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SEARCH MODELS OF MONEY:

ALTERNATIVE MEANS-OF-PAYMENT AND CONSUMER BEHAVIOUR WITH CREDIT

TAN KHENG TAT, MARCUS

A DISSERTATION

In

ECONOMICS

Presented to the Singapore Management University in Partial Fulfilment

of the Requirements for the Degree of PhD in Economics

2023

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Supervisor of Dissertation

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PhD in Economics, Programme Director

Search Models of Money: Alternative Means-of Payment and Consumer Behaviour with Credit

TAN Kheng Tat, Marcus

Submitted to School of Economics

in partial fulfilment of the requirements for the

Degree of Doctor of Philosophy in Economics

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2023

I hereby declare that this PhD dissertation is my original work
and it has been written by me in its entirety.
I have duly acknowledged all the sources of information
which have been used in this dissertation.

This PhD dissertation has also not been submitted for any degree
in any university previously.

TAN Kheng Tat, Marcus

14 July 2023

Abstract

This dissertation consists of three chapters on Search Models of Money.

The first chapter is a review of recent advances in Search Models of Money. It reviews the Lagos and Wright (2005) framework which is the workhorse of many modern search models with applications to models with Competing Media of Exchange to Fiat Currency, and models with Money and Credit. We trace the history of the development of search models of money from the first generation to present day. We highlight recent developments that address puzzles such as the co-existence of money in an environment where an asset serves as both an alternative means-of-payment and a superior store of value. We look at search models of money with credit which address the fact that in the original LW framework, credit could not exist because agents are anonymous in the decentralized market while in the centralized market all agents can work with linear utility in hours rendering credit unnecessary.

The second chapter explores the adoption and acceptance of alternative means-of-payment to fiat currency. We determine the inflation rate and transaction costs of adoption that encourage the adoption of an alternative means-of-payment. However, the buyer's bargaining power must also be high enough for money and the asset to co-exist as means of payment, otherwise buyers will choose to use money only for low inflation and asset only for high inflation. We observe that when inflation is low, for a given fraction of acceptance of the alternative means-of-payment by sellers, and the cost of holding money is not great so the benefit of using the asset as an alternative means-of-payment to the buyer is negative or zero, and buyers will not adopt the asset. At high inflation when the asset is adopted and accepted as an alternative means-of-payment, when acceptance rate is low, welfare

gains are limited because agents do not use too much of the asset as an alternative means-of-payment. However, when the acceptance rate is high, the welfare gains are much higher. In equilibria where money and the asset co-exist as means of payment, increasing the seller's acceptance rate of the asset as means-of-payment encourages the adoption of the asset as means-of-payment at lower inflation rates.

The third chapter investigates consumer behaviour in an environment with two types of credit – secured and unsecured credit, and with four types of agents – (1) low-income agents with high consumption needs, (2) high-income agents with high consumption needs, (3) low-income agents with low consumption needs, and (4) high-income agents with low consumption needs. Given each agent has a strictly less than one probability of access to financial markets or credit, this gives rise to a total of eight heterogenous agents. As inflation increases, the cost of money increases resulting in agents carrying less fiat currency and relying more on credit to finance their consumption needs. Low-income agents with high consumption needs are always the first to require credit while in most situations, high-income agents with low consumption needs never need credit. Credit relaxes liquidity constraints of agents and as inflation increases, welfare decreases because agents carry less money and rely on credit to finance consumption needs. At high levels of inflation, agents start to have insufficient liquidity to obtain the optimal DM quantity of good. Calibrating to US data, we find welfare loss range from 1% to 4% for every 0.1% increase in inflation. Because of our diverse types of agents, we are able to show that inflation affects high consumption agents the most, especially those without access to credit.

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I would like to dedicate this thesis to my late father who unfortunately passed away before I could graduate from the program.

Chapter 1

Advances in Search Models of Money

1.1. Introduction

New Monetarist search models of money show that money is essential as a medium of exchange to facilitate trade in environments where trading frictions and the absence of double coincidence of wants exists, or in environments where the presence of anonymous agents and imperfect record-keeping make credit difficult.

In the first-generation search models of money, to maintain tractability, Kiyotaki and Wright (1989, 1993) had to model both money and goods as indivisible, and agents can hold either money or goods but not both. In addition, only agents without money will want to produce. As a result, prices are exogenous and limit the amount of analytical work that can be carried out on monetary policy, for example, the impact of inflation.

In the second-generation search models of money, Shi (1995) and Trejos and Wright (1995) endogenized prices by introducing divisible goods in their models. However, due to random matching and uncertainty in consumption and production opportunities, agents have different needs and would carry different portfolios making the distribution of money holdings across agents non-degenerate. As a result, money had to remain indivisible to keep the model and results tractable. Like the first-generation search models of money, monetary policy analysis is limited due to the non-degenerate money holdings of agents.

In the third-generation search models of money, Shi (1997) solved the non-degenerate distribution of money holdings by grouping agents into families and having family members consolidate their money holdings at the end of each period.

Lagos and Wright (2005) on the other hand, solved the non-degenerate distribution of money holdings by introducing quasi-linear preferences and periodic access to centralized markets (a decentralized subperiod followed by a centralized subperiod). In both cases, any match-specific risks and uncertainty in consumption among agents are eliminated by the law of large numbers and the agent's portfolio or distribution of money holdings is degenerate in equilibrium. Money can finally be made divisible. The Lagos-Wright framework has since emerged as a workhorse in modern monetary economics, given its ability to address the divisibility of money and goods simultaneously.

Liu (2018) provides a rich overview of recent advances in search models of money. In Section 2, we describe the Lagos and Wright (2005) framework in detail. In Section 3, we look at recent search models of money with competing media of exchange, particularly electronic money. In Section 4, we look at recent search models of money that incorporate credit into the framework. Section 5 then concludes.

1.2. The Lagos and Wright Framework

In this section, we describe the Lagos and Wright (2005) framework which has become the workhorse of many modern search models of money. The framework incorporates many elements of standard search and general equilibrium theory including a large class of pricing mechanisms, such as bargaining, price taking, posting, etc. Its main attractive feature is its tractability and ability to generate useful analytic results, for example, the Friedman rule is the optimal policy for models with Walrasian pricing or Kalai bargaining, but not for models with Nash bargaining.

In the standard LW framework, there is a $[0,1]$ continuum of ex-ante identical and infinitely lived agents. Time is divided into discrete periods with each period sub-divided into day and night subperiods. Agents discount periods with the discount factor $\beta \in (0,1)$, but not between the two subperiods within a period.

During the day, agents are anonymously matched bilaterally in a decentralized market (DM) with probability α. Each agent specializes in production and can turn labour one-for-one into specific DM goods which is non-storable, that is, it may not be carried over to the next subperiod. Agents cannot produce goods that they want to consume. Hence, for any two randomly drawn agents i and j, there are four possible matching – (1) with probability δ, a double-coincidence of wants happens, that is, i produces what j wants and j produces what i wants, (2) with probability σ, a single coincidence of wants, i produces what j wants but not vice versa, (3) with probability σ agent j produces what i wants but not vice versa, and (4) with probability $1 - 2\sigma - \delta$ neither i nor j produces what the other wants. DM trade is carried out through Nash bargaining. Anonymity and limited commitment results in the non-existence of credit in DM trade.

At night, agents interact in a Walrasian centralized market (CM), where all agents produce and consume a general CM good which is non-storable and may not be carried over to the next subperiod. The production of the CM good is linear with real wage $w = 1$ in terms of the CM good.

Let (x, h) and (X, H) represent consumption and labour pairs during the day and night, respectively. The period utility function is

$$\mathcal{U}(x, h, X, H) = u(x) - c(h) + U(X) - H$$

where u, c and U are twice continuously differentiable with $u' > 0 >$, $c' > 0$, $U' > 0$, $u'' < 0$, $c'' > 0$, and $U'' \leq 0$. In addition, $u(0) = c(0) = 0$, $u'(q^*) = c'(q^*)$

for some $q^* \in (0, \infty)$ where q^* is the optimal DM quantity traded that a central planner would aim to achieve, and $U'(X^*) = 1$ for some X^* with $U'(X^*) > X^*$ making U linear in H. Agents aim to maximize the lifetime discounted period utility function.

Fiat money is supplied by the government and its sole purpose is to serve as a medium of exchange to facilitate trade due to the non-existence of credit in DM trade. Fiat money does not pay dividends, is perfectly divisible and storable. The money supply at time t is given by $M_t = (1 + \tau)M_{t-1}$ where τ are taxes via lump-sum monetary transfers at the end of the CM subperiod.

Let $F_t(m)$ denote the distribution of money holdings across agents, and $M_t = \int m_t \, dF_t(m)$ the total amount of money at time t, where m is an individual state variable based on the choice of the agent, while F is an aggregate state variable.

Let $V_t(m)$ be the value function of an agent with m dollars entering the DM at time t, and $W_t(m)$ be the value function of an agent with m dollars entering the CM at time t. Let $q_t(m, \tilde{m})$ denote the amount of goods and $d_t(m, \tilde{m})$ the amount of money exchanged in a single-coincidence meeting, where m denotes the money holdings of the buyer and $\tilde{m}$ denotes the money holdings of the seller. Similarly, let $B_t(m, \tilde{m})$ denote the payoff for an agent holding m who meets another agent with $\tilde{m}$ in a double-coincidence meeting.

In the DM, the agent's value function is

$$V_t(m) = \alpha\sigma \int \{u[q_t(m, \tilde{m})] + W_t[m - d_t(m, \tilde{m})]\} \, dF_t(\tilde{m})$$

$$+ \alpha\sigma \int \{-c[q_t(\tilde{m}, m)] + W_t[m + d_t(\tilde{m}, m)]\} \, dF_t(\tilde{m})$$

$$+ \alpha\delta \int B_t(m, \tilde{m}) \, dF_t(\tilde{m}) + (1 - 2\sigma - \delta)W_t(m)$$

where the first term is the expected payoff from buying in a single-coincidence meeting, the second term is the expected gain from selling in a single-coincidence meeting, the third term is the expected payoff from a double-coincidence meeting, and the last term is the expected value of not trading in the day market and going to the CM with the agent's portfolio m intact.

