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THE SCHOOL OF ECONOMICS, SMU

Non-separable Models with High-dimensional Data*

Liangjun Su[†] Takuya Ura[‡] Yichong Zhang[§]

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Abstract

This paper studies non-separable models with a continuous treatment when the dimension of the control variables is high and potentially larger than the effective sample size. We propose a three-step estimation procedure to estimate the average, quantile, and marginal treatment effects. In the first stage we estimate the conditional mean, distribution, and density objects by penalized local least squares, penalized local maximum likelihood estimation, and penalized conditional density estimation, respectively, where control variables are selected via a localized method of L_1 -penalization at each value of the continuous treatment. In the second stage we estimate the average and the marginal distribution of the potential outcome via the plug-in principle. In the

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third stage, we estimate the quantile and marginal treatment effects by inverting the estimated distribution function and using the local linear regression, respectively. We study the asymptotic properties of these estimators and propose a weighted-bootstrap method for inference. Using simulated and real datasets, we demonstrate the proposed estimators perform well in finite samples.

Keywords: Average treatment effect, High dimension, Least absolute shrinkage and selection operator (Lasso), Nonparametric quantile regression, Nonseparable models, Quantile treatment effect, Unconditional average structural derivative

JEL codes: C21, I19

1 Introduction

Non-separable models without additivity appear frequently in econometric analyses, because economic theory motivates a nonlinear role of the unobserved individual heterogeneity (Altonji and Matzkin, 2005) and its multi-dimensionality (Browning and Carro, 2007; Carneiro, Hansen, and Heckman, 2003; Cunha, Heckman, and Schennach, 2010). A large fraction of the previous literature on non-separable models has used control variables to achieve the unconfoundedness condition (Rosenbaum and Rubin, 1983), that is, the conditional independence between a regressor of interest (or a treatment) and the unobserved individual heterogeneity given the control variables. Although including high-dimensional controls make unconfoundedness more plausible, the estimation and inference become more challenging, as well. It remains unanswered how to select control variables among potentially very many variables and conduct proper statistical inference for parameters of interest in non-separable models with a continuous treatment.

This paper proposes estimation and inference for unconditional parameters, including unconditional means of the potential outcomes, the unconditional cumulative distribution

function, the unconditional quantile function, and the unconditional quantile partial derivative with the presence of both continuous treatment and high-dimensional covariates.¹ The proposed method estimates the parameters of interest in three stages. The first stage selects controls by the method of least absolute shrinkage and selection operator (Lasso) and predicts reduced-form parameters such as the conditional expectation and distribution of the outcome given the variables and treatment level and the conditional density of the treatment given the control variables. We allow for different control variables to be selected at different values of the continuous treatment. The second stage recovers the average and the marginal distribution of the potential outcome by plugging the reduced-form parameters into doubly robust moment conditions. The last stage recovers the quantile of the potential outcome and its derivative with respect to the treatment by inverting the estimated distribution function and using the local linear regression, respectively. The inference is implemented via a multiplier bootstrap without recalculating the first stage variable selections, which saves considerable computation time.

To motivate our parameters of interest, we relate our estimands (the population objects that our procedure aims to recover) with the structural outcome function. Notably, we extend Hoderlein and Mammen (2007) and Sasaki (2015) to demonstrate that the unconditional derivative of the quantile of the potential outcome with respect to the treatment is equal to the weighted average of the marginal effects over individuals with same outcomes and treatments.

This paper contributes to two important strands of the econometric literature. The first is the literature on non-separable models with a continuous treatment, in which previous

¹We focus on unconditional parameters, in which (potentially high-dimensional) covariates are employed to achieve the unconfoundedness but the parameters of interest are unconditional on the covariates. Unconditional parameters are simple to display and the simplicity is crucial especially when the covariates are high dimensional. As emphasized in Frölich and Melly (2013) and Powell (2010), unconditional parameters have two additional attractive features. First, by definition, they capture all the individuals in the sample at the same time instead of investigating the underlying structure separately for each subgroup defined by the covariates X . The treatment effect for the whole population is more policy-relevant. Second, an estimator for unconditional parameters can have better finite/large sample properties.

analyses have focused on a fixed and small number of control variables; see, e.g., Chesher (2003), Chernozhukov, Imbens, and Newey (2007), Hoderlein and Mammen (2007), Imbens and Newey (2009), Matzkin (1994) and Matzkin (2003). The second is a growing literature on recovering the causal effect from the high-dimensional data; see, e.g., Belloni, Chen, Chernozhukov, and Hansen (2012), Belloni, Chernozhukov, and Hansen (2014a), Chernozhukov, Hansen, and Spindler (2015a), Chernozhukov, Hansen, and Spindler (2015b), Farrell (2015), Athey and Imbens (2015), Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, and Newey (2016), Belloni, Chernozhukov, and Hansen (2014b), Wager and Athey (2016), Belloni, Chernozhukov, Fernández-Val, and Hansen (2017a), and Belloni, Chernozhukov, and Wei (2017b). Our paper complements the previous works by studying both the variable selection and post-selection inference of causal parameters in a *non-separable* model with a *continuous* treatment. Recently, Cattaneo, Jansson, and Newey (2016), Cattaneo, Jansson, and Ma (2017a), and Cattaneo, Jansson, and Newey (2017b) considered the semiparametric estimation of the causal effect in a setting with many included covariates and proposed novel bias-correction methods to conduct valid inference. Comparing with them, we deal with the fully nonparametric model with an ultra-high dimension of potential covariates, and rely on the approximate sparsity to reduce dimensionality.

The treatment variable being continuous imposes difficulties in both variable selection and post-selection inference. To address the former, we develop penalized local Maximum Likelihood and Least Square estimations (hereafter, MLE and LS, respectively), which select control variables for each value of the continuous treatment. By relying on kernel smoothing method, we require a different penalty loading than the traditional Lasso method. Chu, Zhu, and Wang (2011) and Ning and Liu (2017) developed general theories of estimation, inference, and hypothesis testing of penalized (Pseudo) MLE. We complement their results by considering the local likelihood with an L_1 penalty term. Belloni, Chernozhukov, Chetverikov, and Wei (2016b) constructed uniformly valid confidence intervals for the Z-

estimators of unconditional moment equalities. Our results are not covered by theirs, either, as our parameters are defined based on conditional moment equalities. To prove the statistical properties of the penalized local MLE and LS, we establish a local version of the compatibility condition (Bühlmann and van de Geer, 2011), which itself is new to the best of our knowledge.

For the post-selection inference, we establish doubly robust moment conditions for the continuous treatment effect model. Our parameters of interest is irregularly identified by the definition in Khan and Tamer (2010), as they are identified by a thin-set. Therefore, by averaging observations only with their treatment levels close to the one of interest, the convergence rates of our estimators are nonparametric, which is in contrast with the \sqrt{n} -rate obtained in Belloni et al. (2017a) and Farrell (2015). Albeit motivated by distinct models, Belloni, Chen, and Chernozhukov (2016a) also estimated the irregular identified parameters in the high-dimensional setting. However, the irregularity faced by Belloni et al. (2016a) is not due to the continuity of the variable of interest. Consequently, Belloni et al. (2016a) did not study the regularized estimator with localization as we do in this paper. To obtain uniformly valid results over values of the continuous treatment, we derive linear expansions of the rearrangement operator for a local process which is not tight, and establish a new maximal inequality for the second order degenerate U-process, extending the existing results in Chernozhukov, Fernández-Val, and Galichon (2010) and Nolan and Pollard (1987), respectively.

We study the finite sample performance of our estimation procedure via Monte Carlo simulations and an empirical application. The simulations suggest that the proposed estimators perform reasonably well in finite samples. In the empirical exercise, we estimate the distributional effect of parental income on son's income and intergenerational elasticity using the 1979 National Longitudinal Survey of Youth (NLSY79). We control for a large dimension of demographic variables. The quartiles of son's potential income are in general

upward slopping with respect to parental income. However, the intergenerational elasticities are not statistically significant. We also found that speaking a foreign language at childhood, years of education, and being born outside U.S. are the leading confounding variables selected by our procedure.

The rest of this paper is organized as follows. Section 2 presents the model and the parameters of interest. Section 3 proposes an estimation method in the presence of high-dimensional covariates. Section 4 demonstrates the validity of a bootstrap inference procedure. Section 5 presents Monte Carlo simulations. Section 6 illustrates the proposed estimator using NLSY79. Section 7 concludes. Proofs of the main theorems and Lemma 3.1 are reported in the appendix. Proofs of the rest of the lemmas are collected in an online supplement.

Throughout this paper, we adopt the convention that the capital letters, such as A , Y , X , denote random elements while their corresponding lower cases and calligraphic letters denote realizations and supports, respectively. C denotes an arbitrary positive constant that may not be the same in different contexts. For a sequence of random variables $\{U_n\}_{n=1}^\infty$ and a random variable U , $U_n \rightsquigarrow U$ indicates weak convergence in the sense of van der Vaart and Wellner (1996). When U_n and U are k -dimensional elements, the space of the sample path is \mathfrak{R}^k equipped with Euclidean norm. When U_n and U are stochastic processes, the space of sample path is $L^\infty(\{v \in \mathfrak{R}^k : |v| < B\})$ for some positive B equipped with sup norm. The calligraphic letters \mathcal{P}_n , \mathcal{P} , and \mathcal{U}_n denote the empirical process, expectation, and U-process, respectively. In particular, \mathcal{P}_n assigns probability $\frac{1}{n}$ to each observation and \mathcal{U}_n assigns probability $\frac{1}{n(n-1)}$ to each pair of observations. \mathbb{E} also denotes expectation. We use \mathcal{P} and \mathbb{E} exchangeably. For any positive (random) sequence (u_n, v_n) , if there exists a positive constant C independent of n such that $u_n \leq Cv_n$, then we write $u_n \lesssim v_n$. $\|\cdot\|_{Q,q}$ denotes L^q norm under measure Q , where $q = 1, 2, \infty$. If measure Q is omitted, the underlying measure is assumed to be the counting measure. For any vector θ , $\|\theta\|_0$ denotes the number

of its nonzero coordinates. $\text{Supp}(\theta)$, the support of a p -dimensional vector θ , is defined as $\{j : \theta_j \neq 0\}$. For $T \subset \{1, 2, \dots, p\}$, let $|T|$ be the cardinality of T , T^c be the complement of T , and θ_T be the vector in \mathfrak{R}^p that has the same coordinates as θ on T and zero coordinates on T^c . Last, let $a \vee b = \max(a, b)$.

2 Model and Parameters of Interest

Econometricians observe an outcome Y , a continuous treatment T , and a set of covariates X , which may be high-dimensional. They are connected by a measurable function $\Gamma(\cdot)$, i.e.,

$$Y = \Gamma(T, X, A),$$

where A is an unobservable random vector and may not be weakly separable from observables (T, X) , and Γ may not be monotone in either T or A .

Let $Y(t) = \Gamma(t, X, A)$. We are interested in the average $\mathbb{E}Y(t)$, the marginal distribution $P(Y(t) \leq u)$ for some $u \in \mathfrak{R}$, and the quantile $q_\tau(t)$, where we denote $q_\tau(t)$ as the τ -th quantile of $Y(t)$ for some $\tau \in (0, 1)$. We are also interested in the causal effect of moving T from t to t' , i.e., $\mathbb{E}(Y(t) - Y(t'))$ and $q_\tau(t) - q_\tau(t')$. Last, we are interested in the average marginal effect $\mathbb{E}[\partial_t \Gamma(t, X, A)]$ and quantile partial derivative $\partial_t q_\tau(t)$. Next, we specify conditions under which the above parameters are identified.

Assumption 1 1. *The sample $\{Y_i, T_i, X_i\}_{i=1}^n$ is i.i.d.*

2. *The random variables A and T are conditionally independent given X .*

Assumption 1.1 can be relaxed at the cost of lengthy arguments, which is not pursued here. Assumption 1.2 is known as the unconfoundedness condition, which is commonly assumed in the treatment effect literature. See Cattaneo (2010), Cattaneo and Farrell (2011),

Hirano, Imbens, and Ridder (2003) and Firpo (2007) for the case of discrete treatment and Graham, Imbens, and Ridder (2014), Graham, Imbens, and Ridder (2016), Galvao and Wang (2015), and Hirano and Imbens (2004) for the case of continuous treatment. It is also called the conditional independence assumption in Hoderlein and Mammen (2007), which is weaker than the full joint independence between A and (T, X) . Note that X can be arbitrarily correlated with the unobservables A . This assumption is more plausible when we control for sufficiently many and potentially high-dimensional covariates.

Theorem 2.1 *Suppose Assumption 1 holds and $\Gamma(\cdot)$ is differentiable in its first argument. Then the marginal distribution of $Y(t)$ and the average marginal effect $\partial_t \mathbb{E}Y(t)$ are identified. In addition, if Assumption 6 in the Appendix holds and X is continuously distributed, then $\partial_t q_\tau(t) = \mathbb{E}_{\mu_{\tau,t}}[\partial_t \Gamma(t, X, A)]$, where, for $f_{(X,A)}$ denoting the joint density of (X, A) , $\mu_{\tau,t}$ is the probability measure on $\{(x, a) : \Gamma(t, x, a) = q_\tau(t)\}$ and proportional to $\frac{f_{(X,A)}}{\|\nabla_{(x,a)} \Gamma(t, \cdot, \cdot)\|}$.*

Several comments are in order. First, because the marginal distribution of $Y(t)$ is identified, so be its average, quantile, average marginal effect, and quantile partial derivative. As pointed out by Imbens and Newey (2009), a non-separable outcome with a general disturbance is equivalent to treatment effect models. Therefore, we can view $Y(t)$ as the potential outcome. Under unconfoundedness, the identification of the marginal distribution of the potential outcome with a continuous treatment has already been established in Hirano and Imbens (2004) and Galvao and Wang (2015). The first part of Theorem 2.1 just re-states their results. Second, the second result indicates that the partial quantile derivative identifies the weighted average marginal effect for the subpopulation with the same potential outcome, i.e., $\{Y(t) = q_\tau(t)\}$. The result is closely related to, but different from Sasaki (2015). We consider the unconditional quantile of $Y(t)$, whereas he considered the conditional quantile of $Y(t)$ given X . Note that $q_\tau(t)$ is not the average of the conditional quantile of $Y(t)$ given X . Third, we require X to be continuous just for the simplicity of derivation. If some elements of X are discrete, a similar result can be established in a conceptually

straightforward manner by focusing on the continuous covariates within samples homogeneous in the discrete covariates, at the expense of additional notation. Finally, we do not require X to be continuous when establishing the estimation and inference results below.

3 Estimation

Let $f_t(X) = f_{T|X}(t|x)$ denote the conditional density of T evaluated at t given $X = x$ and $\delta_t(\cdot)$ denote the Dirac function such that for any function $g(\cdot)$,

$$\int g(s)\delta_t(s)ds = g(t).$$

In addition, let $Y_u(t) = 1\{Y(t) \leq u\}$ and $Y_u = 1\{Y \leq u\}$ for some $u \in \mathfrak{R}$. Then $\mathbb{E}(Y(t))$ and $\mathbb{E}(Y_u(t))$ can be identified by the method of generalized propensity score as proposed in Hirano and Imbens (2004), i.e.,

$$\mathbb{E}(Y(t)) = \mathbb{E}\left(\frac{Y\delta_t(T)}{f_t(X)}\right) \quad \text{and} \quad \mathbb{E}(Y_u(t)) = \mathbb{E}\left(\frac{Y_u\delta_t(T)}{f_t(X)}\right). \quad (3.1)$$

There is a direct analogy between (3.1) for the continuous treatment and $\mathbb{E}(Y_u(t)) = \mathbb{E}\left(\frac{Y_u 1\{T=t\}}{P(T=t|X)}\right)$ when the treatment T is discrete: the indicator function shrinks to a Dirac function and the propensity score is replaced by the conditional density. Following this analogy, Hirano and Imbens (2004) called $f_t(X)$ the generalized propensity.

Belloni et al. (2017a) and Farrell (2015) considered the model with a discrete treatment and high-dimensional control variables, and proposed to use the doubly robust moment for inference. Following their lead, we propose the corresponding doubly robust moment when the treatment status is continuous. Let $\nu_t(x) = \mathbb{E}(Y|X = x, T = t)$ and $\phi_{t,u}(x) = \mathbb{E}(Y_u|X = x, T = t)$, then

$$\mathbb{E}(Y(t)) = \mathbb{E}\left[\left(\frac{(Y - \nu_t(X))\delta_t(T)}{f_t(X)}\right) + \nu_t(X)\right] \quad (3.2)$$

and

$$\mathbb{E}(Y_u(t)) = \mathbb{E}\left[\left(\frac{(Y_u - \phi_{t,u}(X))\delta_t(T)}{f_t(X)}\right) + \phi_{t,u}(X)\right]. \quad (3.3)$$

We propose the following three-stage procedure to estimate $\mu(t) \equiv \mathbb{E}Y(t)$, $\alpha(t, u) \equiv P(Y(t) \leq u)$, $q_\tau(t)$, and $\partial_t q_\tau(t)$:

1. Estimate $\nu_t(x)$, $\phi_{t,u}(x)$, and $f_t(x)$ by $\hat{\nu}_t(x)$, $\hat{\phi}_{t,u}(x)$ and $\hat{f}_t(x)$, respectively.
2. First, estimate $\mu(t)$ and $\alpha(t, u)$ by

$$\hat{\mu}(t) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{(Y_i - \hat{\nu}_t(X_i))}{\hat{f}_t(X_i)h} K\left(\frac{T_i - t}{h}\right) \right) + \hat{\nu}_t(X_i) \right]$$

and

$$\hat{\alpha}(t, u) = \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{(Y_i - \hat{\phi}_{t,u}(X_i))}{\hat{f}_t(X_i)h} K\left(\frac{T_i - t}{h}\right) \right) + \hat{\phi}_{t,u}(X_i) \right], \quad \text{respectively,}$$

where $K(\cdot)$ and h are a kernel function and a bandwidth. Then rearrange $\hat{a}(t, u)$ to obtain $\hat{a}^r(t, u)$, which is monotone in u .

- 3 Estimate $q_\tau(t)$ by inverting $\hat{a}^r(t, u)$ with respect to (w.r.t.) u , i.e., $\hat{q}_\tau(t) = \inf\{u : \hat{a}^r(t, u) \geq \tau\}$; estimate $\partial_t \mu(t) = \mathbb{E}\partial_t \Gamma(t, X, A)$ by $\check{\beta}^1(t)$, which is the estimator of the slope coefficient in the local linear regression of $\hat{Y}(T_i)$ on T_i ; estimate $\partial_t q_\tau(t)$ by $\hat{\beta}_\tau^1(t)$, which is the estimator of the slope coefficient in the local linear regression of $\hat{q}_\tau(T_i)$ on T_i .

3.1 The First Stage Estimation

In this section, we define the first stage estimators and derive their asymptotic properties. Since $\nu_t(x)$, $\phi_{t,u}(x)$, and $f_t(x)$ are local parameters w.r.t. $T = t$, in addition to using L_1 penalty to select relevant covariates, we rely on a kernel function to implement the

localization. In particular, we propose to estimate $\nu_t(x)$, $\phi_{t,u}(x)$, and $f_t(x)$ by a penalized local LS, a penalized local MLE, and a penalized conditional density estimation method, respectively. All three methods are new to the literature and of their own interests.

3.1.1 Penalized Local LS and MLE

Recall $\nu_t(x) = \mathbb{E}(Y|X = x, T = t)$ and $\phi_{t,u}(x) = \mathbb{E}(Y_u|X = x, T = t)$ where $Y_u = 1\{Y \leq u\}$. We approximate $\nu_t(x)$ and $\phi_{t,u}(x)$ by $b(x)'\gamma_t$ and $\Lambda(b(x)'\theta_{t,u})$, respectively, where $\Lambda(\cdot)$ is the logistic CDF and $b(X)$ is a $p \times 1$ vector of basis functions with potentially large p . In the case of high-dimensional covariates, $b(X)$ is just X , while in the case of nonparametric sieve estimation, $b(X)$ is a series of bases of X . The approximation errors for $\nu_t(x)$ and $\phi_{t,u}(x)$ are given by $r_t^\nu(x) = \nu_t(x) - b(x)'\gamma_t$ and $r_{t,u}^\phi(x) = \phi_{t,u}(x) - \Lambda(b(x)'\theta_{t,u})$, respectively.

Note that we only approximate $\nu_t(x)$ and $\phi_{t,u}(x)$ by a linear regression and a logistic regression, respectively, with the approximation errors satisfying Assumption 2 below. However, we do not necessarily require $\nu_t(x)$ and $\Lambda^{-1}(\phi_{t,u}(x))$ to be linear in x . Assumption 2 below will put a sparsity structure on $\nu_t(x)$ and $\phi_{t,u}(x)$ so that the number of effective covariates that can affect them is much smaller than p . If the effective covariates are all discrete, then we can saturate the regressions so that there is no approximate error. If some of the effective covariates are continuous, then we can include sieve bases in the linear regression so that the approximation error can still satisfy Assumption 2. Last, the coefficients γ_t and $\theta_{t,u}$ are both functional parameters that can vary with their indices. This provides additional flexibility of our setup against misspecification.

We estimate $\nu_t(x)$ and $\phi_{t,u}(x)$ by $\widehat{\nu}_t(x) = b(x)'\widehat{\gamma}_t$ and $\widehat{\phi}_{t,u}(x) = \Lambda(b(x)'\widehat{\theta}_{t,u})$, respectively, where

$$\widehat{\gamma}_t = \arg \min_{\gamma} \frac{1}{n} \sum_{i=1}^n (Y_i - b(X_i)'\gamma)^2 K\left(\frac{T_i - t}{h}\right) + \frac{\lambda}{n} \|\widehat{\Xi}_t \gamma\|_1, \quad (3.4)$$

$$\hat{\theta}_{t,u} = \arg \min_{\theta} \frac{1}{n} \sum_{i=1}^n M(1\{Y_i \leq u\}, X_i; \theta) K\left(\frac{T_i - t}{h}\right) + \frac{\lambda}{n} \|\widehat{\Psi}_{t,u}\theta\|_1, \quad (3.5)$$

$\|\cdot\|_1$ denotes the L_1 norm, $\lambda = \ell_n(\log(p \vee n)nh)^{1/2}$ for some slowly diverging sequence ℓ_n , and $M(y, x; g) = -[y \log(\Lambda(b(x)'g)) + (1 - y) \log(1 - \Lambda(b(x)'g))]$. In (3.4) and (3.5), $\widehat{\Xi}_t = \text{diag}(\tilde{l}_{t,1}, \dots, \tilde{l}_{t,p})$ and $\widehat{\Psi}_{t,u} = \text{diag}(l_{t,u,1}, \dots, l_{t,u,p})$ are generic penalty loading matrices to be specified below.

The ideal loading matrices are $\widehat{\Xi}_{t,0} = \text{diag}(\tilde{l}_{t,0,1}, \dots, \tilde{l}_{t,0,p})$ and $\widehat{\Psi}_{t,u,0} = \text{diag}(l_{t,u,0,1}, \dots, l_{t,u,0,p})$ in which

$$\tilde{l}_{t,0,j} = \left\| \left(Y - \nu_t(X) \right) b_j(X) K\left(\frac{T-t}{h}\right) h^{-1/2} \right\|_{P_{n,2}}$$

and

$$l_{t,u,0,j} = \left\| \left(Y_u - \phi_{t,u}(X) \right) b_j(X) K\left(\frac{T-t}{h}\right) h^{-1/2} \right\|_{P_{n,2}},$$

respectively. Since $\nu_t(\cdot)$ and $\phi_{t,u}(\cdot)$ are not known, we propose the following iterative algorithm to obtain the feasible versions of the loading matrices.

Algorithm 3.1 1. Let $\widehat{\Xi}_t^0 = \text{diag}(\tilde{l}_{t,1}^0, \dots, \tilde{l}_{t,p}^0)$ and $\widehat{\Psi}_{t,u}^0 = \text{diag}(l_{t,u,1}^0, \dots, l_{t,u,p}^0)$, where

$\tilde{l}_{t,j}^0 = \|Y b_j(X) K(\frac{T-t}{h}) h^{-1/2}\|_{P_{n,2}}$ and $l_{t,u,j}^0 = \|Y_u b_j(X) K(\frac{T-t}{h}) h^{-1/2}\|_{P_{n,2}}$. Using $\widehat{\Xi}_t^0$ and $\widehat{\Psi}_{t,u}^0$, we can compute $\hat{\gamma}_t^0$ and $\hat{\theta}_{t,u}^0$ by (3.4) and (3.5). Let $\widehat{\nu}_t^0(x) = b(x)' \hat{\gamma}_t^0$ and $\widehat{\phi}_{t,u}^0(x) = \Lambda(b(x)' \hat{\theta}_{t,u}^0)$ for $x = X_1, \dots, X_n$.

2. For $k = 1, \dots, K$ for some fixed positive integer K , we compute $\widehat{\Xi}_t^k = \text{diag}(\tilde{l}_{t,1}^k, \dots, \tilde{l}_{t,p}^k)$ and $\widehat{\Psi}_{t,u}^k = \text{diag}(l_{t,u,1}^k, \dots, l_{t,u,p}^k)$, where

$$\tilde{l}_{t,j}^k = \left\| \left(Y - \widehat{\nu}_t^{k-1}(X) \right) b_j(X) K\left(\frac{T-t}{h}\right) h^{-1/2} \right\|_{P_{n,2}}$$

and

$$l_{t,u,j}^k = \left\| \left(Y_u - \widehat{\phi}_{t,u}^{k-1}(X) \right) b_j(X) K\left(\frac{T-t}{h}\right) h^{-1/2} \right\|_{P_{n,2}}.$$

Using $\widehat{\Xi}_t^k$ and $\widehat{\Psi}_{t,u}^k$, we can compute $\hat{\gamma}_t^k$ and $\hat{\theta}_{t,u}^k$ by (3.4) and (3.5). Let $\widehat{\nu}_t^k(x) = b(x)' \hat{\gamma}_t^k$ and $\widehat{\phi}_{t,u}^k(x) = \Lambda(b(x)' \hat{\theta}_{t,u}^k)$ for $x = X_1, \dots, X_n$. The final penalty loading matrices $\widehat{\Xi}_t^K$ and $\widehat{\Psi}_{t,u}^K$ will be used for $\widehat{\Xi}_t$ and $\widehat{\Psi}_{t,u}$ in (3.4) and (3.5).

Let $\widetilde{\mathcal{S}}_t^\mu$ and $\widetilde{\mathcal{S}}_{t,u}$ contain the supports of $\hat{\gamma}_t$ and $\hat{\theta}_{t,u}$, respectively, such that $|\widetilde{\mathcal{S}}_t^\mu| \lesssim \sup_{t \in \mathcal{T}} \|\hat{\gamma}_t\|_0$, and $|\widetilde{\mathcal{S}}_{t,u}| \lesssim \sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|\hat{\theta}_{t,u}\|_0$. For each $(t, u) \in \mathcal{T}\mathcal{U} \equiv \mathcal{T} \times \mathcal{U}$ where \mathcal{T} and \mathcal{U} are compact subsets of the supports of T and Y , respectively, the post-Lasso estimator of γ_t and $\theta_{t,u}$ based on the set of covariates $\widetilde{\mathcal{S}}_t^\mu$ and $\widetilde{\mathcal{S}}_{t,u}$ are defined as

$$\tilde{\gamma}_t \in \arg \min_{\gamma} \sum_{i=1}^n (Y_i - b(X_i)' \gamma)^2 K\left(\frac{T_i - t}{h}\right), \quad s.t. \quad \text{Supp}(\gamma) \in \widetilde{\mathcal{S}}_t^\mu,$$

and

$$\tilde{\theta}_{t,u} \in \arg \min_{\theta} \sum_{i=1}^n M(1\{Y_i \leq u\}, X_i; \theta) K\left(\frac{T_i - t}{h}\right), \quad s.t. \quad \text{Supp}(\theta) \in \widetilde{\mathcal{S}}_{t,u}.$$

The post-Lasso estimators of $\nu_t(x)$ and $\phi_{t,u}(X)$ are given by $\tilde{\nu}_t(X) = b(X)' \tilde{\gamma}_t$ and $\tilde{\phi}_{t,u}(X) = \Lambda(b(X)' \tilde{\theta}_{t,u})$, respectively.

