{aligned}$$

Proof of Lemma 7: The date-0 marginal valuation of a type-ϵ claim is

$$\frac{\partial\widehat{v}_0}{\partial x_\epsilon}=\frac{1}{1-\rho}\widehat{v}_0^\rho\times\left\{(1-\rho)C_0^{-\rho}\frac{\partial C_0}{\partial x_\epsilon}+\frac{1-\rho}{1-\gamma}\left(\mathbb{E}\left[\widehat{V}_1^{1-\gamma}\right]\right)^{\frac{\gamma-\rho}{1-\gamma}}\times(1-\gamma)\mathbb{E}\left[\widehat{V}_1^{-\gamma}\frac{\partial\widehat{V}_1}{\partial x_\epsilon}\right]\right\}.$$

where

$$\frac{\partial C_0}{\partial x_\epsilon}=-q_\epsilon,$$

and

$$\begin{aligned}\mathbb{E}\left[\widehat{V}_1^{1-\gamma}\right] &= \sum_\omega\pi(\omega)\left\{(1-\sigma)\left(\mathbf{d}(\omega)\cdot\mathbf{x}\right)^{1-\gamma}+\sigma[Z(\omega)\left(\mathbf{d}(\omega)\cdot\mathbf{x}\right)]^{1-\gamma}\right\},\\ &= \mathbb{E}\left[\widehat{\sigma}\left(\mathbf{d}\cdot\mathbf{x}\right)^{1-\gamma}\right],\end{aligned}$$

for

$$\widehat{\sigma}(\omega)\equiv 1+\sigma[(\widehat{Z}(\omega))^{1-\gamma}-1].$$

Furthermore,

$$\begin{aligned}\mathbb{E}\left[\widehat{V}_1^{-\gamma}\frac{\partial\widehat{V}_1}{\partial x_\epsilon}\right] &= \sum_\omega\pi(\omega)\left[(1-\sigma)\left(\mathbf{d}(\omega)\cdot\mathbf{x}\right)^{-\gamma}d_\epsilon(\omega)+\sigma\left(\widehat{Z}(\omega)\mathbf{d}(\omega)\cdot\mathbf{x}\right)^{-\gamma}\widehat{Z}(\omega)d_\epsilon(\omega)\right],\\ &= \mathbb{E}\left[\widehat{\sigma}\left(\mathbf{d}\cdot\mathbf{x}\right)^{-\gamma}\right]-\epsilon\pi\left[\widehat{\sigma}(l)\left(\mathbf{d}(l)\cdot\mathbf{x}\right)^{-\gamma}-\widehat{\sigma}(h)\left(\mathbf{d}(h)\cdot\mathbf{x}\right)^{-\gamma}\right].\end{aligned}$$

We therefore have that $\partial\widehat{v}_0/\partial x_\epsilon=0$ is equivalent to

$$\widehat{v}_0^\rho\times\left(-q_\epsilon C_0^{-\rho}+\left(\mathbb{E}\left[\widehat{\sigma}\left(\mathbf{d}\cdot\mathbf{x}\right)^{1-\gamma}\right]\right)^{\frac{\gamma-\rho}{1-\gamma}}\times\left\{\mathbb{E}\left[\widehat{\sigma}\left(\mathbf{d}\cdot\mathbf{x}\right)^{-\gamma}\right]-\epsilon\pi\left[\widehat{\sigma}(l)\left(\mathbf{d}(l)\cdot\mathbf{x}\right)^{-\gamma}-\widehat{\sigma}(h)\left(\mathbf{d}(h)\cdot\mathbf{x}\right)^{-\gamma}\right]\right\}\right)=0,$$

which proves the result.■

Proof of Proposition 5: Since in a NSE the marginal valuation of all types of claims must be equal for all types of consumers, it must therefore be that for any two types i and j

$$\begin{aligned}(C_0^i)^\rho\widehat{\Delta}^i(\mathbf{x}^i) &= (C_0^j)^\rho\widehat{\Delta}^j(\mathbf{x}^j), \text{ and}\\ (C_0^i)^\rho\widehat{\Gamma}^i(\mathbf{x}^i) &= (C_0^j)^\rho\widehat{\Gamma}^j(\mathbf{x}^j).\end{aligned}$$

Equating the ratios of the left- and right-hand sides it follows that

$$\frac{\left[\widehat{\sigma}^i(l)\left(\mathbf{d}(l)\cdot\mathbf{x}^i\right)^{-\gamma}-\widehat{\sigma}^i(h)\left(\mathbf{d}(h)\cdot\mathbf{x}^i\right)^{-\gamma}\right]}{\mathbb{E}\left[\widehat{\sigma}^i\left(\mathbf{d}\cdot\mathbf{x}^i\right)^{-\gamma}\right]}=\frac{\left[\widehat{\sigma}^j(l)\left(\mathbf{d}(l)\cdot\mathbf{x}^j\right)^{-\gamma}-\widehat{\sigma}^j(h)\left(\mathbf{d}(h)\cdot\mathbf{x}^j\right)^{-\gamma}\right]}{\mathbb{E}\left[\widehat{\sigma}^j\left(\mathbf{d}\cdot\mathbf{x}^j\right)^{-\gamma}\right]},$$

which simplifies to

$$\frac{\widehat{\sigma}^i(h)\left(\mathbf{d}(h)\cdot\mathbf{x}^i\right)^{-\gamma}}{\widehat{\sigma}^i(l)\left(\mathbf{d}(l)\cdot\mathbf{x}^i\right)^{-\gamma}}=\frac{\widehat{\sigma}^j(h)\left(\mathbf{d}(h)\cdot\mathbf{x}^j\right)^{-\gamma}}{\widehat{\sigma}^j(l)\left(\mathbf{d}(l)\cdot\mathbf{x}^j\right)^{-\gamma}},$$

or

$$\frac{\widehat{\sigma}^i(h)\left(\mathbf{1}+\varepsilon^i(h)\right)^{-\gamma}}{\widehat{\sigma}^i(l)\left(\mathbf{1}+\varepsilon^i(l)\right)^{-\gamma}}=\frac{\widehat{\sigma}^j(h)\left(\mathbf{1}+\varepsilon^j(h)\right)^{-\gamma}}{\widehat{\sigma}^j(l)\left(\mathbf{1}+\varepsilon^j(l)\right)^{-\gamma}}.$$

It thus follows that $\varepsilon^i<(=)[>]\varepsilon^j$ if and only if

$$\frac{\widehat{\sigma}^i(h)}{\widehat{\sigma}^i(l)}<(=)[>]\frac{\widehat{\sigma}^j(h)}{\widehat{\sigma}^j(l)},$$

which is equivalent to

$$\sigma^j[(\widehat{Z}(l))^{1-\gamma}-(\widehat{Z}(h))^{1-\gamma}]<(=)[>]\sigma^i[(\widehat{Z}(l))^{1-\gamma}-(\widehat{Z}(h))^{1-\gamma}],$$

which for $\sigma^i>\sigma^j$ is equivalent to

$$(\widehat{Z}(l))^{1-\gamma}>(=)[<](\widehat{Z}(h))^{1-\gamma}.\blacksquare$$

Part 2

This second part of the online appendix contains a number of comparative statics not included in the main text, two extensions of the benchmark model, as well as a discussion of how the model can be used to interpret the 2007-8 subprime financial crisis and the subsequent European sovereign debt crisis.