In the CM, the agent solves

$$W_t(m) = \max_{X,H,m'} \{U(X) - H + \beta V_{t+1}(m' + \tau M)\}$$

subject to

$$X = H + \phi_t m - \phi_t m'$$

$$X \geq 0,\ 0 \leq H \leq \bar{H},\ \text{and}\ m' \geq 0$$

where ϕ_t is the price of money in the CM, and $\bar{H}$ is the upper bound on labor hours (which oftentimes is assumed to be non-binding). Let $m_{t+1} = m_t' + \tau M_t$ be the money holdings taken into the next period by the agent.

Next, we work backwards to solve the monetary equilibrium. In the DM, in double-coincidence meetings, matched pairs give each other the optimal quantity q^* of DM goods with $u'(q^*) = c'(q^*)$ and hence $B_t(m, \widetilde{m}) = u(q^*) - c(q^*) + W_t(m)$. In single-coincidence meetings, the terms of trade (q, d) through Nash Bargaining solves

$$\max_{q,d} [u(q) + W_t(m - d) - W_t(m)]^{\theta} [-c(q) + W_t(\widetilde{m} - d) - W_t(\widetilde{m})]^{1-\theta}$$

subject to

$$d \leq m,\ \text{and}\ q \geq 0$$

where $\theta \in (0,1]$ is the bargaining power of the buyer.

Re-writing the CM value function as

$$W_t(m) = \phi_t m + \max_{X,m'} \{U(X) - X - \phi_t m' + \beta V_{t+1}(m' + \tau M)\}$$

and differentiating with respect to X and solving for X gives $X_t(m) = X^*$ with $U'(X^*) = 1$. Here $m'(m)$ is independent of m, and the continuation value $W_t(m)$ is linear in m with slope ϕ_t. This implies the key result which is the distribution of money $F_t(m)$ is degenerate and each agent leaves the CM and enters the DM with that same money holdings, thus simplifying analysis and analytical work. The lump-sum monetary transfer can also be thus evenly distributed to every agent.

Given the linearity of W_t in m, the DM bargaining problem simplifies to

$$\max_{q,d}[u(q) - \phi_t d]^{\theta}[-c(q) + \phi_t d]^{1-\theta}$$

subject to

$$d \leq m, \text{ and } q \geq 0$$

Solving the DM maximization problem, we get

$$q_t(m, \tilde{m}) = \begin{cases} \hat{q}_t(m) & if\ m < m_t^* \\ q^* & if\ m \geq m_t^* \end{cases}$$

$$d_t(m, \tilde{m}) = \begin{cases} m & if\ m < m_t^* \\ m^* & if\ m \geq m_t^* \end{cases}$$

where $\hat{q}_t(m)$ is the solution to $\phi_t m = z(q_t, \theta)$, with

$$z(q, \theta) = \frac{\theta c(q)u'(q) + (1-\theta)u(q)c'(q)}{\theta u'(q) + (1-\theta)c'(q)}$$

and $m_t^* = z(q^*, \theta)/\phi_t$ where the terms of trade are independent of the seller's money balance. Let $q_t(m, \tilde{m}) = q_t(m)$, and $d_t(m, \tilde{m}) = d_t(m)$ for simplification. The only rational choice in equilibrium is $d = m$ given the curvature assumptions of the DM consumption utility function and because no agent will bring more money than is needed for transactions in the DM due to the cost of holding money and given the access to the CM where all agents can adjust their money holdings by producing the general good.

The DM value function can be simplified to

$$V_t(m) = v_t(m) + \phi_t m + \max_{m'}\{-\phi_t m' + \beta V_{t+1}(m' + \tau M)\}$$

$$= v_t(m) + \phi_t m$$

$$+ \sum_{j=t}^{\infty} \beta^{j-t} \max_{m_{j+1}}\{-\phi_j m_{j+1} + \beta[v_{j+1}(m_{j+1}) + \phi_{j+1} m_{j+1}]\}$$

where

$$v_t(m) = \alpha\sigma\{u[q_t(m)] - \phi_t[d_t(m)]\} + \alpha\sigma \int \{-c[q_t(\widetilde{m})] + \phi_t d_t(\widetilde{m})\} \, dF_t(\widetilde{m})$$

$$+ \alpha\delta[u(q^*) - c(q^*)] + U(X^*) - X^*$$

The optimal choice of m_{t+1} is given by solving the optimization problem

$$\max_{m_{t+1}}\{-\phi_t m_{t+1} + \beta[v_{t+1}(m_{t+1}) + \phi_{t+1} m_{t+1}]\}$$

where the necessary condition for the existence of equilibrium is $\phi_t \geq \beta\phi_{t+1}$.

To derive the difference equation, for m_{t+1}, we differentiate $v_t(m)$ to get $v'_{t+1}(m_{t+1}) = \alpha\sigma[u'(q_{t+1})q'(m_{t+1}) - \phi_{t+1}]$ where $z(q_t, \theta) = \phi_t m_t = \phi_t M_t$ in stationary equilibrium. We then substitute it into the first order condition $\phi_t = \beta[\phi_{t+1} + v'_{t+1}(m_{t+1})]$ to obtain a difference equation in q:

$$\frac{z(q_t, \theta)}{M_t} = \beta \frac{z(q_{t+1}, \theta)}{M_{t+1}} \left[\alpha\sigma \frac{u'(q_{t+1})}{z_q(q_{t+1}, \theta)} + 1 - \alpha\sigma\right]$$

which defines a monetary equilibrium if $q_t \geq 0$ for all t. Since $M_{t+1} = (1 + \tau)M_t$, the equilibrium condition for stationary monetary equilibrium with $\phi_t M_t = \phi_{t+1} M_{t+1}$ is

$$\frac{u'(q)}{z_q(q, \theta)} = 1 + \frac{1 + \tau - \beta}{\alpha\sigma\beta} = 1 + \frac{1 + i}{\alpha\sigma}$$

where $1 + i = (1 + r)(1 + \pi)$, $\pi = \tau$ and $r = (1 - \beta)/\beta$.

The results in the paper found that the Friedman rule is always optimal though not always efficient as there is a holdup problem on money holdings when buyers do not have full bargaining power. This source of inefficiency makes the estimated welfare cost of inflation considerably higher than conventional methods.

1.3. Models with Competing Media of Exchange

In this section, we review models with the coexistence of competing media of exchange in the presence to fiat money. These competing media of exchange are often interest-bearing real or nominal assets. One question to answer is why fiat money is valued in the presence of an interest-bearing competing media of exchange, which serves the function of money in facilitating exchange and is a superior store of value as compared to money. In particular, we focus on recent literature on electronic money given the rise in popularity in e-money. One common theme across these literature in e-money is that the introduction of e-money may not necessarily be welfare enhancing.

Lagos and Rocheteau (2008) modifies the LW framework by allowing the CM goods to be storable, and physical capital to be used directly as an alternative medium of exchange. They show that agents tend to over-accumulate capital when they face a shortage in liquidity, and fiat money helps to alleviate this inefficiency. The Friedman Rule is found to be optimal and efficient. In their model, q_t^b denotes the quantity of special goods consumed by an ex-ante identical agent, q_t^s the quantity of special goods produced, and y_t the net consumption of general goods in period t. The instantaneous utility is $u\left(q_t^b\right) - c\left(q_t^b\right) + y_t$ where $y = X - H$ and $U(X) = X$. Agents have two storage technologies: (1) storing x_{lt} units of general goods at time t that generates dividends $k_{lt+1} = f_l(x_{lt})$ units of general goods before entering the DM of the following period, and (2) storing x_{it} at time t generating dividends $k_{it+1} = f_i(x_{it})$ units of general goods after leaving the DM of the following period. However, the goods stored using the f_i technology cannot be brought into the DM as a medium of exchange. f_l and f_i are assumed to be strictly concave, with $f_l'(0) = f_i'(0) = +\infty$, $f_l(0) = f_i(0) = 0$, $\lim_{x_l \to \infty} f_l'(x_l) < \beta^{-1}$, and $\lim_{x_l \to \infty} f_i'(x_i) < \beta^{-1}$. The value functions $V_t(z, k_l, k_i)$ and $W_t(z, k_l, k_i)$ derived are similar to the $V_t(m)$ and $W_t(m)$ in the LW framework. However, they did not manage to solve the rate-of-return dominance puzzle as liquid capital (k_l) and real balances (z) both earn the same rate of return in monetary equilibrium in their model.

Many factors affect the prices of assets used as competing medium of exchange – (1) their intrinsic properties such as portability, storability, divisibility and recognizability, and (2) extrinsic factors such as informational frictions and subjective beliefs. Lester, Postlewaite and Wright (2012) investigates how information and liquidity affect the prices of assets that are used as alternative means-of-payment. Recognizability leads to acceptability which leads to liquidity by facilitating exchange. The liquidity premium of an asset is $\ell(q) = \frac{u'(q)}{z'(q)} - 1$ where $u(q)$ is the buyer's DM utility and z(q) is the DM payment from trade. The liquidity premium enters the equilibrium via the first-order condition of the DM value function where $\frac{\partial V}{\partial a_j} = (\partial_j + \phi_j)\{1 + \sum \rho_s \ell[q_s(a)]\}$ where ∂_j is the dividend from holding one unit of asset a_j, ϕ_j is the CM price of asset a_j and ρ_s is the recognizability of asset a_j. Recognizability is endogenized by allowing agents to invest in information which allows to recognize and differentiate a good version of the asset (genuine) from a bad version of the asset (fake). They showed that if the asset is in high-demand and if there is not enough asset to go around, the asset can be valued for more than its rate of return due to its ability to relax liquidity constraints.