3.1.2 Penalized Conditional Density Estimation

As in Fan, Yao, and Tong (1996), we can show that $\mathbb{E}(\frac{1}{h} K(\frac{T-t}{h}) | X) = f_t(X) + O(h^2)$ when $K(\cdot)$ is a second order kernel. We approximate $f_t(x)$ by $b(x)' \beta_t$ for some β_t and denote the approximation error as $r_t^f(x) = f_t(x) - b(x)' \beta_t$. Note again that we only approximate the density $f_t(X)$ by a linear regression with the approximation error satisfying Assumption 2 below, but do not necessarily require $f_t(X)$ to be linear in X . The remark in the previous subsection also applies here.

We estimate β_t and $f_t(X)$ by

$$\hat{\beta}_t = \arg \min_{\beta} \left\| \frac{1}{h} K\left(\frac{T-t}{h}\right) - b(X)' \beta \right\|_{P_{n,2}}^2 + \frac{\lambda}{nh} \|\widehat{\Psi}_t \beta\|_1 \quad \text{and} \quad \hat{f}_t(X) = b(X)' \hat{\beta}_t, \quad \text{respectively.} \quad (3.6)$$

Here $\lambda = \ell_n(\log(p \vee n)nh)^{1/2}$ and $\widehat{\Psi}_t = \text{diag}(l_{t,1}, \dots, l_{t,p})$ is a generic penalty loading matrix specified below.

The ideal penalty loading matrix is $\widehat{\Psi}_{t,0} = \text{diag}(l_{t,0,1}, \dots, l_{t,0,p})$ where

$$l_{t,0,j} = h^{1/2} \left\| \left(h^{-1} K\left(\frac{T-t}{h}\right) - f_t(X) \right) b_j(X) \right\|_{P_{n,2}}.$$

Since $f_t(\cdot)$ is not known, we propose to apply the following iterative algorithm to obtain $\widehat{\Psi}_t$.

1. Let $\widehat{\Psi}_t^0 = \text{diag}(l_{t,1}^0, \dots, l_{t,p}^0)$ where $l_{t,j}^0 = \|h^{-1/2} K(\frac{T-t}{h}) b_j(X)\|_{P_{n,2}}$. Using $\widehat{\Psi}_t^0$, we can compute $\widehat{\beta}_t^0$ and $\widehat{f}_t^0(X)$ by the penalized conditional density estimation.
2. For $k = 1, \dots, K$, we compute $\widehat{\Psi}_t^k = \text{diag}(l_{t,1}^k, \dots, l_{t,p}^k)$ where

$$l_{t,j}^k = h^{1/2} \left\| \left(h^{-1} K\left(\frac{T-t}{h}\right) - \widehat{f}_t^{k-1}(X) \right) b_j(X) \right\|_{P_{n,2}}.$$

Using $\widehat{\Psi}_t^k$, we can compute $\widehat{\beta}_t^k$ and $\widehat{f}_t^k(X)$ by the penalized conditional density estimation. The final penalty loading matrix $\widehat{\Psi}_t^K$ will be used for $\widehat{\Psi}_t$ in (3.6).

Let $\widetilde{\mathcal{S}}_t$ contain the support of $\widehat{\beta}_t$ such that $|\widetilde{\mathcal{S}}_t| \lesssim \sup_{t \in \mathcal{T}} \|\widehat{\beta}_t\|_0$. For each $t \in \mathcal{T}$, the post-Lasso estimator of β_t based on the set of covariates $\widetilde{\mathcal{S}}_t$ is defined as

$$\widetilde{\beta}_t \in \arg \min \left\| \frac{1}{h} K\left(\frac{T-t}{h}\right) - b(X)' \beta \right\|_{P_{n,2}}^2, \quad s.t. \quad \text{Supp}(\beta) \in \widetilde{\mathcal{S}}_t.$$

The post-Lasso estimator of $f_t(X)$ is $\widetilde{f}_t(X) = b(X)' \widetilde{\beta}_t$.

3.1.3 Asymptotic Properties of the First Stage Estimators

To study the asymptotic properties of the first stage estimators, we need some assumptions.

Assumption 2 *Uniformly over $(t, u) \in \mathcal{T}\mathcal{U}$,*

1. $\|\max_{j \leq p} |b_j(X)|\|_{P,\infty} \leq \zeta_n$ and $\underline{C} \leq \mathbb{E}b_j(X)^2 \leq 1/\underline{C}$ $j = 1, \dots, p$.

2. $\max(\|\gamma_t\|_0, \|\beta_t\|_0, \|\theta_{t,u}\|_0) \leq s$ for some s which possibly depends on the sample size n .

3. $\|r_t^f(X)\|_{P_n,2} = O_p((s \log(p \vee n)/(nh))^{1/2})$ and

$$\|r_{t,u}^\nu(X)K\left(\frac{T-t}{h}\right)^{1/2}\|_{P_n,2} + \|r_{t,u}^\phi(X)K\left(\frac{T-t}{h}\right)^{1/2}\|_{P_n,2} = O_p((s \log(p \vee n)/n)^{1/2}).$$

4. $\|r_t^f(X)\|_{P,\infty} = O((\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2})$ and

$$\|r_{t,u}^\nu(X)\|_{P,\infty} + \|r_{t,u}^\phi(X)\|_{P,\infty} = O((\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}).$$

5. $\zeta_n^2 s^2 \ell_n^2 \log(p \vee n)/(nh) \rightarrow 0$.

Assumption 2.1 is the same as Assumption 6.1(a) in Belloni et al. (2017a). Assumption 2.2 requires that $\nu_t(x)$, $\phi_{t,u}(x)$, and $f_t(x)$ are approximately sparse, i.e., they can be well-approximated by using at most s elements of $b(x)$. This approximate sparsity condition is common in the literature on high-dimensional data (see, e.g., Belloni et al. (2017a)). Assumption 2.3 and 2.4 specify how well the approximations are in terms of $L_{P_n,2}$ and $L_{P,\infty}$ norms. The exact rates are not the same as those in Belloni et al. (2017a) since the approximations in this paper are local in $T = t$. In the case of nonparametric sieve estimation, Assumptions 2.2 and 2.4 can be verified under some smoothness conditions (see, e.g., Chen (2007)). Assumption 2.5 imposes conditions on the rates at which s , ζ_n , and p grow with sample size n . In particular, we notice that, when all covariates are bounded, i.e., ζ_n is bounded, p can be exponential in n . A similar condition is also imposed in Assumption 6.1(a) in Belloni et al. (2017a).

Assumption 3

1. $K(\cdot)$ is a symmetric probability density function (PDF) with $\int uK(u)du = 0$ and $\int u^2K(u)du < \infty$. There exists a positive constant \bar{C}_K such that $\sup_u u^l K(u) \leq \bar{C}_K$ for $l = 0, 1$.
2. There exists some positive constant $\underline{C} < 1$ such that $\underline{C} \leq f_t(x) \leq 1/\underline{C}$ uniformly over $(t, x) \in \mathcal{TX}$, where \mathcal{TX} is the support of $f_T(X)$ for $T \in \mathcal{T}$.
3. $\nu_t(x)$ and $\phi_{t,u}(x)$ are three times differentiable w.r.t. t , with all three derivatives being bounded uniformly over $(t, x, u) \in \mathcal{TXU}$.
4. For the same \underline{C} as above, $\underline{C} \leq \mathbb{E}(Y_u(t)|X = x) \leq 1 - \underline{C}$ uniformly over $(t, x, u) \in \mathcal{TXU} \equiv \mathcal{TX} \times \mathcal{U}$.

Assumption 3.1 holds for commonly used second-order kernels such as the standard normal PDF. Since $f_T(X)$ was referred to as the generalized propensity by Hirano and Imbens (2004), Assumption 3.2 is analogous to the overlapping support condition commonly assumed in the treatment effect literature; see, e.g., Hirano et al. (2003) and Firpo (2007). Since the conditional density also has the sparsity structure as assumed in Assumption 2, at most s members of X 's affect the conditional density, which makes Assumption 3.2 more plausible. In addition, Assumption 3.2 is only a sufficient condition. In theory, we can allow for the lower bound to decay to zero slowly as the sample size increases. This will affect the convergence rates of our first stage estimators but not the ones in the second and third stages. Assumption 3.3 imposes some smoothness conditions that are widely assumed in nonparametric kernel literature. Assumption 3.4 holds if \mathcal{XU} is a compact subset of the joint support of X and $Y(t)$.

Assumption 4 *There exists a sequence $\ell_n \rightarrow \infty$ such that, with probability approaching one,*

$$0 < \kappa' \leq \inf_{\delta \neq 0, \|\delta\|_0 \leq s\ell_n} \frac{\|b(X)' \delta\|_{P_{n,2}}}{\|\delta\|_2} \leq \sup_{\delta \neq 0, \|\delta\|_0 \leq s\ell_n} \frac{\|b(X)' \delta\|_{P_{n,2}}}{\|\delta\|_2} \leq \kappa'' < \infty.$$

Assumption 4 is the restricted eigenvalue condition commonly assumed in the high-dimensional data literature. We refer interested readers to Bickel, Ritov, and Tsybakov (2009) for more detail and Bühlmann and van de Geer (2011) for a textbook treatment.

Since there is a kernel in the Lasso objective functions in (3.4) and (3.5), the asymptotic properties of $\hat{\gamma}_t$ and $\hat{\theta}_{t,u}$ cannot be established by directly applying the results in Belloni et al. (2017a). The key missing piece is the following local version of the compatibility condition. Let $\mathcal{S}_{t,u}$ be an arbitrary subset of $\{1, \dots, p\}$ such that $\sup_{(t,u) \in \mathcal{TU}} |\mathcal{S}_{t,u}| \leq s$ and $\Delta_{c,t,u} = \{\delta : \|\delta_{\mathcal{S}_{t,u}^c}\|_1 \leq c\|\delta_{\mathcal{S}_{t,u}}\|_1\}$ for some $c < \infty$ independent of (t, u) .

Lemma 3.1 *If Assumptions 1–4 hold, then there exists $\underline{\kappa} = \kappa' \underline{C}^{1/2}/4 > 0$ such that, for n sufficiently large, w.p.a.1,*

$$\inf_{(t,u) \in \mathcal{TU}} \inf_{\delta \in \Delta_{c,t,u}} \frac{\|b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}}{\|\delta_{\mathcal{S}_{t,u}}\|_2 \sqrt{h}} \geq \underline{\kappa}.$$

Note $\mathcal{S}_{t,u}$ in Lemma 3.1 is either the support of $\theta_{t,u}$ or the support of γ_t . For the latter case, the index u is not needed. We refer to Lemma 3.1 as the local compatibility condition because (1) there is a kernel function implementing the localization; and (2) by the Cauchy inequality, Lemma 3.1 implies

$$\inf_{(t,u) \in \mathcal{TU}} \inf_{\delta \in \Delta_{c,t,u}} \frac{s^{1/2} \|b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}}{\|\delta_{\mathcal{S}_{t,u}}\|_1 \sqrt{h}} \geq \underline{\kappa}.$$

Based on Lemma 3.1, we can establish the following asymptotic probability bounds for the first stage estimators.

Theorem 3.1 *Suppose Assumptions 1–2, 3.1–3.3, and 4 hold. Then*

$$\sup_{t \in \mathcal{T}} \|(\widehat{\nu}_t(X) - \nu_t(X))K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}} = O_p(\ell_n(\log(p \vee n)s)^{1/2}n^{-1/2}),$$

$$\sup_{t \in \mathcal{T}} \|\widehat{\nu}_t(X) - \nu_t(X)\|_{P,\infty} = O_p(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}),$$

$$\sup_{t \in \mathcal{T}} \|(\widetilde{\nu}_t(X) - \nu_t(X))K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}} = O_p(\ell_n(\log(p \vee n)s)^{1/2}n^{-1/2}),$$

$$\sup_{t \in \mathcal{T}} \|\widetilde{\nu}_t(X) - \nu_t(X)\|_{P,\infty} = O_p(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}),$$

and $\sup_{t \in \mathcal{T}} \|\widehat{\gamma}_t\|_0 = O_p(s)$. If in addition, Assumption 3.4 holds, then

$$\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|(\widehat{\phi}_{t,u}(X) - \phi_{t,u}(X))K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}} = O_p(\ell_n(\log(p \vee n)s)^{1/2}n^{-1/2}),$$

$$\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|\widehat{\phi}_{t,u}(X) - \phi_{t,u}(X)\|_{P,\infty} = O_p(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}),$$

$$\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|(\widetilde{\phi}_{t,u}(X) - \phi_{t,u}(X))K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}} = O_p(\ell_n(\log(p \vee n)s)^{1/2}n^{-1/2}),$$

$$\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|\widetilde{\phi}_{t,u}(X) - \phi_{t,u}(X)\|_{P,\infty} = O_p(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}),$$

and $\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|\widehat{\theta}_{t,u}\|_0 = O_p(s)$.

Several comments are in order. First, due to the nonlinearity of the logistic link function, Assumption 3.4 is needed for deriving the asymptotic properties of the penalized local MLE estimators $\widehat{\phi}_{t,u}(x)$ and $\widetilde{\phi}_{t,u}(x)$. Second, the $L_{P_{n,2}}$ bounds in Theorem 3.1 are faster than $(nh)^{-1/4}$ by Assumption 5 below. This implies the estimators are sufficiently accurate so that in the second stage, their second and higher order impacts are asymptotically negligible. Last, the numbers of nonzero coordinates of $\widehat{\gamma}_t$ and $\widehat{\theta}_{t,u}$ determine the complexity of our first stage estimators, which are uniformly controlled with a high probability.

For the penalized conditional density estimation, the kernel function will not affect the

bases. Therefore, the usual compatibility condition that

$$\inf_{(t,u) \in \mathcal{TU}} \inf_{\delta \in \Delta_{c,t,u}} \frac{s^{1/2} \|b(X)' \delta\|_{P_{n,2}}}{\|\delta_{\mathcal{S}_{t,u}}\|_1} \geq \underline{\kappa}$$

suffices and we obtain the following results.

Theorem 3.2 *Suppose Assumptions 1–2, 3.1–3.3, and 4 hold. Then*

$$\sup_{t \in \mathcal{T}} \|\hat{f}_t(X) - f_t(X)\|_{P_{n,2}} = O_p(\ell_n(\log(p \vee n)s/(nh))^{1/2}),$$

$$\sup_{t \in \mathcal{T}} \|\hat{f}_t(X) - f_t(X)\|_{P,\infty} = O_p(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}),$$

$$\sup_{t \in \mathcal{T}} \|\tilde{f}_t(X) - f_t(X)\|_{P_{n,2}} = O_p(\ell_n(\log(p \vee n)s/(nh))^{1/2}),$$

$$\sup_{t \in \mathcal{T}} \|\tilde{f}_t(X) - f_t(X)\|_{P,\infty} = O_p(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}),$$

and $\sup_{t \in \mathcal{T}} \|\hat{\beta}_t\|_0 = O_p(s)$.

The penalized conditional density estimation is similar to the common linear Lasso method. The key difference is that the dependent variable here, namely, $\frac{1}{h}K(\frac{T-t}{h})$, is $O_p(h^{-1})$. This affects the rates of our first stage density estimators.

3.2 The Second Stage Estimation

Let $W = \{Y, T, X\}$ and $W_u = \{Y_u, T, X\}$. For three generic functions $\check{\nu}(\cdot)$, $\check{\phi}(\cdot)$ and $\check{f}(\cdot)$ of X , denote

$$\Pi'_t(W, \check{\nu}, \check{f}) = \frac{(Y - \check{\nu}(X))}{\check{f}(X)h} K\left(\frac{T-t}{h}\right) + \check{\nu}(X)$$

and

$$\Pi_{t,u}(W_u, \check{\phi}, \check{f}) = \frac{(Y_u - \check{\phi}(X))}{\check{f}(X)h} K\left(\frac{T-t}{h}\right) + \check{\phi}(X).$$

Then the estimators $\hat{\mu}(t)$ and $\hat{\alpha}(t, u)$ can be written as

$$\hat{\mu}(t) = \mathcal{P}_n \Pi'_t(W, \bar{\nu}_t, \bar{f}_t) \quad \text{and} \quad \hat{\alpha}(t, u) = \mathcal{P}_n \Pi_{t,u}(W_u, \bar{\phi}_{t,u}, \bar{f}_t),$$

where $\bar{\nu}_t(\cdot)$, $\bar{\phi}_{t,u}(\cdot)$, and $\bar{f}_t(\cdot)$ are either the Lasso estimators (i.e., $\hat{\nu}_t(\cdot)$, $\hat{\phi}_{t,u}(\cdot)$, and $\hat{f}_t(\cdot)$) or the post-Lasso estimators (i.e., $\tilde{\nu}_t(\cdot)$, $\tilde{\phi}_{t,u}(\cdot)$, and $\tilde{f}_t(\cdot)$) as defined in Section 3.1.

Assumption 5 *Let $h = C_h n^{-H}$ for some positive constant C_h .*

1. $H \in (1/5, 1/3)$ and $\ell_n^4 s^2 \log(p \vee n)^2 / (nh) \rightarrow 0$.
2. $H \in (1/4, 1/3)$ and $\ell_n^4 s^2 \log(p \vee n)^2 / (nh^2) \rightarrow 0$.

Assumption 5.1 imposes conditions on the bandwidth sequence. As usual, we apply the undersmoothing bandwidth to ensure that the bias term from the first stage kernel estimation does not affect the asymptotic distribution of the last stage estimators. The second condition in Assumption 5.1 is comparable to the corresponding condition in Assumption 6.1 in Belloni et al. (2017a) up to some $\log(n)$ term when nh is replaced with n . Since our objective function is local in $T = t$, the effective sample size for us is nh instead of n . Assumption 5.2 is needed to derive a tighter bound of the remainder term.

Theorem 3.3 *Suppose Assumptions 1–4 and 5.1 hold. Then*

$$\hat{\mu}(t) - \mu(t) = (\mathcal{P}_n - \mathcal{P}) \Pi'_t(W, \nu_t, f_t) + R'_n(t)$$

and

$$\hat{\alpha}(t, u) - \alpha(t, u) = (\mathcal{P}_n - \mathcal{P}) \Pi_{t,u}(W_u, \phi_{t,u}, f_t) + R_n(t, u),$$

where $\sup_{t \in \mathcal{T}} R'_n(t) = o_p((nh)^{-1/2})$ and $\sup_{(t,u) \in \mathcal{T}\mathcal{U}} R_n(t, u) = o_p((nh)^{-1/2})$. If Assumption 5.1 is replaced by Assumption 5.2, then $\sup_{t \in \mathcal{T}} R'_n(t) = o_p(n^{-1/2})$ and $\sup_{(t,u) \in \mathcal{T}\mathcal{U}} R_n(t, u) = o_p(n^{-1/2})$.

Theorem 3.3 presents the Bahadur representations of the nonparametric estimator $\hat{\mu}(t)$ and $\hat{\alpha}(t, u)$ with a uniform control on the remainder terms. For most purposes (e.g., to obtain the asymptotic distributions of these intermediate estimators or to obtain the results below), Assumption 5.1 is sufficient. Occasionally, one needs to impose Assumption 5.2 to have a better control on the remainder terms, say, when one conducts an L_2 -type specification test. See the remark after Theorem 3.4 below.

3.3 The Third Stage Estimation

Recall that $q_\tau(t)$ denotes the τ -th quantile of $Y(t)$ which is the inverse of $\alpha(t, u)$ w.r.t. u . We propose to estimate $q_\tau(t)$ by $\hat{q}_\tau(t)$ where $\hat{q}_\tau(t) = \inf\{u : \hat{\alpha}^r(t, u) \geq \tau\}$, where $\hat{\alpha}^r(t, u)$ is the rearrangement of $\hat{\alpha}(t, u)$.

We rearrange $\hat{\alpha}(t, u)$ to make it monotonically increasing in $u \in \mathcal{U}$. Following Chernozhukov et al. (2010), we define $\bar{Q} = Q \circ \psi^{\leftarrow}$ where ψ is any increasing bijective mapping: $\mathcal{U} \mapsto [0, 1]$. Then the rearrangement \bar{Q}^r of \bar{Q} is defined as

$$\bar{Q}^r(u) = F^{\leftarrow}(u) = \inf\{y : F(y) \geq u\},$$

where $F(y) = \int_0^1 1\{\bar{Q}(u) \leq y\} du$. Then the rearrangement Q^r for Q is $Q^r = \bar{Q}^r \circ \psi(u)$.

The rearrangement and inverse are two functionals operating on the process

$$\{\hat{\alpha}(t, u) : (t, u) \in \mathcal{TU}\}$$

and are shown to be Hadamard differentiable by Chernozhukov et al. (2010) and van der Vaart and Wellner (1996), respectively. However, by Theorem 3.3,

$$\sup_{(t,u) \in \mathcal{TU}} (nh)^{1/2}(\hat{\alpha}(t, u) - \alpha(t, u)) = O_p(\log^{1/2}(n)),$$

which is not asymptotically tight. Therefore, the standard functional delta method used in Chernozhukov et al. (2010) and van der Vaart and Wellner (1996) is not directly applicable. The next theorem overcomes this difficulty and establishes the linear expansion of the quantile estimator. Denote \mathcal{TI} , $\{q_\tau(t) : \tau \in \mathcal{I}\}^\varepsilon$, $\overline{\{q_\tau(t) : \tau \in \mathcal{I}\}^\varepsilon}$, and \mathcal{U}_t as $\mathcal{T} \times \mathcal{I}$, the ε -enlarged set of $\{q_\tau(t) : \tau \in \mathcal{I}\}$, the closure of $\{q_\tau(t) : \tau \in \mathcal{I}\}^\varepsilon$, and the projection of \mathcal{TU} on $T = t$, respectively.

Theorem 3.4 *Suppose that Assumptions 1–4 and 5.1 hold. If $\overline{\{q_\tau(t) : \tau \in \mathcal{I}\}^\varepsilon} \subset \mathcal{U}_t$ for any $t \in \mathcal{T}$, then*

$$\hat{q}_\tau(t) - q_\tau(t) = -(\mathcal{P}_n - \mathcal{P})\Pi_{t,u}(W_{q_\tau(t)}, \phi_{t,q_\tau(t)}, f_t) / f_{Y(t)}(q_\tau(t)) + R_n^q(t, \tau),$$

where $f_{Y(t)}$ is the density of $Y(t)$ and $\sup_{(t,\tau) \in \mathcal{TI}} R_n^q(t, \tau) = o_p((nh)^{-1/2})$. If Assumption 5.1 is replaced by Assumption 5.2, then $\sup_{(t,\tau) \in \mathcal{TI}} R_n^q(t, \tau) = o_p(n^{-1/2})$.

Under Assumption 5.2, the remainder term $R_n^q(t, \tau)$ is $o_p(n^{-1/2})$ uniformly in $(t, \tau) \in \mathcal{TI}$. This result is needed if one wants to establish an L_2 -type specification test of $q_\tau(t)$. For example, one may be interested in testing the null hypotheses of the quantile partial derivative being homogeneous across treatment. In this case, the null hypothesis can be written as

$$H_0 : q_\tau(t) = \beta_0(\tau) + \beta_1(\tau)t \text{ for all } (t, \tau) \in \mathcal{TI},$$

and the alternative hypothesis is the negation of H_0 . One way to conduct a consistent test for the above hypothesis is to employ the residuals of the linear regression of $\hat{q}_\tau(T_i)$ on T_i to construct the test statistic $\Upsilon_n(\tau)$, i.e.,

$$\Upsilon_n(\tau) = \frac{1}{n} \sum_{i=1}^n (\hat{q}_\tau(T_i) - \hat{\beta}_0 - \hat{\beta}_1 T_i)^2 1\{T_i \in \mathcal{T}\},$$

where $(\hat{\beta}_0, \hat{\beta}_1)$ are the linear coefficient estimators. This type of specification test has been

previously studied by Su and Chen (2013), Lewbel, Lu, and Su (2015), Su, Jin, and Zhang (2015), Hoderlein, Su, White, and Yang (2016), and Su and Hoshino (2016) in various contexts. One can follow them and apply the results in Theorem 3.4 to study the asymptotic distribution of $\Upsilon_n(\tau)$ for each τ . In addition, one can also consider either an integrated or a sup-version of $\Upsilon_n(\tau)$ and then study its asymptotic properties. For brevity we do not study such a specification test in this paper.

Given the estimator $\hat{\mu}(t)$ and $\hat{q}_\tau(t)$, we can run local linear regressions of $\hat{\mu}(T_i)$ and $\hat{q}_\tau(T_i)$ on $(1, T_i - t)$ and obtain estimators $\check{\beta}^1(t)$ and $\hat{\beta}_\tau^1(t)$ of $\partial\mu(t)$ and $\partial_t q_\tau(t)$, respectively, as estimators of the linear coefficients in the local linear regression.² Specifically, we define

$$(\check{\beta}^0(t), \check{\beta}^1(t)) = \arg \max_{\beta^0, \beta^1} (\hat{\mu}(T_i) - \beta^0 - \beta^1(T_i - t))^2 K\left(\frac{T_i - t}{h}\right)$$

and

$$(\hat{\beta}_\tau^0(t), \hat{\beta}_\tau^1(t)) = \arg \max_{\beta^0, \beta^1} (\hat{q}_\tau(T_i) - \beta^0 - \beta^1(T_i - t))^2 K\left(\frac{T_i - t}{h}\right).$$

The following theorem shows the asymptotic properties of $\check{\beta}^1(t)$ and $\hat{\beta}_\tau^1(t)$. The asymptotic property of $\check{\beta}^1(t)$ can be established in the same manner.

Theorem 3.5 *Suppose Assumptions 1–4 and 5.1 hold. If $\overline{\{q_\tau(t) : \tau \in \mathcal{I}\}^\varepsilon} \subset \mathcal{U}_t$ for any $t \in \mathcal{T}$, then*

$$\check{\beta}^1(t) - \partial_t \mu(t) = \frac{1}{n} \sum_{j=1}^n (\kappa_2 f_t(X_j) h^2)^{-1} \left[Y_j - \nu_t(X_j) \right] \overline{K}\left(\frac{T_j - t}{h}\right) + \check{R}_n^1(t)$$

and

$$\hat{\beta}_\tau^1(t) - \partial_t q_\tau(t) = \frac{1}{n} \sum_{j=1}^n (\kappa_2 f_{Y(t)}(q_\tau(t)) f_t(X_j) h^2)^{-1} \left[Y_{q_\tau(t), j} - \phi_{t, q_\tau(t)}(X_j) \right] \overline{K}\left(\frac{T_j - t}{h}\right) + R_n^1(t, \tau),$$

²Alternatively, one can consider the local quadratic or cubic regression.

where $\sup_{t \in \mathcal{T}} |\check{R}_n^1(t)| + \sup_{(t, \tau) \in \mathcal{T}\mathcal{I}} |R_n^1(t, \tau)| = o_p((nh^3)^{-1/2})$ and $\bar{K}(v) = \int wK(v-w)K(w)dw$.

Theorem 3.5 presents the Bahadur representations for $\check{\beta}^1$ and $\hat{\beta}_\tau^1(t)$. Since they are estimators for the first order derivatives $\partial_t \mu(t)$ and $\partial_t q_\tau(t)$, respectively, we can show that they converge to the true values at the usual $(nh^3)^{1/2}$ -rate.

4 Inference

In this section, we study the inference for $\mu(t)$, $q_\tau(t)$, and $\partial_t q_\tau(t)$. We follow the lead of Belloni et al. (2017a) and consider the weighted bootstrap inference. Let $\{\eta_i\}_{i=1}^n$ be a sequence of i.i.d. random variables generated from the distribution of η such that it has sub-exponential tails and unit mean and variance.³ For example, η can be a standard exponential random variable or a normal random variable with unit mean and standard deviation. We conduct the bootstrap inference based on the following procedure.

