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**Maximum Likelihood Estimation for the Fractional  
Vasicek Model**

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THE SCHOOL OF ECONOMICS, SMU

# Maximum Likelihood Estimation for the Fractional Vasicek Model\*

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March 3, 2019

## Abstract

This paper is concerned about the problem of estimating the drift parameters in the fractional Vasicek model from a continuous record of observations. Based on the Girsanov theorem for the fractional Brownian motion, the maximum likelihood (ML) method is used. The asymptotic theory for the ML estimates (MLE) is established in the stationary case, the explosive case, and the null recurrent case for the entire range of the Hurst parameter, providing a complete treatment of asymptotic analysis. It is shown that changing the sign of the persistence parameter will change the asymptotic theory for the MLE, including the rate of convergence and the limiting distribution. It is also found that the asymptotic theory depends on the value of the Hurst parameter.

*JEL classification:* C15, C22, C32

*Keywords:* Maximum likelihood estimate; Fractional Vasicek model; Asymptotic distribution; Stationary process; Explosive process; Null recurrent process

## 1 Introduction

Since Vasicek (1977) introduced a model to describe the evolution of short-term interest rates, the so-called Vasicek model has enjoyed a wide range of applications. Jamshidian (1989) used it to price bond options. Scott (1987) used it to model the evolution of instantaneous volatility of stock price and to price European call options.

Many extensions have been made to generalize the specification of Vasicek. For example, motivated by the phenomenon of long-range dependence found in data of hydrology,

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geophysics, climatology, telecommunication, economics and finance, the Brownian motion in the Vasicek model has been replaced by a fractional Brownian motion (fBm), leading to the following fractional Vasicek model (fVm)

$$dX_t = \kappa(\mu - X_t) dt + \sigma dB_t^H, \quad (1.1)$$

where  $\sigma$  is a positive constant,  $\mu, \kappa \in \mathbb{R}$ ,  $B_t^H$  is an fBm with  $H \in (0, 1)$  being the Hurst parameter. An fBm  $B_t^H$  is a zero mean Gaussian process, defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , with the following covariance function

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}). \quad (1.2)$$

The process  $B_t^H$  is self-similar in the sense that  $\forall a \in \mathbb{R}^+$ ,  $B_{at}^H \stackrel{d}{=} a^H B_t^H$ . It becomes the standard Brownian motion  $W_t$  when  $H = 1/2$  and can be represented as a stochastic integral with respect to the standard Brownian motion. It is negatively correlated when  $0 < H < 1/2$ . When  $1/2 < H < 1$ , it has long-range dependence in the sense that  $\sum_{n=1}^{\infty} \mathbb{E}(B_1^H (B_{n+1}^H - B_n^H)) = \infty$ . In this case, the positive (negative) increments are likely to be followed by positive (negative) increments. The parameter  $H$ , which is also called the self similarity parameter, measures the intensity of the long range dependence.

Parameter  $\kappa$  is often referred to as the persistence parameter. When  $\kappa > 0$ ,  $X_t$  is stationary and ergodic. In this case,  $\mu$  is the unconditional mean of  $X_t$  and  $\kappa$  is the mean-reversion parameter. When  $\kappa < 0$ ,  $X_t$  is explosive and hence non-ergodic. When  $\kappa = 0$ ,  $X_t$  is null-recurrent and the drift term  $\kappa(\mu - X_t) dt$  disappears. So  $\mu$  is superfluous in this case. The ergodic fVm has been used to model the evolution of instantaneous volatility in Comte and Renault (1998), the evolution of quadratic variation in Aït-Sahalia and Mancini (2008), the evolution of realized variance in Gatheral et al. (2018), the evolution of VIX in Xiao et al. (2019).

An alternative to and perhaps slightly more general specification than Model (1.1) is

$$dX_t = (\alpha - \kappa X_t) dt + \sigma dB_t^H, \quad (1.3)$$

In Model (1.3), even when  $\kappa = 0$ , the drift term does not vanish and it is  $\alpha dt$ . This alternative specification for the drift term was used in Chan et al. (1992) and Yu and Phillips (2001). When  $\alpha$  in (1.3) is known (without loss of generality, it is assumed to be zero), (1.3) becomes the fractional Ornstein-Uhlenbeck (fOU) process.

Assuming that a continuous record of observations is available for  $X_t$  with  $t \in [0, T]$ , a number of studies have introduced methods to estimate  $\kappa$  and  $\alpha$  (or  $\mu$ ) and developed asymptotic distributions for the proposed estimators under the scheme of  $T \rightarrow \infty$ . When  $H > 1/2$  and  $\kappa > 0$ , borrowing the idea of Hu and Nualart (2010) and Hu et al. (2017), Xiao and Yu (2019a) considered two methods, the least squares (LS) estimates and the ergodic-type

estimates of  $\kappa$  and  $\mu$ . When  $H \geq 1/2$  and  $\kappa = 0$  or  $\kappa < 0$ , Xiao and Yu (2019a) considered the LS method. Xiao and Yu (2019b) extends the results of Xiao and Yu (2019a) from the case where  $H \in (1/2, 1)$  to where  $H \in (0, 1/2)$ . Lohvinenko and Ralchenko (2017) considered the maximum likelihood (ML) estimates of  $\kappa$  and  $\alpha$  when  $\kappa > 0$  and  $H \in (1/2, 1)$ .

Our paper also focuses on the ML estimators (MLE) of  $\kappa$  and  $\alpha$ . We aim to develop the asymptotic distributions for the MLE of  $\kappa$  and  $\alpha$  under the following scenarios: (1)  $\kappa > 0$  and  $H \in (0, 1/2]$ ; (2)  $\kappa = 0$  and  $H \in (0, 1)$ ; (3)  $\kappa < 0$  and  $H \in (0, 1)$ . Therefore, together with Lohvinenko and Ralchenko (2017), a complete coverage of asymptotic theory for all possible cases is provided to the MLE of  $\kappa$  and  $\alpha$ .

The rest of the paper is organized as follows. Section 2 introduces the MLE of  $\kappa$  and  $\alpha$ . Section 3 is devoted to the asymptotic theory for the stationary case (i.e.,  $\kappa > 0$ ) but  $H \in (0, 1/2]$ . Section 4 studies the asymptotic properties of the MLE in the null recurrent case (i.e.,  $\kappa = 0$ ) and for the entire range for the Hurst parameter  $H \in (0, 1)$ . In Section 5, we establish the asymptotic behaviors of the MLE for the non-ergodic case (i.e.,  $\kappa < 0$ ) and for the entire range for the Hurst parameter  $H \in (0, 1)$ . Section 6 contains some concluding remarks and gives directions of further research. All the proofs are collected in the Appendix.

We use the following notations throughout the paper:  $\xrightarrow{p}$ ,  $\xrightarrow{d}$  and  $\sim$  denote convergence in probability, convergence in distribution, and asymptotic equivalence, respectively, as  $T \rightarrow \infty$ . Throughout this paper, the constant  $C$  only depends on  $H$ , whose values can differ at different places.

## 2 ML Estimation

Following Kleptsyna et al. (2000) and Lohvinenko and Ralchenko (2017), by applying the Girsanov theorem for the fBm developed in Norros et al. (1999), one can get the expression for the continuous-record log-likelihood function for Model (1.3) as follows:

$$\ell(\kappa, \alpha) = \int_0^T Q_H(t) dM_t^H + \frac{1}{2} \int_0^T (Q_H(t))^2 d\omega_t^H,$$

where

$$Q_H(t) = \frac{1}{\sigma} \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) (\alpha - \kappa X_s) ds, \quad (2.1)$$

$$k_H(t, s) = \frac{1}{k_H} (s(t-s))^{\frac{1}{2}-H}, \quad k_H = 2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(H+\frac{1}{2}\right), \quad (2.2)$$

$$\omega_t^H = \frac{1}{\lambda_H} t^{2-2H}, \quad (2.3)$$

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \quad (2.4)$$

$$M_t^H = \int_0^t k_H(t, s) dB_s^H. \quad (2.5)$$

Taking the derivatives of the log-likelihood function with respect to  $\kappa$  and  $\alpha$  and setting them to zero, Lohvinenko and Ralchenko (2017) obtained the following expressions for the MLE of  $\alpha$  and  $\kappa$ :

$$\tilde{\alpha}_T = \frac{S_T \int_0^T P_H^2(t) d\omega_t^H - \int_0^T P_H(t) dS_t \int_0^T P_H(t) d\omega_t^H}{\omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2} \sigma, \quad (2.6)$$

$$\tilde{\kappa}_T = \frac{S_T \int_0^T P_H(t) d\omega_t^H - \omega_T^H \int_0^T P_H(t) dS_t}{\omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2}, \quad (2.7)$$

where

$$S_t = \frac{1}{\sigma} \int_0^t k_H(t, s) dX_s, \quad (2.8)$$

$$P_H(t) = \frac{1}{\sigma} \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) X_s ds, \quad (2.9)$$

Combining (1.3), (2.2) with (2.9), we deduce that

$$P_H(t) = \frac{1}{\sigma} \frac{\alpha}{\kappa} + \frac{1}{\sigma} \left( X_0 - \frac{\alpha}{\kappa} \right) V_H(t) + \tilde{P}_H(t), \quad (2.10)$$

where

$$V_H(t) = \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) e^{-\kappa s} ds, \quad (2.11)$$

$$\tilde{P}_H(t) = \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) U_s ds, \quad (2.12)$$

$$U_t = \int_0^t e^{-\kappa(t-s)} dB_s^H. \quad (2.13)$$

Using the idea of Kleptsyna and Le Breton (2002), Lohvinenko and Ralchenko (2017) obtained the following results

$$Q_H(t) = \frac{\alpha}{\sigma} - \kappa P_H(t), \quad (2.14)$$

$$S_t = \int_0^t Q_H(s) d\omega_s^H + M_t^H = \frac{\alpha}{\sigma} \omega_t^H - \kappa \int_0^t P_H(s) d\omega_s^H + M_t^H, \quad (2.15)$$

$$dS_t = \frac{\alpha}{\sigma} d\omega_t^H - \kappa P_H(t) d\omega_t^H + dM_t^H. \quad (2.16)$$

The process  $M_t^H$ , the so-called fundamental martingale, is a Gaussian martingale with the variance function being  $\omega_t^H$ . Moreover, the natural filtration of the martingale  $M^H$  coincides with the natural filtration of the fBm. Based on (2.15) and (2.16), the MLE of  $\alpha$  and  $\kappa$  can be represented as

$$\tilde{\alpha}_T = \alpha + \frac{M_T^H \int_0^T P_H^2(t) d\omega_t^H - \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H}{\omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2} \sigma, \quad (2.17)$$

$$\tilde{\kappa}_T = \kappa + \frac{M_T^H \int_0^T P_H(t) d\omega_t^H - \omega_T^H \int_0^T P_H(t) dM_t^H}{\omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2}, \quad (2.18)$$

When a continuous record of observations of  $X_t$  is available, Lohvinenko and Ralchenko (2017) studied the consistency and the asymptotic normality of the MLE defined by (2.6) and (2.7) when  $H > 1/2$  and  $\kappa > 0$ . The goal of the present paper is to establish asymptotic theory for the MLE of  $\alpha$  and  $\kappa$  for all the other cases, including  $H < 1/2$  and  $\kappa > 0$ ,  $H \in (0, 1)$  and  $\kappa = 0$ ,  $H \in (0, 1)$  and  $\kappa < 0$ .

### 3 Asymptotic Theory When $\kappa > 0$

In this section, inspired by Lohvinenko and Ralchenko (2017), we extend the asymptotic properties of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  from the case of  $H \in (1/2, 1)$  to the case of  $H \in (0, 1/2]$ . For the sake of comparison, we first introduce the main result of Lohvinenko and Ralchenko (2017). When  $H > 1/2$ , Lohvinenko and Ralchenko (2017) obtained the asymptotic normality for the MLE of  $\alpha$  and  $\kappa$ , i.e.,

$$T^{1-H} (\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}(0, \lambda_H \sigma^2), \quad (3.1)$$

$$\sqrt{T} (\tilde{\kappa}_T - \kappa) \xrightarrow{d} \mathcal{N}(0, 2\kappa). \quad (3.2)$$

The objective of this section is to obtain the consistency and the asymptotic normality of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  when  $H \in (0, 1/2]$ . Since the asymptotic laws of  $\tilde{\alpha}_T$  are different when  $H \in (0, 1/2]$  from those when  $H = 1/2$ , we need to treat them separately.