Chiu and Wong (2014) develops a search model with indivisible electronic and investigates optimal policies. The problem they tried to tackle is that the use of money normally results in cash-in-advance economies where buyers hold too little cash (due to discounting, inflation, liquidity shocks, etc.) and are liquidity constrained in decentralized trading, leading to inefficient allocation. E-money can restore efficiency with targeted redistribution of trade surplus between buyers and seller since policy makers can know the exact balances in agent's e-wallets. In their model, each period consists of three subperiods – (1) loading day subperiod, (2) DM trading subperiod, and (3) CM night subperiod. Money and e-money have three distinct differences – (1) money circulates forever and the supply is constant while e-money is retired at the end of each period and new e-money is issued at the start of each period, (2) money is not redeemable on demand while e-money is guaranteed at face-value, and (3) money is transferable between agents in both subperiods, e-money is only transferable within the subperiod and not across the loading and trading subperiods. Sellers accept both fiat money and electronic money with probability α_B and fiat money only with probability $1-\alpha_B$. DM goods however is indivisible to maintain tractability of the equilibrium solution. The CM value function W_L is

$$\max_{\varepsilon,e,z^{m'},z^{e'}} -l-\varepsilon\kappa_C-e\kappa_B+\beta W_T\left(\varepsilon z^{m'},ez^{e'}\right)$$

where l is the labour hours worked, $\varepsilon\in\{0,1\}$ is the decision whether to carry money, κ_C is the cost for using money, $e\in\{0,1\}$ is the decision whether to carry e-money, κ_B is the cost for using e-money, and W_T is the DM value function. Because of the indivisibility of the DM good, the DM value function is given by

$$W_T = \begin{cases} u + W_L(z^m - d, z^e) & if\ z^m \geq d \\ \alpha_B[u + W_L(0, z^e + z^m - d) + (1 - \alpha_B)W_L(z^m, z^e)] & if\ z^e + z^m \geq d > z^m \\ W_L(z^m, z^e) & if\ d > z^e + z^m \end{cases}$$

They found that when the usage of money is costly, the introduction of e-money is always enhances welfare. However, when money is widely accepted, the introduction of e-money is not necessarily welfare enhancing. Efficiency depends on several factors related to liquidity constraints faced by consumers, market powers between consumers and merchants, the network externality in adoption, and monopoly distortion in e-money issuance. When money is not a viable alternative to e-money (e.g. online transaction), both public and private issuance of e-money leads to under-adoption. Efficiency can be restored by providing positive incentives to consumers and merchants.

Dovoodalhosseini (2018) studies central bank issued digital currency (CBDC) to better understand the interactions between cash and CBDC. However, cash gives agents the advantage of anonymity. The paper aimed to answer the question as to whether banks should eliminate cash from circulation and also determine the optimal monetary policy under the following scenarios- (1) when only cash is available to agents, (2) only CBDC is available to agents, or (3) both cash and CBDC are available to agents.

In his model, CBDC is differentiated from cash by being taxable, having the possibility of bearing interest, and that CBDC transfers can be tailored based on the CBDC balances of the agent while lump sum transfers can only be accomplished for cash. Buyers have an i.i.d. preference shock $w_t \in [w_{min}, w_{max}]$ to their DM utility function. There is a cost $c_e(z_e)$ to carrying CBDC from the CM to the DM incurred by the buyer, while there is no cost for cash. The CM function of buyer of type w is W_w and is given by

$$\max_{X,Y,z_c,z_e} \{X - Y - c_e\big(z_e + t_e(z_e, w)\big) + \beta V_w\big(z_c + t_c, z_e + t_e(z_e, w)\big)\}$$

where X is the CM good produce, Y is the labour hours worked, and t_e transfers. The DM value function V_w is

$$V_w(z_c, z_e) = \mathbb{E}W_w(z_c + z_e) + \sigma\Big(wu\big(q_w(z_c, z_e)\big) + \mathbb{E}W_w\Big(z_c + z_e - d_{c,w}(z_c, z_e) - d_{e,w}(z_c, z_e)\Big) - \mathbb{E}W_w(z_c, z_e)\Big)$$

where $d_{c,w}(z_c, z_e)$ is the DM payment in cash and $d_{e,w}(z_c, z_e)$ is the DM payment in CBDC.

For cash and CBDC to be used by agents, the cash inflation must be strictly positive. A negative cash inflation rate can be implemented through open market operations where cash is traded for CBDC, but this would induce CBDC users to use cash instead and thus CBDC would not be adopted under a negative cash inflation rate.

If the cost of carrying CBDC is sufficiently small, and if CBDC is interest bearing, the central bank is able to achieve better allocations with CBDC than with cash only. It is possible to achieve the first-best level of production by using CBDC if agents are patient enough and if the bargaining power of buyers is sufficiently high, while it is never possible to achieve the first best by using cash. When cash and CBDC are both available to agents and valued in equilibrium, the monetary policy may be more constrained, i.e., welfare may be lower compared to the case when only CBDC is available or only cash is available to agents.

In an economy where only cash is available, the optimal inflation in the economy is zero. A positive inflation would lead agents to allocate their real balances relative to the first best. If only CBDC is available, the set of implementable allocations is larger because the CBDC is transferrable and the first-best level of production can be achieved even with positive inflation. However, there is welfare loss resulting from the cost of carrying CBDC. The optimal policy would then to compare the trade-off between the first best under the cash-only scheme and the welfare loss when agents incur the cost of carrying CBDC under the CBDC-only scheme.

In an economy where both cash and CBDC are available, agents with lower transaction needs endogenously choose to use cash, and agents with higher transaction needs choose to use CBDC. Agents may also endogenously choose cash to evade CBDC tax. To discourage these agents from using cash, the central bank could target a high cash inflation rate, but it would hurt cash users. Therefore, the availability of cash in the presence of CBDC imposes a constraint for the central bank's maximization problem. Whether or not the co-existence scheme is optimal (i.e., leading to higher welfare) relative to cash-only or CBDC-only schemes depends on how tight this constraint is. If the constraint is too tight, the central bank would prefer to have only one means of payment used by agents. In this case, if the cost of carrying CBDC is not too high, the central bank eliminates cash, and if the cost is too high, the central bank eliminates CBDC. On the other hand, if this constraint is relatively relaxed, the central bank would prefer having both cash and CBDC circulating in the economy.

He found that having both cash and CBDC available to agents sometimes results in lower welfare than in cases where only cash or only CBDC is available. However, CBDC provides more flexibility for the central bank to conduct monetary policy through targeted transfers and improve monetary policy effectiveness. This is because the central bank can monitor agents' portfolios of CBDC and can cross-subsidize between different types of agents, which is not possible if agents use cash. The welfare gains of introducing CBDC are estimated at up to 0.64% for Canada.

Lotz and Vasselin (2019) studies the tension between cash (fiat money) and e-money, and explore why cash remains the most widely used means of payment for everyday transactions. Their aim is to better understand why some economies find it difficult to replace cash with e-money, or have them coexist, while others are more successful. For example, in Europe, e-money in the form of e-purses were introduced in the 1990s but adoption was muted and many of these e-purses have ceased to exist. On the other hand, the adoption of e-money has been successful in Asia and the United States with the trend of adoption increasing.

The focus of the paper is on the adoption of an alternative means of payment to fiat money and whether this new means of payment will replace fiat money in all or some transactions. For cash and e-money to co-exist, it is essential that one means of payment has an advantage over the other (e.g. security, investment cost, and acceptability) so that agents will choose one over the other under specific conditions. E-money has the advantage over fiat money in that it is immune to theft. Using the LW framework, agents choose their money holding composition $m = m_f + m_e$ upon leaving the CM where m_f is the fiat money holdings and m_e is the e-money holdings. Due to risk of theft α, the DM trade payment d in terms of fiat money is discounted $(1 - \alpha)d_f$, whereas for e-money it is undiscounted d_e. Adoption of e-money depends on three variables – (1) safety level of the monetary instrument used as a medium of exchange, (2) the cost of investment in a new e-payment terminal, and (3) the seller's e-money adoption rate.

If all sellers accept e-money, when there is no risk of theft of fiat money, and when both currencies are accepted by all sellers, buyers are indifferent to holding cash or using e-money. An additional unit of one or the other type of money involves the same marginal increase in the buyers' surplus. However, when the risk of theft of cash α is strictly positive, it is not rational for a buyer to use fiat money given that e-money, which is safer, is always accepted. Since e-money has a higher return, and a lower holding cost than fiat money, the equilibrium is such that only e-money is used. In this equilibrium, fiat money is no longer used and valued, and the quantity exchanged with e-money is higher than the quantity that would have been exchanged with cash. However, when $i > 0$, the quantity traded is less than the optimal quantity q^*. The monetary authority can drive the holding cost of e-money to zero. Indeed, when $\gamma = \beta$, the opportunity cost of holding money i is zero, whereas the cost of insecurity of fiat money remains positive for all $\alpha > 0$. Therefore, if the opportunity cost of e-money is zero $i = 0$, at the Friedman rule buyers will hold enough e-money to buy the optimal quantity of output q^*.

If no seller accepts e-money then, even if e-money is less costly to hold, buyers neither hold nor trade e-money.

If some sellers but not all accept e-money, payment with e-money allows exchanging a larger amount of goods than cash. Therefore, if buyers anticipate that e-money may be accepted by some sellers, all of them will decide to possess e-money in addition to cash. When the risk of theft is low enough $\alpha < \bar{\alpha}$, different multiplicities may appear, depending on the value of the investment cost. Three monetary equilibria may coexist, where no sellers, all sellers, or a fraction of them choose to invest in the new technology, so long as the investment cost is not too high. There is also a region where the investment cost is within an intermediate range such that a mixed monetary equilibrium does not exist, but such that a pure fiat money and a pure e-money \equilibrium coexist. However, if the investment cost is too high for the pure e-money equilibrium to exist, then the economy will end up in a pure fiat money equilibrium. If the investment cost is zero, then e-money dominates fiat money, and the only equilibrium is with e-money.

The adoption of e-money may improve welfare compared to the exclusive use of fiat money, or reduce it, depending on the risk of theft, the investment cost, and the number of sellers who accept e-money. By introducing e-money, welfare may be higher or lower than an economy where only fiat money exists depending on the three factors mentioned earlier. Due to multiplicity of equilibria, entire replacement of cash with e-money is unfeasible. Low inflation can facilitate the adoption of e-money in parallel with fiat money.