1. Obtain $\hat{\nu}_t(x)$, $\hat{\phi}_{t,u}(x)$, $\hat{f}_t(x)$, $\tilde{\nu}_t(x)$, $\tilde{\phi}_{t,u}(x)$ and $\tilde{f}_t(x)$ from the first stage.
2. For the b -th bootstrap sample:
 - Generate $\{\eta_i\}_{i=1}^n$ from the distribution of η .
 - Compute $\hat{\mu}^b(t) \equiv \frac{1}{n} \sum_{i=1}^n \eta_i \Pi'_t(W_i, \bar{\nu}_t, \bar{f}_t)$ and $\hat{\alpha}^b(t, u) \equiv \frac{1}{n} \sum_{i=1}^n \eta_i \Pi_{t,u}(W_{ui}, \bar{\phi}_{t,u}, \bar{f}_t)$, where $(\bar{\phi}_{t,u}(\cdot), \bar{f}_t(\cdot))$ are either $(\hat{\phi}_{t,u}(\cdot), \hat{f}_t(\cdot))$ or $(\tilde{\phi}_{t,u}(\cdot), \tilde{f}_t(\cdot))$.
 - Rearrange $\hat{\alpha}^b(t, u)$ and obtain $\hat{\alpha}^{br}(t, u)$.
 - Invert $\hat{\alpha}^{br}(t, u)$ w.r.t. u and obtain $\hat{q}_\tau^b(t) = \inf\{u : \hat{\alpha}^{br}(t, u) \geq \tau\}$.
 - Compute $\check{\beta}^{b1}(t)$ and $\hat{\beta}_\tau^{b1}(t)$ as the slope coefficients of local linear regressions of $\eta_i \hat{\mu}^b(T_i)$ on $(\eta_i, \eta_i(T_i - t))$ and $\eta_i \hat{q}_\tau^b(T_i)$ on $(\eta_i, \eta_i(T_i - t))$, respectively.

³A random variable η has sub-exponential tails if $P(|\eta| > x) \leq K \exp(-Cx)$ for every x and some constants K and C .

3. We repeat the above step for $b = 1, \dots, B$ and obtain a bootstrap sample of

$$\{\hat{\mu}^b(t), \hat{q}_\tau^b(t), \check{\beta}^{b1}(t), \hat{\beta}_\tau^{b1}(t)\}_{b=1}^B.$$

4. Obtain $\hat{Q}^\mu(\alpha)$, $\hat{Q}^0(\alpha)$, $\hat{Q}^{\mu^1}(\alpha)$, and $\hat{Q}^1(\alpha)$ as the α -th quantile of the sequences $\{\hat{\mu}^b(t) - \hat{\mu}(t)\}_{b=1}^B$, $\{\hat{q}_\tau^b(t) - \hat{q}_\tau(t)\}_{b=1}^B$, $\{\check{\beta}^{b1}(t) - \check{\beta}^1(t)\}_{b=1}^B$, and $\{\hat{\beta}_\tau^{b1}(t) - \hat{\beta}_\tau^1(t)\}_{b=1}^B$, respectively.

The following theorem summarizes the main results in this section.

Theorem 4.1 *Suppose that Assumptions 1–4 and 5.1 hold. Then*

$$P(\hat{Q}^\mu(\alpha/2) \leq \mu(t) \leq \hat{Q}^\mu(1 - \alpha/2)) \rightarrow 1 - \alpha,$$

$$P(\hat{Q}^0(\alpha/2) \leq q_\tau(t) \leq \hat{Q}^0(1 - \alpha/2)) \rightarrow 1 - \alpha,$$

$$P(\hat{Q}^{\mu^1}(\alpha/2) \leq \partial_t \mu(t) \leq \hat{Q}^{\mu^1}(1 - \alpha/2)) \rightarrow 1 - \alpha,$$

and

$$P(\hat{Q}^1(\alpha/2) \leq \partial_t q_\tau(t) \leq \hat{Q}^1(1 - \alpha/2)) \rightarrow 1 - \alpha.$$

Theorem 4.1 says that the $100(1 - \alpha)\%$ bootstrap confidence intervals for $\mu(t)$, $q_\tau(t)$, $\partial_t \mu(t)$, and $\partial_t q_\tau(t)$ have the correct asymptotic coverage probability $1 - \alpha$. With more complicated notations and arguments, we can show that the convergences hold uniformly in (t, τ) .

5 Monte Carlo Simulations

This section presents the results of Monte Carlo simulations, which demonstrate the finite sample performance of the estimation and inference procedure. We modify the simulation

designs of Belloni et al. (2017a) to the case with a continuous treatment. In particular, Y is generated via

$$Y = \Lambda \left(U - 2 \left(\sum_{j=1}^p j^{-2} X_j - \pi^2/12 \right) + (T/2)^2 \right)$$

while T solves the following equation:

$$V = (T/2 + 0.1 \left(\sum_{j=1}^p j^{-2} X_j \right) (0.5 - \cos(\pi T/2)/2 - T/2)),$$

where U and V are two independent standard logistic random variables, $\Lambda(\cdot)$ is the logistic CDF, $p = 200$, X is a p -dimensional normal random variables with mean $\mathbf{0}$ and covariance $[0.2^{|j-k|}]_{jk}$. T ranges from 0 to 2. The parameters of interest are $q_\tau(t)$ and $\partial_t q_\tau(t)$, where $\tau = 0.25, 0.5, 0.75$ and $t \in (0.2, 1.8)$.

We have two tuning parameters: h and ℓ_n . First, we build our rule-of-thumb h based on $h_{rt}(\tau)$, which is the rule-of-thumb bandwidth for the local τ -th quantile regression suggested by Yu and Jones (1998). In particular, $h_{rt}(\tau) = C(\tau) \times 1.08 \times sd(T) \times n^{-1/5}$, where $C(\tau)$ is a constant dependent on τ , and $C(0.5) = 1.095$ and $C(0.25) = C(0.75) = 1.13$. We refer interested readers to (Yu and Jones, 1998, Table 1) for more detail on $C(\tau)$. In order to achieve under-smoothing, we define $h = n^{-1/10} \times h_{rt}$, where our choice of the factor $n^{-1/10}$ follows Cai and Xiao (2012, p.418). For ℓ_n , we use $\ell_n = 0.75\sqrt{\log(\log(n))}$ for the penalized conditional density estimation and $\ell_n = 0.5\sqrt{\log(\log(n))}$ for the penalized local MLE. Based on our simulation experiences, the choice of ℓ_n does have an impact on the first stage variable selections, but does not significantly affect the finite sample performances of the second and third stage estimators.

We repeat the bootstrap inference 500 times. The sample size we use is $n = 1,000$.

Although the sample size is quite large compared to p , in this DGP, the bandwidth is as small as 0.083, which leads to an effective sample size of $nh \approx 83 \ll 200$. In fact, we obtained warning signs of potential multi-collinearity and were unable to estimate the model when implementing the traditional estimation procedures without variable selection (i.e., without penalization).

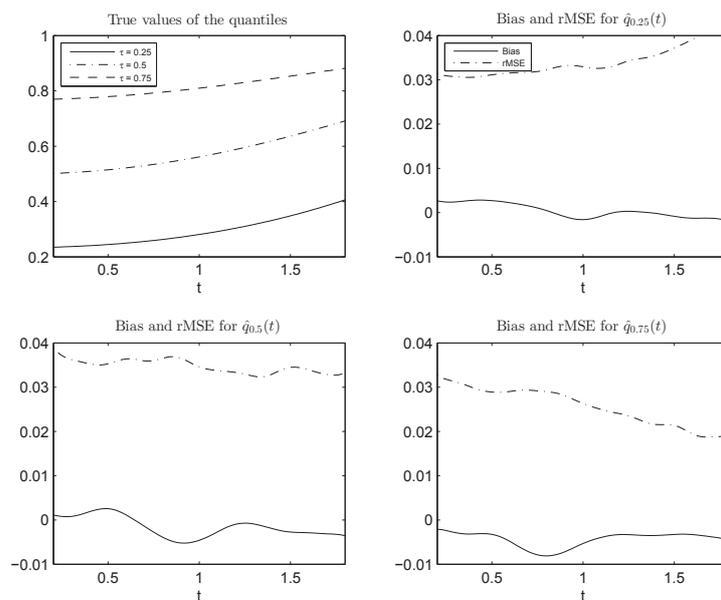


Figure 1: Finite sample performances of $\hat{q}_r(t)$

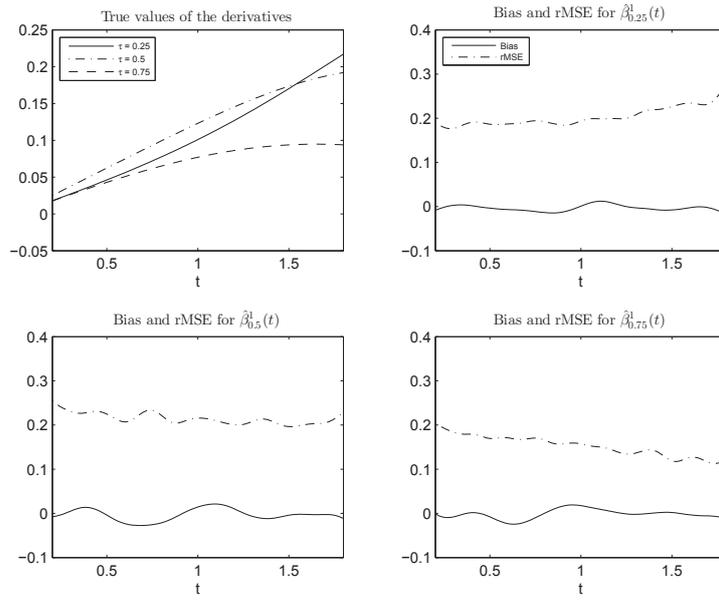


Figure 2: Finite sample performances of $\hat{\beta}_\tau^1(t)$

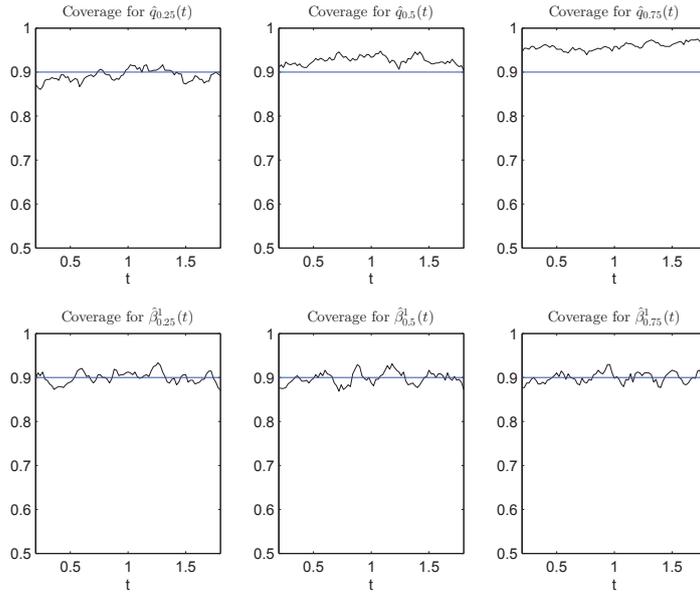


Figure 3: Coverage probability

The upper-right subplot of Figures 1 and 2 report the true functions of $q_\tau(t)$ and $\partial_t q_\tau(t)$ for $\tau = 0.25, 0.5, 0.75$ and $t \in (0.2, 1.8)$. Both $q_\tau(t)$ and $\partial_t q_\tau(t)$ are heterogeneous across τ and t , which imposes difficulties for estimation and inference. The rest of the subplots in

Figures 1 and 2 show that all the biases of our estimators are of smaller order of magnitude of the root mean squared error (RMSE), which indicates the doubly robust moments effectively remove the selection bias induced by the Lasso method. The estimators of the quantile functions are very accurate. The estimators of the quantile partial derivatives are less so because they have slower convergence rates. Figure 3 shows the 90% bootstrap confidence intervals have reasonable performances for both the quantile functions and their derivatives, across all τ and t values considered, with slight over-covers for the quantile function when $\tau = 0.75$. The results of variable selections depend on the values of t and (t, u) for the penalized conditional density estimation and penalized local MLE, respectively, which are tedious to report, and thus are omitted for brevity. However, overall, about 7 covariates for the density estimations and 1 to 8 covariates for the MLE method are selected.

6 Empirical Illustration

To investigate our proposed estimation and inference procedures, we use the 1979 National Longitudinal Survey of Youth (NLSY79) and consider the effect of father's income on son's income in the presence of many control variables. Our analysis is based on Bhattacharya and Mazumder (2011). The data consists of a nationally representative sample of individuals with age 14-22 years old as of 1979. We use only white and black males and discard the individuals with missing values in the covariates we use. The resulting sample size is 1,795, out of which 1,272 individuals are white and 523 individuals are black.

The treatment variable of interest is the logarithm of father's income, in which father's income is computed as the average family income for 1978, 1979, and 1980. The outcome variable is the logarithm of son income, in which son income is computed as the average family income for 1997, 1999, 2001 and 2003. We create control variables by interacting a list of demographic variables with the cubic splines of the AFQT score and the years of

education. The list includes the mother’s education level, the father education level, the indicators of (i) living in urban areas at age 14, (ii) living in the south, (iii) speaking a foreign language at childhood, and (iv) being born outside the U.S. The resulting number of control variables is 112.

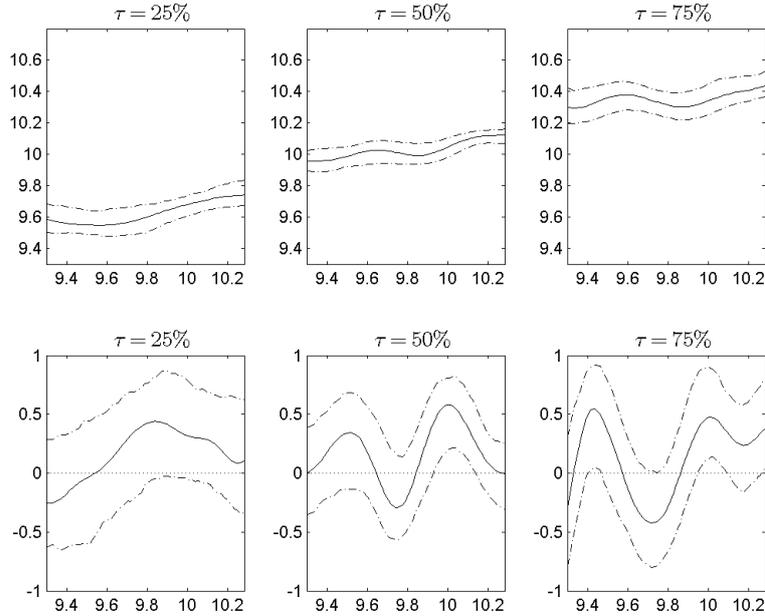


Figure 4: Whites. First row: the father’s log income (X-axis), the son’s log income (Y-axis), the estimated unconditional quantile function at τ (solid line), and its 90% confidence interval (dot-dash line). Second row: the father’s log income (X-axis), the intergenerational elasticity (Y-axis), the estimated derivative of the unconditional quantile function at τ (solid line), and its 90% confidence interval (dot-dash line).

We apply the proposed estimation and inference procedures for black and white individuals separately. We use the same bandwidth choices as in the previous section, such that our effective sample size is $nh = 106$ for whites and $nh = 60$ for blacks. The number of control variables, 112, is larger than the effective sample size. Figures 4 and 5 show the estimated unconditional quantile functions and the estimated derivative, as well as the point-wise confidence intervals for $\tau = 25\%$, 50% , and 75% . Under the context of intergenerational income mobility, the unconditional quantile and its derivative represent the

quantile of son’s potential log income indexed by father’s log income and the intergenerational elasticity, respectively. The unconditional quantile functions have a slight upward trend and the estimated derivative is positive in most part of father’s log income. The confidence intervals for the unconditional quantile functions are quite narrow. However, we cannot reject the (locally) zero intergenerational elasticity for most of the values of father’s log income, except for the whites with father’s log income around 10 and $\tau = 50\%$ and 75% . This is considered as the cost of our fully nonparametric specification.

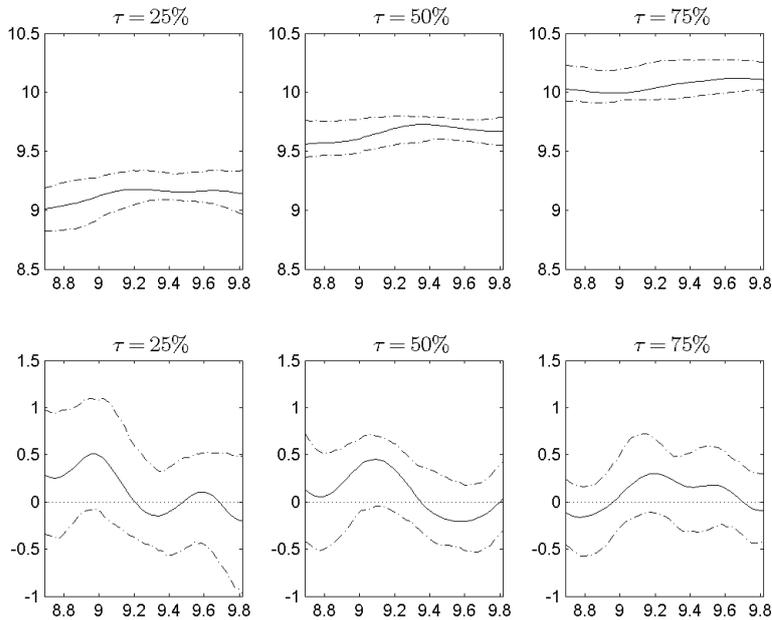


Figure 5: Blacks. First row: the father’s log income (X-axis), the son’s log income (Y-axis), the estimated unconditional quantile function at τ (solid line), and its 90% confidence interval (dot-dash line). Second row: the father’s log income (X-axis), the intergenerational elasticity (Y-axis), the estimated derivative of the unconditional quantile function at τ (solid line), and its 90% confidence interval (dot-dash line).

It is worthwhile to mention the variable selection in this application. For whites, on average, about 7 and 5 control variables are selected for the density estimations and the penalized local MLE, respectively. The indicator of speaking a foreign language at childhood and the (linear term of) years of education are the two leading control variables selected. For

blacks, on average, about 10 and 7 control variables are selected for the density estimations and the penalized local MLE, respectively. The interaction of the indicator of speaking a foreign language at childhood and (the quadratic term of) the years of education and the indicator of being born outside the U.S. are the two leading variables selected.

7 Conclusion

This paper studies non-separable models with a continuous treatment and high-dimensional control variables. It extends the existing results on the causal inference in non-separable models to the case with both continuous treatment and high-dimensional covariates. It develops a method based on localized L_1 -penalization to select covariates at each value of the continuous treatment. It then proposes a multi-stage estimation and inference procedure for average, quantile, and marginal treatment effects. The simulation and empirical exercises support the theoretical findings in finite samples.

Appendix

A Proof of the Main Results in the Paper

Before proving the theorem, we first introduce some additional notation and Assumption 6, which is a restatement of Sasaki (2015, Assumptions 1 and 2) in our framework. Denote by \dim_X (resp. \dim_A) the dimensionality of X (resp. A). We define $\partial V(y, t) = \{(x, a) : \Gamma(t, x, a) = y\}$ and $\partial V(y, t)$ can be parametrized as a mapping from a $(\dim_X + \dim_A - 1)$ -dimensional rectangle, denoted by Σ , to $\partial V(y, t)$. $H^{\dim_X + \dim_A - 1}$ is the $(\dim_X + \dim_A - 1)$ -dimensional Hausdorff measure restricted from $\mathbb{R}^{\dim_X + \dim_A}$ to $(\partial V(y, t), \mathcal{B}(y, t))$, where $\mathcal{B}(y, t)$ is the set of the interactions between $\partial V(y, t)$ and a Borel set in $\mathbb{R}^{\dim_X + \dim_A}$. $\partial v(y, \cdot; u)/\partial y$ (resp. $\partial v(\cdot, t; u)/\partial t$) is the velocity of $\partial V(y, t)$ at u with respect to y (resp. t).

Assumption 6 1. Γ is continuously differentiable.

2. $\|\nabla_{(x,a)}\Gamma(t, \cdot, \cdot)\| \neq 0$ on $\partial V(y, x)$.

3. The conditional distribution of (X, A) given T is absolutely continuous with respect to the Lebesgue measure, and $f_{(X,A)|T}$ is a continuously differentiable function of \mathcal{T} to $L^1(\mathbb{R}^{\dim_X + \dim_A})$.

4. $\int_{\partial V(y,t)} f_{(X,A)|T} dH^{\dim_X + \dim_A - 1}(u) > 0$.

5. $t \mapsto \partial V(y, t)$ is a continuously differentiable function of $\Sigma \times \mathcal{T}$ to $\mathbb{R}^{\dim_X + \dim_A}$ for every y and $y \mapsto \partial V(y, t)$ is a continuously differentiable function of $\Sigma \times \mathcal{Y}$ to $\mathbb{R}^{\dim_X + \dim_A}$ for every t .

6. The mapping $\partial v(y, \cdot; \cdot)/\partial t$ is a continuously differentiable function of \mathcal{T} to $\mathbb{R}^{\dim_X + \dim_A}$ and $\partial v(\cdot, t; \cdot)/\partial y$ is a continuously differentiable function of \mathcal{Y} to $\mathbb{R}^{\dim_X + \dim_A}$.

7. There is $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that the mapping $(x, a) \mapsto \|\nabla_{(x,a)}\Gamma(t, x, a)\|^{-1}$ is bounded in $L^p(\partial V(y, t), H^{\dim_X + \dim_A})$ and that the mapping $(x, a) \mapsto f_{(X,A)=(x,a)}$ is bounded in $L^q(\partial V(y, t), H^{\dim_X + \dim_A})$.

Assumption 6 is a combination of Assumptions 1 and 2 in Sasaki (2015). We refer the readers to the latter paper for detailed explanation.

Proof of Theorem 2.1. For the marginal distribution of $Y(t)$, we note that, by Assumption 1, $P(Y(t) \leq u) = \mathbb{E}[\mathbb{E}(1\{Y(t) \leq u\}|X, T = t)]$. The first result follows as $\mathbb{E}(1\{Y(t) \leq u\}|X, T = t)$ is identified.

For the second result, consider a random variable T^* which has the same marginal distribution as T and is independent of (X, A) . Define

$$Y^* = \Gamma(T^*, X, A). \tag{A.1}$$

Note that the (i) (X, A) and T^* are independent, and (ii) the τ -th quantile of Y^* given $T^* = t$ is $q_\tau(t)$ for all t , because $P(Y^* \leq q_\tau(t) | T^* = t) = P(\Gamma(t, X, A) \leq q_\tau(t)) = \tau$. Assumption 6 implies Assumptions 1 and 2 in Sasaki (2015) for (Y^*, T^*, U^*) with $U^* = (X, A)$, and then his Theorem 1 implies that the derivative of the τ -th quantile of Y^* given $T^* = t$ is equal to $\mathbb{E}_{\mu_{\tau,t}}[\partial_t \Gamma(t, X, A)]$.

■

Lemma 3.1 is the local version of the compatibility condition, which is one of the key building blocks for Lemma A.1. Then, Lemma A.1 is used to prove Theorem 3.1.

Proof of Lemma 3.1. By Assumption 4, we can work on the set

$$\left\{ \{X\}_{i=1}^n : \sup_{|\delta|_0 \leq s\ell_n} \frac{\|b(X)'\delta\|_{P_n,2}}{\|\delta\|_2} \leq \kappa'' < \infty. \right\}$$

We use the same partition as in Bickel et al. (2009). Let $\mathcal{S}_0 = \mathcal{S}_{t,u}$ and $m \geq s$ be an integer which will be specified later. Partition $\mathcal{S}_{t,u}^c$, the complement of $\mathcal{S}_{t,u}$, as $\sum_{l=1}^L \mathcal{S}_l$ such that $|\mathcal{S}_l| = m$ for $1 \leq l < L$, $|\mathcal{S}_L| \leq m$, where \mathcal{S}_l , for $l < L$, contains the indices corresponding to m largest

coordinates (in absolute value) of δ outside $\cup_{j=0}^{l-1} \mathcal{S}_j$ and \mathcal{S}_L collects the remaining indices. Further denote $\delta_j = \delta_{\mathcal{S}_j}$ and $\delta_{01} = \delta_{\mathcal{S}_0 \cup \mathcal{S}_1}$. Then

$$\|b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}} \geq \|b(X)' \delta_{01} K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}} - \sum_{l=2}^L \|b(X)' \delta_l K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}. \quad (\text{A.2})$$

For the first term on the right hand side (r.h.s.) of (A.2), we have

$$\begin{aligned} & \|b(X)' \delta_{01} K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}^2 \\ & \geq \|b(X)' \delta_{01} K(\frac{T-t}{h})^{1/2}\|_{P,2}^2 - |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2 K(\frac{T-t}{h})| \\ & \geq \underline{C}h \|b(X)' \delta_{01}\|_{P,2}^2 - |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2 K(\frac{T-t}{h})| \quad (\text{A.3}) \\ & \geq \underline{C}h \|b(X)' \delta_{01}\|_{P_{n,2}}^2 - \underline{C}h |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2| - |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2 K(\frac{T-t}{h})| \\ & \geq \underline{C}h \|\delta_{01}\|_2^2 (\kappa')^2 - \underline{C}h |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2| - |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2 K(\frac{T-t}{h})| \end{aligned}$$

where the second inequality holds because

$$\mathbb{E}(b(X)' \delta_{01})^2 K(\frac{T-t}{h}) = h \mathbb{E}(b(X)' \delta_{01})^2 \int f_{t+hv}(X) K(v) dv \geq \underline{C}h \mathbb{E}(b(X)' \delta_{01})^2.$$

We next bound the last term on the r.h.s. of (A.2). The second term can be bounded in the same manner. Let $\tilde{\delta}_{01} = \delta_{01} / \|\delta_{01}\|_2$. Then we have

$$|(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2 K(\frac{T-t}{h})| = \|\delta_{01}\|_2^2 |(\mathcal{P}_n - \mathcal{P})(b(X)' \tilde{\delta}_{01})^2 K(\frac{T-t}{h})|.$$

Let $\{\eta_i\}_{i=1}^n$ be a sequence of Rademacher random variables which is independent of the data and $\mathcal{F} = \{\eta b(X)' \delta K(\frac{T-t}{h}) : \|\delta\|_0 = m + s, \|\delta\|_2 = 1, t \in \mathcal{T}\}$ with envelope $F = \overline{C}_K \zeta_n(m + s)^{1/2}$.