1 Comparative Statics

1.1 A Decrease in the Stock of Claims

The following propositions shows that, according to the model, financial crises are unlikely to be created solely by the sudden realization that some assets are actually worthless, in particular because such shocks imply that the price of all claims rises, as in Shi (2015).

Proposition 1 *Assume that Assumption 1 holds and that a positive mass of claims become worthless without changing the average risk of the market portfolio. Assume further that the equilibrium pair $(X^1,\mathcal{E}^1)$ before and after the change is such that no segmentation takes place. Then $\widetilde{\mathcal{E}}^1=\mathcal{E}^1$, and $\widetilde{q}_\epsilon>q_\epsilon$ for all ϵ.*

Proof. If the stock of claims decreases from E to $\widetilde{E}$, then $(\widetilde{X}^1,\mathcal{E}^1)$ solves the system (A4)-(A5) in the main appendix, where $\widetilde{X}^1$ is such that

$$\widetilde{X}^1=\frac{\widetilde{E}}{E}X^1<X^1.$$

Since an equilibrium is essentially unique and we assume a new equilibrium is a NSE, the new NSE is characterized by $(\widetilde{X}^1, \mathcal{E}^1)$.
Since

$$\widetilde{q}_\epsilon = \widetilde{v}^i_\epsilon(\widetilde{X}^i, \widetilde{\mathcal{E}}^i)) = \pi(1+\epsilon)\widetilde{\sigma}^1(h)((1+\mathcal{E}^1(h))\widetilde{X}^1)^{-\gamma} + (1-\pi)(1-\frac{\pi}{1-\pi}\epsilon)\widetilde{\sigma}^1(l)((1+\mathcal{E}^1(l))\widetilde{X}^1))^{-\gamma} \text{ for } i = 1, 2,$$

it follows directly that $\widetilde{q}_\epsilon > q_\epsilon$ for all ϵ. ■

This result is intuitive. The price of a claim is a weighted average of the marginal value that the consumers who hold this type of claims attach to consumption in each state, and this marginal valuation in a given state is made up of the marginal utility of consumption at maturity augmented by the gains from the liquidity services. When the size of the pool of claims decreases and there are no changes in the market conditions in the middle period, the amount of resources for consumption in the middle and last periods decrease in each state, which raises the marginal value of consumption in both states. And because the utility function and the marginal utility function are homogeneous, the high- and low-liquidity-need consumers are affected in the same way by the change and they both adjust their portfolio by reducing the size in proportion without any change in the average risk.

We thus obtain that when a mass of claims becomes worthless and changes the size of market portfolio, but not its average risk, then the price of all claims increases, while the average risk of the portfolio of type-1 consumers is unchanged. This implies that the velocity of claims does not change. These predictions are clearly is at odds with what we typically observe during a financial crisis, especially because there tend to be a broad-based drop in the price of assets.

1.2 Safe Assets and Flight-to-Safety

In this section I show that the model can be used to define what constitutes a safe asset as well as understand phenomena known as "flight-to-safety," where investors' demand for safe assets increases. In the interest of brevity the analysis is carried out assuming that the equilibrium is a NSE.

The analysis was so far carried out assuming that the distribution of risk across claims has a given cdf G and an associated level of aggregate risk $\mathcal{E}$. Assume now that there is an increase in the individual risk of a positive mass of claims, so that there is a new distribution of risk with cdf $\widetilde{G}$, which first-order stochastically dominates G, and the new level of aggregate risk is $\widetilde{\mathcal{E}} > \mathcal{E}$. If we define $q_\epsilon(\mathcal{E})$ as being the price of a claim with risk ϵ when the portfolio risk is $\mathcal{E}$, we have the following proposition:

Proposition 2 *Assume that u satisfies Assumption 1 and that $\bar{\epsilon} = (1-\pi)/\pi$. Suppose that the equilibrium is initially a NSE, that there is an increase in aggregate risk from $\mathcal{E}$ to $\widetilde{\mathcal{E}}$, and that the new equilibrium is still a NSE. Then:*
(i) $\mathcal{E}^1$ and $\mathcal{E}^2$ both increase, $\Delta\mathcal{E}^1/\Delta\mathcal{E} < (=)[>]\Delta\mathcal{E}^2/\Delta\mathcal{E}$ when $(A(l))^{1-\gamma}\theta(l) > (=)[<](A(h))^{1-\gamma}\theta(h)$, while $X^1(\widetilde{\mathcal{E}}) = X^1(\mathcal{E})$ and $X^2(\widetilde{\mathcal{E}}) = X^2(\mathcal{E})$.
(ii) Furthermore, when $(A(l))^{1-\gamma}\theta(l) \geq (A(h))^{1-\gamma}\theta(h)$ there exists a cutoff $\widehat{\epsilon} \in (0, (1-\pi)/\pi)$ such that $q_\epsilon(\widetilde{\mathcal{E}}) > q_\epsilon(\mathcal{E})$ for all $\epsilon < \widehat{\epsilon}$, while $q_\epsilon(\widetilde{\mathcal{E}}) \leq q_\epsilon(\mathcal{E})$ for all $\epsilon \geq \widehat{\epsilon}$, with strict inequality for $\epsilon > \widehat{\epsilon}$.

Proof. (i) We know that there always exists a unique pair $(X^1, \mathcal{E}^1)$ associated with a NSE for a given set of parameters. However, X^1 and $\widetilde{\mathcal{E}}^1$ such that $\Phi(X^1, \widetilde{\mathcal{E}}^1; \widetilde{\mathcal{E}}) = 0$ are the unique solution to the system of two equations (A4) and (A5) in the main appendix when the portfolio risk is $\widetilde{\mathcal{E}}$. This implies that the portfolio sizes for type-1 and type-2 consumers do not change when portfolio risk changes. Let $\widetilde{\mathcal{E}}^i \equiv \mathcal{E}^i + \Delta\mathcal{E}^i$. From $\Psi(X^1, \mathcal{E}^1; \mathcal{E}, h) = \Psi(X^1, \mathcal{E}^1; \widetilde{\mathcal{E}}, h) = 0$ we have that

$$\frac{\Delta\mathcal{E}^1}{\Delta\mathcal{E}^2} = \frac{1+\mathcal{E}^1}{1+\mathcal{E}^2}.$$

Hence, we have that $\Delta\mathcal{E}^i > 0$ for $i = 1, 2$. Moreover, since we have that $\mathcal{E}^1 < (=)[>]\mathcal{E}^2$ for $(A(l))^{1-\gamma}\theta(l) > (=)[<](A(h))^{1-\gamma}\theta(h)$, it follows that $\Delta\mathcal{E}^1 < (=)[>]\Delta\mathcal{E}^2$ whenever $(A(l))^{1-\gamma}\theta(l) > (=)[<](A(h))^{1-\gamma}\theta(h)$. ■

