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Limit Theory for Moderate Deviation from Integrated GARCH Processes

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Abstract

This paper develops the limit theory of the GARCH(1,1) process that moderately deviates from IGARCH process towards both stationary and explosive regimes. The asymptotic theory extends Berkes et al. (2005) by allowing the parameters to have a slower rate of convergence. The results can be applied to unit root test for processes with mildly-integrated GARCH innovations (e.g. Boswijk (2001), Cavaliere & Taylor (2007, 2009)) and deriving limit theory of estimators for models involving mildly-integrated GARCH processes (e.g. Jensen & Rahbek (2004), Francq & Zakoïan (2012, 2013)).

Keywords: Central Limit Theorem, Limiting Process, Localization, Explosive GARCH, Volatility Process
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1. Introduction

The model considered in this paper is a GARCH(1,1) process:

$$\text{(Return Process)} \qquad u_t = \sigma_t \varepsilon_t,$$
$$\text{(Volatility Process)} \qquad \sigma_t^2 = \omega + \alpha_n u_{t-1}^2 + \beta_n \sigma_{t-1}^2, \quad \omega > 0,\ \alpha_n \geq 0, \text{ and } \beta_n \geq 0,$$

where $\{\varepsilon_t\}_{t=0}^n$ is a sequence of independent identically distributed (i.i.d) variables such that $E\varepsilon_0 = 0$ and $E\varepsilon_0^2 = 1$.

Unlike conventional GARCH(1,1) process, the innovation process considered in this paper is a mildly-integrated GARCH process whose key parameters, α_n and β_n, are changing with the sample size, viz.

$$\alpha_n = O(n^{-p}), \quad \beta_n = 1 + O(n^{-q}), \text{ where } p, q \in (0,1),$$

and

$$\gamma_n = \alpha_n + \beta_n - 1 = O(n^{-\kappa}), \quad \kappa = \min\{p, q\}.$$

The limiting process of this GARCH process is first derived in Berkes et al. (2005) by imposing the assumption $\kappa \in (1/2, 1)$. Extending their results, we obtain the limiting process that applies to parameter values that covers the whole range of $(0,1)$. This is a non-trivial extension because when the process deviates further

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from the integrated GARCH process, the approximation errors in Berkes et al. (2005) diverges and thus a different normalization is needed.

2. Main Results

The main results are summarized in the following one proposition and three theorems. The first proposition modifies the additive representation for σ_t^2 in Berkes et al. (2005) to accommodate $\kappa \in (0,1)$. Based on the proposition, we establish three theorems to describe the asymptotic behaviours of σ_t^2 and u_t under the cases $\gamma_n \lesseqgtr 0$ respectively.

To establish the additive representation of σ_t^2, we make the following assumptions on the distribution of the innovations $\{\varepsilon_t\}_{t=0}^n$ and the convergence rates of the GARCH coefficients, α_n and β_n.

Assumption 1. *$\{\varepsilon_t\}_{t=0}^n$ is an i.i.d sequence with $E\varepsilon_0^2 = 1$ and $E|\varepsilon_0|^{4+\delta} < \infty$, for some $\delta > 0$.*

Assumption 2. *$\alpha_n \log\log n \to 0$, $n\alpha_n \to \infty$ and $\beta_n \to 1$.*

Assumption 1 imposes a non-degeneracy condition on the distribution of ε_t^2 and thus ensures its applicability to the central limit theorem. Assumption 2 bounds the convergence rate of α_n so that the normalized sequence could converge to a proper limit. Based on these assumptions, we obtain a modified additive representation for σ_t^2 in Proposition 1 on the top of Berkes et al. (2005).

Proposition 1 (Additive Representation). *Under Assumption 1 and 2, we have the additive representation for σ_t^2 as*

$$\sigma_t^2 = \sigma_0^2 t^{t/2} e^{\sqrt{t}\gamma_n} \left(1 + \frac{\alpha_n}{\sqrt{t}} \sum_{j=1}^{t} \xi_{t-j} + R_t^{(1)}\right) + \omega \left[1 + t^{t/2} \sum_{j=1}^{t} e^{\frac{j\gamma_n}{\sqrt{t}}} \left(1 + \frac{\alpha_n}{\sqrt{t}} \sum_{i=1}^{j} \xi_{t-i} + R_{t,j}^{(2)}\right)\left(1 + R_{t,j}^{(3)}\right)\right]$$

where $\xi_t = \varepsilon_t^2 - 1$ and the remainder terms satisfy

$$\left|R_t^{(1)}\right| = O_p\left(\alpha_n^2 + \gamma_n^2\right), \quad \max_{1\le j\le t}\left|R_{t,j}^{(2)}\right| = O_p\left(\alpha_n^2\right)$$
$$\max_{1\le j\le t} \frac{1}{j\log\log j}\left|R_{t,j}^{(2)}\right| = O_p\left(\frac{\alpha_n^2}{t}\right), \quad \max_{1\le j\le t}\frac{1}{j}\left|R_{t,j}^{(3)}\right| = O_p\left(\frac{\alpha_n^2+\gamma_n^2}{t}\right)$$

Remark 1. *The key difference between our results and Berkes et al. (2005) is the convergence rate of the approximation errors. In Berkes et al. (2005), the approximation error $|R_t^{(p)}|$, $\forall p = \{1,2,3\}$ is of order $t(\alpha_n^2+\gamma_n^2)$ or $t\alpha_n^2$ asymptotically. Hence, these errors are negligible only when $\kappa \in (1/2,1)$. We relax this restrictive assumption by normalizing the original terms with $\sqrt{t}$. Under this new normalization, all the approximation errors remains negligible when $\kappa \in (0,1)$.*

To formulate the theorems below, I introduce the following notations. For $0 < t_1 < t_2 < \cdots < t_N < 1$ define $k(m) = \lfloor nt_m \rfloor$, $1 \le m \le N$. Further, we need the assumptions for relative convergence rate between α_n and γ_n to regulate the asymptotic behaviours of returns and volatilities for near-stationary case.

