5-2019

Equal-quantile rules in resource allocation with uncertain needs

Yan LONG

Jay SETHURAMAN

Jingyi XUE
Singapore Management University, JYXUE@smu.edu.sg

SMU ECONOMICS & STATISTICS

Equal-quantile rules in resource allocation with uncertain needs

Yan Long, Jay Sethuraman, Jingyi Xue

December 2020

Paper No. 11-2019

Equal-quantile rules in resource allocation with uncertain needs*

Yan Long
Huazhong University of Science and Technology
ylong@hust.edu.cn

Jay Sethuraman
Columbia University
jay@ieor.columbia.edu

Jingyi Xue
Singapore Management University
jyxue@smu.edu.sg

December 8, 2020

Abstract

A group of agents have uncertain needs on a resource, which must be allocated before uncertainty resolves. We propose a parametric class of division rules we call *equal-quantile rules*. The parameter λ of an equal-quantile rule is the maximal probability of satiation imposed on agents — for each agent, the probability that his assignment is no less than his realized need is at most λ. It determines the extent to

*We are grateful for useful comments and suggestions from Christopher P. Chambers, Shurojit Chatterji, Johannes Hörner, Atsushi Kajii, Asen Kochov, Juan Moreno-Ternero, Hervé Moulin, Thomas Sargent, Maxwell B. Stinchcombe, Satoru Takahashi, William Thomson, John Weymark, and Yongchao Zhang. In addition, we thank seminar participants at the Meeting of the Society for Social Choice and Welfare in Seoul, NUS Workshop on Game Theory in Singapore, SAET Conference on Current Trends in Economics in Faro, Economic Design Conferences in York and Budapest, GRODE Workshop on Claims Problems in Reus, SMU Workshop on Mechanism Design in Singapore, SMU Economic Theory Workshop in Singapore, Wang Yanan Institute for Studies in Economics at Xiamen University in Xiamen, Department of Economics at University of Hawaii in Hawaii, and the Asia Meeting of the Econometric Society at Xiamen University in Xiamen for their feedback and critical comments.

which the resource should be used to satiate agents. If the resource is no more than the sum of the agents' λ-quantile assignments, it is fully allocated and the rule equalizes the probabilities of satiation across agents. Otherwise, each agent just receives his λ-quantile assignment. The equal-quantile class is characterized by four axioms, *conditional strict ranking*, *continuity*, *double consistency*, and *coordinality*. All are variants of familiar properties in the literature on deterministic fair division problems. Moreover, the rules are optimal with respect to two utilitarian objectives. The optimality results not only provide welfare interpretations of λ, but also show how the rules balance the concerns for generating waste and deficit across agents.

Keywords: *Resource allocation, Fair division, Uncertain needs, Equal-quantile rules, Utilitarian social welfare function, Waste, Deficit, Coordinality*

***JEL* classification:** D44, D63, D71, D82.

1 Introduction

How to divide a resource among agents who have conflicting claims on the resource has been extensively studied since O'Neill (1982). However, the studies have been largely limited to situations with deterministic claims. Little has been done regarding situations in which agents have uncertain claims that arise from their uncertain needs. If a resource can be divided after uncertainty resolves, then the division rules developed in the deterministic context can be applied to the realized needs. But in many real-life situations, a resource has to be divided ex-ante, and ex-post reallocation is not an option.

For example, an international emergency management organization distributes rescue forces/medical supplies among assistance centers across the world in preparation for random emergency strikes. Since emergency response is time sensitive, transferring unused rescue supplies from one area to another area may have little effect. A government allocates budgets to local authorities to finance local public facilities (public hospitals or roads) with a rough knowledge of local public demands (Copas, 1993). Due to technological constraints, it could be costly to downsize an underutilized public facility or expand an over-demanded one. An academic institute divides grants among various departments to support research activities (seminars or conferences) based on an estimate of the number of participants. Due to institutional constraints, there may be no flexibility in reallocating grants among the departments to match the realized numbers of participants in the various activities.

Formally, there is a one-dimensional and perfectly divisible resource. By agent's "need" for the resource we mean his *satiation point*. When he gets less than he needs, he is better off by consuming more of the resource; as soon as his need is satisfied, he is indifferent to any increase of the resource. The uncertain need of an agent is represented by a cumulative distribution function (CDF), called a *claim*. The minimal value of its support is the *sure need*, and the maximal value the *maximal need*. Agents' claims are objectively verifiable and non-manipulable. A *problem* for a population of agents consists of the profile of claims and a non-negative endowment of the resource. An *allocation* is a vector of assignments such that no agent gets more than his maximal need, and the sum of the assignments is *no more than* the endowment. A *rule* chooses, for each population and each problem this population may face, an allocation. We search for desirable rules.

Difficulties arise when needs are uncertain and division is committed. If the amount of the resource assigned to an agent turns out to exceed his realized need, some of the resource is wasted; if the assigned amount turns out to fall short of his realized need, he still lacks for some of the resource. Reducing the risk of generating waste necessarily increases the risk of generating deficit, and vice versa. This raises two important questions.

First, to what extent should the resource be used to satiate agents? Full use of the resource is required in the deterministic fair division literature because of efficiency considerations (Thomson (2017)).[1] But when satiation points are uncertain, waste is typically inevitable, and it could generates an opportunity cost, impairing efficiency (see Section 5). Thus, our feasibility only requires the sum of the assignments not to exceed the endowment, and a rule should recommend the extent to which the resource is used to satiate agents.

Second, how should the concerns for waste and deficit be balanced across agents with different claims? Due to differences in the probability distributions of agents' needs, the same assignment to different agents induces different risks of waste and deficit. Reducing the risk of waste/deficit for one agent may well increase that for another. Thus, a rule should also provide a way to deal with these trade-offs across agents.

To address the two questions, we introduce a class of rules we call *equal-quantile rules*,

[1]In particular, all agents should be satiated if there is enough of the resource, and the resource should be fully allocated if it is limited.

parameterized by $\lambda \in [0, 1]$. The equal-quantile rule with λ works as follows. When the endowment is no more than the sum of the sure needs, the classical "constrained equal awards" method is applied to the profile of the sure needs.[2] The rule fully allocates the resource and makes assignments as equal as possible, subject to the constraint that no agent receives more than his sure need. When the endowment exceeds the sum of the sure needs, each agent receives an α-quantile assignment, where α is the same for all agents and is maximized subject to the feasibility constraint and to the constraint that it not exceed λ. Loosely, α is a common "probability of satiation" — for each agent, the probability of his need being covered by his assignment is α. The maximal common probability of satiation λ determines the extent to which the resource should be used to satiate agents. Each agent's maximal assignment is his λ-quantile assignment. The resource is fully allocated only if it is no more than the sum of the maximal assignments; otherwise, each agent just receives his maximal assignment. Generally, agents are never satiated for sure when $\lambda < 1$.

Four axioms characterize the equal-quantile class. All are variants of properties that have been discussed in the literature on deterministic fair division problems. The first is *conditional strict ranking*. It says that if an agent's claim is larger than another agent's according to a strict first-order stochastic dominance (FSD) criterion,[3] and if the latter agent is assigned a positive amount, then the former agent should be assigned a larger amount. The second is *continuity*. It requires the allocation not to change too much if the claims and the endowment do not change too much. The other two axioms, *double consistency* and *coordinality*, are invariance properties.

Double consistency, like the standard consistency property, pertains to the possibility that after an allocation is chosen for a problem, some agents come first and take away their assignments. At that point, if the remaining agents, who have not received their assignments yet, re-divide what is left, then each of them should receive his initial assignment. In our model, allowing partial use of the resource leads to an interesting twist in the standard property. Indeed, depending on whether the unassigned resource has been given away to an alternative use, what is left is either the sum of the remaining agents' initial assignments or the difference between the initial endowment and the sum of the assignments of those who leave. *Double consistency* requires the assignments to the remaining agents to be invariant

[2]The constrained equal awards method is also known as the uniform gains method in the literature.

[3]There are different versions of strict FSD, depending on in which sense a CDF is defined to be larger than another.

in both situations since both are plausible.

Coordinality requires the way of allocating the resource to be invariant with respect to a common increasing and continuous transformation of claims and endowment. It is often the case in real life that there is an underlying production technology that converts the resource to some output and agents' needs for the resource come from their needs for the output. Suppose that due to a technology shock, the production function has undergone an increasing and continuous transformation, which gives each agent a correspondingly transformed claim on the resource. Assume also that the endowment is reset to be the sum of the agents' initial assignments subjected to the same transformation, so that their needs for the output can be met as they were initially. In this case, we require that agents receive their transformed assignments and have their needs for the output met as initially.

Going back to our first question, while the axioms characterize the rules that set a maximal probability of satiation λ to determine the extent to which a resource should be used to satiate agents, a further examination of the optimality of the rules suggests how λ should be chosen. Assume that each agent obtains a common and constant marginal utility $u > 0$ from each unit of his assignment capped at his realized need. Assume also that there is an alternative way of using the resource outside the model, and this outside agent obtains a constant marginal utility $v \in [0, u]$ from each unit of the leftover resource. The equal-quantile rule with $\lambda = \frac{u-v}{u}$ selects, for each problem, an allocation that maximizes the expected sum of the utilities of all agents including the outside agent. As the ratio of v to u gets smaller, using more of the resource within the model improves the social welfare, so the maximal probability of satiation λ gets larger. As long as $v > 0$, agents should never be satiated for sure, i.e., $\lambda < 1$. This suggests that to choose λ, a planner should explore, outside the data of a problem, more information on alternative uses of the resource.

Going back to our second question, an equivalent cost-minimization objective explicitly shows how the rules balance the concerns for waste and deficit across agents, although their characterizing axioms, originating in the deterministic fair division literature, have nothing to do with waste or balance. Assume that each unit of waste incurs a constant marginal cost $c^w \geq 0$ and deficit $c^d \geq 0$, where $c^w + c^d > 0$. The costs can be understood as opportunity costs generated by an allocation (see Section 5). The equal-quantile rule with $\lambda = \frac{c^d}{c^w + c^d}$ minimizes the sum of the aggregate expected waste and the aggregate expected deficit, weighted, respectively, by c^w and c^d. The ratio of c^w to c^d reflects the balance between waste and deficit, and determines the maximal probability of satiation as analogous to the ratio of v and u. The objective function also shows that the trade-offs across agents are achieved in a utilitarian manner.

Lastly, an important result that helps to establish our characterization is that three familiar axioms, when extended to the uncertain context, provide a guideline on assessing the extent to which the resource should be used to satiate agents. They are, respectively, *symmetry*, which requires agents with equal claims to receive equal assignments; *endowment continuity*, which requires a rule to be continuous in endowment; and *double consistency*. They together imply that for each agent, a maximal assignment that depends only on his claim should be imposed, and the resource be fully allocated when and only when it does not exceed the sum of the agents' maximal assignments.

1.1 Related literature

Resource allocation with deterministic claims has been extensively studied from a normative perspective in the rationing/bankruptcy literature.[4] However, there are only a few axiomatic studies on uncertain needs. The most closely related papers are Xue (2018a,b) and Long and Xue (2019). Xue (2018a,b) focuses on the issue of waste and proposes axioms that explicitly address how the risk of generating waste should affect the resource allocation. In contrast, we address the trade-offs between waste and deficit, and our axioms are extensions of existing ones in the deterministic fair division literature. Moreover, while Xue (2018a,b) assumes that the endowment never exceeds the sum of the maximal needs and has to be fully allocated, we allow the endowment to be arbitrarily large and partially allocated.

Long and Xue (2019) extend the class of "parametric" rules, an important class introduced by Young (1987a), to the uncertain context.[5] Generalizing Young's (1987a) results

[4] For example, the constrained equal awards rule and the constrained equal losses rule are studied by Sprumont (1991), Dagan (1996), Herrero and Villar (2001, 2002), Yeh (2004, 2006, 2008), Martínez (2008), and Marchant (2008). The proportional rule is studied by Banker (1981), O'Neill (1982), Moulin (1987), and Chun (1988). The class of equal-sacrifice rules is studied by Young (1987b, 1988, 1990); a class of generalized equal-sacrifice rules is further studied by Chambers and Moreno-Ternero (2017). The TAL-family is studied by Moreno-Ternero and Villar (2006); the ICI and CIC families are studied by Thomson (2008). Excellent surveys are provided by Moulin (2002) and Thomson (2003, 2015).

[5] Kaminski (2000, 2006) extends Young (1987a) to a setting where agents have abstract types. Different

on the deterministic domain, they characterize the extended class and show that each parametric rule minimizes a cost function. Our equal-quantile class is a subclass of the extended parametric class, and is characterized with two additional axioms (*conditional strict ranking* and *coordinality*). Moreover, the optimality of the equal-quantile rules is not a corollary of their result, since the cost-minimization objective of an equal-quantile rule, which has an interpretation in terms of balancing waste and deficit, cannot be constructed using their method.

We are also aware of the following axiomatic studies on resource allocation with uncertain needs. Yager and Kreinovich (2000) model uncertain needs as intervals and characterize a version of the proportional rule (see also Branzei, Dimitrov, Pickl and Tijs (2004), Woeginger (2006)). Ertemel and Kumar (2018) study state-contingent needs, and characterize the "ex-ante" and "ex-post" proportional rules. Those works, having different focuses, do not study the conflict between waste and deficit under uncertainty.

In contrast with the limited number of normative studies on resource allocation with uncertain needs, there is a rich operations research literature on this subject, especially in the areas of inventory management, emergency control, project management, and network design (e.g., Arrow, Harris, and Marschak (1951), Qin, Wang, Vakharia, Chen, and Seref (2011), Johansson and Sternad (2005), Rawls and Turnquist (2010), Turnquist and Nozick (2003), Wex, Schryen, and Neumann (2012)). There, an objective function is typically assumed, and the focus is on finding algorithms to solve for optimal allocations.

Among these studies, the closest to ours is the newsvendor problem. It consists of a probabilistic demand on a perishable good (e.g., newspaper) represented by a CDF F_i, a unit purchasing price c, and a unit selling price $p \geq c$. A manager selects an amount to stock to maximize expected profit. The optimal stock, given by what is known as Littlewood's rule (Littlewood (1972)), is $F_i^{-1}\left(\frac{p-c}{p}\right)$. It depends on the marginal cost of overstocking, c, and that of understocking, $p-c$, in the same way as the maximal assignments to our agents depend on the marginal costs of waste and deficit. The main difference is that we focus on the axiomatic foundation of our rules. Moreover, our optimization problem generalizes the newsvendor problem from an unconstrained single-agent problem to a constrained multi-agent one.

from Long and Xue (2019), both Young (1987a) and Kaminski (2000, 2006) assume that each agent has an exogenously given upper bound on his assignment, and the resource, never exceeding the sum of the upper bounds, has to be fully allocated.

Our characterization is mostly related to Chambers (2009). In an abstract environment where the primitive is the space of CDFs, Chambers (2009) studies functions on this space and characterizes the quantile functions of CDFs by the axioms of *monotonicity*, *upper semicontinuity*, and *ordinal covariance*.[6] These axioms resemble our *conditional strict ranking*, *continuity*, and *coordinality*. Besides, our equal-quantile rules, when restricted to single-agent problems with sufficiently large endowments, are the quantile functions of CDFs. But, in general, we deal with the problem of dividing a limited resource among a group of agents with different claims, which is beyond the scope of Chambers (2009)'s application. Moreover, due to the difference between the two models, the restrictions imposed by the axioms are different. For example, *ordinal covariance* of Chambers (2009) deals with transformations of one individual CDF; our *coordinality* deals with *common* transformations of all claims and the endowment. While each continuous CDF can be transformed to any other CDF in an increasing and continuous manner, not every profile of CDFs can be transformed to any other profile. Thus, the techniques used in deriving the implications of the axioms are different.

Our model and rules are also related to Moreno-Ternero and Roemer (2006, 2012). They study how to divide a resource among agents who are equipped with output functions that convert the assigned resource into outputs. They characterize, respectively in the two papers, a class of "index-egalitarian" rules,[7] and two extreme rules in this class, one of which equalizes the outputs of agents. Our agents' claims are CDFs which play the role of their output functions, and each equal-quantile rule equalizes the probabilities of satiation among agents. But claims and output functions are different objects and there are two main differences between their characterization and ours. First, their characterizations invoke a no-domination and an additivity axiom ("*priority*" and "*composition down*", respectively), while ours does not invoke any no-domination- and additivity-type requirements. Second, in their model, in which there is no uncertainty, a resource is always fully allocated and no upper bound is imposed on an agent's assignment. Our agents have uncertain satiation points, and our rules recommend how much of the resource should be used to satiate agents.

[6]One application is related to risk measures in the insurance literature (Wang, Young, and Panjer (1997) and Artzner, Delbaen, Eber, and Heath (1999)). Thomson (1979) also studies a problem in which distribution functions are agents' private information, and characterizes all the payment schemes that incentivize an agent to truthfully reveal a pre-specified quantile of his distribution.

[7]Chun, Jang, and Ju (2014) provide alternative characterizations in a fixed-population setting.

Quantile decision rules are also used in individual choices under uncertainty. For example, Rostek (2010) axiomatizes the quantile-maximization rule in a Savage setting; de Castro and Galvao (2019) axiomatize a class of recursive quantile preferences in a dynamic model. Moreover, distributive justice under uncertainty has also been explored in the other contexts. For example, cost sharing of risky projects is studied by Hougaard and Moulin (2018), ex-ante egalitarian division of a stochastic cost is studied by Koster and Boonen (2019), and social welfare orderings that assess risky social situations are studied by Fleurbaey (2010) and Fleurbaey, Gajdos and Zuber (2015).

2 The model

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ non-negative real numbers, $\mathbb{R}_{++}$ positive real numbers, and $\mathbb{N}$ positive integers. Let $\mathcal{N}$ be the set of all finite subsets of $\mathbb{N}$. Potential agents are labeled by elements of $\mathbb{N}$. A **population** is an element of $\mathcal{N}$.

A one-dimensional and perfectly divisible resource is to be allocated among a population of agents, each of whom needs some of the resource. By the "need" of an agent we mean his **satiation point**. When the resource that he obtains falls short of his need, he is better off with more of the resource, and as soon as he obtains what he needs, he becomes indifferent to any further increase of the resource. The need of an agent is uncertain in general and is represented by a CDF, called the **claim** of the agent. We assume that the support of each claim is a bounded interval in $\mathbb{R}_+$. Our main results remain true if we drop the boundedness assumption; interested readers are referred to the online appendix for details. The interval assumption restricts our attention to CDFs that are increasing on their support; it simplifies our analysis.[8] Let $\mathcal{F}$ be the set of such claims. We denote a typical element of $\mathcal{F}$ by F_i. Given $F_i \in \mathcal{F}$, we denote, respectively, by c_i and C_i the minimal and maximal values of the support of F_i, omitting the dependent variable F_i in the notation. Note that c_i is the amount of the resource that agent i needs for sure, and is called the **sure need** of agent i; C_i is the maximal amount of the resource that agent i would possibly need, and is called the **maximal need** of agent i. Given $I \in \mathcal{N}$, we denote by F a typical claim profile for population I, i.e., $F = (F_i)_{i \in I}$ where for each $i \in I$, $F_i \in \mathcal{F}$. For each $I \in \mathcal{N}$, we denote

[8]This assumption excludes an important case of discrete needs (except for deterministic needs), which deserves further research.

by $\mathcal{F}^I$ the set of claim profiles for population I.

Let $I \in \mathcal{N}$. A claims problem, or simply a **problem**, for population I is a pair (F, T), where $F \in \mathcal{F}^I$ is a claim profile and $T \in \mathbb{R}_+$ is an **endowment** of the resource.[9] Let $\mathcal{P}^I$ be the set of all problems for population I. An **allocation** for $(F, T) \in \mathcal{P}^I$ is a vector $t \in \mathbb{R}^I_+$ such that for each $i \in I$, t_i, which is called the **assignment** to agent i, is no larger than his maximal need C_i, and $\sum t_i \leq T$. An original feature of the model in which agents have uncertain satiation points is that it may no longer be desirable to make full use of the resource to satiate the agents.[10] Our rules will advise a planner, who faces a feasibility constraint, on whether and to what extent agents should be satiated.

Let $i \in \mathbb{N}$ and $F_i \in \mathcal{F}$. Let $t_i \in [0, C_i]$ be an assignment to agent i. When t_i exceeds agent i's realized need x_i, agent i only uses x_i units of the resource and the extra amount $t_i - x_i$ is called the **waste** generated by agent i. When x_i exceeds t_i, he uses t_i units of the resource, and $x_i - t_i$ is called the **deficit** of agent i. We call $F_i(t_i)$, the probability that agent i's need is no more than his assignment, agent i's **probability of satiation**. When $t_i = C_i$, agent i is said to be **satiated for sure**. A division rule, or simply a **rule**, is a function $\boldsymbol{r}$ that specifies for each problem in $\bigcup_{I \in \mathcal{N}} \mathcal{P}^I$ an allocation. For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, we denote by $r_i(F, T)$ agent i's assignment given by r.

Our problems do not contain information on the underlying joint distributions of agents' needs. There is no loss when agents' needs are subject to idiosyncratic risk. When their needs are correlated, it is an open question how the resource allocation should depend on the correlation (see Section 6), but still, our model applies in the following scenarios. First, an allocation has to be chosen based on marginal distributions if the underlying joint distribution is too complicated to obtain compared with the marginal distributions. Second, when a planner has a utilitarian objective function, the joint distribution is not needed; the optimal allocations depend only on the marginal distributions (see Section 5). Third, if a planner thinks that an agent should not be responsible for things that are beyond his individual claim, how claims are correlated is of no concern to the planner.

[9]In the literature of deterministic claims problems, it is commonly assumed that T is no more than the sum of the agents' deterministic claims. We do not impose this restriction in our model, and we allow the endowment not to be fully allocated.

[10]See the Introduction and Section 5.

3 Equal-quantile rules

We propose a class of rules which we call equal-quantile rules. Each equal-quantile rule is associated with a parameter $\lambda \in [0, 1]$, and is composed of two parts depending on the endowment. First, when the endowment does not exceed the sum of the agents' sure needs, the well-known constrained equal awards method is applied to the profile of sure needs. That is, the rule makes assignments as equal as possible subject to the constraint that no agent receives more than his sure need. In particular, when all agents have deterministic claims, the rule selects the constrained equal awards allocation. Second, when the endowment exceeds the sum of the agents' sure needs, a new method that we propose is applied. Loosely speaking, the allocation is calculated by equating probabilities of satiation across agents, and the common probability of satiation is maximized subject to the feasibility constraint and to the constraint that it not exceed the parameter λ. Thus, λ serves as an upper bound on the common probability of satiation; it has no impact on the first part of the rule.

Example 1. We illustrate the equal-quantile rule with $\lambda = 0.75$ in Figure 1 using the following problem. There are two agents, 1 and 2, whose claims are, respectively,

$$F_1(x_1) = \begin{cases} 0 & x_1 < 1.75 \\ \frac{1}{7}(x_1 - 1.75) + 0.25 & 1.75 \le x_1 < 6.5 \\ 1 & x_1 \ge 6.5 \end{cases} \text{ and } F_2(x_2) = \begin{cases} 0 & x_2 < 2 \\ \frac{1}{2}(x_2 - 2)^{\frac{1}{2}} & 2 \le x_2 < 6 \\ 1 & x_2 \ge 6 \end{cases}.$$

Thus, agent 1 has a sure need of 1.75 and agent 2 has a sure need of 2.

When the endowment does not exceed the sum of their sure needs, 1.75 + 2, the constrained equal awards method is applied to the sure needs. If $T = 2$, the assignments are $t_1 = t_2 = 1$. When the endowment exceeds 1.75 + 2, loosely speaking, the rule chooses the maximal allocation that equates probabilities of satiation across agents, subject to the feasibility constraint and to the upper-bound constraint on the probabilities of satiation. If $T = 4$, the assignments are $t'_1 = 1.75$ and $t'_2 = 2.25$, since $F_1(t'_1) = F_2(t'_2) = 0.25 < \lambda$ and $t'_1 + t'_2 = 4$. Since agent 1's claim is discontinuous at 1.75, when T increases from 1.75 + 2 to 4, his assignment remains unchanged at 1.75 and his probability of satiation is constant and equal to 0.25; all the resource increment goes to agent 2, and agent 2's probability of satiation increases from 0 to 0.25.[11] If $T = 6.5$, the assignments are $t''_1 = 3.5$ and $t''_2 = 3$,

[11]When $1.75 + 2 \le T < 4$, the rule gives the agents different probabilities of satiation, but still, there is a sense in which their probabilities of satiation are "approximately" the same, as is to be elaborated later.