Carli and Uras (2022) modelled e-money into the LW framework by modelling agents belonging to a two-member family with different idiosyncratic income shocks spatially separated into different DM markets. They aim to answer the following questions – (1) What are the welfare implications of introducing e-money products for consumers, who demand both fiat money and e-money payment instruments for their transactions? And (2) Should the provision of e-money be regulated by public authorities; and if so, what are the effective means of policy instruments that would improve consumer welfare? In their model, e-money solves spatial separation frictions that fiat money is subject to, but its usage comes with electronic transaction fees – set by monopolistic technology providers with private profit incentives.

E-money has the property that it can travel across space to induce financial integration among spatially separated individuals while fiat money is not transferable across space. To do this, agents receive an endowment $\epsilon \in \{\epsilon_L, \epsilon_H\}$ units of the CM good at the beginning of each DM where $\epsilon_H > \epsilon_L = 0$. The endowment can be stored until the next subperiod (CM) and perished if not consumed by then. There are two possible states of nature $\tilde{s} \in \{s_1, s_2\}$, determining the endowment profile of each family. When $\tilde{s} = s_1$, the endowment profile of the family is $\{\epsilon_H, 0\}$, and if $\tilde{s} = s_2$, the endowment profile of the family is $\{0, \epsilon_H\}$. E-money allows the family member receiving ϵ_H to transfer excess cash balances after trade to the family member receiving $\epsilon_L = 0$. They found that as long as the e-money provider is a monopolist with private profit incentives, having e-money reduces the equilibrium price of fiat money and welfare as compared to an economy with only fiat money because buyers have to work harder in acquiring fiat money balances to be utilized when purchasing goods from the market and e-money units from the provider.

E-money could improve the net welfare of consumers by helping to mobilize their insurance agreements. However, as a surprising key finding, we also observe that the positive welfare effect could only prevail when the scope of insurance is not so large among the members of a family. The technology provider could extract all the surplus when the dispersion in income shocks is large enough by charging a large e-money transaction fee. E-money adoption has real effects on consumption allocations and improves consumer welfare when the equilibrium conversion fee is such that buyers benefit from saving idle cash balances. The reason is that buyers could always replicate the equilibrium consumption allocation by acquiring enough cash balances and not resorting to the e-money technology. When the equilibrium conversion fee makes buyers indifferent between adopting e-money and only using fiat money, consumption allocations are identical to the benchmark case (of no e-money in place). Differently, when buyers strictly prefer to make use of e-money, their consumption improves relative to the benchmark case.

Lump-sum taxation of the technology provider can move the economy with e-money to a Pareto superior allocation if and only if the scope of insurance is small enough. In that particular case, their analysis shows that it is possible to redistribute profits to increase consumer welfare and overcome the pecuniary externality due to e-money adoption. However, when the scope of insurance is large, the most that taxation can do is to achieve the same allocation efficiency of the economy with fiat money only. In this respect, their findings are highly relevant for e-money development policies that aim to stimulate financial inclusion of low income households. If the scope of insurance is large, monopolistic provision of e-money may cause welfare losses and redistributive taxes are ineffective to reduce the welfare losses, arguing for the regulation of the e-money sector and influencing its degree of competitiveness.

1.4. Models with Money and Credit

In this section, we review search models of money with credit considering the fact that in the original LW framework, credit could not exist because agents are anonymous in the decentralized market while in the centralized market all agents can work and produce rendering credit unnecessary.

Berentsen et al. (2007) modified the LW framework to include banks, which can record financial transaction history at no cost, but cannot record goods trade history. Banks provide uncollateralized credit in the form of bank loans to ease agents' liquidity constraints. The DM in the LW is replaced by a perfectly competitive market to simplify pricing. At the beginning of the day market, there is a preference shock such that with probability $1 - N$ an agent can consume but cannot produce while with probability N the agent can produce but cannot consume.

Money balances is defined by $M_{t+1} = (1+\tau)M_t$, where $\tau = \tau_1 + \tau_2$, where $\tau_1 M_1$ denotes the lump-sum monetary transfers in the day market and $\tau_2 M_2$ the transfers in the night market. In addition, $\tau_1 = (1-N)\tau_b + N\tau_s$, where τ_b and τ_s represent the shares of DM transfer going to buyers and sellers, respectively. $V_t(m)$ denotes the value function for an agent with m dollars when entering the day market, and $W_t(m, L, D)$ the value function for an agent entering the night market with m dollars, L loans, and D deposits at time t. The CM value function is modified as

$$W_t(m, L, D) = \max_{X,H,m'} \{U(\beta) - H + \beta V_{t+1}(m')\}$$

such that

$$X + \phi_t m' = H + \phi_t(m + \tau_2 M_{t-1}) + \phi_t(1 + i_d)D - \phi_t(1 + i_l)L$$

where i_l is the nominal loan rate, and i_d the nominal deposit rate. Notably, $W_t(m, L, D)$ is linear in m, L and D. The DM value function is modified as

$$V_t(m) = (1-N)[u(q^b) + W_t(m + \tau_b M_{t-1} + L - pq^b, L, 0)]$$
$$+ N[-c(q^s) + W_t(m + \tau_s M_{t-1} + D - pq^s, 0, D)]$$

where p is nominal price of goods in the day market, q^b and q^s the corresponding quantities consumed by a buyer and produced by a seller. Note that buyers will never deposit money in the bank and sellers will never take out loans, and that sellers cannot deposit receipts of cash pq^s, since the bank closes before the onset of goods trading in the day market. Buyers solve

$$\max_{q^b,L} [u(q^b) + W_t(m + \tau_b M_{t-1} + L - pq^b, L, 0)]$$

$$\text{s.t. } pq^b \leq m + \tau_b M_{t-1} + L - pq^b, L \leq \bar{L}$$

where $\bar{L}$ is buyers' borrowing constraint. A seller faces the problem

$$\max_{L} (i_l - i_d)L$$

$$\text{s.t. } L \leq \bar{L}, u(q^b) - \phi_t(1 + i_l)L \geq \Gamma$$

where Γ is a borrower's surplus by accepting a loan from another bank.

They showed that credit extended by banks is welfare-improving as money is reallocated across agents who have heterogeneous preferences for consumption and production.

Gu et. al. (2016) investigates a similar model whose primary goal is to develop a framework that can be used to study the relationship between money and credit in their roles as competing payment instruments. They found that in a variety of environments, in equilibrium where money is valued, credit is inessential and changes in credit conditions are neutral, that is, the set of equilibria, or the set of incentive-feasible allocations, is bigger or better with an institution than without it. In monetary equilibrium, tightening the debt limit is neutral — it has no impact on allocations or welfare and, as a special case, shutting down credit does not matter, making it inessential. The real value of money adjusts endogenously to changes in debt limits so that total liquidity remains the same—something one would miss if one concentrated solely on models without money. Whenever money is valued, credit is inessential and changes in the debt limit are policy neutral, as real balances adjust endogenously to changes in the debt limit while keeping the total liquidity the same. This result holds for both secured and unsecured credit, exogenous and endogenous debt limits, and any general DM pricing mechanism.

Lotz and Zhang (2016) develops a similar model but endogenizes seller's decision to accept credit. The seller's problem is

$$\max\{-\kappa + \sigma(1-\theta)S(z+\bar{b}) - S(z)\}$$

where κ is the investment cost to accepting credit, σ the probability of matching in the DM, θ the buyer's bargaining power, S the surplus from DM trade, z the real balances of the buyer and $\bar{b}$ the debt of the buyer. They aim to address if improved record-keeping can drive out money, and in economies where both money and credit are used, how does monetary policy affect output and welfare through the credit channel.

One key feature of their paper is the endogenizing of the credit limit. When the government's ability to enforce repayment is limited, borrowers may have an incentive to renege on their debt obligations. To support trade in a credit economy, the punishment for default is the permanent exclusion from the credit system. In that case, a borrower who defaults can only use money for all future transactions. The equilibrium credit limit, b, is determined so the buyer voluntarily repays his debt.

In an equilibrium where money is not valued, if credit is tight, the flow cost of default increases with the size of the loan. Since a higher credit line makes default more tempting, a harsher punishment is needed to ensure credit is incentive-feasible. When credit is not tight, credit alone is sufficient to finance the first best, and the flow cost of default becomes constant. Default is less costly at the margin when money is valued than when money is not valued since in the former, the buyer can still use money for future transactions.

They showed that imperfect record keeping is a necessary but not sufficient condition for money and credit to co-exist. However, inflation is a necessary condition for money and credit to co-exist as high inflation lowers the rate of return of money and makes default more costly. But if inflation is too high, money is not valued, while if inflation is too low, agents default on their debts. By raising the cost of default and lowering the rate of return on money, higher inflation relaxes credit constraints and agents shift from money to credit. This is important as in an environment when money and credit are used, policy changes in debt limits will not crowd out money completely.

When record-keeping is perfect, there can be trades with credit only or trades with money only, but generally not trades with both. This special case also points to the fundamental difficulty of getting money and credit to coexist when all trades are identical and record-keeping is perfect: either only credit is used as money becomes inessential, or only money is used since the incentive to renege on debt repayment is too high.

Limited commitment yields an endogenous debt limit that depends on monetary policy. Money and credit coexist for a range of parameters, and bargaining related hold-up problems can lead to inefficiencies in the adoption of monitoring technologies. Changes in monetary policy generate multiplier effects in the credit market due to complementarities between consumer borrowing and the adoption of credit by merchants.