(ii) Since we assume that the equilibrium is initially and remains a NSE, we can use the valuation of type-1 agents to price all claims. The size of their portfolio does not change, while the risk of their portfolio increases. This implies that $((1+\mathcal{E}^1(l))X^1))^{-\gamma}$ $(((1+\mathcal{E}^1(h))X^1)^{-\gamma})$, their marginal valuation of consumption in state l (h) increases (decreases) since they have less (more) resources in that state, both as an early and as a late consumer. Furthermore, the change in marginal utility of consumption in the two states are such that $|\Delta(1+\mathcal{E}^1(l))X^1))^{-\gamma}| > |\Delta((1+\mathcal{E}^1(h))X^1)^{-\gamma}|$. Since in a NSE

$$q_\epsilon = \widetilde{v}^i_\epsilon(X^i, \mathcal{E}^i) = \pi(1+\epsilon)\widetilde{\sigma}^i(h)((1+\mathcal{E}^1(h))X^1)^{-\gamma} + (1-\pi)(1-\frac{\pi}{1-\pi}\epsilon)\widetilde{\sigma}^i(l)((1+\mathcal{E}^1(l))X^1))^{-\gamma} \text{ for } i = 1, 2,$$

it follows that $q_\epsilon(\widetilde{\mathcal{E}}) < q_\epsilon(\mathcal{E})$ for $\epsilon = (1-\pi)/\pi$, and that when $\widetilde{\sigma}^i(l) \geq \widetilde{\sigma}^i(h)$, then $\widetilde{\sigma}^i(l)|\Delta(1+\mathcal{E}^1(l))X^1))^{-\gamma}| > \widetilde{\sigma}^i(h)|\Delta((1+\mathcal{E}^1(h))X^1)^{-\gamma}|$ and therefore $q_0(\widetilde{\mathcal{E}}) > q_0(\mathcal{E})$. Hence, if $\widetilde{\sigma}^i(l) \geq \widetilde{\sigma}^i(h)$, then there exists a cutoff $\widehat{\epsilon} \in (0, (1-\pi)/\pi)$ such that $q_\epsilon(\widetilde{\mathcal{E}}) > q_\epsilon(\mathcal{E})$ for all $\epsilon < \widehat{\epsilon}$, while $q_\epsilon(\widetilde{\mathcal{E}}) \geq q_\epsilon(\mathcal{E})$ for all $\epsilon \geq \widehat{\epsilon}$, with strict inequality for $\epsilon > \widehat{\epsilon}$.■

Asset Price Changes and Safe Assets - Part (ii) of proposition 2 establishes that, following an increase in the risk of the market portfolio, the asset pricing function rotates when there is no or positive sorting. Note also that the price of claims whose risk increases can also increase, depending on (i) how their initial risk compares with $\widehat{\epsilon}$, (ii) the extent of the increase in their risk, and (iii) the extent of the change in portfolio risk. In fact, *ceteris paribus*, an increase in the risk of an asset pushes its price down. But at the same time, if the risk of the claim is and stays below the cutoff $\widehat{\epsilon}$, its price is, *ceteris paribus*, pushed up by the increase in portfolio risk. But the larger its initial risk, the more unlikely it is that this asset's price increases, because the portfolio risk effect is stronger for riskier assets. And similarly, the larger the increase in the individual risk of a claim, the less likely it is that the price of this asset increases. Note that this part of the proposition would hold with one type of consumers.

The rotation of the asset pricing function and the fact that the price of low ϵ can increase even if their risk increases highlights that in this set up safe assets are *doubly* safe, in that (i) their payoffs do not fluctuate much, or even are negatively correlated to aggregate risk, for a given level of risk and (ii) their payoffs fluctuate in a counter-cyclical fashion with respect to the level of market risk. This can be useful when thinking about risk and liquidity management.[2]

This part of the proposition would hold even if there were a representative consumer, as the intuition suggests. In fact, the price of a claim is a weighted average of the marginal value that the consumers who hold this type of claims attach to consumption in each state, with the weight being given by the payoffs in

[2]For instance, the fact that safe and risky assets are respectively more expensive and cheaper when portfolio risk increases means there are gains from arbitrage for an institution that does not have to worry about short-term liquidity constraints. This suggests that asset purchase programs like the US Treasury's TARP, the Federal Reserve's rounds of Quantitative Easing (QE), the European Stability Mechanism (ESM), and the European Central Bank's extended Asset Purchase Programme (APP) are not bound to suffer from large losses (at least if assets are not purchased and sold at prices well above and well below market prices).

each state while the marginal valuation in a given state is made up of the marginal utility of consumption at maturity augmented with the value of liquidity times the probability she is an early consumer. When the risk of the market portfolio increases, there are less resources available at maturity in state l and more in state h, which all else being equal leads to an increase in the marginal utility of consumption at maturity for state l and a decrease for state h, and the magnitude of the impact is greater for state l than state h because of the curvature of the utility function. Since the safer the claim, the larger the fraction of the payout coming from state l, the more (less) their expected valuation is positively (negatively) affected by the increase (decrease) in the marginal utility of consumption in the last period for state l (h). Whenever $(A(l))^{1-\gamma}\theta(l) \geq (A(h))^{1-\gamma}\theta(h)$ the combined effects of the terms of trade and of the intensity of liquidity needs in the middle period reinforce the effect that the greater portfolio risk has on the marginal utility of consumption at maturity. So that in this case the safest assets' price goes up while the riskier claim see their price decrease. When $(A(l))^{1-\gamma}\theta(l) < (A(h))^{1-\gamma}\theta(h)$, however, the combined effects of the terms of trade and of the intensity of liquidity needs in the middle period go dampen the effect of the fall in consumption in state l relative to the effect of the increase in consumption in state h at maturity, and if these effects are strong enough it can be that the price of any claim goes down.

Velocity Changes and Flight-to-Safety - Proposition 2 also establishes that when the utility function satisfies Assumption 1, then there is no change in the portfolio sizes, and the average risk of the portfolio of agents with high liquidity needs increases, but by less than the market portfolio if there is no or positive sorting. A consequence of these properties of the model is that, following an increase in portfolio risk, assets with low risk are held, on average, in even greater proportion by agents with high liquidity needs if the initial and new equilibria display positive sorting. And *vice versa* for assets at the right end of the distribution of risk. This in turn implies that the velocity of safe assets increases on average, whereas the velocity of the risky assets decreases on average, because of the clientele effect.

We thus obtain that an increase in the risk of the market portfolio can trigger a flight-to-safety, for the price and velocity of the safer assets increases while they decrease for the riskier assets. Moreover, this also shows that a flight-to-safety can be mistaken for a flight-to-liquidity, where investors wish to change the composition of their portfolio towards more liquid assets. In fact, one could misinterpret the changes in velocity of the different types of assets arising from the clientele effect as indicating that there are changes in the willingness to trade when in fact there is no change in the intrinsic liquidity properties of assets.

1.3 Changes in Market Conditions

I now investigate the impact of changes in the market conditions in the middle period, by which I mean that the terms of trade or the intensity of the liquidity needs change. In order to simplify the presentation I once again focus my attention on NSE. Let $z(\omega) \equiv (A(\omega))^{1-\gamma}\theta(\omega)$ and denote by $X^1(z(l), z(h))$ and $\mathcal{E}^1(z(l), z(h))$ the equilibrium portfolio size and average risk for type-1 agents given $z(l)$ and $z(h)$. We then have the following proposition.