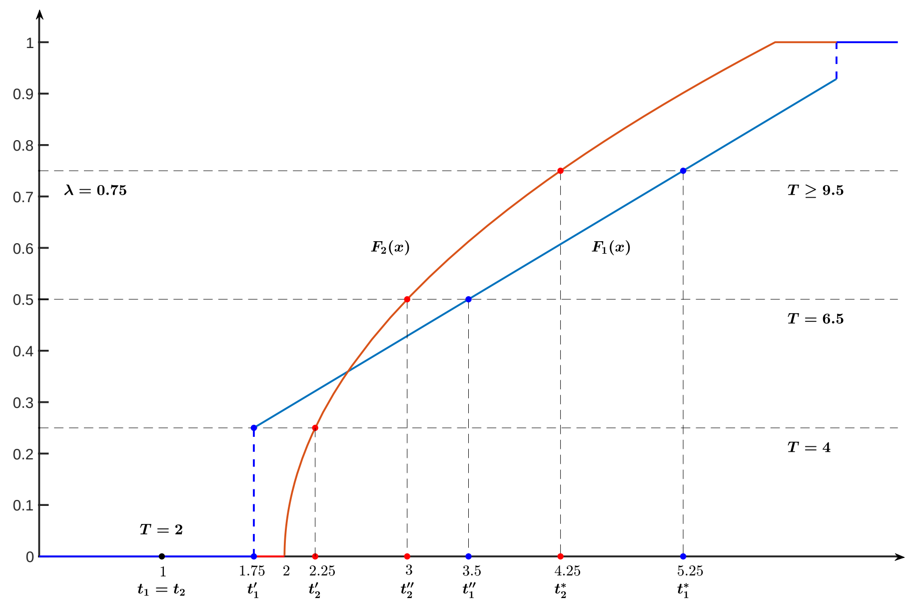

Figure 1: The equal-quantile rule with parameter $\lambda = 0.75$

since $F_1(t_1'') = F_2(t_2'') = 0.5 < \lambda$ and $t_1'' + t_2'' = 6.5$. When the endowment exceeds 9.5, the assignments remain constant at $t_1^* = 5.25$ and $t_2^* = 4.25$, since $F_1(t_1^*) = F_2(t_2^*) = \lambda$ and λ is the maximal probability of satiation imposed by the rule. Thus, the parameter of the rule determines the maximal assignments of the agents.

To define equal-quantile rules, we introduce, for each claim that an agent may have, a quantile function that returns for each probability $\alpha \in [0, 1]$ an assignment that gives an α probability of satiation to the agent (possibly in an approximate sense as will be elaborated later). Formally, for each $F_i \in \mathcal{F}$, the **quantile function** $Q_{F_i} : [0, 1] \to \mathbb{R}$ is defined by setting, for each $\alpha \in [0, 1]$,

$$Q_{F_i}(\alpha) := \begin{cases} c_i & \alpha = 0 \\ \min\{x_i \in \mathbb{R} : F_i(x_i) \geq \alpha\} & \alpha \in (0, 1] \end{cases}. \tag{1}$$

Since F_i is right-continuous and non-decreasing, and since F_i has bounded support, when $\alpha \in (0, 1]$, the minimum operator in (1) is well defined. Note that for each $\alpha \in [0, 1]$, $Q_{F_i}(\alpha) \in [c_i, C_i]$, and thus, $Q_{F_i}(\alpha)$ is a valid assignment. Moreover, it can be verified that

Q_{F_i} is non-decreasing and continuous.[12] If F_i is continuous, Q_{F_i} returns for each $\alpha \in [0,1]$ the assignment in $[c_i, C_i]$ that induces an α probability of satiation, i.e., Q_{F_i} is the inverse function F_i^{-1} with $F_i^{-1}(0) := c_i$ and $F_i^{-1}(1) := C_i$. If F_i is discontinuous at $Q_{F_i}(\alpha)$ for some $\alpha \in [0,1)$, the assignment $Q_{F_i}(\alpha)$ may induce a probability of satiation larger than α. In example 1, when $\alpha = 0.2$, agent 1's probability of satiation given by $Q_{F_1}(\alpha)$ is 0.25. In this case, α is an "approximate" probability of satiation in the sense that any smaller assignment induces a probability of satiation smaller than α, and any larger assignment a probability larger than α.[13]

Equal-quantile rules: For each $\lambda \in [0,1]$, let $\boldsymbol{r}^{\boldsymbol{\lambda}}$ denote the **equal-quantile rule with parameter $\boldsymbol{\lambda}$**. For each $I \in \mathcal{N}$ and each $(F, T) \in \mathcal{P}^I$, when $T \leq \sum c_i$, for each $j \in I$,

$$r_j^\lambda(F,T) := \min\{c^*, c_j\}, \text{ where } c^* \in \mathbb{R}_+ \text{ satisfies } \sum \min\{c^*, c_i\} = T;$$

when $T > \sum c_i$, for each $j \in I$,

$$r_j^\lambda(F,T) := Q_{F_j}(\alpha^*), \text{ where } \alpha^* \in [0,\lambda] \text{ satisfies } \sum Q_{F_i}(\alpha^*) = \min\left\{T, \sum Q_{F_i}(\lambda)\right\}.$$

The rule r^λ is well defined. When $T \leq \sum c_i$, there is $c^* \in \mathbb{R}_+$ satisfying $\sum \min\{c^*, c_i\} = T$, and all values of c^* satisfying the condition give rise to the same allocation since the minimum operator is non-decreasing in c^*. When $T > \sum c_i$, note the following. First, if $T \geq \sum Q_{F_i}(\lambda)$, then the condition $\sum Q_{F_i}(\alpha^*) = \min\{T, \sum Q_{F_i}(\lambda)\}$ is satisfied with $\alpha^* = \lambda$. Second, if $T < \sum Q_{F_i}(\lambda)$, since $\sum Q_{F_i}(0) = \sum c_i < T < \sum Q_{F_i}(\lambda)$ and $\sum Q_{F_i}$ is continuous, there is $\alpha^* \in (0, \lambda)$ satisfying the condition. Lastly, all values of α^* satisfying the condition give rise to the same allocation since quantile functions are non-decreasing in α^*.

Each equal-quantile rule sets a parameter $\lambda \in [0,1]$ to determine the extent to which agents should be satiated. The maximal assignment to each agent i is $Q_{F_i}(\lambda)$. When $\lambda < 1$, in general, agents are never satiated for sure, even if there is enough of the resource.[14] Moreover, even if the resource is limited, i.e., even if it falls short of the sum of the agents' maximal needs, as long as it exceeds the sum of the agents' maximal assignments, it is not fully allocated. How to choose the parameter is further addressed in Section 5.

[12] The continuity of Q_{F_i} relies on the fact that F_i is increasing on $[c_i, C_i]$.

[13] In Example 1, when $T = Q_{F_1}^{-1}(0.2) + Q_{F_2}^{-1}(0.2)$, the rule gives agent 1 an approximately 0.2 probability of satiation and agent 2 an exactly 0.2 probability of satiation; the probabilities are approximately the same.

[14] An agent could be satiated for sure when his claim is discontinuous at his maximal need.

Equal-quantile rules belong to Young's (1987a) parametric class appropriately extended to the uncertain context. According to the definition of Long and Xue (2019), each extended parametric rule determines, in addition to a parametric way of rationing as in the deterministic context, a maximal assignment to each agent.[15] To be precise, let $\underline{\alpha}, \overline{\alpha} \in \mathbb{R} \cup \{-\infty, \infty\}$ with $\underline{\alpha} < \overline{\alpha}$, and let $f : \mathcal{F} \times [\underline{\alpha}, \overline{\alpha}] \to \mathbb{R}$ be such that for each $F_i \in \mathcal{F}$, $f(F_i, \cdot)$ is non-decreasing and continuous with $f(F_i, \underline{\alpha}) = 0$ and $f(F_i, \overline{\alpha}) \leq C_i$. The **parametric rule with** $\boldsymbol{f}$, denoted by $\boldsymbol{r^f}$, is defined by setting for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $j \in I$,

$$r_j^f(F, T) := f(F_j, \alpha^*), \text{ where } \alpha^* \in [\underline{\alpha}, \overline{\alpha}] \text{ satisfies } \sum f(F_i, \alpha^*) = \min\Big\{T, \sum f(F_i, \overline{\alpha})\Big\}.$$

For each $\lambda \in [0, 1]$, define $f_\lambda : \mathcal{F} \times [-1, \lambda] \to \mathbb{R}_+$ by setting for each $F_i \in \mathcal{F}$ and each $\alpha \in [-1, \lambda]$,

$$f_\lambda(F_i, \alpha) := \begin{cases} \min\Big\{-\dfrac{1}{\alpha} - 1, c_i\Big\} & \alpha \in [-1, 0) \\ Q_{F_i}(\alpha) & \alpha \in [0, \lambda] \end{cases}.$$

Note that f_λ is non-decreasing and continuous in α, $f_\lambda(F_i, -1) = 0$, and $f_\lambda(F_i, \lambda) = Q_{F_i}(\lambda) \leq C_i$. It can be readily verified that the equal-quantile rule r^λ is the parametric rule r^{f_λ}.

4 Axiomatic foundation

4.1 Axioms

A basic fairness principle is Aristotle's "equal treatment of equals". It requires that agents who have equal claims receive equal assignments (Thomson (2003)).

Symmetry: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $i, j \in I$, if $F_i = F_j$, then $r_i(F, T) = r_j(F, T)$.

A familiar order-preservation principle is that a rule should respect the sizes of agents' claims: If an agent's claim is at most as large as another agent's, then he should receive at most as much as the other does (Aumann and Maschler (1985), Thomson (2003)). To compare the sizes of claims in the uncertain context, we adopt the commonly-used FSD

[15] Based on this definition, Long and Xue (2019) extend the results of Young (1987a).

criterion. Formally, for each pair $F_i, F_j \in \mathcal{F}$, we say that F_i is **no smaller than** F_j **in the FSD sense**, denoted by $\boldsymbol{F_i \succsim_{FSD} F_j}$, if for each $c \in \mathbb{R}$, $F_i(c) \le F_j(c)$.[16]

Ranking: For each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{P}^I$, and each pair $i, j \in I$, if $F_i \succsim_{FSD} F_j$, then $r_i(F,T) \ge r_j(F,T)$.

If two claims are equal, then either of them is no smaller than the other in the FSD sense. Thus, *ranking* implies *symmetry*.

A strict version of *ranking*, called *strict ranking*, requires that if an agent's claim is larger than another agent's, then whenever there is something to divide, he receive more than the other does (Young (1988), Thomson (2003)). *Strict ranking* is known to be demanding.[17] In our context, if using the resource on the two agents is less worthwhile — according to some criterion (e.g., expected waste minimization) — than using it on a third agent or outside the model, then it is reasonable to assign neither of the two agents a positive amount. To accommodate this situation, we propose a conditional version of *strict ranking*, requiring the agent who has a larger claim to receive more of the resource only when the agent who has a smaller claim is assigned a positive amount.[18]

The criterion that we invoke for the comparison of claims is a strong version of strict FSD.[19] Formally, for each pair $F_i, F_j \in \mathcal{F}$, we say that F_i is **larger than** F_j **in the strict FSD sense**, denoted by $\boldsymbol{F_i \succ_{FSD} F_j}$, if either $C_j = 0 < C_i$, or $C_j > 0$ and for each $c \in (0, C_j]$, $F_i(c) < \lim_{c' \uparrow c} F_j(c')$. When $C_j = 0$, F_j is a zero claim, and F_i is larger than F_j as long as F_i is non-zero. Consider the case $C_j > 0$. When F_j is continuous, the condition reduces to saying that for each $c \in (0, C_j]$, $F_i(c) < F_j(c)$, i.e., the graph of F_i lies strictly below that of F_j on $(0, C_j]$, without overlap. When F_j is not continuous, to visualize the condition, we first make the graph of each claim connected by filling the vertical gaps at its discontinuity points.[20] See Figure 2. Then $F_i \succ_{FSD} F_j$ holds if and only if the connected

[16] One may also consider an order-preservation requirement with respect to some other order of stochastic dominance. For example, a risk-averse planner would assign a no larger amount to an agent whose claim is riskier in the second-order stochastic dominance sense. This is studied by Xue (2018a).

[17] In the deterministic context, it excludes compelling rules such as the constrained equal awards rule, the constrained equal losses rule, and the Talmud rule.

[18] Similar weakenings of axioms such as *resource monotonicity* and *population monotonicity* have also been studied in the literature (Dagan, Serrano, and Volij (1997), Thomson (2003)).

[19] As pointed out in footnote 3, there are different versions of strict FSD.

[20] That is, for each $k \in \{i, j\}$, the connected graph of F_k is the graph of the correspondence $f_k : \mathbb{R} \rightrightarrows [0, 1]$

graph of F_i lies strictly below that of F_j on $(0, C_j]$; in contrast, $F_i \succsim_{FSD} F_j$ holds as long as the former graph lies weakly below the latter on $(0, C_j]$, no matter whether they overlap or not. The non-overlapping condition ensures that for each positive need c of agent j, not only the probability that agent i needs more than c is larger than that probability for agent j (i.e., $F_i(c) < F_j(c)$), but also this relationship is robust to measurement error in claims.[21] Despite being a strong condition, it makes our axiom weak. In particular, claims with positive sure needs are deemed not to be comparable,[22] and consequently, our axiom imposes no requirement on the deterministic domain.

Conditional strict ranking: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $i, j \in I$, if $F_i \succ_{FSD} F_j$ and $r_j(F, T) > 0$, then $r_i(F, T) > r_j(F, T)$.

Ranking and *conditional strict ranking* are not logically related. However, *conditional strict ranking*, together with a list of additional axioms that includes the following continuity requirement, implies *ranking* (Lemma 2).

The continuity requirement is that small changes in a problem not lead to large changes in the chosen allocation. It ensures that errors in specifying the data of a problem, or corrections of these errors, do not radically affect the recommendation (Young (1987a, 1988), Thomson (2003)). The topology that we adopt to evaluate changes in a problem is based on the following notion of convergence. For each $F_i \in \mathcal{F}$ and each sequence $\{F_i^n\}_{n=1}^{\infty}$ of elements of $\mathcal{F}$, we say that F_i^n **converges to** F_i if F_i^n converges weakly to F_i, $\lim c_i^n = c_i$, and $\lim C_i^n = C_i$.[23] For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each sequence $\{(F^n, T^n)\}_{n=1}^{\infty}$ of elements of $\mathcal{P}^I$, we say that (F^n, T^n) **converges to** (F, T), denoted by $\boldsymbol{(F^n, T^n) \to (F, T)}$, if for each $i \in I$, F_i^n converges to F_i, and $\lim T^n = T$.[24]

Continuity: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each sequence $\{(F^n, T^n)\}_{n=1}^{\infty}$ of elements of $\mathcal{P}^I$, if $(F^n, T^n) \to (F, T)$, then $\lim r(F^n, T^n) = r(F, T)$.

---

defined by setting for each $x_k \in \mathbb{R}$, $f_k(x_k) := [\lim_{x'_k \uparrow x_k} F_k(x'_k), F_k(x_k)]$.

[21]For each $c \in (0, C_j]$, $F_i(c) < \lim_{c' \uparrow c} F_j(c')$ if and only if for any $F'_i, F'_j \in \mathcal{F}$ that are sufficiently close to F_i and F_j, respectively, in the weak topology, $F'_i(c) < F'_j(c)$.

[22]Note that $F_i \succ_{FSD} F_j$ implies $c_j = 0$. This is because if $C_j = 0$, then $c_j = 0$, and if $C_j > 0$, then for each $c \in (0, C_j]$, $0 \leq F_i(c) < \lim_{c' \uparrow c} F_j(c') \leq F_j(c)$, and thus, $c_j = 0$.

[23]This notion of convergence of claims is equivalent to the convergence in a metric that is defined based on the Lévy-Prokhorov metric. See Long and Xue (2019).

[24]This is equivalent to saying that (F^n, T^n) converges to (F, T) in the product topology in the space $\mathcal{F}^I \times \mathbb{R}_+$.

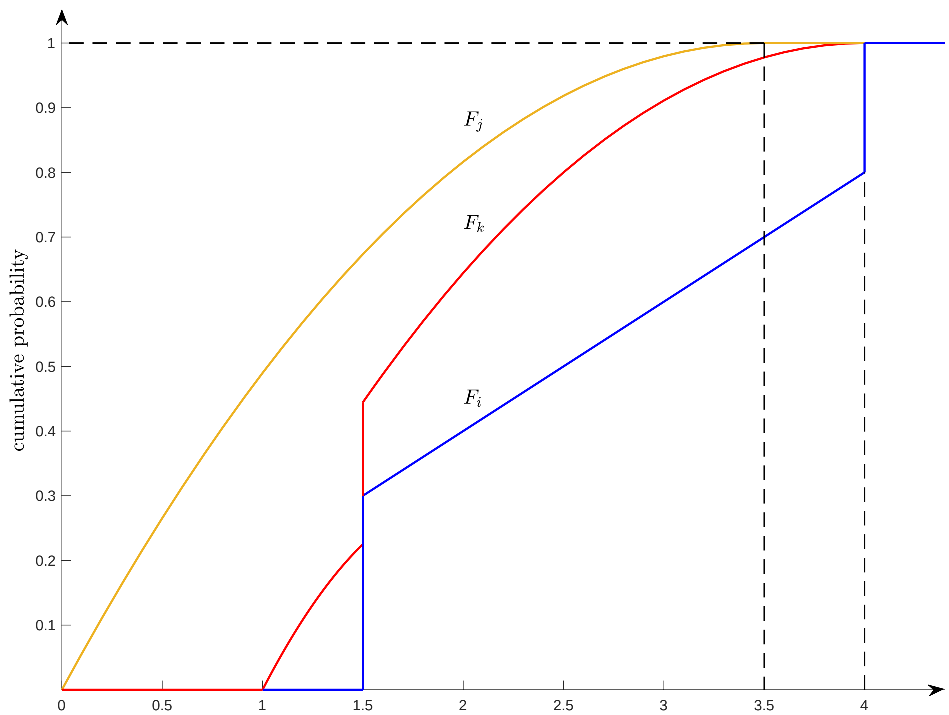

Figure 2: The connected graph of F_i lies strictly below that of F_j on $(0, 3.5]$, which implies $F_i \succ_{FSD} F_j$; it overlaps that of F_k at $(0, 1] \cup \{1.5, 4\}$, and the overlap on any of these points implies that $F_i \succ_{FSD} F_k$ does not hold.

Note that discontinuity may occur when the sure needs or the maximal needs fail to converge. This is a feature of a large class of rules under uncertainty and is allowed by our *continuity*. For example, consider two agents. Agent 1 needs 10 for sure. For agent 2, with probability α, his need is uniformly distributed on $[0, 10]$, and with probability $1-\alpha$, it is uniformly distribution on $(10, 20]$. Pick any division method in the deterministic context that assigns positive amounts to agents who have positive claims. Consider any rule that applies such a deterministic method to the profile of the sure needs whenever the endowment is no more than the sum of the sure needs. When the endowment is 10, as α goes to 0, agent 2's sure need jumps from 0 to a positive amount, and thus, so does his assignment. Moreover, consider any rule that satiates agents for sure whenever feasible. When the endowment is 30, as α goes to 1, agent 2's maximal need jumps from 20 to 10, and thus, so does his assignment.

A weaker *continuity* requirement considers only small changes of the endowment.

Endowment continuity: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each sequence $\{(F, T^n)\}_{n=1}^{\infty}$ of elements of $\mathcal{P}^I$, if $\lim T^n = T$, then $\lim r(F, T^n) = r(F, T)$.

The next axiom states an invariance principle that has played a central role in resource allocation with a variable population (Aumann and Maschler (1985), Young (1987a), Thomson (1988, 2012), Moulin (2000)).[25] Imagine that after an allocation has been chosen for a problem, some agents come first and take away their assignments. The invariance principle says that if at that point the situation is re-evaluated from the viewpoint of the remaining agents, who have not received anything yet, then each of them should still be assigned the same amount as initially (Thomson (2011)).

The issue is how to define the problem faced by the remaining agents, called the "reduced problem". Whereas more than one definition has been found plausible in other contexts, there is a most natural one in the context of deterministic claims problems. It is to divide what is left among the remaining agents. Since the endowment is required to be fully allocated in the deterministic context, what is left of the resource is also the sum of the amounts initially assigned to the remaining agents.

In the uncertain context, a subtlety arises in specifying the amount to be divided in the reduced problem. In our model, the endowment is the maximal amount that a planner

[25]It has also been adopted in various types of models that have rationing structures (see e.g., Moulin and Sethuraman (2003), Sprumont (2018)). Maschler (1990) and Thomson (2017) provide excellent surveys.

may use to satiate agents, and it might be optimal for a planner to reserve a part of it for some alternative use (see Section 5). Thus, after some agents leave with their assignments, the remainder of the resource may well differ from the amount assigned in total to the remaining agents. This gives rise to two ways of evaluating the situation. First, after the departure of some agents, the reserved resource has been given away to an alternative use, and thus, the endowment in the reduced problem is the sum of the amounts initially assigned to the remaining agents. Second, the reserved resource has not been given away and is still available for the remaining agents, and thus, the amount to be divided is the difference between the initial endowment and the sum of the assignments to the agents who leave. Since both ways of specifying the endowment in the reduced problem are reasonable, we require the invariance principle to hold in both cases.[26]

Double consistency: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $J \subseteq I$ with $J \neq \emptyset$, $r_J(F, T) = r(F_J, \sum_{i \in J} r_i(F, T)) = r(F_J, T - \sum_{i \in I \setminus J} r_i(F, T))$, where $r_J(F, T)$ and F_J are, respectively, the restrictions of $r(F, T)$ and F onto J.

Different ways of defining the endowment in the reduced problem have also led to variations of the consistency property in related contexts. For the problem of allocating indivisible goods or "objects" to agents who have unit demands, when there are more objects than agents, after some agents leave with their assignments, the set of objects assigned to those who stay differ from the set of remaining objects. Depending on whether or not the unassigned objects have been disposed of when the reduced problem is defined, the consistency property is formulated as *post-disposal consistency* or *pre-disposal consistency* (Ehlers and Klaus (2006, 2007), Thomson (2017)).[27] For economies with production, solutions specify production plans and allocations of commodity bundles to agents. After some agents leave, depending on whether production has occurred or not, the options open to those who stay are either to divide their collective assignments, or to produce and divide the remaining outputs after delivering to the agents who leave their assignments. The consistency property is formulated as *post-production consistency* and *pre-production consistency*, respectively, in the two scenarios (Thomson (1998)).

The last axiom is also an invariance property. Roughly, it requires that in a "transformed problem", all agents receive their "transformed assignments". Imagine that there is a pro-

[26]Part of Theorem 1 holds with the invariance requirement imposed only in the first case. See footnote 29.

[27]Ehlers and Klaus (2006, 2007) call the two axioms *reallocation consistency* and *consistency*, respectively.

duction technology converting the resource into some output. Agents' (uncertain) claims on the resource come from their (uncertain) needs for the output. Our axiom pertains to the possibility that after an allocation has been chosen for a problem, due to a technology shock, different amounts of the resource are required to produce designated quantities of the output. Then each agent's need for the output is transformed into a new claim on the resource. Assume that the endowment is also revised so that agents' needs for the output can be met as they were initially, i.e., the new endowment is obtained by first applying the same transformation to each agent's initial assignment and then summing up the transformed assignments. In this case, agents should receive their transformed assignments and have their needs met as initially. In other words, the division principle should depend only on agents' ultimate needs for output, and allocations should be calculated independently of the underlying production technology. The familiar *scale invariance* axiom requires the same thing when the transformation is a scalar multiplication (Moulin (1987, 2000), Young (1988), Marchant (2008)). Our axiom requires invariance for all increasing and continuous transformations, and thus, is stronger than *scale invariance*.

Formally, a **transformation** is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Let $\mathbf{\Phi}$ be the set of all increasing and continuous transformations. For each $F_i \in \mathcal{F}$ and each $\phi \in \Phi$, let F_i^ϕ denote the transformed claim in $\mathcal{F}$ defined by setting for each $x_i \in \mathbb{R}$,

$$F_i^\phi(x_i) := \begin{cases} 0 & x_i \in (-\infty, \phi(0)) \\ F_i(\phi^{-1}(x_i)) & x_i \in \phi([0,\infty)) \\ 1 & x_i \in [\lim_{c\to\infty} \phi(c), \infty) \end{cases}.$$

Put differently, each need $x_i \in \mathbb{R}_+$ is transformed to a new need $\phi(x_i)$, so that $F_i^\phi(\phi(x_i)) = F_i(x_i)$. For each $I \in \mathcal{N}$, each $F \in \mathcal{F}^I$, and each $\phi \in \Phi$, let F^ϕ denote the transformed claim profile in $\mathcal{F}^I$, namely, for each $i \in I$, $(F^\phi)_i = F_i^\phi$.