Assumption 3. $\frac{\sqrt{|\gamma_n|}}{\alpha_n n^{1/4}} \to \infty$, *while* $\frac{\sqrt{|\gamma_n|^3}}{\alpha_n n^{1/4}} \to 0$, *as* $n \to \infty$.

Assumption 3 imposes a rate condition on the localized parameters α_n and γ_n. This condition is less restrictive than that in Berkes et al. (2005) in the sense that instead of requiring $|\gamma_n|^{3/2}/\alpha_n$ to converge to 0, we allow it to diverge slowly at a rate of $n^{1/4}$. The relaxation of the assumption also attributes to the change of the normalization.

Theorem 1 (Near-stationary Case)**.** *Suppose* $\gamma_n < 0$, *then under Assumption 1-3, the random variables*

$$\frac{\sqrt{2|\gamma_n|^3}}{\alpha_n k(m)^{1/4}} \frac{1}{\sqrt{E\xi_0^2}} \left(\frac{\sigma^2_{k(m)}}{\omega k(m)^{k(m)/2}} - \sum_{j=1}^{k(m)-1} e^{\frac{j\gamma_n}{\sqrt{k(m)}}} \right) \xrightarrow{d} \mathcal{N}(0,1).$$

In addition, the random variables

$$\left(\frac{|\gamma_n|}{\omega k(m)^{(k(m)+1)/2}} \right)^{1/2} u_{k(m)}$$

are asymptotically independent, each with the asymptotic distribution equals to the distribution of ε_0.

Theorem 2 (Integrate Case)**.** *Suppose* $\gamma_n = 0$, *then under Assumption 1 and 2, the volatility has the asymptotic distribution*

$$\frac{k(m)^{1/2}}{n^{3/2}\alpha_n} \frac{1}{\sqrt{E\xi_0^2}} \left(\frac{\sigma^2_{k(m)}}{\omega k(m)^{k(m)/2}} - k(m) \right) \xrightarrow{d} \int_0^{t_m} x dW(x)$$

In addition, the random variables

$$\left(\omega k(m)^{k(m)/2+1} \right)^{-1/2} u_{k(m)}$$

are asymptotically independent, each with the asymptotic distribution equals to the distribution of ε_0.

Similar to the near-stationary case, we have to impose additional assumption on the relative speed of converging to zero between α_n and γ_n.

Assumption 4. $\gamma_n/\alpha_n \to 0$, *as* $n \to \infty$.

Theorem 3 (Near-explosive Case)**.** *Suppose* $\gamma_n > 0$, *then under Assumption 1, 2 and 4, the volatility has the asymptotic distribution*

$$\frac{\gamma_n e^{-\sqrt{k(m)}\gamma_n}}{\alpha_n \sqrt{k(m)}} \frac{1}{\sqrt{E\xi_0^2}} \left(\frac{\sigma^2_{k(m)}}{\omega k(m)^{k(m)/2}} - \sum_{j=1}^{k(m)-1} e^{\frac{j\gamma_n}{\sqrt{k(m)}}} \right) \Rightarrow W(t_m).$$

In addition, the random variables

$$\left(\frac{\gamma_n e^{-\sqrt{k(m)}\gamma_n}}{\omega k(m)^{(k(m)+1)/2}} \right)^{1/2} u_{k(m)}.$$

are asymptotically independent, each with the asymptotic distribution equals to the distribution of ε_0.

Remark 2. *As one may notice, the rate of convergence for both volatility process and return process in all three cases decreases to 0 asymptotically. These seemingly awkward results are reasonable in the sense that the convergence rate is a part of the normalization which reflects the order of the process. In other words, when we compute a partial sum of Xs in form of $\sum_{i=1}^{n} a_i X_i$, the normalization just plays the role of a_i which is usually required to decrease to 0 for applying a central limit theorem.*

3. Proofs

In this section, I present detailed proofs for all the propositions and the theorems listed in the previous section. For readers' convenience, I provide a roadmap for understanding the proofs of the theorems. In general, the proofs are done in three steps:

Step 1: We decompose the volatility process into 4 components, $\sigma^2_{k,s}$, $s = 1, \cdots, 4$, by expanding the multiplicative form provided in Proposition 1.

Step 2: We show the first 3 volatility components are negligible after normalization, and the last term converges to a proper limit by using Cramer-Wold device and Liapounov central limit theorem or Donsker's theorem.

Step 3: We figure out a normalization to make the normalized volatility converges to 1. Then, applying this normalization to the return process, we complete the proof.

Proof of Proposition 1. First, note the GARCH(1,1) model can be written into the following multiplicative form:

$$\begin{aligned}\sigma_t^2 &= \sigma_0^2 \prod_{i=1}^{t} \left(\beta_n + \alpha_n \varepsilon_{t-i}^2\right) + \omega \left[1 + \sum_{j=1}^{t-1} \prod_{i=1}^{j} \left(\beta_n + \alpha_n \varepsilon_{t-i}^2\right)\right] \\ &= \sigma_0^2 t^{t/2} \prod_{i=1}^{t} \frac{\left(\beta_n + \alpha_n \varepsilon_{t-i}^2\right)}{\sqrt{t}} + \omega \left[1 + t^{t/2} \sum_{j=1}^{t-1} \prod_{i=1}^{j} \frac{\left(\beta_n + \alpha_n \varepsilon_{t-i}^2\right)}{\sqrt{t}}\right]\end{aligned}$$

Note that

$$\max_{1 \le i \le t} \frac{\left|\beta_n + \alpha_n \varepsilon_{t-i}^2 - 1\right|}{\sqrt{t}} \le \frac{|\gamma_n|}{\sqrt{t}} + \alpha_n \max_{1 \le i \le t} \frac{|\varepsilon_{t-i}^2 - 1|}{\sqrt{t}} = \frac{|\gamma_n|}{\sqrt{t}} + \alpha_n \max_{1 \le i \le t-1} \frac{|\varepsilon_i^2 - 1|}{\sqrt{t}}$$