Denote π_{1n} as $(\frac{\log(p \vee n)(s+m)^2 \zeta_n^2}{nh})^{1/2}$ with $m = s\ell_n^{1/2}$. Then,

$$\begin{aligned}
& \mathbb{E} \sup_{\|\tilde{\delta}_{01}\|_0 \leq m+s, \|\tilde{\delta}_{01}\|_2=1, t \in \mathcal{T}} |(\mathcal{P}_n - \mathcal{P})(b(X)' \tilde{\delta}_{01})^2 K(\frac{T-t}{h})| \\
& \leq 2\mathbb{E} \sup_{\|\tilde{\delta}_{01}\|_0 \leq m+s, \|\tilde{\delta}_{01}\|_2=1, t \in \mathcal{T}} |\mathcal{P}_n \eta(b(X)' \tilde{\delta}_{01})^2 K(\frac{T-t}{h})| \\
& \leq 2\zeta_n \left(\sup_{\|\tilde{\delta}_{01}\|_0 \leq m+s, \|\tilde{\delta}_{01}\|_2=1} \|\tilde{\delta}_{01}\|_1 \right) \left(\mathbb{E} \sup_{f \in \mathcal{F}} |\mathcal{P}_n \eta f| \right) \\
& \lesssim 2\zeta_n (m+s)^{1/2} \left[\left(\frac{\log(p \vee n)(s+m)h}{n} \right)^{1/2} + \frac{\bar{C}_K \zeta_n (m+s)^{1/2} \log(p \vee n)(s+m)}{n} \right] \\
& \lesssim \left(\frac{\log(p \vee n)(s+m)^2 h \zeta_n^2}{n} \right)^{1/2} = h\pi_{1n},
\end{aligned}$$

where the first inequality is by Lemma 2.3.1 in van der Vaart and Wellner (1996), the second inequality is by Theorem 4.4 in Ledoux and Talagrand (2013), and the third one is by applying Corollary 5.1 of Chernozhukov, Chetverikov, and Kato (2014) with $\sigma^2 = \sup_{f \in \mathcal{F}} \mathbb{E} f^2 \lesssim h$ and, for some $A \geq e$,

$$\sup_Q N(\mathcal{F}, e_Q, \varepsilon \|F\|_{Q,2}) \leq \binom{p}{s+m} \left(\frac{A}{\varepsilon}\right)^{s+m} \lesssim \left(\frac{Ap}{\varepsilon}\right)^{s+m}.$$

By Assumption 2, $\pi_{1n} \rightarrow 0$. Then we have, w.p.a.1.,

$$|(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2 K(\frac{T-t}{h})| \leq 3h\underline{C}(\kappa')^2 \|\delta_{01}\|_2^2 / 8. \tag{A.4}$$

By the same token we can show that

$$\mathbb{E} \sup_{\|\tilde{\delta}_{01}\|_0 \leq m+s, \|\tilde{\delta}_{01}\|_2=1, t \in \mathcal{T}} |(\mathcal{P}_n - \mathcal{P})(b(X)' \tilde{\delta}_{01})^2| \lesssim \sqrt{h}\pi_{1n} \rightarrow 0.$$

Therefore, we have, w.p.a.1.,

$$|(\mathcal{P}_n - \mathcal{P})(b(X)' \delta_{01})^2| \leq 3(\kappa')^2 \|\delta_{01}\|_2^2 / 8. \tag{A.5}$$

Combining (A.3), (A.4), and (A.5) yields that w.p.a.1.,

$$\|b(X)' \delta_{01} K \left(\frac{T-t}{h} \right)^{1/2} \|_{P_{n,2}}^2 \geq \|\delta_{01}\|_2^2 h (\kappa')^2 \underline{C}/4.$$

Analogously, we can show that, w.p.a.1.,

$$\|b(X)' \delta_l K \left(\frac{T-t}{h} \right)^{1/2} \|_{P_{n,2}}^2 \leq 4 \|\delta_l\|_2^2 \underline{C}^{-1} h (\kappa'')^2.$$

Following (A.2), we have, w.p.a.1.,

$$\begin{aligned} \|b(X)' \delta K \left(\frac{T-t}{h} \right)^{1/2} \|_{P_{n,2}} &\geq h^{1/2} \|\delta_{01}\|_2 \kappa' \underline{C}^{1/2} / 2 - h^{1/2} \sum_{l=2}^L 2 \|\delta_l\|_2 \kappa'' \underline{C}^{-1/2} \\ &\geq h^{1/2} \|\delta_{01}\|_2 \kappa' \underline{C}^{1/2} / 2 - h^{1/2} \sum_{l=2}^L 2 \kappa'' \underline{C}^{-1/2} (\|\delta_{l-1}\|_1 \|\delta_l\|_1)^{1/2} / \sqrt{m} \\ &\geq h^{1/2} \|\delta_{01}\|_2 \kappa' \underline{C}^{1/2} / 2 - 2 h^{1/2} \kappa'' \underline{C}^{-1/2} \|\delta_{T^c}\|_1 / \sqrt{m} \\ &\geq h^{1/2} \|\delta_{01}\|_2 \kappa' \underline{C}^{1/2} / 2 - 2 h^{1/2} \kappa'' \underline{C}^{-1/2} c^{1/2} \|\delta_0\|_1 / \sqrt{m} \\ &\geq h^{1/2} \|\delta_{01}\|_2 \kappa' \underline{C}^{1/2} / 2 - 2 h^{1/2} \kappa'' \underline{C}^{-1/2} c^{1/2} \|\delta_0\|_2 \sqrt{s} / \sqrt{m} \\ &\geq h^{1/2} \|\delta_0\|_2 \left[\kappa' \underline{C}^{1/2} / 2 - 2 \kappa'' \underline{C}^{-1/2} c^{1/2} \sqrt{s} / \sqrt{m} \right], \end{aligned}$$

where the second inequality holds because, by construction, $\|\delta_l\|_2^2 \leq \|\delta_{l-1}\|_1 \|\delta_l\|_1 / \sqrt{m}$. Since $m = s \ell_n^{1/2}$, $s/m = \ell_n^{-1/2} \rightarrow 0$, and thus, for n large enough, the constant inside the brackets is greater than $\kappa' \underline{C}^{1/2} / 4$ which is independent of (t, u, n) . Therefore, we can conclude that, for n large enough,

$$\inf_{(t,u) \in \mathcal{T}\mathcal{U}} \inf_{\delta \in \Delta_{2\bar{c},t,u}} \frac{\|b(X)' \delta K \left(\frac{T-t}{h} \right)^{1/2} \|_{P_{n,2}}}{\|\delta_{\mathcal{S}_{t,u}}\|_2 \sqrt{h}} \geq \kappa' \underline{C}^{1/2} / 4 \equiv \underline{\kappa}.$$

This completes the proof of the lemma. \blacksquare

We aim to prove the results with regard to $\hat{\phi}_{t,u}(X)$ and $\hat{\theta}_{t,u}$ in Theorem 3.1. The derivations for the results regarding $\tilde{\phi}_{t,u}(X)$ and $\tilde{\theta}_{t,u}$ are exactly the same. We do not need to deal with the nonlinear logistic link function when deriving the results regarding $\hat{\nu}_t(X)$, $\tilde{\nu}_t(X)$, $\hat{\gamma}_t$, and $\tilde{\gamma}_t$.

Therefore, the corresponding results can be shown by following the same proving strategy as below and treating $\omega_{t,u}$ defined below as 1. The proofs for results regarding $\widehat{\nu}_t(X)$, $\widetilde{\nu}_t(X)$, $\widehat{\gamma}_t$, and $\widetilde{\gamma}_t$ are omitted for brevity.

Let $\widetilde{r}_{t,u}^\phi = \Lambda^{-1}(\mathbb{E}(Y_u|X, T=t)) - b(X)' \theta_{t,u}$, $\delta_{t,u} = \widehat{\theta}_{t,u} - \theta_{t,u}$, $\widehat{s}_{t,u} = \|\widehat{\theta}_{t,u}\|_0$, $\omega_{t,u} = \mathbb{E}(Y_u(t)|X)(1 - \mathbb{E}(Y_u(t)|X))$, and $\widehat{\mathcal{S}}_{t,u}$ be the support of $\widehat{\theta}_{t,u}$. We need the following four lemmas, whose proofs are relegated to the online supplement.

Lemma A.1 *If Assumptions 1–4 hold, then*

$$\sup_{(t,u) \in \mathcal{TU}} \|\omega_{t,u}^{1/2} b(X)' \delta_{t,u} K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}} = O_p((\log(p \vee n) s)^{1/2} n^{-1/2})$$

and

$$\sup_{(t,u) \in \mathcal{TU}} \|\delta_{t,u}\|_1 = O_p((\log(p \vee n) s^2)^{1/2} (nh)^{-1/2}).$$

Lemma A.2 *Suppose Assumptions 1–4 hold. Let $\xi_{t,u} = Y_u - \phi_{t,u}(X)$. Then*

$$\sup_{(t,u) \in \mathcal{TU}} \left\| \widehat{\Psi}_{t,u}^{-1} \mathcal{P}_n \left[\xi_{t,u} K(\frac{T-t}{h}) b(X) \right] \right\|_\infty = O_p((\log(p \vee n) h/n)^{1/2}).$$

Lemma A.3 *If the assumptions in Theorem 3.1 hold, then there exists a constant $C_\psi \in (0, 1)$ such that w.p.a.1,*

$$C_\psi/2 \leq \inf_{(t,u) \in \mathcal{TU}} \|\widehat{\Psi}_{t,u,0}\|_\infty \leq \sup_{(t,u) \in \mathcal{TU}} \|\widehat{\Psi}_{t,u,0}\|_\infty \leq 2/C_\psi. \quad (\text{A.6})$$

For any $k = 0, 1, \dots, K$ and $\widehat{\Psi}_{t,u}^k$ defined in Algorithm 2, there exists a constant $C_k \in (0, 1)$ such that, w.p.a.1,

$$C_k/2 \leq \inf_{(t,u) \in \mathcal{TU}} \|\widehat{\Psi}_{t,u}^k\|_\infty \leq \sup_{(t,u) \in \mathcal{TU}} \|\widehat{\Psi}_{t,u}^k\|_\infty \leq 2C_k. \quad (\text{A.7})$$

In addition, for any $k = 0, 1, \dots, K$ and $\widehat{\Psi}_{t,u}^k$ defined in Algorithm 2, there exist constants $l < 1 < L$ independent of n , (t, u) , and k such that, uniformly in $(t, u) \in \mathcal{TU}$ and w.p.a.1,

$$l\widehat{\Psi}_{t,u,0} \leq \widehat{\Psi}_{t,u}^k \leq L\widehat{\Psi}_{t,u,0}. \quad (\text{A.8})$$

Lemma A.4 *If the assumptions in Theorem 3.1 hold, then w.p.a.1,*

$$\sup_{t \in \mathcal{T}, \|\delta\|_2=1, \|\delta\|_0 \leq s\ell_n} \|b(X)' \delta K \left(\frac{T-t}{h} \right)^{1/2}\|_{P_n, 2} h^{-1/2} \leq 2\underline{C}^{-1/2} \kappa''.$$

Proof of Theorem 3.1. By the mean value theorem, there exist $\underline{\theta}_{t,u} \in (\theta_{t,u}, \hat{\theta}_{t,u})$ and $\bar{r}_{t,u}^\phi \in (0, \tilde{r}_{t,u}^\phi)$ such that

$$|\phi_{t,u}(X) - \widehat{\phi}_{t,u}(X)| \leq \Lambda(b(X)'\underline{\theta}_{t,u} + \bar{r}_{t,u}^\phi)(1 - \Lambda(b(X)'\underline{\theta}_{t,u} + \bar{r}_{t,u}^\phi))(b(X)'\delta_{t,u} + \tilde{r}_{t,u}^\phi),$$

where $\delta_{t,u} = \hat{\theta}_{t,u} - \theta_{t,u}$. By the proof of Lemma A.1, we have, w.p.a.1,

$$|\tilde{r}_{t,u}^\phi| \leq [\underline{C}/2(1 - \underline{C}/2)]^{-1} |r_{t,u}^\phi|.$$

Therefore, by Lemma A.1 and Assumptions 3.4-3.5 we have

$$\begin{aligned} & \sup_{(t,u) \in \mathcal{TU}} |b(X)'\underline{\theta}_{t,u} + \bar{r}_{t,u}^\phi - b(X)'\theta_{t,u} - \tilde{r}_{t,u}^\phi| \\ & \lesssim \sup_{(t,u) \in \mathcal{TU}} |b(X)'\delta_{t,u}| + \sup_{(t,u) \in \mathcal{TU}} |r_{t,u}^\phi| \\ & \lesssim \zeta_n \sup_{(t,u) \in \mathcal{TU}} \|\delta_{t,u}\|_1 + O((\log(p \vee n)s^2\zeta_n^2/(nh))^{-1/2}) = o_p(1), \end{aligned}$$

where the last equality is because $\sup_{(t,u) \in \mathcal{TU}} \|\delta_{t,u}\|_1 = O_p((\log(p \vee n)s^2)^{1/2}(nh)^{-1/2})$ by Lemma

A.1 and $\log(p \vee n)s^2\zeta_n^2/(nh) \rightarrow 0$ by Assumption 3.5. In addition, under Assumption 4.4 we have

$$\Lambda(b(X)' \theta_{t,u} + \tilde{r}_{t,u}^\phi) = \mathbb{E}(Y_u | X, T = t) \in [\underline{C}, 1 - \underline{C}].$$

Hence, there exist some positive constants c and c' only depending on \underline{C} such that, w.p.a.1,

$$\Lambda(b(X)' \underline{\theta}_{t,u} + \tilde{r}_{t,u}^\phi)(1 - \Lambda(b(X)' \underline{\theta}_{t,u} + \tilde{r}_{t,u}^\phi)) \leq c$$

and uniformly over $(t, u) \in \mathcal{TU}$,

$$|\phi_{t,u}(X) - \hat{\phi}_{t,u}(X)| \leq c(b(X)' \delta_{t,u} + \tilde{r}_{t,u}^\phi) \leq c'(b(X)' \delta_{t,u} + r_{t,u}^\phi). \quad (\text{A.9})$$

By Assumptions 3.3-3.4, Lemma A.1, and the fact that $\omega_{t,u}$ is bounded and bounded away from zero uniformly over \mathcal{TU} , we have, w.p.a.1,

$$\begin{aligned} & \sup_{(t,u) \in \mathcal{TU}} \|(\phi_{t,u}(X) - \hat{\phi}_{t,u}(X))K(\frac{T-t}{h})^{1/2}\|_{P_n,2} \\ & \leq \sup_{(t,u) \in \mathcal{TU}} c \left[\|b(X)' \delta_{t,u} K(\frac{T-t}{h})^{1/2}\|_{P_n,2} + \|r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}\|_{P_n,2} \right] \\ & = O_p((\log(p \vee n)s/n)^{1/2}) \end{aligned}$$

and

$$\begin{aligned} \sup_{(t,u) \in \mathcal{TU}} \|\phi_{t,u}(X) - \hat{\phi}_{t,u}(X)\|_{P,\infty} & \lesssim \zeta_n \sup_{(t,u) \in \mathcal{TU}} \|\delta_{t,u}\|_1 + O((\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}) \\ & = O_p((\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}). \end{aligned}$$

This gives the first and second results in Theorem 3.1.

Next, recall that $\lambda = \ell_n(\log(p \vee n)nh)^{1/2}$. By the first order conditions (FOC), for any $j \in \hat{\mathcal{S}}_{t,u}$, we have

$$\left| \mathcal{P}_n \left[(Y_u - \Lambda(b(X)' \hat{\theta}_{t,u})) b_j(X) K(\frac{T-t}{h}) \right] \right| = \hat{\Psi}_{t,u,jj} \frac{\lambda}{n}.$$

Denote $\xi_{t,u} = Y_u - \phi_{t,u}(X)$. By Lemmas A.1, A.2 and A.3, for any $\varepsilon > 0$, with probability greater than $1 - \varepsilon$, there exist positive constants C_λ and C , which only depend on ε and are independent of (t, u, n) , such that

$$\begin{aligned}
\frac{\lambda \hat{s}_{t,u}^{1/2}}{n} &= \left\| \widehat{\Psi}_{t,u}^{-1} \left\{ \mathcal{P}_n \left[(Y_u - \Lambda(b(X)' \hat{\theta}_{t,u})) b(X) K\left(\frac{T-t}{h}\right) \right] \right\} \right\|_{\hat{s}_{t,u}, 2} \\
&\leq \sup_{\|\theta\|_0 \leq \hat{s}_{t,u}, \|\theta\|_2 = 1} \|\theta\|_1 \sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|\widehat{\Psi}_{t,u}^{-1}(\mathcal{P}_n \xi_{t,u} b(X) K\left(\frac{T-t}{h}\right))\|_\infty \\
&\quad + \frac{\|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{l} \sup_{\|\theta\|_0 \leq \hat{s}_{t,u}, \|\theta\|_2 = 1} \left| \left\{ \mathcal{P}_n \left(\Lambda(b(X)' \hat{\theta}_{t,u}) - \Lambda(b(X)' \theta_{t,u}) - r_{t,u}^\phi \right) b(X)' \theta K\left(\frac{T-t}{h}\right) \right\} \right| \\
&\leq \frac{C_\lambda \lambda \hat{s}_{t,u}^{1/2}}{n l_n} + \frac{c' \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{l} \|(b(X)' \delta_{t,u} + r_{t,u}^\phi) K\left(\frac{T-t}{h}\right)^{1/2}\|_{P_n, 2} \\
&\quad \times \sup_{\|\theta\|_0 \leq \hat{s}_{t,u}, \|\theta\|_2 = 1} \|b(X)' \theta K\left(\frac{T-t}{h}\right)^{1/2}\|_{P_n, 2} \\
&\leq \frac{\lambda \hat{s}_{t,u}^{1/2}}{2n} + C(\log(p \vee n) s/n)^{1/2} \phi_{max}^{1/2}(\hat{s}_{t,u}) \\
&\leq \frac{\lambda \hat{s}_{t,u}^{1/2}}{2n} + \frac{C\lambda}{nh^{1/2}} \phi_{max}^{1/2}(\hat{s}_{t,u})
\end{aligned}$$

where $\phi_{max}(s) = \sup_{\|\theta\|_0 \leq s, \|\theta\|_2 = 1} \|b(X)' \theta K\left(\frac{T-t}{h}\right)^{1/2}\|_{P_n, 2}^2$ and $r_{t,u}^\phi = r_{t,u}^\phi(X)$. This implies that there exists a constant C only depending on ε , such that, with probability greater than $1 - \varepsilon$,

$$\hat{s}_{t,u} \leq C s \phi_{max}(\hat{s}_{t,u}) / h. \quad (\text{A.10})$$

Let $\mathcal{M} = \{m \in \mathbb{Z} : m > 2Cs\phi_{max}(m)/h\}$. By (A.10), for any $m \in \mathcal{M}$, $\hat{s}_{t,u} \leq m$. In addition, by Lemma A.4, we can choose $C_s > 4C\underline{C}^{-1}(\kappa'')^2$, which is independent of (t, u, n) , such that

$$2Cs\phi_{max}(C_s s)/h \leq 4C\underline{C}^{-1}(\kappa'')^2 s < C_s s.$$

This implies $C_s s \in \mathcal{M}$ and thus with probability greater than $1 - \varepsilon$, $\hat{s}_{t,u} \leq C_s s$. This result holds uniformly over $(t, u) \in \mathcal{T}\mathcal{U}$. ■

To prove Theorem 3.2, we need the following two lemmas, whose proofs are relegated to the

online supplement.

Lemma A.5 *If the assumptions in Theorem 3.2 hold, then there exists a constant $C_f \in (0, 1)$ such that w.p.a.1,*

$$C_f/2 \leq \inf_{t \in \mathcal{T}} \|\widehat{\Psi}_{t,0}\|_\infty \leq \sup_{t \in \mathcal{T}} \|\widehat{\Psi}_{t,0}\|_\infty \leq 2/C_f. \quad (\text{A.11})$$

For any $k = 0, 1, \dots, K$ and $\widehat{\Psi}_t^k$ defined in Algorithm 2, there exists a constant $C_k \in (0, 1)$ such that, w.p.a.1,

$$C_k/2 \leq \inf_{t \in \mathcal{T}} \|\widehat{\Psi}_t^k\|_\infty \leq \sup_{t \in \mathcal{T}} \|\widehat{\Psi}_t^k\|_\infty \leq 2C_k. \quad (\text{A.12})$$

In addition, for any $k = 0, 1, \dots, K$ and $\widehat{\Psi}_t^k$ defined in Algorithm 2, there exist constants $l < 1 < L$ independent of n, t , and k such that, uniformly in $t \in \mathcal{T}$ and w.p.a.1,

$$l\widehat{\Psi}_{t,0} \leq \widehat{\Psi}_t^k \leq L\widehat{\Psi}_{t,0}. \quad (\text{A.13})$$

Lemma A.6 *Let $\xi_t(X) = \frac{1}{h}K(\frac{T-t}{h}) - f_t(X)$. Suppose that the assumptions in Theorem 3.2 hold. Then $\sup_{t \in \mathcal{T}} \|\mathcal{P}_n \widehat{\Psi}_{t_0}^{-1} \xi_t(X) b(X)\|_\infty = O_p((\log(p \vee n)/(nh))^{1/2})$.*

Proof of Theorem 3.2. Recall that $\lambda = \ell_n(\log(p \vee n)nh)^{1/2}$, and some C_λ large enough such that $C_\lambda^{-1} < l < 1 < L$. In addition, denote

$$E_1 = \{\|r_t^f(X)\|_{P_n,2} \leq C_r(\log(p \vee n)s/(nh))^{1/2}\},$$

$$E_2 = \{\sup_{t \in \mathcal{T}} C_\lambda \|\mathcal{P}_n \widehat{\Psi}_{t_0}^{-1} \xi_t(X) b(X)\|_\infty \leq \frac{\lambda}{nh}\},$$

and

$$E_3 = \{l\widehat{\Psi}_{t,0} \leq \widehat{\Psi}_t \leq L\widehat{\Psi}_{t,0} \quad \text{and} \quad C_f/2 \leq \inf_{t \in \mathcal{T}} \|\widehat{\Psi}_{t,0}\|_\infty \leq \sup_{t \in \mathcal{T}} \|\widehat{\Psi}_{t,0}\|_\infty \leq 2/C_f\}$$

where C_r and C_f are some positive constants which will be defined later in this proof and in

Lemma A.5, respectively. $\widehat{\Psi}_{t,0} = \text{Diag}(l_{t,0,1}, \dots, l_{t,0,p})$ with

$$l_{t,0,j} = \left[\mathcal{P}_n h \left(\frac{1}{h} K\left(\frac{T-t}{h}\right) - f_t(X) \right)^2 f_j^2(X) \right]^{1/2}$$

is the ideal weight.

By Assumption 3.4, for any $\varepsilon > 0$, there exists a constant C_r (possibly dependent on ε) such that E_1 holds with probability greater than $1 - \varepsilon$. Lemma A.6 and the fact that $\ell_n \rightarrow \infty$ imply that, for any $\varepsilon > 0$ and any $C_\lambda > 0$ such that $C_\lambda^{-1} < l$ and E_2 holds with probability greater than $1 - \varepsilon$. By Lemma A.5, there exists $l < 1 < L$ independent of n, t such that E_3 holds uniformly over $t \in \mathcal{T}$ with probability greater than $1 - \varepsilon_n$ for some $\varepsilon_n \rightarrow 0$. Under E_1, E_2 , and E_3 , which occur with probability greater than $1 - 2\varepsilon - \varepsilon_n$, by Lemma J.3 and J.4 in Belloni et al. (2017a) with their λ replaced by $\ell_n(\log(p \vee n)nh)^{1/2}/h$ (i.e., our λ divided by h) and their c_r replaced by $(\log(p \vee n)s/(nh))^{1/2}$, we have

$$\sup_{t \in \mathcal{T}} \|\hat{f}_t(X) - f_t(X)\|_{P_n,2} = O_p(\ell_n(\log(p \vee n)s/(nh))^{1/2}),$$

$$\hat{s} = \sup_{t \in \mathcal{T}} \|\hat{\beta}_t\|_0 = O_p(s),$$

and

$$\sup_{t \in \mathcal{T}} \|\hat{\beta}_t - \beta_t\|_1 = O_p(\ell_n(\log(p \vee n)s^2/(nh))^{1/2}).$$

Following the analysis of $\|\widehat{\phi}_{t,u}(X) - \phi_{t,u}(X)\|_{P,\infty}$ in the proof of Theorem 3.1 we can show that

$$\sup_{t \in \mathcal{T}} \|\hat{f}_t(X) - f_t(X)\|_{P,\infty} = O_p(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}).$$

In addition, by Lemma J.5 in Belloni et al. (2017a) with their λ and c_r replaced by $\ell_n(\log(p \vee n)nh)^{1/2}/h$ and $(\log(p \vee n)s/(nh))^{1/2}$, respectively,

$$\sup_{t \in \mathcal{T}} \|\tilde{f}_t(X) - f_t(X)\|_{P_n,2} = O_p(\ell_n(\log(p \vee n)s/(nh))^{1/2}),$$

and

$$\sup_{t \in \mathcal{T}} \|\tilde{f}_t(X) - f_t(X)\|_{P, \infty} = O(\ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}).$$

This completes the proof of the theorem. ■

Proof of Theorem 3.3. Let $\hat{\alpha}^*(t, u) = \mathcal{P}_n \eta \Pi_{t,u}(W_u, \hat{\phi}_{t,u}, \hat{f}_t)$ where either $\eta = 1$ or η is a random variable that has sub-exponential tails with unit mean and variance. When $\eta = 1$, $\hat{\alpha}^*(t, u) = \hat{\alpha}(t, u)$, which is our original estimator. When η is random, $\hat{\alpha}^*(t, u)$ is the bootstrap estimator. In the following, we establish the linear expansion of both the original and bootstrap estimators together.

Let $\varepsilon_n = \ell_n(\log(p \vee n)s/(nh))^{1/2}$ and $\delta_n = \ell_n(\log(p \vee n)s^2\zeta_n^2/(nh))^{1/2}$. For some $M > 0$, which will be specified later, denote

$$\mathcal{G}_t = \left\{ \begin{array}{l} b(X)' \beta : \|\beta\|_0 \leq Ms, \quad \|b(X)' \beta - f_t(X)\|_{P, \infty} \leq M\delta_n, \\ \|b(X)' \beta - f_t(X)\|_{P_n, 2} \leq M\varepsilon_n \end{array} \right\}$$

and

$$\mathcal{H}_{t,u} = \left\{ \begin{array}{l} \Lambda(b(x)'\theta) : \|\theta\|_0 \leq Ms, \|\Lambda(b(X)'\theta) - \phi_{t,u}(X)\|_{P_n, 2} \leq M\varepsilon_n h^{1/2}, \\ \|\Lambda(b(X)'\theta) - \phi_{t,u}(X)\|_{P, \infty} \leq M\delta_n. \end{array} \right\}$$

By Theorem 3.1 and 3.2, for any $\varepsilon > 0$, there exists a constant M such that, with probability greater than $1 - \varepsilon$, $\hat{f}_t(\cdot) \in \mathcal{G}_t$ uniformly in $t \in \mathcal{T}$ and $\hat{\phi}_{t,u}(\cdot) \in \mathcal{H}_{t,u}$ uniformly in $(t, u) \in \mathcal{T}\mathcal{U}$. We focus on the case in which $(\hat{\phi}_{t,u}, \hat{f}_t) \in \mathcal{G}_t \times \mathcal{H}_{t,u}$. Then

$$\begin{aligned} \hat{\alpha}^*(t, u) - \alpha(t, u) &= (\mathcal{P}_n - \mathcal{P})\eta \Pi_{t,u}(W_u, \phi_{t,u}, f_t) + (\mathcal{P}_n - \mathcal{P}) \left[\eta \Pi_{t,u}(W_u, \bar{\phi}, \bar{f}) - \eta \Pi_{t,u}(W_u, \phi_{t,u}, f_t) \right] \\ &\quad + \mathcal{P} \left[\eta \Pi_{t,u}(W_u, \bar{\phi}, \bar{f}) - \eta \Pi_{t,u}(W_u, \phi_{t,u}, f_t) \right] + \left[\mathcal{P} \eta \Pi_{t,u}(W_u, \phi_{t,u}, f_t) - \alpha(t, u) \right] \\ &=: I + II + III + IV, \end{aligned}$$

where $(\bar{\phi}, \bar{f}) = (\hat{\phi}_{t,u}, \hat{f}_t)$.