Coordinality: For each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{P}^I$, each $\phi \in \Phi$, and each $i \in I$, $r_i(F^\phi, \sum \phi(r_j(F,T))) = \phi(r_i(F,T))$.

The invariance of the choice of a social alternative with respect to increasing transformations of individual characteristics has been well studied in various contexts. In the deterministic claims problems, besides *scale invariance*, there are invariance requirements related to different types of transformations of agents' claims and endowment (Marchant

(2008)). In the problem of aggregating a distribution of individual utilities into a social utility, *ordinal covariance* requires the social utility of a transformed distribution of individual utilities to be the same as the transformed social utility of the initial distribution of individual utilities (Chambers (2007, 2009)).[28] In the cost sharing problem, an axiom called *ordinality* requires that cost shares be invariant under increasing transformations of the scales in which agents demands are measured (Sprumont (1998)). It is stronger than *coordinality* in that transformations are allowed to differ across agents. In the axiomatic bargaining problem, solutions can be required to be invariant under common increasing transformations of individual utility representations that preserve preference orderings and interpersonal comparisons (Nielsen (1983)). A stronger requirement related to agent-specific transformations is considered by Shapley (1967) and Roth (1979). In social choice theory, analogous requirements are also considered for social welfare orderings (d'Aspremont and Gevers (1977), Gevers (1979)).

Note that *coordinality* does not deal with situations in which agents are subject to possibly idiosyncratic technology shocks. To accommodate those situations, invariance should be imposed with respect to all lists of possibly different transformations of agents' claims, analogous to *ordinality* in the cost sharing problem (Sprumont (1998)). The implications of *ordinality* in the resource allocation problem deserve further exploration.

4.2 Characterization

As discussed in the Introduction, there are two important questions regarding resource allocation with uncertain needs. The first is how much of the resource should be used to satiate agents. Surprisingly, a combination of standard axioms, when extended to the uncertain context, provides a guideline for assessing the extent to which the resource should be used to satiate agents. Our first result says that a planner, who agrees with the axioms, should impose on each agent a maximal assignment that depends on his claim, fully allocate the resource whenever it falls short of the sum of the maximal assignments, and otherwise, simply give each agent his maximal assignment.

Theorem 1. *Let r be symmetric, endowment continuous, and doubly consistent. There is an $M : \mathcal{F} \to \mathbb{R}_+$ such that for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, (1)*

[28] Chambers (2009) studies functions defined on the space of CDFs. His *ordinal covariance* requires a function to be invariant under increasing and continuous transformations of CDFs. See Section 1.1.

$T < \sum M(F_j) \Rightarrow \sum r_j(F,T) = T$ and $r_i(F,T) \leq M(F_i)$, and (2) $T \geq \sum M(F_j) \Rightarrow r_i(F,T) = M(F_i)$.[29]

It is under our weaker feasibility constraint that the implications of these axioms are uncovered. In the deterministic claims problems, the sum of the agents' claims is assumed to exceed the amount of the resource available, and the resource is required to be fully allocated. As a result, each agent's maximal assignment is trivially his (deterministic) claim.

When *conditional strict ranking* and *coordinality* are further imposed, a specific way of determining agents' maximal assignments stands out and the class of equal-quantile rules emerges.[30] In the online appendix, we extend the characterization to allow for unbounded claims.

Theorem 2. *A rule satisfies conditional strict ranking, continuity, double consistency, and coordinality if and only if it is an equal-quantile rule.*

Theorem 2 provides a more specific solution to the question regarding the extent to which a resource should be used to satiate agents. It says that a planner, who agrees with the axioms, should set a common maximal probability of satiation λ, fully allocate the resource if it falls short of the sum of the agents' λ-quantile assignments, and otherwise, give each agent exactly his λ-quantile assignment. In Section 5, we provide welfare interpretations of λ, which further suggest that the choice of λ depends on some information outside the data of a problem. In particular, the planner should explore the uses of the resource outside the model, or equivalently, evaluate the cost of generating waste relative to deficit.

The second question is how the concerns for waste and deficit should be balanced across agents with different claims. While the axioms, originating in the deterministic fair division literature, have nothing to do with waste or balance, they surprisingly characterize a class of rules that achieves such a balance in a reasonable way. This will be seen in Section 5, where we show that each equal-quantile rule minimizes a social cost function that aggregates the costs of waste and deficit.

[29] Statement (1) holds with a weaker consistency property that requires invariance only in the case where the collective assignment of the remaining agents is divided in the reduced problem. It is the proof of statement (2) that invokes our stronger consistency property.

[30] *Symmetry* becomes redundant in this case. Moreover, the characterization does not need the full power of *coordinality*. Only specific types of transformations are involved. See Appendix A.2.1 for the details of such transformations.

The characterization in Theorem 2 is tight. Dropping *conditional strict ranking*, a sequential priority rule that gives a higher priority to an agent labeled by a larger integer satisfies *continuity*, *double consistency*, and *coordinality*. The rule first satiates the agent labeled by the largest integer for sure, if feasible; if not, it assigns the entire endowment to him. Then it satiates the agent labeled by the second largest integer for sure, if feasible; if not, it assigns all of the remaining endowment to him; so on and so forth, until either all agents are satiated for sure or the endowment is fully allocated.

Dropping *continuity*, a rule that favors agents who have larger maximal needs satisfies *conditional strict ranking*, *double consistency*, and *coordinality*. The rule first satiates the agents who have the largest maximal need for sure, if feasible; if not, it divides the endowment equally among them. Then it satiates the agents who have the second largest maximal need for sure, if feasible; if not, it divides all of the remaining endowment equally among them; so on and so forth, until either all agents are satiated for sure or the endowment is fully allocated.

Dropping *double consistency*, consider an equal-quantile rule relating to some $\lambda \in [0, 1]$ for two-agent problems and a different $\lambda' \in [0, 1]$ for the other problems. Such a rule satisfies *conditional strict ranking*, *continuity*, and *coordinality*.

Dropping *coordinality*, a version of the proportional rule satisfies *conditional strict ranking*, *continuity*, and *double consistency*. When the endowment exceeds the sum of the agents' maximal needs, each agent is assigned his maximal need. Otherwise, the rule fully allocates the endowment by applying a "constrained proportional method" to the profile of the expected claims. Precisely, each agent is assigned either his maximal need or c times his expected claim, whichever is smaller, where $c \geq 0$ is determined by the binding feasibility constraint.

5 Optimality

One central principle in social choice theory is that collective decisions should be made in accordance with the optimization of an ordering over alternatives. An application of this principle in fair division problems is that the recommended allocations should be those maximizing some collective measure of "welfare" or minimizing some collective measure of "cost". In the deterministic claims problems, each parametric rule minimizes a social cost function (Young (1987a)).[31] In the uncertain context, the optimality of parametric rules remains true (Long and Xue (2019)).

The difficulty with the optimization approach lies in choosing an appropriate measure of welfare or cost (Young (1987a)). While each parametric rule can be rationalized by many objectives, our contribution here is to discover, for equal-quantile rules, two relevant measures that explicitly show how the rules respond to the two important questions regarding resource allocation under uncertainty raised in the Introduction.[32]

Each equal-quantile rule maximizes a utilitarian social welfare function. To define the function, imagine that each agent, after his need is realized, obtains a constant marginal utility of $u \in \mathbb{R}_{++}$ from each unit of the assigned resource that he needs, and zero utility from any amount of the resource that exceeds his need, if any. Thus, if agent i is assigned t_i when he needs x_i, he obtains utility $u \cdot \min\{x_i, t_i\}$. Note that u is the same across agents. This is typically the case when agents use the resource for the same purpose, as in the applications in the Introduction. For example, in the case of a government's budget allocation, the utility of a local authority usually depends on the number of citizens served by its public facility, and a marginal increase in the number yields a constant and common marginal utility for all local authorities.[33]

Besides the agents in the model, imagine another agent outside the model that can make alternative use of the resource. In each problem, after an allocation is chosen, all the leftover resource is directed to the alternative use. This outside agent can "absorb" any amount of the leftover and obtain a constant marginal utility $v \in [0, u]$ from each unit of it.[34] Thus, if an allocation t is chosen for a problem (F, T), the outside agent obtains utility $v \cdot (T - \sum t_i)$. For example, an emergency management organization can assign its rescue personnel to a non-emergency activity (e.g., educational activity) that has a lower priority

[31] The same result is obtained by Stovall (2014) for asymmetric parametric rules.

[32] Our optimality result is not implied by Long and Xue (2019). The cost function that they construct is strictly convex, and thus, admits a unique optimal allocation. Ours is not strictly convex and there is more than one optimal allocations when the endowment is positive and no more than the sum of the sure needs.

[33] Similarly, in the case of emergency management, the utility of an assistance center usually depends on the number of lives saved by its rescue force, and an marginal increase in the number yields a constant and common marginal utility for all assistance centers. In the case of an institute's grant allocation, it is also arguable that the utility of a department is in proportion to the number of participants in its research activity.

[34] This does not necessarily mean that the outside agent surely needs the resource. Our analysis accommodates the case that v is a constant expected marginal utility.

than emergency rescue. This is a valuable use of the rescue personnel if there turns out to be no emergency.[35] Alternatively, the outside agent could represent the option of saving the resource for future use, which yields a discounted marginal utility.

Let $I \in \mathcal{N}$ and $(F, T) \in \mathcal{P}^I$. We abuse notation and denote by F a joint distribution of agents' needs whose marginal distributions are their claims. The choice of such a joint distribution can be arbitrary since our result depends only on marginal distributions.

A utilitarian planner, someone who cares only about the sum of individual utilities, chooses an allocation t to maximize

$$\int \Big[\sum u \min\{x_i, t_i\} + v(T - \sum t_i) \Big] dF. \tag{2}$$

The expected sum of individual utilities is equal to the sum of expected individual utilities

$$\sum \int u \min\{x_i, t_i\} dF_i + v(T - \sum t_i),$$

which depends only on marginal distributions.[36]

Proposition 1. *Let $u \in \mathbb{R}_{++}$, $v \in [0, u]$, and $\lambda := \frac{u-v}{u}$. For each $I \in \mathcal{N}$ and each $(F, T) \in \mathcal{P}^I$, $r^\lambda(F, T)$ maximizes the utilitarian social welfare function (2).*

To see that $r^\lambda(F, T)$ maximizes (2) when $\lambda = \frac{u-v}{u}$, assume for simplicity that the F_i's are continuous. For each $i \in I$, the partial derivative of (2) with respect to t_i is

$$u(1 - F_i(t_i)) - v. \tag{3}$$

Intuitively, a marginal increase in agent i's assignment t_i benefits agent i only when he needs more than t_i, which occurs with probability $1 - F_i(t_i)$. Thus, the benefit of a marginal increase in t_i is the expected marginal increase in agent i's utility, which is $u(1 - F_i(t_i))$. Moreover, the cost is the definite marginal loss in the outside agent's utility, which is v. For each $i \in I$, the partial derivative (3) is non-increasing in t_i, and (3) is equal to zero when

[35] A government can allocate a budget to an alternative project (e.g., expanding government office space) that has a lower priority than building a public facility. This is beneficial if there turns out to be insufficient demand for the public facility. A similar example can be found in the case of an institute's grant allocation.

[36] The sum over the population and the expectation over uncertainty are interchangeable since the integrand in (2) is additively separable across agents (Fishburn (1970) and Al-Najjar and Pomatto (2016)). The interchangeability of the sum and the expectation does not depend on the linearity of individual utilities. It holds for each utilitarian social welfare function with general individual utilities.

$t_i = F_i^{-1}(\frac{u-v}{u})$. Thus, when the endowment is sufficiently large, i.e., when $T \geq \sum F_i^{-1}(\frac{u-v}{u})$, it is optimal to assign to each agent i an amount equal to $F_i^{-1}(\frac{u-v}{u})$. When the endowment is limited, i.e., when $T < \sum F_i^{-1}(\frac{u-v}{u})$, an allocation is optimal if and only if the endowment is fully allocated and the agents' probabilities of satiation, the $F_i(t_i)$'s, are equalized.[37]

Proposition 1 provides a welfare interpretation of λ. It is the optimal upper bound on agents' probabilities of satiation with respect to the social welfare function (2) and it depends on the ratio of v to u. As the ratio gets smaller, allocating more of the resource to the agents in the model would result in an increase in their expected utilities that exceeds the concurrent decrease in the outside agent's utility. Thus, allowing a larger maximal probability of satiation improves the social welfare. When $v = 0$, it is optimal to fully use the resource within the model, no matter how uncertain their needs are, and the optimal maximal probability of satiation is 1. When $v = u$, it is optimal to satisfy only the sure needs of the agents in the model, no matter how large their maximal needs are.

Bringing the outside agent into the picture is the key to understanding the first question concerning the extent to which a resource should be used *within* the model.[38] In fact, it would be ideal to include the information of the outside agent into the data of a problem.[39] But it may often be the case that the planner is only given information on agents' claims and a budget limit and is asked to recommend an allocation. Theorem 2 shows that even without complete information, a specific class of rules can be recommended based on four axioms; and to choose one rule from this class, Proposition 1 further suggests that the planner explore the benefits from alternative uses of the resource outside the model.

Moreover, an equivalent cost-minimization objective provides an explicit answer to the second question on how to balance the concerns for waste and deficit across agents: Each equal-quantile rule balances the trade-offs by minimizing a utilitarian social cost function which linearly depends on the expected waste and deficit of agents. Formally, given a unit

[37]When $T < \sum c_i$, all allocations that fully use the resource and assign no agent more than his sure need are optimal, since they give all agents the same (zero) probability of satiation; the constrained equal awards allocation chosen by equal-quantile rules is one of them. When $T \geq \sum c_i$, the optimal allocation is unique.

[38]Considering such an outside agent is also consistent with the assumption of full resource allocation in the deterministic context. When the agents in the model have deterministic satiation points, the outside agent loses his advantage, and it is optimal to fully use the resource within the model.

[39]A more precise recommendation of rules can be obtained when our model is enriched with information on the outside agent and some additional characteristics of agents such as their utility functions. Implications of new axioms that incorporate such information deserves future research.

waste cost $c^w \in \mathbb{R}_+$ and a unit deficit cost $c^d \in \mathbb{R}_+$, the social cost at an allocation t is

$$c^w \sum \int_{x_i < t_i} (t_i - x_i) dF_i + c^d \sum \int_{x_i > t_i} (x_i - t_i) dF_i. \tag{4}$$

Proposition 2. *Let $c^w, c^d \in \mathbb{R}_+$ be such that $c^w + c^d > 0$. Let $\lambda := \frac{c^d}{c^w + c^d}$. For each $I \in \mathbb{N}$ and each $(F, T) \in \mathcal{P}^I$, $r^\lambda(F, T)$ minimizes the utilitarian social cost function (4).*

When $c^w = v$ and $c^d = u - v$, maximizing the social welfare (2) is equivalent to minimizing the social cost (4), since it can be shown that (4) is essentially the opportunity cost generated by an allocation under uncertainty.[40] Intuitively, the marginal opportunity cost of waste is the utility that could have been obtained by the outside agent had a unit wasted by an agent in the model been assigned to him. Thus, $c^w = v$. The marginal opportunity cost of deficit is the additional utility that could have been experienced by an unsatiated agent in the model had a unit been assigned to him instead of to the outside agent. Thus, $c^d = u - v$.

Proposition 2 provides an equivalent welfare interpretation of λ. It is the optimal maximal probability of satiation with respect to the social cost function (4). The ratio of c^w to c^d, reflecting how to balance waste and deficit, determines λ. As waste gets less costly relative to deficit, allowing a larger maximal probability of satiation reduces the social cost. Thus, in addition to providing an answer to the second question, Proposition 2 also suggests an answer to the first question: to pick one rule from the equal-quantile class, a planner should investigate the cost of waste relative to deficit.

Note that a utilitarian planner always gives priority to sure needs. For example, suppose that 100 units of a resource are to be divided between two agents. The sure need of each agent is 50, while the maximal need of one agent is 200 and that of the other is 51. All equal-quantile rules assign 50 to each agent, ignoring difference in their maximal needs. This would not be desirable, say, if a planner aims to maximize the number of agents whose maximal needs are satisfied. But still, utilitarianism is a long-standing welfare criterion and there are situations where it is relevant. For example, an emergency management agency should arguably give priority to the assistance centers that surely need the resources to carry out rescue operations.

When claims are allowed to be unbounded, Proposition 1 remains true (see the online appendix), and Proposition 2 is meaningful only if all claims have finite expected values:[41]

[40] See Appendix A.3.1 for more details about the opportunity cost.

[41] In this case, it can be readily seen that our proof of Proposition 2 remains valid without changes.

If the expected value of a claim is infinite, then the aggregate expected deficit will be infinite no matter how the resource is allocated, and Proposition 2 becomes vacuous.

6 Concluding remarks

In this paper, we study resource allocation with uncertain needs. We address two questions. To what extent should a resource be used to satiate agents? How should the trade-offs between waste and deficit be balanced across agents? We introduce the class of equal-quantile rules, provide an axiomatic justification of it, and show that each equal-quantile rule is optimal with respect to two utilitarian objectives. An equal-quantile rule determines the extent to which a resource should be used by setting a common maximal probability of satiation. It balances the trade-offs between waste and deficit across agents by minimizing a utilitarian social cost function that linearly aggregates the costs of waste and deficit.

One important open question is how to allocate a resource when agents' needs are correlated. In the end of Section 2, we provide three scenarios where it is reasonable to focus on individual claims, i.e., marginal distributions. However, there are also good reasons to challenge our objective functions and axioms when agents' needs are correlated.

For example, a dollar is to be divided among three agents, 1, 2, and 3, whose needs depend on two states, "sunny" and "rainy". Each state occurs with probability $\frac{1}{2}$. When sunny, both agents 1 and 2 need a dollar, and agent 3 needs zero. When rainy, both agents 1 and 2 need zero, and agent 3 needs a dollar. Thus, all agents claim zero and a dollar with equal probabilities. Our *symmetry* requires the agents to be assigned the same amount. Imagine that each agent obtains a marginal utility of $u > 0$ from each unit of money assigned to him if he needs it, and a marginal utility of 0 if he does not. Consider the allocations $t = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $t' = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Aggregate utility at t is $\frac{1}{3}u + \frac{1}{3}u = \frac{2}{3}u$ when sunny, and $\frac{1}{3}u$ when rainy, and at t' is $\frac{1}{4}u + \frac{1}{4}u = \frac{1}{2}u$ when sunny, and $\frac{1}{2}u$ when rainy. Although the two allocations yield the same aggregate utility in expectation, the variance of the utilities is greater at t than at t'. A risk averse planner would choose t' over t, violating *symmetry*. Indeed, two agents with equal claims may not have the same need in each state, so that they should not necessarily be treated equally. To accommodate the planner's choice of t', one would invoke a weaker version of *symmetry*, or a social welfare function that imposes a concave transformation on aggregate utility before taking the expectation over states. This example suggests that resource allocation based on joint distributions of needs deserves further investigation.

A Appendix

Given $I, I' \in \mathcal{N}$, $F \in \mathcal{F}^I$, and $F' \in \mathcal{F}^{I'}$ with $I \cap I' = \emptyset$, let (F, F') denote the claim profile in $\mathcal{F}^{I \cup I'}$ defined by setting for each $i \in I$, agent i's claim F_i, and for each $j \in I'$, agent j's claim F'_j.

A.1 Proof of Theorem 1

Let r be a *symmetric*, *endowment continuous*, and *consistent* rule. For each $i \in \mathbb{N}$ and each $F_i \in \mathcal{F}$, let $t^*_{F_i} := \sup\{r(F_i, T) : T \in \mathbb{R}_+\}$. It is readily seen that $t^*_{F_i} \in [0, C_i]$. We will show that for each $i \in \mathbb{N}$ and each $F_i \in \mathcal{F}$, $t^*_{F_i}$ depends only on an agent's claim, not his label. Moreover, define $M : \mathcal{F} \to \mathbb{R}_+$ by setting for each $F_i \in \mathcal{F}$, $M(F_i) := t^*_{F_i}$. We will show that the function M satisfies the two conditions stated in the theorem.

Step 1: Full use of sufficiently small resources in single-agent problems. In each single-agent problem, the resource is fully allocated as long as it does not exceed the agent's maximal assignment. Formally, for each $i \in \mathbb{N}$, each $F_i \in \mathcal{F}$, and each $T \in [0, t^*_{F_i}]$, $r(F_i, T) = T$.

Let $i \in \mathbb{N}$, $F_i \in \mathcal{F}$, and $T \in [0, t^*_{F_i}]$. We first show that $r(F_i, t^*_{F_i}) = t^*_{F_i}$. By the definition of $t^*_{F_i}$, there is a sequence $\{T^n\}_{n=1}^{\infty}$ of elements of $\mathbb{R}_+$ such that $\lim r(F_i, T^n) = t^*_{F_i}$. By *double consistency*, for each $n \in \mathbb{N}$, $r(F_i, r(F_i, T^n)) = r(F_i, T^n)$. Then $\lim r(F_i, r(F_i, T^n)) = \lim r(F_i, T^n) = t^*_{F_i}$. By *endowment continuity*, $r(F_i, \lim r(F_i, T^n)) = \lim r(F_i, r(F_i, T^n))$. Thus, $r(F_i, t^*_{F_i}) = r(F_i, \lim r(F_i, T^n)) = \lim r(F_i, r(F_i, T^n)) = t^*_{F_i}$.

Since $r(F_i, 0) = 0$, $r(F_i, t^*_{F_i}) = t^*_{F_i}$, and $T \in [0, t^*_{F_i}]$, by *endowment continuity*, there is $T' \in [0, t^*_{F_i}]$ such that $r(F_i, T') = T$. Then by *double consistency*, $r(F_i, T) = T$.

Step 2: Maximal assignments and full use of sufficiently small resources in multi-agent problems. The maximal assignment to an agent in a problem depends only on the agent and his claim. Moreover, in each multi-agent problem, the resource is full allocated as long as it does not exceed the sum of the agents' maximal assignments. Formally, for each $I \in \mathcal{N}$ and each $(F, T) \in \mathcal{P}^I$, (1) for each $i \in I$, $r_i(F, T) \leq t^*_{F_i}$, (2) when $T \leq \sum t^*_{F_i}$, $\sum r_i(F, T) = T$, and thus (3) for each $i \in I$, $r_i(F, \sum t^*_{F_j}) = t^*_{F_i}$.

Let $I \in \mathcal{N}$ and $(F, T) \in \mathcal{P}^I$. For each $i \in I$, by *double consistency* and the definition of $t^*_{F_i}$, $r_i(F, T) = r(F_i, r_i(F, T)) \leq t^*_{F_i}$. Hence, (1) holds.

Let $T^* := \sup\{\sum r_i(F, T') : T' \in \mathbb{R}_+\}$. By (1), for each $T' \in \mathbb{R}_+$, $\sum r_i(F, T') \leq \sum t^*_{F_i}$, and thus $T^* \in [0, \sum t^*_{F_i}]$. To show (2), we first show that $\sum r_i(F, T^*) = T^*$. By the definition of T^*, there is a sequence $\{T^n\}_{n=1}^{\infty}$ of elements of $\mathbb{R}_+$ such that $\lim \sum r_i(F, T^n) = T^*$. By *double consistency*, for each $n \in \mathbb{N}$, $r(F, T^n) = r(F, \sum r_i(F, T^n))$. By *endowment continuity*, $r(F, \lim \sum r_i(F, T^n)) = \lim r(F, \sum r_i(F, T^n))$. Thus, $r(F, T^*) = r(F, \lim \sum r_i(F, T^n)) = \lim r(F, \sum r_i(F, T^n)) = \lim r(F, T^n)$. Hence, $\sum r_i(F, T^*) = \lim \sum r_i(F, T^n) = T^*$.

Next, we show that $\sum t^*_{F_i} = T^*$. Suppose to the contrary that $\sum t^*_{F_i} \neq T^*$. Since $T^* \in [0, \sum t^*_{F_i}]$ and $\sum t^*_{F_i} \neq T^*$, $T^* < \sum t^*_{F_i}$. Since $\sum r_i(F, T^*) = T^* < \sum t^*_{F_i}$, there is $j \in I$ such that $T^* - \sum\limits_{i \in I \setminus \{j\}} r_i(F, T^*) = r_j(F, T^*) < t^*_{F_j}$. By *endowment continuity*, there is $T' \in (T^*, \infty)$ such that $T' - \sum\limits_{i \in I \setminus \{j\}} r_i(F, T') < t^*_{F_j}$. Then by Step 1, $r(F_j, T' - \sum\limits_{i \in I \setminus \{j\}} r_i(F, T')) = T' - \sum\limits_{i \in I \setminus \{j\}} r_i(F, T')$. By *double consistency*, $r_j(F, T') = r(F_j, T' - \sum\limits_{i \in I \setminus \{j\}} r_i(F, T'))$. Thus, $r_j(F, T') = T' - \sum\limits_{i \in I \setminus \{j\}} r_i(F, T')$. Hence, $\sum r_i(F, T') = T' > T^*$. However, by the definition of T^*, $T^* \geq \sum r_i(F, T')$, which contradicts the inequality $\sum r_i(F, T') > T^*$.