Proposition 3 *Assume that $\epsilon_{\min} = -1$ and $\epsilon_{\max} = (1-\pi)/\pi$, and that the equilibrium is a NSE for all the parameter values considered.*
(i) If $\widetilde{z}(l) > z(l)$, then $X^1(\widetilde{z}(l), z(h)) > X^1(z(l), z(h))$ and $\mathcal{E}^1(\widetilde{z}(l), z(h)) < \mathcal{E}^1(z(l), z(h))$, and $q_\epsilon(\widetilde{z}(l), z(h)) \geq q_\epsilon(z(l), z(h))$ for all ϵ, with strict inequality for $\epsilon < \epsilon_{\max}$;
(ii) If $\widetilde{z}(h) < z(h)$, then $X^1(z(l), \widetilde{z}(h)) < X^1(z(l), z(h))$ and $\mathcal{E}^1(z(l), \widetilde{z}(h)) < \mathcal{E}^1(z(l), z(h))$, and $q_\epsilon(z(l), \widetilde{z}(h)) \leq$

$q_\epsilon(z(l), z(h))$ for all ϵ, with strict inequality for $\epsilon > \epsilon_{\min}$;
(iii) If $\widetilde{z}(l) > z(l)$ and $\widetilde{z}(h) < z(h)$, then $\mathcal{E}^1(\widetilde{z}(l), \widetilde{z}(h)) < \mathcal{E}^1(z(l), z(h))$, and there exists $\widetilde{\epsilon} \in (\epsilon_{\min}, \epsilon_{\max})$ such that $q_\epsilon(\widetilde{z}(l), \widetilde{z}(h)) > (=)[<] q_\epsilon(z(l), z(h))$ for all $\epsilon < (=)[>] \widetilde{\epsilon}$.

Proof. Remember first that a NSE is such that $(X^1, \mathcal{E}^1)$ solves (A4) and (A5). Moreover, if z^l increases and/or z^h decreases, we have that $(\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)) \times (\widetilde{\sigma}^2(h)/\widetilde{\sigma}^2(l))$ increases, and therefore Φ decreases for given $(X^1, \mathcal{E}^1)$. It must therefore be that the new equilibrium portfolio for type-1 agents is characterized by $(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ such that $\widetilde{\mathcal{E}}^1/\widetilde{\mathcal{E}}^2 < \mathcal{E}^1/\mathcal{E}^2$, which given that there is no change in the risk of the market portfolio implies that $\mathcal{E}^1$ decreases.
(i) Assume $z(l)$ increases to $\widetilde{z}(l)$ but $z(h)$ does not change. This means that $\widetilde{\sigma}^i(l)$ increases to $\widetilde{\widetilde{\sigma}}^i(l) > \widetilde{\sigma}^i(l)$ for both i, but that $\widetilde{\sigma}^i(h)$ does not change. Then, since the new portfolio of type-1 agents must be characterized by $(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ such that $\widetilde{\mathcal{E}}^1 < \mathcal{E}^1$ and $\Psi(\widetilde{X}^1, \widetilde{\mathcal{E}}^1; h) = 0$, it follows that $\widetilde{X}^1 > X^1$.
Furthermore, since there are no changes in the amount of resources available at date $t = 2$, $\Psi(\widetilde{X}^1, \widetilde{\mathcal{E}}^1; h) = 0$ also implies that the amounts consumed in state h by late consumers of type-1 and type-2 do not change, and therefore the marginal value attached to consumption in state h for each type does not change, and therefore for both i,

$$\widetilde{v}^i_{(1-\pi)/\pi}(\widetilde{X}^i, \widetilde{\mathcal{E}}^i) = \widetilde{\sigma}^i(h) U'((1 + \widetilde{\mathcal{E}}^i)\widetilde{X}^i) = \widetilde{\sigma}^i(h) U'((1 + \mathcal{E}^i)X^i) = \widetilde{v}^i_{(1-\pi)/\pi}(X^i, \mathcal{E}^i).$$

And hence,

$$q_{(1-\pi)/\pi}(\widetilde{z}(l), z(h)) = q_{(1-\pi)/\pi}(z(l), z(h)).$$

And since late consumers of type 1 consume more in state l, late consumers of type 2 consume less in this state. Moreover, $\widetilde{\widetilde{\sigma}}^i(l) > \widetilde{\sigma}^i(l)$ for both i, and therefore $\widetilde{\widetilde{\sigma}}^2(l) U'((1+\widetilde{\mathcal{E}}^2(l))\widetilde{X}^2) > \widetilde{\sigma}^2(l) U'((1+\mathcal{E}^2(l))X^2)$. In a NSE the same inequality must also hold for type-1 consumers, so that for both i and all $\epsilon < \epsilon_{\max} = (1-\pi)/\pi$

$$\begin{aligned}\widetilde{\widetilde{v}}^i_\epsilon(\widetilde{X}^i, \widetilde{\mathcal{E}}^i) &= \pi(1+\epsilon)\widetilde{\sigma}^i(h) U'((1 + \widetilde{\mathcal{E}}^i(h))\widetilde{X}^i) + (1-\pi)(1 - \frac{\pi}{1-\pi}\epsilon)\widetilde{\widetilde{\sigma}}^i(l) U'((1 - \frac{\pi}{1-\pi}\widetilde{\mathcal{E}}^i)\widetilde{X}^i) \\ &> \widetilde{v}^i_\epsilon(X^i, \mathcal{E}^i) = \pi(1+\epsilon)\widetilde{\sigma}^i(h) U'((1 + \mathcal{E}^i)X^i) + (1-\pi)(1 - \frac{\pi}{1-\pi}\epsilon)\widetilde{\sigma}^i(l) U'((1 - \frac{\pi}{1-\pi}\mathcal{E}^i)X^i),\end{aligned}$$

and therefore $q_\epsilon(\widetilde{z}(l), z(h)) > q_\epsilon(z(l), z(h))$ for all $\epsilon < \epsilon_{\max}$.
(ii) Assume $z(h)$ decreases to $\widetilde{z}(h)$ but $z(l)$ does not change. This means that $\widetilde{\sigma}^i(h)$ decreases to $\widetilde{\widetilde{\sigma}}^i(h) < \widetilde{\sigma}^i(h)$ for both i, but that $\widetilde{\sigma}^i(l)$ does not change. Then, since the new portfolio of type-1 agents must be characterized by $(\widetilde{X}^1, \widetilde{\mathcal{E}}^1)$ such that $\widetilde{\mathcal{E}}^1 < \mathcal{E}^1$ and $\Psi(\widetilde{X}^1, \widetilde{\mathcal{E}}^1; l) = 0$, it follows that $\widetilde{X}^1 < X^1$.
Furthermore, since there are no changes in the amount of resources available at date $t = 2$, $\Psi(\widetilde{X}^1, \widetilde{\mathcal{E}}^1; l) = 0$ also implies that the amounts consumed in state l by late consumers of type-1 and type-2 do not change, and therefore the marginal value attached to consumption in state l for each type does not change, and therefore for both i,

$$\widetilde{v}^i_{-1}(\widetilde{X}^i, \widetilde{\mathcal{E}}^i) = \widetilde{\sigma}^i(l) U'((1 - \frac{\pi}{1-\pi}\widetilde{\mathcal{E}}^i)\widetilde{X}^i) = \widetilde{\sigma}^i(l) U'(1 - \frac{\pi}{1-\pi}\mathcal{E}^i)X^i) = \widetilde{v}^i_{-1}(X^i, \mathcal{E}^i).$$