Then by Assumption 1 and Chow & Teicher (2012), we have the almost sure convergence of

$$\max_{1 \le j \le t-1} |\varepsilon_i^2 - 1| = O(\sqrt{t})$$

Therefore, the term above is

$$\max_{1 \le i \le t} \frac{\left|\beta_n + \alpha_n \varepsilon_{t-i}^2 - 1\right|}{\sqrt{t}} = o_p(1).$$

Now consider the sequence of events

$$A_n = \left\{ \max_{1 \le i \le t} \frac{|\beta_n + \alpha_n \varepsilon_{t-i}^2 - 1|}{\sqrt{t}} \le \frac{1}{2} \right\}.$$

From the previous result we know $\lim_{n\to\infty} P(A_n) = 1$. Then by Taylor expansion, $|\log(1+x) - x| \le 2x^2$, $|x| \le 1/2$ on the event A_n, which implies

$$\begin{aligned}
\left|R_{t,j}^{(3)}\right| &= \left|\sum_{i=1}^{j} \log \frac{\left(\beta_n + \alpha_n \varepsilon_{t-i}^2\right)}{\sqrt{t}} - \sum_{i=1}^{j} \frac{(\gamma_n + \alpha_n \xi_{t-i})}{\sqrt{t}}\right| \\
&= \left|\sum_{i=1}^{j} \log \frac{(\gamma_n + \alpha_n \xi_{t-i} + 1)}{\sqrt{t}} - \sum_{i=1}^{j} \frac{(\gamma_n + \alpha_n \xi_{t-i})}{\sqrt{t}}\right| \\
&\le \sum_{i=1}^{j} \left|\log\left(\frac{\gamma_n + \alpha_n \xi_{t-i}}{\sqrt{t}} + 1\right) - \frac{(\gamma_n + \alpha_n \xi_{t-i})}{\sqrt{t}}\right| \\
&\le 2\sum_{i=1}^{j} \frac{(\gamma_n + \alpha_n \xi_{t-i})^2}{t} \le \frac{4j\gamma_n^2}{t} + \frac{4\alpha_n^2 \sum_{i=1}^{j} \xi_{t-i}^2}{t}.
\end{aligned}$$

By Assumption 1 and law of large numbers (LLN), we know

$$\max_{1\le j\le t} \frac{1}{j}\left|\sum_{i=1}^{j} \xi_{t-i}^2\right| \sim \max_{1\le j\le t} \frac{1}{j}\left|\sum_{i=1}^{j} \xi_i^2\right| = O_p(1)$$

Then by the equation above, we have

$$\max_{1\le i\le j} \frac{1}{j}|R_{t,j}^{(3)}| = O_p\left(\frac{\gamma_n^2 + \alpha_n^2}{t}\right)$$

Now by direct plugging into the key multiplicative term we care about, we have

$$\begin{aligned}
\prod_{i=1}^{j} \frac{\left(\beta_n + \alpha_n \varepsilon_{t-i}^2\right)}{\sqrt{t}} &= \exp\left\{\sum_{i=1}^{j} \log\left(\frac{\beta_n + \alpha_n \varepsilon_{t-i}^2}{\sqrt{t}}\right)\right\} \\
&= \exp\left\{\frac{j\gamma_n}{\sqrt{t}}\right\} \exp\left\{\frac{\alpha_n \sum_{i=1}^{j} \xi_{t-i}}{\sqrt{t}}\right\} \exp\left\{R_{t,j}^{(3)}\right\} \\
&= e^{\frac{j\gamma_n}{\sqrt{t}}} \exp\left\{\frac{\alpha_n \sum_{i=1}^{j} \xi_{t-i}}{\sqrt{t}}\right\}\left(1 + R_{t,j}^{(3)}\right)
\end{aligned}$$

Further, note $\{\xi_t\}_{t=1}^n$ is an i.i.d sequence with $E\xi_0^2 < \infty$, then we know

$$\max_{1\le j\le t} \left|\sum_{i=1}^{j} \xi_{t-i}\right| = O_p(\sqrt{t})$$

which implies

$$\max_{1\le j\le t} \left|\frac{\alpha_n}{\sqrt{t}} \sum_{i=1}^{j} \xi_{t-i}\right| = O_p(\alpha_n) = o_p(1)$$

Similarly, we define the sequence of events

$$B_n = \left\{\max_{1\le j\le t} \left|\frac{\alpha_n}{\sqrt{t}} \sum_{i=1}^{j} \xi_{t-i}\right| \le \frac{1}{2}\right\}$$

which is known to have the property $\lim_{n\to\infty} P(B_n) = 1$. Then by Taylor expansion, $|\exp(x) - (1+x)| \le \sqrt{e}x^2/2$ when $|x| \le 1/2$, on the event B_n

$$\left|R_{t,j}^{(2)}\right| = \left|\exp\left\{\frac{\alpha_n}{\sqrt{t}} \sum_{i=1}^{j} \xi_{t-i}\right\} - \left(1 + \frac{\alpha_n}{\sqrt{t}} \sum_{i=1}^{j} \xi_{t-i}\right)\right| \le \frac{\sqrt{e}}{2}\left(\frac{\alpha_n}{\sqrt{t}} \sum_{i=1}^{j} \xi_{t-i}\right)^2 = O_p\left(\alpha_n^2\right)$$

and by law of iterated logarithm, we know

$$\max_{1\le j\le t}\frac{1}{j\log\log j}\left(\frac{\alpha_n}{\sqrt{t}}\sum_{i=1}^{j}\xi_{t-i}\right)^2=O_p\left(\frac{\alpha_n^2}{t}\right)$$

Combining the results above, we have thus showed that

$$\prod_{i=1}^{j}\left(\frac{\beta_n+\alpha_n\varepsilon_{t-i}^2}{\sqrt{t}}\right)=e^{\frac{j\gamma_n}{\sqrt{t}}}\left(1+\frac{\alpha_n}{\sqrt{t}}\sum_{i=1}^{j}\xi_{t-i}+R_{t,j}^{(2)}\right)\left(1+R_{t,j}^{(3)}\right)$$