He et. al. (2015) develops an interesting spin to the credit model by allowing assets, (e.g. housing) to be pledged as collateral for credit relaxing credit frictions. The CM value function $W_t(d_t, h_t)$ is

$$\max_{x_t, l_t, h_{t+1}} \{U(x_t, h_t) - l_t + \beta V_{t+1}(h_{t+1})\}$$

$$\text{s.t. } x_t + \psi_t h_{t+1} = l_t + \psi_t h_t - d_t + T_t$$

where x_t is the CM good produce, l_t is the labour hours worked, ψ_t is the price of the asset (housing), h_t housing, d_t the debt from previous DM, and T_t transfers. Debt is subject to a debt limit where $d_t \leq D(e_t) = D_0 + D_1 e_t$ with $e_t = \psi_t h_t$. The DM quantity y traded is thus given by

$$y(D) = \begin{cases} f(D) & if\ D < d^* \\ y^* & otherwise \end{cases}$$

where f is strictly increasing with $f(0) = 0$, and d^* the debt limit that renders the constraint slack, $f(d^*) = y^*$. The debt taken on $d(D)$ is

$$d(D) = \begin{cases} D & if\ D < d^* \\ d^* & otherwise \end{cases}$$

The DM value function V_t is given by

$$V_t(h_t) = (1 - \alpha)W(0, h_t) + \alpha[u(y_t) + W(d_t, h_t)]$$

with α the probability of matching in the DM.

Solving the FOCs, the Euler equation obtained is

$$r\psi_t = U_2(x_{t+1}, h_{t+1}) + (\psi_{t+1} - \psi_t) + \alpha D_1 \psi_{t+1} \lambda(y_t)$$

where $\lambda(y_t)$ is the liquidity premium. The RHS of the Euler equation says that the price of an asset is determined by (1) the utility it provides, (ii) potential capital gains, and (iii) liquidity value in relaxing credit restrictions.

They showed the existence of 2-cycles. They also showed the existence of 3-cycles in asset pricing which implies that n-cycles are inherently built into the model by the Sarkovskii theorem and the Li-Yorke theorem on chaotic dynamics whereby if a 3-cycle exists then there are n-cycles for all n. Hence volatility in prices can emerge because of self-fulfilling prophecies. For example, for a 2-cycle for housing, at time t, agents expect the price of housing in the next period ψ_{t+1} will be high. This relaxes liquidity constraints and home equity and liquidity will be relatively plentiful in $t + 1$, which lowers the amount people are willing to pay for it at time t. Thus, low ψ_t can be consistent with market clearing given high ψ_{t+1}. On the other hand, high ψ_{t+1} is consistent with low ψ_{t+2}. Agents are willing to pay more for H when they know the price is about to fall, because liquidity will be scare the next period given the low price of the asset. Hence prices for liquid assets alternates leading to a 2-cycle based on beliefs.

They then introduced banking and a new subperiod for access to banks before the DM subperiod. The CM function is $W_t(d_t, h_t, m_t)$ now

$$\max_{x_t, l_t, h_{t+1}, m_{t+1}} \{U(x_t, h_t) - l_t + \beta J_{t+1}(h_{t+1}, m_{t+1})\}$$

$$\text{s.t. } x_t + \psi_t h_{t+1} + \phi_t m_{t+1} = l_t + \psi_t h_t + \phi_t m_t - d_t + T_t$$

where ϕ_t is the value of money. The banking subperiod value function J_t is

$$\alpha \max_{\hat{m}_t} V_t[(1 + \rho_t)(\hat{m}_t - m_t)\phi_t, h_t, \hat{m}_t] + (1 - \alpha)W_t[-(1 + \rho_t)m_t\phi_t, h_t, 0]$$

$$\text{s.t. } (1 + \rho_t)(\hat{m}_t - m_t)\phi_t \leq D(\psi_t h_t)$$

where ρ_t is the bank interest rate. The DM value function is straight-forward.

Three types of equilibria results – (i) aggregate and individual limits are slack, and housing is priced fundamentally, (ii) individual limit binds but the aggregate is slack and $\rho_t > 0$, and (iii) both bind and $\rho_t = 0$. Two conditions determine which equilibrium follows – (1) individual debt limit binds with liquidity bearing a premium, and (2) the aggregate condition where if there are more deposits than bowers can borrow relaxing borrowing constraints.

1.5. Conclusion

Lagos and Wright (2005) offer a tractable search model for analysis of many aspects of money together with other forms of payment such as credit or assets as collateral facilitate trade in the presence of various DM meeting frictions. The literature spun off from the LW framework is enormous and actively researched. Search models of money based on the LW framework allows us to have a better understanding about the essentiality of money, the relationship among money, credit and banking, the mechanisms by which policy can affect allocations and welfare, liquidity and asset pricing, and about economic growth in monetary economies.

We focus on recent literature related to Chapter 2 (alternative means of payment) and Chapter 3 (consumer behavior with credit) of this paper. For an alternative means of payment to co-exist with fiat money, it is essential that one means of payment has an advantage over the other (e.g. security, investment cost, and acceptability) so that agents will choose one over the other under specific conditions. The literature found that the introduction of an alternative means of payment may not always be welfare improving. The introduction of credit enables agents to consume beyond what they would normally with only money. The ability of assets to be pledged as collateral for credit eases liquidity constraints which gives rise to interesting results on their prices based on their scarcity and the extent to which it can ease liquidity constraints.

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Chapter 2

Adoption and Acceptance of Alternative Means-of-Payment

2.1 Introduction

This chapter aims to understand what determines a buyer's decision to adopt an alternative means-of-payment in an economy where fiat currency is already widely accepted. It also aims to understand what determines a seller's decision to accept this alternative means-of-payment. We extend the framework developed by Lagos and Wright (2005), by introducing an alternative asset as an alternative means-of-payment to fiat money which is universally accepted. This asset can be foreign currency, bonds, T-bills, equity shares, mortgage-backed securities, digital currencies (issued by either a private company or central bank), debit accounts, or commodities like gold or silver. Note that this alternative asset may not be universally accepted as a means-of-payment.

In an economy where fiat money is universally accepted, why would buyers adopt, and sellers accept an alternative means-of-payment? One reason comes to mind is that both money and the alternative means-of-payment are perfect substitutes, that is, both are universally accepted by sellers and hence buyers carry both. However, in an environment where there are adoption and acceptance costs to the alternative means-of-payment, it is not so obvious why would agents use the alternative means-of-payment especially when it is assumed that there are no adoption and acceptance costs to using money. As our results show, as inflation increases, the cost to using money makes adopting and accepting the alternative means-of-payment more attractive.

To differentiate the asset that is used as an alternative means-of-payment from money, the asset has the potential to earn dividends. The dividend η may be thought of as the fixed interest payments on bonds or T-bills, or simply the dividend payments of equity shares and mortgage-backed securities. In the case of digital currencies or gold which do not pays dividends, η can be thought of as price appreciation. To further differentiate the alternative asset from money, an adoption cost is introduced for buyers who want to adopt the asset as a means-of-payment. Similarly, an acceptance cost is imposed on sellers who wish to accept the asset as a means-of-payment. On contrast, there are no adoption or acceptance cost to using money as a means-of-payment. The holder to the asset earns dividends across periods which makes the asset a desirable store of value. Hence the asset may be valued for its rate of return and liquidity, and in scenarios where it is scarce, may even command a liquidity premium on its price.

Our main contribution is the analysis of simultaneous endogenous adoption of the asset as a means-of-payment by buyers and the endogenous acceptance by sellers. This has the potential to generate multiple equilibria because as more sellers accept the asset as a means-of-payment, more buyers adopt it which in turn result in more sellers willing to pay the acceptance cost. The stability of equilibrium points in environments with multiple equilibria are analysed.

Analytical results show that in a low inflation rate environment, buyers are unlikely to adopt an alternative means-of-payment unless the alternative means-of-payment is almost as widely accepted as money. In contrast, in a high inflation environment, buyers are more ready to adopt an alternative means-of-payment due to the high cost of holding money. Similarly, in a low inflation rate environment, because the mass of buyers willing to adopt and the amount of the asset they carry as a means-of-payment is low, the cost of accepting the asset as means-of-payment outweighs the benefit from DM trade in accepting the asset as a means-of-payment resulting in sellers unlikely to accept the asset as means-of-payment to begin with. In contrast, in a high inflation environment, where a sufficient mass of buyers adopts and carry the asset as a means-of-payment, more sellers are willing to invest in the acceptance cost to accept the asset as a means-of-payment. This sheds light on the policies necessary for the successful adoption and acceptance of an asset as an alternative means-of-payment such as low adoption and acceptance costs, sufficient quantity of the asset to provide liquidity, and the buyer’s bargaining power in DM trade.

In terms of welfare, it is found that a positive adoption cost deters buyers from obtaining the optimal quantity in DM trade if they would with money only. This is because they substitute the asset from a means-of-payment to a store of value. However, in general, the introduction of an alternative means-of-payment increases DM trade.

The paper is organized as follows. Section 2 reviews literature related to alternative means of payment. Section 3 describes the model and environment, including the trading mechanisms and terms of trade. Section 4 describes the general equilibrium and investigates how the various equilibrium regions vary with parameters as well as adoption decision of the buyer and endogenizes the acceptance decision of the seller. Section 5 describes welfare and policy recommendations, and Section 6 concludes.

2.2 Literature Review

We present a brief literature review in this chapter as extensive review of the literature is covered in Chapter 1.

Lester, Postlewaite and Wright (2012) lays out the basic framework and investigates how information and liquidity affect the prices of assets that are used as alternative means-of-payment. Recognizability leads to acceptability which in turn leads to liquidity by facilitating exchange. Assets can be valued for more than their rate of return if they provide liquidity, such as fiat currency whose price should be zero since it pays zero dividends. In their paper, multiplicity in equilibria arise because as more sellers recognize the asset, the asset becomes more liquid and more buyers use it, making more valuable resulting in more sellers willing to pay the investment cost to recognize and distinguish high-quality assets from their low-quality counterparts. Our paper similarly generates multiple equilibria as more sellers accept the asset as an alternative means-of-payment, the more buyers use it, resulting in more benefit for sellers to invest in the acceptance cost.

Li (2011) introduces checking as an alternative means-of-payment with fixed costs incurred in the centralized market whenever an individual uses bank deposits or checking accounts to make payments in the decentralized market. As long as the fixed cost is not too large or not to small, there exists an equilibrium where money is used for all transactions while checks are used only for large transactions.