#### 3.1 Asymptotic theory when $H \in (0, 1/2)$

Before presenting asymptotic properties of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  for  $H \in (0, 1/2)$ , we first state a useful technical lemma.

**Lemma 3.1** *For  $\kappa > 0$  and  $H \in (0, 1)$  in Model (1.3), and for any  $\varepsilon > 0$ , as  $T \rightarrow \infty$ , we have*

$$V_H(T) = O\left(T^{H-\frac{3}{2}}\right), \quad (3.3)$$

$$\int_0^T V_H(t) d\omega_t^H = O\left(T^{\frac{1}{2}-H}\right), \quad (3.4)$$

$$\int_0^T \tilde{P}_H(t) dM_t^H = O_p\left(\sqrt{T}\right), \quad (3.5)$$

$$\int_0^T \tilde{P}_H^2(t) d\omega_t^H = O_p(T), \quad (3.6)$$

$$\int_0^T V_H^2(t) d\omega_t^H = O(1), \quad (3.7)$$

$$\int_0^T \tilde{P}_H(t) d\omega_t^H = O_p\left(T^{1-H+\varepsilon}\right), \quad (3.8)$$

$$\int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H = O_p\left(\sqrt{T}\right). \quad (3.9)$$

We can now describe the asymptotic laws of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  as  $T \rightarrow \infty$ .

**Theorem 3.1** For  $\kappa > 0$  and  $H \in (0, 1/2)$  in Model (1.3), as  $T \rightarrow \infty$ , we have

$$\sqrt{T}(\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}\left(0, \frac{2\alpha^2}{\kappa}\right), \quad (3.10)$$

$$\sqrt{T}(\tilde{\kappa}_T - \kappa) \xrightarrow{d} \mathcal{N}(0, 2\kappa). \quad (3.11)$$

**Remark 3.1** Comparing the asymptotic theory with that obtained in Lohvinenko and Ralchenko (2017), the asymptotic normality continues to hold for both estimators. Moreover, comparing (3.11) with (3.2), we can see that the asymptotic theory for  $\tilde{\kappa}_T$  is the same regardless of  $H \in (0, 1/2)$  or  $H \in (1/2, 1)$ . Comparing (3.10) with (3.1), we can see that the asymptotic variance of  $\tilde{\alpha}_T$  depends on  $H$ . The asymptotic variance is  $\lambda_H \sigma^2$  with the consistency order  $T^{1-H}$  if  $H \in (1/2, 1)$  whereas it does not depend on  $H$  with the consistency order  $\sqrt{T}$  as  $T$  becomes large if  $H \in (0, 1/2)$ .

**Remark 3.2** The asymptotic theory for the MLE of  $\kappa$  in the fOU when  $H \in (0, 1/2)$  has been developed in the literature; see, for example, Theorem 2 in Brouste and Kleptsyna (2010). It is the same as in (3.11). So having to estimate an additional parameter  $\alpha$ , there is no efficiency loss in estimating  $\kappa$  asymptotically.

**Remark 3.3** Following the idea of Hu et al. (2017), Xiao and Yu (2019b) obtained the asymptotic distribution of the LS estimate and the ergodic-type estimate of  $\kappa$  for  $H \in (0, 1/2)$ . The LS estimator of  $\kappa$  is given by

$$\hat{\kappa}_{LS} = \frac{(X_T - X_0) \int_0^T X_t dt - T \int_0^T X_t dX_t}{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \quad (3.12)$$

where the stochastic integral  $\int_0^T X_t dX_t$  is interpreted as a divergence integral. The ergodic-type estimate of  $\kappa$  is given by

$$\hat{\kappa}_{HN} = \left( \frac{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}{T^2 \sigma^2 H \Gamma(2H)} \right)^{-\frac{1}{2H}}. \quad (3.13)$$

Moreover, Xiao and Yu (2019b) showed that

$$\sqrt{T}(\hat{\kappa}_{LS} - \kappa) \xrightarrow{d} \mathcal{N}(0, \kappa \delta_{LS}^2) \text{ as } T \rightarrow \infty, \quad (3.14)$$

where  $\delta_{LS}^2 = (4H - 1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}$  and that

$$\sqrt{T}(\hat{\kappa}_{HN} - \kappa) \xrightarrow{d} \mathcal{N}(0, \kappa \delta_{HN}^2) \text{ as } T \rightarrow \infty, \quad (3.15)$$



where  $\delta_{HN}^2 = \frac{1}{4H^2} \left[ (4H - 1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)} \right]$ . Setting  $\kappa = 1$ , Figure 1 compares the efficiency of ML, LS and ergodic-type estimates of  $\kappa$  by plotting  $\delta_{LS}^2$ ,  $\delta_{HN}^2$  against  $2$  when  $H$  takes a value between  $(0, 0.5)$ . It can be seen that the LS estimate is the most efficient, followed by the MLE and then by the ergodic-type estimate. The efficiency gap is larger for a smaller value of  $H$  and disappears when  $H = 1/2$ .

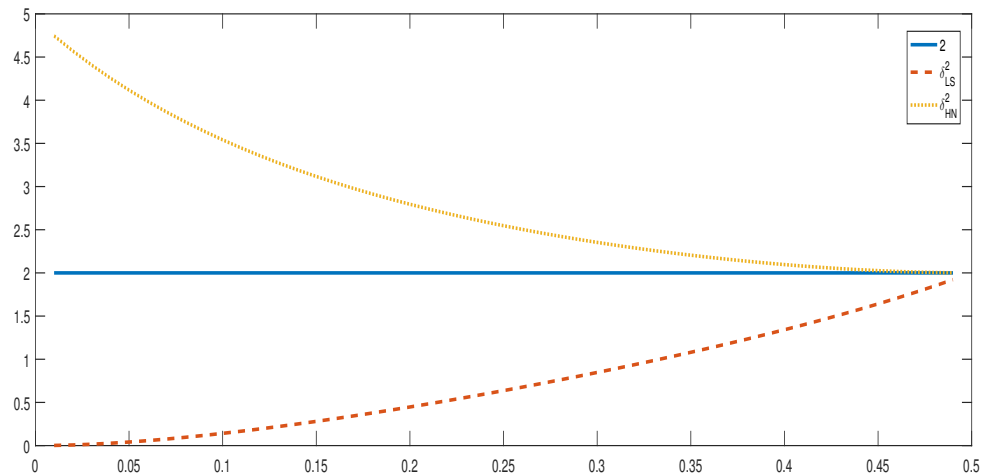


Figure 1. Plots of  $\delta_{LS}^2$  and  $\delta_{HN}^2$  against  $2$  as functions of  $H$

### 3.2 Asymptotic theory when $H = 1/2$

When  $H = 1/2$ ,  $B_t^{1/2} = W_t$  which is a standard Brownian motion and the fVm becomes the standard Vasicek model. In this case, it can be shown that fundamental martingale  $M_t^H$  becomes a standard Brownian motion and the MLE reduces to the LS estimate. The model has been extensively studied in the literature; see, for example, Kubilius et al. (2018) and Xiao and Yu (2019a). Since our model is slightly different from that in Xiao and Yu (2019a) (i.e.,  $\mu$  versus  $\alpha$ ), before we report our asymptotic theory, we review asymptotic theory of the LS estimate of  $\kappa$  and  $\mu$  in the Vasicek model given in Xiao and Yu (2019a).

**Lemma 3.2** For  $\kappa > 0$  and  $H = 1/2$  in Model (1.1), as  $T \rightarrow \infty$ , we have

$$\sqrt{T}(\hat{\kappa}_T - \kappa) \xrightarrow{d} \mathcal{N}(0, 2\kappa), \quad (3.16)$$

$$\sqrt{T}(\hat{\mu}_T - \mu) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right), \quad (3.17)$$

where

$$\hat{\kappa}_T = \frac{(X_T - X_0) \int_0^T X_t dt - T \int_0^T X_t dX_t}{T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2}, \quad (3.18)$$

$$\hat{\mu}_T = \frac{(X_T - X_0) \int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{(X_T - X_0) \int_0^T X_t dt - T \int_0^T X_t dX_t}, \quad (3.19)$$

and  $\int_0^T X_t dX_t$  is interpreted as an Itô integral.

While it was not shown,  $\hat{\kappa}_T$  and  $\hat{\mu}_T$  are independent asymptotically. Using the results of Lemma 3.2 and the independence, we can obtain the asymptotic laws of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  defined by (2.6) and (2.7).

**Theorem 3.2** For  $\kappa > 0$  and  $H = 1/2$  in Model (1.3), as  $T \rightarrow \infty$ , we have

$$\sqrt{T}(\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2 + \frac{2\alpha^2}{\kappa}\right), \quad (3.20)$$

$$\sqrt{T}(\tilde{\kappa}_T - \kappa) \xrightarrow{d} \mathcal{N}(0, 2\kappa). \quad (3.21)$$

**Remark 3.4** When  $\alpha \neq 0$ , we can summarize the three sets of asymptotic theory for the MLE of  $\alpha$  as follows:

$$\text{If } H \in (0, 1/2), \sqrt{T}(\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}\left(0, \frac{2\alpha^2}{\kappa}\right),$$

$$\text{If } H = 1/2, \sqrt{T}(\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2 + \frac{2\alpha^2}{\kappa}\right),$$

$$\text{If } H \in (1/2, 1), T^{1-H}(\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}(0, \lambda_H \sigma^2),$$

where the last asymptotic theory was obtained in Theorem 3.4 of Lohvinenko and Ralchenko (2017). While the three sets of asymptotic theory for  $\tilde{\kappa}_T$  are identical, the three sets of asymptotic theory for  $\tilde{\alpha}_T$  are different. When  $H$  changes from a value in  $(0, 1/2)$  to  $1/2$ , while the rate of convergence stays the same (i.e.,  $\sqrt{T}$ ), the asymptotic variance changes from  $\frac{2\alpha^2}{\kappa}$  to  $\sigma^2 + \frac{2\alpha^2}{\kappa}$ . When  $H$  changes from a value in  $(0, 1/2]$  to  $(1/2, 1)$ , both the rate of convergence and the asymptotic variance change.

**Remark 3.5** When  $\alpha$  is known and assumed to be zero and  $H = 1/2$ , the asymptotic theory for the MLE of  $\kappa$  was obtained in Brown and Hewitt (1975) and in Feigin (1976). The two sets of asymptotic theory are the same, suggesting that there is no efficiency loss in estimating  $\kappa$  when  $\alpha$  is estimated or not.

## 4 Asymptotic Theory When $\kappa = 0$

In this section, we consider the asymptotic laws of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  for the entire range for the Hurst parameter, i.e.,  $H \in (0, 1)$ . Note that when  $\kappa = 0$ , we have

$$X_t = X_0 + \alpha t + \sigma B_t^H. \quad (4.1)$$

For the model  $dU_t = -\kappa U_t dt + dB_t^H$ , it is well known that the MLE of  $\kappa$  can be expressed as

$$\hat{\kappa}_T - \kappa = \frac{-\int_0^T \hat{P}_H(t) dM_t^H}{\int_0^T \hat{P}_H^2(t) d\omega_t^H}, \quad (4.2)$$

where  $\hat{P}_H(t) = \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) B_s^H ds$ .

Before considering asymptotic properties of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$ , we first introduce a lemma, which will be used to derive the asymptotic theory.

**Lemma 4.1** *For  $\kappa = 0$  and  $H \in (0, 1)$  in Model (1.1), as  $T \rightarrow \infty$ , we have*

$$\int_0^T \hat{P}_H(t) dM_t^H = O_p(T), \quad (4.3)$$

$$\int_0^T \hat{P}_H^2(t) d\omega_t^H = O_p(T^2), \quad (4.4)$$

$$\int_0^T t d\omega_t^H = \frac{1}{\lambda_H} \frac{2-2H}{3-2H} T^{3-2H}, \quad (4.5)$$

$$\int_0^T t^2 d\omega_t^H = \frac{1}{\lambda_H} \frac{1-H}{2-H} T^{4-2H}, \quad (4.6)$$

$$\int_0^T \hat{P}_H(t) d\omega_t^H = O_p(T^{2-H}), \quad (4.7)$$

$$\int_0^T t \hat{P}_H(t) d\omega_t^H = O_p(T^{3-H}), \quad (4.8)$$

$$V_H(T) = \frac{\lambda_H}{k_H} B\left(\frac{3}{2} - H, \frac{3}{2} - H\right), \quad (4.9)$$

$$\frac{d}{d\omega_t^H} \int_0^t k_H(t, s) ds = a_H t, \quad (4.10)$$

where  $B(\cdot, \cdot)$  is the Beta function,  $\lambda_H$  is defined by (2.4) and  $a_H = \frac{3-2H}{4(1-H)}$ .