Below we fix $(\bar{\phi}, \bar{f}) \in \mathcal{G}_t \times \mathcal{H}_{t,u}$. Term *IV* is $O(h^2)$ uniformly in $(t, u) \in \mathcal{TU}$. For term *III*, uniformly over $(t, u) \in \mathcal{TU}$, we have

$$\begin{aligned}
& \mathcal{P}\eta \left[\Pi_{t,u}(W_u, \bar{\phi}, \bar{f}) - \Pi_{t,u}(W_u, \phi_{t,u}, f_t) \right] \\
&= \mathbb{E} \left(\bar{\phi}(X) - \phi_{t,u}(X) \right) \left(1 - \frac{\mathbb{E}(K(\frac{T-t}{h})|X)}{hf_t(X)} \right) + \mathbb{E} \left(\frac{Y_u - \bar{\phi}(X)}{\bar{f}_t(X)f_t(X)} \right) \left(\frac{f_t(X) - \bar{f}_t(X)}{h} \right) K\left(\frac{T-t}{h}\right) \\
&= O(\varepsilon_n h^2) + \mathbb{E} \left(\frac{Y_u - \bar{\phi}(X)}{\bar{f}_t(X)f_t(X)} \right) \left(\frac{f_t(X) - \bar{f}_t(X)}{h} \right) K\left(\frac{T-t}{h}\right) \\
&= O(\delta_n h^2) + O(h^{-1} \|(\phi_{t,u}(X) - \bar{\phi}(X))K(\frac{T-t}{h})^{1/2}\|_{P,2} \| (f_t(X) - \bar{f}_t(X))K(\frac{T-t}{h})^{1/2}\|_{P,2}) \\
&= O(\varepsilon_n^2 + h^2).
\end{aligned} \tag{A.14}$$

The second equality of (A.14) follows because that there exists a constant c independent of n such that

$$\sup_{(t,u) \in \mathcal{TU}} \left| 1 - \frac{\mathbb{E}(K(\frac{T-t}{h})|X)}{hf_t(X)} \right| \leq ch^2$$

and then

$$\begin{aligned}
& \mathbb{E} \left(\bar{\phi}(X) - \phi_{t,u}(X) \right) \left(1 - \frac{\mathbb{E}(K(\frac{T-t}{h})|X)}{hf_t(X)} \right) \\
&\leq ch^{3/2} \|(\bar{\phi}(X) - \phi_{t,u}(X))K(\frac{T-t}{h})^{1/2}\|_{P,2} \\
&= ch^{3/2} [\mathbb{E}\|(\bar{\phi}(X) - \phi_{t,u}(X))K(\frac{T-t}{h})^{1/2}\|_{P_n,2}^2]^{1/2} = O(\varepsilon_n h^2).
\end{aligned}$$

The third equality of (A.14) holds by the fact that $\|\bar{f}_t(X) - f_t(X)\|_{P,\infty} = O(\delta_n) = o(1)$, $f_t(x)$ is assumed to be bounded away from zero uniformly over t, τ , the Cauchy inequality, and

$$\sup_{(t,u) \in \mathcal{TU}} \mathbb{E}((Y_u - \phi_{t,u}(X))K(\frac{T-t}{h})|X) = O(h^3).$$

The fourth inequality of (A.14) holds because

$$\|(\phi_{t,u}(X) - \bar{\phi}(X))K(\frac{T-t}{h})^{1/2}\|_{P,2} = [\mathbb{E}\|(\phi_{t,u}(X) - \bar{\phi}(X))K(\frac{T-t}{h})^{1/2}\|_{P_n,2}^2]^{1/2} = O(\varepsilon_n h^{1/2})$$

and for some constant $c > 0$ independent of (t, u, n) ,

$$\|(f_t(X) - \bar{f}_t(X))K(\frac{T-t}{h})^{1/2}\|_{P,2} \leq c[h\mathbb{E}\|f_t(X) - \bar{f}_t(X)\|_{P_n,2}^2]^{1/2} = O(\varepsilon_n h^{1/2}).$$

For the term II , we have

$$\mathbb{E}(\mathcal{P}_n - \mathcal{P})\eta \left[\Pi_{t,u}(W_u, \bar{\phi}, \bar{f}_t) - \Pi_{t,u}(W_u, \phi_{t,u}, f_t) \right] \leq \mathbb{E}\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}$$

where

$$\mathcal{F} = \cup_{(t,u) \in \mathcal{TU}} \mathcal{F}_{t,u} \quad \text{and} \quad \mathcal{F}_{t,u} = \left\{ \eta \left[\Pi_{t,u}(W_u, \bar{\phi}, \bar{f}_t) - \Pi_{t,u}(W_u, \phi_{t,u}, f_t) \right] : \bar{\phi} \in \mathcal{H}_{t,u}, \bar{f}_t \in \mathcal{G}_t \right\}.$$

Note \mathcal{F} has envelope $|\frac{\eta}{h}|$,

$$\begin{aligned} \sigma^2 &\equiv \sup_{f \in \mathcal{F}} \mathbb{E} f^2 \\ &\lesssim \sup_{(t,u) \in \mathcal{TU}} \mathbb{E} \left[(\bar{\phi}(X) - \phi(X))^2 \left(1 - \frac{K(\frac{T-t}{h})}{f_t(X)h} \right)^2 \right] \\ &\quad + \sup_{(t,u) \in \mathcal{TU}} \mathbb{E} \left[\frac{Y_u - \bar{\phi}(X)}{\bar{f}_t(X)f_t(X)h} K(\frac{T-t}{h}) \left(f_t(X) - \bar{f}_t(X) \right) \right]^2 \\ &\lesssim \sup_{(t,u) \in \mathcal{TU}} \mathbb{E} \left[(\bar{\phi}(X) - \phi(X))^2 \right] h^{-1} + h^{-1} \sup_{(t,u) \in \mathcal{TU}} \mathbb{E} \left[f_t(X) - \bar{f}_t(X) \right]^2 \lesssim h^{-1} \varepsilon_n^2, \end{aligned}$$

and \mathcal{F} is nested by

$$\bar{\mathcal{F}} = \left\{ \begin{array}{l} \Pi_{t,u}(W_u, \Lambda(b(X)' \theta), b(X)' \beta) - \Pi_{t,u}(W_u, \phi_{t,u}, f_t) : (t, u) \in \mathcal{TU}, \\ \|\theta\|_0 \leq Ms, \|\beta\|_0 \leq Ms \end{array} \right\},$$

where

$$\sup_Q \log N(\mathcal{F}, e_Q, \varepsilon \|F\|_{Q,2}) \lesssim s \log(p \vee n) + s \log\left(\frac{1}{\varepsilon}\right) \vee 0.$$

In addition, we claim $\|\max_{1 \leq i \leq n} |\eta_i/h|\|_{p,2} \lesssim \log(n)h^{-1}$. When $\eta = 1$, the above claim holds

trivially. When η has sub-exponential tail, and the claim holds by Lemma 2.2.2 in van der Vaart and Wellner (1996). Therefore, by Corollary 5.1 of Chernozhukov et al. (2014), we have

$$\mathbb{E}\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} \lesssim \varepsilon_n (nh)^{-1/2} s^{1/2} \log^{1/2}(p \vee n) + \log(n) (nh)^{-1} s \log(p \vee n) \lesssim \varepsilon_n^2.$$

This implies that $\sup_{(t,u) \in \mathcal{TU}} II = O_p(\varepsilon_n^2)$. Combining the bounds for *II*, *III*, and *IV*, we have

$$\hat{\alpha}^*(t, u) - \alpha(t, u) = (\mathcal{P}_n - \mathcal{P})\eta \Pi_{t,u}(W_u, \phi_{t,u}, f_t) + R_n(t, u) \quad \text{and} \quad \sup_{(t,u) \in \mathcal{TU}} |R_n(t, u)| = O_p(\varepsilon_n^2 + h^2).$$

Then, Assumption 5 implies the desired results. ■

Proof of Theorem 3.4. We first derive the linear expansion of the rearrangement of $\hat{\alpha}^*(t, u)$ as defined in the proof of Theorem 3.3. For $z \in (0, 1)$, let

$$F(t, z) = \int_0^1 1\{\alpha(t, \psi^{\leftarrow}(v)) \leq z\} dv, \quad F(t, z|h_n) = \int_0^1 1\{\hat{\alpha}^*(t, \psi^{\leftarrow}(v)) \leq y\} dv.$$

Then, by Lemma B.2, we have

$$\frac{F(t, z|h_n) - F(t, z)}{s_n} + \frac{h_n(t, \psi(q_z(t)))\psi'(q_z(t))}{f_{Y(t)}(q_z(t))} = o_p(\delta_n) \quad (\text{A.15})$$

and

$$\frac{\hat{\alpha}^{*r}(t, u) - \alpha(t, u)}{s_n} + \frac{F(t, \alpha(t, u)|h_n) - F(t, \alpha(t, u))f_{Y(t)}(u)}{s_n \psi'(u)} = o_p(\delta_n). \quad (\text{A.16})$$

where $s_n = (nh)^{-1/2}$, $h_n(t, v) = (nh)^{1/2}(\hat{\alpha}^*(t, \psi^{\leftarrow}(v)) - \alpha(t, \psi^{\leftarrow}(v)))$, $f_{Y(t)}(\cdot)$ is the density of $Y(t)$, $q_z(t)$ is the z -th quantile of $Y(t)$, and δ_n equals to either 1 or $h^{1/2}$, depending on either Assumption 5.1 or 5.2 is in place.

Combining (A.15) and (A.16), we have

$$(nh)^{1/2}(\hat{\alpha}^{*r}(t, u) - \alpha(t, u)) = h_n(t, \psi(u)) + o_p(\delta_n) = (nh)^{1/2}(\hat{\alpha}^*(t, u) - \alpha(t, u)) + o_p(\delta_n) \quad (\text{A.17})$$

uniformly over $(t, u) \in \mathcal{TU}$.

We can apply Lemma B.2 on $\hat{\alpha}^{*r}(t, u)$ again with $J_n(t, u) = (nh)^{1/2}(\hat{\alpha}^{*r}(t, u) - \alpha(t, u))$, $F(t, u) = P(Y(t) \leq u) = \alpha(t, u)$, $f(t, u) = f_{Y(t)}(u)$, and $F^{\leftarrow}(t, \tau) = q_\tau(t)$. Then, for δ_n equals 1 or $h^{1/2}$ under either Assumption 5.1 or 5.2, respectively, we have,

$$\frac{\hat{q}_\tau^*(t) - q_\tau(t)}{s_n} = -\frac{J_n(t, q_\tau(t))}{f_{Y(t)}(q_\tau(t))} + o_p(\delta_n) = -\frac{(nh)^{1/2}(\hat{\alpha}^{*r}(t, q_\tau(t)) - \tau)}{f_{Y(t)}(q_\tau(t))} + o_p(\delta_n) \quad (\text{A.18})$$

uniformly over $(t, \tau) \in \mathcal{TL}$.

Combining (A.17), (A.18), and Theorem 3.3, we have

$$\hat{q}_\tau^*(t) - q_\tau(t) = -(\mathcal{P}_n - \mathcal{P})\eta\Pi_{t,u}(W_{q_\tau(t)}, \phi_{t,q_\tau(t)}, f_t)/f_{Y(t)}(q_\tau(t)) + R_n(t, \tau) + o_p(\delta_n(nh)^{-1/2}).$$

By taking $\delta_n = 1$ and $\delta_n = h^{1/2}$ under Assumptions 5.1 and 5.2, respectively, we have establish the desired results. ■

Proof of Theorem 3.5. We consider the general case in which the observations are weighted by $\{\eta_i\}_{i=1}^n$ as above. For brevity, denote $\hat{\delta} \equiv (\hat{\delta}_0, \hat{\delta}_1)' = (\hat{\beta}_\tau^{*0}(t), \hat{\beta}_\tau^{*1}(t))'$ and $\delta \equiv (\delta_0, \delta_1)' = (\beta_\tau^0(t), \beta_\tau^1(t))$. Then $\hat{\delta} = \hat{\Sigma}_2^{-1}\hat{\Sigma}_1$, where

$$\hat{\Sigma}_1 = \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right)\eta_i \hat{q}_\tau^*(T_i) \\ \frac{1}{n} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right)(T_i-t)\eta_i \hat{q}_\tau^*(T_i) \end{pmatrix}$$

and

$$\hat{\Sigma}_2 = \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right)\eta_i & \frac{1}{n} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right)(T_i-t)\eta_i \\ \frac{1}{n} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right)(T_i-t)\eta_i & \frac{1}{n} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right)(T_i-t)^2\eta_i \end{pmatrix}.$$

Let $\Sigma_2 = \begin{pmatrix} f(t) & 0 \\ \kappa_2 f^{(1)}(t) & \kappa_2 f(t) \end{pmatrix}$ and $G = \begin{pmatrix} h^{-1} & 0 \\ 0 & h^{-3} \end{pmatrix}$. Then we have

$$G\hat{\Sigma}_2 - \Sigma_2 = O_p^*(\log^{1/2}(n)(nh^3)^{-1/2}).$$

In addition, note

$$\hat{q}_\tau^*(T_i) = \delta_0 + \delta_1(T_i - t) + (q_\tau(T_i) - \delta_0 - \delta_1(T_i - t)) + (\hat{q}_\tau^*(T_i) - q_\tau(T_i))$$

and

$$\left(\begin{array}{c} \frac{1}{nh} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right) \eta_i \left(q_\tau(T_i) - \delta_0 - \delta_1(T_i - t) \right) \\ \frac{1}{nh^3} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right) (T_i - t) \eta_i \left(q_\tau(T_i) - \delta_0 - \delta_1(T_i - t) \right) \end{array} \right) = O_p^*\left(\sqrt{\frac{\log(n)h}{n}} + h^2\right)$$

Therefore,

$$G\hat{\Sigma}_1 = G\hat{\Sigma}_2\delta + \left(\begin{array}{c} \frac{1}{nh} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right) \left(\hat{q}_\tau(T_i) - q_\tau(T_i) \right) \eta_i \\ \frac{1}{nh^3} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right) (T_i - t) \left(\hat{q}_\tau(T_i) - q_\tau(T_i) \right) \eta_i \end{array} \right) + O_p^*\left(\sqrt{\frac{\log(n)h}{n}} + h^2\right). \quad (\text{A.19})$$

By Theorem 3.4, the fact that $\mathbb{E} \frac{Y_u - \phi_{t,u}(X)}{f_t(X)h} K\left(\frac{T-t}{h}\right) = O(h^2)$, and that $\mathbb{E} \phi_{t,q_\tau(t)}(X) = \tau$, we have

$$\hat{q}_\tau^*(t) - q_\tau(t) = \frac{-1}{f_{Y_t}(q_\tau(t))} \frac{1}{n} \sum_{j=1}^n \eta_j \left(\frac{Y_{q_\tau(t),j} - \phi_{t,q_\tau(t)}(X_j)}{f_t(X_j)h} K\left(\frac{T_j-t}{h}\right) + \phi_{t,q_\tau(t)}(X_j) - \tau \right) + o_p^*((nh)^{-1/2}), \quad (\text{A.20})$$

where $o_p^*((nh)^{-1/2})$ term is uniform over $(t, \tau) \in \mathcal{TL}$. Let $\Upsilon_i = (Y_i, T_i, X_i, \eta_i)$. Then, by plugging (A.20) in (A.19) and noticing that

$$\left\{ \begin{array}{c} \sup_{t \in \mathcal{T}} \frac{1}{nh} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right) \eta_i \\ \sup_{t \in \mathcal{T}} \frac{1}{nh^3} \sum_{i=1}^n K\left(\frac{T_i-t}{h}\right) |T_i - t| \eta_i \end{array} \right\} = \left\{ \begin{array}{c} O_p(1) \\ O_p(h^{-1}) \end{array} \right\},$$

we have

$$G\hat{\Sigma}_1 = G\hat{\Sigma}_2\delta + \frac{1}{n(n-1)} \sum_{i \neq j} \eta_i \eta_j \Gamma(\Upsilon_i, \Upsilon_j; t, \tau) + \left\{ \begin{array}{c} o_p^*((nh)^{-1/2}) \\ o_p^*((nh^3)^{-1/2}) \end{array} \right\}$$

where $\Gamma(\Upsilon_i, \Upsilon_j; t, \tau) = (\Gamma_0(\Upsilon_i, \Upsilon_j; t, \tau), \Gamma_1(\Upsilon_i, \Upsilon_j; t, \tau))'$, and

$$\Gamma_\ell(\Upsilon_i, \Upsilon_j; t, \tau) = \frac{(T_i - t)^\ell}{h^{1+2\ell} f_{Y_{T_i}}(q_\tau(T_i))} K\left(\frac{T_i - t}{h}\right) \left(\frac{Y_{q_\tau(T_i),j} - \phi_{T_i, q_\tau(T_i)}(X_j)}{f_{T_i}(X_j)h} K\left(\frac{T_j - T_i}{h}\right) + \phi_{T_i, q_\tau(T_i)}(X_j) - \tau \right)$$

for $\ell = 0, 1$. Let $\Gamma^s(\Upsilon_i, \Upsilon_j; t, \tau) = (\Gamma(\Upsilon_i, \Upsilon_j; t, \tau) + \Gamma(\Upsilon_j, \Upsilon_i; t, \tau))/2$. Then we have

$$\hat{\beta}_\tau^{1*}(t) - \beta_\tau^1(t) = e_2'(G\widehat{\Sigma}_2)^{-1}U_n(t, \tau) + o_p^*((nh^3)^{-1/2}), \quad (\text{A.21})$$

where $e_2 = (0, 1)'$ and $U_n(t, \tau) = (C_n^2)^{-1} \sum_{1 \leq i < j \leq n} \eta_i \eta_j \Gamma^s(\cdot, \cdot; t, \tau)$ is a U-process indexed by (t, τ) .

By Lemma B.3,

$$e_2'(G\widehat{\Sigma}_2)^{-1}U_n(t, \tau) = \frac{1}{n} \sum_{j=1}^n \eta_j (\kappa_2 f_{Y(t)}(q_\tau(t)) f_t(X_j) h^2)^{-1} \left[Y_{q_\tau(t), j} - \phi_{t, q_\tau(t)}(X_j) \right] \overline{K}\left(\frac{T_j - t}{h}\right) + o_p^*((nh^3)^{-1/2}). \quad (\text{A.22})$$

Combining (A.21) and (A.22), we have

$$\hat{\beta}_\tau^{1*}(t) - \beta_\tau^1(t) = \frac{1}{n} \sum_{j=1}^n \eta_j (\kappa_2 f_{Y(t)}(q_\tau(t)) f_t(X_j) h^2)^{-1} \left[Y_{q_\tau(t), j} - \phi_{t, q_\tau(t)}(X_j) \right] \overline{K}\left(\frac{T_j - t}{h}\right) + o_p^*((nh^3)^{-1/2}).$$

■

Proof of Theorem 4.1. By the proofs of Theorem 3.4 and 3.5, we have

$$\hat{q}_\tau^b(t) - \hat{q}_\tau(t) = -(\mathcal{P}_n - \mathcal{P})(\eta - 1) \Pi_{t, u}(W_{q_\tau(t)}, \phi_{t, q_\tau(t)}, f_t) / f_t(q_\tau(t)) + o_p((nh)^{-1/2})$$

and

$$\hat{\beta}_\tau^{1b}(t) - \hat{\beta}_\tau^1(t) = \frac{1}{n} \sum_{j=1}^n (\eta_j - 1) (\kappa_2 f_{Y(t)}(q_\tau(t)) f_t(X_j) h^2)^{-1} \left[Y_{q_\tau(t), j} - \phi_{t, q_\tau(t)}(X_j) \right] \overline{K}\left(\frac{T_j - t}{h}\right) + o_p^*((nh^3)^{-1/2}).$$

Then, it is straightforward to show that $\sqrt{nh}(\hat{q}_\tau^b(t) - \hat{q}_\tau(t))$ and $(nh^3)^{1/2}(\hat{\beta}_\tau^{1b}(t) - \hat{\beta}_\tau^1(t))$ converge weakly to the limiting distribution of $\sqrt{nh}(\hat{q}_\tau(t) - q_\tau(t))$ and $(nh^3)^{1/2}(\hat{\beta}_\tau^1(t) - \beta_\tau^1(t))$, respectively, conditional on data in the sense of van der Vaart and Wellner (1996, Section 2.9). The desired results then follow. ■

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Supplementary Material for
“Non-separable Models with High-dimensional Data”

[NOT INTENDED FOR PUBLICATION]

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This supplement is composed of three parts. Appendix B provides the proofs of some technical lemmas used in the proofs of the main results in the paper. Appendix C studies the rearrangement operator on a local process. Appendix D derives a new bound for the second order degenerate U-process.

B Proofs of the Technical Lemmas

Lemma A.1 and Lemma B.1 below are closely related to Lemmas J.6 and O.2 in Belloni et al. (2017a) with one major difference: we have an additional kernel function which affects the rate of convergence. We follow the proof strategies in Belloni et al. (2017a) in general, but use the local compatibility condition established in Lemma 3.1 when needed. We include these proofs mainly for completeness. Lemma A.2 and A.6 are proved without referring to the theory of moderate deviations for self-normalized sums, in contrast to the proof of Lemma J.1 in Belloni et al. (2017a). Consequently, we have the additional ℓ_n term but avoid one constraint on the rates of p , s , and n , as well.

Proof of Lemma A.1. We define the following three events.

$$E_1 = \{C_r(\log(p \vee n)s/n)^{1/2} \geq \sup_{(t,u) \in \mathcal{TU}} \left\| \frac{r_{t,u}^\phi}{\omega_{t,u}^{1/2}} K\left(\frac{T-t}{h}\right)^{1/2} \right\|_{P_n, 2}\},$$

$$E_2 = \left\{ \frac{\lambda}{n} \geq \sup_{(t,u) \in \mathcal{TU}} C_\lambda \left\| \widehat{\Psi}_{t,u,0}^{-1} \mathcal{P}_n \left[\xi_{t,u} K\left(\frac{T-t}{h}\right) b(X) \right] \right\|_\infty \right\},$$

and

$$E_3 = \{l\widehat{\Psi}_{t,u,0} \leq \widehat{\Psi}_{t,u} \leq L\widehat{\Psi}_{t,u,0} \quad \text{and} \quad C_\psi/2 \leq \inf_{(t,u) \in \mathcal{TU}} \|\widehat{\Psi}_{t,u,0}\|_\infty \leq \sup_{(t,u) \in \mathcal{TU}} \|\widehat{\Psi}_{t,u,0}\|_\infty \leq 2/C_\psi\}$$

where l , L , and C_ψ are defined in the statement of Lemma A.3 and the generic penalty loading matrix is $\widehat{\Psi}_{t,u} = \widehat{\Phi}_{t,u}^k$ for $k = 0, \dots, K$.

By Assumption 2.3, for an arbitrary $\varepsilon > 0$, we can choose C_r and n sufficiently large so that $P(E_1) \geq 1 - \varepsilon$. By Lemma A.2 below and the fact that $\ell_n \rightarrow \infty$, for any $\varepsilon > 0$ and any $C_\lambda > 0$, for n sufficiently large, we have $P(E_2) \geq 1 - \varepsilon$. In particular, we choose C_λ such that $C_\lambda l > 1$. Last, by Lemma A.3 below, $P(E_3) > 1 - \varepsilon_n$ for some deterministic sequence $\varepsilon_n \downarrow 0$.

From now on we assume E_1 , E_2 , and E_3 hold with constants C_r , C_λ , l , and L , which occurs with probability greater than $1 - 2\varepsilon - \varepsilon_n$. Let $\delta_{t,u} = \hat{\theta}_{t,u} - \theta_{t,u}$ and $\mathcal{S}_{t,u}^0 = \text{Supp}(\theta_{t,u})$. Let

$$\Gamma_{t,u} = \|\omega_{t,u}^{1/2} b(X)' \delta_{t,u} K\left(\frac{T-t}{h}\right)^{1/2}\|_{P_n, 2},$$

and

$$\tilde{c} = \max(4(LC_\lambda + 1)(lC_\lambda - 1)^{-1}C_\psi^{-2}, 1).$$

Then, under E_3 ,

$$\tilde{c} \geq \max((LC_\lambda + 1)/(lC_\lambda - 1) \sup_{(t,u) \in \mathcal{TU}} \|\widehat{\Psi}_{t,u,0}\|_\infty \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty, 1) \geq 1.$$

Let $Q_{t,u}(\theta) = \mathcal{P}_n M(Y_u, X; \theta) K\left(\frac{T-t}{h}\right)$. By the fact that $\hat{\theta}_{t,u}$ solves the minimization problem in (3.5), we have

$$\begin{aligned} Q_{t,u}(\hat{\theta}_{t,u}) - Q_{t,u}(\theta_{t,u}) &\leq \frac{\lambda}{n} \|\widehat{\Psi}_{t,u} \theta_{t,u}\|_1 - \frac{\lambda}{n} \|\widehat{\Psi}_{t,u} \hat{\theta}_{t,u}\|_1 \\ &\leq \frac{\lambda}{n} \|\widehat{\Psi}_{t,u}(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 - \frac{\lambda}{n} \|\widehat{\Psi}_{t,u}(\hat{\theta}_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 \\ &= \frac{\lambda}{n} \|\widehat{\Psi}_{t,u}(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 - \frac{\lambda}{n} \|\widehat{\Psi}_{t,u}(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 \\ &\leq \frac{\lambda L}{n} \|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 - \frac{\lambda l}{n} \|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1. \end{aligned} \tag{B.1}$$

Because the kernel function $K(\cdot)$ is nonnegative, $Q_{t,u}(\theta)$ is convex in θ . It follows that $Q_{t,u}(\hat{\theta}_{t,u}) - Q_{t,u}(\theta_{t,u}) \geq \partial_\theta Q_{t,u}(\theta_{t,u})' \delta_{t,u}$.

Let $D_{t,u} = -\mathcal{P}_n b(X) \xi_{t,u} K(\frac{T-t}{h})$ and $\xi_{t,u} = Y_u - \phi_{t,u}(X)$. Then,

$$\begin{aligned}
& |\partial_\theta Q_{t,u}(\theta_{t,u})' \delta_{t,u}| \\
&= |\mathcal{P}_n(\Lambda(b(X)' \theta_{t,u}) - Y_u) K(\frac{T-t}{h}) b(X)' \delta_{t,u}| \\
&= |\mathcal{P}_n r_{t,u}^\phi b(X)' \delta_{t,u} K(\frac{T-t}{h}) + D'_{t,u} \delta_{t,u}| \\
&\leq \|\widehat{\Psi}_{t,u,0}^{-1} D_{t,u}\|_\infty \|\widehat{\Psi}_{t,u,0} \delta_{t,u}\|_1 + \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}} \\
&\leq \frac{\lambda}{nC_\lambda} \|\widehat{\Psi}_{t,u,0} \delta_{t,u}\|_1 + \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}} \\
&\leq \frac{\lambda}{nC_\lambda} \|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 + \frac{\lambda}{nC_\lambda} \|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 + \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}},
\end{aligned} \tag{B.2}$$

where $r_{t,u}^\phi = r_{t,u}^\phi(X)$. Combining (B.1) and (B.2), we have

$$\frac{\lambda(lC_\lambda - 1)}{nC_\lambda} \|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 \leq \frac{\lambda(LC_\lambda + 1)}{nC_\lambda} \|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 + \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}}.$$

Then

$$\begin{aligned}
\|(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 &\leq \frac{LC_\lambda + 1}{lC_\lambda - 1} \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty \|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 + \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}} \\
&\leq \tilde{c} \|(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 + \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}}.
\end{aligned}$$

We will consider two cases: $\delta_{t,u} \notin \Delta_{2\tilde{c}, t, u}$ and $\delta_{t,u} \in \Delta_{2\tilde{c}, t, u}$.

First, if $\delta_{t,u} \notin \Delta_{2\tilde{c}, t, u}$, i.e., $\|(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 \geq 2\tilde{c} \|(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1$, then

$$\begin{aligned}
\|\delta_{t,u}\|_1 &\leq (1 + \frac{1}{2\tilde{c}}) \|(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 \\
&\leq (\tilde{c} + \frac{1}{2}) \|(\delta_{t,u})_{\mathcal{S}_{t,u}^0}\|_1 + (1 + \frac{1}{2\tilde{c}}) \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}} \\
&\leq (\frac{1}{2} + \frac{1}{4\tilde{c}}) \|(\delta_{t,u})_{\mathcal{S}_{t,u}^{0c}}\|_1 + (1 + \frac{1}{2\tilde{c}}) \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}} \\
&\leq (\frac{1}{2} + \frac{1}{4\tilde{c}}) \|\delta_{t,u}\|_1 + (1 + \frac{1}{2\tilde{c}}) \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_n, 2\Gamma_{t,u}}.
\end{aligned}$$

Noting that $\tilde{c} \geq 1$, we have

$$\begin{aligned} \|\delta_{t,u}\|_1 &\leq \left[\frac{4\tilde{c} + 2}{2\tilde{c} - 1} \right] \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_{n,2}} \Gamma_{t,u} \\ &\leq 6 \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_{n,2}} \Gamma_{t,u} =: I_{t,u}. \end{aligned}$$

Now, we consider the case where $\delta_{t,u} \in \Delta_{2\tilde{c},t,u}$. By Lemma 3.1, we have

$$\underline{\kappa} \leq \inf_{(t,u) \in \mathcal{T}\mathcal{U}} \min_{\delta \in \Delta_{2\tilde{c},t,u}} \frac{\|b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}}{\sqrt{h} \|\delta_{S_{t,u}^0}\|_2}.$$

In addition, $\omega_{t,u} \in (\underline{C}(1 - \underline{C}), 1/4)$. If $\delta_{t,u} \in \Delta_{2\tilde{c},t,u}$, then

$$\|(\delta_{t,u})_{S_{t,u}^0}\|_1 \leq \frac{\sqrt{s}}{\underline{\kappa} \sqrt{h} \omega_{t,u}^{1/2}} \Gamma_{t,u} =: II_{t,u}.$$

In this case, $\|\delta_{t,u}\|_1 \leq (1 + 2\tilde{c})II_{t,u}$.