Now suppose that $T \leq \sum t^*_{F_i}$. Since $\sum r_i(F, T^*) = T^*$ and $\sum t^*_{F_i} = T^*$, $\sum r_i(F, \sum t^*_{F_i}) = \sum t^*_{F_i}$. Since $\sum r_i(F, 0) = 0$, $\sum r_i(F, \sum t^*_{F_i}) = \sum t^*_{F_i}$, and $T \in [0, \sum t^*_{F_i}]$, by *endowment continuity*, there is $T' \in [0, \sum t^*_{F_i}]$ such that $\sum r_i(F, T') = T$. By *double consistency*, $r(F, T') = r(F, \sum r_i(F, T'))$. Then $r(F, T') = r(F, T)$, and thus $\sum r_i(F, T) = \sum r_i(F, T') = T$. Hence, (2) holds.

Lastly, it is readily seen that (3) follows from (1) and (2).

Step 3: Anonymity of maximal assignments. The maximal assignment to an agent in a problem depends only on the agent's claim, not his label. Formally, for each pair $i, j \in \mathbb{N}$ and each pair $F_i, F_j \in \mathcal{F}$ such that $F_i = F_j$, $t^*_{F_i} = t^*_{F_j}$.

Let $i, j \in \mathbb{N}$ and $F_i, F_j \in \mathcal{F}$ be such that $F_i = F_j$. By Step 2, $r((F_i, F_j), t^*_{F_i} + t^*_{F_j}) = (t^*_{F_i}, t^*_{F_j})$. By *symmetry*, $t^*_{F_i} = t^*_{F_j}$.

Step 4: Constant assignment in single-agent problems with sufficiently large resources. In each single-agent problem, the agent always receives his maximal assignment when the resource exceeds his maximal assignment. Formally, for each $i \in \mathbb{N}$, each $F_i \in \mathcal{F}$, and each $T \in (t^*_{F_i}, \infty)$, $r(F_i, T) = t^*_{F_i}$.

Let $i \in \mathbb{N}$, $F_i \in \mathcal{F}$, $T \in (t^*_{F_i}, \infty)$, and $t_i := r(F_i, T)$. By the definition of $t^*_{F_i}$, $t_i \leq t^*_{F_i}$. To show that $t_i = t^*_{F_i}$, suppose to the contrary that $t_i < t^*_{F_i}$.

Let $I \in \mathcal{N}$ be such that $i \notin I$ and $|I|(t^*_{F_i} - t_i) + t^*_{F_i} > T$. Let $F \in \mathcal{F}^I$ be such that for each $j \in I$, $F_j = F_i$. Define $f : [(|I| + 1)t^*_{F_i}, \infty) \to \mathbb{R}_+$ by setting for each $T' \in [(|I| + 1)t^*_{F_i}, \infty)$,

$$f(T') := T' - |I| r_i((F_i, F), T').$$

For each $T' \in \mathbb{R}_+$, by *symmetry*, $\sum_{j \in I} r_j((F_i, F), T') = |I| r_i((F_i, F), T')$, and thus by *double consistency*,

$$r_i((F_i, F), T') = r(F_i, f(T')). \tag{5}$$

We claim that there is $T'' \in [(|I| + 1)t^*_{F_i}, \infty)$ such that $f(T'') = T$. To see this, note first that for each $j \in I$, since $F_j = F_i$, by Step 3, $t^*_{F_j} = t^*_{F_i}$. Thus, by Step 2, $r_i((F_i, F), (|I| + 1)t^*_{F_i}) = t^*_{F_i}$. Hence, $f((|I| + 1)t^*_{F_i}) = (|I| + 1)t^*_{F_i} - |I| t^*_{F_i} = t^*_{F_i} < T$. Moreover, by Step 2, $r_i((F_i, F), T + |I| t^*_{F_i}) \leq t^*_{F_i}$. Hence, $f(T + |I| t^*_{F_i}) \geq T + |I| t^*_{F_i} - |I| t^*_{F_i} = T$. By *endowment continuity*, f is continuous. Since $f((|I| + 1)t^*_{F_i}) < T \leq f(T + |I| t^*_{F_i})$ and f is continuous, there is $T'' \in [(|I| + 1)t^*_{F_i}, T + |I| t^*_{F_i}]$ such that $f(T'') = T$.

By (5), $r_i((F_i, F), T'') = r(F_i, f(T''))$. Since $f(T'') = T$ and $r(F_i, T) = t_i$, $r(F_i, f(T'')) = t_i$. Thus, $r_i((F_i, F), T'') = t_i$. Since $T'' \in [(|I| + 1)t^*_{F_i}, \infty)$ and $r_i((F_i, F), T'') = t_i$, $f(T'') = T'' - |I| r_i((F_i, F), T'') \geq (|I|+1)t^*_{F_i} - |I| t_i = |I|(t^*_{F_i} - t_i) + t^*_{F_i}$. By the choice of I, $|I|(t^*_{F_i} - t_i) + t^*_{F_i} > T$. Thus, $f(T'') > T$, which contradicts $f(T'') = T$.

Step 5: Constant assignment in multi-agent problems with sufficiently large resources. In each multi-agent problem, the agents always receive their maximal assignments when the resource exceeds the sum of their maximal assignments. Formally, for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$ such that $T > \sum t^*_{F_j}$, and each $i \in I$, $r_i(F, T) = t^*_{F_i}$.

Let $I \in \mathcal{N}$ and $(F, T) \in \mathcal{P}^I$ be such that $T > \sum t^*_{F_j}$. Let $i \in I$. By *double consistency*, $r_i(F, T) = r(F_i, T - \sum_{j \in I \setminus \{i\}} r_j(F, T))$. For each $j \in I$, by Step 2, $r_j(F, T) \leq t^*_{F_j}$. Then $T - \sum_{j \in I \setminus \{i\}} r_j(F, T) > \sum t^*_{F_j} - \sum_{j \in I \setminus \{i\}} t^*_{F_j} = t^*_{F_i}$. Thus, by Step 4, $r(F_i, T - \sum_{j \in I \setminus \{i\}} r_j(F, T)) = t^*_{F_i}$. Hence, $r_i(F, T) = t^*_{F_i}$. □

A.2 Characterization of equal-quantile rules

A.2.1 Additional axioms

Anonymity: For each $I \in \mathcal{N}$ and each $\pi : I \to \mathbb{N}$ that is injective, if $(F, T) \in \mathcal{P}^I$ and $(F', T) \in \mathcal{P}^{\pi(I)}$ are such that for each $i \in I$, $F_i = F'_{\pi(i)}$, then for each $i \in I$, $r_i(F, T) =$

$r_{\pi(i)}(F', T)$.

Endowment monotonicity: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $T' \in (T, \infty)$, $r(F, T) \leq r(F, T')$.

The above axioms are familiar. The next two axioms weaken *coordinality* by restricting transformations to specific types. We call $\phi \in \Phi$ an **upper transformation** if either (i) there are $d_1, d_2, d_3, d_4 \in \mathbb{R}_+$ such that $d_1 < d_2 < d_3 < d_4$ and for each $c \in \mathbb{R}_+$,

$$\phi(c) = \begin{cases} c & c \in [0, d_1) \cup [d_4, \infty) \\ d_1 + \frac{d_3 - d_1}{d_2 - d_1}(c - d_1) & c \in [d_1, d_2) \\ d_3 + \frac{d_4 - d_3}{d_4 - d_2}(c - d_2) & c \in [d_2, d_4) \end{cases}, \tag{6}$$

or (ii) ϕ is the continuous extension of the limit of (6) as d_1, d_2 approach zero,[42] i.e., there are $d_3, d_4 \in \mathbb{R}_{++}$ such that $d_3 < d_4$ and for each $c \in \mathbb{R}_+$,

$$\phi(c) = \begin{cases} d_3 + \frac{d_4 - d_3}{d_4} c & c \in [0, d_4) \\ c & c \in [d_4, \infty) \end{cases}. \tag{7}$$

In other words, either ϕ expands the interval $[d_1, d_2]$ to $[d_1, d_3]$, squeezes $[d_2, d_4]$ to $[d_3, d_4]$, and leaves c outside $[d_1, d_4]$ unchanged, or ϕ squeezes $[0, d_4]$ to $[d_3, d_4]$ and leaves c outside $[0, d_4]$ unchanged. Figure 3 shows examples of upper transformations (6) and (7) and the correspondingly transformed claims. Let $\boldsymbol{\Phi^u}$ denote the set of upper transformations.

Upper coordinality: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, each $\phi \in \Phi^u$, and each $i \in I$, $r_i(F^\phi, \sum \phi(r_j(F, T))) = \phi(r_i(F, T))$.

Analogously, we call $\phi \in \Phi$ a **lower transformation** if either (i) there are $d_1, d_2, d_3, d_4 \in \mathbb{R}_+$ such that $d_1 < d_2 < d_3 < d_4$ and for each $c \in \mathbb{R}_+$,

$$\phi(c) = \begin{cases} c & c \in [0, d_1) \cup [d_4, \infty) \\ d_1 + \frac{d_2 - d_1}{d_3 - d_1}(c - d_1) & c \in [d_1, d_3) \\ d_2 + \frac{d_4 - d_2}{d_4 - d_3}(c - d_3) & c \in [d_3, d_4) \end{cases}, \tag{8}$$

[42] The limit of (6) as d_1, d_2 approach zero is discontinuous at the point $c = 0$.

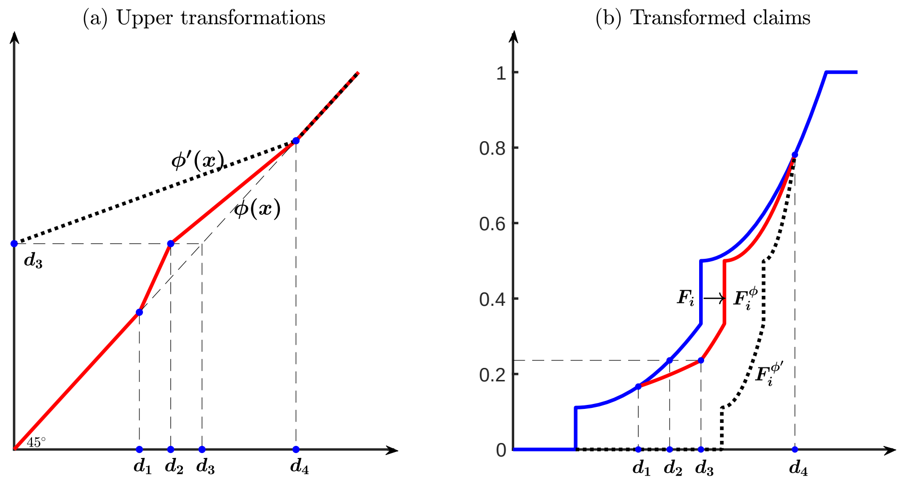

Figure 3: Part (a) shows upper transformations ϕ and ϕ', where ϕ' is the continuous extension of the limit of ϕ as d_1, d_2 approach zero; part (b) shows the transformed claims.

or (ii) ϕ is the limit of (8) as d_1 approaches zero and d_4 approaches infinity, i.e., there are $d_2, d_3 \in \mathbb{R}_{++}$ such that $d_2 < d_3$ and for each $c \in \mathbb{R}_+$,

$$\phi(c) = \begin{cases} \frac{d_2}{d_3} c & c \in [0, d_3) \\ c + d_2 - d_3 & c \in [d_3, \infty) \end{cases}. \tag{9}$$

In other words, either ϕ squeezes $[d_1, d_3]$ to $[d_1, d_2]$, expands $[d_3, d_4]$ to $[d_2, d_4]$, and leaves c outside $[d_1, d_4]$ unchanged, or ϕ squeezes $[0, d_3]$ to $[0, d_2]$ and shifts each $c \in [d_3, \infty)$ down by $d_3 - d_2$. Figure 4 shows examples of lower transformations (8) and (9) and the correspondingly transformed claims. Let $\boldsymbol{\Phi}^{\boldsymbol{l}}$ denote the set of lower transformations.

Lower coordinality: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, each $\phi \in \Phi^l$, and each $i \in I$, $r_i(F^{\phi}, \sum \phi(r_j(F, T))) = \phi(r_i(F, T))$.

A.2.2 Lemmata

Lemma 1. *If a rule is symmetric, endowment continuous, and consistent, then it is anonymous.*[43]

[43]The proof of our Lemma 1 is similar to that of Lemma 3 of Chambers and Thomson (2002). The difference is that we allow the resource to be partially allocated and divide the proof into two cases depending

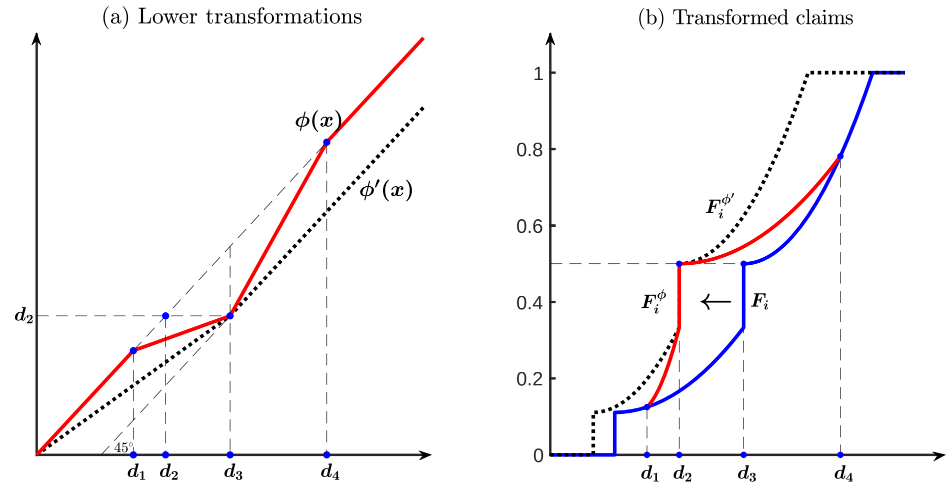

Figure 4: Part (a) shows lower transformations ϕ and ϕ', where ϕ' is the limit of ϕ as d_1 approaches zero and d_4 approaches infinity; part (b) shows the transformed claims.

Proof. Let r be a rule that is *symmetric*, *endowment continuous*, and *consistent*. By Theorem 1, there is a function $M : \mathcal{F} \to \mathbb{R}_+$ such that for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, (1) $T < \sum M(F_j) \Rightarrow \sum r_j(F, T) = T$ and $r_i(F, T) \leq M(F_i)$, and (2) $T \geq \sum M(F_j) \Rightarrow r_i(F, T) = M(F_i)$.

Let $I \in \mathcal{N}$ and $\pi : I \to \mathbb{N}$ be such that π is injective. Let $(F, T) \in \mathcal{P}^I$ and $(F', T) \in \mathcal{P}^{\pi(I)}$ be such that for each $i \in I$, $F_i = F'_{\pi(i)}$. We want to show that for each $i \in I$, $r_i(F, T) = r_{\pi(i)}(F', T)$. For each $i \in I$, since $F_i = F'_{\pi(i)}$, $M(F_i) = M(F'_{\pi(i)})$. Thus, $\sum M(F_i) = \sum M(F'_{\pi(i)})$.

Assume that $T \geq \sum M(F_i)$. Then $T \geq \sum M(F'_{\pi(i)})$. Thus, for each $i \in I$, $r_i(F, T) = M(F_i)$ and $r_{\pi(i)}(F', T) = M(F'_{\pi(i)})$. Hence, for each $i \in I$, $r_i(F, T) = M(F_i) = M(F'_{\pi(i)}) = r_{\pi(i)}(F', T)$.

Assume that $T < \sum M(F_i)$. Then $T < \sum M(F'_{\pi(i)})$. Suppose first that $\pi(I) \cap I = \emptyset$. Let $t := r((F, F'), 2T)$. Since $2T < \sum M(F_i) + \sum M(F'_{\pi(i)})$, $\sum t_i + \sum t_{\pi(i)} = 2T$. For each $i \in I$, since $F_i = F'_{\pi(i)}$, by *symmetry*, $t_i = t_{\pi(i)}$. Thus, $\sum t_i = \sum t_{\pi(i)} = T$. Then by *double consistency*, for each $i \in I$, $t_i = r_i(F, T)$ and $t_{\pi(i)} = r_{\pi(i)}(F', T)$. Hence, for each $i \in I$, $r_i(F, T) = t_i = t_{\pi(i)} = r_{\pi(i)}(F', T)$. Suppose now that $\pi(I) \cap I \neq \emptyset$. Let $\pi' : I \to \mathbb{N}$ be an injective function such that $\pi'(I) \cap I = \pi'(I) \cap \pi(I) = \emptyset$. Let $F'' \in \mathcal{F}^{\pi'(I)}$ be such that for each $i \in I$, $F_i = F''_{\pi'(i)}$, and thus $F'_{\pi(i)} = F''_{\pi'(i)}$. By the previous arguments, for each $i \in I$,

on whether the resource is fully allocated or not. To do that, we impose *endowment continuous* in addition and invoke Theorem 1 to understand when the resource is fully allocated.

$r_i(F,T) = r_{\pi'(i)}(F'',T) = r_{\pi(i)}(F',T)$. $\square$

Lemma 2. *If a rule satisfies conditional strict ranking, continuity, double consistency, and coordinality, then it satisfies ranking.*

Proof. Let r be a rule satisfying *conditional strict ranking*, *continuity*, *double consistency*, and *coordinality*. Let $I \in \mathcal{N}$, $(F,T) \in \mathcal{P}^I$, and $\{i,j\} \subseteq I$ be such that $F_i \succsim_{FSD} F_j$. Then for each $c \in \mathbb{R}$, $F_i(c) \leq F_j(c)$. Thus, $c_i \geq c_j$ and $C_i \geq C_j$. Let $t := r(F,T)$. We want to show that $t_i \geq t_j$. We divide the proof into the following four cases.

Case 1: $c_j = 0$ and F_j is continuous on $\mathbb{R}_{++}$. If $C_j = 0$, then by the definition of a rule, $t_i \geq 0 = t_j$. Assume that $C_j > 0$. For each $n \in \mathbb{N}$ with $n \geq 2$, define $F_j^n : \mathbb{R} \to [0,1]$ by setting for each $x_j \in \mathbb{R}$,

$$F_j^n(x_j) := \begin{cases} 0 & x_j \in (-\infty, 0) \\ \frac{1}{n} + (1-\frac{1}{n})F_j(x_j) & x_j \in [0, \frac{n-1}{n}C_j) \\ 1 & x_j \in [\frac{n-1}{n}C_j, \infty) \end{cases}.$$

For each $n \in \mathbb{N}$ with $n \geq 2$, since $c_j = 0 < C_j$, F_j^n is a well-defined CDF whose support is $[0, \frac{n-1}{n}C_j]$. It is readily seen that F_j^n converges to F_j. For each $n \in \mathbb{N}$ with $n \geq 2$, $\frac{n-1}{n}C_j > 0$, and for each $c \in (0, \frac{n-1}{n}C_j]$, since $F_j(c) < 1$ and F_j is continuous at c, $F_j(c) < \frac{1}{n} + (1-\frac{1}{n})F_j(c) = \frac{1}{n} + (1-\frac{1}{n})\lim_{c' \uparrow c} F_j(c') = \lim_{c' \uparrow c} F_j^n(c')$, and thus, $F_i(c) \leq F_j(c) < \lim_{c' \uparrow c} F_j^n(c')$. Hence, for each $n \in \mathbb{N}$ with $n \geq 2$, $F_i \succ_{FSD} F_j^n$, and thus, by *conditional strict ranking*, if $r_j((F_j^n, F_{I\setminus\{j\}}), T) > 0$, then $r_i((F_j^n, F_{I\setminus\{j\}}), T) > r_j((F_j^n, F_{I\setminus\{j\}}), T)$. Moreover, for each $n \in \mathbb{N}$ with $n \geq 2$, if $r_j((F_j^n, F_{I\setminus\{j\}}), T) = 0$, then by the definition of a rule, $r_i((F_j^n, F_{I\setminus\{j\}}), T) \geq 0 = r_j((F_j^n, F_{I\setminus\{j\}}), T)$. Since F_j^n converges to F_j and for each $n \in \mathbb{N}$ with $n \geq 2$, $r_i((F_j^n, F_{I\setminus\{j\}}), T) \geq r_j((F_j^n, F_{I\setminus\{j\}}), T)$, by *continuity*, $t_i \geq t_j$.

Case 2: $c_j = 0$ and F_j is not continuous on $\mathbb{R}_{++}$. For each $n \in \mathbb{N}$ and each $k \in \{i,j\}$, define $F_k^n : \mathbb{R} \to [0,1]$ by setting for each $c \in \mathbb{R}$,

$$F_k^n(c) := \begin{cases} 0 & c \in (-\infty, 0) \\ \dfrac{\frac{(m+1)C_i}{2^n} - c}{\frac{C_i}{2^n}} F_k(\frac{mC_i}{2^n}) + \dfrac{c - \frac{mC_i}{2^n}}{\frac{C_i}{2^n}} F_k(\frac{(m+1)C_i}{2^n}) & c \in [\frac{mC_i}{2^n}, \frac{(m+1)C_i}{2^n}), m \in \{0,1,\cdots,2^n-1\} \\ 1 & c \in [C_i, \infty) \end{cases}.$$

For each $n \in \mathbb{N}$, since $F_i, F_j \in \mathcal{F}$ and $c_j = 0 < C_j \leq C_i$, it can be readily seen that for each $k \in \{i, j\}$, F_k^n is a well-defined CDF whose support is a bounded interval, $C_i^n = C_i$, $c_j^n = 0$, and F_j^n is continuous on $\mathbb{R}_{++}$. Moreover, since for each $c \in \mathbb{R}$, $F_i(c) \leq F_j(c)$, it can be readily seen that for each $n \in \mathbb{N}$ and each $c \in \mathbb{R}$, $F_i^n(c) \leq F_j^n(c)$, and thus, $F_i^n \succsim_{FSD} F_j^n$. For each $n \in \mathbb{N}$, since $F_i^n \succsim_{FSD} F_j^n$, $c_j^n = 0$, and F_j^n is continuous on $\mathbb{R}_{++}$, by the result in Case 1, $r_i((F_i^n, F_j^n, F_{I\setminus\{i,j\}}), T) \geq r_j((F_i^n, F_j^n, F_{I\setminus\{i,j\}}), T)$. We claim that for each $k \in \{i, j\}$, F_k^n converges to F_k. Then by *continuity*, $t_i \geq t_j$.

To prove our claim, we first check that for each $k \in \{i, j\}$, F_k^n weakly converges to F_k. Let $k \in \{i, j\}$. Let $c \in \mathbb{R}$ be such that F_k is continuous at c. We check that $\lim F_k^n(c) = F_k(c)$. Clearly, this is true when $c < 0$. Since $C_j \leq C_i$, this is also true when $c \geq C_i$. Assume that $c \in [0, C_i)$. For each $n \in \mathbb{N}$, there is $m_n \in \{0, 1, \cdots, 2^n - 1\}$ such that $c \in [\frac{m_n C_i}{2^n}, \frac{(m_n+1)C_i}{2^n})$, and thus, $F_k(\frac{m_n C_i}{2^n}) \leq F_k^n(c) \leq F_k(\frac{(m_n+1)C_i}{2^n})$. Since for each $n \in \mathbb{N}$, $c \in [\frac{m_n C_i}{2^n}, \frac{(m_n+1)C_i}{2^n})$ and $|\frac{(m_n+1)C_i}{2^n} - \frac{m_n C_i}{2^n}| = \frac{1}{2^n}$, as n approaches infinity, $\lim \frac{m_n C_i}{2^n} = \lim \frac{(m_n+1)C_i}{2^n} = c$. Since F_k is continuous at c and $\lim \frac{m_n C_i}{2^n} = \lim \frac{(m_n+1)C_i}{2^n} = c$, $\lim F_k(\frac{m_n C_i}{2^n}) = \lim F_k(\frac{(m_n+1)C_i}{2^n}) = F_k(c)$, and since for each $n \in \mathbb{N}$, $F_k(\frac{m_n C_i}{2^n}) \leq F_k^n(c) \leq F_k(\frac{(m_n+1)C_i}{2^n})$, $\lim F_k^n(c) = F_k(c)$.