We can now describe the asymptotic behavior of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  as  $T \rightarrow \infty$ .

**Theorem 4.1** *For  $\kappa = 0$ ,  $\alpha \neq 0$  and  $H \in (0, 1)$  in Model (1.1), as  $T \rightarrow \infty$ , we have*

$$T^{1-H} (\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}(0, \sigma^2 \rho_H), \quad (4.11)$$

$$T^{2-H} (\tilde{\kappa}_T - \kappa) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\alpha^2} \phi_H\right), \quad (4.12)$$

where  $\rho_H = \lambda_H (3-2H)^2$ ,  $\phi_H = \frac{32H(1-H)(2-H)\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}$  and  $\lambda_H$  is defined by (2.4).

**Remark 4.1** In the case of  $H = 1/2$  and  $\alpha \neq 0$ , we can see that  $X_t = X_0 + \alpha t + \sigma W_t$ . A straightforward algebraic calculation shows  $\omega_t^H = t$ ,  $P_H(t) = \frac{1}{\sigma} X_t$ ,  $M_T^H = W_t$ ,  $\hat{P}_H(t) = W_t$  and that

$$\frac{1}{T^3} \int_0^T X_t^2 dt = \frac{\alpha^2}{3} + o_p(1), \quad (4.13)$$

$$\frac{1}{T^2} \int_0^T X_t dt = \frac{\alpha}{2} + o_p(1), \quad (4.14)$$

$$\frac{1}{T\sqrt{T}} \int_0^T X_t dW_t = \alpha \int_0^T t dW_t + o_p(1). \quad (4.15)$$

Then, by the scaling properties of the Brownian motion, (2.17), (4.13), (4.14) and (4.15), we deduce that

$$\begin{aligned} \sqrt{T}(\tilde{\alpha}_T - \alpha) &= \sqrt{T} \left( \frac{W_T \int_0^T X_t^2 dt - \int_0^T X_t dW_t \int_0^T X_t dt}{T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2} \sigma \right) \\ &= \sqrt{T} \left( \frac{1}{\sqrt{T}} \frac{\frac{1}{\sqrt{T}} W_T \frac{1}{T^3} \int_0^T X_t^2 dt - \frac{1}{T\sqrt{T}} \int_0^T X_t dW_t \frac{1}{T^2} \int_0^T X_t dt}{\frac{1}{T^3} \int_0^T X_t^2 dt - \left( \frac{1}{T^2} \int_0^T X_t dt \right)^2} \sigma \right) \\ &= \frac{\frac{\alpha^2}{3} \frac{W_T}{\sqrt{T}} - \frac{\alpha^2}{2} \frac{1}{T\sqrt{T}} \int_0^T t dW_t + o_p(1)}{\frac{\alpha^2}{3} - \left( \frac{\alpha}{2} \right)^2 + o_p(1)} \\ &= 12\sigma \left( \frac{W_T}{3\sqrt{T}} - \frac{1}{2T\sqrt{T}} \int_0^T t dW_t \right) + o_p(1) \\ &\xrightarrow{d} \mathcal{N}(0, 4\sigma^2), \end{aligned}$$

which is identical to (4.11) with  $H = 1/2$ . Moreover, using (2.18), (4.13), (4.14) and (4.15), we can write

$$\begin{aligned} T\sqrt{T}(\tilde{\kappa}_T - \kappa) &= T\sqrt{T} \left[ \frac{\left( W_T \int_0^T X_t dt - T \int_0^T X_t dW_t \right) \sigma}{T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2} \right] \\ &= \frac{\left[ \frac{W_T}{\sqrt{T}} \frac{1}{T^2} \int_0^T X_t dt - \frac{1}{T\sqrt{T}} \int_0^T X_t dW_t \right] \sigma}{\frac{1}{T^3} \int_0^T X_t^2 dt - \left( \frac{1}{T^2} \int_0^T X_t dt \right)^2} \\ &\xrightarrow{d} \mathcal{N} \left( 0, \frac{12\sigma^2}{\alpha^2} \right), \end{aligned}$$

which is identical to (4.12) with  $H = 1/2$  being assumed.

**Remark 4.2** In the case of  $H = 1/2$  and  $\alpha = 0$ , with  $\alpha$  and  $\kappa$  being estimated, by the scaling

properties of the Brownian motion, we have

$$\begin{aligned}\sqrt{T}(\tilde{\alpha}_T - \alpha) &\xrightarrow{d} \frac{W_1 \left( \int_0^1 W_t dt \right)^2 - \int_0^1 W_t dt \int_0^1 W_t dW_t}{\int_0^1 W_t^2 dt - \left( \int_0^1 W_t dt \right)^2} \sigma, \\ T(\tilde{\kappa}_T - \kappa) &\xrightarrow{d} \frac{W_1 \int_0^1 W_t dt - \int_0^1 W_t dW_t}{\int_0^1 W_t^2 dt - \left( \int_0^1 W_t dt \right)^2}.\end{aligned}$$

Thus, the limiting distributions of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  are not normal. In particular, the asymptotic distribution of  $\tilde{\kappa}_T$  is a Dickey-Fuller-Phillips type distribution with the rate of convergence being  $T$ . Hence, when  $\kappa = 0$  is unknown, the value of  $\alpha$  plays an important role in the study of asymptotic laws for the MLE.

## 5 Asymptotic Theory When $\kappa < 0$

When  $\kappa < 0$ , the model given by (1.3) is non-ergodic or explosive. Since the proofs of the asymptotic theory of  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  when  $H = 1/2$  are different from those when  $H \in (0, 1/2) \cup (1/2, 1)$ , we first consider the case of  $H = 1/2$ . For the sake of notational simplicity, we introduce the process  $\xi_t = \sigma \int_0^t e^{\kappa s} dW_s$  for  $t \geq 0$ . Obviously,  $\xi_\infty \sim \mathcal{N}\left(0, -\frac{\sigma^2}{2\kappa}\right)$ . Moreover, using (2.13) and the definition of  $\xi_t$ , we can easily obtain

$$\sigma e^{\kappa T} \int_0^T U_t dt = e^{\kappa T} \int_0^T e^{-\kappa t} \xi_t dt \xrightarrow{p} -\frac{\xi_\infty}{\kappa}, \quad (5.1)$$

$$\sigma e^{2\kappa T} \int_0^T e^{-\kappa t} U_t dt \xrightarrow{p} -\frac{\xi_\infty}{2\kappa}. \quad (5.2)$$

### 5.1 Asymptotic theory when $H = 1/2$

Now, we can state the key results of the asymptotic theory for  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  when  $H = 1/2$ .

**Theorem 5.1** *For  $\kappa < 0$ ,  $H = 1/2$  in Model (1.1), as  $T \rightarrow \infty$ , we have*

$$\sqrt{T}(\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad (5.3)$$

$$\frac{e^{-\kappa T}}{2\kappa} (\tilde{\kappa}_T - \kappa) \xrightarrow{d} \frac{\eta_\infty}{X_0 - \frac{\alpha}{\kappa} + \xi_\infty}, \quad (5.4)$$

where  $\xi_\infty$  and  $\eta_\infty$  are two independent  $\mathcal{N}(0, -\sigma^2/(2\kappa))$  random variables.

**Remark 5.1** *In (5.4), if we set  $X_0 = \frac{\alpha}{\kappa}$ , the limiting distribution of  $\frac{e^{-\kappa T}}{2\kappa} (\tilde{\kappa}_T - \kappa)$  becomes a standard Cauchy variate. This limiting distribution is the same as that in the Vasicek model driven by a standard Brownian motion (see, e.g., Feigin, 1976). The asymptotic theory in (5.4) is similar to that in the explosive discrete-time and continuous-time models when discretely-sampled data are available (see e.g., White, 1958; Anderson, 1959; Phillips and Magdalinos, 2007; Wang and Yu, 2015, 2016).*

## 5.2 Asymptotic theory when $H \in (0, 1/2) \cup (1/2, 1)$

We now turn to the case when  $H \in (0, 1/2) \cup (1/2, 1)$  assuming  $\kappa < 0$ . The limiting theory is the most difficult to derive in our paper. Let  $A = \frac{\lambda_H \sqrt{\pi} (-\kappa)^{H-1} \Gamma(3/2-H)}{k_H (2-2H) t^{1-2H}}$ . Applying (2.10) and (2.11), we can obtain

$$\begin{aligned}
V_H(t) &= \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) e^{-\kappa s} ds \\
&= \frac{\frac{d}{dt} \int_0^t k_H(t, s) e^{-\kappa s} ds}{\frac{d\omega_t^H}{dt}} \\
&= A \left[ (1-H) t^{-H} e^{-\frac{\kappa t}{2}} I_{1-H} \left( -\frac{\kappa t}{2} \right) - \frac{\kappa}{2} t^{1-H} e^{-\frac{\kappa t}{2}} I_{1-H} \left( -\frac{\kappa t}{2} \right) \right. \\
&\quad \left. + t^{1-H} e^{-\frac{\kappa t}{2}} \frac{1}{2} \left( -\frac{\kappa}{2} \right) \left( I_{2-H} \left( -\frac{\kappa t}{2} \right) + I_{-H} \left( -\frac{\kappa t}{2} \right) \right) \right] \\
&= \frac{\lambda_H \sqrt{\pi} (-\kappa)^{H-1} \Gamma\left(\frac{3}{2}-H\right)}{k_H (2-2H)} \left[ (1-H) t^{-1+H} e^{-\frac{\kappa t}{2}} I_{1-H} \left( -\frac{\kappa t}{2} \right) \right. \\
&\quad \left. - \frac{\kappa}{2} t^H e^{-\frac{\kappa t}{2}} I_{1-H} \left( -\frac{\kappa t}{2} \right) - \frac{\kappa}{4} t^H e^{-\frac{\kappa t}{2}} I_{2-H} \left( -\frac{\kappa t}{2} \right) - \frac{\kappa}{4} t^H e^{-\frac{\kappa t}{2}} I_{-H} \left( -\frac{\kappa t}{2} \right) \right] \\
&= \frac{\lambda_H \sqrt{\pi} (-\kappa)^{H-1} \Gamma\left(\frac{3}{2}-H\right)}{k_H (2-2H)} \left[ (1-H) t^{-1+H} \frac{e^{-\kappa t}}{\sqrt{-\pi \kappa t}} (1 + O(t^{-1})) \right. \\
&\quad \left. - \frac{\kappa}{2} t^H \frac{e^{-\kappa t}}{\sqrt{-\pi \kappa t}} (1 + O(t^{-1})) - \frac{\kappa}{4} t^H \frac{e^{-\kappa t}}{\sqrt{-\pi \kappa t}} (1 + O(t^{-1})) \right. \\
&\quad \left. - \frac{\kappa}{4} t^H \frac{e^{-\kappa t}}{\sqrt{-\pi \kappa t}} (1 + O(t^{-1})) \right] \\
&= \frac{\lambda_H (-\kappa)^{H-1} \Gamma\left(\frac{3}{2}-1\right)}{k_H (2-2H)} e^{-\kappa t} \left[ (1-H) (-\kappa)^{-\frac{1}{2}} t^{-\frac{3}{2}+H} (1 + O(t^{-1})) \right. \\
&\quad \left. + \frac{1}{2} (-\kappa)^{\frac{1}{2}} t^{H-\frac{1}{2}} (1 + O(t^{-1})) \right. \\
&\quad \left. + \frac{1}{4} t^{H-\frac{1}{2}} (-\kappa)^{\frac{1}{2}} (1 + O(t^{-1})) + \frac{1}{4} (-\kappa)^{\frac{1}{2}} t^{H-\frac{1}{2}} (1 + O(t^{-1})) \right] \\
&= O\left(t^{H-\frac{1}{2}} e^{-\kappa t}\right). \tag{5.5}
\end{aligned}$$

where  $I_\nu(z)$  is the modified Bessel function of the first kind defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)},$$

and we used the asymptotic expansion that, as  $z \rightarrow \infty$ ,

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left( 1 + O(z^{-1}) \right).$$

Consequently, we can state the following lemma.