Lastly, by the equation above, we know

$$\begin{aligned}\prod_{i=1}^{t}\left(\frac{\beta_n+\alpha_n\varepsilon_{t-i}^2}{\sqrt{t}}\right)&=e^{\frac{t\gamma_n}{\sqrt{t}}}\left(1+\frac{\alpha_n}{\sqrt{t}}\sum_{i=1}^{t}\xi_{t-i}+O_p(\alpha_n^2)\right)\left(1+O_p(\gamma_n^2+\alpha_n^2)\right)\\&=e^{\sqrt{t}\gamma_n}\left(1+\frac{\alpha_n}{\sqrt{t}}\sum_{i=1}^{t}\xi_{t-i}+O_p(\gamma_n^2+\alpha_n^2)\right)\end{aligned}$$

and this establishes $R_t^{(1)}$. □

Proof of Theorem 1. First, we focus on the volatilities. Denote $k=\lfloor nt\rfloor$, $0<t\le 1$,

$$\begin{aligned}\sigma_k^2&=\omega+\sigma_0^2k^{k/2}e^{\sqrt{k}\gamma_n}\left(1+\frac{\alpha_n}{\sqrt{k}}\sum_{j=1}^{k}\xi_{k-j}+R_k^{(1)}\right)+\omega k^{k/2}\sum_{j=1}^{k-1}e^{\frac{j\gamma_n}{\sqrt{k}}}\left(1+\frac{\alpha_n}{\sqrt{k}}\sum_{i=1}^{j}\xi_{k-i}+R_{k,j}^{(2)}\right)R_{k,j}^{(3)}\\&\quad+\omega k^{k/2}\sum_{j=1}^{k-1}e^{\frac{j\gamma_n}{\sqrt{k}}}R_{k,j}^{(2)}+\omega k^{k/2}\sum_{j=1}^{k-1}e^{\frac{j\gamma_n}{\sqrt{k}}}\left(1+\frac{\alpha_n}{\sqrt{k}}\sum_{i=1}^{j}\xi_{k-i}\right)\\&=\omega+\sigma_{k,1}^2+\sigma_{k,2}^2+\sigma_{k,3}^2+\sigma_{k,4}^2\end{aligned}$$

For $\sigma_{k,1}^2$, note $k^{-1/2}\sum_{j=1}^{k}\xi_{k-j}$ is asymptotically normal, then by Proposition 1,

$$\frac{\alpha_n}{\sqrt{k}}\sum_{j=1}^{k}\xi_{k-j}+R_k^{(1)}=o_p(1)$$

and this implies

$$\left|\sigma_{k,1}^2\right|=O_p\left(k^{k/2}e^{\sqrt{k}\gamma_n}\right)$$

For $\sigma_{k,2}^2$, note by Lemma 4.1 in Berkes et al. (2005), we have

$$\sum_{j=1}^{k}je^{\frac{j\gamma_n}{\sqrt{k}}}\sim\frac{k}{|\gamma_n|^2}\Gamma(2)\tag{1}$$

and note that

$$\max_{1\le j\le k-1}\left|\frac{\alpha_n}{\sqrt{k}}\sum_{i=1}^{j}\xi_{k-i}+R_{k,j}^{(2)}\right|=o_p(1)\tag{2}$$

Then by equation (1), (2) and Proposition 1 we have

$$\begin{aligned}\left|\sigma_{k,2}^2\right| &= \left|\omega k^{k/2}\sum_{j=1}^{k-1} j e^{\frac{j\gamma_n}{\sqrt{k}}}\left(1+\frac{\alpha_n}{\sqrt{k}}\sum_{i=1}^{j}\xi_{k-i}+R_{k,j}^{(2)}\right)\frac{1}{j}R_{k,j}^{(3)}\right| \\ &= O_p(1)\omega k^{k/2}\frac{\alpha_n^2+\gamma_n^2}{k}\frac{k}{|\gamma_n|^2} \\ &= O_p\left(\frac{k^{k/2}\left(\alpha_n^2+\gamma_n^2\right)}{\gamma_n^2}\right)\end{aligned}$$

For $\sigma_{k,3}^2$, similarly, by Proposition 1 and Lemma 4.1 in Berkes et al. (2005), we have

$$\begin{aligned}\left|\sigma_{k,3}^2\right| &= \left|\omega k^{k/2}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}R_{k,j}^{(2)} = O_p(1)\omega k^{k/2}\frac{\alpha_n^2}{k}\sum_{j=1}^{k-1} j e^{\frac{j\gamma}{\sqrt{k}}}\log\log j\right| \\ &= O_p(1)\omega k^{k/2}\left(\alpha_n^2\log\log k\right)|\gamma_n|^{-2} \\ &= O_p\left(\frac{k^{k/2}\left(\alpha_n^2\log\log k\right)}{\gamma_n^2}\right)\end{aligned}$$

Lastly, for $\sigma_{k,4}^2$, by Lemma 4.1 in 1 we have

$$\begin{aligned}\sigma_{k,4}^2 &= \omega k^{k/2}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} + \omega k^{k/2}\frac{\alpha_n}{\sqrt{k}}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\sum_{i=1}^{j}\xi_{k-i} \\ &= O_p\left(\frac{k^{k/2}k^{1/2}}{|\gamma_n|}\right) + \omega k^{k/2}\frac{\alpha_n}{\sqrt{k}}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\sum_{i=1}^{j}\xi_{k-i}\end{aligned}$$

Therefore, we only have to consider the last term in the above equation. Define

$$\tau_m = k(m)^{-1/4}\sum_{j=1}^{k(m)-1} e^{\frac{j\gamma_n}{\sqrt{k(m)}}}\xi_{k(m)-j}, \quad 1\le m\le N$$

and

$$\tau_m^* = k(m)^{-1/2}\sum_{j=1}^{k(m)-1} e^{\frac{j\gamma_n}{\sqrt{k(m)}}}\sum_{i=1}^{j}\xi_{k(m)-i}, \quad 1\le m\le N$$