Second, we check that $\lim c_i^n = c_i$ and $\lim C_i^n = C_i$. If $c_i = 0$, then for each $n \in \mathbb{N}$, $c_i^n = 0$, and thus, $\lim c_i^n = c_i$. Assume that $c_i > 0$. Let $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N$, there is $m_n \in \{0, 1, \cdots, 2^n - 1\}$ satisfying that $c_i - \epsilon < \frac{m_n C_i}{2^n} < c_i$, and thus, $F_i^n(c_i - \epsilon) \leq F_i^n(\frac{m_n C_i}{2^n}) = F_i(\frac{m_n C_i}{2^n}) = 0$. Moreover, if $c_i + \epsilon \geq C_i$, then for each $n \in \mathbb{N}$, $F_i^n(c_i + \epsilon) = 1 > 0$; if $c_i + \epsilon < C_i$, then there is $N' \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N'$, there is $m_n \in \{0, 1, \cdots, 2^n - 1\}$ satisfying that $c_i < \frac{m_n C_i}{2^n} < c_i + \epsilon$, and thus, $F_i^n(c_i + \epsilon) \geq F_i^n(\frac{m_n C_i}{2^n}) = F_i(\frac{m_n C_i}{2^n}) > 0$. Hence, for each $n \in \mathbb{N}$ with $n \geq \max\{N, N'\}$, $c_i^n \in [c_i - \epsilon, c_i + \epsilon]$. Since ϵ can be arbitrarily small, $\lim c_i^n = c_i$. Moreover, since for each $n \in \mathbb{N}$, $C_i^n = C_i$, $\lim C_i^n = C_i$.

Lastly, we check that $\lim c_j^n = c_j$ and $\lim C_j^n = C_j$. Since for each $n \in \mathbb{N}$, $c_j^n = 0 = c_j$, $\lim c_j^n = c_j$. If $C_j = C_i$, then for each $n \in \mathbb{N}$, $C_j^n = C_i$, and thus, $\lim C_j^n = C_j$. Assume that $C_j < C_i$. Let $\epsilon > 0$. If $C_j - \epsilon < 0$, then for each $n \in \mathbb{N}$, $F_j^n(C_j - \epsilon) = 0 < 1$; if $C_j - \epsilon \geq 0$, then there is $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N$, there is $m_n \in \{0, 1, \cdots, 2^n - 1\}$ satisfying that $C_j - \epsilon < \frac{m_n C_i}{2^n} < C_j$, and thus, $F_j^n(C_j - \epsilon) \leq F_j^n(\frac{m_n C_i}{2^n}) = F_j(\frac{m_n C_i}{2^n}) < 1$. Moreover, if $C_j + \epsilon \geq C_i$, for each $n \in \mathbb{N}$, $F_j^n(C_j + \epsilon) = 1$; if $C_j + \epsilon < C_i$, there is $N' \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ with $n \geq N'$, there is $m_n \in \{0, 1, \cdots, 2^n - 1\}$ satisfying that $C_j < \frac{m_n C_i}{2^n} < C_j + \epsilon$, and thus, $F_j^n(C_j + \epsilon) \geq F_j^n(\frac{m C_i}{2^n}) = F_j(\frac{m_n C_i}{2^n}) = 1$. Hence, for each $n \in \mathbb{N}$ with $n \geq \max\{N, N'\}$,

$C_j^n \in [C_j - \epsilon, C_j + \epsilon]$. Since ϵ can be arbitrarily small, $\lim C_j^n = C_j$.

Case 3: $c_j > 0$ and $t_j > c_j$. Suppose to the contrary that $t_i < t_j$. By *double consistency*, $r((F_i, F_j), t_i + t_j) = (t_i, t_j)$. For each $k \in \{i, j\}$, define $F'_k : \mathbb{R} \to [0,1]$ by setting for each $c \in \mathbb{R}$,

$$F'_k(c) := \begin{cases} 0 & c \in (-\infty, 0) \\ F_k(c + c_j) & c \in [0, \infty) \end{cases}.$$

It can be readily seen that $F'_i, F'_j \in \mathcal{F}$, $F'_i \succsim_{FSD} F'_j$, and $c'_j = 0$. For each $n \in \mathbb{N}$ with $\frac{1}{n} < c_j$, let $\phi^n \in \Phi^l$ be the lower transformation defined by (9) with $d_2 = \frac{1}{n}$ and $d_3 = c_j$. For each $n \in \mathbb{N}$ with $\frac{1}{n} < c_j$, since $r((F_i, F_j), t_i + t_j) = (t_i, t_j)$, by *lower coordinality*, $r((F_i^{\phi^n}, F_j^{\phi^n}), \phi^n(t_i) + \phi^n(t_j)) = (\phi^n(t_i), \phi^n(t_j))$. It can be shown by routine arguments that for each $k \in \{i, j\}$, $F_k^{\phi^n}$ converges to F'_k.[44] By *continuity*, $r((F'_i, F'_j), \lim \phi^n(t_i) + \lim \phi^n(t_j)) = (\lim \phi^n(t_i), \lim \phi^n(t_j))$. For each $n \in \mathbb{N}$ with $\frac{1}{n} < c_j$ and each $c \in \mathbb{R}_+$,

$$\phi^n(c) = \begin{cases} \frac{c}{nc_j} & c \in [0, c_j) \\ c + \frac{1}{n} - c_j & c \in [c_j, \infty) \end{cases},$$

and thus,

$$\lim \phi^n(c) = \begin{cases} 0 & c \in [0, c_j) \\ c - c_j & c \in [c_j, \infty) \end{cases}.$$

Since $t_j > c_j$, $\lim \phi^n(t_j) = t_j - c_j$. Assume that $t_i < c_j$. Then $\lim \phi^n(t_i) = 0$. Thus, $r((F'_i, F'_j), t_j - c_j) = (0, t_j - c_j)$. Since $t_j > c_j$, $r_i((F'_i, F'_j), t_j - c_j) < r_j((F'_i, F'_j), t_j - c_j)$. Assume that $t_i \geq c_j$. Then $\lim \phi^n(t_i) = t_i - c_j$. Thus, $r((F'_i, F'_j), t_i + t_j - 2c_j) = (t_i - c_j, t_j - c_j)$. Since $t_i < t_j$, $r_i((F'_i, F'_j), t_i + t_j - 2c_j) < r_j((F'_i, F'_j), t_i + t_j - 2c_j)$. However, no matter whether $t_i < c_j$ or $t_i \geq c_j$, since $F'_i \succsim_{FSD} F'_j$ and $c'_j = 0$, by the results in Cases 1 and 2, it is not possible for agent i to receive a smaller assignment than agent j.

Case 4: $c_j > 0$ and $t_j \leq c_j$. Suppose to the contrary that $t_i < t_j$. By *double consistency*, $r((F_i, F_j), t_i + t_j) = (t_i, t_j)$. Let $\phi \in \Phi^u$ be the upper transformation defined by (7) with $d_3 = \frac{t_i + t_j}{2}$ and $d_4 = c_j$. Since $r((F_i, F_j), 0) = (0, 0)$, by *upper coordinality*, $r((F_i^\phi, F_j^\phi), 2\phi(0)) = (\phi(0), \phi(0))$, and since $\phi(0) = \frac{t_i + t_j}{2}$, $r((F_i^\phi, F_j^\phi), t_i + t_j) = (\frac{t_i + t_j}{2}, \frac{t_i + t_j}{2})$.

We claim that for each $k \in \{i, j\}$, $F_k = F_k^\phi$. To see it, let $k \in \{i, j\}$. For each $x_k \in (-\infty, \frac{t_i + t_j}{2})$, $F_k^\phi(x_k) = 0$, and since $t_i < t_j \leq c_j \leq c_i$, $x_k < c_k$, so that $F_k(x_k) = 0 = F_k^\phi(x_k)$.

[44] The proof is available upon request.

For each $x_k \in [\frac{t_i+t_j}{2}, c_j)$, $x_k < c_j \le c_k$ and $\phi^{-1}(x_k) < c_j \le c_k$, and thus, $F_k(x_k) = 0 = F_k(\phi^{-1}(x_k)) = F_k^{\phi}(x_k)$. For each $x_k \in [c_j, \infty)$, $x_k = \phi^{-1}(x_k)$, and thus, $F_k(x_k) = F_k(\phi^{-1}(x_k)) = F_k^{\phi}(x_k)$. Therefore, $F_k = F_k^{\phi}$.

Since for each $k \in \{i, j\}$, $F_k = F_k^{\phi}$, $r((F_i, F_j), t_i + t_j) = r((F_i^{\phi}, F_j^{\phi}), t_i + t_j) = (\frac{t_i+t_j}{2}, \frac{t_i+t_j}{2})$, and since $r((F_i, F_j), t_i + t_j) = (t_i, t_j)$, $t_i = t_j = \frac{t_i+t_j}{2}$, which contradicts $t_i < t_j$. □

Lemma 3. *If a rule r is symmetric, endowment continuous, and consistent, then it is endowment monotonic.*[45]

Proof. Let r be a rule that is *symmetric*, *endowment continuous*, and *consistent*. By Theorem 1, there is a function $M : \mathcal{F} \to \mathbb{R}_+$ such that for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each $i \in I$, (1) $T < \sum M(F_j) \Rightarrow \sum r_j(F, T) = T$ and $r_i(F, T) \le M(F_i)$, and (2) $T \ge \sum M(F_j) \Rightarrow r_i(F, T) = M(F_i)$.

Let $I \in \mathcal{N}$, $(F, T) \in \mathcal{P}^I$, and $T' \in (T, \infty)$. Let $t := r(F, T)$ and $t' := r(F, T')$. We want to show that $t \le t'$. If $T' \ge \sum M(F_i)$, then for each $i \in I$, $t'_i = M(F_i) \ge t_i$, as desired. In the following, we assume that $T' < \sum M(F_i)$. Suppose to the contrary that there is $j \in I$ such that $t_j > t'_j$. Since $T < T' < \sum M(F_i)$, $\sum t_i = T$ and $\sum t'_i = T'$, and thus $\sum t_i < \sum t'_i$. Then there is $k \in I \setminus \{j\}$ such that $t_k < t'_k$.

Assume that $t_j + t_k \ge t'_j + t'_k$. Let $n \in \mathbb{N}$ be such that $t_j + nt_k < t'_j + nt'_k$. Let $I' \in \mathcal{N}$ and $F' \in \mathcal{F}^{I'}$ be such that $j, k \in I'$, $|I'| = n + 1$, $F'_j = F_j$, and for each $i \in I' \setminus \{j\}$, $F'_i = F_k$. Define $f : [0, \sum M(F'_i)] \to \mathbb{R}_+$ by setting for each $c \in [0, \sum M(F'_i)]$,

$$f(c) := r_j(F', c) + r_k(F', c).$$

By *endowment continuity*, f is continuous. Note that $f(0) = 0$. Moreover, $t_j + t_k \le M(F_j) + M(F_k)$, and for each $i \in I'$, $r_i(F', \sum M(F'_i)) = M(F'_i)$, so that $f(\sum M(F'_i)) = M(F_j) + M(F_k)$. Then $t_j + t_k \in [f(0), f(\sum M(F'_i))]$. Thus, by the continuity of f, there is $\hat{T} \in [0, \sum M(F'_i)]$ such that $f(\hat{T}) = t_j + t_k$. By *double consistency*, restricting $(F', \hat{T})$ to $(F'_{\{j,k\}}, f(\hat{T}))$ yields $r_{\{j,k\}}(F', \hat{T}) = r((F_j, F_k), f(\hat{T}))$, and restricting (F, T) to $(F_{\{j,k\}}, t_j + t_k)$ yields $(t_j, t_k) = r((F_j, F_k), t_j + t_k)$. Then $r_{\{j,k\}}(F', \hat{T}) = (t_j, t_k)$, and thus by *symmetry*, for

[45] Lemma 3 extends Lemma 1 of Young (1987a) from the deterministic context to the uncertain one. The only difference between our proof and Young's (1987a) is that Young (1987a) assumes that the endowment is always fully allocated, whereas we invoke Theorem 1 to ensure that the endowment is fully allocated if it is less than some threshold amount, and otherwise each agent receives his maximal assignment no matter how large the endowment is.

each $i \in I' \setminus \{j\}$, $r_i(F', \hat{T}) = r_k(F', \hat{T}) = t_k$, so that $\sum r_i(F', \hat{T}) = t_j + nt_k$. Since $\hat{T} \leq \sum M(F'_i)$, $\hat{T} = \sum r_i(F', \hat{T}) = t_j + nt_k$. Recall that $t'_j + t'_k \leq t_j + t_k = f(\hat{T})$. Since $t'_j + t'_k \in [f(0), f(\hat{T})]$ and f is continuous, there is $\tilde{T} \in [0, \hat{T}]$ such that $f(\tilde{T}) = t'_j + t'_k$. By similar arguments, $\tilde{T} = t'_j + nt'_k$. Since $t_j + nt_k < t'_j + nt'_k$, $\hat{T} < \tilde{T}$, which contradicts $\hat{T} \geq \tilde{T}$.

Assume that $t_j + t_k < t'_j + t'_k$. Let $n \in \mathbb{N}$ be such that $nt_j + t_k > nt'_j + t'_k$. Then by similar arguments as in the last paragraph, we obtain a contradiction. □

Lemma 4. *If a rule is symmetric, endowment continuous, and consistent, then for each pair $i, j \in \mathbb{N}$ and each pair $(F,T),(F,T') \in \mathcal{F}^{\{i,j\}}$, $r_i(F,T) = r_j(F,T) \implies (r_i(F,T') - \frac{T}{2})(r_j(F,T') - \frac{T}{2}) \geq 0$.*

Proof. Let r be a rule that is *symmetric*, *endowment continuous*, and *consistent*. By Theorem 1, there is a function $M : \mathcal{F} \to \mathbb{R}_+$ such that for each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{P}^I$, and each $i \in I$, (1) $T < \sum M(F_j) \Rightarrow \sum r_j(F,T) = T$ and $r_i(F,T) \leq M(F_i)$, and (2) $T \geq \sum M(F_j) \Rightarrow r_i(F,T) = M(F_i)$. Moreover, by Lemma 3, r is *endowment monotonic*.

Let $i, j \in \mathbb{N}$ and $(F,T),(F,T') \in \mathcal{F}^{\{i,j\}}$ be such that $r_i(F,T) = r_j(F,T)$. To show that $(r_i(F,T') - \frac{T}{2})(r_j(F,T') - \frac{T}{2}) \geq 0$, suppose to the contrary and without loss of generality that $r_i(F,T') < \frac{T}{2} < r_j(F,T')$. Since $r_i(F,T) = r_j(F,T)$, $r_i(F,T) = r_j(F,T) \leq \frac{T}{2}$. Assume that $r_i(F,T) = r_j(F,T) = \frac{T}{2}$. Then $r_i(F,T') < \frac{T}{2} = r_i(F,T)$ and $r_j(F,T) = \frac{T}{2} < r_j(F,T')$, which contradicts *endowment monotonicity*. Assume that $r_i(F,T) = r_j(F,T) < \frac{T}{2}$. Note that $r_j(F,T') \leq M(F_j)$. Then $r_j(F,T) < \frac{T}{2} < r_j(F,T') \leq M(F_j)$. Since $r_j(F,T) < M(F_j)$, $T < M(F_i) + M(F_j)$, and thus, $r_i(F,T) + r_j(F,T) = T$. Since $r_i(F,T) + r_j(F,T) = T$ and $r_i(F,T) = r_j(F,T)$, $r_i(F,T) = r_j(F,T) = \frac{T}{2}$, which contradicts $r_i(F,T) = r_j(F,T) < \frac{T}{2}$. □

A.2.3 Proof of Theorem 2

The proof of the "if" direction is standard, and thus, is put in the online appendix.[46] To show the "only if" direction, let r be a rule satisfying *conditional strict ranking*, *continuity*, *double consistency*, and *coordinality*. By Lemma 2, r satisfies *ranking*, and thus *symmetry*. Since r is *continuous*, r is *endowment continuous*. By Lemma 1, r is *anonymous*. By Lemma 3, r is *endowment monotonic*. By *coordinality*, r is both *upper ordinal* and *lower ordinal*. By Theorem 1, there is a function $M : \mathcal{F} \to \mathbb{R}_+$ such that for each $I \in \mathcal{N}$, each

[46]In the online appendix, we prove both the "if" and the "only if" directions in a richer environment that allows for unbounded claims.

$(F,T) \in \mathcal{P}^I$, and each $i \in I$, (1) $T < \sum M(F_j) \Rightarrow \sum r_j(F,T) = T$ and $r_i(F,T) \leq M(F_i)$, and (2) $T \geq \sum M(F_j) \Rightarrow r_i(F,T) = M(F_i)$. We show, via the following steps, that r is an equal-quantile rule.

Step 1: Head symmetry. If each need that does not exceed a certain amount occurs with equal probabilities for two agents, then each endowment that does not exceed twice that amount is equally divided between them. Formally, for each pair $\{i, j\} \subseteq \mathbb{N}$, each $(F,T) \in \mathcal{P}^{\{i,j\}}$, and each $c \in \mathbb{R}_+$, if F_i and F_j agree on $(-\infty, c]$ and $T \leq 2c$, then $r_i(F,T) = r_j(F,T)$.

Let $\{i, j\} \subseteq \mathbb{N}$, $(F,T) \in \mathcal{P}^{\{i,j\}}$, and $c \in \mathbb{R}_+$ be such that F_i and F_j agree on $(-\infty, c]$ and $T \leq 2c$. To show that $r_i(F,T) = r_j(F,T)$, suppose to the contrary and without loss of generality that $r_i(F,T) < r_j(F,T)$. Since $r_i(F,T)+r_j(F,T) \leq T \leq 2c$ and $r_i(F,T) < r_j(F,T)$, $r_i(F,T) < \frac{T}{2} \leq c$. Let $a \in (r_i(F,T), \min\{c, r_j(F,T)\})$ and $b \in (\max\{C_i, C_j\}, \infty)$. Since F_i and F_j agree on $(-\infty, c]$ and $a < c$, and since $b > \max\{C_i, C_j\}$, F_i and F_j agree on $(-\infty, a) \cup [b, \infty)$. We now modify F_i and F_j so that the modified claims agree on $\mathbb{R}$. Specifically, since $r_i(F,T) < a < b$, we can enlarge each agent's needs in $(r_i(F,T), a)$ proportionally to those in $(r_i(F,T), b)$, transfer the probabilities of the original needs to the enlarged needs, and then transfer all the probability on $[a, b)$ to b. Formally, for each $k \in \{i, j\}$, define $F'_k : \mathbb{R} \to [0, 1]$ by setting for each $x_k \in \mathbb{R}$,

$$F'_k(x_k) := \begin{cases} F_k(x_k) & x_k \in (-\infty, r_i(F,T)) \cup [b, \infty) \\ F_k(r_i(F,T) + \frac{(x_k - r_i(F,T))(a - r_i(F,T))}{b - r_i(F,T)}) & x_k \in [r_i(F,T), b) \end{cases}.$$

It can be readily seen that for each $k \in \{i, j\}$, F'_k is a well-defined CDF whose support is a bounded interval, and $F'_i = F'_j$. Thus, by *symmetry*, $r_i((F'_i, F'_j), r_i(F,T) + b) = r_j((F'_i, F'_j), r_i(F,T) + b)$. On the other hand, we claim that $r_i((F'_i, F'_j), r_i(F,T) + b) = r_i(F,T)$ and $r_j((F'_i, F'_j), r_i(F,T) + b) = b$. Since $r_i(F,T) < b$, if our claim is true, then it contradicts the equality $r_i((F'_i, F'_j), r_i(F,T) + b) = r_j((F'_i, F'_j), r_i(F,T) + b)$.

To prove our claim, we construct for each agent $k \in \{i, j\}$ a sequence of claims that converges to F'_k and such that each claim in the sequence is obtained by applying some transformation on agent k's needs. Formally, for each $n \in \mathbb{N}$ with $n \geq 2$, let $\phi^n \in \Phi^u$ be the upper transformation defined by (6) with $d_1 = r_i(F,T)$, $d_2 = a$, $d_3 = \frac{1}{n}a + (1 - \frac{1}{n})b$, and $d_4 = b$. By *upper coordinality*, for each $n \in \mathbb{N}$ with $n \geq 2$ and each $k \in \{i, j\}$, $r_k((F_i^{\phi^n}, F_j^{\phi^n}), \phi^n(r_i(F,T)) + \phi^n(r_j(F,T))) = \phi^n(r_k(F,T))$. For each $k \in \{i, j\}$, it can be shown

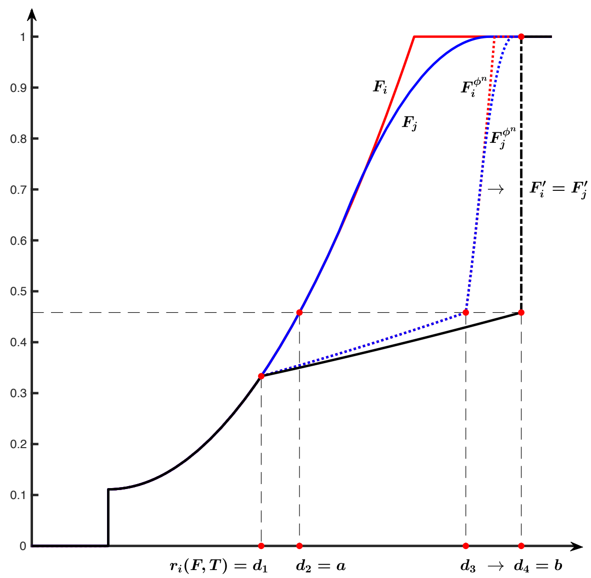

Figure 5: Convergence of transformed claims

by routine arguments that $F_k^{\phi^n}$ converges to F'_k.[47] See Figure 5 for an illustration. Since for each $n \in \mathbb{N}$ with $n \geq 2$, $\phi^n(r_i(F,T)) = r_i(F,T)$, $\lim \phi^n(r_i(F,T)) = r_i(F,T)$. Moreover, since $a < r_j(F,T) < b$, for each $n \in \mathbb{N}$ with $n \geq 2$, $\frac{1}{n}a + (1 - \frac{1}{n})b < \phi^n(r_j(F,T)) < b$, and thus $\lim \phi^n(r_j(F,T)) = b$. Then by *continuity*, $r_i((F'_i, F'_j), r_i(F,T) + b) = r_i(F,T)$ and $r_j((F'_i, F'_j), r_i(F,T) + b) = b$, as desired.

Step 2: Tail symmetry. If each need that exceeds a certain amount occurs with equal probabilities for two agents, then each endowment that exceeds twice that amount is equally divided between them. Formally, for each pair $\{i,j\} \subseteq \mathbb{N}$, each $(F,T) \in \mathcal{P}^{\{i,j\}}$, and each $c \in [0, \max\{M(F_i), M(F_j)\})$, if F_i and F_j agree on $[c, \infty)$ and $T \geq 2c$, then $r_i(F,T) = r_j(F,T)$.

Let $\{i,j\} \subseteq \mathbb{N}$, $(F,T) \in \mathcal{P}^{\{i,j\}}$, and $c \in [0, \max\{M(F_i), M(F_j)\})$ be such that F_i and F_j agree on $[c, \infty)$ and $T \geq 2c$. To show that $r_i(F,T) = r_j(F,T)$, suppose to the contrary and without loss of generality that $r_i(F,T) < r_j(F,T)$.

Assume that $r_j(F,T) \leq c$. Then $r_i(F,T) < r_j(F,T) \leq c < \max\{M(F_i), M(F_j)\}$, so that either $r_i(F,T) < M(F_i)$ or $r_j(F,T) < M(F_j)$. Then $T < M(F_i) + M(F_j)$, and thus, $r_i(F,T) + r_j(F,T) = T$. Since $r_i(F,T) + r_j(F,T) = T$ and $r_i(F,T) < r_j(F,T)$, $r_j(F,T) > \frac{T}{2} \geq$

[47] The proof is available upon request.

c, which contradicts $r_j(F,T) \leq c$.

Assume that $c < r_j(F,T)$. Let $a \in (\max\{c, r_i(F,T)\}, r_j(F,T))$. Since $F_i, F_j \in \mathcal{F}$, and since F_i and F_j agree on $[c,\infty)$ and $c < a$, F_i and F_j agree on $(-\infty,0) \cup [a,\infty)$. We now modify F_i and F_j so that the modified claims agree on $\mathbb{R}$. Specifically, since $0 < a < r_j(F,T)$, we can reduce each agent's ex-post needs in $(a, r_j(F,T))$ proportionally to those in $(0, r_j(F,T))$, transfer the probabilities of the original needs to the reduced needs, and then transfer all the probability on $(0,a]$ to 0. Formally, for each $k \in \{i,j\}$, define $F'_k : \mathbb{R} \to [0,1]$ by setting for each $x_k \in \mathbb{R}$,

$$F'_k(x_k) := \begin{cases} F_k(x_k) & x_k \in (-\infty,0) \cup [r_j(F,T),\infty) \\ F_k(r_j(F,T) - \frac{(r_j(F,T)-x_k)(r_j(F,T)-a)}{r_j(F,T)}) & x_k \in [0, r_j(F,T)) \end{cases}.$$

It is readily seen that for each $k \in \{i,j\}$, F'_k is a well-defined CDF whose support is a bounded interval, and $F'_i = F'_j$. Thus, by *symmetry*, $r_i((F'_i,F'_j), r_j(F,T)) = r_j((F'_i,F'_j), r_j(F,T))$. On the other hand, we claim that $r_i((F'_i,F'_j), r_j(F,T)) = 0$ and $r_j((F'_i,F'_j), r_j(F,T)) = r_j(F,T)$. Since $0 < r_j(F,T)$, if our claim is true, then it contradicts the equality $r_i((F'_i,F'_j), r_j(F,T)) = r_j((F'_i,F'_j), r_j(F,T))$.