**Lemma 5.1** For  $\kappa < 0$  and  $H \in (0, 1)$  in Model (1.1), we have

$$\int_0^T \tilde{P}_H^2(t) d\omega_t^H = O_p(e^{-2\kappa T}), \quad (5.6)$$

$$\int_0^T \tilde{P}_H(t) dM_t^H = O_p(e^{-\kappa T}), \quad (5.7)$$

$$\int_0^T V_H(t) d\omega_t^H = O\left(T^{\frac{1}{2}-H} e^{-\kappa T}\right), \quad (5.8)$$

$$\int_0^T \tilde{P}_H(t) d\omega_t^H = O_p\left(e^{-\kappa T} T^{\frac{1}{2}-H}\right), \quad (5.9)$$

$$\int_0^T V_H^2(t) d\omega_t^H = O(e^{-2\kappa T}), \quad (5.10)$$

$$\int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H = O_p(e^{-2\kappa T}), \quad (5.11)$$

$$\int_0^T V_H(t) dM_t^H = O_p(e^{-\kappa T}). \quad (5.12)$$

Now, we can state the asymptotic theory for  $\tilde{\alpha}_T$  and  $\tilde{\kappa}_T$  for  $\kappa < 0$  and  $H \in (0, 1/2) \cup (1/2, 1)$ .

**Theorem 5.2** When  $\kappa < 0$ ,  $H \in (0, 1/2) \cup (1/2, 1)$ , and  $X_0 = \frac{\alpha}{\kappa}$  in Model (1.1), as  $T \rightarrow \infty$ , we have

$$T^{1-H} (\tilde{\alpha}_T - \alpha) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad (5.13)$$

$$\frac{e^{-\kappa T}}{2\kappa} (\tilde{\kappa}_T - \kappa) \xrightarrow{d} \frac{X \sqrt{\sin(\pi H)}}{Y}, \quad (5.14)$$

where  $X$  and  $Y$  are two independent  $\mathcal{N}(0, 1)$  random variables.

**Remark 5.2** For the entire range of  $H \in (0, 1)$ , the asymptotic distribution of  $\tilde{\alpha}_T$  is normal with the rate of convergence of  $T^{1-H}$  and variance  $\sigma^2$ . This asymptotic distribution is the same as that of the LS estimate (see Theorem 3.5 in Xiao and Yu (2019a) and Section 3 in Xiao and Yu (2019b)).

**Remark 5.3** According to (5.14), the asymptotic law of  $\frac{e^{-\kappa T}}{2\kappa} (\tilde{\kappa}_T - \kappa)$  is the standard Cauchy times  $\sqrt{\sin(\pi H)}$ . For  $H \in (0, 1/2) \cup (1/2, 1)$ ,  $\sqrt{\sin(\pi H)} \in (0, 1)$ , suggesting that as  $H$  draws further away from  $1/2$ ,  $\kappa$  is estimated with higher accuracy.

**Remark 5.4** When  $X_0 \neq \frac{\alpha}{\kappa}$ , using Lemma 5.1, we can obtain

$$\frac{e^{-\kappa T}}{2\kappa} (\tilde{\kappa}_T - \kappa) = \frac{-2\kappa e^{\kappa T} \int_0^T \left[ \frac{1}{\sigma} \left( X_0 - \frac{\alpha}{\kappa} \right) V_H(t) + \tilde{P}_H(t) \right] dM_t^H + o_p(1)}{4\kappa^2 e^{2\kappa T} \int_0^T \left[ \frac{1}{\sigma} \left( X_0 - \frac{\alpha}{\kappa} \right) V_H(t) + \tilde{P}_H(t) \right]^2 d\omega_t^H + o_p(1)}.$$

In this case, to obtain the asymptotic distribution of  $e^{-\kappa T} (\tilde{\kappa}_T - \kappa) / (2\kappa)$ , one needs to calculate the Laplace transform of  $\int_0^T \left[ \frac{1}{\sigma} (X_0 - \frac{\alpha}{\kappa}) V_H(t) + \tilde{P}_H(t) \right]^2 d\omega_t^H$ . This is complicated and we leave it in our future work.

## 6 Concluding Remarks and Future Directions

The fVm has found more and more applications in practice. In this paper, we consider the MLE of parameters in the drift term when a continuous record of observations is available. The ML estimation is made possible due to the presence of the fundamental martingale and the generalized Girsanov theorem. The asymptotic theory is based on the assumption that  $T \rightarrow \infty$ .

It is shown that the MLE of  $\alpha$  is asymptotically normal regardless of the sign of  $\kappa$ . However, the asymptotic law of the MLE of  $\kappa$  critically depends on the sign of  $\kappa$ . More precisely, when  $\kappa > 0$  and  $H \in (0, 1)$ , we have shown that the asymptotic distribution of the MLE of  $\kappa$  is normal with the rate of convergence being  $\sqrt{T}$ . The asymptotic variance is  $2\kappa$ , which is independent of  $H$ . When  $\kappa = 0$  and  $\alpha \neq 0$ , the asymptotic distribution of the MLE of  $\kappa$  is normal with the rate of convergence being  $T^{2-H}$ . The asymptotic variance depends on  $H$ . When  $\kappa = 0$  and  $\alpha = 0$ , the asymptotic distribution of the MLE of  $\kappa$  is a Dickey-Fuller-Phillips distribution with the rate of convergence being  $T$ . When  $\kappa < 0$ , it is shown that the limiting distribution is a Cauchy-type with the rate of convergence being  $e^{-\kappa T}$ . If one further assumes that  $X_0 = \alpha/\kappa$ , the limiting distribution becomes a standard Cauchy variate multiplied by  $\sqrt{\sin(\pi H)}$ .

This study also suggests several important directions for future research. First, it is worth investigating to generalize the results in this paper to nonlinear stochastic differential equations driven by the fBm. The ergodic theorem, fractional calculus and Malliavin calculus will be employed for obtaining the asymptotic properties of both the MLE and the LS estimators.

Second, in this paper,  $H$  and  $\sigma$  are assumed to be known. In practice, both  $H$  and  $\sigma$  are almost always unknown. Although many approaches have been proposed to estimate the Hurst coefficient and the volatility parameter from discrete time observations, how to estimate  $H$  and  $\sigma$  in fVm with a continuous record of observations is an open question. It is interesting to realize that we can use the generalized quadratic variation to estimate both the Hurst parameter and the volatility parameter in fVm. For  $T > 0$  and any  $\epsilon \neq \xi$ ,

$$H = \lim_{\epsilon \downarrow 0, \xi \downarrow 0} \frac{1}{2} \log \left( \frac{\epsilon}{\xi} \right) \log \left( \frac{\int_0^T (X_{t+\xi} - X_t)^2 dt}{\int_0^T (X_{t+\epsilon} - X_t)^2 dt} \right), \quad \sigma^2 = \frac{\lim_{\epsilon \downarrow 0} \int_0^T (X_{t+\epsilon} - X_t)^2 dt}{\epsilon^{2HT}}$$

It would be interesting to study the asymptotic properties of these estimators mentioned above, which will be reported in later work.

Third, this paper assumes that a continuous record of an increasing time span is available for the development of asymptotic theory. In practice, data is typically observed at discrete



time points with  $(0, h, 2h, \dots, Nh(:= T))$  where  $h$  is the sampling interval and  $T$  is the time span. When high frequency data over a long span of time period is available, one may consider using a double asymptotic scheme by assuming  $h \rightarrow 0$  and  $T \rightarrow \infty$ . The discretized model corresponding to (1.3) is given by

$$y_{th} = \mu + \exp(-\kappa h) (y_{(t-1)h} - \mu) + u_t, \quad (1 - L)^d u_t = \varepsilon_t, \quad t = 1, \dots, N,$$

where  $L$  is the lag operator,  $d = H - 1/2$ . As shown in Wang and Yu (2016), under the double asymptotic scheme,  $\exp(-\kappa h) = \exp\{-\kappa/k_N\} = 1 - \kappa/k_N + O(k_N^{-2}) \rightarrow 1$  where  $k_N := 1/h \rightarrow \infty$  as  $h \rightarrow 0$  and  $k_N/N = 1/T \rightarrow 0$  as  $T \rightarrow \infty$ . This implies an autoregressive (AR) model with the AR root being moderately deviated from unity and with a fractionally integrated error term with  $d \in (-1/2, 0)$ . This model is closely related to a model considered in Magdalinos (2012) where it is assumed that  $d \in (0, 1/2)$ . Developing double asymptotic theory based on discretely sampled data will allow one to extend the results of Magdalinos (2012) to the case where  $d \in (-1/2, 1/2)$ . The development of the MLE and the asymptotic theory is beyond the scope of this paper and will be reported in later work.

## 7 Appendix

### 7.1 Proof of Theorem 3.1

We first consider (3.3). Using (2.11), (2.2) and the properties of the modified Bessel function of the first kind, for  $T$  tending to infinity, we get

$$\begin{aligned} V_H(T) &= \frac{d}{d\omega_T^H} \int_0^T k_H(T, s) e^{-\kappa s} ds \\ &= \frac{d}{d\omega_T^H} \left[ \frac{\sqrt{\pi} \kappa^{H-1} \Gamma(\frac{3}{2} - H)}{k_H} T^{1-H} e^{-\frac{\kappa T}{2}} I_{1-H} \left( \frac{\kappa T}{2} \right) \right] \\ &= \frac{\kappa^{H-\frac{3}{2}} \Gamma(2 - 2H)}{\Gamma(\frac{1}{2} - H)} T^{H-\frac{3}{2}} + O\left(T^{H-\frac{5}{2}}\right), \end{aligned} \quad (7.1)$$

which is (3.3).

Then, as  $T \rightarrow \infty$ , using Lemma 4.2 of Lohvinenko and Ralchenko (2017), we can obtain

$$\int_0^T V_H(t) d\omega_t^H = \int_0^T k_H(T, s) e^{-\kappa s} ds = O\left(T^{\frac{1}{2}-H}\right),$$

which yields (3.4).

By the proof of Theorem 3 in Tanaka (2013), we can easily obtain (3.5) and (3.6). The result of (3.7) follows directly from  $\int_0^1 V_H^2(t) d\omega_t^H < \infty$  and  $\int_1^T V_H^2(t) d\omega_t^H < \infty$  (see the proof of Lemma 4.7 in Lohvinenko and Ralchenko, 2017). Applying Lemma 4.5 in Lohvinenko and Ralchenko (2017), we can easily obtain (3.8).

Now, we are left with (3.9). Using the Cauchy-Schwarz inequality, (3.6) and (3.7), we obtain

$$\begin{aligned} \int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H &\leq \sqrt{\int_0^T V_H^2(t) d\omega_t^H \int_0^T \tilde{P}_H^2(t) d\omega_t^H} \\ &= \sqrt{O(1)O_p(T)}, \end{aligned}$$

which implies (3.9).