In sum, we have

$$\|\delta_{t,u}\|_1 \leq I_{t,u} + (1 + 2\tilde{c})II_{t,u} \leq \left(6 \frac{nC_\lambda \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty}{\lambda(C_\lambda l - 1)} \left\| \frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_{n,2}} + \frac{(1 + 2\tilde{c})\sqrt{s}}{\underline{\kappa} \sqrt{h} \omega_{t,u}^{1/2}} \right) \Gamma_{t,u} \quad (\text{B.3})$$

and $\delta_{t,u} \in A_{t,u} \equiv \Delta_{2\tilde{c},t,u} \cup \{\delta : \|\delta\|_1 \leq I_{t,u}\}$.

Recall $\tilde{r}_{t,u}^\phi = \Lambda^{-1}(\Lambda(b(X)'\theta_{t,u}) + r_{t,u}^\phi) - b(X)'\theta_{t,u}$ and denote

$$\bar{q}_{A_{t,u}} = \inf_{\delta \in A_{t,u}} \frac{[\mathcal{P}_n \omega_{t,u} |b(X)'\delta|^2 K(\frac{T-t}{h})]^{3/2}}{\mathcal{P}_n [\omega_{t,u} |b(X)'\delta|^3 K(\frac{T-t}{h})]}.$$

Then, w.p.a.1., for some $\bar{r}_{t,u}^\phi$ between 0 and $r_{t,u}^\phi$,

$$\begin{aligned} |\tilde{r}_{t,u}^\phi| &= \{[\Lambda(b(X)'\theta_{t,u}) + \bar{r}_{t,u}^\phi][1 - \Lambda(b(X)'\theta_{t,u}) - \bar{r}_{t,u}^\phi]\}^{-1} |r_{t,u}^\phi| \\ &\in [4|r_{t,u}^\phi|, \{(\underline{C}/2)(1 - \underline{C}/2)\}^{-1} |r_{t,u}^\phi|], \end{aligned}$$

where the second line holds because $\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \|r_{t,u}^\phi\|_{P_\infty} \xrightarrow{p} 0$. In addition, by Lemma B.1 below

and equations (B.1)–(B.3), we have

$$\begin{aligned}
& \min\left(\frac{1}{3}\Gamma_{t,u}^2, \frac{\bar{q}_{A_{t,u}}}{3}\Gamma_{t,u}\right) \\
& \leq Q_{t,u}(\theta_{t,u} + \delta_{t,u}) - Q_{t,u}(\theta_{t,u}) - \partial_{\theta}Q_{t,u}(\theta_{t,u})'\delta_{t,u} + 2\left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}}\Gamma_{t,u} \\
& \leq \frac{\lambda}{n}\left(L + \frac{1}{C_{\lambda}}\right)\|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{S_{t,u}^0}\|_1 - \frac{\lambda}{n}\left(l - \frac{1}{C_{\lambda}}\right)\|\widehat{\Psi}_{t,u,0}(\delta_{t,u})_{S_{t,u}^{0c}}\|_1 + 3\left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}}\Gamma_{t,u} \\
& \leq \frac{\lambda}{n}\left(L + \frac{1}{C_{\lambda}}\right)\|\widehat{\Psi}_{t,u,0}\|_{\infty}\|\delta_{t,u}\|_1 + 3\left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}}\Gamma_{t,u} \\
& \leq \left(9\tilde{c}\left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}} + \frac{\lambda}{n}\left(L + \frac{1}{C_{\lambda}}\right)\|\widehat{\Psi}_{t,u,0}\|_{\infty}\frac{(1+2\tilde{c})\sqrt{s}}{\underline{\kappa}\sqrt{h}}\right)\Gamma_{t,u},
\end{aligned}$$

where the last inequality holds because $|r_{t,u}^{\phi}| \leq |\tilde{r}_{t,u}^{\phi}|$. If

$$\bar{q}_{A_{u,r}} > 3\left\{9\tilde{c}\left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}} + \frac{\lambda}{n}\left(L + \frac{1}{C_{\lambda}}\right)\|\widehat{\Psi}_{t,u,0}\|_{\infty}\frac{(1+2\tilde{c})\sqrt{s}}{\underline{\kappa}\sqrt{h}}\right\}, \quad (\text{B.4})$$

then

$$\Gamma_{t,u} \leq 3\left\{9\tilde{c}\left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}} + \frac{\lambda}{n}\left(L + \frac{1}{C_{\lambda}}\right)\|\widehat{\Psi}_{t,u,0}\|_{\infty}\frac{(1+2\tilde{c})\sqrt{s}}{\underline{\kappa}\sqrt{h}}\right\} \quad (\text{B.5})$$

and

$$\begin{aligned}
\|\delta_{t,u}\|_1 & \leq \left(6\frac{nC_{\lambda}\|\widehat{\Psi}_{t,u,0}^{-1}\|_{\infty}}{\lambda(C_{\lambda}l-1)}\left\|\frac{r_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}} + \frac{(1+2\tilde{c})\sqrt{s}}{\underline{\kappa}\sqrt{h}}\right) \\
& \quad \times 3\left\{9\tilde{c}\left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}} + \frac{\lambda}{n}\left(L + \frac{1}{C_{\lambda}}\right)\|\widehat{\Psi}_{t,u,0}\|_{\infty}\frac{(1+2\tilde{c})\sqrt{s}}{\underline{\kappa}\sqrt{h}}\right\}.
\end{aligned} \quad (\text{B.6})$$

Since E_1 holds,

$$\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \left\|\frac{\tilde{r}_{t,u}^{\phi}K\left(\frac{T-t}{h}\right)^{1/2}}{\omega_{t,u}^{1/2}}\right\|_{P_{n,2}} \leq [C/2(1-C/2)]^{-1}C_r\left(\sqrt{\frac{\log(p \vee n)s}{n}}\right).$$

Further note that $\lambda = \ell_n(\log(p \vee n)nh)^{1/2}$. Hence, if (B.4) holds, then (B.5) and (B.6) imply that

$$\sup_{(t,u) \in \mathcal{T}\mathcal{U}} \Gamma_{t,u} \leq C\Gamma\ell_n(\log(p \vee n)s)^{1/2}n^{-1/2}$$

with $C_\Gamma = 3(9\tilde{c}[\underline{C}/2(1 - \underline{C}/2)]^{-1}C_r + (LC_\lambda + 1)2C_\psi(1 + 2\tilde{c})/\underline{\kappa}$ and

$$\sup_{(t,u) \in \mathcal{TU}} \|\delta_{t,u}\|_1 \leq C_1 \ell_n (\log(p \vee n) s^2)^{1/2} (nh)^{-1/2}$$

with $C_1 = \frac{2(1+2\tilde{c})}{\underline{\kappa}} C_\Gamma$, which are the desired results.

Last, we verify (B.4). By Lemma B.1, since $\ell_n^2 \log(p \vee n) s^2 \zeta_n^2 / (nh) \rightarrow 0$,

$$\frac{\bar{q}_{A_{u,r}}}{3 \left\{ 9\tilde{c} \left\| \frac{\tilde{r}_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_{n,2}} + \frac{\lambda}{n} (L + \frac{1}{C_\lambda}) \|\widehat{\Psi}_{t,u,0}\|_\infty \frac{(1+2\tilde{c})\sqrt{s}}{\underline{\kappa}\sqrt{h}} \right\}} \geq c \sqrt{\frac{nh}{\log(p \vee n) s^2 \zeta_n^2 \ell_n^2}} \rightarrow \infty.$$

This concludes the proof. ■

Proof of Lemma A.2. By Lemma A.3 below, $\widehat{\Psi}_{t,u}^{-1}$ is bounded away from zero w.p.a.1, uniformly over (t, u) . Therefore, we can just focus on bounding

$$\sup_{(t,u) \in \mathcal{TU}} \left\| \mathcal{P}_n \left[\xi_{t,u} K\left(\frac{T-t}{h}\right) b(X) \right] \right\|_\infty.$$

For j -th element, $1 \leq j \leq p$,

$$|\mathbb{E}[\xi_{t,u} K\left(\frac{T-t}{h}\right) b_j(X)]| \leq c \mathbb{E}|b_j(X)| h^3 \leq c \|b_j(X)\|_{P,2} h^3 \leq ch^3.$$

where c is a universal constant independent of (j, t, u, n) . In addition,

$$nh^3 / (\log(p \vee n) hn)^{1/2} = (nh^5 / \log(p \vee n))^{1/2} \rightarrow 0.$$

So $\sup_{(t,u) \in \mathcal{TU}} \|\mathbb{E}[\xi_{t,u} K\left(\frac{T-t}{h}\right) b(X)]\|_\infty$ is asymptotically negligible. We focus on the centered term: $\sup_{g \in \mathcal{G}} |(\mathbb{P}_n - \mathcal{P})g|$, where $\mathcal{G} = \{\xi_{t,u} b_j(X) K\left(\frac{T-t}{h}\right) : (t, u) \in \mathcal{TU}, 1 \leq j \leq p\}$ with envelope $G = \bar{C}_K \zeta_n$. Note that $\sup_{g \in \mathcal{G}} \mathbb{E}g^2 \lesssim h$ and $\sup_Q N(\mathcal{G}, e_Q, \varepsilon \|G\|) \leq p \left(\frac{A}{\varepsilon}\right)^v$ for some $A > e$ and $v > 0$. So by Corollary 5.1 of Chernozhukov et al. (2014), we have

$$\mathbb{E} \sup_{g \in \mathcal{G}} |(\mathbb{P}_n - \mathcal{P})g| \leq (\log(p \vee n) h/n)^{1/2} + \log(p \vee n) \zeta_n / n \lesssim (\log(p \vee n) h/n)^{1/2}$$

because $\log(p \vee n) \zeta_n^2 / (nh) \rightarrow 0$. ■

Proof of Lemma A.3. We only show the first result. The rest can be derived in a similar manner. In addition, analogous to the proof of Lemma A.5 below, we only need to compute

$$\mathbb{E}(Y_u - \phi_{t,u}(X))^2 b_j^2(X) K\left(\frac{T-t}{h}\right)^2 h^{-1}.$$

Let $\kappa_1 = \int K(u)^2 du$. Then,

$$\begin{aligned}
& \mathbb{E}(Y_u - \phi_{t,u}(X))^2 b_j^2(X) K\left(\frac{T-t}{h}\right)^2 h^{-1} \\
&= \mathbb{E} \int \left[\phi_{t+hv,u}(X) - 2\phi_{t+hv,u}(X)\phi_{t,u}(X) + \phi_{t,u}^2(X) \right] f_{t+hv}(X) K(v)^2 dv b_j^2(X) \\
&\geq \underline{C} \mathbb{E} \int \left[\phi_{t,u}(X)(1 - \phi_{t,u}(X)) - h|\partial_t \phi_{t,u}(X)v| \right] K(v)^2 dv b_j^2(X) \\
&\geq \kappa_1 \underline{C}^2 (1 - \underline{C}) \mathbb{E} b_j^2(X) / 2 \geq C_\psi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \mathbb{E}(Y_u - \phi_{t,u}(X))^2 b_j^2(X) K\left(\frac{T-t}{h}\right)^2 h^{-1} \\
&= \mathbb{E} \int \left[\phi_{t+hv,u}(X) - 2\phi_{t+hv,u}(X)\phi_{t,u}(X) + \phi_{t,u}^2(X) \right] f_{t+hv}(X) K(v)^2 dv b_j^2(X) \\
&\leq \underline{C} \mathbb{E} \int \left[\phi_{t,u}(X)(1 - \phi_{t,u}(X)) + h|\partial_t \phi_{t,u}(X)v| \right] K(v)^2 dv b_j^2(X) \\
&\leq 2\kappa_1 \underline{C} \mathbb{E} b_j^2(X) \leq 1/C_\psi.
\end{aligned}$$

■

Proof of Lemma A.4. Following the same arguments as used in the proof of Lemma 3.1 and by Assumption 5, we have, w.p.a.1,

$$\begin{aligned}
& \sup_{t \in \mathcal{T}, \|\delta\|_2=1, \|\delta\|_0 \leq s_{l_n}} \|b(X)' \delta K\left(\frac{T-t}{h}\right)^{1/2}\|_{P_{n,2}}^2 \\
&\leq \sup_{t \in \mathcal{T}, \|\delta\|_2=1, \|\delta\|_0 \leq s_{l_n}} |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta)^2 K\left(\frac{T-t}{h}\right)| \\
&\quad + \sup_{t \in \mathcal{T}, \|\delta\|_2=1, \|\delta\|_0 \leq s_{l_n}} |\mathcal{P}(b(X)' \delta)^2 K\left(\frac{T-t}{h}\right)| \\
&\leq O_p(h\pi_{n1}) + \underline{C}^{-1} h \sup_{t \in \mathcal{T}, \|\delta\|_2=1, \|\delta\|_0 \leq s_{l_n}} |\mathcal{P}(b(X)' \delta)^2| \\
&\leq o_p(h) + \underline{C}^{-1} h \left(\sup_{t \in \mathcal{T}, \|\delta\|_2=1, \|\delta\|_0 \leq s_{l_n}} |(\mathcal{P}_n - \mathcal{P})(b(X)' \delta)^2| + \sup_{t \in \mathcal{T}, \|\delta\|_2=1, \|\delta\|_0 \leq s_{l_n}} |\mathcal{P}_n(b(X)' \delta)^2| \right) \\
&\leq o_p(h) + \underline{C}^{-1} h (O_p(\pi_{n2}) + \kappa''^2) \\
&\leq 2\underline{C}^{-1} \kappa''^2 h
\end{aligned}$$

where the second inequality holds because

$$\mathbb{E}(b(X)' \delta)^2 K\left(\frac{T-t}{h}\right) = \mathbb{E}(b(X)' \delta)^2 \int f_{t+hu}(X) K(u) du \leq \frac{\mathbb{E}(b(X)' \delta)^2}{\underline{C}}.$$

■

Proof of Lemma A.5. For (A.11), we denote $\mathcal{F} = \{h(\frac{1}{h}K(\frac{T-t}{h}) - f_t(X))^2 b_j^2(X) : t \in \mathcal{T}, j = 1, \dots, p\}$ with envelope $2\overline{C}_K^2 \zeta_n^2/h$. By the proof of Lemma A.6 below, the entropy of \mathcal{F} is bounded by $p(\frac{A}{\varepsilon})^v$. In addition, $\sup_{f \in \mathcal{F}} \mathbb{E} f^2 \lesssim \zeta_n^2$. Therefore,

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}^2 \lesssim O_p(\log(p \vee n) \zeta_n^2 / (nh)).$$

In addition,

$$h \mathbb{E} \left(\frac{1}{h} K\left(\frac{T-t}{h}\right) - f_t(X) \right)^2 b_j^2(X) = \mathbb{E} \int \left[K^2(u) - 2hK(u) + h^2 f_t^2(X) \right] f_{t+hu}(X) du b_j^2(X).$$

Note that $f_t(x)$ is bounded and bounded away from zero uniformly over $(t, x) \in \mathcal{TX}$. Therefore, there exists some positive constant C_f such that,

$$\begin{aligned} C_f &\leq h \inf_{t \in \mathcal{T}, j=1, \dots, p} \mathbb{E} \left(\frac{1}{h} K\left(\frac{T-t}{h}\right) - f_t(X) \right)^2 b_j^2(X) \\ &\leq h \sup_{t \in \mathcal{T}, j=1, \dots, p} \mathbb{E} \left(\frac{1}{h} K\left(\frac{T-t}{h}\right) - f_t(X) \right)^2 b_j^2(X) \leq 1/C_f. \end{aligned}$$

Therefore, w.p.a.1,

$$\begin{aligned} C_f/2 &\leq h \inf_{t \in \mathcal{T}, j=1, \dots, p} \mathcal{P}_n \left(\frac{1}{h} K\left(\frac{T-t}{h}\right) - f_t(X) \right)^2 b_j^2(X) \\ &\leq h \sup_{t \in \mathcal{T}, j=1, \dots, p} \mathcal{P}_n \left(\frac{1}{h} K\left(\frac{T-t}{h}\right) - f_t(X) \right)^2 b_j^2(X) \leq 2/C_f. \end{aligned}$$

For $k = 0$, we let $\mathcal{F} = \{\frac{1}{h} K^2(\frac{T-t}{h}) f_j^2(X) : t \in \mathcal{T}, j = 1, \dots, p\}$ with envelope $\overline{C}_K^2 \zeta_n^2/h$. By the same argument as above, we can show that, w.p.a.1,

$$C_0/2 \leq \inf_{t \in \mathcal{T}, j=1, \dots, p} \mathcal{P}_n \frac{1}{h} K^2\left(\frac{T-t}{h}\right) b_j^2(X) \leq \sup_{t \in \mathcal{T}, j=1, \dots, p} \mathcal{P}_n \frac{1}{h} K^2\left(\frac{T-t}{h}\right) b_j^2(X) \leq 2/C_0.$$

For $k \geq 1$, by Theorem 3.2 with $\widehat{\Psi}_t = \widehat{\Psi}_t^{k-1}$, we have, w.p.a.1,

$$\begin{aligned}
& \sup_{t \in \mathcal{T}, j=1, \dots, p} |\mathcal{P}_n h(\frac{1}{h} K(\frac{T-t}{h}) - \hat{f}_t^k(X))^2 b_j^2(X)| \\
& \leq 2 \sup_{t \in \mathcal{T}, j=1, \dots, p} |\mathcal{P}_n h(\frac{1}{h} K(\frac{T-t}{h}) - f_t(X))^2 b_j^2(X)| + \sup_{t \in \mathcal{T}, j=1, \dots, p} |\mathcal{P}_n h(\hat{f}_t^k(X) - f_t(X))^2 b_j^2(X)| \\
& \leq 2 \sup_{t \in \mathcal{T}, j=1, \dots, p} l_{t,0,j}^2 + O_p(\ell_n \log(p \vee n) s \zeta_n^2 h/n) \\
& \leq 2/C_f.
\end{aligned}$$

Similarly, we can show that w.p.a.1. $\inf_{t \in \mathcal{T}, j=1, \dots, p} \|\mathcal{P}_n h(\frac{1}{h} K(\frac{T-t}{h}) - \hat{f}_t^k(X))^2 b_j^2(X)\|_\infty \geq C_f/2$. This concludes (A.12) with $C_k = C_f$ for $k = 1, \dots, K$. Last, (A.13) holds with $l = \min(C_0 C_f/4, \dots, C_k C_f/4, 1)$ and $L = \max(4/(C_0 C_f), \dots, 4/(C_k C_f), 1)$. ■

Proof of Lemma A.6. Let $\mathcal{F} = \{\xi_t(X) b_j(X) : t \in \mathcal{T}, j = 1, \dots, p\}$ with envelope $F = 2\overline{C}_K \zeta_n/h$. Then

$$\begin{aligned}
\sup_{t \in \mathcal{T}} \|\mathcal{P}_n \widehat{\Psi}_{t,0}^{-1} \xi_t(X) b(X)\|_\infty & \leq \sup_{t \in \mathcal{T}} \|\widehat{\Psi}_{t,0}^{-1}\|_\infty \sup_{t \in \mathcal{T}} \|\mathcal{P}_n \xi_t(X) b(X)\|_\infty \\
& \leq O_p(1) (\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} + \|\mathcal{P}\|_{\mathcal{F}}) \\
& \leq O_p(1) (\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} + O(h^2))
\end{aligned}$$

where the second inequality because $\sup_{t \in \mathcal{T}} \|\widehat{\Psi}_{t,0}^{-1}\|_\infty = O_p(1)$ by Lemma A.5.

Next, we bound the term $\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}}$. Note that there exist some constants A and v independent of n such that the entropy of \mathcal{F} is bounded by

$$\sup_Q N(\mathcal{F}, e_Q, \varepsilon \|F\|_{Q,2}) \leq p \left(\frac{A}{\varepsilon} \right)^v.$$

In addition, $\sup_{f \in \mathcal{F}} \mathbb{E} f^2 \lesssim h^{-1}$. Therefore, by Corollary 5.1 of Chernozhukov et al. (2014),

$$\|\mathcal{P}_n - \mathcal{P}\|_{\mathcal{F}} \lesssim O_p \left((\log(p \vee n)/(nh))^{1/2} + \zeta_n^2 \log(p \vee n)/(nh) \right) = O_p((\log(p \vee n)/(nh))^{1/2}).$$

This concludes the proof. ■

Lemma B.1 Recall that $Q_{t,u}(\theta) = \mathcal{P}_n M(Y_u, X; \theta) K(\frac{T-t}{h})$. Let $\bar{q}_{A_{t,u}} = \inf_{\delta \in A_{t,u}} \frac{[\mathcal{P}_n \omega_{t,u} |b(X)' \delta|^2 K(\frac{T-t}{h})]^{3/2}}{\mathcal{P}_n \omega_{t,u} |b(X)' \delta|^3 K(\frac{T-t}{h})}$, $\Gamma_{t,u}^\delta = \|\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}$, and $s_{t,u} = \|\theta_{t,u}\|_0$. Let events E_1 , E_2 , and E_3 defined in the

proof of Lemma A.1 hold. Then, for any $(t, u) \in \mathcal{TU}$ and $\delta \in A_{t,u}$, we have

$$\begin{aligned} F_{t,u}(\delta) &=: Q_{t,u}(\theta_{t,u} + \delta) - Q_{t,u}(\theta_{t,u}) - \partial_{\theta} Q_{t,u}(\theta_{t,u})' \delta + 2 \left\| \frac{\tilde{r}_{t,u}^{\phi} K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_{n,2}} \Gamma_{t,u}^{\delta} \\ &\geq \min\left(\frac{1}{3} \|\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}^2, \frac{1}{3} \bar{q}_{A_{t,u}} \Gamma_{t,u}^{\delta}\right) \end{aligned}$$

and w.p.a.1,

$$\bar{q}_{A_{t,u}} \geq \frac{1}{\zeta_n} \min\left(\frac{\underline{\kappa} \sqrt{h}}{\sqrt{s_{t,u}}(1+2\tilde{c})}, \frac{(\lambda/n)(lC_{\lambda}-1)}{6c \|\widehat{\Psi}_{t,u,0}^{-1}\|_{\infty} \left\| \frac{r_{t,u}^{\phi} K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_{n,2}}}\right).$$

Proof. The proof follows closely from that of Lemma O.2 in Belloni et al. (2017a). Note that

$$Q_{t,u}(\theta_{t,u} + \delta) - Q_{t,u}(\theta_{t,u}) - \partial_{\theta} Q_{t,u}(\theta_{t,u})' \delta = \mathcal{P}_n[\tilde{g}_{t,u}(1) - \tilde{g}_{t,u}(0) - \tilde{g}'_{t,u}(0)],$$

where $\tilde{g}_{t,u}(s) = \log[1 + \exp(b(X)'(\theta_{t,u} + s\delta))] K(\frac{T-t}{h})$. Let $g_{t,u}(s) = \log[1 + \exp(b(X)'(\theta_{t,u} + s\delta) + \tilde{r}_{t,u}^{\phi})] K(\frac{T-t}{h})$. Then

$$g'_{t,u}(0) = (b(X)' \delta) \mathbb{E}(Y_u | X, T = t) K(\frac{T-t}{h}),$$

$$g''_{t,u}(0) = (b(X)' \delta)^2 \mathbb{E}(Y_u | X, T = t) (1 - \mathbb{E}(Y_u | X, T = t)) K(\frac{T-t}{h}),$$

and

$$g'''_{t,u}(0) = (b(X)' \delta)^3 \mathbb{E}(Y_u | X, T = t) (1 - \mathbb{E}(Y_u | X, T = t)) (1 - 2\mathbb{E}(Y_u | X, T = t)) K(\frac{T-t}{h}).$$

By Lemmas O.3 and O.4 in Belloni et al. (2017a),

$$g_{t,u}(1) - g_{t,u}(0) - g'_{t,u}(0) \geq \omega_{t,u} K(\frac{T-t}{h}) \left[\frac{(b(X)' \delta)^2}{2} - \frac{|b(X)' \delta|^3}{6} \right].$$

Let $\Upsilon_{t,u}(s) = \tilde{g}_{t,u}(s) - g_{t,u}(s)$. Then

$$|\Upsilon'_{t,u}(s)| \leq \|\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}\| \left| \frac{\tilde{r}_{t,u}^{\phi} K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right|.$$

It follows that

$$\begin{aligned}
& \mathcal{P}_n |\tilde{g}_{t,u}(1) - g_{t,u}(1) - (\tilde{g}_{t,u}(0) - g_{t,u}(0)) - (\tilde{g}'_{t,u}(0) - g'_{t,u}(0))| \\
&= \mathcal{P}_n |\Upsilon_{t,u}(1) - \Upsilon_{t,u}(0) - \Upsilon'_{t,u}(0)| \\
&\leq 2\mathcal{P}_n |\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}| \left| \frac{\tilde{r}_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right| \\
&\leq 2\Gamma_{t,u}^\delta \left\| \frac{\tilde{r}_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}} \right\|_{P_{n,2}},
\end{aligned}$$

and

$$F_{t,u}(\delta) \geq \frac{1}{2} \mathcal{P}_n \omega_{t,u} (b(X)' \delta)^2 K(\frac{T-t}{h}) - \frac{1}{6} \mathcal{P}_n \omega_{t,u} |b(X)' \delta|^3 K(\frac{T-t}{h}).$$

We consider two cases: $\Gamma_{t,u}^\delta \leq \bar{q}_{A_{t,u}}$ and $\Gamma_{t,u}^\delta > \bar{q}_{A_{t,u}}$.

First, if $\Gamma_{t,u}^\delta \leq \bar{q}_{A_{t,u}}$, we have

$$\mathcal{P}_n \omega_{t,u} |b(X)' \delta|^3 K(\frac{T-t}{h}) \leq \|\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}^2$$

and

$$F_{t,u}(\delta) \geq \frac{1}{3} (\Gamma_{t,u}^\delta)^2.$$

When $\Gamma_{t,u}^\delta > \bar{q}_{A_{t,u}}$, we let $\tilde{\delta} = \delta \bar{q}_{A_{t,u}} / \Gamma_{t,u}^\delta \in A_{t,u}$. Then by the convexity of $F_{t,u}(\delta)$ and the fact that $F_{t,u}(0) = 0$, we have

$$F_{t,u}(\delta) \geq \frac{\Gamma_{t,u}^\delta}{\bar{q}_{A_{t,u}}} F_{t,u}(\tilde{\delta}) \geq \frac{\Gamma_{t,u}^\delta}{\bar{q}_{A_{t,u}}} \left(\frac{1}{3} \|\omega_{t,u}^{1/2} b(X)' \tilde{\delta} K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}^2 \right) = \frac{1}{3} \bar{q}_{A_{t,u}} \Gamma_{t,u}^\delta.$$

Consequently, we have $F_{t,u}(\delta) \geq \min(\frac{1}{3} (\Gamma_{t,u}^\delta)^2, \frac{\bar{q}_{A_{t,u}}}{3} \Gamma_{t,u}^\delta)$.