To prove our claim, we construct for each agent $k \in \{i,j\}$ a sequence of claims that converges to F'_k and such that each claim in the sequence is obtained by applying some transformation on agent k's needs. Formally, for each $n \in \mathbb{N}$ with $n \geq 2$, let $\phi^n \in \Phi^l$ be a lower transformation defined by (8) with $d_1 = 0$, $d_2 = \frac{1}{n}a$, $d_3 = a$, and $d_4 = r_j(F,T)$. By *lower coordinality*, for each $n \in \mathbb{N}$ with $n \geq 2$ and each $k \in \{i,j\}$, $r_k((F_i^{\phi^n}, F_j^{\phi^n}), \phi^n(r_i(F,T)) + \phi^n(r_j(F,T))) = \phi^n(r_k(F,T))$. For each $k \in \{i,j\}$, it can be shown by routine arguments that $F_k^{\phi^n}$ converges to F'_k.[48] See Figure 6 for an illustration. Since $0 \leq r_i(F,T) < a$, for each $n \in \mathbb{N}$ with $n \geq 2$, $0 \leq \phi^n(r_i(F,T)) < \frac{1}{n}a$, and thus $\lim \phi^n(r_i(F,T)) = 0$. Moreover, since for each $n \in \mathbb{N}$ with $n \geq 2$, $\phi^n(r_j(F,T)) = r_j(F,T)$, $\lim \phi^n(r_j(F,T)) = r_j(F,T)$. Then by *continuity*, $r_i((F'_i,F'_j), r_j(F,T)) = 0$ and $r_j((F'_i,F'_j), r_j(F,T)) = r_j(F,T)$, as desired.

Step 3: Head irrelevance. Changing the probability distributions of agents' needs that are smaller than their assignments does not affect the allocation. Formally, for each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{P}^I$, and each $F' \in \mathcal{F}^I$, if for each $i \in I$, $\lim_{x_i \uparrow r_i(F,T)} F'_i(x_i) \leq \lim_{x_i \uparrow r_i(F,T)} F_i(x_i)$, $C'_i \geq r_i(F,T)$, and F_i and F'_i agree on $[r_i(F,T),\infty)$, then $r(F,T) = r(F',T)$.

[48] The proof is available upon request.

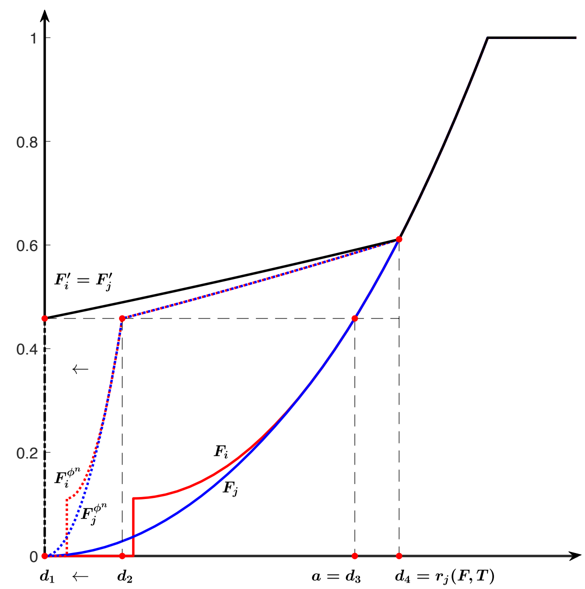

Figure 6: Convergence of transformed claims

Let $I \in \mathcal{N}$ and $(F,T) \in \mathcal{P}^I$. Let $i \in I$ and $F_i' \in \mathcal{F}$ be such that $\lim_{x_i \uparrow r_i(F,T)} F_i'(x_i) \leq \lim_{x_i \uparrow r_i(F,T)} F_i(x_i)$, $C_i' \geq r_i(F,T)$, and F_i and F_i' agree on $[r_i(F,T), \infty)$. To prove Step 3, it suffices to prove that $r(F,T) = r((F_i', F_{I\setminus\{i\}}), T)$. Let $j \in \mathbb{N} \setminus I$ and $G \in \mathcal{F}^{I\cup\{j\}}$ be such that for each $k \in I$, $G_k = F_k$, and $G_j = F_i'$. Let $t := r(G, T + r_i(F,T))$. By *double consistency* and *anonymity*, it suffices to show that $t_j = r_i(F,T)$. We divide the proof into the following two cases.

Case 1: There is $\epsilon > 0$ such that F_i and F_i' agree on $[r_i(F,T) - \epsilon, \infty)$. To show that $t_j = r_i(F,T)$, suppose to the contrary that $t_j \neq r_i(F,T)$. We derive a contradiction either when $t_j > r_i(F,T)$ or when $t_j < r_i(F,T)$.

Assume first that $t_j > r_i(F,T)$. Then $T + r_i(F,T) - t_j < T$. By *double consistency* and *endowment monotonicity*, $t_i = r_i(F, T + r_i(F,T) - t_j) \leq r_i(F,T)$. Let $c \in (r_i(F,T), t_j)$. Since F_i and F_i' agree on $[r_i(F,T), \infty)$, G_i and G_j agree on $[c, \infty)$. Note that $t_j \leq M(G_j)$. Since $c \in (r_i(F,T), t_j)$ and $t_j \leq M(G_j)$, $c \in (0, \max\{M(G_i), M(G_j)\})$. Thus by Step 2, $r_i((G_i, G_j), 2c) = r_j((G_i, G_j), 2c)$. By *double consistency*, $r((G_i, G_j), t_i + t_j) = (t_i, t_j)$. Thus by Lemma 4, $(t_i - c)(t_j - c) \geq 0$. Since $t_i \leq r_i(F,T)$ and $c \in (r_i(F,T), t_j)$, $t_i < c < t_j$, which contradicts $(t_i - c)(t_j - c) \geq 0$.

Assume now that $t_j < r_i(F,T)$. Then $T + r_i(F,T) - t_j > T$. By *double consistency* and *endowment monotonicity*, $t_i = r_i(F, T + r_i(F,T) - t_j) \geq r_i(F,T)$. Let $c \in (\max\{r_i(F,T) - \epsilon, t_j\}, r_i(F,T))$. Since F_i and F'_i agree on $[r_i(F,T) - \epsilon, \infty)$, G_i and G_j agree on $[c, \infty)$. Note that $r_i(F,T) \leq M(F_i)$, and thus $r_i(F,T) \leq M(G_i)$. Since $c \in (\max\{r_i(F,T) - \epsilon, t_j\}, r_i(F,T))$ and $r_i(F,T) \leq M(G_i)$, $c \in (0, \max\{M(G_i), M(G_j)\})$. Thus by Step 2, $r_i((G_i, G_j), 2c) = r_j((G_i, G_j), 2c)$. By *double consistency*, $r((G_i, G_j), t_i + t_j) = (t_i, t_j)$. Thus by Lemma 4, $(t_i - c)(t_j - c) \geq 0$. Since $t_i \geq r_i(F,T)$ and $c \in (\max\{r_i(F,T) - \epsilon, t_j\}, r_i(F,T))$, $t_j < c < t_i$, which contradicts the inequality $(t_i - c)(t_j - c) \geq 0$.

Case 2: There is no $\epsilon > 0$ such that F_i and F'_i agree on $[r_i(F,T) - \epsilon, \infty)$. In this case, we construct a sequence $\{F'^n_i\}_{n=1}^{\infty}$ of agent i's claims that converges to F'_i and such that for each $n \in \mathbb{N}$, F'^n_i agrees with F_i on $[r_i(F,T) - \epsilon, \infty)$ for some $\epsilon > 0$, so that we can use *continuity* and the result in Case 1 to show that $t_j = r_i(F,T)$.

To construct such a sequence $\{F'^n_i\}_{n=1}^{\infty}$, we first construct two sequences $\{a^n\}_{n=1}^{\infty}$ and $\{b^n\}_{n=1}^{\infty}$ of elements of $\mathbb{R}$ such that (1) $\lim a^n = r_i(F,T)$ and (2) for each $n \in \mathbb{N}$, $a^n < b^n < r_i(F,T)$ and $F'_i(a^n) < F_i(b^n)$. Condition (2) ensures that for each $n \in \mathbb{N}$, F'^n_i can be constructed such that it agrees with F'_i on $(-\infty, a^n]$ and with F_i on $[b^n, \infty)$. Formally, for each $n \in \mathbb{N}$, let $a^n := \frac{1}{n}c_i + (1 - \frac{1}{n})r_i(F,T)$. Clearly, condition (1) is satisfied. For each $n \in \mathbb{N}$, to see that we can find b^n such that a^n and b^n satisfy condition (2), it suffices to show that $c_i < r_i(F,T)$ and $F'_i(a^n) < \lim_{x_i \uparrow r_i(F,T)} F_i(x_i)$.

To see that $c_i < r_i(F,T)$, suppose to the contrary that $r_i(F,T) \leq c_i$. Then $\lim_{x_i \uparrow r_i(F,T)} F_i(x_i) = 0$. Since $0 \leq \lim_{x_i \uparrow r_i(F,T)} F'_i(x_i) \leq \lim_{x_i \uparrow r_i(F,T)} F_i(x_i) = 0$, and since F_i and F'_i agree on $[r_i(F,T), \infty)$, F_i and F'_i agree on $\mathbb{R}$. This contradicts our assumption that there is no $\epsilon > 0$ such that F_i and F'_i agree on $[r_i(F,T) - \epsilon, \infty)$.

For each $n \in \mathbb{N}$, to see that $F'_i(a^n) < \lim_{x_i \uparrow r_i(F,T)} F_i(x_i)$, suppose to the contrary that there is $m \in \mathbb{N}$ such that $\lim_{x_i \uparrow r_i(F,T)} F_i(x_i) \leq F'_i(a^m)$. Since $c_i < r_i(F,T)$, by definition, $a^m < a^{m+1} < r_i(F,T)$. Thus, $F'_i(a^m) \leq F'_i(a^{m+1}) \leq \lim_{x_i \uparrow r_i(F,T)} F'_i(x_i) \leq \lim_{x_i \uparrow r_i(F,T)} F_i(x_i) \leq F'_i(a^m)$. Hence, $F'_i(a^m) = F'_i(a^{m+1}) = \lim_{x_i \uparrow r_i(F,T)} F_i(x_i)$. Since $C'_i \geq r_i(F,T) > a^{m+1} > a^m$ and $F'_i(a^m) = F'_i(a^{m+1})$, and since F'_i is increasing on $[c'_i, C'_i]$, $a^m < c'_i$, and thus $F'_i(a^m) = 0$. Then $\lim_{x_i \uparrow r_i(F,T)} F_i(x_i) = F'_i(a^m) = 0$, which contradicts $c_i < r_i(F,T)$.

For each $n \in \mathbb{N}$, define $F_i'^n : \mathbb{R} \to [0,1]$ by setting for each $x_i \in \mathbb{R}$,

$$F_i'^n(x_i) := \begin{cases} F_i'(x_i) & x_i \in (-\infty, a^n) \\ F_i'(a^n) + \frac{[F_i(b^n) - F_i'(a^n)](x_i - a^n)}{b^n - a^n} & x_i \in [a^n, b^n) \\ F_i(x_i) & x_i \in [b^n, \infty) \end{cases}.$$

For each $n \in \mathbb{N}$, since $a^n < b^n$ and $F_i'(a^n) < F_i(b^n)$, $F_i'^n$ is a well-defined CDF, and it can be readily seen that the support of $F_i'^n$ is a bounded interval. We claim that $F_i'^n$ converges to F_i'. We first check that $F_i'^n$ weakly converges to F_i'. Note that for each $x_i \in (-\infty, r_i(F,T))$ and for sufficiently large $n \in \mathbb{N}$, $x_i < a^n$, and thus, $F_i'^n(x_i) = F_i'(x_i)$. Moreover, for each $x_i \in [r_i(F,T), \infty)$ and each $n \in \mathbb{N}$, $F_i'^n(x_i) = F_i(x_i) = F_i'(x_i)$. Hence, for each $x_i \in \mathbb{R}$, $\lim F_i'^n(x_i) = F_i'(x_i)$. We then check $\lim c_i'^n = c_i'$. Since $c_i < r_i(F,T)$ and $F_i(r_i(F,T)) = F_i'(r_i(F,T))$, $c_i' \leq r_i(F,T)$. Assume $c_i' < r_i(F,T)$. Then for sufficiently large $n \in \mathbb{N}$, $c_i' < a^n$. For sufficiently large $n \in \mathbb{N}$, since $F_i'^n$ and F_i' agree on $(-\infty, a^n)$ and $c_i' < a^n$, $c_i'^n = c_i'$. Assume $c_i' = r_i(F,T)$. Then for each $n \in \mathbb{N}$, $a^n < c_i'$. For each $n \in \mathbb{N}$, since $F_i'^n$ and F_i' agree on $(-\infty, a^n)$ and $a^n < c_i'$, and since $F_i'^n$ is increasing on $[a^n, b^n)$, $c_i'^n = a^n$. In either case, $\lim c_i'^n = c_i'$. Lastly, we check $\lim C_i'^n = C_i'$. By the definition of a rule, $C_i \geq r_i(F,T)$. Assume $C_i > r_i(F,T)$. For each $n \in \mathbb{N}$, since $F_i'^n$, F_i, and F_i' agree on $[r_i(F,T), \infty)$ and $C_i > r_i(F,T)$, $C_i'^n = C_i = C_i'$. Assume $C_i = r_i(F,T)$. Since F_i and F_i' agree on $[r_i(F,T), \infty)$ and $C_i' \geq r_i(F,T) = C_i$, $C_i' = C_i$. For each $n \in \mathbb{N}$, since $F_i'^n$ and F_i agree on $[b^n, \infty)$ and $C_i = r_i(F,T) > b^n$, $C_i'^n = C_i$. Thus, for each $n \in \mathbb{N}$, $C_i'^n = C_i = C_i'$. In either case, $\lim C_i'^n = C_i'$. Hence, $F_i'^n$ converges to F_i'.

For each $n \in \mathbb{N}$, let $G^n \in \mathcal{F}^{I \cup \{j\}}$ be such that for each $k \in I$, $G_k^n = F_k$, and $G_j^n = F_i'^n$. For each $n \in \mathbb{N}$, since $b^n < r_i(F,T)$ and F_i and $F_i'^n$ agree on $[b^n, \infty)$, by applying the same arguments as in Case 1, $r_j(G^n, T + r_i(F,T)) = r_i(F,T)$. Since for each $n \in \mathbb{N}$, $r_j(G^n, T + r_i(F,T)) = r_i(F,T)$, and since $F_i'^n$ converges to F_i', by *continuity*, $t_j = r_i(F,T)$, as desired.

Step 4: Tail irrelevance. Changing the probability distributions of agents' needs that are larger than their assignments does not affect the allocation. Formally, for each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{P}^I$, and each $F' \in \mathcal{F}^I$, if for each $i \in I$, $c_i = r_i(F,T)$ implies $c_i' = r_i(F,T)$, and F_i and F_i' agree on $(-\infty, r_i(F,T)]$, then $r(F,T) = r(F',T)$.

Let $I \in \mathcal{N}$ and $(F,T) \in \mathcal{P}^I$. Let $i \in I$ and $F_i' \in \mathcal{F}$ be such that $c_i = r_i(F,T)$ implies $c_i' = r_i(F,T)$, and F_i and F_i' agree on $(-\infty, r_i(F,T)]$. Since F_i and F_i' agree on $(-\infty, r_i(F,T)]$,

$C_i' \geq r_i(F,T)$. To prove Step 4, it suffices to show that $r(F,T) = r((F_i', F_{I\setminus\{i\}}), T)$. Let $j \in \mathbb{N} \setminus I$. Let $G \in \mathcal{F}^{I\cup\{j\}}$ be such that for each $k \in I$, $G_k = F_k$, and $G_j = F_i'$. Let $t := r(G, T + r_i(F,T))$. By *double consistency* and *anonymity*, it suffices to show that $t_j = r_i(F,T)$. We divide the proof into the following two cases.

Case 1: There is $\epsilon > 0$ such that F_i and F_i' agrees on $(-\infty, r_i(F,T) + \epsilon]$. To show that $t_j = r_i(F,T)$, suppose to the contrary that $t_j \neq r_i(F,T)$. We will derive contradictions, respectively, when $t_j > r_i(F,T)$ and when $t_j < r_i(F,T)$.

Assume first that $t_j > r_i(F,T)$. Then $T + r_i(F,T) - t_j < T$. By *double consistency* and *endowment monotonicity*, $t_i = r_i(F, T + r_i(F,T) - t_j) \leq r_i(F,T)$. Let $c \in (r_i(F,T), \min\{t_j, r_i(F,T) + \epsilon\})$. Since F_i and F_i' agree on $(-\infty, r_i(F,T) + \epsilon]$, G_i and G_j agree on $(-\infty, c]$. By Step 1, $r_i((G_i, G_j), 2c) = r_j((G_i, G_j), 2c)$. By *double consistency*, $r((G_i, G_j), t_i + t_j) = (t_i, t_j)$. Thus by Lemma 4, $(t_i - c)(t_j - c) \geq 0$. Since $t_i \leq r_i(F,T)$ and $c \in (r_i(F,T), \min\{t_j, r_i(F,T) + \epsilon\})$, $t_i < c < t_j$, which contradicts $(t_i - c)(t_j - c) \geq 0$.

Assume now that $t_j < r_i(F,T)$. Then $T + r_i(F,T) - t_j > T$. By *double consistency* and *endowment monotonicity*, $t_i = r_i(F, T + r_i(F,T) - t_j) \geq r_i(F,T)$. Let $c \in (t_j, r_i(F,T))$. Since F_i and F_i' agree on $(-\infty, r_i(F,T)]$, G_i and G_j agree on $(-\infty, c]$. By Step 1, $r_i((G_i, G_j), 2c) = r_j((G_i, G_j), 2c)$. By *double consistency*, $r((G_i, G_j), t_i + t_j) = (t_i, t_j)$. Thus by Lemma 4, $(t_i - c)(t_j - c) \geq 0$. Since $t_i \geq r_i(F,T)$ and $c \in (t_j, r_i(F,T))$, $t_j < c < t_i$, which contradicts $(t_i - c)(t_j - c) \geq 0$.

Case 2: There is no $\epsilon > 0$ such that F_i and F_i' agrees on $(-\infty, r_i(F,T) + \epsilon]$. In this case, we are going to construct a sequence $\{F_i'^n\}_{n=1}^\infty$ of agent i's claims that converges to F_i' and such that for each $n \in \mathbb{N}$, $F_i'^n$ agrees with F_i on $(-\infty, r_i(F,T) + \epsilon]$ for some $\epsilon > 0$, so that we can use *continuity* and the result in Case 1 to show that $t_j = r_i(F,T)$.

To construct such a sequence $\{F_i'^n\}_{n=1}^\infty$, we first construct two sequences $\{a^n\}_{n=1}^\infty$ and $\{b^n\}_{n=1}^\infty$ of elements of $\mathbb{R}$ such that (1) $\lim b^n = r_i(F,T)$ and (2) for each $n \in \mathbb{N}$, $r_i(F,T) < a^n < b^n$ and $F_i(a^n) < F_i'(b^n)$. Condition (2) ensures that for each $n \in \mathbb{N}$, $F_i'^n$ can be constructed such that it agrees with F_i on $(-\infty, a^n]$ and with F_i' on $[b^n, \infty)$. Formally, for each $n \in \mathbb{N}$, let $b^n := r_i(F,T) + \frac{1}{n}$. Clearly, condition (1) is satisfied. For each $n \in \mathbb{N}$, to see that we can find a^n such that a^n and b^n satisfy condition (2), it suffices to show that $F_i(r_i(F,T)) < F_i'(b^n)$. To do this, we first show that $c_i' \leq r_i(F,T) < C_i$.

To see that $r_i(F,T) < C_i$, suppose to the contrary that $r_i(F,T) \geq C_i$. Then $F_i(r_i(F,T)) = 1$. Since $F_i(r_i(F,T)) = 1$ and F_i and F_i' agree on $(-\infty, r_i(F,T)]$, F_i and F_i' agree on $\mathbb{R}$, which contradicts our assumption that there is no $\epsilon > 0$ such that F_i and F'_i agrees on $(-\infty, r_i(F,T) + \epsilon]$.

To see that $r_i(F,T) \geq c'_i$, suppose to the contrary that $r_i(F,T) < c'_i$. Since $r_i(F,T) < c'_i$ and F_i and F'_i agree on $(-\infty, r_i(F,T)]$, $r_i(F,T) \leq c_i$. Since $r_i(F,T) < c'_i$ and $c_i = r_i(F,T)$ implies $c'_i = r_i(F,T)$, $c_i \neq r_i(F,T)$. Thus, $r_i(F,T) < c_i$. The fact that $r_i(F,T) < c_i$ and $r_i(F,T) < c'_i$ contradicts our assumption that there is no $\epsilon > 0$ such that F_i and F'_i agrees on $(-\infty, r_i(F,T) + \epsilon]$.

Now we show that for each $n \in \mathbb{N}$, $F_i(r_i(F,T)) < F'_i(b^n)$. Let $n \in \mathbb{N}$. If $F'_i(b^n) = 1$, since $r_i(F,T) < C_i$, $F_i(r_i(F,T)) < 1 = F'_i(b^n)$. Assume that $F'_i(b^n) < 1$. Then $C'_i > b^n$. Since $C'_i > b^n > r_i(F,T) \geq c'_i$ and F'_i is increasing on $[c'_i, C'_i]$, $F'_i(b^n) > F'_i(r_i(F,T))$. Since F_i and F'_i agree on $(-\infty, r_i(F,T)]$, $F_i(r_i(F,T)) = F'_i(r_i(F,T))$. Thus, $F'_i(b^n) > F_i(r_i(F,T))$.

For each $n \in \mathbb{N}$, define $F'^n_i : \mathbb{R} \to [0,1]$ by setting for each $x_i \in \mathbb{R}$,

$$F'^n_i(x_i) := \begin{cases} F_i(x_i) & x_i \in (-\infty, a^n) \\ F_i(a^n) + \frac{[F'_i(b^n) - F_i(a^n)](x_i - a^n)}{b^n - a^n} & x_i \in [a^n, b^n) \\ F'_i(x_i) & x_i \in [b^n, -\infty) \end{cases} .$$

For each $n \in \mathbb{N}$, since $a^n < b^n$ and $F_i(a^n) < F'_i(b^n)$, F'^n_i is a well-defined CDF, and it can be readily seen that the support of F'^n_i is a bounded interval. We claim that F'^n_i converges to F'_i. We first check that F'^n_i weakly converges to F'_i. Note that for each $x_i \in (-\infty, r_i(F,T)]$ and each $n \in \mathbb{N}$, $F'^n_i(x_i) = F_i(x_i) = F'_i(x_i)$. Moreover, for each $x_i \in (r_i(F,T), \infty)$ and for sufficiently large $n \in \mathbb{N}$, $x_i \geq b^n$, and thus, $F'^n_i(x_i) = F'_i(x_i)$. Hence, for each $x_i \in \mathbb{R}$, $\lim F'^n_i(x_i) = F_i(x_i)$. We then check that $\lim c'^n_i = c'_i$. Recall that $r_i(F,T) \geq c'_i$. Assume that $r_i(F,T) > c'_i$. Since F_i and F'_i agree on $(-\infty, r_i(F,T)]$, for each $n \in \mathbb{N}$, $c'^n_i = c'_i$. Assume that $r_i(F,T) = c'_i$. For each $n \in \mathbb{N}$, since F_i and F'_i agree on $(-\infty, r_i(F,T)]$ and $F'^n_i(b^n) = F'_i(b^n) > F_i(a^n) \geq 0$, $r_i(F,T) \leq c'^n_i \leq b^n = r_i(F,T) + \frac{1}{n}$. In either case, $\lim c'^n_i = c'_i$. Lastly, we check $\lim C'^n_i = C'_i$. Since $r_i(F,T) < C_i$ and $F'_i(r_i(F,T)) = F_i(r_i(F,T))$, $r_i(F,T) < C'_i$. Then for sufficiently large $n \in \mathbb{N}$, $b^n < C'_i$, and thus $C'^n_i = C'_i$. Hence, $\lim C'^n_i = C'_i$. Thus, F'^n_i converges to F'_i.