## 7.2 Proof of Theorem 3.1

To simplify notations, let  $\tilde{X}_0 := X_0 - \frac{\alpha}{\kappa}$ . Using (2.10), we have

$$\begin{aligned} \int_0^T P_H^2(t) d\omega_t^H &= \int_0^T \left[ \frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma} \tilde{X}_0 V_H(t) + \tilde{P}_H(t) \right]^2 d\omega_t^H \\ &= \frac{\alpha^2}{\sigma^2\kappa^2} \omega_T^H + \frac{1}{\sigma^2} \tilde{X}_0^2 \int_0^T V_H^2(t) d\omega_t^H + \int_0^T \tilde{P}_H^2(t) d\omega_t^H \\ &\quad + \frac{2\alpha}{\sigma^2\kappa} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + \frac{2\alpha}{\sigma\kappa} \int_0^T \tilde{P}_H(t) d\omega_t^H \\ &\quad + \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H. \end{aligned} \tag{7.2}$$

Using (2.10) again, we obtain

$$\begin{aligned} \left( \int_0^T P_H(t) d\omega_t^H \right)^2 &= \left[ \int_0^T \left( \frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma} \tilde{X}_0 V_H(t) + \tilde{P}_H(t) \right) d\omega_t^H \right]^2 \\ &= \frac{\alpha^2}{\alpha^2\kappa^2} (\omega_T^H)^2 + \frac{1}{\sigma^2} \tilde{X}_0^2 \left( \int_0^T V_H(t) d\omega_t^H \right)^2 + \left( \int_0^T \tilde{P}_H(t) d\omega_t^H \right)^2 \\ &\quad + \frac{2\alpha}{\sigma\kappa} \omega_T^H \int_0^T \tilde{P}_H(t) d\omega_t^H + \frac{2\alpha}{\sigma^2\kappa} \omega_T^H \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H \\ &\quad + \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H \int_0^T \tilde{P}_H(t) d\omega_t^H. \end{aligned} \tag{7.3}$$

Using (7.2), (7.3) and Lemma 3.1, we deduce that

$$\begin{aligned}
& \omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2 \\
&= \frac{\omega_T^H}{\sigma^2} \tilde{X}_0^2 \int_0^T V_H^2(t) d\omega_t^H + \omega_T^H \int_0^T \tilde{P}_H^2(t) d\omega_t^H + \omega_T^H \frac{2\alpha}{\sigma^2 \kappa} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H \\
& \quad + \omega_T^H \frac{2\alpha}{\sigma \kappa} \int_0^T \tilde{P}_H^2(t) d\omega_t^H + \omega_T^H \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H \\
& \quad - \frac{1}{\sigma^2} \tilde{X}_0^2 \left( \int_0^T V_H(t) d\omega_t^H \right)^2 - \left( \int_0^T \tilde{P}_H(t) d\omega_t^H \right)^2 - \frac{2\alpha}{\sigma^2 \kappa} \omega_T^H \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H \\
& \quad - \frac{2\alpha}{\sigma \kappa} \omega_T^H \int_0^T \tilde{P}_H(t) d\omega_t^H - \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H \int_0^T \tilde{P}_H(t) d\omega_t^H \\
&= \omega_T^H \int_0^T \tilde{P}_H^2(t) d\omega_t^H + o_p(T^{3-2H}). \tag{7.4}
\end{aligned}$$

Moreover, using (2.10), we get

$$\begin{aligned}
& \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H \\
&= \left[ \frac{\alpha}{\sigma \kappa} M_T^H + \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) dM_t^H + \int_0^T \tilde{P}_H(t) dM_t^H \right] \times \\
& \quad \left[ \frac{\alpha}{\sigma \kappa} W_T^H + \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + \int_0^T \tilde{P}_H(t) d\omega_t^H \right] \\
&= \frac{\alpha^2}{\sigma^2 \kappa^2} M_T^H \omega_T^H + \frac{\alpha}{\sigma \kappa} M_T^H \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + \frac{\alpha}{\sigma \kappa} M_T^H \int_0^T \tilde{P}_H(t) d\omega_t^H \\
& \quad + \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) dM_t^H \frac{\alpha}{\sigma \kappa} \omega_T^H + \frac{1}{\sigma^2} \tilde{X}_0^2 \int_0^T V_H(t) dM_t^H \int_0^T V_H(t) d\omega_t^H \\
& \quad + \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) dM_t^H \int_0^T \tilde{P}_H(t) d\omega_t^H + \frac{\alpha}{\sigma \kappa} \omega_T^H \int_0^T \tilde{P}_H(t) dM_t^H \\
& \quad + \frac{1}{\sigma} \tilde{X}_0 \int_0^T \tilde{P}_H(t) dM_t^H \int_0^T V_H(t) d\omega_t^H + \int_0^T \tilde{P}_H(t) dM_t^H \int_0^T \tilde{P}_H(t) d\omega_t^H. \tag{7.5}
\end{aligned}$$

By combining (7.2), (7.5) and Lemma 3.1, we have

$$\begin{aligned}
& M_T^H \int_0^T P_H^2(t) d\omega_t^H - \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H \\
= & \frac{\alpha^2}{\sigma^2 \kappa^2} M_T^H \omega_T^H + \frac{M_T^H}{\sigma^2} \tilde{X}_0^2 \int_0^T V_H^2(t) d\omega_t^H + M_T^H \int_0^T \tilde{P}_H^2(t) d\omega_t^H \\
& + M_T^H \frac{2\alpha}{\sigma^2 \kappa} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + M_T^H \frac{2\alpha}{\sigma \kappa} \int_0^T \tilde{P}_H(t) d\omega_t^H \\
& + M_T^H \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H - \frac{\alpha^2}{\sigma^2 \kappa^2} M_T^H \omega_T^H \\
& - \frac{\alpha}{\sigma \kappa} M_T^H \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H - \frac{\alpha}{\sigma \kappa} M_T^H \int_0^T \tilde{P}_H(t) d\omega_t^H \\
& - \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) dM_t^H \frac{\alpha}{\sigma \kappa} \omega_T^H - \frac{1}{\sigma^2} \tilde{X}_0^2 \int_0^T V_H(t) dM_t^H \int_0^T V_H(t) d\omega_t^H \\
& - \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) dM_t^H \int_0^T \tilde{P}_H(t) d\omega_t^H - \frac{\alpha}{\sigma \kappa} \omega_T^H \int_0^T \tilde{P}_H(t) dM_t^H \\
& - \frac{1}{\sigma} \tilde{X}_0 \int_0^T \tilde{P}_H(t) dM_t^H \int_0^T V_H(t) d\omega_t^H - \int_0^T \tilde{P}_H(t) dM_t^H \int_0^T \tilde{P}_H(t) d\omega_t^H \\
= & -\frac{\alpha}{\sigma \kappa} \omega_T^H \int_0^T \tilde{P}_H(t) dM_t^H + o_p\left(T^{\frac{5}{2}-2H}\right). \tag{7.6}
\end{aligned}$$

Combining (2.17), (7.6), (7.4), (4.4), (4.5) in Lohvinenko and Ralchenko (2017) with Slutsky's theorem, we obtain

$$\begin{aligned}
\sqrt{T}(\tilde{\alpha}_T - \alpha) &= \frac{\sqrt{T} \left[ M_T^H \int_0^T P_H^2(t) d\omega_t^H - \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H \right]}{\omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2} \sigma \\
&= \frac{-\frac{\alpha}{\sigma \kappa} \omega_T^H \frac{1}{\sqrt{T}} \int_0^T \tilde{P}_H(t) dM_t^H + o_p(T^{2-2H})}{\omega_T^H \frac{1}{T} \int_0^T \tilde{P}_H^2(t) d\omega_t^H + o_p(T^{2-2H})} \sigma \\
&\xrightarrow{d} \mathcal{N}\left(0, \frac{2\alpha^2}{\kappa}\right).
\end{aligned}$$

Now, we consider (3.11). Using (2.10) and (7.3), we have

$$\begin{aligned}
& M_T^H \int_0^T P_H(t) d\omega_t^H - \omega_T^H \int_0^T P_H(t) dM_t^H \\
= & M_T^H \frac{\alpha}{\sigma \kappa} \omega_T^H + M_T^H \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + M_T^H \int_0^T \tilde{P}_H(t) d\omega_t^H \\
& - \left[ \omega_T^H \frac{\alpha}{\sigma \kappa} M_T^H + \omega_T^H \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) dM_t^H + \omega_T^H \int_0^T \tilde{P}_H^2(t) dM_t^H \right] \\
= & -\omega_T^H \int_0^T \tilde{P}_H(t) dM_t^H + o_p\left(T^{\frac{5}{2}-2H}\right). \tag{7.7}
\end{aligned}$$

Finally, combining (2.18), (7.7), (7.4), (4.4), (4.5) in Lohvinenko and Ralchenko (2017)

with Slutsky's theorem, we have

$$\sqrt{T}(\tilde{\kappa}_T - \kappa) = \frac{-\omega_T^H \frac{1}{\sqrt{T}} \int_0^T \tilde{P}_H(t) dM_t^H + o_p(T^{2-2H})}{\omega_T^H \frac{1}{T} \int_0^T \tilde{P}_H^2(t) d\omega_t^H + o_p(T^{2-2H})} \xrightarrow{d} \mathcal{N}(0, 2\kappa).$$

### 7.3 Proof of Lemma 3.2

For  $H = 1/2$ , using arguments similar to the proof Theorem 3.1 in Xiao and Yu (2019a), we can easily obtain

$$\frac{1}{T} \int_0^T X_t^2 dt \xrightarrow{p} \frac{\alpha^2}{\kappa^2} + \frac{\sigma^2}{2\kappa}, \quad (7.8)$$

$$\frac{1}{T} \int_0^T X_t dt \xrightarrow{p} \frac{\alpha}{\kappa}, \quad (7.9)$$

$$\frac{1}{T} \int_0^T X_t dW_t = \frac{\alpha}{\kappa\sqrt{T}} W_T + \frac{\sigma}{\sqrt{T}} \int_0^T e^{-\kappa t} \int_0^t e^{\kappa s} dW_s dW_t + o_p(1). \quad (7.10)$$

Now, we consider the second term on the right-hand side of (7.10). For convenience, let  $F_T = \frac{\sigma}{\sqrt{T}} \int_0^T \int_0^t e^{-\kappa(t-s)} dW_s dW_t$ . By direct computations,

$$\lim_{T \rightarrow \infty} \mathbb{E}[F_T^2] = \lim_{T \rightarrow \infty} \frac{\sigma^2}{T} \int_0^T \int_0^t e^{-2\kappa(t-s)} ds dt = \frac{\sigma^2}{2\kappa}. \quad (7.11)$$

Moreover, using some basic facts on the Malliavin calculus for Gaussian processes (for details, see Nualart, 2006), we obtain

$$D_s F_T = \frac{\sigma}{\sqrt{T}} \int_0^s e^{-\kappa(s-u)} dW_u + \frac{\sigma}{\sqrt{T}} \int_s^T e^{-\kappa(t-s)} dW_t.$$

Consequently, we have

$$\begin{aligned} \|DF_T\|_{\mathcal{H}}^2 &= \frac{\sigma^2}{T} \int_0^T \left[ \left( \int_0^s e^{-\kappa(s-u)} dW_u \right)^2 + \left( \int_s^T e^{-\kappa(t-s)} dW_t \right)^2 \right. \\ &\quad \left. + 2 \int_0^s e^{-\kappa(s-u)} dW_u \int_s^T e^{-\kappa(t-s)} dW_t \right] ds \\ &= J_1 + J_2 + J_3, \end{aligned} \quad (7.12)$$

where

$$\begin{aligned} J_1 &= \frac{\sigma^2}{T} \int_0^T \left( \int_0^s e^{-\kappa(s-u)} dW_u \right)^2 ds, \\ J_2 &= \frac{\sigma^2}{T} \int_0^T \left( \int_s^T e^{-\kappa(t-s)} dW_t \right)^2 ds, \\ J_3 &= \frac{2\sigma^2}{T} \int_0^T \left( \int_0^s e^{-\kappa(s-u)} dW_u \int_s^T e^{-\kappa(t-s)} dW_t \right) ds. \end{aligned}$$

A standard calculation yields

$$\begin{aligned}
\mathbb{E}[J_1] &= \frac{\sigma^2}{T} \int_0^T \int_0^s e^{-2\kappa(s-u)} du ds \\
&= \frac{\sigma^2}{T} \int_0^T \int_0^s e^{2\kappa u} du e^{-2\kappa s} ds \\
&= \frac{\sigma^2}{T} \int_0^T \frac{1}{2\kappa} (e^{2\kappa s} - 1) e^{-2\kappa s} ds \\
&\rightarrow \frac{\sigma^2}{2\kappa}, \text{ as } T \rightarrow \infty.
\end{aligned} \tag{7.13}$$

Moreover, a standard calculation implies

$$\begin{aligned}
\mathbb{E}[J_1]^2 &= \mathbb{E} \frac{\sigma^4}{T^2} \left[ \int_0^T \left( \int_s^T e^{-\kappa(s-u)} dW_u \right)^2 ds \right]^2 \\
&= \frac{\sigma^4}{T^2} \int_{\{s < t \leq T\}} \mathbb{E} \left[ \left( \int_0^s e^{-\kappa(s-u)} dW_u \right)^2 \left( \int_0^t e^{-\kappa(t-v)} dW_v \right)^2 \right] ds dt \\
&= \frac{\sigma^4}{T^2} \int_s^T \int_0^t \frac{1}{2\kappa} (1 - e^{-2\kappa s}) ds e^{-2\kappa t} dt \\
&= \frac{\sigma^8}{2\kappa T^2} \int_s^T t e^{-2\kappa t} dt - \frac{\sigma^8}{4\kappa^2 T^2} \int_s^T (e^{-4\kappa t} - e^{-2\kappa t}) dt \\
&\rightarrow 0, \text{ as } T \rightarrow \infty.
\end{aligned} \tag{7.14}$$

By combining (7.13) with (7.14), we can obtain that  $J_1$  converges in  $L^2$  to  $\frac{\sigma^2}{2\kappa}$  as  $T \rightarrow \infty$ .