Then by Cramer-Wold device, we have

$$\begin{aligned}\sum_{m=1}^{N}\mu_m\tau_m &= \sum_{i=1}^{k(1)-1}\sum_{m=1}^{N} k(m)^{-1/4}\mu_m e^{\frac{(k(m)-i)\gamma_n}{\sqrt{k(m)}}} + \sum_{i=k(1)}^{k(2)-1}\sum_{m=2}^{N} k(m)^{-1/4}\mu_m e^{\frac{(k(m)-i)\gamma_n}{\sqrt{k(m)}}} \\ &\quad + \cdots + \sum_{i=k(N-1)}^{k(N)-1} k(N)^{-1/4}\mu_N e^{\frac{(k(N)-i)\gamma_n}{\sqrt{k(N)}}} \\ &= S_1+S_2+\cdots+S_N\end{aligned}$$

Observe that

$$
\begin{aligned}
ES_1^2 &= E\xi_0^2 \left(\sum_{i=1}^{k(1)-1} \sum_{m=1}^{N} k(m)^{-1/4} \mu_m e^{\frac{(k(m)-i)\gamma_n}{\sqrt{k(m)}}} \right)^2 \\
&= E\xi_0^2 \sum_{m=1}^{N} \frac{\mu_m^2}{\sqrt{k(m)}} \sum_{i=1}^{k(1)-1} e^{\frac{2(k(m)-i)\gamma_n}{\sqrt{k(m)}}} + E\xi_0^2 \sum_{1\le m\neq l\le N} (k(m)k(l))^{-1/4} \mu_m \mu_l \sum_{i=1}^{k(1)-1} e^{\frac{(k(m)-i)\gamma_n}{\sqrt{k(m)}} + \frac{(k(l)-i)\gamma_n}{\sqrt{k(l)}}} \\
&= E\xi_0^2 \frac{\mu_1^2}{\sqrt{k(1)}} \sum_{i=1}^{k(1)-1} e^{\frac{2(k(1)-i)\gamma_n}{\sqrt{k(1)}}} + E\xi_0^2 \sum_{m=2}^{N} \frac{\mu_m^2}{\sqrt{k(m)}} \sum_{i=1}^{k(1)-1} e^{\frac{2(k(m)-i)\gamma_n}{\sqrt{k(m)}}} \\
&\quad + E\xi_0^2 \sum_{1\le m\neq l\le N} (k(m)k(l))^{-1/4} \mu_m \mu_l \sum_{i=1}^{k(1)-1} e^{\frac{(k(m)-i)\gamma_n}{\sqrt{k(m)}} + \frac{(k(l)-i)\gamma_n}{\sqrt{k(l)}}} \\
&= E\xi_0^2 \frac{\mu_1^2}{\sqrt{k(1)}} \sum_{i=1}^{k(1)-1} e^{\frac{2i\gamma_n}{\sqrt{k(1)}}} + E\xi_0^2 \sum_{m=2}^{N} \frac{\mu_m^2}{\sqrt{k(m)}} e^{\frac{2(k(m)-k(1))\gamma_n}{\sqrt{k(m)}}} \sum_{i=1}^{k(1)-1} e^{\frac{2i\gamma_n}{\sqrt{k(m)}}} \\
&\quad + E\xi_0^2 \sum_{1\le m\neq l\le N} (k(m)k(l))^{-1/4} \mu_m \mu_l e^{\frac{(k(m)-k(1))\gamma_n}{\sqrt{k(m)}} + \frac{(k(l)-k(1))\gamma_n}{\sqrt{k(l)}}} \sum_{i=1}^{k(1)-1} e^{\frac{i\gamma_n}{\sqrt{k(m)}} + \frac{i\gamma_n}{\sqrt{k(l)}}} \\
&\sim E\xi_0^2 \mu_1^2 \frac{1}{2|\gamma_n|} + E\xi_0^2 \sum_{m=2}^{N} \mu_m^2 e^{\frac{2(k(m)-k(1))\gamma_n}{\sqrt{k(m)}}} \frac{1}{2|\gamma_n|} \\
&\quad + E\xi_0^2 \sum_{1\le m\neq l\le N} \mu_m \mu_l e^{\frac{(k(m)-k(1))\gamma_n}{\sqrt{k(m)}} + \frac{(k(l)-k(1))\gamma_n}{\sqrt{k(l)}}} \frac{1}{(\sqrt{k(m)} + \sqrt{k(l)})|\gamma_n|} \\
&= E\xi_0^2 \mu_1^2 \frac{1}{2|\gamma_n|} + o\left(\frac{1}{|\gamma_n|}\right)
\end{aligned}
$$

Therefore, we have

$$
E\left(\sum_{m=1}^{N} \mu_m \tau_m\right)^2 = \left(\sum_{m=1}^{N} \mu_m^2\right) E\xi_0 \frac{1}{2|\gamma_n|} + o\left(\frac{1}{|\gamma_n|}\right)
$$

Observe also that, for some c_i, $1 \le i \le k(N)-1$, we have

$$
\sum_{m=1}^{N} \mu_m \tau_m = \sum_{i=1}^{k(N)-1} c_i \xi_i
$$

and by Jensen's inequality, we know for some $\delta > 0$,

$$
\begin{aligned}
|c_i|^{2+\delta} &= \left| k(1)^{-1/4} \mu_1 e^{\frac{(k(1)-i)\gamma_n}{\sqrt{k(1)}}} + k(2)^{-1/4} \mu_1 e^{\frac{(k(2)-i)\gamma_n}{\sqrt{k(2)}}} + \cdots + k(N)^{-1/4} \mu_1 e^{\frac{(k(N)-i)\gamma_n}{\sqrt{k(N)}}} \right|^{2+\delta} \\
&\le C_1(N) \left[\frac{|\mu_1|^{2+\delta}}{k(1)^{1/2+\delta/4}} e^{\frac{(k(1)-i)(2+\delta)\gamma_n}{\sqrt{k(1)}}} + \cdots + \frac{|\mu_N|^{2+\delta}}{k(N)^{1/2+\delta/4}} e^{\frac{(k(N)-i)(2+\delta)\gamma_n}{\sqrt{k(N)}}} \right]
\end{aligned}
$$