Chiu and Wong (2014) develops a micro-founded, dynamic, general equilibrium model of e-money for policy analysis and investigates optimal policies for indivisible electronic money. The introduction of e-money is not necessarily welfare enhancing, especially when money is widely accepted. Efficiency depends on several factors related to liquidity constraints faced by consumers, market powers between consumers and merchants, the network externality in adoption, and monopoly distortion in e-money issuance. In theory, efficiency can be restored by providing positive or negative incentives to consumers and merchants.

Dovoodalhosseini (2018) studies central bank issued digital currency which is taxable while cash is not. Having both cash and CBDC available to agents sometimes results in lower welfare than in cases where only cash or only CBDC is available. However, CBDC provides more flexibility for the central bank to conduct monetary policy and improve monetary policy effectiveness. This is because the central bank can monitor agents' portfolios of CBDC and can cross-subsidize between different types of agents, but these actions are not possible if agents use cash. The welfare gains of introducing CBDC are estimated as up to 0.64% for Canada.

Lotz and Vasselin (2019) studies electronic means-of-payment which provides better security compared to cash. The adoption of e-money may improve welfare compared to the exclusive use of fiat money, or reduce it, depending on the risk of theft, the investment cost, and the number of sellers who accept e-money.

Our model aims to incorporate the features of the above models of alternative means of payment, mainly the fixed cost incurred for adoption and acceptance of the alternative means of payment, both to the buyer and seller respectively. We also investigate the stability of the equilibria given the amount of assets that buyers carry and fraction of sellers who accept the alternative means of payment, which is often missing in the analysis in the above papers.

2.3 Model

2.3.1 Environment

As in the framework developed by Lagos and Wright (2005), time is discrete and divided into periods. Each period is further divided into two subperiods called day and night. The time horizon is infinite, and agents live forever. Agents apply a discount factor $\beta \in (0,1)$ across periods but not between subperiods. We assume that there is limited commitment and imperfect record keeping across period ruling out credit, making a medium of exchange of means-of-payment essential for trade across periods. There are two means-of-payment, money and an alternative asset.

During the day, agents meet bilaterally in a decentralized market (DM) with matching probability σ that a buyer meets a seller in the DM. To rule out barter trade, only sellers can produce $q \in \mathbb{R}_+$ units of the DM good at cost $c(q)$ which only buyers want to consume with utility $u(q)$ but cannot produce. The optimal consumption and production q^* is given by $u'(q^*) = c'(q^*)$. It is assumed that $u(0) = 0$, $u'(q) > 0$, $u'(0) = \infty$, $u''(q) < 0$, and $c(0) = 0$, $c'(q) > 0$, $c'(0) = 0$, $c''(q) > 0$. The DM good is assumed to be perfectly divisible and non-storable and cannot be carried over to the night subperiod, bringing the quantity of DM good traded as $q \in [0, q^*]$. For simplicity, it is assumed that the group of buyers in the DM are homogeneous.

At night, agents trade in a Walrasian centralized market (CM). Here, agents can choose to work h units of labour to produce a general CM good where for simplicity it is assumed that h is non-binding and it is normalised that 1 unit of labour produces 1 unit of the CM good. Agents can also choose to consume x units of the CM good at utility $U(x)$ by producing the good themselves or buying from an agent that produces the CM good. It Is assumed that $U(0) = 0$, $U'(x) > 0$ and

$U''(x) \leq 0$**.** It is also assumed that the CM good is perfectly divisible, perishable, and non-storable and may not be carried over to the day subperiod. At the end of the CM subperiod, agents decide the quantity of money and assets they wish to bring into the next subperiod DM.

Money or fiat currency (typically issued by a central bank) is assumed to be accepted by all sellers in the DM. Money is priced at ϕ in terms of the CM good. The money supply in the economy at any time is M, and the next period $\widehat{M}$. The growth rate of money, γ_m, is given by $\widehat{M} = \gamma_m M$. Stationary equilibrium in terms of real balances require that $\hat{\phi}\widehat{M} = \phi M$ which gives $\frac{\phi}{\hat{\phi}} = \frac{\widehat{M}}{M} = \gamma_m$. Changes in the money supply are accomplished in the CM by means of lump-sum transfers if $\gamma_m > 1$ and taxes if $\gamma_m < 1$. For simplicity, we rule out counterfeiting (so that money is widely accepted by all sellers).

There is an asset which can serve as an alternative means-of-payment in the DM. To model the asset to represent a large class of assets each with different characteristics, such as bonds, T-bills, equity shares, mortgage-backed securities, digital currencies (issued by either a private company or central bank), debit accounts, or commodities like gold or silver, we allow the asset to pay dividends, $\eta \geq 0$. The dividend η may be thought of as the fixed interest payments on bonds or T-bills, or simply the dividend payments of equity shares and mortgage-backed securities. In the case of digital currencies or gold which do not pays dividends, η can be thought of as price appreciation. For this paper, we take the asset to represent perpetual government bonds with fixed payments every period or stocks with real rate of returns. The asset supply in the economy at any time is A, and the next period $\hat{A}$, but to simplify analysis, we take the asset to be fixed at $A = \hat{A}$ at equilibrium. The asset may not be as widely accepted as money where a fraction $\alpha \in [0,1]$ of sellers accept the asset as an alternative means-of-payment in the DM. Agents incurs a transaction or adoption cost $\xi(a)$ payable in the CM to use the asset as a means-of-payment. $\xi(a)$ can be thought of as credit card annual fees or simply purchasing a stored value card, or the brokerage account cost for bonds and shares. Here we assume $\xi'(a) > 0$ so that the adoption increases as the quantity a used for DM trade increase. We assume that $\xi(0) = 0$ so that an agent will only pay the cost if and only if he intends to bring positive amounts of the asset a into the next period's DM. Thus, an agent with portfolio $A = a + b$ units of the asset uses a for trade in the DM only needs pay $\xi(a)$ for the transaction, while keeping and earning dividends on portion, b. The seller who receives a earns dividends on a. The price of the asset in terms of the CM good is ψ and dividends are paid at the start of the CM.

2.3.2 DM Terms of Trade

In the DM, in a match where a buyer meets a seller, the terms of trade are determined by Kalai (1977) proportional bargaining. Let p be the payment handed over to the seller for quantity q of the DM goods if an agreement is reached. The proportional solution is then given by solving

$$\max_{p,q}\{u(q) - p\}$$

$$\text{s.t.} \quad u(q) - p = \theta[u(q) - c(q)]$$

where $\theta \in [0,1]$ is the buyer's bargaining power**.** Define

$$\omega(q) = (1 - \theta)u(q) + \theta c(q) \tag{2.1}$$

Let y^* denote the liquid wealth required to acquire the optimal consumption, that is, $y^* = \omega(q^*)$ where q^* is the optimal production and consumption given by

$u'(q^*) = c'(q^*)$. Hence buyers will only want to consume up to the optimal quantity q^*, and we have the quantity of DM good traded $q \in [0, q^*]$.

Depending on the liquid wealth y of the buyer, if the buyer has sufficient liquid wealth, i.e. $y \geq y^*$, he pays only $p = y^*$, consumes $q = q^*$ units of the DM good and keeps the rest of his money and asset holdings. If the buyer has insufficient wealth, i.e. $y < y^*$, he exhausts all his wealth holdings and pays $p = y$ to consume $q < q^*$ units of the DM good where q solves $\omega(q) = y$. This shows that p and q are functions of the amount of liquid wealth the buyer can use in a meeting which is dependent on the composition of his portfolio. We note that $p \in [0, y^*]$.

The wealth of a buyer available for trade in a meeting depends on his portfolio and the means-of-payment that the seller accepts in the meeting. Let us denote a meeting between a buyer and a seller who only accepts money and does not accept the asset as a means-of-payment as a type 1 meeting, and the meeting between a buyer and a seller who accepts both money and the asset as a means-of-payment as a type 2 meeting.

In a type 1 meeting, a buyer with portfolio (m, a, b) can only use the money portion of his portfolio to trade. His equivalent liquid wealth in a type 1 meeting is $y_1(m) = \phi m$. In a type 2 meeting, a buyer with portfolio (m, a, b) can only use both money and the liquid asset portion of his portfolio to trade. His equivalent liquid wealth in a type 2 meeting is $y_2(m, a, b) = \phi m + (\psi + \eta)a$.

2.3.3 Optimal Portfolio Conditions

In the CM, the value function of an agent with portfolio (m, a, b) is $W(m, a, b)$ such that

$$W(m, a, b) = \max_{x,h,\hat{m},\hat{a},\hat{b}} \{U(x) - h + \beta V(\hat{m}, \hat{a}, \hat{b})\} \tag{2.2}$$

s.t. $\quad x + \phi\hat{m} + \psi\left(\hat{a} + \hat{b}\right) = h + \phi m + (\psi + \eta)(a + b) - \xi(a) + T$

where $U(x)$ is the utility from consuming x units of the CM good, h is the labour worked and T are taxes or transfers. ϕ is the price of money in terms of the CM good and ψ the price of the asset in the current period. η is the dividend paid per unit of the asset to the holder of the asset and $\xi(a)$ is the adoption cost paid in order to use a units of the asset for trade in the DM.