For  $J_2$ , we can easily obtain

$$J_2 = \frac{\sigma^2}{T} \int_0^T \left( \int_s^T e^{-\kappa(t-s)} dW_t \right)^2 ds = \frac{\sigma^2}{T} \int_0^T \left( \int_0^u e^{-\theta(u-v)} dW_v \right)^2 du.$$

Hence  $J_2$  also converges to  $\frac{\sigma^2}{2\kappa}$  in  $L^2$  as  $T \rightarrow \infty$ .

Finally, we consider  $J_3$ . A standard calculation yields

$$\mathbb{E}[J_3] = 0. \tag{7.15}$$

Then, a simple calculation shows that

$$\begin{aligned}
\mathbb{E}[J_3]^2 &= \frac{4\sigma^4}{T^2} \int_{\{s < u \leq T\}} \mathbb{E} \left[ \int_0^s e^{-\kappa(s-v)} dW_v \int_0^u e^{-\kappa(u-w)} dW_w \right. \\
&\quad \left. \cdot \int_s^T e^{-\kappa(t-s)} dW_t \int_u^T e^{-\kappa(t-u)} dW_t \right] ds du \\
&= \frac{4\sigma^4}{T^2} \left[ \int_{\{s < u \leq T\}} \int_0^s e^{-\kappa(s+u-2v)} dv \int_u^T e^{-\kappa(2t-s-u)} dt \right] ds du \\
&= \frac{4\sigma^4}{4\kappa^2 T^2} (e^{2\kappa s} - 1) (e^{-2\kappa u} - e^{-2\kappa T}) ds du \\
&\rightarrow 0, \text{ as } T \rightarrow \infty.
\end{aligned} \tag{7.16}$$

From (7.10)-(7.16), we obtain

$$\|DF_T\|_{\mathcal{H}}^2 \xrightarrow{L^2} \frac{\sigma^2}{\kappa}, \quad (7.17)$$

where  $\xrightarrow{L^2}$  denotes convergence in mean square.

Using (7.11), (7.17) and Theorem 4 in Nualart and Ortiz-Latorre (2008), we have

$$F_T = \frac{\sigma}{\sqrt{T}} \int_0^T \int_0^t e^{-\kappa(t-s)} dW_s dW_t \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{2\kappa}\right). \quad (7.18)$$

On the other hand, from (2.17), we have

$$\sqrt{T}(\tilde{\alpha}_T - \alpha) = \frac{\frac{1}{\sqrt{T}} W_T \frac{1}{T} \int_0^T X_t^2 dt - \frac{1}{\sqrt{T}} \int_0^T X_t dW_t \frac{1}{T} \int_0^T X_t dt}{\frac{1}{T} \int_0^T X_t^2 dt - \left(\frac{1}{T} \int_0^T X_t dt\right)^2} \sigma. \quad (7.19)$$

Finally, combining (7.8), (7.9), (7.10), (7.18), (7.19) with Slutsky's theorem, we obtain (3.20). The proof of (3.21) is analogous to the proof of (3.4) in Xiao and Yu (2019a) and omitted.

#### 7.4 Proof of Lemma 4.1

From the proof of Theorem 2 in Tanaka (2013), we can easily obtain (4.3) and (4.4). A simple calculation shows that

$$\int_0^T t d\omega_t^H = \int_0^T t \frac{1}{\lambda_H} (2 - 2H) t^{1-2H} dt = \frac{1}{\lambda_H} \frac{2 - 2H}{3 - 2H} T^{3-2H}.$$

Similarly, a standard calculation yields

$$\int_0^T t^2 d\omega_t^H = \int_0^T t^2 \frac{1}{\lambda_H} (2 - 2H) t^{1-2H} dt = \frac{1}{\lambda_H} \frac{1 - H}{2 - H} T^{4-2H}.$$

Combining (4.4) with the Cauchy-Schwarz inequality, we have

$$\int_0^T \hat{P}_H(t) d\omega_t^H \leq \sqrt{\omega_T^H \int_0^T \hat{P}_H^2(t) d\omega_t^H} = O_p(T^{2-H}).$$

Using (4.4), (4.6) and the Cauchy-Schwarz inequality, we obtain

$$\int_0^T t \hat{P}_H(t) d\omega_t^H \leq \sqrt{\int_0^T t^2 d\omega_t^H \int_0^T \hat{P}_H^2(t) d\omega_t^H} = O_p(T^{3-H}).$$

Now, we consider (4.9). Form the definition of  $V_H(T)$ , we conclude that

$$\begin{aligned}
V_H(T) &= \frac{d}{d\omega_T^H} \int_0^T k_H(T, s) e^{-\kappa s} ds = \frac{d}{d\omega_T^H} \int_0^T k_H(T, s) ds \\
&= \frac{d}{d\omega_T^H} \left[ \frac{1}{k_H} \int_0^T (s(T-s))^{\frac{1}{2}-H} ds \right] \\
&= \frac{d}{d\omega_T^H} \left[ \frac{1}{k_H} \int_0^1 T^{1-2H} (u(1-u))^{\frac{1}{2}-H} T du \right] \\
&= \frac{d}{dT} \left[ \frac{1}{k_H} T^{2-2H} B\left(\frac{3}{2}-H, \frac{3}{2}-H\right) \right] \Big/ \frac{d\omega_T^H}{dT} \\
&= \frac{\lambda_H}{k_H} B\left(\frac{3}{2}-H, \frac{3}{2}-H\right).
\end{aligned}$$

Finally, we deal with (4.10). A standard calculation yields

$$\begin{aligned}
\frac{d}{d\omega_t^H} \int_0^t k_H(t, s) s ds &= \frac{d}{d\omega_t^H} \left[ \frac{1}{k_H} \int_0^t (s(t-s))^{\frac{1}{2}-H} s ds \right] \\
&= \frac{d}{d\omega_t^H} \left[ \frac{1}{k_H} \int_0^t s^{\frac{3}{2}-H} (t-s)^{\frac{1}{2}-H} ds \right] \\
&= \frac{d}{d\omega_t^H} \left[ \frac{1}{k_H} \int_0^1 (vt)^{\frac{3}{2}-H} (t-vt)^{\frac{1}{2}-H} t dv \right] \\
&= \frac{d}{d\omega_t^H} \left[ \frac{1}{k_H} t^{3-2H} \int_0^1 v^{\frac{3}{2}-H} (1-v)^{\frac{1}{2}-H} dv \right] \\
&= \frac{d}{d\omega_t^H} \left[ \frac{1}{k_H} t^{3-2H} B\left(\frac{5}{2}-H, \frac{3}{2}-H\right) \right] \\
&= a_H t,
\end{aligned}$$

where  $a_H = (3 - 2H)/(4 - 4H)$  and the proof of this lemma is completed.

## 7.5 Proof of Theorem 4.1

Using (2.9), (4.1) and (4.10), we have

$$\begin{aligned}
P_H(t) &= \frac{1}{\sigma} \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) X_s ds \\
&= \frac{1}{\sigma} \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) [X_0 + \alpha s + \sigma B_s^H] ds \\
&= \frac{X_0}{\sigma} + \frac{\alpha}{\sigma} \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) s ds + \hat{P}_H(t) \\
&= \frac{X_0}{\sigma} + \frac{\alpha}{\sigma} a_H t + \hat{P}_H(t), \tag{7.20}
\end{aligned}$$



where  $\hat{P}_H(t) = \frac{d}{d\omega_t^H} \int_0^t k_H(t, s) B_s^H ds$ . Using (4.4)–(4.8) and (7.20), we have

$$\begin{aligned}
\int_0^T P_H^2(t) d\omega_t^H &= \int_0^T \left[ \frac{X_0}{\sigma} + \frac{\alpha}{\sigma} a_H t + \hat{P}_H(t) \right]^2 d\omega_t^H \\
&= \frac{X_0^2}{\sigma^2} \omega_T^H + \frac{\alpha^2}{\sigma^2} a_H^2 \int_0^T t^2 d\omega_t^H + \int_0^T \hat{P}_H^2(t) d\omega_t^H \\
&\quad + 2 \frac{X_0}{\sigma^2} \alpha a_H \int_0^T t d\omega_t^H + \frac{2X_0}{\sigma} \int_0^T \hat{P}_H(t) d\omega_t^H \\
&\quad + \frac{2\alpha}{\sigma} a_H \int_0^T t \hat{P}_H(t) d\omega_t^H \\
&= \frac{\alpha^2}{\sigma^2} a_H^2 \int_0^T t^2 d\omega_t^H + o_p(T^{4-2H}). \tag{7.21}
\end{aligned}$$

Similarly, combining (4.5) with (4.7) leads to

$$\begin{aligned}
\int_0^T P_H(t) d\omega_t^H &= \int_0^T \left[ \frac{X_0}{\sigma} + \frac{\alpha}{\sigma} a_H t + \hat{P}_H(t) \right] d\omega_t^H \\
&= \frac{X_0}{\sigma} \omega_T^H + \frac{\alpha}{\sigma} a_H \int_0^T t d\omega_t^H + \int_0^T \hat{P}_H(t) d\omega_t^H \\
&= \frac{\alpha}{\sigma} a_H \int_0^T t d\omega_t^H + o_p(T^{3-2H}). \tag{7.22}
\end{aligned}$$

Moreover, using (4.3) and (7.20), we have

$$\begin{aligned}
\int_0^T P_H(t) dM_t^H &= \int_0^T \left[ \frac{X_0}{\sigma} + \frac{\alpha}{\sigma} a_H t + \hat{P}_H(t) \right] dM_t^H \\
&= \frac{X_0}{\sigma} M_T^H + \frac{\alpha}{\sigma} a_H \int_0^T t dM_t^H + \int_0^T \hat{P}_H(t) dM_t^H \\
&= \frac{X_0}{\sigma} M_T^H + \frac{\alpha}{\sigma} a_H \int_0^T t dM_t^H + O_p(T). \tag{7.23}
\end{aligned}$$

According to (7.21) and (7.22), we get

$$\begin{aligned}
&\omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2 \\
&= \frac{1}{\lambda_H} T^{2-2H} \frac{\alpha^2}{\sigma^2} a_H^2 \int_0^T t^2 d\omega_t^H - \frac{\alpha^2}{\sigma^2} a_H^2 \frac{1}{\lambda_H^2} \frac{(2-2H)^2}{(3-2H)^2} T^{6-4H} + o_p(T^{6-4H}) \\
&= \frac{T^{6-4H}}{\lambda_H^2} \frac{\alpha^2}{\sigma^2} a_H^2 \left( \frac{1-H}{2-H} - \frac{(2-2H)^2}{(3-2H)^2} \right) + o_p(T^{6-4H}) \\
&= \frac{T^{6-4H}}{\sigma^2 \lambda_H^2} \alpha^2 a_H^2 \frac{1-H}{(2-H)(3-2H)^2} + o_p(T^{6-4H}). \tag{7.24}
\end{aligned}$$