For the second result, note that

$$\bar{q}_{A_{t,u}} \geq \inf_{\delta \in A_{t,u}} \frac{\|\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}}{\zeta_n \|\delta\|_1}.$$

If $\delta \in \Delta_{2\tilde{c}, t, u}$, then by Lemma 3.1

$$\frac{\Gamma_{t,u}^\delta}{\zeta_n \|\delta\|_1} \geq \frac{\|\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}}{\zeta_n \|\delta_{S_{t,u}^0}\|_2 (1 + 2\tilde{c}) s_{t,u}^{1/2}} \geq \frac{1}{\zeta_n} \frac{\underline{\kappa} \sqrt{h}}{\sqrt{s_{t,u}} (1 + 2\tilde{c})}.$$

If $\|\delta\|_1 \leq I_{t,u}$, where $I_{t,u}$ is defined in the proof of Lemma A.1, then

$$\frac{\Gamma_{t,u}^\delta}{\zeta_n \|\delta\|_1} \geq \frac{\|\omega_{t,u}^{1/2} b(X)' \delta K(\frac{T-t}{h})^{1/2}\|_{P_{n,2}}}{\zeta_n I_{t,u}} \geq \frac{1}{\zeta_n} \frac{(\lambda/n)(lC_\lambda - 1)}{6c \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty \|\frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}}\|_{P_{n,2}}}.$$

Combining the above two results, we obtain that

$$\bar{q}_{A_{t,u}} \geq \frac{1}{\zeta_n} \min\left(\frac{\underline{\kappa}\sqrt{h}}{\sqrt{s_{t,u}}(1+2\tilde{c})}, \frac{(\lambda/n)(lC_\lambda - 1)}{6c \|\widehat{\Psi}_{t,u,0}^{-1}\|_\infty \|\frac{r_{t,u}^\phi K(\frac{T-t}{h})^{1/2}}{\omega_{t,u}^{1/2}}\|_{P_{n,2}}}\right).$$

■

Lemma B.2 Let $q_y(t)$ be the y -th quantile of $Y(t)$, $f_{Y(t)}(\cdot)$ the unconditional density of $Y(t)$,

$$F(t, y) = \int_0^1 1\{\alpha(t, \psi^{\leftarrow}(v)) \leq y\} dv, \quad F(t, y|h_n) = \int_0^1 1\{\hat{\alpha}^*(t, \psi^{\leftarrow}(v)) \leq y\} dv,$$

$s_n = (nh)^{-1/2}$, $h_n(t, v) = (nh)^{1/2}(\hat{\alpha}^*(t, \psi^{\leftarrow}(v)) - \alpha(t, \psi^{\leftarrow}(v)))$, and $J_n(t, y) = \frac{F(t, y|h_n) - F(t, y)}{s_n}$. Then, for δ_n being either 1 or $h^{1/2}$, depending on either Assumption 5.1 or 5.2 is in place,

$$\frac{F(t, y|h_n) - F(t, y)}{s_n} + \frac{h_n(t, \psi(q_y(t)))\psi'(q_y(t))}{f_{Y(t)}(q_y(t))} = o_p(\delta_n) \quad (\text{B.7})$$

and

$$\frac{\hat{\alpha}^{*r}(t, u) - \alpha(t, u)}{s_n} + \frac{F(t, \alpha(t, u)|h_n) - F(t, \alpha(t, u))f_{Y(t)}(u)}{s_n \psi'(u)} = o_p(\delta_n). \quad (\text{B.8})$$

uniformly over $(t, y) \in \{(t, y) : y = \alpha(t, \psi^{\leftarrow}(v)), (t, v) \in \mathcal{T} \times [0, 1]\}$.

Proof. Let $Q(t, v) = \alpha(t, \psi^{\leftarrow}(v))$ for $v \in [0, 1]$. Then, we have

$$F(t, y) = \int_0^1 1\{Q(t, v) \leq y\} dv \quad \text{and} \quad F(t, y|h_n) = \int_0^1 1\{Q(t, v) + s_n h_n \leq y\} dv.$$

We prove the lemma by applying Propositions C.1 and C.2 in Appendix C.

First, we verify Assumption 7 with $(\delta_n, \varepsilon_n) = (1, (nh)^{-1/2} \log(n))$ and $(\delta_n, \varepsilon_n) = (h^{1/2}, (nh)^{-1/2} \log(n))$ under Assumptions 5.1 and 5.2, respectively, in order to apply Proposition C.1 to prove (B.7). We only consider the case in which $\delta_n = h^{1/2}$ as the $\delta_n = 1$ case can be studied similarly. Note that $Q(t, v) = \alpha(t, \psi^{\leftarrow}(v))$, $\partial_u \alpha(t, u) = f_{Y(t)}(u) > 0$ uniformly over $(t, u) \in \mathcal{T}\mathcal{U}$, and $\psi(\cdot)$ can be chosen such that $\partial_v \psi^{\leftarrow}(v) > 0$ uniformly over $v \in [0, 1]$. This verifies Assumption 7.1.

For Assumption 7.2, by Theorem 3.3, $\sup_{(t,v) \in \mathcal{T} \times [0,1]} |h_n(t,v)| = O_p(\log^{1/2}(n))$. So we can take $\varepsilon_n = (nh)^{-1/2} \log(n)$. In addition, $\sup_{(t,v) \in \mathcal{T} \times [0,1]} |h_n^2(t,v)| s_n = O_p(\log(n)(nh)^{-1/2}) = o_p(h^{1/2})$ because $nh^2/\log^2(n) \rightarrow \infty$. So we only need to show

$$\sup_{(t,v,v') \in \mathcal{T} \times [0,1]^2, |v-v'| \leq \varepsilon_n} |h_n(t,v) - h_n(t,v')| = o_p(h^{1/2}). \quad (\text{B.9})$$

Let

$$\mathcal{G} = \left\{ \begin{array}{l} \eta \Pi_{t,u}(W_u, \phi_{t,u}, f_t) - \Pi_{t,u'}(W'_u, \phi_{t,u'}, f_t) : u = \psi^{\leftarrow}(v), u' = \psi^{\leftarrow}(v'), \\ (t,v,v') \in \mathcal{T} \times [0,1]^2, |v-v'| \leq \varepsilon_n \end{array} \right\}$$

with envelope $c\eta h^{-1}$. By Theorem 3.3, we have

$$h_n(t,v) - h_n(t,v') = (\mathcal{P}_n - \mathcal{P})g + R_n(t, \psi^{\leftarrow}(v)) - R_n(t, \psi^{\leftarrow}(v')).$$

$\sup_{(t,v) \in \mathcal{T} \times [0,1]} R_n(t, \psi^{\leftarrow}(v)) = o_p(\delta_n)$. So we only have to show that

$$\sup_{g \in \mathcal{G}} |(\mathcal{P}_n - \mathcal{P})g| = o_p(h^{1/2}).$$

We know that \mathcal{G} is VC-type with fixed VC index and that $\sup_{g \in \mathcal{G}} \mathbb{E}g^2 \leq \varepsilon_n h^{-1}$. In addition, as shown in the proof of Theorem 3.4, $\|\max_{1 \leq i \leq n} |\eta_i h^{-1}|\|_{P,2} \leq \log(n)$. Therefore, by Corollary 5.1 of Chernozhukov et al. (2014), we have

$$(nh)^{1/2} \|(\mathcal{P}_n - \mathcal{P})g\|_{\mathcal{G}} = O_p((\log(n)\varepsilon_n)^{1/2}).$$

Given $\varepsilon_n = (nh)^{-1/2} \log(n)$, $(\log(n)\varepsilon_n)^{1/2} = o(h^{1/2})$ because $h = C_h n^{-H}$ for some $H < 1/3$. This establishes (B.9). Then (B.7) follows by Proposition C.1.

To prove (B.8), we apply Proposition C.2 by verifying Assumption 8. We note that $\hat{\alpha}^{*r}(t,u) = F^{\leftarrow}(t, \psi(u)|h_n)$ and $J_n(t,y) = \frac{F(t,y|h_n) - F(t,y)}{s_n}$. Furthermore, notice that $\alpha^{*r}(t,u) = \alpha(t,u) = F^{\leftarrow}(t, \psi(u))$, $F^{\leftarrow}(t,v) = \alpha(t, \psi^{\leftarrow}(v))$,

$$F(t,y) = \int_0^1 1\{Q(t,v) \leq y\} dv = \int_0^1 1\{v \leq \psi(q_y(t))\} dv = \psi(q_y(t)),$$

and

$$\partial_y F(t,y) = \psi'(q_y(t))/f_{Y(t)}(q_y(t)).$$

Because $f_{Y(t)}(q_y(t))$ is bounded and bounded away from zero uniformly over $(t,y) \in \mathcal{T}\mathcal{Y}$, so be $\partial_y F(t,y)$. In addition,

$$\partial_{yy}^2 F(t,y) = f''(q_y(t))/f_{Y(t)}^2(q_y(t)) - \phi'(q_y(t))f'_{Y(t)}(q_y(t))/f_{Y(t)}^3(q_y(t)),$$

which is bounded because $f'_{Y(t)}(q_y(t))$ is bounded. This verifies Assumption 8.2.

For Assumption 8.3, we note that

$$J_n(t, y) = \frac{F(t, y|h_n) - F(t, y|h_n)}{s_n} = -\frac{h_n(t, \psi(q_y(t)))\psi'(q_y(t))}{f_{Y(t)}(q_y(t))} + o_p(\delta_n),$$

where the $o_p(\delta_n)$ is uniform over $(t, y) \in \mathcal{TY}$. In addition, by definition, $(t, q_y(t)) \in \mathcal{TU}$, $f_{Y(t)}(q_y(t))$ is bounded away from zero, and we can choose ψ such that $\psi'(q_y(t))$ is bounded. Therefore, by Theorem 3.3 ,

$$\sup_{(t,y) \in \mathcal{TY}} |J_n(t, y)| = O_p\left(\sup_{(t,u) \in \mathcal{TU}} |h_n(t, \psi(u))|\right) + o_p(\delta_n) = O_p(\log^{1/2}(n)).$$

We can choose $\varepsilon_n = s_n \log(n)$. In addition, $\sup_{(t,y) \in \mathcal{TY}} |J_n(t, y)|^2 s_n = o_p(h^{1/2})$ because $nh^3 \rightarrow \infty$. So we only need to show that

$$\sup_{(t,y,y') \in \mathcal{TY}\mathcal{Y}, |y-y'| \leq \max(\varepsilon_n, s_n \delta_n)} |J_n(t, y) - J_n(t, y')| = o_p(\delta_n).$$

Note that, for $v = \psi(Q_{Y_t}(y))$ and $v' = \psi(Q_{Y_t}(y'))$

$$|J_n(t, y) - J_n(t, y')| \lesssim |h_n(t, v) - h_n(t, v')| + o_p(\delta_n).$$

In addition, $\phi(Q_{Y_t}(y))$ is Lipschitz uniformly over $(t, y) \in \mathcal{TY}$. Thus,

$$\begin{aligned} & \sup_{(t,y,y') \in \mathcal{TY}\mathcal{Y}, |y-y'| \leq \max(\varepsilon_n, s_n \delta_n)} |J_n(t, y) - J_n(t, y')| \\ & \leq \sup_{(t,v,v') \in \mathcal{T} \times [0,1]^2, |v-v'| \leq C\varepsilon_n} |h_n(t, v) - h_n(t, v')| = o_p(\delta_n), \end{aligned}$$

given that $h = C_h n^{-H}$ for some $H < 1/3$. This completes the verification of Assumption 8.2.

Last, it is essentially the same as above to verify Assumption 8 for $J_n(t, u) = (nh)^{1/2}(\hat{\alpha}^{*r}(t, u) - \alpha(t, u))$. The proof is omitted. ■

Lemma B.3 *Suppose the conditions in Theorem 3.5 hold. Then*

$$\begin{aligned} & e'_2(G\hat{\Sigma}_2)^{-1}U_n(t, \tau) \\ & = \frac{1}{n} \sum_{j=1}^n \eta_j (\kappa_2 f_{Y(t)}(q_\tau(t)) f_t(X_j) h^2)^{-1} \left[Y_{q_\tau(t), j} - \phi_{t, q_\tau(t)}(X_j) \right] \bar{K}\left(\frac{T_j - t}{h}\right) + o_p^*((nh^3)^{-1/2}). \end{aligned}$$

Proof. Note that

$$U_n(t, \tau) = \frac{2}{n} \sum_{j=1}^n \eta_j \mathcal{P}\Gamma^s(\cdot, \Upsilon_j; t, \tau) + \mathcal{U}_n H(\cdot, \cdot; t, \tau), \quad (\text{B.10})$$

where \mathcal{U}_n assigns probability $\frac{1}{n(n-1)}$ to each pair of observations and

$$H(\Upsilon_i, \Upsilon_j; t, \tau) = \eta_i \eta_j \Gamma^s(\Upsilon_i, \Upsilon_j; t, \tau) - \eta_i \mathcal{P}\Gamma^s(\cdot, \Upsilon_j; t, \tau) - \eta_j \mathcal{P}\Gamma^s(\Upsilon_i, \cdot; t, \tau) + \mathcal{P}\Gamma^s(\cdot, \cdot; t, \tau).$$

Further denote $\tilde{\eta}_i = \eta_i 1\{|\eta_i| \leq C' \log(n)\}$, $\chi_n = 1 - \mathbb{E}\tilde{\eta}_i$, and

$$\begin{aligned} \tilde{H}(\Upsilon_i, \Upsilon_j; t, \tau) &= \tilde{\eta}_i \tilde{\eta}_j \Gamma^s(\Upsilon_i, \Upsilon_j; t, \tau) - (1 - \chi_n) \tilde{\eta}_i \mathcal{P}\Gamma^s(\cdot, \Upsilon_j; t, \tau) \\ &\quad - (1 - \chi_n) \tilde{\eta}_j \mathcal{P}\Gamma^s(\Upsilon_i, \cdot; t, \tau) + (1 - \chi_n)^2 \mathcal{P}\Gamma^s(\cdot, \cdot; t, \tau). \end{aligned}$$

Since η has a sub-exponential tail, we can choose C' large enough such that

$$|\chi_n| \leq n^{-2} \quad \text{and} \quad \lim_{n \rightarrow \infty} P\left(\max_{1 \leq i \leq n} |\eta_i| > C' \log(n)\right) = 0.$$

This means that, w.p.a.1,

$$\max_{(t, \tau) \in \mathcal{TI}} \left| \mathcal{U}_n \left[H(\Upsilon_i, \Upsilon_j; t, \tau) - \tilde{H}(\Upsilon_i, \Upsilon_j; t, \tau) \right] \right| \lesssim n^{-1},$$

because

$$\left| K\left(\frac{T_i - t}{h}\right)(T_i - t) \right| \leq Ch, \quad (\text{B.11})$$

$\sup_{(t, \tau) \in \mathcal{TI}} |\Gamma^s(\Upsilon_i, \Upsilon_j; t, \tau)| \lesssim h^{-3}$, and $nh^3 \rightarrow \infty$.

We next bound the U-process $\mathcal{U}_n \tilde{H}(\cdot, \cdot; t, \tau) = (C_n^2)^{-1} \sum_{1 \leq i < j \leq n} \tilde{H}(\Upsilon_i, \Upsilon_j; t, \tau)$. Let $\mathcal{H} = \{\tilde{H}(\cdot, \cdot; t, \tau), (t, \tau) \in \mathcal{TI}\}$. Then \mathcal{H} is Euclidean and has envelop $(C \log^2(n)h^{-2}, C \log^2(n)h^{-3})'$ for some large constant C . Denote $g(\Upsilon; t, \tau) = \mathbb{E}H^2(\Upsilon, \cdot; t, \tau)$ with envelope $(C \log^2(n)h^{-3}, C \log^2(n)h^{-5})'$.

By simple moment calculations, we have

$$\sup_{(t, \tau) \in \mathcal{TI}} \mathbb{E}g(\Upsilon; t, \tau) \lesssim (Ch^{-2}, Ch^{-4})'$$

and

$$\sup_{(t, \tau) \in \mathcal{TI}} \mathbb{E}g^2(\Upsilon; t, \tau) \lesssim (Ch^{-5}, Ch^{-9})'.$$

Then, by Proposition D.1 in Appendix D, we have

$$\sup_{(t, \tau) \in \mathcal{TI}} \mathcal{U}_n \tilde{H}(\cdot, \cdot; t, \tau) = (O_p(\log(n)(nh)^{-1}), O_p(\log(n)(nh^2)^{-1}))'$$

and thus

$$\sup_{(t,\tau) \in \mathcal{TI}} \mathcal{U}_n H(\cdot, \cdot; t, \tau) = (O_p(\log(n)(nh)^{-1}), O_p(\log(n)(nh^2)^{-1}))'. \quad (\text{B.12})$$

We now compute $\mathcal{P}\Gamma^s(\cdot, \Upsilon_j; t, \tau)$.

$$\begin{aligned} & \mathcal{P}\Gamma^s(\cdot, \Upsilon_j; t, \tau) \\ = & \left\{ \int \frac{f_T(t+hw)}{\bar{f}_{Y_{t+hw}}(q_\tau(t+hw))} \left(\frac{Y_{q_\tau(t+hw),j} - \phi_{t+hw,q_\tau(t+hw)}(X_j)}{f_{t+hw}(X_j)h} K\left(\frac{T_j-t-hv}{h}\right) + \phi_{t+hw,q_\tau(t+hw)}(X_j) - \tau \right) K(v) dv \right. \\ & \left. \int \frac{vf_T(t+hw)}{hf_{Y_{t+hw}}(q_\tau(t+hw))} \left(\frac{Y_{q_\tau(t+hw),j} - \phi_{t+hw,q_\tau(t+hw)}(X_j)}{f_{t+hw}(X_j)h} K\left(\frac{T_j-t-hv}{h}\right) + \phi_{t+hw,q_\tau(t+hw)}(X_j) - \tau \right) K(v) dv \right\}. \end{aligned}$$

By the usual maximal inequality,

$$\begin{aligned} & \sup_{(t,\tau) \in \mathcal{TI}} \left| \frac{2}{n} \sum_{j=1}^n \eta_j \int \frac{f_T(t+hw)}{f_{Y_{t+hw}}(q_\tau(t+hw))} \left(\frac{Y_{q_\tau(t+hw),j} - \phi_{t+hw,q_\tau(t+hw)}(X_j)}{f_{t+hw}(X_j)h} K\left(\frac{T_j-t-hv}{h}\right) \right. \right. \\ & \left. \left. + \phi_{t+hw,q_\tau(t+hw)}(X_j) - \tau \right) K(v) dv \right| = O_p(\log^{1/2}(n)(nh)^{-1/2}). \end{aligned}$$

For the second element in $\mathcal{P}\Gamma^s(\cdot, \Upsilon_j; t, \tau)$, we first note that

$$\sup_{(t,\tau) \in \mathcal{TI}} \left| \int \frac{vf_T(t+hw)}{hf_{Y_{t+hw}}(q_\tau(t+hw))} \left(\phi_{t+hw,q_\tau(t+hw)}(X_j) - \tau \right) K(v) dv \right| = O_p(h^2).$$

So we can focus on

$$\int v \bar{f}(X_j; t+hw) \frac{Y_{q_\tau(t+hw),j} - \phi_{t+hw,q_\tau(t+hw)}(X_j)}{h^2} K\left(\frac{T_j-t-hv}{h}\right) K(v) dv,$$

where $\bar{f}(X_j; t) = \frac{f_T(t)}{\bar{f}_{Y_t}(q_\tau(t))f_t(X_j)}$. Since

$$\sup_{(t,\tau) \in \mathcal{TI}} \mathbb{E} \int v [\bar{f}(X_j; t+hw) - \bar{f}(X_j; t)] \frac{Y_{q_\tau(t+hw),j} - \phi_{t+hw,q_\tau(t+hw)}(X_j)}{h^2} K\left(\frac{T_j-t-hv}{h}\right) K(v) dv = O(h^2)$$

and

$$\begin{aligned} & \sup_{(t,\tau) \in \mathcal{TI}} \mathbb{E} \left[\int v [\bar{f}(X_j; t+hw) - \bar{f}(X_j; t)] \frac{Y_{q_\tau(t+hw),j} - \phi_{t+hw,q_\tau(t+hw)}(X_j)}{h^2} K\left(\frac{T_j-t-hv}{h}\right) K(v) dv \right]^2 \\ & \lesssim \sup_{t \in \mathcal{T}} \int h^{-2} v^2 \mathbb{E} K^2\left(\frac{T_j-t-hv}{h}\right) K(v) dv \lesssim h^{-1}, \end{aligned}$$

we have

$$\begin{aligned} & \sup_{(t,\tau) \in \mathcal{TI}} \frac{1}{n} \sum_{j=1}^n \eta_j \int v [\bar{f}(X_j; t+ hv) - \bar{f}(X_j; t)] \frac{Y_{q_\tau(t+ hv),j} - \phi_{t+ hv, q_\tau(t+ hv)}(X_j)}{h^2} K\left(\frac{T_j - t - hv}{h}\right) K(v) dv \\ & = O_p(\log^{1/2}(n)(nh)^{-1/2}). \end{aligned}$$

Furthermore, uniformly over $(t, \tau) \in \mathcal{TI}$,

$$\int \frac{v \bar{f}(X_j; t)}{h^2} \left[Y_{q_\tau(t+ hv),j} - \phi_{t+ hv, q_\tau(t+ hv)}(X_j) - (Y_{q_\tau(t),j} - \phi_{t, q_\tau(t)}(X_j)) \right] K\left(\frac{T_j - t - hv}{h}\right) K(v) dv$$

has $O(h^2)$ bias and

$$\begin{aligned} & \mathbb{E} \left\{ \int \frac{v \bar{f}(X_j; t)}{h^2} \left[Y_{q_\tau(t+ hv),j} - \phi_{t+ hv, q_\tau(t+ hv)}(X_j) - (Y_{q_\tau(t),j} - \phi_{t, q_\tau(t)}(X_j)) \right] K\left(\frac{T_j - t - hv}{h}\right) K(v) dv \right\}^2 \\ & \lesssim \int \mathbb{E} v^2 h^{-4} \left(| \phi_{T_j, q_\tau(t+ hv)}(X_j) - \phi_{T_j, q_\tau(t)}(X_j) | + (\phi_{t+ hv, q_\tau(t+ hv)}(X_j) - \phi_{t, q_\tau(t)}(X_j))^2 \right) \\ & \quad \times K^2\left(\frac{T_j - t - hv}{h}\right) K(v) dv \lesssim h^{-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{1}{n} \sum_{j=1}^n \eta_j \int \frac{v \bar{f}(X_j; t)}{h^2} [Y_{q_\tau(t+ hv),j} - \phi_{t+ hv, q_\tau(t+ hv)}(X_j) - (Y_{q_\tau(t),j} - \phi_{t, q_\tau(t)}(X_j))] K\left(\frac{T_j - t - hv}{h}\right) K(v) dv \\ & = O_p^*(\log^{1/2}(n)(nh^2)^{-1/2}). \end{aligned}$$

Combining the above results and denoting $\bar{K}(u) = \int v K(u - v) K(v) dv$, we have

$$\begin{aligned} & \frac{1}{n} \sum_{j=1}^n \eta_j \int v \bar{f}(X_j; t+ hv) \frac{Y_{q_\tau(t+ hv),j} - \phi_{t+ hv, q_\tau(t+ hv)}(X_j)}{h^2} K\left(\frac{T_j - t - hv}{h}\right) K(v) dv \\ & = \frac{1}{n} \sum_{j=1}^n \eta_j \frac{\bar{f}(X_j; t)}{h^2} [Y_{q_\tau(t),j} - \phi_{t, q_\tau(t)}(X_j)] \bar{K}\left(\frac{T_j - t}{h}\right) = O_p^*(\log^{1/2}(n)(nh^3)^{-1/2}) \end{aligned}$$

and

$$\frac{2}{n} \sum_{j=1}^n \eta_j \mathcal{P}\Gamma^s(\cdot, \Upsilon_j; t, \tau) = \left\{ \begin{array}{l} O_p^*(\log^{1/2}(n)(nh)^{-1/2}) \\ \frac{1}{n} \sum_{j=1}^n \eta_j \frac{\bar{f}(X_j; t)}{h^2} [Y_{q_\tau(t),j} - \phi_{t, q_\tau(t)}(X_j)] \bar{K}\left(\frac{T_j - t}{h}\right) + O_p^*(\log^{1/2}(n)(nh^2)^{-1/2}) \end{array} \right\} \quad (\text{B.13})$$

Combining (B.10), (B.12), and (B.13), we have the desired results. \blacksquare

C Rearrangement Operator on A Local Process

The rearrangement operator has been previously studied by Chernozhukov et al. (2010), in which they required the underlying process to be tight to apply the continuous mapping theorem. However, the local processes encountered in our paper are not tight due to the presence of the kernel function. Therefore, the original results on the rearrangement operator cannot directly apply to our case. Instead, in this section, we extend the results in Chernozhukov et al. (2010) and show that the linear expansion of the rearrangement operator is valid under general conditions, allowing for the underlying process not to be tight.

Let $Q(t, v)$ be a generic monotonic function in $v \in [0, 1]$. The functional Ψ maps $Q(t, v)$ to $F(t, y)$ as follows:

$$\Psi(Q)(t, y) \equiv F(t, y) = \int_0^1 1\{Q(t, v) \leq y\} dv.$$

We want to derive a linear expansion of $\Psi(Q + s_n h_n) - \Psi(Q)$ where $s_n \downarrow 0$ as the sample size $n \rightarrow \infty$ and $h_n(t, v)$ is some perturbation function.

Assumption 7

1. $Q(t, v)$ is twice differentiable w.r.t. v with both derivatives bounded. In addition, $\partial_v Q(t, v) > c$ for some positive constant c , uniformly over $(t, v) \in \mathcal{T} \times [0, 1]$.
2. There exist two vanishing sequences ε_n and δ_n such that

$$\sup_{(t, v, v') \in \mathcal{T} \times [0, 1]^2, |v - v'| \leq \varepsilon_n} |h_n(t, v) - h_n(t, v')| = o(\delta_n),$$

$$\sup_{(t, v) \in \mathcal{T} \times [0, 1]} |h_n(t, v)| s_n = o(\varepsilon_n), \quad \text{and} \quad \sup_{(t, v) \in \mathcal{T} \times [0, 1]} |h_n(t, v)|^2 s_n = o(\delta_n).$$

The following proposition extends the first part of Proposition 2 in Chernozhukov et al. (2010).

Proposition C.1 *Let $(t, y) \in \mathcal{TY} \equiv \{(t, y) : y = Q(t, v), (t, v) \in \mathcal{T} \times [0, 1]\}$, $F(t, y|h_n) = \int_0^1 1\{Q(t, v) + s_n h_n(t, v) \leq y\} dv$, and $y = Q(t, v^y)$. If Assumption 7 holds, then*

$$\frac{F(t, y|h_n) - F(t, y)}{s_n} - \left(\frac{-h_n(t, v^y)}{\partial_v Q(t, v^y)} \right) = o(\delta_n)$$

uniformly over $(t, y) \in \mathcal{TY}$.