And hence,

$$q_{-1}(z(l), \widetilde{z}(h)) = q_{-1}(z(l), z(h)).$$

And since late consumers of type 1 consume less in state h, late consumers of type 2 consume more in this state. Moreover, $\widetilde{\widetilde{\sigma}}^i(h) < \widetilde{\sigma}^i(h)$ for both i, and therefore $\widetilde{\widetilde{\sigma}}^2(h)U'((1+\widetilde{\mathcal{E}}^2(h))\widetilde{X}^2) < \widetilde{\sigma}^2(h)U'((1+\mathcal{E}^2(h))X^2)$. In a NSE the same inequality must also hold for type-1 consumers, so that for both i and all $\epsilon < \epsilon_{\max} = (1-\pi)/\pi$

$$\begin{aligned}\widetilde{\widetilde{v}}^i_\epsilon(\widetilde{X}^i, \widetilde{\mathcal{E}}^i) &= \pi(1+\epsilon)\widetilde{\widetilde{\sigma}}^i(h)U'((1+\widetilde{\mathcal{E}}^i)\widetilde{X}^i) + (1-\pi)(1-\frac{\pi}{1-\pi}\epsilon)\widetilde{\sigma}^i(l)U'((1-\frac{\pi}{1-\pi}\widetilde{\mathcal{E}}^i)\widetilde{X}^i) \\ &< \widetilde{v}^i_\epsilon(X^i, \mathcal{E}^i) = \pi(1+\epsilon)\widetilde{\sigma}^i(h)U'((1+\mathcal{E}^i)X^i) + (1-\pi)(1-\frac{\pi}{1-\pi}\epsilon)\widetilde{\sigma}^i(l)U'((1-\frac{\pi}{1-\pi}\mathcal{E}^i)X^i),\end{aligned}$$

and therefore $q_\epsilon(z(l), \widetilde{z}(h)) < q_\epsilon(z(l), z(h))$ for all $\epsilon > \epsilon_{\min}$.

(iii) Assume $z(h)$ decreases to $\widetilde{z}(h)$ and that $z(l)$ increases to $\widetilde{z}(l)$. Because there is no change in the amount of resources available at maturity, then:

- either $(1+\widetilde{\mathcal{E}}^i)\widetilde{X}^i = (1+\mathcal{E}^i)X^i$ for both i or $(1+\widetilde{\mathcal{E}}^i)\widetilde{X}^i > (1+\mathcal{E}^i)X^i$ for some i. This means that for some i, $U'((1+\widetilde{\mathcal{E}}^i)\widetilde{X}^i) < U'((1+\mathcal{E}^i)X^i)$, and therefore that $\widetilde{\widetilde{\sigma}}^i(h)U'((1+\widetilde{\mathcal{E}}^i)\widetilde{X}^i) < \widetilde{\sigma}^i(h)U'((1+\mathcal{E}^i)X^i)$ for some i. Since we are considering a NSE, this relationship is true for the other type of consumers; and
- either $(1+\widetilde{\mathcal{E}}^i(l))\widetilde{X}^i = (1+\mathcal{E}^i(l))X^i$ for both i or $(1+\widetilde{\mathcal{E}}^i(l))\widetilde{X}^i < (1+\mathcal{E}^i(l))X^i$ for some i. This means that $U'((1+\widetilde{\mathcal{E}}^i(l))\widetilde{X}^i) > U'((1+\mathcal{E}^i(l))X^i)$ for some i, and therefore that $\widetilde{\widetilde{\sigma}}^i(l)U'((1+\widetilde{\mathcal{E}}^i(l))\widetilde{X}^i) > \widetilde{\sigma}^i(l)U'((1+\mathcal{E}^i(l))X^i)$ for some i. Since we are considering a NSE, this relationship is true for the other type of consumers;

It thus follows that there exists $\widetilde{\epsilon} \in (\epsilon_{\min}, \epsilon_{\max})$ such that $q_\epsilon(\widetilde{z}(l), \widetilde{z}(h)) > (=)[<] q_\epsilon(z(l), z(h))$ for all $\epsilon < (=)[>]\widetilde{\epsilon}$. ■

The results regarding the pricing of claims in this proposition are intuitive. If the terms of trade in a state worsen (improve) when consumers' willingness to substitute consumption is low, or if the intensity of liquidity needs as early consumer increases (decreases) in one state, then the liquidity services provided by claims in the middle period become more (less) important to consumers in that state, which raises (reduces) the value of holding claims that pay in that state. And since consumers with the greatest likelihood to benefit from these liquidity services are the consumers of type 1, the average risk of their portfolio decreases (and that of type-2 consumer therefore increases). And it also follows that all claims delivering goods in that state see their price increase (decrease). And when one combines a decrease of the marginal value of liquidity in one state with an increase of this marginal value in the other state, then the price of claims that pay most in the latter state increases, while the price of the claims that pay most in the former state decreases.

We thus obtain that such a change in the market conditions in the middle period can replicate the features of a flight to safety just as well as an increase in the risk of the market portfolio. Interestingly, if the two types of changes happen simultaneously, they would reinforce each other. Lastly, the model is currently setup such that the terms of trade in the middle period are determined exclusively by the disutility of effort of producers, but if they were to depend on the portfolio of consumers one could introduce feedback effects.

2 Early Sale of Claims by Late Consumers

In the basic setup it is assumed that claims are the only non-perishable items, which implies that late consumers would never sell claims in the middle period, and in particular they cannot take advantage of favorable terms of trade in the middle period. In this section I instead assume that there exists a storage technology between dates $t = 1$ and $t = 2$, which enables agents to obtain $s(\omega) > 0$ units of consumption good at date $t = 2$ for each unit stored at date $t = 1$ when the state is ω.

When the state is ω, a late consumer can now either sell her portfolio in the middle period for $A(\omega)\mathbf{d}(\omega)\cdot\mathbf{x}$ and store it to obtain obtain $s(\omega)A(\omega)\mathbf{d}(\omega)\cdot\mathbf{x}$ units of consumption in the last period; or she can wait until the last period to sell her portfolio to obtain $\mathbf{d}(\omega)\cdot\mathbf{x}$. Hence, the problem at date 0 for a consumer of type i is now

$$\max_{(c,h,\mathbf{x})}\{c-h+\sum_{\omega\in\{h,l\}}\pi(\omega)\left[\sigma^i\theta(\omega)u(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{x})+(1-\sigma^i)\max\{U(s(\omega)A(\omega)\mathbf{d}(\omega)\cdot\mathbf{x});U(\mathbf{d}(\omega)\cdot\mathbf{x})\}\right],$$

subject to the budget constraint $c+\mathbf{q}\cdot\mathbf{x}\leq h+\mathbf{q}\cdot\mathbf{e}$, the feasibility constraints $c,h\geq 0$, and short-selling constraints $x_\epsilon\geq 0$ for all ϵ. It is clear that a late consumer sells her entire portfolio whenever $s(\omega)A(\omega)>1$, holds onto it when $s(\omega)A(\omega)<1$, and is indifferent if $s(\omega)A(\omega)=1$.