Similarly, applying (7.21) and (7.23), we have

$$\begin{aligned}
& M_T^H \int_0^T P_H^2(t) d\omega_t^H - \int_0^T P_H(t) d\omega_t^H \int_0^T P_H(t) dM_t^H \\
&= M_T^H \frac{\alpha^2}{\sigma^2} a_H^2 \int_0^T t^2 d\omega_t^H - \frac{\alpha}{\sigma} a_H \int_0^T t d\omega_t^H \frac{\alpha}{\sigma} a_H \int_0^T t dM_t^H + o_p(T^{5-3H}) \\
&= \frac{\alpha^2}{\sigma^2} a_H^2 \left[ M_T^H \frac{T^{4-2H}}{\lambda_H} \frac{1-H}{2-H} - \frac{T^{3-2H}}{\lambda_H} \frac{2-2H}{3-2H} \int_0^T t dM_t^H \right] + o_p(T^{5-3H}) \\
&= \frac{\alpha^2 a_H^2 T^{5-3H}}{\lambda_H \sigma^2} \left[ \frac{1-H}{2-H} \frac{M_T^H}{T^{1-H}} - \frac{2-2H}{3-2H} \frac{1}{T^{2-H}} \int_0^T t dM_t^H \right] + o_p(T^{5-3H}). \quad (7.25)
\end{aligned}$$

Consequently, combining (2.17), (7.24), (7.25) with Slutsky's theorem, we have

$$\begin{aligned}
T^{1-H} (\tilde{\alpha}_T - \alpha) &= \frac{\frac{1}{T^{5-3H}} \left[ M_T^H \int_0^T P_H^2(t) d\omega_t^H - \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H \right]}{\frac{1}{T^{6-4H}} \left[ \omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2 \right]} \sigma \\
&\xrightarrow{d} \mathcal{N}(0, \sigma^2 \rho_H).
\end{aligned}$$

By (7.22), (7.23), (2.3) and the fact  $M_T^H = O_p(T^{1-H})$ , we obtain

$$\begin{aligned}
& M_T^H \int_0^T P_H(t) d\omega_t^H - \omega_T^H \int_0^T P_H(t) dM_t^H \\
&= M_T^H \frac{\alpha}{\sigma} a_H \int_0^T t d\omega_t^H - \omega_T^H \frac{\alpha}{\sigma} a_H \int_0^T t dM_t^H + o_p(T^{4-3H}) \\
&= \frac{\alpha}{\sigma} a_H \left[ M_T^H \frac{1}{\lambda_H} \frac{2-2H}{3-2H} T^{3-2H} - \frac{1}{\lambda_H} T^{2-2H} \int_0^T t dM_t^H \right] + o_p(T^{4-3H}) \\
&= \frac{\alpha}{\sigma \lambda_H} a_H T^{4-3H} \left[ \frac{M_T^H}{T^{1-H}} \frac{2-2H}{3-2H} - \frac{1}{T^{2-H}} \int_0^T t dM_t^H \right] + o_p(T^{4-3H}). \quad (7.26)
\end{aligned}$$

Using (2.18), (7.24), (7.26) and Slutsky's theorem, we can see that

$$\begin{aligned}
T^{2-H} (\tilde{\kappa}_T - \kappa) &= \frac{\frac{1}{T^{4-3H}} \left[ M_T^H \int_0^T P_H(t) d\omega_t^H - \omega_T^H \int_0^T P_H(t) dM_t^H \right]}{\frac{1}{T^{6-4H}} \left[ \omega_T^H \int_0^T P_H^2(t) d\omega_t^H - \left( \int_0^T P_H(t) d\omega_t^H \right)^2 \right]} \\
&\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\alpha^2} \phi_H\right).
\end{aligned}$$

## 7.6 Proof of Theorem 5.1

Using (2.13), (5.1) and (5.2), we can obtain

$$\begin{aligned}
\int_0^T X_t^2 dt &= \int_0^T \left[ \frac{\alpha}{\kappa} + e^{-\kappa t} \tilde{X}_0 + \sigma U_t \right]^2 dt \\
&= \frac{\alpha^2}{\kappa^2} T + \tilde{X}_0^2 \int_0^T e^{-2\kappa t} dt + \sigma^2 \int_0^T U_t^2 dt + \frac{2\alpha}{\kappa} \tilde{X}_0 \int_0^T e^{-\kappa t} dt \\
&\quad + \frac{2\alpha\sigma}{\kappa} \int_0^T U_t dt + 2\tilde{X}_0\sigma \int_0^T e^{-\kappa t} U_t dt \\
&= \tilde{X}_0^2 \int_0^T e^{-2\kappa t} dt + \int_0^T e^{-2\kappa t} \xi_t^2 dt + 2\tilde{X}_0 \int_0^T e^{-2\kappa t} \xi_t dt + o_p(e^{-2\kappa T}) \\
&= \int_0^T e^{-2\kappa t} (\tilde{X}_0 + \xi_t)^2 dt + o_p(e^{-2\kappa T}). \tag{7.27}
\end{aligned}$$

Similarly, using (2.13), (5.1) and (5.2) again, we can easily have

$$\begin{aligned}
\int_0^T X_t dt &= \int_0^T \left[ \frac{\alpha}{\kappa} + e^{-\kappa t} \tilde{X}_0 + \sigma U_t \right] dt \\
&= \frac{\alpha}{\kappa} T + \tilde{X}_0 \frac{1}{\kappa} (1 - e^{-\kappa T}) + \sigma \int_0^T U_t dt \\
&= O_p(e^{-\kappa T}). \tag{7.28}
\end{aligned}$$

A straightforward calculation shows

$$\begin{aligned}
\int_0^T X_t dW_t &= \int_0^T \left[ \frac{\alpha}{\kappa} + e^{-\kappa t} \tilde{X}_0 + \sigma U_t \right] dW_t \\
&= \frac{\alpha}{\kappa} W_T + \tilde{X}_0 \int_0^T e^{-\kappa t} dW_t + \sigma \int_0^T U_t dW_t \\
&= O_p(e^{-\kappa T}). \tag{7.29}
\end{aligned}$$

From the definition of  $\xi_t$ , we can rewrite  $X_t$  as  $X_t = \frac{\alpha}{\kappa} + e^{-\kappa t} \tilde{X}_0 + e^{-\kappa t} \xi_t$ . As a consequence, using (7.27), we can see that

$$e^{2\kappa T} \int_0^T X_t^2 dt = \frac{\int_0^T e^{-2\kappa t} (\tilde{X}_0 + \xi_t)^2 dt}{e^{-2\kappa T}} + o_p(1), \tag{7.30}$$

$$\sigma e^{\kappa T} \int_0^T X_t dW_t = \tilde{X}_0 \sigma \int_0^T e^{\kappa(T-t)} dW_t + \sigma \int_0^T e^{\kappa(T-t)} \xi_t dW_t + o_p(1). \tag{7.31}$$

Now, applying (2.17), (7.28), (7.29), (7.30) and Slutsky's theorem, we deduce

$$\begin{aligned}
\sqrt{T}(\tilde{\alpha}_T - \alpha) &= \frac{\frac{W_T}{\sqrt{T}} e^{2\kappa T} \int_0^T X_t^2 dt - \frac{e^{2\kappa T}}{\sqrt{T}} \int_0^T X_t dW_t \int_0^T X_t dt}{e^{2\kappa T} \left( \int_0^T X_t^2 dt - \frac{1}{T} \left( \int_0^T X_t dt \right)^2 \right)} \sigma \\
&= \frac{\frac{W_T}{\sqrt{T}} e^{2\kappa T} \int_0^T X_t^2 dt + o_p(1)}{e^{2\kappa T} \int_0^T X_t^2 dt + o_p(1)} \sigma = \frac{\sigma W_T}{\sqrt{T}} + o_p(1),
\end{aligned}$$

which implies (5.3).

Finally, using (2.18), (7.28), (7.30), (7.31) and Slutsky's theorem, we have

$$\begin{aligned} e^{-\kappa T} (\tilde{\kappa}_T - \kappa) &= e^{-\kappa T} \left( \frac{\frac{W_T}{T} \int_0^T X_t dt - \int_0^T X_t dW_t}{\int_0^T X_t^2 dt - \frac{1}{T} \left( \int_0^T X_t dt \right)^2 \sigma} \right) \\ &= \frac{-\sigma e^{\kappa T} \int_0^T X_t dW_t + o_p(1)}{e^{2\kappa T} \int_0^T X_t^2 dt + o_p(1)} \xrightarrow{d} \frac{-(X_0 - \frac{\alpha}{\kappa} + \xi_\infty) \eta_\infty}{-\frac{1}{2\kappa} (X_0 - \frac{\alpha}{\kappa} + \xi_\infty)^2}, \end{aligned}$$

which yields (5.4) and the proof is done.

## 7.7 Proof of Lemma 5.1

Let us observe that (5.6) can be obtained easily from Theorem 2 in Tanaka (2015) and the details are omitted here. For (5.7), using the Cauchy-Schwarz inequality, we have

$$\mathbb{E} \left[ \left( \int_0^T \tilde{P}_H(t) dM_t^H \right)^2 \right] = \int_0^T \tilde{P}_H^2(t) d\omega_t^H = O(e^{-2\kappa T}),$$

which implies (5.7) directly.

Let  ${}_1F_1(\cdot, \cdot, \cdot)$  be the confluent hypergeometric function of the first kind. From (5.5), and the well known result of the confluent hypergeometric function (see for example, Eq. 3.383 (1) in Gradshteyn and Ryzhik, 2007), we have

$$\begin{aligned} \frac{\int_0^T V_H(t) d\omega_t^H}{T^{\frac{1}{2}-H} e^{-\kappa T}} &= C \int_0^T \left( \frac{t}{T} (T-t) \right)^{\frac{1}{2}-H} e^{\kappa(T-t)} dt \\ &= CT \int_0^1 (u(1-u))^{\frac{1}{2}-H} e^{\kappa T(1-u)} du \\ &= CT \int_0^1 (v(1-v))^{\frac{1}{2}-H} e^{\kappa T v} dv \\ &= CT {}_1F_1 \left( \frac{3}{2} - H, 3 - 2H, \kappa T \right) \\ &= O(1), \end{aligned}$$

which yields (5.8).

We now deal with (5.9). Let  $\zeta_t = \sigma \int_0^t e^{\kappa s} dB_s^H$ . Then, as  $T \rightarrow \infty$ , we have

$$\zeta_T \xrightarrow{p} \zeta_\infty \sim \mathcal{N} \left( 0, \frac{H\Gamma(2H)}{(-\kappa)^{2H}} \sigma^2 \right). \quad (7.32)$$

Using (2.2), (2.13), (7.32) and the property of the confluent hypergeometric function (see

for example, Eq. 3.383 (1) in Gradshteyn and Ryzhik, 2007), we have

$$\begin{aligned}
\int_0^T \tilde{P}_H(t) d\omega_t^H &= \int_0^T k_H(T, t) U_t dt \\
&= \frac{1}{k_H} \int_0^T (t(T-t))^{\frac{1}{2}-H} \frac{e^{-\kappa t}}{\sigma} \zeta_t dt \\
&= CT^{2-2H} \int_0^1 (u(1-u))^{\frac{1}{2}-H} e^{-\kappa T u} \zeta_{Tu} du \\
&= O_p(1) T^{2-2H} \int_0^1 (u(1-u))^{\frac{1}{2}-H} e^{-\kappa T u} du \\
&= O_p(1) T^{2-2H} {}_1F_1\left(\frac{3}{2}-H, 3-2H, -\kappa T\right) \\
&= O_p(1) T^{2-2H} O_p(T^{H-\frac{3}{2}} e^{-\kappa T}) \\
&= O_p(T^{\frac{1}{2}-H} e^{-\kappa T}),
\end{aligned}$$

which implies (5.9).

We now turn to the term (5.10). Using (5.5), we can easily obtain

$$\int_0^T V_H^2(t) d\omega_t^H = C \int_0^T t^{2H-1} e^{-2\kappa t} t^{1-2H} dt = O(e^{-2\kappa T}),$$

which yields (5.10).

Using the Cauchy-Schwarz inequality, (5.6) and (5.10), we obtain

$$\left( \int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H \right)^2 \leq \int_0^T V_H^2(t) d\omega_t^H \int_0^T \tilde{P}_H^2(t) d\omega_t^H = O_p(e^{-4\kappa T}),$$

which implies (5.11).