This implies that

$$
\sum_{i=1}^{k(N)-1} |c_i|^{2+\delta} \sim C_1(N) |\mu_1|^{2+\delta} \frac{1}{k(1)^{\delta/4} (2+\delta) |\gamma_n|} + O\left(\frac{1}{k(2)^{\delta/4} |\gamma_n|}\right) = o\left(\frac{1}{|\gamma_n|}\right)
$$

Now we can easily check the Liapounov condition, where

$$
\frac{\left(\sum_{i=1}^{k(N)-1} |c_i|^{2+\delta} E|\xi_i|^{2+\delta}\right)^{1/(2+\delta)}}{\left(\sum_{i=1}^{k(N)-1} c_i^2 E\xi_i^2\right)^{1/2}} = o\left(|\gamma_n|^{1/2-1/(2+\delta)}\right) = o(1)
$$

Then by Liapounov central limit theorem, we have

$$\sqrt{2|\gamma_n|}\,[\tau_1, \tau_2, \cdots, \tau_N] \xrightarrow{d} \sqrt{E\xi_0^2}\,[\eta_1, \eta_2, \cdots, \eta_N]$$

where $\eta_1, \eta_2, \cdots, \eta_N$ are independent standard normal random variables.

Now we have to check the relationship between τ_m and τ_m^*. Note by $k^{-1/2}\left(e^{\frac{\gamma_n}{\sqrt{k}}} - 1\right)^{-1} = (\gamma_n + o(1))^{-1}$, we have

$$\begin{aligned}
\frac{1}{\sqrt{k}}\sum_{j=i}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} - |\gamma_n|^{-1}e^{\frac{i\gamma_n}{\sqrt{k}}} &= \frac{1}{\sqrt{k}}\frac{e^{\frac{k\gamma_n}{\sqrt{k}}} - e^{\frac{i\gamma_n}{\sqrt{k}}}}{e^{\frac{\gamma_n}{\sqrt{k}}} - 1} - |\gamma_n|^{-1}e^{\frac{i\gamma_n}{\sqrt{k}}} \\
&= (\gamma_n + o(1))^{-1}\left(e^{\frac{k\gamma_n}{\sqrt{k}}} - e^{\frac{i\gamma_n}{\sqrt{k}}}\right) - |\gamma_n|^{-1}e^{\frac{i\gamma_n}{\sqrt{k}}} \\
&= \left(\gamma_n^{-1} + O(1)\right)e^{\frac{k\gamma_n}{\sqrt{k}}} - e^{\frac{i\gamma_n}{\sqrt{k}}}O(1)
\end{aligned}$$

Then, we know

$$\begin{aligned}
E\left[\sqrt{2|\gamma_n|^3}\tau_m^* - \sqrt{2|\gamma_n|}\tau_m\right]^2 &= \frac{2|\gamma_n|^3}{\sqrt{k}}E\left[\frac{1}{\sqrt{k}}\sum_{i=1}^{k-1}\left(\sum_{j=i}^{k-1} e^{\frac{j\gamma}{\sqrt{k}}}\right)\xi_{k-i} - |\gamma_n|^{-1}\sum_{i=1}^{k-1} e^{\frac{i\gamma_n}{\sqrt{k}}}\xi_{k-i}\right]^2 \\
&= \frac{2|\gamma_n|^3}{\sqrt{k}}E\xi_0^2\sum_{i=1}^{k-1}\left(\frac{1}{\sqrt{k}}\sum_{j=i}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} - |\gamma_n|^{-1}e^{\frac{i\gamma_n}{\sqrt{k}}}\right)^2 \\
&\sim \frac{2|\gamma_n|^3}{\sqrt{k}}E\xi_0^2\left(k\gamma_n^{-2}e^{\sqrt{k}\gamma_n} + \frac{\sqrt{k}}{2|\gamma|} - 2\gamma_n^{-1}e^{\sqrt{k}\gamma_n}\frac{\sqrt{k}}{|\gamma|}\right) \\
&= 2E\xi_0^2 O\left(\sqrt{k}|\gamma_n|e^{\sqrt{k}\gamma_n}\right) + o_p(1) \\
&= o_p(1)
\end{aligned}$$

where the last equality comes from the well known limits of xe^{-x},

$$\lim_{x\to\infty}\frac{x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0, \quad \lim_{x\to 0}\frac{x}{e^x} = 0$$

Therefore, we have

$$\sqrt{2|\gamma_n|^3}\,[\tau_1^*, \tau_2^*, \cdots, \tau_N^*] \xrightarrow{d} \sqrt{E\xi_0^2}\,[\eta_1, \eta_2, \cdots, \eta_N],$$

Now combine the results above, we have, for each $k = \lfloor nt_m \rfloor$, $m \in [1, N]$

$$\frac{\sqrt{2|\gamma_n|^3}}{\alpha_n}\frac{1}{\sqrt{E\xi_0^2}}\left(\frac{\sigma_k^2}{\omega k^{(2k+1)/4}} - \frac{1}{k^{1/4}}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\right) \sim \mathcal{N}(0,1)$$

Now, for returns, we know from the above result that

$$\frac{|\gamma_n|\sigma_k^2}{\omega k^{(k+1)/2}} - 1 = O_p\left(\frac{\alpha_n n^{1/4}}{\sqrt{|\gamma_n|}}\right) = o_p(1)$$

Therefore, by the return equation, we have

$$\left(\frac{|\gamma_n|}{\omega k^{(k+1)/2}}\right)^{1/2} u_k = \left(\frac{|\gamma_n|\sigma_k^2}{\omega k^{(k+1)/2}}\right)^{1/2}\varepsilon_k \sim \varepsilon_k$$