Assume, for the sake of illustration, that $A(l)>1>A(h)$ and that $s(l)=s(h)=1$, so that $s(l)A(l)<1<s(h)A(h)$. In this case late consumers sell their portfolio early in state h, but not in state l. We thus get that the marginal value of a claim is given by

$$\begin{aligned}v_\epsilon^i(\mathbf{x}) &= \pi\left\{A(h)d_\epsilon(h)U'(\mathbf{d}(h)\cdot\mathbf{x})\times\left[1+\sigma^i\left(\theta(h)-1\right)\right]\right\}\\ &\quad+(1-\pi)\left\{d_\epsilon(l)U'(\mathbf{d}(l)\cdot\mathbf{x})\times\left[1+\sigma^i\left((A(l))^{1-\gamma}\theta(l)-1\right)\right]\right\},\end{aligned}$$

which can be rewritten as

$$v_\epsilon^i(\mathbf{x}^i)=\sum_\omega\pi(\omega)\widetilde{\sigma}^i(\omega)U'(\mathbf{d}(\omega)\cdot\mathbf{x}^i)-\epsilon\pi(\widetilde{\sigma}^i(l)U'(\mathbf{d}(l)\cdot\mathbf{x}^i)-\widetilde{\sigma}^i(h)U'(\mathbf{d}(h)\cdot\mathbf{x}^i)),$$

but where this time

$$\begin{aligned}\widetilde{\sigma}^i(l) &\equiv 1+\sigma^i[(A(l))^{1-\gamma}\theta(l)-1],\text{ and}\\ \widetilde{\sigma}^i(h) &\equiv A(h)[1+\sigma^i(\theta(h)-1)].\end{aligned}$$

And one can easily show that $\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)>(=)[<]\widetilde{\sigma}^2(l)/\widetilde{\sigma}^2(h)$ if and only if $(A(l))^{1-\gamma}\theta(l)>(=)[<]\theta(h)$. Since the relationship between $\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)$ and $\widetilde{\sigma}^2(l)/\widetilde{\sigma}^2(h)$ is what is driving sorting, it follows that the terms of trade effects for state h no longer matters for sorting, and this is intuitive. In fact, in this case late consumers can take advantage of the relatively high price of claims in the middle period, which implies that the only difference between an early and a late consumer in that state is the urgency of the liquidity needs as measured by $\theta(h)$.

More generally, we have that $\widetilde{\sigma}^1(l)/\widetilde{\sigma}^1(h)>(=)[<]\widetilde{\sigma}^2(l)/\widetilde{\sigma}^2(h)$ if and only if $(\max\{1;s(l)A(l)\})^{1-\gamma}\theta(l)>(=)[<](\max\{1;s(h)A(h)\})^{1-\gamma}\theta(h)$. Here again there is no fundamental change to the way sorting takes place relative to the benchmark case. There is one important difference, however, with the basic setup, and again the difference concerns the velocity of claims in the middle period: in this modified setup the velocity of claims is higher when terms of trade are good in the middle period, because late consumers also sell their claims then.

3 Frictional Illiquidity

I have so far assumed that all assets can be traded with equal ease in the middle period. In this section I instead allow claims to differ in how easy it is to trade them in the middle period by introducing a parameter which controls what I call the degree of frictional illiquidity of a claim. More specifically, I now assume that a claim is characterized by two parameters, the risk parameter ϵ, and a new parameter δ, which measures the fraction of one's holding of a claim that can actually be traded in the middle period, very much like in Kiyotaki and Moore (2005) or Lagos (2010). The remainder $1-\delta$ is only available in the last period. I denote the joint distribution of these two parameters by $\mathcal{G}$.

The claim-specific parameter δ is introduced to capture, in a reduced form fashion, the fact that it is easier to dispose of some assets relative to others, and that one can hold onto assets that cannot easily be disposed off until maturity. Differences in liquidity across assets can be, for instance, because it is easier to find potential buyers or agents who accept them as collateral than for other assets. Alternatively, investors might prefer to hold onto some assets more than others when in need of funds because of differences in the terms of trade they can obtain. There are several reasons why some assets might be easier to trade than others, and among them are market thickness effects (as in Vayanos and Weill, 2007, and Weill, 2008) and asymmetry of information (adverse selection as in Rocheteau, 2011; Lester *et al.*, 2012; or moral hazard as in Li *et al.*, 2012). It would be interesting to endogenize the parameter in the framework of this model, but it is beyond the scope of this paper.

The difference in frictional properties of claims implies that an agent effectively chooses two distributions of risk for a portfolio: one at maturity, and one in the middle period. In order to reduce the complexity of the task I consider the case where consumers are risk-neutral with respect to consumption in the last period with $U(c)=c$, for in this case consumers do not care about the distribution of risk at maturity.

3.1 Asset Pricing and Implications for Sorting and Velocity

From my earlier analysis we have that $Q_\epsilon(\delta)$, the price of a claim risk ϵ and frictional illiquidity parameter δ, is

$$Q_\epsilon(\delta)=1+\delta\max_i\left\{\sigma_i\left[\sum_{\omega\in\{h,l\}}\pi(\omega)A(\omega)d_\epsilon(\omega)\left(\theta(\omega)u'\left(A(\omega)\mathbf{d}(\omega)\cdot\mathbf{x}\right)-1\right)\right]\right\}.$$

Hence, if we define q_ϵ such that

$$q_\epsilon-1\equiv\delta^{-1}\left(Q_\epsilon(\delta)-1\right),$$

we have that the pricing function $\mathbf{q}$ inherits all the properties of the pricing function that we derived in the absence of frictional illiquidity by making the proper normalization to the distribution of risks and stock of asset. Given $\mathcal{G}$ defined over $[\epsilon_{\min},\epsilon_{\max}]\times[\delta_{\min},\delta_{\max}]$, then we need to define the cdf G for the marginal distribution over the risk parameter ϵ such that for all ϵ

$$G(\epsilon)=\int_{\delta_{\min}}^{\delta_{\max}}d\mathcal{G}(\epsilon,\delta). \tag{AA5}$$

And associated with G is an effective level of aggregate risk defined as before, i.e., $\mathcal{E}=\int_{\epsilon_{\min}}^{\epsilon_{\max}}\epsilon dG(\epsilon)$. If we denote by E^e the mass of claims, the effective stock of asset, that is the stock of asset that can effectively be used to trade in the middle period, is

$$E=E\int_{\epsilon_{\min}}^{\epsilon_{\max}}\int_{\delta_{\min}}^{\delta_{\max}}\delta d^2\mathcal{G}(\epsilon,\delta)<E^e. \tag{AA6}$$

Then, we get that

$$Q_\epsilon(\delta)=\delta q_\epsilon+(1-\delta), \tag{AA7}$$

where q is derived as in previous sections in the absence of trading frictions with an effective mass of assets E as given in (AA6), G as defined in (AA5) for the effective distribution of risk, and given the associated effective level of aggregate risk $\mathcal{E}$. Hence, the price of a claim is a weighted average, weighted by its degree of illiquidity, of the price of a perfectly liquid asset with the same risk ϵ, i.e., a claim with $\delta = 1$, and the price of a perfectly illiquid risk-free claim issued in the first period and maturing in the last period.