Proof. Consider $(t_n, y_n) \rightarrow (t_0, y_0)$ and denote v_n as $y_n = Q(t_n, v_n)$. Note that

$$\begin{aligned} F(t_n, y_n | h_n) &= \int_0^1 1\{Q(t_n, v) + s_n h_n(t_n, v) \leq y_n\} dv \\ &= \int_0^1 1\{Q(t_n, v) + s_n(h_n(t_n, v_n) + h_n(t_n, v) - h_n(t_n, v_n)) \leq y_n\} dv. \end{aligned}$$

Let $\mathbb{B}_\varepsilon(v) = \{v' : |v - v'| \leq \varepsilon\}$. For fixed n , if $v \in \mathbb{B}_{\varepsilon_n}(v_n) \cap [0, 1]$, by Assumption 7,

$$h_n(t_n, v) - h_n(t_n, v_n) = o(\delta_n).$$

Then for any $\delta > 0$, there exists n_1 such that if $n \geq n_1$, $|h_n(t_n, v) - h_n(t_n, v_n)| \leq \delta \delta_n$ and

$$F(t_n, y_n | h_n) \leq \int_0^1 1\{Q(t_n, v) + s_n(h_n(t_n, v_n) - \delta \delta_n) \leq y_n\} dv.$$

If $v \notin \mathbb{B}_{\varepsilon_n}(v_n)$, then there exists n_2 such that for $n \geq n_2$,

$$|Q(t_n, v) - y_n| \geq c\varepsilon_n. \tag{C.1}$$

Furthermore, by Assumption 7,

$$s_n h_n(t_n, v) \leq \sup_{(t,v) \in \mathcal{T} \times [0,1]} |h_n(t, v)| s_n = o(\varepsilon_n).$$

Therefore,

$$F(t_n, y_n | h_n) = \int_0^1 1\{Q(t_n, v) \leq y_n\} dv$$

and

$$\begin{aligned} & \frac{F(t_n, y_n | h_n) - F(t_n, y_n)}{s_n} - \left(\frac{-h_n(t_n, v_n)}{\partial_v Q(t_n, v_n)} \right) \\ & \leq \int_{\mathbb{B}_{\varepsilon_n}(v_n)} \frac{1}{s_n} \left(1\{Q(t_n, v) + s_n(h_n(t_n, v_n) - \delta \delta_n) \leq y_n\} - 1\{Q(t_n, v) \leq y_n\} \right) dv + \left(\frac{h_n(t_n, v_n)}{\partial_v Q(t_n, v_n)} \right) \\ & = \int_{\mathbb{J}_n \cap [y_n, y_n - s_n(h_n(t_n, v_n) - \delta \delta_n)]} \frac{dy}{s_n \partial_v Q(t_n, v_n(y))} + \frac{h_n(t_n, v_n)}{\partial_v Q(t_n, v_n)}, \end{aligned} \tag{C.2}$$

where the equality follows by the change of variables: $y = Q(t_n, v)$, $v_n(y) = Q^{\leftarrow}(t_n, \cdot)(y)$, and \mathbb{J}_n is the image of $\mathbb{B}_{\varepsilon_n}(v_n)$. By (C.1) again, we know that $[y_n, y_n - s_n(h_n(t_n, v_n) - \delta \delta_n)]$ is nested by \mathbb{J}_n for n sufficiently large. In addition, since $\partial_v Q(t, v) > c$ uniformly over $\mathcal{T} \times [0, 1]$, for

$y \in [y_n, y_n - s_n(h_n(t_n, v_n) - \delta\delta_n)]$,

$$|v_n(y) - v_n| = |Q^\leftarrow(t_n, \cdot)(y) - Q^\leftarrow(t_n, \cdot)(y_n)| \leq C s_n \left(\sup_{(t,v) \in \mathcal{T} \times [0,1]} |h_n(t, v)| + \delta\delta_n \right).$$

Then the r.h.s. of (C.2) is bounded from above by

$$\begin{aligned} & \frac{\delta\delta_n}{\partial_v Q(t_n, v_n)} + \int_{[y_n, y_n - s_n(h_n(t_n, v_n) - \delta\delta_n)]} \left(\frac{1}{\partial_v Q(t_n, v_n(y))} - \frac{1}{\partial_v Q(t_n, v_n)} \right) \frac{dy}{s_n} \\ & \leq C\delta\delta_n + C s_n \left(\sup_{(t,v) \in \mathcal{T} \times [0,1]} |h_n^2(t, v)| + \delta^2\delta_n^2 \right) \leq C'\delta\delta_n. \end{aligned}$$

Since δ is arbitrary, by letting $\delta \rightarrow 0$, we obtain that

$$\frac{F(t_n, y_n | h_n) - F(t_n, y_n)}{s_n} - \left(\frac{-h_n(t_n, v_n)}{\partial_v Q(t_n, v_n)} \right) \leq o(\delta_n).$$

Similarly, we can show that

$$\frac{F(t_n, y_n | h_n) - F(t_n, y_n)}{s_n} - \left(\frac{-h_n(t_n, v_n)}{\partial_v Q(t_n, v_n)} \right) \geq o(\delta_n).$$

Therefore, we have proved that

$$\frac{F(t_n, y_n | h_n) - F(t_n, y_n)}{s_n} - \left(\frac{-h_n(t_n, v_n)}{\partial_v Q(t_n, v_n)} \right) = o(\delta_n).$$

Since the above result holds for any sequence of (t_n, y_n) , then by Lemma 1 Chernozhukov et al. (2010), we have that uniformly over $(t, y) \in \mathcal{TV}$,

$$\frac{F(t, y | h_n) - F(t, y)}{s_n} - \left(\frac{-h_n(t, v^y)}{\partial_v Q(t, v^y)} \right) = o(\delta_n).$$

This completes the proof of the proposition. \blacksquare

Let $F(t, y)$ and $F^\leftarrow(t, u)$ be a monotonic function and its inverse w.r.t. y , respectively. Next, we consider the linear expansion of the inverse functional:

$$(F + s_n J_n)^\leftarrow - F^\leftarrow$$

where $s_n \downarrow 0$ as the sample size $n \rightarrow \infty$ and $J_n(t, y)$ is some perturbation function.

Assumption 8

1. $F(t, y)$ has a compact support $\mathcal{TV} = \{(t, y) : y = Q(t, v), (t, v) \in \mathcal{TV} \equiv \mathcal{T} \times \mathcal{V}\}$. Denote \mathcal{V}_ε , \mathcal{TV}_ε , $\mathcal{Y}_{t\varepsilon}$, and \underline{y}_t as a compact subset of \mathcal{V} , $\{(t, y) : y = Q(t, v), (t, v) \in \mathcal{T} \times \mathcal{V}_\varepsilon\}$, the

projection of $\mathcal{T}\mathcal{Y}_\varepsilon$ on $T = t$, and the lower bound of $\overline{(\mathcal{Y}_{\varepsilon t})^\varepsilon}$, respectively. Then for any $t \in \mathcal{T}$, $\underline{y}_t > -\infty$ and $\overline{(\mathcal{Y}_{\varepsilon t})^\varepsilon} \subset \mathcal{Y}_t$.

2. $F(t, y)$ is monotonic and twice continuously differentiable w.r.t. y . The first and second derivatives are denoted as $f(t, y)$ and $f'(t, y)$ respectively. Then both $f(t, y)$ and $f'(t, y)$ are bounded and $f(t, y)$ is also bounded away from zero, uniformly over $\mathcal{T}\mathcal{Y}$.
3. Let $\mathcal{T}\mathcal{Y}\mathcal{Y} = \{(t, y, y') : y = Q(t, v), y' = Q(t, v'), (t, v, v') \in \mathcal{T} \times \mathcal{V} \times \mathcal{V}\}$. Then, there exist two vanishing sequences ε_n and δ_n such that

$$\sup_{(t, y, y') \in \mathcal{T}\mathcal{Y}\mathcal{Y}, |y - y'| \leq \max(\varepsilon_n, s_n \delta_n)} |J_n(t, y) - J_n(t, y')| = o(\delta_n),$$

$$\sup_{(t, y) \in \mathcal{T}\mathcal{Y}} |J_n(t, y)| s_n = o(\varepsilon_n), \quad \text{and} \quad \sup_{(t, y) \in \mathcal{T}\mathcal{Y}} |J_n(t, y)|^2 s_n = o(\delta_n).$$

Proposition C.2 *If Assumption 8 holds, then*

$$\frac{(F + s_n J_n)^\leftarrow(t, v) - F^\leftarrow(t, v)}{s_n} + \frac{J_n(t, F^\leftarrow(t, v))}{f(t, F^\leftarrow(t, v))} = o(\delta_n)$$

uniformly over $(t, v) \in \mathcal{T}\mathcal{V}_\varepsilon$.

Proof. Without loss of generality, we assume $F(t, y)$ is monotonically increasing in y . Let $\xi(t, v) = F^\leftarrow(t, v)$ and $\xi_n(t, v) = (F + s_n J_n)^\leftarrow(t, v)$. Since for n sufficiently large, $\sup_{(t, v) \in \mathcal{T}\mathcal{V}_\varepsilon} s_n |J_n^\leftarrow(t, v)| < \varepsilon$ and by the definition of \mathcal{V}_ε , we can choose $\xi(t, v) \in \mathcal{Y}_t$ and $\xi_n(t, v) \in \mathcal{Y}_t$. In addition, since F is differentiable, we have $F(t, \xi(t, v)) = v$. Denote $\eta_n(t, v) = \min(s_n \delta_n^2, \xi_n(t, v) - \underline{y}_t)$. Then, the definition of the inverse function implies that

$$(F + s_n J_n)(t, \xi_n(t, v) - \eta_n(t, v)) \leq v \leq (F + s_n J_n)(t, \xi_n(t, v)). \quad (\text{C.3})$$

Since $f(t, y)$ is bounded uniformly in $(t, y) \in \mathcal{T}\mathcal{Y}$, we have

$$F(t, \xi_n(t, v) - \eta_n(t, v)) - v = F(t, \xi_n(t, v)) - F(t, \xi(t, v)) + o(s_n \delta_n)$$

and

$$|s_n J_n(t, \xi_n(t, v) - \eta_n(t, v))| \leq \sup_{(t, y) \in \mathcal{T}\mathcal{Y}} s_n |J_n(t, y)|.$$

Therefore,

$$- \sup_{(t, y) \in \mathcal{T}\mathcal{Y}} s_n |J_n(t, y)| \leq F(t, \xi_n(t, v)) - F(t, \xi(t, v)) \leq \sup_{(t, y) \in \mathcal{T}\mathcal{Y}} s_n |J_n(t, y)| + o(s_n \delta_n).$$

Since $f(t, y)$ is bounded and bounded away from zero, we have

$$|\xi_n(t, v) - \xi(t, v)| = O\left(\sup_{(t, y) \in \mathcal{TV}} s_n |J_n(t, y)|\right) + o(s_n \delta_n) = o(\max(\varepsilon_n, s_n \delta_n)).$$

Then,

$$\begin{aligned} & F(t, \xi_n(t, v) - \eta_n(t, v)) - F(t, \xi(t, v)) + s_n J_n(t, \xi_n(t, v) - \eta_n(t, v)) \\ & \geq F(t, \xi_n(t, v)) - F(t, \xi(t, v)) + o(s_n \delta_n) + s_n J_n(t, \xi(t, v)) - s_n \sup |J_n(t, y) - J_n(t, y')| \\ & \geq f(t, \xi(t, v))(\xi_n(t, v) - \xi(t, v)) + s_n J_n(t, \xi(t, v)) - O\left(\sup_{(t, y) \in \mathcal{TV}} s_n^2 |J_n(t, y)|^2\right) - o(s_n^2 \delta_n^2) - o(s_n \delta_n) \\ & \geq f(t, \xi(t, v))(\xi_n(t, v) - \xi(t, v)) + s_n J_n(t, \xi(t, v)) - o(s_n \delta_n), \end{aligned}$$

where the supremum in the second line is taken over $(t, y, y') \in \mathcal{TVY}$, $|y - y'| \leq \max(\varepsilon_n, s_n \delta_n)$,

and the third line is because $f'(t, y)$ is bounded uniformly in $(t, y) \in \mathcal{TV}$.

On the other hand, by (C.3),

$$F(t, \xi_n(t, v) - \eta_n(t, v)) - F(t, \xi(t, v)) + s_n J_n(t, \xi_n(t, v) - \eta_n(t, v)) \leq 0.$$

Therefore, we have

$$\frac{(\xi_n(t, v) - \xi(t, v))}{s_n} + \frac{J_n(t, \xi(t, v))}{f(t, \xi(t, v))} \leq o(\delta_n). \quad (\text{C.4})$$

Similarly, we can show that

$$\begin{aligned} & F(t, \xi_n(t, v)) - F(t, \xi(t, v)) + s_n J_n(t, \xi_n(t, v)) \\ & \leq f(t, \xi(t, v))(\xi_n(t, v) - \xi(t, v)) + s_n J_n(t, \xi(t, v)) + o(s_n \delta_n). \end{aligned}$$

The r.h.s. of (C.3) implies that

$$F(t, \xi_n(t, v)) - F(t, \xi(t, v)) + s_n J_n(t, \xi_n(t, v)) \geq 0.$$

Therefore,

$$\frac{(\xi_n(t, v) - \xi(t, v))}{s_n} + \frac{J_n(t, \xi(t, v))}{f(t, \xi(t, v))} \geq -o(\delta_n). \quad (\text{C.5})$$

(C.4) and (C.5) imply that

$$\frac{(\xi_n(t, v) - \xi(t, v))}{s_n} + \frac{J_n(t, \xi(t, v))}{f(t, \xi(t, v))} = o(\delta_n)$$

uniformly over $(t, v) \in \mathcal{TV}$. ■

D A Maximal Inequality for the Second Order P-degenerate U-process

In this section, we derive a new bound for the second order P-degenerate U-process, which sharpens the results in Nolan and Pollard (1987). We combine the symmetrization, the exponential inequality, and the entropy bound of the U-process established in Nolan and Pollard (1987) with the innovative proving strategy used in Chernozhukov et al. (2014) for the empirical process. In addition, we establish a contraction principle for the second order U-process in the same manner as (Ledoux and Talagrand, 2013, Theorem 4.4) in our Lemma D.2, which seems to be new to the literature.

Let $\{X_i\}_{i=1}^n$ be a sequence of i.i.d. random variables taking values in a measurable space $(\mathcal{X}, \mathcal{X})$ with common distribution P . Let \mathcal{F} be a class of real-valued, symmetric functions with a nonnegative envelope F , i.e., $\sup_{f \in \mathcal{F}} |f(\cdot, \cdot)| \leq F(\cdot, \cdot)$. Let $M \equiv \sup_{(x_1, x_2) \in \mathcal{X} \times \mathcal{X}} F(x_1, x_2) < \infty$, $l(x) = \mathbb{E}f^2(x, \cdot)$, $\mathcal{L} = \{l : f \in \mathcal{F}\}$ with envelope $L(x)$, and $M_L = \sup_{x \in \mathcal{X}} L(x) < \infty$. In addition, denote

$$\sigma_2^2 = \sup_{f \in \mathcal{F}} \mathbb{E}f^2 \quad \text{and} \quad \sigma_1^2 = \sup_{l \in \mathcal{L}} \mathbb{E}l^2.$$

f is P -degenerate, i.e., $\mathcal{P}f(x_1, \cdot) = 0$ for all $x_1 \in \mathcal{X}$. $S_n(f) = \sum_{1 \leq i \neq j \leq n} f(X_i, X_j)$. \mathcal{U}_n is the empirical U-process that place probability mass $\frac{1}{n(n-1)}$ on each pair of (X_i, X_j) , i.e., $\mathcal{U}_n f = \frac{1}{n(n-1)} S_n(f)$.

Proposition D.1 *If there exist constants A and v (potentially dependent on n) such that*

$$\sup_Q N(\varepsilon \|F\|_Q, e_Q, \mathcal{F}) \leq \left(\frac{A}{\varepsilon}\right)^v,$$

then

$$\sup_{f \in \mathcal{F}} |\mathcal{U}_n f| = O_p\left(\frac{(\sigma_2 + \pi)v}{n} \log\left(\frac{A\|F\|_{P,2}}{\sigma_2}\right) + \frac{M}{n^2} v^2 \log^2\left(\frac{A\|F\|_{P,2}}{\sigma_2}\right)\right),$$

where $\pi^2 \equiv \sqrt{\frac{v\sigma_1^2}{n} \log\left(\frac{2A\|L\|_{P,2}}{\sigma_1}\right)} + \frac{v\|M_L\|_{P,2}}{n} \log\left(\frac{2A\|L\|_{P,2}}{\sigma_1}\right)$.

Proof of Proposition D.1.

Step 1: Symmetrization. We symmetrize the U-process using Rademacher random variables.

Take an independent copy Y_1, Y_2, \dots, Y_n of X_1, X_2, \dots, X_n from P and define

$$T_n^0(f) = \sum_{1 \leq i \neq j \leq n} \eta_i \eta_j (f(X_i, X_j) - f(X_i, Y_j) - f(Y_i, X_j) + f(Y_i, Y_j)),$$

and

$$S_n^0(f) = \sum_{1 \leq i \neq j \leq n} \eta_i \eta_j f(X_i, X_j),$$

where $\{\eta_i\}_{i=1}^n$ is a sequence of Rademacher random variables. Then, by Lemma 1 of Nolan and Pollard (1987),

$$\mathbb{E} \sup_{f \in \mathcal{F}} |S_n(f)| \leq \mathbb{E} \sup_{f \in \mathcal{F}} |T_n^0(f)| \leq 4 \mathbb{E} \sup_{f \in \mathcal{F}} |S_n^0(f)|.$$

Define $\mathcal{U}_n^0(f) = S_n^0(f)/(n(n-1))$. We have

$$\mathbb{E} \sup_{f \in \mathcal{F}} |\mathcal{U}_n(f)| \leq 4 \mathbb{E} \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0(f)|. \quad (\text{D.1})$$

Therefore, we can focus on bounding $\mathbb{E} |\mathcal{U}_n^0(f)|$.

Step 2: Entropy integral inequality. Following Corollary 4 of Nolan and Pollard (1987) with T_n^0 replaced by S_n^0 , we have

$$P_\eta(S_n^0(f) > x) \leq 2 \exp\left(-\frac{x}{13(S_n f^2)^{1/2}}\right).$$

This means, for a realization (x_1, \dots, x_n) ,

$$\left\| \sum_{1 \leq i \neq j \leq n} \eta_i \eta_j f(x_i, x_j) \right\|_{\psi_1} \lesssim \left[\sum_{1 \leq i \neq j \leq n} f^2(x_i, x_j) \right]^{1/2}$$

where $\|\cdot\|_{\psi_1}$ is the Orlicz norm with $\psi_1(x) = \exp(x) - 1$. Let $\sigma_n^2 = \sup_{f \in \mathcal{F}} \mathcal{U}_n f^2$ and $J(\delta) = \int_0^\delta \lambda(\varepsilon) d\varepsilon$ where $\lambda(\varepsilon) = 1 + \sup_Q \log N(\varepsilon \|F\|_{Q,2}, e_Q, \mathcal{F})$, which is nonincreasing in ε . Then by Lemma 5 in Nolan and Pollard (1987), we have

$$\begin{aligned} \mathcal{P}_\eta n \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0(f)| &\lesssim \int_0^{\sigma_n} (1 + \log N(\varepsilon, e_{U_n}, \mathcal{F})) d\varepsilon \\ &\lesssim \|F\|_{\mathcal{U}_n, 2} \int_0^{\frac{\sigma_n}{\|F\|_{\mathcal{U}_n, 2}}} (1 + \log N(\varepsilon \|F\|_{\mathcal{U}_n, 2}, e_{U_n}, \mathcal{F})) d\varepsilon \\ &\lesssim \|F\|_{\mathcal{U}_n, 2} \int_0^{\frac{\sigma_n}{\|F\|_{\mathcal{U}_n, 2}}} (1 + \sup_Q \log N(\varepsilon \|F\|_{Q,2}, e_Q, \mathcal{F})) d\varepsilon \\ &\lesssim \|F\|_{\mathcal{U}_n, 2} J\left(\frac{\sigma_n}{\|F\|_{\mathcal{U}_n, 2}}\right). \end{aligned}$$

Taking expectation w.r.t. (X_1, X_2, \dots, X_n) on both sides, we obtain that

$$\mathcal{P}_n \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0(f)| \leq \mathbb{E} \left[\|F\|_{\mathcal{U}_n, 2} J \left(\frac{\sigma_n}{\|F\|_{\mathcal{U}_n, 2}} \right) \right].$$

The following lemma is borrowed from Chernozhukov et al. (2014):

Lemma D.1 *Write $J(\delta)$ for $J(\delta, \mathcal{F}, F)$ and suppose that $J(1) < \infty$. Then*

1. $\delta \mapsto J(\delta)$ is concave.
2. $J(c\delta) \leq cJ(\delta)$, $\forall c \geq 1$.
3. $\delta \mapsto J(\delta)/\delta$ is non-increasing.
4. The map $(x, y) \mapsto J(\sqrt{x/y})\sqrt{y}$ is concave for $(x, y) \in [0, +\infty) \times (0, +\infty)$.

Although Chernozhukov et al. (2014) proved this lemma for

$$\lambda(\varepsilon) = \sqrt{1 + \sup_Q \log N(\varepsilon \|F\|_{Q, 2}, e_Q, \mathcal{F})},$$

their proofs work for $\lambda(\varepsilon) = 1 + \sup_Q \log N(\varepsilon \|F\|_{Q, 2}, e_Q, \mathcal{F})$, as well. Therefore, by Jensen's inequality, we have

$$Z =: \mathcal{P}_n \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0(f)| \leq \|F\|_{P, 2} J \left(\frac{\sqrt{\mathbb{E}\sigma_n^2}}{\|F\|_{P, 2}} \right). \quad (\text{D.2})$$

Step 3: Bound $\mathbb{E}\sigma_n^2$. To bound $\mathbb{E}\sigma_n^2$, let $\sigma_2^2 = \sup_{f \in \mathcal{F}} \mathbb{E}f^2$, $l(f)(x) = \mathbb{E}f^2(x, \cdot)$, $\mathcal{L} = \{l(f) : f \in \mathcal{F}\}$ with envelope $L(x) = \mathbb{E}F^2(x, \cdot)$, $h(x_1, x_2) = f^2(x_1, x_2) - l(x_1) - l(x_2) - \mathbb{E}f^2(X_1, X_2) + \mathbb{E}l(X_1) + \mathbb{E}l(X_2)$, and $\mathcal{H} = \{h : f \in \mathcal{F}\}$. In the following, we omit the dependence of l on f and simply write $l(x)$. Then we have

$$f^2(x_1, x_2) = \mathbb{E}f^2(X_1, X_2) + l(x_1) - \mathbb{E}l(X_1) + l(x_2) - \mathbb{E}l(X_2) + h(x_1, x_2).$$

By the triangle inequality,

$$\begin{aligned}
\mathbb{E}\sigma_n^2 &= \mathbb{E} \sup_{f \in \mathcal{F}} |\mathcal{U}_n h + \mathbb{E} f^2(X_1, X_2) + 2(\mathcal{P}_n - \mathcal{P})l(\cdot)| \\
&\leq \sigma_2^2 + \mathbb{E} \sup_{h \in \mathcal{H}} |\mathcal{U}_n h| + 2\mathbb{E} \sup_{l \in \mathcal{L}} |(\mathcal{P}_n - \mathcal{P})l(\cdot)| \\
&\leq \sigma_2^2 + \frac{1}{n(n-1)} \mathbb{E} \sup_{h \in \mathcal{H}} |T_n^0 h| + 2\mathbb{E} \sup_{l \in \mathcal{L}} |(\mathcal{P}_n - \mathcal{P})l(\cdot)| \\
&= \sigma_2^2 + \frac{1}{n(n-1)} \mathbb{E} \sup_{f \in \mathcal{F}} |T_n^0 f^2| + 2\mathbb{E} \sup_{l \in \mathcal{L}} |(\mathcal{P}_n - \mathcal{P})l(\cdot)| \\
&\leq \sigma_2^2 + 4\mathbb{E} \sup_{f \in \mathcal{F}} |U_n^0 f^2| + 2\mathbb{E} \sup_{l \in \mathcal{L}} |(\mathcal{P}_n - \mathcal{P})l(\cdot)|.
\end{aligned}$$

Step 4: A contraction inequality.

Lemma D.2 *If $\sup_{f \in \mathcal{F}} |f| \leq M$, then*

$$\mathbb{E} \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f^2| \leq M \mathbb{E} \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f|.$$

Proof of Lemma D.2. Note that $\{\eta_i \eta_j\}_{1 \leq i < j \leq n}$ takes each value of $\{a_{i,j}^0\}_{1 \leq i < j \leq n} \in \{1, -1\}^{n(n-1)/2}$ with equal probability. For $j < i$, let $a_{j,i}^0 = a_{i,j}^0$. Then, for any symmetric $\{b_{i,j}\}_{1 \leq i \neq j \leq n}$, $\{\eta_i \eta_j a_{i,j}^0 b_{i,j}\}_{1 \leq i \neq j \leq n}$ and $\{\eta_i \eta_j b_{i,j}\}_{1 \leq i \neq j \leq n}$ have the same distribution. Let (x_1, \dots, x_n) be an arbitrary realization of (X_1, \dots, X_n) and $f_{i,j} = f(x_i, x_j)$. Then

$$\mathbb{E}_\eta \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 M f_{i,j}| = \mathbb{E}_\eta \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f_{i,j} M a_{i,j}^0|. \quad (\text{D.3})$$

In addition, $\mathbb{E} \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f_{i,j} M a_{i,j}|$ is convex in $\{a_{i,j}\}_{1 \leq i \neq j \leq n}$. So for any $|a_{i,j}| \leq 1$, $a_{i,j} = a_{j,i}$,

$$\mathbb{E}_\eta \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f_{i,j} M a_{i,j}|$$

obtains its maximal at extreme points, i.e., some $\{a_{i,j}^0\}_{1 \leq i \neq j \leq n}$ with $a_{i,j}^0 = a_{j,i}^0$. This and (D.3) imply that, for any any $|a_{i,j}| \leq 1$, $a_{i,j} = a_{j,i}$,

$$\mathbb{E}_\eta \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f_{i,j} M a_{i,j}| \leq \mathbb{E}_\eta \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f_{i,j} M|.$$

Letting $a_{i,j} = f_{i,j}/M$, we have $\mathbb{E}_\eta \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f_{i,j}^2| \leq M \mathbb{E}_\eta \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0 f_{i,j}|$. Taking expectations w.r.t. (X_1, \dots, X_n) , we have the desired result. ■

Step 5: The upper bound. Since

$$\sup_Q N(\varepsilon \|F\|_Q, e_Q, \mathcal{F}) \leq \left(\frac{A}{\varepsilon}\right)^v,$$

by Lemma L.2 in Belloni et al. (2017a),

$$\sup_Q N(\varepsilon \|L\|_Q, e_Q, \mathcal{L}) \leq \left(\frac{2A}{\varepsilon}\right)^v.$$

Denote $\sigma_1^2 = \sup_{l \in \mathcal{L}} \mathbb{E} l^2$, $L(x) = \mathcal{P}F(x, \cdot)$, and $M_L = \max_{1 \leq i \leq n} L(X_i)$, then we have

$$\mathbb{E} \sup_{l \in \mathcal{L}} |(\mathcal{P}_n - \mathcal{P})l(\cdot)| \lesssim \pi^2 \equiv \sqrt{\frac{v\sigma_1^2}{n} \log\left(\frac{2A\|L\|_{P,2}}{\sigma_1}\right) + \frac{v\|M_L\|_{P,2}}{n} \log\left(\frac{2A\|L\|_{P,2}}{\sigma_1}\right)}$$

and thus

$$\mathbb{E}\sigma_n^2 \lesssim \sigma_2^2 + \pi^2 + MZ/n \lesssim \|F\|_{P,2}^2 \max(\Delta^2, DZ) \quad (\text{D.4})$$

where $\Delta^2 = \max(\sigma_2^2, \pi^2)/\|F\|_{P,2}^2$, $Z \equiv \mathcal{P}n \sup_{f \in \mathcal{F}} |\mathcal{U}^0(f)|$, and $D = M/(n\|F\|_{P,2}^2)$. Plugging (D.4) in (D.2), we have

$$Z = \|F\|_{P,2} J(C \max(\Delta, \sqrt{DZ})) \lesssim \|F\|_{P,2} J(\max(\Delta, \sqrt{DZ})).$$

We discuss two cases.

1. $\sqrt{DZ} \leq \Delta$. We have $Z \lesssim \|F\|_{P,2} J(\Delta)$. In addition, recall $\delta = \sigma_2/\|F\|_{P,2}$, then $\Delta \geq \delta$ and by Lemma D.1,

$$J(\Delta) = \Delta \frac{J(\Delta)}{\Delta} \leq \Delta \frac{J(\delta)}{\delta} = \max\left(J(\delta), \frac{\pi J(\delta)}{\delta \|F\|_{P,2}}\right).$$

2. $\sqrt{DZ} > \Delta$. Then by Lemma D.1

$$J(\Delta, \sqrt{DZ}) \leq J(\sqrt{DZ}) = \sqrt{DZ} \frac{J(\sqrt{DZ})}{\sqrt{DZ}} \leq \sqrt{DZ} \frac{J(\Delta)}{\Delta} \leq \sqrt{DZ} \frac{J(\delta)}{\delta}.$$

It follows that $Z \leq C\|F\|_{P,2} \sqrt{DZ} \frac{J(\delta)}{\delta}$, or equivalently,

$$Z \lesssim \|F\|_{P,2}^2 D \frac{J^2(\delta)}{\delta^2} \lesssim \|F\|_{P,2} \frac{M}{n\|F\|_{P,2}} \frac{J^2(\delta)}{\delta^2}.$$

So overall,

$$Z \lesssim \|F\|_{P,2} \max\left(J(\delta), \frac{\pi J(\delta)}{\delta \|F\|_{P,2}}, \frac{M}{n\|F\|_{P,2}} \frac{J^2(\delta)}{\delta^2}\right).$$

Note that $J(\delta) \leq C\delta v \log(\frac{A}{\delta})$. So

$$Z \equiv \mathcal{P}n \sup_{f \in \mathcal{F}} |\mathcal{U}_n^0(f)| \lesssim (\sigma_2 + \pi)v \log\left(\frac{A\|F\|_{P,2}}{\sigma_2}\right) + \frac{M}{n}v^2 \log^2\left(\frac{A\|F\|_{P,2}}{\sigma_2}\right).$$

Then (D.1) implies the desired result. ■