□

Proof of Theorem 2. Similar to Theorem 1, when $\gamma_n = 0$, the volatility admits the additive decomposition. Then, for $\sigma^2_{k,1}$, by central limit theorem, we know

$$\frac{\alpha_n}{\sqrt{k}} \sum_{j=1}^{k} \xi_{k-j} = O_p(\alpha_n) = o_p(1)$$

which, combining with Proposition 1, implies that

$$\left|\sigma^2_{k,1}\right| = O_p\left(k^{k/2}\right)$$

For $\sigma^2_{k,2}$, note that we have established equation (2), then by Proposition 1, we have

$$\left|\sigma^2_{k,2}\right| = O_p\left(k^{k/2}\alpha_n^2\right)$$

For $\sigma^2_{k,3}$, by Proposition 1 we have

$$\left|\sigma^2_{k,3}\right| = O_p(k^{k/2}\alpha_n^2)$$

Lastly, for $\sigma^2_{k,4}$, note by Lemma 5.1 in Berkes et al. (2005), for $k = \lfloor nt \rfloor$, $t \in (0,1)$, we have

$$\frac{1}{n^{3/2}} \sum_{j=1}^{\lfloor nt \rfloor - 1} \sum_{i=1}^{j} \xi_{k-i} \xrightarrow{d} \sqrt{E\xi_0^2} \int_0^t x dW(x)$$

where $W(x)$ is a Wiener process.

Therefore, for $k(m) = \lfloor nt_m \rfloor$, $m \in [1, N]$

$$\frac{1}{n^{3/2}\alpha_n} \left(\frac{\sigma^2_{k(m)}}{\omega k(m)^{(k(m)-1)/2}} - k(m)^{3/2} \right) = \frac{1}{n^{3/2}} \sum_{j=1}^{\lfloor nt_m \rfloor - 1} \sum_{i=1}^{j} \xi_{k(m)-i} + o_p(1) \xrightarrow{d} \sqrt{E\xi_0^2} \int_0^{t_m} x dW(x).$$

Further, note the results above implies that

$$\frac{\sigma^2_{k(m)}}{\omega k(m)^{k(m)/2+1}} - 1 = O_p\left(\left(\frac{n}{k}\right)^{3/2} \alpha_n \right) = o_p(1).$$

Hence, by return equation, we obtain

$$\left(\frac{1}{\omega k(m)^{k(m)/2+1}} \right)^{1/2} u_{k(m)} = \left(\frac{\sigma^2_{k(m)}}{\omega k(m)^{k(m)/2+1}} \right)^{1/2} \varepsilon_{k(m)} \xrightarrow{d} \varepsilon_{k(m)}.$$

□

Proof of Theorem 3. By Proposition 1, we know when $\gamma_n > 0$, the volatility admits the additive representation. Then, for $\sigma^2_{k,1}$, similar to that in Theorem 1,

$$\left|\sigma^2_{k,1}\right| = O_p\left(k^{k/2} e^{\sqrt{k}\gamma_n}\right)$$

For $\sigma_{k,2}^2$, by Proposition 1 and equation (2), we have the relation

$$\begin{aligned}
\left|\sigma_{k,2}^2\right| &= \left|\omega k^{k/2}\sum_{j=1}^{k-1} j e^{\frac{j\gamma_n}{\sqrt{k}}}\left(1+o_p(1)\right)\frac{1}{j}R_{k,j}^{(3)}\right| \\
&= O_p(1)\omega k^{k/2}(\alpha_n^2+\gamma_n^2)\frac{e^{\frac{k\gamma_n}{\sqrt{k}}}-e^{\frac{\gamma_n}{\sqrt{k}}}}{e^{\frac{\gamma_n}{\sqrt{k}}}-1} \\
&< O_p\left(k^{k/2}\left(\alpha_n^2+\gamma_n^2\right)\frac{\sqrt{k}e^{\sqrt{k}\gamma_n}}{\gamma_n}\right)
\end{aligned}$$

where the last inequality comes from the fact that

$$\frac{e^{\frac{k\gamma_n}{\sqrt{k}}}-e^{\frac{\gamma_n}{\sqrt{k}}}}{e^{\frac{\gamma_n}{\sqrt{k}}}-1} < \frac{e^{\sqrt{k}\gamma_n}}{\gamma_n/\sqrt{k}}$$

For $\sigma_{k,3}^2$, by Proposition 1, we have

$$\begin{aligned}
\left|\sigma_{k,3}^2\right| &= \left|\omega k^{k/2}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\left(j\log\log j\right)\frac{1}{j\log\log j}R_{k,j}^{(2)}\right| \\
&= O_p(1)\omega k^{k/2}\left(k\log\log k\right)\frac{\alpha_n^2}{k}\frac{e^{\frac{k\gamma_n}{\sqrt{k}}}-e^{\frac{\gamma_n}{\sqrt{k}}}}{e^{\frac{\gamma_n}{\sqrt{k}}}-1} \\
&< O_p\left(k^{k/2}\left(\alpha_n^2\log\log k\right)\frac{\sqrt{k}e^{\sqrt{k}\gamma_n}}{\gamma_n}\right)
\end{aligned}$$

Lastly, for $\sigma_{k,4}^2$, we have

$$\sigma_{k,4}^2 = \omega k^{k/2}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} + \omega k^{k/2}\frac{\alpha_n}{\sqrt{k}}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\sum_{i=1}^{j}\xi_{k-i}$$

Now, we introduce the following lemma to assist the proof.

Lemma 1. *If Assumption 1 and 2 hold, then*

$$\frac{\gamma_n^2}{k}e^{-2\sqrt{k}\gamma_n}E\left(\frac{1}{\sqrt{k}}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\sum_{i=1}^{j}\xi_{k-i} - \frac{e^{\sqrt{k}\gamma_n}}{\gamma_n}\sum_{i=1}^{k-1}\xi_i\right)^2 \to 0$$

Then by Lemma 1, we have

$$\frac{\gamma_n e^{-\sqrt{k}\gamma_n}}{\sqrt{k}\alpha_n}\left(\frac{\sigma_{k,4}^2}{\omega k^{k/2}} - \sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\right) = \frac{\gamma_n e^{-\sqrt{k}\gamma_n}}{\sqrt{k}}\frac{1}{\sqrt{k}}\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}}\sum_{i=1}^{j}\xi_{k-i} + o_p(1) = \frac{1}{\sqrt{k}}\sum_{i=1}^{k-1}\xi_i + o_p(1).$$

Therefore, by Donsker's theorem, we obtain that, for $k(m)=\lfloor nt_m\rfloor$, $t_m\in(0,1)$ and $m=1,2,\cdots,N$,

$$\frac{\gamma_n e^{-\sqrt{k(m)}\gamma_n}}{\sqrt{k(m)}\alpha_n}\frac{1}{\sqrt{E\xi_0^2}}\left(\frac{\sigma_{k(m)}^2}{\omega k(m)^{k(m)/2}} - \sum_{j=1}^{k(m)-1} e^{\frac{j\gamma_n}{\sqrt{k(m)}}}\right) \Rightarrow W(t_m)$$

where $W(t)$ is a finite dimensional Wiener process.