Substituting the budget constraint into the CM value function and assuming there exists a x^* such that maximizing with respect to x gives $U'(x^*) - 1 = 0$, we maximize with respect to x to obtain

$$W(m, a, b) = \phi m + (\psi + \eta)(a + b) - \xi(a) + T + U(x^*) - x^* + \max_{\hat{m},\hat{a},\hat{b}}\{-\phi\hat{m} - \psi\left(\hat{a} + \hat{b}\right) + \beta V\left(\hat{m}, \hat{a}, \hat{b}\right)\} \tag{2.3}$$

where (2.3) shows that $W(m, a, b)$ is linear in the agent's wealth $y(m, a, b) = \phi m + (\psi + \eta)(a + b)$ and independent of the next period's holding $\left(\hat{m}, \hat{a}, \hat{b}\right)$. The envelope conditions are given by

$$\frac{\partial W(m,a,b)}{\partial m} = \phi \tag{2.4}$$

$$\frac{\partial W(m,a,b)}{\partial a} = \psi + \eta - \frac{\partial \xi(a)}{\partial a} \tag{2.5}$$

$$\frac{\partial W(m,a,b)}{\partial b} = \psi + \eta \tag{2.6}$$

The first-order conditions are given by

$$\phi \geq \beta \frac{\partial V\left(\hat{m},\hat{a},\hat{b}\right)}{\partial \hat{m}} \tag{2.7}$$

$$\psi \geq \beta \frac{\partial V\left(\hat{m},\hat{a},\hat{b}\right)}{\partial \hat{a}} \tag{2.8}$$

$$\psi \geq \beta \frac{\partial V\left(\hat{m},\hat{a},\hat{b}\right)}{\partial \hat{b}} \tag{2.9}$$

where (2.7) holds with equality if $\hat{m} > 0$, (2.8) holds with equality if $\hat{a} > 0$ and (2.9) holds with equality if $\hat{b} > 0$.

In the DM, an agent with portfolio (m, a, b) has liquid wealth $y_1(m, a, b) = \phi m$ in a type 1 meeting and $y_2(m, a, b) = \phi m + (\psi + \eta)a$ in a type 2 meeting. The terms of trade from Kalai bargaining for a type 1 meeting are given by

$$p_1(m, a, b) = \begin{cases} y^* & if \;\; y_1(m, a, b) \geq y^* \\ y_1(m, a, b) & if \;\; y_1(m, a, b) < y^* \end{cases} \tag{2.10}$$

$$q_1(m, a, b) = \begin{cases} q^* & if \;\; y_1(m, a, b) \geq y^* \\ \omega^{-1}\big(y_1(m, a, b)\big) & if \;\; y_1(m, a, b) < y^* \end{cases} \tag{2.11}$$

And for a type 2 meeting

$$p_2(m, a, b) = \begin{cases} y^* & if \;\; y_2(m, a, b) \geq y^* \\ y_2(m, a, b) & if \;\; y_2(m, a, b) < y^* \end{cases} \tag{2.12}$$

$$q_2(m, a, b) = \begin{cases} q^* & if \;\; y_2(m, a, b) \geq y^* \\ \omega^{-1}\big(y_2(m, a, b)\big) & if \;\; y_2(m, a, b) < y^* \end{cases} \tag{2.13}$$

Here (2.10) and (2.11) says that if an agent has enough liquid wealth to obtain q^* in a type 1 meeting, he pays only $p_1 = y^*$ and consumes $q_1 = q^*$ units of the DM good, otherwise he exhausts all his wealth to consume $q_1 < q^*$ units of the DM good. Similarly, (2.12) and (2.13) says that if an agent has enough liquid wealth to obtain q^* in a type 2 meeting, he pays only $p_2 = y^*$ and consumes $q_2 = q^*$ units of the DM good, otherwise he exhausts all his wealth to consume $q_2 < q^*$ units of the DM good.

Using the linearity of $W(m, a, b)$, the DM value function of a buyer with portfolio (m, a, b) is given as

$$\begin{aligned} V(m, a, b) = {} & (1 - \sigma)W(m, a, b) \\ & + \sigma\big\{(1 - \alpha)\big[u\big(q_1(m, a, b)\big) - \; p_1(m, a, b) + W(m, a, b)\big] \\ & + \alpha\big[u\big(q_2(m, a, b)\big) - p_2(m, a, b) + W(m, a, b)\big]\big\} \end{aligned}$$

where the first term on the RHS says that with probability $1-\sigma$ the buyer does not meet any seller while the second term says that with probability σ, he meets a seller. If the buyer meets a seller, with probability $1-\alpha$ the seller does not accept the asset as means-of-payment and with probability α the seller accepts the asset as means-of-payment. Here because of the linearity of $W(m,a,b)$, we extracted the terms of payment $p_1(m,a,b)$ and $p_2(m,a,b)$ from the continuation value of the CM function of buyers after going through a Type 1 meeting and Type 2 meeting respectively.

The DM value function can be simplified to

$$V(m,a,b) = W(m,a,b) + \sigma(1-\alpha)\left[u\left(q_1(m,a,b)\right) - p_1(m,a,b)\right] + \sigma\alpha\left[u\left(q_2(m,a,b)\right) - p_2(m,a,b)\right] \quad (2.14)$$

Differentiating (2.10), (2.11), (2.12) and (2.13) with respect to m, we have

$$\frac{\partial p_1(m,a,b)}{\partial m} = \begin{cases} 0 & if\ y_1(m,a,b) \geq y^* \\ \phi & if\ y_1(m,a,b) < y^* \end{cases} \quad (2.15)$$

$$\frac{\partial q_1(m,a,b)}{\partial m} = \begin{cases} 0 & if\ y_1(m,a,b) \geq y^* \\ \frac{\phi}{\omega'\left(y_1(m,a,b)\right)} & if\ y_1(m,a,b) < y^* \end{cases} \quad (2.16)$$

$$\frac{\partial p_2(m,a,b)}{\partial m} = \begin{cases} 0 & if\ y_2(m,a,b) \geq y^* \\ \phi & if\ y_2(m,a,b) < y^* \end{cases} \quad (2.17)$$

$$\frac{\partial q_2(m,a,b)}{\partial m} = \begin{cases} 0 & if\ y_2(m,a,b) \geq y^* \\ \frac{\phi}{\omega'\left(y_2(m,a,b)\right)} & if\ y_2(m,a,b) < y^* \end{cases} \quad (2.18)$$

Differentiating (2.14) with respect to m we get

$$\frac{\partial V(m,a,b)}{\partial m} = \frac{\partial W(m,a,b)}{\partial m}$$

$$+ \sigma(1-\alpha)\left[u'\big(q_1(m,a,b)\big)\frac{\partial q_1(m,a,b)}{\partial m} - \frac{\partial p_1(m,a,b)}{\partial m}\right]$$

$$+ \sigma\alpha\left[u'\big(q_2(m,a,b)\big)\frac{\partial q_2(m,a,b)}{\partial m} - \frac{\partial p_2(m,a,b)}{\partial m}\right]$$

Substituting (2.4), (2.15), (2.16), (2.17) and (2.18) into the above equation we get

$$\frac{\partial V(m,a,b)}{\partial m} = \phi\big[1 + \sigma(1-\alpha)L\big(q_1(m,a,b)\big) + \sigma\alpha L\big(q_2(m,a,b)\big)\big] \quad (2.19)$$

where

$$L(q) = \frac{u'(q)}{\omega'(q)} - 1 = \begin{cases} 0 & if\ y \geq y^* \\ \frac{\theta[u'(q)-c'(q)]}{(1-\theta)u'(q)+\theta c'(q)} & if\ y < y^* \end{cases} \quad (2.20)$$

is the liquidity premium $L(q)$. As $u'(q) - c'(q) > 0$, we have $L(q) > 0$ only if bringing an additional unit of money or the asset into the DM provides liquidity to the buyer, that is, the agent can use it to trade to acquire some strictly positive amount of the DM good. Similarly, $u'(q) = c'(q)$ implies that $L(q^*) = 0$, where buyers have sufficient wealth to acquire the optimal consumption quantity and any additional unit of money beyond will not be used to acquire additional units of the DM good.

Advancing (2.19) one period and substituting into (2.7), we get

$$\phi \geq \beta\hat{\phi}\left[1 + \sigma(1-\alpha)L\left(q_1(\hat{m},\hat{a},\hat{b})\right) + \sigma\alpha L\left(q_2(\hat{m},\hat{a},\hat{b})\right)\right] \quad (2.21)$$

with equality if $\hat{m} > 0$.

Similarly differentiating (10), (11), (12) and (13) with respect to a, we have

$$\frac{\partial p_1(m,a,b)}{\partial a} = \begin{cases} 0 & if\ y_1(m,a,b) \geq y^* \\ 0 & if\ y_1(m,a,b) < y^* \end{cases} \quad (2.22)$$

$$\frac{\partial q_1(m,a,b)}{\partial a} = \begin{cases} 0 & if\ y_1(m,a,b) \geq y^* \\ 0 & if\ y_1(m,a,b) < y^* \end{cases} \quad (2.23)$$

$$\frac{\partial p_2(m,a,b)}{\partial a} = \begin{cases} 0 & if\ y_2(m,a,b) \geq y^* \\ \psi & if\ y_2(m,a,b) < y^* \end{cases} \tag{2.24}$$

$$\frac{\partial q_2(m,a,b)}{\partial a} = \begin{cases} 0 & if\ y_2(m,a,b) \geq y^* \\ \frac{\psi}{\omega'(y_2(m,a,b))} & if\ y_2(m,a,b) < y^* \end{cases} \tag{2.25}$$

Differentiating (2.14) with respect to a, we get

$$\frac{\partial V(m,a,b)}{\partial a} = \frac{\partial W(m,a,b)}{\partial a}$$
$$+ \sigma(1-\alpha)\left[u'\big(q_1(m,a,b)\big)\frac{\partial q_1(m,a,b)}{\partial a} - \frac{\partial p_1(m,a,b)}{\partial a}\right]$$
$$+ \sigma\alpha\left[u'\big(q_2(m,a,b)\big)\frac{\partial q_2(m,a,b)}{\partial a} - \frac{\partial p_2(m,a,b)}{\partial a}\right]$$

Substituting (2.5), (22), (23), (24) and (25) into the above equation we get

$$\frac{\partial V(m,a,b)}{\partial a} = (\psi + \eta)\big[1 + \sigma\alpha L\big(q_2(m,a,b)\big)\big] - \frac{\partial \xi(a)}{\partial a} \tag{2.26}$$

where $L(q)$ is as defined in (2.20).

Advancing (2.26) one period and substituting into (2.8), we get

$$\psi \geq \beta\big(\hat{\psi} + \hat{\eta}\big)\left[1 + \sigma\alpha L\left(q_2\big(\hat{m},\hat{a},\hat{b}\big)\right)\right] - \frac{\partial \xi(\hat{a})}{\partial \hat{a}} \tag{2.27}$$

with equality if $\hat{a} > 0$.