There are a number of interesting implications coming out of the model with frictional illiquidity. First, the degree of frictional illiquidity of an asset matters for its price, but does not matter for the type(s) of agents who hold it. This is because the price adjusts to the degree of frictional illiquidity: if an agent wants to consume c units of the special good in the middle period in both states she can achieve this by purchasing c units of a perfectly liquid risk-free asset, i.e., an asset with $\epsilon = 0$ and $\delta = 1$, or she can purchase $\delta^{-1}c$ units of a risk-free asset with frictional illiquidity parameter $\delta < 1$.

A corollary to the above result is that a claim with frictional illiquidity index δ can have a higher velocity than another claim with frictional illiquidity $\delta' > \delta$. In fact, assume for simplicity that there are two types of agents and that the equilibrium is a SE. If the former asset has a risk-index ϵ lower than the risk-index ϵ' of the latter, and their risk indices are such that the former asset belongs to the first class while the latter asset belongs to the second class, then the intrinsically less liquid asset is traded more often than the intrinsically more liquid asset, and thus it is possible to find parameter values for which $\sigma_1\delta > \sigma_2\delta'$. This can reconcile the model with the fact that many equity shares appear a lot more liquid than safer assets like some US government bonds or high-grade corporate bonds. The differences in velocity for these different types of assets are affected by the their individual risk, but also by other factors such as differences in markets' microstructures. This highlights once again that it is important to distinguish between the intrinsic liquidity of an asset and its measured liquidity.

3.2 Frictional Liquidity Shocks

In this section I study the impact on asset prices of a negative liquidity shock taking the form of an increase in the level of frictional illiquidity of a mass of claims. I show that such a shock is unlikely to generate asset price movements consistent with those of a flight to safety, unless it is accompanied by an increase in portfolio risk.

Assume a liquidity shock affects the supply of liquidity in the form of a fall of δ for a mass of claims, so that there is a new joint distribution of claims over risk and frictional illiquidity $\widetilde{\mathcal{G}}$ (and a new associated marginal distribution $\widetilde{G}$). Suppose first that the marginal distribution is unchanged, so that the effective level of portfolio risk $\mathcal{E}$ stays constant.[3] First note that a fall of δ for a mass of claims translates into a fall in E, the effective supply of liquidity to carry out transactions in the middle period. As proven earlier, this triggers a rise in the price of all perfectly liquid claims. Since the price of a claim with risk ϵ and degree of frictional illiquidity of δ is given by (AA7), its price after the shock becomes

$$\widetilde{Q}_\epsilon\left(\delta\right) = 1 + \delta(\widetilde{q}_\epsilon - 1),$$

where the "$\sim$" indicates that these expressions are for the new distribution $\widetilde{\mathcal{G}}$ and marginal distribution G. Hence, if we consider claims whose degree of illiquidity does not change, their price necessarily increases.

[3] The portfolio risk can be different from the effective risk of the portfolio obtained from the marginal distribution with cdf G.

Considering now claims whose frictional illiquidity increases, the new price of a claim with risk ϵ and degree of frictional illiquidity of δ is given by (AA7), and therefore its price after the shock becomes

$$\widetilde{Q}_{\epsilon}(\widetilde{\delta}) = 1 + \widetilde{\delta}(\widetilde{q}_{\epsilon} - 1),$$

and therefore there is another force at work: since $\widetilde{\delta} < \delta$, these assets provide, keeping q constant, less liquidity services, which pushes down their liquidity premium and their price. The overall effect on the price of these assets is therefore unclear.

It follows from the above analysis that the model requires a broad-based increase in frictional illiquidity if it is to generate a broad-based fall in asset prices in the absence of a change in the effective amount of aggregate risk. In particular, this implies that disruptions in the functioning of markets for specific assets, which increases the degree of frictional illiquidity on these markets, can generate effects in line with characteristics of a flight to safety only if these disruptions spread to a significant proportion of the markets for other assets. If, however, the equilibrium displays positive sorting and the changes in frictional liquidity properties of claims leads to a fall in the effective risk of the market portfolio that can be traded in the middle period, then the model is more likely to be able to generate the changes in prices and velocity observed during a flight to safety. A decrease in intrinsic liquidity of low-ϵ assets can thus generate price movements consistent with those of a flight-to-safety, even if there is no change in the individual level of risk of each claim.

3.3 Sensitivity to Information and Changes in Frictional Liquidity

I have so far assumed that there is no particular relationship between an asset's risk level and its degree of frictional illiquidity. However, if we assume that asset prices and their liquidity are sensitive to information, then it is possible to generate a positive relationship between risk and frictional illiquidity, at least for assets whose risk is positively correlated with market risk.

Assume that when the state of the world is revealed at date $t = 1$, not all information about an asset with risk ϵ is disclosed publicly. In particular, assume that if the state of the world is ω, then the asset yields a high payoff $\overline{D}_{\epsilon}$ with some probability $\lambda(\omega)$ and a low payoff $\underline{D}_{\epsilon}$ with probability $1 - \lambda(\omega)$, so that the expected payoff in that state is

$$(1 - \lambda(\omega))\,\underline{D}_{\epsilon} + \lambda(\omega)\overline{D}_{\epsilon} = d_{\epsilon}(\omega),$$

and such that the high and low payoffs are still given by

$$d_{\epsilon}(h) = 1 + \epsilon, \text{ and } d_{\epsilon}(l) = 1 - \frac{\pi}{1-\pi}\epsilon.$$

This implies that

$$\overline{D}_{\epsilon} = \underline{D}_{\epsilon} + \frac{\epsilon}{(\lambda(h) - \lambda(l))\,(1-\pi)}.^{4}$$

Assume further that at date $t = 1$ only the agent holding a claim in her portfolio knows whether the payoff at date $t = 2$ will be the high or the low payoff.[5] Rocheteau (2011) shows that in such an environment with two assets and where one asset is safe and the other is risky and subject to adverse selection, the latter asset is less liquid than the safe asset, in that an agent buying goods in the middle period will choose to use

[5] If neither agents in a match know whether the payoff is high or low, this model is isomorphic to the model studied previously.

assets whose payoffs are high but not publicly known last. And the larger the asymmetry of information, i.e., the larger ϵ in the context of this paper, the greater is the fraction of her holding that the buyer keeps.