Further, note that

$$\frac{\gamma_n}{\sqrt{k}} e^{-\sqrt{k}\gamma_n} \left(\sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} - \frac{\sqrt{k} e^{\sqrt{k}\gamma_n}}{\gamma_n} \right) = o(1)$$

then by the result above we know

$$\frac{\gamma_n e^{-\sqrt{k(m)}\gamma_n}}{\sqrt{k(m)}} \left(\frac{\sigma^2_{k(m)}}{\omega k(m)^{k(m)/2}} - \sum_{j=1}^{k(m)-1} e^{\frac{j\gamma_n}{\sqrt{k(m)}}} \right) = O_p(\alpha_n) = o_p(1)$$

Hence, by return equantion, we derive

$$\left(\frac{\gamma_n e^{-\sqrt{k}\gamma_n}}{\omega k^{(k+1)/2}} \right)^{1/2} u_k = \left(\frac{\gamma_n e^{-\sqrt{k}\gamma_n}}{\omega k^{(k+1)/2}} \sigma^2_k \right)^{1/2} \varepsilon_k \sim \varepsilon_k$$

□

Proof of Lemma 1. Note that

$$\frac{1}{\sqrt{k}} \sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} \sum_{i=1}^{j} \xi_{k-i} = \frac{1}{\sqrt{k}} \sum_{i=1}^{k-1} \left(\sum_{j=i}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} \right) \xi_{k-i} \quad \text{and} \quad \sum_{i=1}^{k-1} \xi_i = \sum_{i=1}^{k-1} \xi_{k-i},$$

Then,

$$\begin{aligned} E\left(\frac{1}{\sqrt{k}} \sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} \sum_{i=1}^{j} \xi_{k-i} - \frac{e^{\sqrt{k}\gamma_n}}{\gamma_n} \sum_{i=1}^{k-1} \xi_i \right)^2 &= E\xi_0^2 \sum_{i=1}^{k-1} \left(\frac{1}{\sqrt{k}} \sum_{j=i}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} - \frac{\sqrt{k} e^{\sqrt{k}\gamma_n}}{\gamma_n} \right)^2 \\ &= \frac{E\xi_0^2}{k} \sum_{i=1}^{k-1} \left(\frac{e^{\frac{k\gamma_n}{\sqrt{k}}} - e^{\frac{i\gamma_n}{\sqrt{k}}}}{e^{\frac{\gamma_n}{\sqrt{k}}} - 1} - \frac{\sqrt{k} e^{\sqrt{k}\gamma_n}}{\gamma_n} \right)^2 \end{aligned}$$

Note by Taylor expansion,

$$\left| \frac{e^{\frac{k\gamma_n}{\sqrt{k}}} - e^{\frac{i\gamma_n}{\sqrt{k}}}}{e^{\frac{\gamma_n}{\sqrt{k}}} - 1} - \frac{\sqrt{k} e^{\sqrt{k}\gamma_n}}{\gamma_n} \right| \leq C_1 \left(\frac{\sqrt{k} e^{\frac{i\gamma_n}{\sqrt{k}}}}{\gamma_n} + e^{\sqrt{k}\gamma_n} \right)$$

which implies that

$$\begin{aligned} \sum_{i=1}^{k-1} \left(\frac{e^{\frac{k\gamma_n}{\sqrt{k}}} - e^{\frac{i\gamma_n}{\sqrt{k}}}}{e^{\frac{\gamma_n}{\sqrt{k}}} - 1} - \frac{\sqrt{k} e^{\sqrt{k}\gamma_n}}{\gamma_n} \right)^2 &\leq 2C_1^2 \left(\sum_{i=1}^{k-1} \frac{k e^{\frac{2i\gamma_n}{\sqrt{k}}}}{\gamma_n^2} + k e^{2\sqrt{k}\gamma_n} \right) \\ &= O(1) \left(\frac{k}{\gamma_n^2} \frac{e^{\frac{2k\gamma_n}{\sqrt{k}}} - e^{\frac{2\gamma_n}{\sqrt{k}}}}{e^{\frac{2\gamma_n}{\sqrt{k}}} - 1} + k e^{2\sqrt{k}\gamma_n} \right) \\ &= O(1) \left(\frac{k}{\gamma_n^3} e^{2\sqrt{k}\gamma_n} + k e^{2\sqrt{k}\gamma_n} \right) \end{aligned}$$

Now we can see that

$$\begin{aligned} &\frac{\gamma_n^2}{k} e^{-2\sqrt{k}\gamma_n} E\left(\frac{1}{\sqrt{k}} \sum_{j=1}^{k-1} e^{\frac{j\gamma_n}{\sqrt{k}}} \sum_{i=1}^{j} \xi_{k-i} - \frac{e^{\sqrt{k}\gamma_n}}{\gamma_n} \sum_{i=1}^{k-1} \xi_i \right)^2 \\ &= O(1) \frac{\gamma_n^2}{k} e^{-2\sqrt{k}\gamma_n} \frac{E\xi_0^2}{k} \left(\frac{k}{\gamma_n^3} e^{2\sqrt{k}\gamma_n} + k e^{2\sqrt{k}\gamma_n} \right) \\ &= O(1) \left(\frac{1}{k\gamma_n} + \frac{\gamma_n^2}{k} \right) \\ &= o_p(1). \end{aligned}$$

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