For each $n \in \mathbb{N}$, let $G^n \in \mathcal{F}^{I \cup \{j\}}$ be such that for each $k \in I$, $G^n_k = F_k$, and $G^n_j = F'^n_i$. For each $n \in \mathbb{N}$, since $a^n > r_i(F,T)$ and F_i and F'^n_i agree on $(-\infty, a^n]$, by applying the same arguments as in Case 1, $r_j(G^n, T + r_i(F,T)) = r_i(F,T)$. Since for each $n \in \mathbb{N}$, $r_j(G^n, T + r_i(F,T)) = r_i(F,T)$, and since F'^n_i converges to F'_i, by *continuity*, $t_j = r_i(F,T)$, as desired.

Step 5: No domination. No agent is assigned a larger amount of the resource than another agent while having a larger probability of generating waste than the other agent. Formally, for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $\{i, j\} \subseteq I$, if $r_i(F, T) > r_j(F, T)$, then $\lim_{x_i \uparrow r_i(F,T)} F_i(x_i) \leq F_j(r_j(F, T))$.

Let $I \in \mathcal{N}$, $(F, T) \in \mathcal{P}^I$, and $\{i, j\} \subseteq I$. Let $t := r(F, T)$. Suppose to the contrary that $t_i > t_j$ and $\lim_{x_i \uparrow t_i} F_i(x_i) > F_j(t_j)$. We construct two claims $F'_i, F'_j \in \mathcal{F}$, respectively, of agents i and j such that $F'_j \succsim_{FSD} F'_i$ and $r((F'_i, F'_j, F_{I\setminus\{i,j\}}), T) = t$. Then by *ranking*, $t_i \leq t_j$, which contradicts that $t_i > t_j$, as desired.

We construct F'_i and F'_j by changing the "head" of F_i and the "tail" of F_j. Formally, define $F'_i : \mathbb{R} \to [0, 1]$ and $F'_j : \mathbb{R} \to [0, 1]$ by setting for each $c \in \mathbb{R}$,

$$F'_i(c) := \begin{cases} 0 & c \in (-\infty, 0) \\ \lim_{x_i \uparrow t_i} F_i(x_i) - \frac{t_i - c}{2t_i}[\lim_{x_i \uparrow t_i} F_i(x_i) - F_j(t_j)] & c \in [0, t_i) \\ F_i(c) & c \in [t_i, \infty) \end{cases}$$

and

$$F'_j(c) := \begin{cases} F_j(c) & c \in (-\infty, t_j) \\ F_j(t_j) + \frac{c - t_j}{2(C_i - t_j)}[\lim_{x_i \uparrow t_i} F_i(x_i) - F_j(t_j)] & c \in [t_j, C_i) \\ 1 & c \in [C_i, \infty) \end{cases}.$$

Since $F_j(t_j) < \lim_{x_i \uparrow t_i} F_i(x_i) \leq F_i(t_i) \leq 1$ and $0 \leq t_j < t_i \leq C_i$, F'_i, F'_j are well-defined CDFs and the support of each of these CDFs is a bounded interval.

We check that $F'_j \succsim_{FSD} F'_i$. For each $c \in (-\infty, 0)$, $F'_i(c) = 0 = F_j(c) = F'_j(c)$. For each $c \in [0, t_i)$, $F'_i(c) \geq F'_i(0) = \frac{1}{2} \lim_{x_i \uparrow t_i} F_i(x_i) + \frac{1}{2} F_j(t_j) = \lim_{x_j \uparrow C_i} F'_j(x_j) \geq F'_j(c)$. For each $c \in [t_i, C_i)$, $F'_i(c) = F_i(c) \geq F_i(t_i) \geq \lim_{x_i \uparrow t_i} F_i(x_i) > \frac{1}{2} \lim_{x_i \uparrow t_i} F_i(x_i) + \frac{1}{2} F_j(t_j) = \lim_{x_j \uparrow C_i} F'_j(x_j) \geq F'_j(c)$. For each $c \in [C_i, \infty)$, $F'_i(c) = F_i(c) = 1 = F'_j(c)$. Hence, $F'_j \succsim_{FSD} F'_i$.

Lastly, we check that $r((F'_i, F'_j, F_{I\setminus\{i,j\}}), T) = t$. It is readily seen that $\lim_{x_i \uparrow t_i} F'_i(x_i) = \lim_{x_i \uparrow t_i} F_i(x_i)$, $C'_i \geq t_i$, and F_i and F'_i agree on $[t_i, \infty)$. Thus, by Step 3, $r((F'_i, F_{I\setminus\{i\}}), T) = t$. Moreover, if $c_j = t_j$, since F_j and F'_j agree on $(-\infty, t_j]$ and for each $c > t_j$, $F'_j(c) > 0$, then $c'_j = t_j$. Since $c_j = t_j$ implies $c'_j = t_j$ and F_j and F'_j agree on $(-\infty, t_j]$, by Step 4, $r((F'_i, F'_j, F_{I\setminus\{i,j\}}), T) = t$.

Step 6: No vertical domination. No agent is assigned an equal amount of the resource as another agent while having a larger probability of generating waste than the other agent.

Formally, for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $\{i, j\} \subseteq I$, if $r_i(F, T) = r_j(F, T)$, then

$$[\lim_{x_i \uparrow r_i(F,T)} F_i(x_i), F_i(r_i(F, T))] \cap [\lim_{x_j \uparrow r_j(F,T)} F_j(x_j), F_j(r_j(F, T))] \neq \emptyset.$$

Let $I \in \mathcal{N}$, $(F, T) \in \mathcal{P}^I$, $\{i, j\} \subseteq I$, and $t := r(F, T)$. Suppose to the contrary and without loss of generality that $t_i = t_j$ and $F_i(t_i) < \lim_{x_j \uparrow t_j} F_j(x_j)$. Then $t_j > 0$. We construct two claims $F'_i, F'_j \in \mathcal{F}$, respectively, of agents i and j such that $F'_i \succ_{FSD} F'_j$ and $r((F'_i, F'_j, F_{I\setminus\{i,j\}}), T) = t$. Then by *conditional strict ranking*, $t_i > t_j$, which contradicts that $t_i = t_j$, as desired.

We construct F'_i by first changing the head of F_i and then the tail. First, define $F''_i : \mathbb{R} \to [0, 1]$ by setting for each $x_i \in \mathbb{R}$,

$$F''_i(x_i) := \begin{cases} 0 & x_i \in (-\infty, t_i) \\ F_i(x_i) & x_i \in [t_i, \infty) \end{cases}.$$

It can be readily seen that F''_i is a well-defined CDF whose support is a bounded interval. Moreover, $\lim_{x_i \uparrow t_i} F''_i(x_i) = 0 \leq \lim_{x_i \uparrow t_i} F_i(x_i)$, $C''_i \geq t_i$, and F_i and F''_i agree on $[t_i, \infty)$. Thus, by Step 3, $r((F''_i, F_{I\setminus\{i\}}), T) = t$.

Then define $F'_i : \mathbb{R} \to [0, 1]$ by setting for each $x_i \in \mathbb{R}$,

$$F'_i(x_i) := \begin{cases} 0 & x_i \in (-\infty, t_i) \\ F_i(t_i) + \frac{x_i - t_i}{C_j + 1 - t_i}[\lim_{x_j \uparrow t_j} F_j(x_j) - F_i(t_i)] & x_i \in [t_i, C_j + 1) \\ 1 & x_i \in (C_j + 1, \infty) \end{cases}.$$

Since $t_i = t_j < C_j + 1$ and $F_i(t_i) < \lim_{x_j \uparrow t_j} F_j(x_j) \leq 1$, F'_i is a well-defined CDF, and it can be readily seen that the support of F'_i is $[t_i, C_j + 1]$. Since $c'_i = t_i$ and F''_i and F'_i agree on $(-\infty, t_i]$, by Step 4, $r((F'_i, F_{I\setminus\{i\}}), T) = t$.

We construct F'_j by changing the head of F_j. Since $\lim_{x_j \uparrow t_j} F_j(x_j) > F_i(t_i)$, $\lim_{x_j \uparrow t_j} F_j(x_j) > 0$. Thus, we can pick $t'_j \in (0, t_j)$ such that $F_j(t'_j) > 0$. Define $F'_j : \mathbb{R} \to [0, 1]$ by setting for each $x_j \in \mathbb{R}$,

$$F'_j(x_j) := \begin{cases} 0 & x_j \in (-\infty, 0) \\ \frac{t'_j + x_j}{2t'_j} F_j(t'_j) & x_j \in [0, t'_j) \\ F_j(x_j) & x_j \in [t'_j, \infty) \end{cases}.$$

Since $t'_j > 0$ and $F_j(t'_j) > 0$, F'_j is a well-defined CDF, and it can be readily seen that the support of F'_j is $[0, C_j]$. Since $t_j > t'_j$, $\lim_{x_j \uparrow t_j} F'_j(x_j) = \lim_{x_j \uparrow t_j} F_j(x_j)$ and F_j and F'_j agree on $[t_j, \infty)$. Moreover, $C'_j = C_j \geq t_j$. Thus, by Step 3, $r((F'_i, F'_j, F_{I \setminus \{i,j\}}), T) = t$.

Lastly, we check that $F'_i >_{FSD} F'_j$. Since $C'_j = C_j \geq t_j > 0$, we check that for each $c \in (0, C_j]$, $F'_i(c) < \lim_{c' \uparrow c} F'_j(c')$. For each $c \in (0, t_i)$, $F'_i(c) = 0 < \frac{1}{2} F_j(t'_j) = F'_j(0) \leq \lim_{c' \uparrow c} F'_j(c')$. For each $c \in [t_i, C_j]$, since $t'_j < t_j = t_i \leq c < C_j + 1$, $F'_i(c) < \lim_{x_i \uparrow C_j + 1} F'_i(x_i) = \lim_{x_j \uparrow t_j} F_j(x_j) \leq \lim_{x_j \uparrow c} F_j(x_j) = \lim_{c' \uparrow c} F'_j(c')$. Hence, $F'_i >_{FSD} F'_j$.

Step 7: No "unjustifiable" domination. Consider an arbitrary two-agent problem with agents, say, i and j. We say that agent j's probability of generating waste in the two-agent problem is unjustifiably large if it is larger than a third agent's, say, k's, probability of generating waste in a three-agent problem, with agents i, j, k, in which agent i is assigned the same amount as in the two-agent problem. Agent j cannot receive a larger amount of the resource than agent i in the two-agent problem while having an unjustifiably large probability of generating waste. Formally, let $\{i, j, k\} \subseteq \mathbb{N}$, and let for each $h \in \{i, j, k\}$, $F_h \in \mathcal{F}$ and $t_h \in [0, C_h]$ be such that $r((F_i, F_j), t_i + t_j) = (t_i, t_j)$, $t_i < t_j$, and for each $T \in [0, t_i + t_j)$, $r_i((F_i, F_j), T) < t_i$. If $r((F_i, F_k), t_i + t_k) = (t_i, t_k)$, then $\lim_{x_j \uparrow t_j} F_j(x_j) \leq F_k(t_k)$.

Let $\{i, j, k\} \subseteq \mathbb{N}$ and let for each $h \in \{i, j, k\}$, $F_h \in \mathcal{F}$ and $t_h \in [0, C_h]$ be such that $r((F_i, F_j), t_i + t_j) = (t_i, t_j)$, $t_i < t_j$, and for each $T \in [0, t_i + t_j)$, $r_i((F_i, F_j), T) < t_i$. Suppose to the contrary that $r((F_i, F_k), t_i + t_k) = (t_i, t_k)$ and $\lim_{x_j \uparrow t_j} F_j(x_j) > F_k(t_k)$.

Let $t'_j > \max\{t_j, t_k\}$. Let $\phi \in \Phi^u$ be an upper transformation defined by (6) with $d_1 = t_i$, $d_2 = t_j$, $d_3 = t'_j$, and $d_4 > \max\{t'_j, C_j\}$. By *upper coordinality*, $r((F^\phi_i, F^\phi_j), \phi(t_i) + \phi(t_j)) = (\phi(t_i), \phi(t_j))$. Thus, $r((F^\phi_i, F^\phi_j), t_i + t'_j) = (t_i, t'_j)$. We claim that $r((F_i, F^\phi_j), t_i + t'_j) = (t_i, t'_j)$. To see this, note first that since $t_j > t_i \geq 0$ and for each $T \in [0, t_i + t_j)$, $r_i((F_i, F_j), T) < t_i$, $t_i > 0$. We then check that $c^\phi_i = t_i$ implies $c_i = t_i$. Assume that $c^\phi_i = t_i$. Then for each $x_i \in [0, t_i)$, since $\phi(x_i) < t_i$, $F_i(x_i) = F^\phi_i(\phi(x_i)) = 0$, and for each $x_i \in (t_i, \infty)$, since $\phi(x_i) > t_i$, $F_i(x_i) = F^\phi_i(\phi(x_i)) > 0$. Thus, $c_i = t_i$. Since $c^\phi_i = t_i$ implies $c_i = t_i$ and F^ϕ_i and F_i agree on $(-\infty, t_i]$, by Step 4, $r((F_i, F^\phi_j), t_i + t'_j) = (t_i, t'_j)$. Let $t'' := r((F_i, F^\phi_j, F_k), t_i + t'_j + t_k)$. We will derive a contradiction, respectively, when $t''_j \geq t'_j$ and when $t''_j < t'_j$.

Assume first that $t''_j \geq t'_j$. Then $t_i + t'_j + t_k - t''_j \leq t_i + t_k$. By *double consistency* and *endowment monotonicity*, $t''_k = r_k((F_i, F_k), t_i + t'_j + t_k - t''_j) \leq r_k((F_i, F_k), t_i + t_k) = t_k$. Thus, $F_k(t''_k) \leq F_k(t_k) < \lim_{x_j \uparrow t_j} F_j(x_j) = \lim_{x_j \uparrow t'_j} F^\phi_j(x_j) \leq \lim_{x_j \uparrow t''_j} F^\phi_j(x_j)$, where the strict inequality holds by our initial hypothesis. On the other hand, $t''_j \geq t'_j > t_k \geq t''_k$, where the strict inequality follows from our choice of t'_j. Since $t''_j > t''_k$, by Step 5, $\lim_{x_j \uparrow t''_j} F^\phi_j(x_j) \leq F_k(t''_k)$, which contradicts $F_k(t''_k) < \lim_{x_j \uparrow t''_j} F^\phi_j(x_j)$.

Assume now that $t''_j < t'_j$. By *double consistency*, $r((F_i, F^\phi_j), t''_i + t''_j) = (t''_i, t''_j)$. Since $r((F_i, F^\phi_j), t''_i + t''_j) = (t''_i, t''_j)$ and $r((F_i, F^\phi_j), t_i + t'_j) = (t_i, t'_j)$, and since $t''_j < t'_j$, by *endowment monotonicity*, $t''_i \leq t_i$. By the definition of ϕ, we can pick $c \in [0, t_j)$ such that $\phi(c) \in (t''_j, t'_j)$. Since $c < t_j$, $t_i + c < t_i + t_j$. Since $t_i + c \in [0, t_i + t_j)$ and for each $T \in [0, t_i + t_j)$, $r_i((F_i, F_j), T) < t_i$, $r_i((F_i, F_j), t_i + c) < t_i$. Then $\phi(r_i((F_i, F_j), t_i + c)) = r_i((F_i, F_j), t_i + c) < t_i$. By *upper coordinality*, $r_i((F^\phi_i, F^\phi_j), \phi(t_i) + \phi(c)) = \phi(r_i((F_i, F_j), t_i + c))$. Thus, $r_i((F^\phi_i, F^\phi_j), t_i + \phi(c)) < t_i$. Since $r_i((F^\phi_i, F^\phi_j), t_i + \phi(c)) < t_i$ and F^ϕ_i and F_i agree on $(-\infty, t_i]$, $c^\phi_i = r_i((F^\phi_i, F^\phi_j), t_i + \phi(c))$ implies $c_i = r_i((F^\phi_i, F^\phi_j), t_i + \phi(c))$. Thus, by Step 4, $r_i((F_i, F^\phi_j), t_i + \phi(c)) = r_i((F^\phi_i, F^\phi_j), t_i + \phi(c))$. Since $t''_i \leq t_i$ and $t''_j < \phi(c)$, by *endowment monotonicity*, $r_i((F_i, F^\phi_j), t''_i + t''_j) \leq r_i((F_i, F^\phi_j), t_i + \phi(c))$. Then $t''_i = r_i((F_i, F^\phi_j), t''_i + t''_j) \leq r_i((F_i, F^\phi_j), t_i + \phi(c)) = r_i((F^\phi_i, F^\phi_j), t_i + \phi(c)) < t_i$. Note that $t_i \leq M(F_i)$. Since $t''_i < t_i \leq M(F_i)$, $t''_i + t''_j + t''_k = t_i + t'_j + t_k$. Since $t''_j < t'_j$ and $t''_i + t''_j + t''_k = t_i + t'_j + t_k$, $t''_i + t''_k > t_i + t_k$. Then by *endowment monotonicity*, $r_i((F_i, F_k), t''_i + t''_k) \geq r_i((F_i, F_k), t_i + t_k)$. By *double consistency*, $r_i((F_i, F_k), t''_i + t''_k) = t''_i$. Thus, $t''_i \geq t_i$, which contradicts $t''_i < t_i$.

Step 8: Equal quantiles. All agents have equal probabilities of generating waste. Formally, for each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{P}^I$, and each pair $\{i, j\} \subseteq I$,

$$[\lim_{x_i \uparrow r_i(F,T)} F_i(x_i), F_i(r_i(F, T))] \cap [\lim_{x_j \uparrow r_j(F,T)} F_j(x_j), F_j(r_j(F, T))] \neq \emptyset.$$

Let $I \in \mathcal{N}$, $(F, T) \in \mathcal{P}^I$, $\{i, j\} \subseteq I$, and $t := r(F, T)$. Suppose to the contrary and without loss of generality that $F_i(t_i) < \lim_{x_j \uparrow t_j} F_j(x_j)$. By *double consistency*, $r((F_i, F_j), t_i + t_j) = (t_i, t_j)$. Let $k \in \mathbb{N} \setminus I$. We will pick $F_k \in \mathcal{F}$ and $t_k \in [0, C_k]$ such that $r((F_j, F_k), t_j + t_k) = (t_j, t_k)$, $t_j < t_k$, and for each $T' \in [0, t_j + t_k)$, $r_j((F_j, F_k), T') < t_j$. Then we will show that agent k has an unjustifiably large probability of generating waste, i.e., $F_i(t_i) < \lim_{x_k \uparrow t_k} F_k(x_k)$, which contradicts Step 7.

Since $F_i(t_i) < \lim_{x_j \uparrow t_j} F_j(x_j)$, we can pick $F_k \in \mathcal{F}$ such that $F_i(t_i) < F_k(t_j) < \lim_{x_j \uparrow t_j} F_j(x_j)$. Note that $t_j \leq M(F_j) = r_j((F_j, F_k), M(F_j) + M(F_k))$. Since $t_j \in [r_j((F_j, F_k), 0), r_j((F_j, F_k), M(F_j) + M(F_k)]$, by *endowment continuity*, there is a smallest endowment $T^* \in [0, M(F_j) + M(F_k)]$ satisfying $r_j((F_j, F_k), T^*) = t_j$. Thus, by *endowment* *monotonicity*, for each $T' \in [0, T^*)$, $r_j((F_j, F_k), T') < t_j$. Let $t_k := r_k((F_j, F_k), T^*)$. Thus, $t_k \in [0, C_k]$. Since $T^* \in [0, M(F_j) + M(F_k)]$ and $r((F_j, F_k), T^*) = (t_j, t_k)$, $t_j + t_k = T^*$. Then $r((F_j, F_k), t_j + t_k) = (t_j, t_k)$ and for each $T' \in [0, t_j + t_k)$, $r_j((F_j, F_k), T') < t_j$.

We check that $t_j < t_k$. Suppose to the contrary that $t_k \leq t_j$. Then $F_k(t_k) \leq F_k(t_j) < \lim_{x_j \uparrow t_j} F_j(x_j)$, where the strict inequality follows from the choice of F_k. Since $r((F_j, F_k), t_j + t_k) = (t_j, t_k)$, if $t_k < t_j$, by Step 5, $\lim_{x_j \uparrow t_j} F_j(x_j) \leq F_k(t_k)$, and if $t_k = t_j$, by Step 6, $[\lim_{x_k \uparrow t_k} F_k(x_k), F_k(t_k)] \cap [\lim_{x_j \uparrow t_j} F_j(x_j), F_j(t_j)] \neq \emptyset$. In either case, we get a contradiction to $F_k(t_k) < \lim_{x_j \uparrow t_j} F_j(x_j)$. Hence, $t_j < t_k$.

Lastly, we check that $F_i(t_i) < \lim_{x_k \uparrow t_k} F_k(x_k)$. By the choice of F_k, $F_i(t_i) < F_k(t_j)$. Since $t_j < t_k$, $F_k(t_j) \leq \lim_{x_k \uparrow t_k} F_k(x_k)$. Thus, $F_i(t_i) < \lim_{x_k \uparrow t_k} F_k(x_k)$.

Step 9: A common and positive maximal probability of satiation. All agents have equal maximal probabilities of being satiated, and this common maximal probability is positive. Formally, there is $\lambda \in [0, 1]$ such that for each $F_i \in \mathcal{F}$, $M(F_i) = Q_{F_i}(\lambda)$, and thus, $M(F_i) \geq c_i$.

Let $F_j \in \mathcal{F}$ be such that F_j is atomless and $c_j = 0$. Let $\lambda := F_j(M(F_j))$. Since F_j is atomless, $\lim_{x_j \uparrow M(F_j)} F_j(x_j) = F_j(M(F_j)) = \lambda$. For each $F_i \in \mathcal{F}$, consider a two-agent problem in which the agents claim respectively F_i and F_j and the endowment is $M(F_i) + M(F_j)$; since $r((F_i, F_j), M(F_i) + M(F_j)) = (M(F_i), M(F_j))$, by Step 8, $[\lim_{x_i \uparrow M(F_i)} F_i(x_i), F_i(M(F_i))] \cap [\lim_{x_j \uparrow M(F_j)} F_j(x_j), F_j(M(F_j))] \neq \emptyset$. Hence, for each $F_i \in \mathcal{F}$, $\lambda \in [\lim_{x_i \uparrow M(F_i)} F_i(x_i), F_i(M(F_i))]$. We prove that for each $F_i \in \mathcal{F}$, $M(F_i) = Q_{F_i}(\lambda)$, and then $M(F_i) \geq c_i$ follows immediately from $Q_{F_i}(\lambda) \geq c_i$. Let $F_i \in \mathcal{F}$. We divide the proof into the following two cases.

Case 1: $\lambda = 0$. Then $\lim_{x_i \uparrow M(F_i)} F_i(x_i) \leq 0$, which implies $M(F_i) \leq c_i$. Since $M(F_i) \leq c_i = Q_{F_i}(0)$, to show $M(F_i) = Q_{F_i}(0)$, we suppose to the contrary that $M(F_i) < c_i$.

First, assume that $M(F_i) > 0$. For each $n \in \mathbb{N}$, let $\phi^n \in \Phi^u$ be the upper transformation defined by (6) with $d_1 = 0$, $d_2 = M(F_i)$, $d_3 = c_i$, and $d_4 = c_i + \frac{1}{n}$. For each $n \in \mathbb{N}$, since $r(F_i, M(F_i)) = M(F_i)$, by *upper coordinality*, $r(F_i^{\phi^n}, \phi^n(M(F_i))) = \phi^n(M(F_i))$, and since $\phi^n(M(F_i)) = c_i$, $r(F_i^{\phi^n}, c_i) = c_i$. It can be shown by routine arguments that $F_i^{\phi^n}$ converges to F_i.[49] By *continuity*, $\lim r(F_i^{\phi^n}, c_i) = r(F_i, c_i)$. Hence, $r(F_i, c_i) = c_i$. Since $c_i > M(F_i)$, $r(F_i, c_i) = M(F_i) < c_i$, which contradicts $r(F_i, c_i) = c_i$.