In terms of the model in this paper, the impact of this sensitivity to information can be captured in a reduced form way by the parameter of frictional illiquidity δ by assuming that $\delta(0) = 0$, and that δ decreases as ϵ increases for $\epsilon > 0$ and that δ increases as ϵ increases for $\epsilon < 0$. This implies that assets with high risk are also assets for which the asymmetry of information is more severe, and are therefore assets with the highest level of frictional illiquidity. If we assume that the mass of assets whose risk is negatively correlated with aggregate risk is small, then it is possible to obtain a positive relationship between risk and frictional illiquidity. We also have in this case that frictional illiquidity reinforces the clientele effect for the velocity of assets for $\epsilon \geq 0$.

4 An application: The 2007-8 subprime financial crisis and the European sovereign debt crisis

It is commonly believed that the 2007-8 subprime financial crisis was triggered by an unexpected sharp increase in the risk of certain classes of mortgage-linked financial products known as Mortgage-Backed Securities (MBSs) and Collateralized Debt Obligation (CDOs). For instance, mortgage-based CDOs are structured financial products where mortgages are grouped into diversified pools, which are then split into several tranches. The so-called senior tranch of a CDO is the first one to be paid and is therefore the safest. However, the senior tranches can be impacted as the delinquency rate on mortgages rises. And in the late 2000's the delinquency rate in the US on single-family home mortgages rose to levels well beyond recent historical levels and what most investors projected,[6] and to such an extent that the senior tranches of CDOs that were deemed to be very safe became perceived as being not so safe. As a result, one witnessed a dramatic drop in the price and volume of trade for mortgage-based CDOs, while the price of Treasury securities, which are considered to be the safest of assets, rose.

In the case of the European sovereign debt crisis, the 2007-8 subprime financial crisis and the Great Recession that ensued triggered a deterioration of the public finances of member countries of the European Monetary Union (EMU). Portugal, Ireland, Greece, and Spain, the so-called PIGS,[7] suffered from sharp increases in public deficits,[8] which lead to large increases in their debt-to-GDP ratio and in interest rates on their respective public debts from 2008.[9] The situation deteriorated so much that Ireland and Greece ended up requesting bailouts in 2010, while Portugal asked for assistance in 2011. In short, there was a sharp deterioration in the confidence of investors that certain countries member of the EMU would be able

[6] The delinquency rate on single-family homes, which had been falling from 2001 to 2006 from about 2.4% to about 1.5%, started to rise sharply at the end of 2006. It reached 3.69% in the first quarter of 2008, a level exceeding the highest level on record since 1991 (which was 3.36% for the third and fourth quarters of 1991). The situation got worse still, with the delinquency rate reaching 7.85% in the first quarter of 2008, to finally reach a peak of 11.26% in the first quarter of 2010.

[7] Some include Italy in the group, which then become known as PIIGS.

[8] The reasons for the worsening of public finances are varied. Ireland and Spain suffered from sharp turn-arounds of their housing markets, which severly impacted their respective financial sectors, and led to worries about public finances because of explicit or implicit guarantees of financial institutions by the two governments. The problem in Greece, and to a lesser extent Portugal, were for the most part rooted in the poor management of public finances before the crisis.

[9] Focusing on Greece and Ireland as examples, the yield on 10-year bonds went from slightly above 4% in early 2007 to close to 6% in late-2008 to early-2009. Although the situation improved temporarily in 2009, the yield on 10-year Greek debt shot up again in late-2009, crossing the 10% mark in 2010, while the yield on 10-year Irish debt rose above 6% again in late-2010.

to service their debt, and some of these countries actually needed a bailout. During the crisis, the price of the public debt of the PIGS dropped sharply, while, taking Greece as an example, the volume of trade on the secondary markets for public debt plummeted.[10] In the mean time the price of debt issued by the French and Dutch governments and German federal government rose.

It thus appears that the events of the 2007-8 subprime financial crisis and the subsequent European sovereign debt crisis can be modeled as situation where the average risk of the market portfolio increases. In light of proposition 5 in the main, the sharp drops in prices of the senior tranches of CDOs observed during the 2007-8 subprime financial crisis can have been the consequence of the combination of two factors. First, the increased in risk associated with CDOs would itself have pushed down the price. But a general equilibrium effect might also have been at work. If the new risk level of these CDOs was above the cutoff $\widehat{\epsilon}$, then the price decrease can also be explained by the rotation of the asset price function which lowered the price of all assets with risk above the cutoff.

The properties of the change in the asset price function following an increase in portfolio risk established in proposition 5 can also help explain a number of features observed during the European sovereign debt crisis. The model suggests, and this is not surprising, that the significant increase in the price of the debt of the PIGS fell sharply because of a very large increase in ϵ from a low or moderately high level (in the case of Ireland, Spain, and Portugal), or a very high level (in the case of Greece). But in addition, the rotation of the asset price function triggered by the increase in portfolio risk which pushed further down the price of asset beyond a certain level of risk. In fact, if we take the evolution of the debt-to-GDP ratio of a country as a measure of the risk associated with the country's sovereign debt,[11] the increase in the risk of the public debts of the PIGS can be interpreted as being due to large increases in public deficits that would lead to very large increases in debt-to-GDP ratios.[12] On the other hand, the fall of the yield on the public debts of countries like France, Germany or the Netherlands, despite an increase in their risk as signaled by the rise in their debt-to-GDP ratios,[13] can be explained by the fact that these countries' risk levels were considered relatively low, that is below the cutoff $\widehat{\epsilon}$ in the model. Hence, these countries' public debt benefited from the flight to safety that drove up all asset prices with risk level below the cutoff.

It is also worth highlighting that changes in the spreads between the yields on two assets in response to the increase in portfolio risk come from two sources. First, if the risk associated with the sovereign debt of a nation rises more than that of another nation, as has been the case with most country members of the EMU relative to Germany, then the increase in the difference in their individual risk leads to an increase in the price difference between the two types of claims. In addition, the general equilibrium effect that triggers the rotation of the asset pricing function leads to an increase in spreads for a given difference in risk. This can help explain why a country like Italy, whose debt-to-GDP worsened less than that of France both in relative

[10] Data from HDAT (Electronic Secondary Securities Market for government or government linked fixed-income debt securities) indicate that the volume of trade averaged close to 53 trillion euros from 2001 to 2007, dropped to less than 30 trillion euros in 2008 and 2009, 8 trillion euros in 2010, and less than 1 trillion euros since.

[11] The debt-to-GDP ratio is clearly an imperfect indicator of the risk of the sovereign debt of a country, but this analysis is only meant to be illustrative.

[12] The debt-to-GDP levels of public debt for Portugal, Ireland, Greece, and Spain were respectively 63.9%, 24.8%, 106%, and 39.6% in 2007, while in 2014 they were 129.7%, 123.2%, over 170%, and 92.1%. For the Eurozone as a whole the change was from 68.5% to 90.9% of GDP, a 32.7% increase.

[13] The debt-to-GDP ratios for France, Germany, and the Netherlands increased respectively from 63.7%, 67.6%, and 47.4% in 2007 to 92.3%, 77.1%, and 68.6%, in 2014.

and absolute terms,[14] saw the spread on its sovereign debt relative to that of Germany rise significantly more than that of France.

[14] Italy's debt-to-GDP ratio started high at 106.6% in 2007, and was "only" 20.5% higher in 2014 at 128.5%.