Similarly, using (5.10), we have

$$\mathbb{E} \left[ \left( \int_0^T V_H(t) dM_t^H \right)^2 \right] = \int_0^T V_H^2(t) d\omega_t^H = O_p(e^{-2\kappa T}),$$

which yields (5.12) and we complete the proof.

## 7.8 Proof of Theorem 5.2

Using (2.10), (5.6), (5.8)-(5.10) and (5.11), we can obtain

$$\begin{aligned}
\int_0^T P_H^2(t) d\omega_t^H &= \int_0^T \left[ \frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma} \tilde{X}_0 V_H(t) + \tilde{P}_H(t) \right]^2 d\omega_t^H \\
&= \frac{\alpha^2}{\sigma^2\kappa^2} \omega_T^H + \frac{1}{\sigma^2} \tilde{X}_0^2 \int_0^T V_H^2(t) d\omega_t^H + \int_0^T \tilde{P}_H^2(t) d\omega_t^H \\
&\quad + \frac{2\alpha}{\sigma^2\kappa} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + \frac{2\alpha}{\sigma\kappa} \int_0^T \tilde{P}_H(t) d\omega_t^H \\
&\quad + \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H \\
&= \frac{1}{\sigma^2} \tilde{X}_0^2 \int_0^T V_H^2(t) d\omega_t^H + \int_0^T \tilde{P}_H^2(t) d\omega_t^H \\
&\quad + \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) \tilde{P}_H(t) d\omega_t^H + o_p(e^{-2\kappa T}) \\
&= \int_0^T \left( \frac{1}{\sigma} \tilde{X}_0 V_H(t) + \tilde{P}_H(t) \right)^2 d\omega_t^H + o_p(e^{-2\kappa T}). \tag{7.33}
\end{aligned}$$

According to (2.10), (5.6), (5.8) and (5.9), we obtain

$$\begin{aligned}
\frac{1}{\omega_T^H} \left( \int_0^T P_H(t) d\omega_t^H \right)^2 &= \frac{1}{\omega_T^H} \left[ \int_0^T \left( \frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma} \tilde{X}_0 V_H(t) + \tilde{P}_H(t) \right) d\omega_t^H \right]^2 \\
&= \frac{1}{\omega_T^H} \left[ \frac{\alpha}{\sigma\kappa} \omega_T^H + \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + \int_0^T \tilde{P}_H(t) d\omega_t^H \right]^2 \\
&= \frac{1}{\omega_T^H} \left[ \frac{\alpha^2}{\sigma^2\kappa^2} (\omega_T^H)^2 + \frac{1}{\sigma^2} \tilde{X}_0^2 \left( \int_0^T V_H(t) d\omega_t^H \right)^2 \right. \\
&\quad \left. + \left( \int_0^T \tilde{P}_H(t) d\omega_t^H \right)^2 + \frac{2\alpha}{\sigma^2\kappa} \omega_T^H \int_0^T V_H(t) d\omega_t^H \right. \\
&\quad \left. + \frac{2\alpha}{\sigma\kappa} \omega_T^H \int_0^T \tilde{P}_H(t) d\omega_t^H + \frac{2}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H \int_0^T \tilde{P}_H(t) d\omega_t^H \right] \\
&= o_p(e^{-2\kappa T}). \tag{7.34}
\end{aligned}$$

From (2.10), (5.7) and (5.12), we can see that

$$\begin{aligned}
\int_0^T P_H(t) dM_t^H &= \int_0^T \left[ \frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma} \tilde{X}_0 V_H(t) + \tilde{P}_H(t) \right] dM_t^H \\
&= \frac{\alpha}{\sigma\kappa} M_T^H + \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) dM_t^H + \int_0^T \tilde{P}_H(t) dM_t^H \\
&= \frac{\tilde{X}_0}{\sigma} \int_0^T V_H(t) dM_t^H + \int_0^T \tilde{P}_H(t) dM_t^H + o_p(e^{-\kappa T}). \tag{7.35}
\end{aligned}$$

From (2.10) and the definition of  $\omega_t^H$ , we can obtain

$$\begin{aligned}
\int_0^T P_H(t) d\omega_t^H &= \int_0^T \left[ \frac{\alpha}{\sigma\kappa} + \frac{1}{\sigma} \tilde{X}_0 V_H(t) + \tilde{P}_H(t) \right] d\omega_t^H \\
&= \frac{\alpha}{\sigma\kappa} \omega_T^H + \frac{1}{\sigma} \tilde{X}_0 \int_0^T V_H(t) d\omega_t^H + \int_0^T \tilde{P}_H(t) d\omega_t^H \\
&= \frac{\tilde{X}_0}{\sigma} \int_0^T V_H(t) d\omega_t^H + \int_0^T \tilde{P}_H(t) d\omega_t^H + O(T^{2-2H}). \quad (7.36)
\end{aligned}$$

Now, combining (2.17), (7.33), (7.34), Lemma 5.1 with Slutsky's theorem, we have

$$\begin{aligned}
T^{1-H} (\tilde{\alpha}_T - \alpha) &= \frac{T^{1-H} \frac{M_T^H}{\omega_T^H} \int_0^T P_H^2(t) d\omega_t^H - \frac{T^{1-H}}{\omega_T^H} \int_0^T P_H(t) dM_t^H \int_0^T P_H(t) d\omega_t^H}{\int_0^T P_H^2(t) d\omega_t^H - \frac{1}{\omega_T^H} \left( \int_0^T P_H(t) d\omega_t^H \right)^2} \sigma \\
&\xrightarrow{d} \mathcal{N}(0, \sigma^2).
\end{aligned}$$

Now, let  $X$  and  $Y$  be two independent  $\mathcal{N}(0, 1)$  random variables. Then using (2.18), (7.33)-(7.36), (2.10), Lemma 5.1, Slutsky's theorem and Eq. (33) in Tanaka (2015), we can see that

$$\begin{aligned}
\frac{e^{-\kappa T}}{2\kappa} (\tilde{\kappa}_T - \kappa) &= \frac{\frac{e^{-\kappa T}}{2\kappa} \left[ \frac{M_T^H}{\omega_T^H} \int_0^T P_H(t) d\omega_t^H - \int_0^T P_H(t) dM_t^H \right]}{\int_0^T P_H^2(t) d\omega_t^H - \frac{1}{\omega_T^H} \left( \int_0^T P_H(t) d\omega_t^H \right)^2} \\
&= \frac{-2\kappa e^{\kappa T} \int_0^T P_H(t) dM_t^H + o_p(1)}{4\kappa^2 e^{2\kappa T} \int_0^T P_H^2(t) d\omega_t^H + o_p(1)} \\
&= \frac{-2\kappa e^{\kappa T} \int_0^T \left[ \frac{\tilde{X}_0}{\sigma} V_H(t) + \tilde{P}_H(t) \right] dM_t^H + o_p(1)}{4\kappa^2 e^{2\kappa T} \int_0^T \left[ \frac{\tilde{X}_0}{\sigma} V_H(t) + \tilde{P}_H(t) \right]^2 d\omega_t^H + o_p(1)} \\
&\xrightarrow{d} \frac{X \sqrt{\sin(\pi H)}}{Y},
\end{aligned}$$

with  $\tilde{X}_0 = 0$ .

## References

- Ait-Sahalia, Y., Mancini, T. S. (2008). Out of sample forecasts of quadratic variation. *Journal of Econometrics*, 147(1):17–33.
- Anderson, T. W. (1959). On asymptotic distributions of estimates of parameters of stochastic difference equations. *Annals of Mathematical Statistics*, 30(3): 676–687.
- Brouste, A., Kleptsyna, M. (2010). Asymptotic properties of MLE for partially observed fractional diffusion system. *Statistical Inference for Stochastic Processes*, 13(1): 1–13.

- Brown, B. M., Hewitt, J. I. (1975). Asymptotic likelihood theory for diffusion processes. *Journal of Applied Probability*, 12(2): 228–238.
- Chan, K. C., Karolyi, G. A., Longstaff, F. A., Sanders, A. B. (1992). An empirical comparison of alternative models of the short-term interest rate. *Journal of Finance*, 47(3): 1209–1227.
- Comte, F., Renault, E. (1998). Long memory continuous-time stochastic volatility models. *Mathematical Finance*, 8(4):291–323.
- Feigin, P. D. (1976). Maximum likelihood estimation for continuous-time stochastic processes. *Advances in Applied Probability*, 8(4): 712–736.
- Gatheral, J., Jaisson, T., Rosenbaum, M. (2018). Volatility is rough. *Quantitative Finance*, 18(6):933–949.
- Gradshteyn, I.S., Ryzhik, I.M. (2007). Jeffrey, A., Zwillinger, D. (Eds.). *Table of Integrals, Series, and Products*. Elsevier.
- Hu, Y., Nualart, D. (2010). Parameter estimation for fractional Ornstein–Uhlenbeck processes. *Statistics and Probability Letters*, 80(11):1030–1038.
- Hu, Y., Nualart, D., Zhou, H. (2017). Parameter estimation for fractional Ornstein–Uhlenbeck processes of general Hurst parameter. to appear in *Statistical Inference for Stochastic Processes*, <https://doi.org/10.1007/s11203-017-9168-2>.
- Jamshidian, F. (1989). An exact bond option formula. *Journal of Finance*, 44(1): 205–209.
- Kleptsyna, M. L., Le Breton, A. (2002). Statistical analysis of the fractional Ornstein–Uhlenbeck type process. *Statistical Inference for Stochastic Processes*, 5(3):229–248.
- Kleptsyna, M. L., Le Breton, A., Roubaud, M. C. (2000). Parameter estimation and optimal filtering for fractional type stochastic systems. *Statistical Inference for Stochastic Processes*, 3(1-2): 173–182.
- Kubilius, K., Mishura, Y., Ralchenko, K. (2018). *Parameter Estimation in Fractional Diffusion Models*. Springer.
- Lohvinenko, S., Ralchenko, K. (2017). Maximum likelihood estimation in the fractional Vasicek model. *Lithuanian Journal of Statistics*, 56(1): 77–87.
- Magdalinos, T. (2012) Mildly explosive autoregression under weak and strong dependence. *Journal of Econometrics*, 169(2): 179–187.



- Norros, I., Valkeila, E., Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, 5(4): 571–587.
- Nualart, D. (2006) *The Malliavin Calculus and Related Topics*. Second Edition. Springer.
- Nualart, D., Ortiz-Latorre, S. (2008). Central limit theorems for multiple stochastic integrals and Malliavin calculus. *Stochastic Processes and their Applications*, 118(4): 614–628.
- Phillips, P. C. B., Magdalinos, T. (2007). Limit theory for moderate deviations from a unit root. *Journal of Econometrics*, 136(1): 115–130.
- Scott, L. (1987). Option pricing when the variance changes randomly: theory, estimation, and an application. *Journal of Financial and Quantitative Analysis*, 22(4): 419–438.
- Tanaka, K. (2013). Distributions of the maximum likelihood and minimum contrast estimators associated with the fractional Ornstein–Uhlenbeck process. *Statistical Inference for Stochastic Processes*, 16(3):173–192.
- Tanaka, K. (2015). Maximum likelihood estimation for the non-ergodic fractional Ornstein–Uhlenbeck process. *Statistical Inference for Stochastic Processes*, 18(3):315–332.
- Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2): 177–188.
- White, J. S. (1958). The limiting distribution of the serial correlation coefficient in the explosive case. *Annals of Mathematical Statistics*, 29(4): 1188–1197.
- Wang, X., Yu, J. (2015). Limit theory for an explosive autoregressive process. *Economics Letters*, 126: 176–180.
- Wang, X., Yu, J. (2016). Double asymptotics for explosive continuous time models. *Journal of Econometrics*, 193(1): 35–53.
- Xiao, W., Wang, X., Yu, J. (2019). Estimation and inference in fractional Ornstein–Uhlenbeck model with discrete-sampled data. Manuscript.
- Xiao, W., Yu, J. (2019a). Asymptotic theory for estimating the drift parameters in the fractional Vasicek model. *Econometric Theory*, 35(1): 198–231.
- Xiao, W., Yu, J. (2019b). Asymptotic theory for rough fractional Vasicek models. *Economics Letters*, 177: 26–29.
- Yu, J., Phillips, P. C. B. (2001). A Gaussian approach for continuous time models of the short–term interest rate. *Econometrics Journal*, 4(2): 210–224.