Second, assume that $M(F_i) = 0$. Since $c_i > M(F_i)$, $c_i > 0$. Let $\phi \in \Phi^u$ be the

[49]The proof is available upon request.

upper transformation defined by (7) with $d_3 = \frac{1}{4}c_i$ and $d_4 = \frac{1}{2}c_i$. Since $r(F_i, 0) = 0$, by *upper coordinality*, $r(F_i^\phi, \phi(0)) = \phi(0)$, and since $\phi(0) = \frac{1}{4}c_i$, $r(F_i^\phi, \frac{1}{4}c_i) = \frac{1}{4}c_i$. For each $x_i \in (-\infty, \frac{1}{4}c_i)$, $F_i^\phi(x_i) = 0 = F_i(x_i)$; for each $x_i \in [\frac{1}{4}c_i, \frac{1}{2}c_i)$, $\phi^{-1}(x_i) \in [0, \frac{1}{2}c_i)$, and thus, $F_i^\phi(x_i) = F_i(\phi^{-1}(x_i)) = 0 = F_i(x_i)$; for each $x_i \in [\frac{1}{2}c_i, \infty)$, $\phi^{-1}(x_i) = x_i$, and thus, $F_i^\phi(x_i) = F_i(\phi^{-1}(x_i)) = F_i(x_i)$. Hence, $F_i^\phi = F_i$. Thus, $r(F_i, \frac{1}{4}c_i) = r(F_i^\phi, \frac{1}{4}c_i) = \frac{1}{4}c_i > 0$, which contradicts $M(F_i) = 0$.

Case 2: $\lambda > 0$. Let $i \in \mathbb{N}$ and $F_i \in \mathcal{F}$. Since $\lambda \leq F_i(M(F_i))$, by the definition of Q_{F_i}, to show $M(F_i) = Q_{F_i}(\lambda)$, it suffices to show that for each $x_i \in (-\infty, M(F_i))$, $F_i(x_i) < \lambda$. Let $x_i' \in (-\infty, M(F_i))$. Consider the following three cases. First, $\lim_{x_i \uparrow M(F_i)} F_i(x_i) = 0$. Since $\lim_{x_i \uparrow M(F_i)} F_i(x_i) = 0$ and $\lambda > 0$, $F_i(x_i') = 0 < \lambda$. Second, $0 < \lim_{x_i \uparrow M(F_i)} F_i(x_i) < 1$. Since $0 < \lim_{x_i \uparrow M(F_i)} F_i(x_i) < 1$ and F_i is increasing on its support, $F_i(x_i') < \lim_{x_i \uparrow M(F_i)} F_i(x_i)$. Moreover, since $\lim_{x_i \uparrow M(F_i)} F_i(x_i) \leq \lambda$, $F_i(x_i') < \lambda$. Third, $\lim_{x_i \uparrow M(F_i)} F_i(x_i) = 1$. Since $1 = \lim_{x_i \uparrow M(F_i)} F_i(x_i) \leq \lambda \leq 1$, $\lambda = 1$. Note that $M(F_i) \leq C_i$. Then $F_i(x_i') < 1$, and thus, $F_i(x_i') < \lambda$.

Step 10: Constrained equal awards for sure needs. When the endowment is no larger than the sum of the agents' sure needs, all agents receive equal assignments capped at their sure needs. Formally, for each $I \in \mathcal{N}$ and each $(F, T) \in \mathcal{P}^I$, if $T \leq \sum c_i$, then for each $j \in I$, $r_j(F, T) = \min\{c^*, c_j\}$ where c^* satisfies $\sum \min\{c^*, c_i\} = T$.

Let $I \in \mathcal{N}$ and $(F, T) \in \mathcal{P}^I$ be such that $T \leq \sum c_i$. Let $c^* \in \mathbb{R}_+$ be such that $\sum \min\{c^*, c_i\} = T$. Let $t := r(F, T)$. Suppose to the contrary that there is $j \in I$ such that $t_j \neq \min\{c^*, c_j\}$. By Step 9, $\sum M(F_i) \geq \sum c_i$. Then $\sum M(F_i) \geq T$, and thus, $\sum t_i = T$. Since $\sum t_i = T = \sum \min\{c^*, c_i\}$ and $t_j \neq \min\{c^*, c_j\}$, we can assume without loss of generality that $t_j > \min\{c^*, c_j\}$, and for some $k \in I \setminus \{j\}$, $t_k < \min\{c^*, c_k\}$. Since $t_k < \min\{c^*, c_k\} \leq c_k$, $\lim_{x_k \uparrow t_k} F_k(x_k) = F_k(t_k) = 0$. By Step 8, $[\lim_{x_j \uparrow t_j} F_j(x_j), F_j(t_j)] \cap [\lim_{x_k \uparrow t_k} F_k(x_k), F_k(t_k)] \neq \emptyset$. Thus, $\lim_{x_j \uparrow t_j} F_j(x_j) \leq 0$, and hence, $t_j \leq c_j$. Since $\min\{c^*, c_j\} < t_j \leq c_j$, $c^* < c_j$ and $c^* < t_j$. Since $t_k < \min\{c^*, c_k\}$, we can pick $c \in (t_k, \min\{c^*, c_k\})$. Since $c < \min\{c^*, c_k\}$ and $c^* < c_j$, $c < \min\{c_j, c_k\}$. Then F_j and F_k agree on $(-\infty, c]$, and thus, by Step 1, $r_j((F_j, F_k), 2c) = r_k((F_j, F_k), 2c)$. By *double consistency*, $r((F_j, F_k), t_j + t_k) = (t_j, t_k)$. Thus, by Lemma 4, $(t_j - c)(t_k - c) \geq 0$. However, $c < \min\{c^*, c_k\} \leq c^* < t_j$ and $c > t_k$, which contradicts $(t_j - c)(t_k - c) \geq 0$.

Step 11: Equal-quantile rule. The rule r is an equal-quantile rule. Formally, there is

$\lambda \in [0,1]$ such that for each $I \in \mathcal{N}$ and each $(F,T) \in \mathcal{P}^I$, if $T \leq \sum c_i$, for each $j \in I$,

$$r_j(F,T) = \min\{c^*, c_j\}, \text{ where } c^* \in \mathbb{R}_+ \text{ satisfies } \sum \min\{c^*, c_i\} = T, \tag{10}$$

and if $T > \sum c_i$, for each $j \in I$,

$$r_j(F,T) = Q_{F_j}(\alpha^*), \text{ where } \alpha^* \in [0,\lambda] \text{ satisfies } \sum Q_{F_i}(\alpha^*) = \min\left\{T, \sum Q_{F_i}(\lambda)\right\}. \tag{11}$$

Let $\lambda \in [0,1]$ be given by Step 9. Then for each $F_i \in \mathcal{F}$, $M(F_i) = Q_{F_i}(\lambda) \geq c_i$. Let $I \in \mathcal{N}$, $(F,T) \in \mathcal{P}^I$, and $t := r(F,T)$. Assume that $T \leq \sum c_i$. By Step 10, for each $j \in I$, t_j is given by (10). Assume that $T > \sum c_i$. We divide the proof into the following two cases.

Case 1: $T \geq \sum Q_{F_i}(\lambda)$. Then $\sum Q_{F_i}(\lambda) = \min\{T, \sum Q_{F_i}(\lambda)\}$. Since $T \geq \sum Q_{F_i}(\lambda) = \sum M(F_i)$, for each $j \in I$, $t_j = M(F_j)$, and thus, $t_j = Q_{F_j}(\lambda)$. Hence, (11) holds with $\alpha^* = \lambda$.

Case 2: $\sum c_i < T < \sum Q_{F_i}(\lambda)$. Then $\lambda > 0$ and $T = \min\{T, \sum Q_{F_i}(\lambda)\}$. Since $\sum Q_{F_i}$ is continuous on $[0,\lambda]$ and $\sum Q_{F_i}(0) = \sum c_i < T < \sum Q_{F_i}(\lambda)$, there is $\alpha^* \in (0,\lambda)$ such that $\sum Q_{F_i}(\alpha^*) = T$. Thus, $\sum Q_{F_i}(\alpha^*) = \min\{T, \sum Q_{F_i}(\lambda)\}$.

To show that (11) holds, it remains to show that for each $j \in I$, $t_j = Q_{F_j}(\alpha^*)$. Suppose to the contrary that there is $j \in I$ such that $t_j \neq Q_{F_j}(\alpha^*)$. Since $T < \sum Q_{F_i}(\lambda) = \sum M(F_i)$, $\sum t_i = T$. Thus, $\sum t_i = T = \sum Q_{F_i}(\alpha^*)$. Since $\sum t_i = \sum Q_{F_i}(\alpha^*)$ and $t_j \neq Q_{F_j}(\alpha^*)$, we can assume without of generality that $t_j > Q_{F_j}(\alpha^*)$, and for some $k \in I \setminus \{j\}$, $t_k < Q_{F_k}(\alpha^*)$. By Step 8, $[\lim_{x_j \uparrow t_j} F_j(x_j), F_j(t_j)] \cap [\lim_{x_k \uparrow t_k} F_k(x_k), F_k(t_k)] \neq \emptyset$. Since $\alpha^* > 0$, by the definition of Q_{F_j}, $F_j(Q_{F_j}(\alpha^*)) \geq \alpha^*$. Since $t_j > Q_{F_j}(\alpha^*)$ and $F_j(Q_{F_j}(\alpha^*)) \geq \alpha^*$, $\lim_{x_j \uparrow t_j} F_j(x_j) \geq \alpha^*$. Since $t_k < Q_{F_k}(\alpha^*)$ and $\alpha^* > 0$, by the definition of Q_{F_k}, $F_k(t_k) < \alpha^*$. Since $F_k(t_k) < \alpha^* \leq \lim_{x_j \uparrow t_j} F_j(x_j)$, $F_k(t_k) < \lim_{x_j \uparrow t_j} F_j(x_j)$, which contradicts $[\lim_{x_j \uparrow t_j} F_j(x_j), F_j(t_j)] \cap [\lim_{x_k \uparrow t_k} F_k(x_k), F_k(t_k)] \neq \emptyset$, as desired. □

A.3 Optimality of equal-quantile rules

As discussed in Section 5, if claims are represented by continuous CDFs, then Proposition 1 can be proved by checking first-order conditions. The extension of the proof to the general case is standard, and thus, is put in the online appendix.[50]

[50] In the online appendix, we prove Proposition 1 in a richer environment that allows for unbounded claims.

A.3.1 Proof of Proposition 2

Let $c^w, c^d \in \mathbb{R}_+$ be such that $c^w + c^d > 0$. Let $\lambda := \frac{c^d}{c^w+c^d}$, $v := c^w$, and $u := c^w + c^d$. Thus, $u \geq v \geq 0$, $u > 0$, and $\lambda = \frac{u-v}{u}$. Let $I \in \mathbb{N}$ and $(F, T) \in \mathcal{P}^I$. Let t be an arbitrary allocation for (F, T). The opportunity cost generated by t is the difference between the maximal social welfare that would be obtained if the resource could be allocated after uncertainty resolves and the actual social welfare (2). Since $u \geq v \geq 0$, the maximal social welfare is

$$\int [u \min\{\textstyle\sum x_i, T\} + v(T - \min\{\textstyle\sum x_i, T\})]dF, \tag{12}$$

achieved by assigning to the agents in the model all of the resource up to their realized needs and to the outside agent what remains. We will show that the opportunity cost (12)-(2) is

$$v \sum \int_0^{t_i} (t_i - x_i)dF_i + (u - v) \sum \int_{t_i}^{\infty} (x_i - t_i)dF_i - \int_{\sum x_i > T} (u - v)(\textstyle\sum x_i - T)dF. \tag{13}$$

Since both (12) and the last term of (13) do not depend on t, maximizing (2) is equivalent to minimizing the sum of the first two terms of (13). Thus, by Proposition 1 and the fact that $v = c^w$ and $u = c^w + c^d$, $r^\lambda(F, T)$ minimizes (4).

We now show that the opportunity cost (12)-(2) is (13). Intuitively, when the endowment T is sufficiently large that the last term of (13) vanishes, as explained in the paragraph after Proposition 2, the marginal opportunity cost of waste is v and that of deficit $u - v$. When the endowment T is limited so that the last term of (13) is positive, deficit arises not only from resource misallocation under uncertainty but also from a shortage of the resource. Hence, the opportunity cost of deficit is adjusted down by the endowment's falling short of the agent's needs, i.e., the last term of (13). Formally,

$$
\begin{aligned}
(12)-(2) &= \int [u(\min\{\sum x_i, T\} - \sum \min\{x_i, t_i\}) + v(\sum t_i - \min\{\sum x_i, T\})]dF \\
&= \int_{\sum x_i \le T} \Big[u(\sum x_i - \sum \min\{x_i, t_i\}) + v(\sum t_i - \sum x_i)\Big]dF \\
&\quad + \int_{\sum x_i > T} \Big[u(T - \sum \min\{x_i, t_i\}) + v(\sum t_i - T)\Big]dF \\
&= \int_{\sum x_i \le T} \Big[u(\sum x_i - \sum \min\{x_i, t_i\}) + v(\sum t_i - \sum x_i)\Big]dF \\
&\quad + \int_{\sum x_i > T} \Big[u(\sum x_i - \sum \min\{x_i, t_i\}) + v(\sum t_i - \sum x_i)\Big]dF \\
&\quad - \int_{\sum x_i > T} (u - v)(\sum x_i - T)dF \\
&= \int \Big[u(\sum x_i - \sum \min\{x_i, t_i\}) + v(\sum t_i - \sum x_i)\Big]dF \\
&\quad - \int_{\sum x_i > T} (u - v)(\sum x_i - T)dF \\
&= \sum \int [u(x_i - \min\{x_i, t_i\}) + v(t_i - x_i)]dF_i - \int_{\sum x_i > T} (u - v)(\sum x_i - T)dF \\
&= (13)
\end{aligned} \tag{14}
$$

where the second last equality holds since the sum and the expectation are interchangeable when the integrand is additively separable.[51] □

References

[1] Al-Najjar, N. I. and Pomatto, L. Choice under aggregate uncertainty, *Theory and Decision*, **80(2)** (2016), 187-209

[2] Arrow, K. J., Harris, T., and Marschak, J. Optimal inventory policy, *Econometrica*, **19** (1951), 250-272

[3] Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D., Coherent Measures of Risk, *Mathematical Finance* **9(3)** (1999), 203-228

[51] See, for example, the proof of Theorem 2 of Al-Najjar and Pomatto (2016).

[4] Aumann, R. and Maschler, M. Game theoretic analysis of a bankruptcy problem from the Talmud, *Journal of Economic Theory*, **36(2)** (1985), 195-213

[5] Banker, R. Equity consideration in traditional full-cost allocation practices: an axiomatic perspective, In: Moriarity, S. (Ed.), *Joint Cost Allocations*, University of Oklahoma Press (1981), 110-130

[6] Branzei, R., Dimitrov, D., Pickl, S., and Tijs, S. How to cope with division problems under interval uncertainty of claims? *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **12(2)** (2004), 191-200

[7] Chambers, C. P. Ordinal aggregation and quantiles, *Journal of Economic Theory*, **137(1)** (2007), 416-431

[8] Chambers, C. P. An axiomatization of quantiles on the domain of distribution functions, *Mathematical Finance*, **19(2)** (2009), 335-342

[9] Chambers, C. P. and Moreno-Ternero, J. D. Taxation and poverty, *Social Choice and Welfare*, **48(1)** (2017), 153-175

[10] Chambers, C. P. and Thomson, W. Group order preservation and the proportional rule for the adjudication of conflicting claims, *Mathematical Social Sciences*, **44(3)** (2002), 235-252

[11] Chun, Y. The proportional solution for rights problems, *Mathematical Social Sciences*, **15(3)** (1988), 231-246

[12] Chun, Y., Jang, I., and Ju, B-G. Priority, solidarity and egalitarianism, *Social Choice and Welfare*, **43(3)** (2014), 577-589

[13] Copas, J. B. A formula for the allocation of resources based on uncertain predictions of need, *Journal of the Royal Statistical Society, Series A*, **156(1)** (1993), 107-113

[14] Dagan, N. New characterizations of old bankruptcy rules, *Social Choice and Welfare*, **13** (1996), 51-59

[15] Dagan, N., Serrano, R., and Volij, O. A non-cooperative view of consistent bankruptcy rules. *Games and Economic Behavior*, **18** (1997), 55-72

[16] d'Aspremont, C. and Gevers, L. Equity and the Informational Basis of Collective Choice, *Review of Economic Studies*, **44(2)** (1977), 199-209

[17] de Castro, L. I. and Galvao, A. F. Dynamic Quantile Models of Rational Behavior, *Mimeo* (2019)

[18] Ehlers, L. and Klaus, B. Efficient priority rules, *Games and Economic Behavior*, **55(2)** (2006), 372-384

[19] Ehlers, L. and Klaus, B. Consistent house allocation, *Economic Theory*, **30(3)** (2007), 561-574

[20] Ertemel, S. and Kumar, R. Proportional rules for state contingent claims, *International Journal of Game Theory*, **47(1)** (2018), 229-246

[21] Fishburn, P. Utility theory for decision making, New York: Wiley (1970)

[22] Fleurbaey, M. Assessing risky social situations, *Journal of Political Economy*, **118(4)** (2010), 649-680

[23] Fleurbaey, M., Gajdos, T., and Zuber, S. Social rationality, separability, and equity under uncertainty, *Mathematical Social Sciences*, **73(C)** (2015), 13-22

[24] Gevers, L. On interpersonal comparability and social welfare orderings, *Econometrica*, **47(1)** (1979), 75-89

[25] Herrero, C. and Villar, A. The three musketeers: four classical solutions to bankruptcy problems, *Mathematical Social Sciences*, **39** (2001), 307-328

[26] Herrero, C. and Villar, A. Sustainability in bankruptcy problems, *Top*, **10(2)** (2002), 261-273

[27] Hougaard, J. L. and Moulin, H. Sharing the cost of risky projects, *Economic Theory*, **65(3)** (2018), 663-679

[28] Johansson, M. and Sternad, M. Resource allocation under uncertainty using the maximum entropy principle, *IEEE Transactions on Information Theory*, **51(12)** (2005), 4103-4117

[29] Kaminski, M. M. ‘Hydraulic’ rationing, *Mathematical Social Sciences*, **40(2)** (2000), 131-155

[30] Kaminski, M. M. Parametric rationing methods, *Games and Economic Behavior*, **54(1)** (2006), 115-133

[31] Koster, M. and Boonen, T. J. Constrained stochastic cost allocation, *Mathematical Social Sciences*, **101** (2019), 20-30

[32] Littlewood, K. Forecasting and Control of Passenger Bookings, *Proceedings of the 12th AGIFORS Symposium*, (1972), reprinted in *Journal of Revenue and Pricing Management*, **4(2)** (2005), 111-123

[33] Long, Y. and Xue, J. Parametric rationing with uncertain needs, *Mimeo*, (2019)

[34] Marchant, T. Scale invariance and similar invariance conditions for bankruptcy problems, *Social Choice and Welfare*, **31** (2008), 693-707. Erratum in *Social Choice and Welfare*, **31** (2008), 709-710.

[35] Martínez, R. Strong composition down. Characterizations of new and classical bankruptcy rules, *Mimeo*, (2008)

[36] Maschler, M. Consistency, In: Ichiishi T, Neyman A, Tauman Y (Eds.), *Game Theory and Applications*, San Diego California Academic Press (1990), 183-186

[37] Moreno-Ternero, J. D. and Roemer, J. E. Impartiality, priority and solidarity in the theory of justice, *Econometrica*, **74(5)** (2006), 1419-1427

[38] Moreno-Ternero, J. D. and Roemer, J. E. A common ground for resource and welfare egalitarianism, *Games and Economic Behavior*, **75(2)** (2012), 832-841

[39] Moreno-Ternero, J. D. and Villar, A. The TAL-family of rules for bankruptcy problems, *Social Choice and Welfare*, **27(2)** (2006), 231-249

[40] Moulin, H. Equal or proportional division of a surplus, and other methods, *International Journal of Game Theory*, **16** (1987), 161-186

[41] Moulin, H. Priority rules and other asymmetric rationing methods, *Econometrica*, **68(3)** (2000), 643-684

[42] Moulin, H. Axiomatic cost and surplus-sharing, Chapter 6 of K. Arrow, A. Sen and K. Suzumura (Eds.), *The Handbook of Social Choice and Welfare, Vol. 1.*, North-Holland (2002)

[43] Moulin, H. and Sethuraman J. The bipartite rationing problem, *Operations Research*, **61(5)** (2013)

[44] Neilsen, L. T. Ordinal interpersonal comparisons in bargaining, *Econometrica*, **51(1)** (1983), 219-221

[45] O'Neill, B. A problem of rights arbitration from the Talmud, *Mathematical Social Sciences*, **2(4)** (1982), 345-371

[46] Qin, Y., Wang, R., Vakharia, A. J., Chen, Y., and Seref, M. M. H. The newsvendor problem: Review and directions for future research, *European Journal of Operational Research*, **213(2)** (2011), 361-374

[47] Rawls, C. G. and Turnquist, M. A. Pre-positioning of emergency supplies for disaster response, *Transportation Research Part B*, **44(4)** (2010), 521-534

[48] Rostek, M. Quantile maximization in decision theory, *Review of Economic Studies*, **77** (2010), 339-371

[49] Roth, A. E. *Axiomatic models of bargaining*, Lecture Notes in Economics and Mathematical Systems 170, Springer Verlag (1979)

[50] Shapley, L. S. Utility comparison and the theory of games, Santa Monica, CA: RAND Corporation (1967)

[51] Sprumont, Y. The division problem with single-peaked preferences: a characterization of the uniform allocation rule, *Econometrica*, **59(2)** (1991), 509-519

[52] Sprumont, Y. Ordinal cost sharing, *Journal of Economic Theory*, **81(1)** (1998), 126-162

[53] Sprumont, Y. Ranking by rating, *Theoretical Economics*, **13** (2018), 1-18

[54] Stovall, J. E. Asymmetric parametric division rules, *Games and Economic Behavior*, **(84)** (2014), 87-110

[55] Thomson, W. Eliciting production possibilities from a wellinformed manager, *Journal of Economic Theory*, **20(3)** (1979), 360-380

[56] Thomson, W. A study of choice correspondences in economies with a variable number of agents, *Journal of Economic Theory*, **46(2)** (1988), 247-259

[57] Thomson, W. Theory of Fair Allocation, Princeton University Press (1998)

[58] Thomson, W. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey, *Mathematical Social Sciences*, **45** (2003), 249-297

[59] Thomson, W. Two families of rules for the adjudication of conflicting claims, *Social Choice and Welfare*, **31(4)**, (2008), 667-692

[60] Thomson, W. Consistency and its converse: an introduction, *Review of Economic Design*, **15(4)** (2011), 257-291

[61] Thomson, W. On the axiomatics of resource allocation: interpreting the consistency principle, *Economics and Philosophy*, **28(3)** (2012), 385-421

[62] Thomson, W. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update, *Mathematical Social Sciences*, **74** (2015), 41-59

[63] Thomson, W. *Consistent allocation rules* (forthcoming ed.), Cambridge University Press (2017)

[64] Turnquist, M. A. and Nozick, L. K. Allocating time and resources in project management under uncertainty, *Proceedings of the 36th Hawaii International Conference on System Sciences (HICSS'03)*, **1** (2003), 187-195

[65] Wang, S. S., Young, V. R., and Panjer, H. H. Axiomatic characterization of insurance prices, *Insurance: Mathematics and Economics*, **21(2)** (1997), 173-183

[66] Wex, F., Schryen, G., and Neumann, D. A fuzzy decision support model for natural disaster response under informational uncertainty, *International Journal of Information Systems for Crisis Response and Management*, **4(3)** (2012), 23-41

[67] Woeginger, G. J. A note on fair division under interval uncertainty, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **14(6)** (2006), 753-756

[68] Xue, J. Fair division with uncertain needs, *Social choice and welfare*, **51(1)** (2018a), 105-136

[69] Xue, J. Waste constrained uniform gains rules, *Mimeo*, (2018b)

[70] Yager, R. and Kreinovich, V. Fair division under interval uncertainty, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **8(5)** (2000), 611-618

[71] Yeh, C-H. Sustainability, exemption, and the constrained equal award rule: a note, *Mathematical Social Sciences*, **47** (2004), 103-110

[72] Yeh, C-H. Protective properties and the constrained equal awards rule for claims problems: a note, *Social Choice and Welfare*, **27** (2006), 221-230

[73] Yeh, C-H. Secured lower bound, composition up, and minimal rights rst for bankruptcy problems, *Journal of Mathematical Economics*, **44** (2008), 925-932

[74] Young, P., On dividing an amount according to individual claims or liabilities, *Mathematics of Operations Research*, **32** (1987a), 398-414

[75] Young, P. Progressive taxation and the equal sacrifice principle, *Journal of Public Economics*, **15(3)** (1987b), 203-214

[76] Young, P. Distributive justice in taxation, *Journal of Economic Theory*, **43** (1988), 321-335

[77] Young, P. Progressive taxation and equal sacrifice, *American Economic Review*, **80(1)** (1990), 253-266