Lastly differentiating (2.10), (2.11), (2.12) and (2.13) with respect to b, we have

$$\frac{\partial p_1(m,a,b)}{\partial b} = \begin{cases} 0 & if\ y_1(m,a,b) \geq y^* \\ 0 & if\ y_1(m,a,b) < y^* \end{cases} \tag{2.28}$$

$$\frac{\partial q_1(m,a,b)}{\partial b} = \begin{cases} 0 & if\ y_1(m,a,b) \geq y^* \\ 0 & if\ y_1(m,a,b) < y^* \end{cases} \tag{2.29}$$

$$\frac{\partial p_2(m,a,b)}{\partial b} = \begin{cases} 0 & if\ y_2(m,a,b) \geq y^* \\ 0 & if\ y_2(m,a,b) < y^* \end{cases} \tag{2.30}$$

$$\frac{\partial q_2(m,a,b)}{\partial b} = \begin{cases} 0 & if\ y_2(m,a,b) \geq y^* \\ 0 & if\ y_2(m,a,b) < y^* \end{cases} \tag{2.31}$$

Differentiating (2.14) with respect to b and substituting (2.6), (2.28), (2.29), (2.30) and (2.31) we get

$$\frac{\partial V(m,a,b)}{\partial b} = \psi + \eta \tag{2.32}$$

Advancing (2.32) one period and substituting into (2.9), we get

$$\psi \geq \beta(\hat{\psi} + \hat{\eta}) \tag{2.33}$$

with equality if $\hat{b} > 0$.

2.4 General Equilibrium

In this section, we solve for the general equilibrium conditions for which money and/or the asset is used as a means-of-payment. For this section, buyers take the acceptance rate α as given. In the next section we will endogenize α. We investigate how parameters such as the acceptance rate α and the buyer's bargaining power θ affect these regions. Throughout this section, the following assumptions are made:

Assumption 2.1 (A2.1). The economy is in stationary equilibrium, i.e., $\hat{\phi}\hat{M} = \phi M$ which gives $\frac{\phi}{\hat{\phi}} = \frac{\hat{M}}{M} = \gamma_m$, and $\psi = \hat{\psi}$.

Assumption 2.2 (A2.2). $u(0) = 0$, $u'(q) > 0$, $u'(0) = \infty$, $u''(q) < 0$, and $c(0) = 0$, $c'(q) > 0$, $c'(0) = 0$, $c''(q) > 0$.

A2.2 says that we have $L(0) = \frac{\theta}{1-\theta}$ and $L(q^*) = 0$. Differentiating (2.20) with respect to q, we get

$$L'(q) = \frac{\theta[u''(q)c'(q) - u'(q)c''(q)]}{[(1-\theta)u'(q) + \theta c'(q)]^2} \tag{2.34}$$

and substituting A2.2 into (2.34), we have $L'(q) < 0$ for $q \in [0, q^*]$ with $L'(0) = 0$. That is, we have $L(q) \in \left[0, \frac{\theta}{1-\theta}\right]$ and $L'(q) < 0$ for $q \in [0, q^*]$ as depicted in the Figure 2.1.

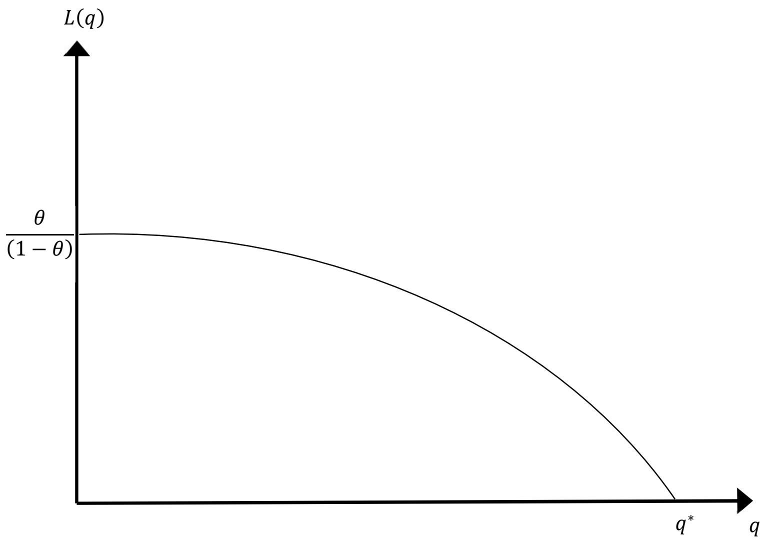


Figure 2.1: Curvature of Liquidity Premium $L(q)$

Assumption 2.3 (A2.3). Let $1 + i = \frac{\phi}{\beta\hat{\phi}}$ where i is the inflation rate.

Substituting $1 + i = \frac{\phi}{\beta\hat{\phi}}$ into (2.21) and dropping notations for time steps, we get

$$i \geq \sigma(1-\alpha)L\big(q_1(m,a,b)\big) + \sigma\alpha L\big(q_2(m,a,b)\big) \qquad (2.35)$$

with equality if $m > 0$.

Assumption 2.4 (A2.4). Let $\xi(\hat{a}) = \kappa\big(\hat{\psi} + \hat{\eta}\big)\hat{a}$ where $\kappa \in [0,1]$. Then $\frac{\partial \xi(\hat{a})}{\partial \hat{a}} = \kappa\big(\hat{\psi} + \hat{\eta}\big)$.

Since the economy is stationary by A2.1, substituting $\psi = \hat{\psi}$ and $\frac{\partial \xi(\hat{a})}{\partial \hat{a}} = \kappa\left(\hat{\psi} + \hat{\eta}\right)$ into (2.27), dropping notations for time steps, we have

$$\psi \geq \beta(\psi + \eta)\left[1 + \sigma\alpha L\left(q_2(m, a, b)\right)\right] - \kappa(\psi + \eta) \qquad (2.36)$$

with equality if $a > 0$.

Similarly substituting $\psi = \hat{\psi}$ into (2.33) and dropping notations for time steps, re-arranging we get

$$\psi \geq \frac{\beta\eta}{1-\beta} \qquad (2.37)$$

with equality if $b > 0$.

Note that (2.37) at equality gives the fundamental price of the asset in a stationary equilibrium, i.e., the fundamental price of the asset is $\underline{\psi} = \frac{\beta\eta}{1-\beta}$ which is also the infinite sum of the discounted dividend stream of the asset of future periods.

Proposition 2.1. Assuming A2.1, A2.3 and A2.4, if $\kappa > 0$ and $q_1 < q^*$, then $q_2 < q^*$.

Proof: Since $q_1 < q^*$, the agent does not have enough money holdings to obtain the optimal quantity of DM goods. If $q_2 = q^*$, then the agent must carry a positive amount of the asset, i.e. $a > 0$, and (2.36) holds at equality, re-arranging we get

$$\psi \geq \frac{\beta\eta\left[1 + \sigma\alpha L\left(q_2(m, a, b)\right)\right] - \kappa\eta}{1 - \beta\left[1 + \sigma\alpha L\left(q_2(m, a, b)\right)\right] + \kappa} \qquad (2.36a)$$

Substituting $L(q^*) = 0$, we get

$$\psi = \frac{\beta\eta - \kappa\eta}{1 - \beta + \kappa}$$

Next, observe that both (2.36a) and (2.37) define the stationary price of the asset and markets clear in the CM, but there can only be one stationary price of the asset. Since $\kappa > 0$ and $\eta > 0$, we have $\frac{\beta\eta - \kappa\eta}{1-\beta+\kappa} < \frac{\beta\eta}{1-\beta}$ which means that (2.36a) can't hold at equality if (2.37) holds at equality for $q_2 = q^*$. So we must have $L(q_2) > 0$ which implies that $q_2 < q^*$. *End of proof.*

What Proposition 2.1 says is that if the adoption cost is strictly positive, and that money holdings alone are insufficient to obtain the optimal quantity, then even with an alternative means-of-payment, buyers will not consume the optimal quantity because of the positive adoption cost.

Equating the RHS of (2.36a) to the RHS of (2.37), we can solve for the liquidity premium agents are willing to pay before redirecting the remaining asset as a store of value. We get

$$\frac{\beta\eta[1 + \sigma\alpha L(q_2)] - \kappa\eta}{1 - \beta[1 + \sigma\alpha L(q_2)] + \kappa} = \frac{\beta\eta}{1 - \beta}$$

Solving for $L(q_2)$, we have

$$L(q_2) = \frac{\kappa}{\beta\sigma\alpha}$$

Hence agents will use the asset to trade until $q_2 = L^{-1}\left(\frac{\kappa}{\beta\sigma\alpha}\right)$. Beyond which the excess asset will be carried as a store of value. Also note that if $L(q_2) > \frac{\kappa}{\beta\sigma\alpha}$, we have $\psi > \underline{\psi}$ and (2.36) holds at equality while (2.37) holds at inequality, meaning that all the asset is held for trade and there are insufficient money and asset holding to obtain $q_2 = L^{-1}\left(\frac{\kappa}{\beta\sigma\alpha}\right)$. The asset is priced above its fundamental value. Otherwise if $L(q_2) = \frac{\kappa}{\beta\sigma\alpha}$, both (2.36) and (2.37) hold at equality, meaning that the agent obtains $q_2 = L^{-1}\left(\frac{\kappa}{\beta\sigma\alpha}\right)$ in a type 2 meeting and the excess asset is not traded but held as a store of value and $\psi = \underline{\psi}$.

In summary, $\psi \geq \underline{\psi} = \frac{\beta\eta}{1-\beta}$ and $0 \leq q_1 \leq q_2 \leq L^{-1}\left(\frac{\kappa}{\beta\sigma\alpha}\right)$.

2.4.1 Stationary Monetary Equilibrium

Definition 2.1. Define $\bar{i}(\kappa)$ as the inflation rate above which money is not valued.

Definition 2.2. Define $\bar{\kappa}(i)$ as the adoption cost above which the asset is not used as a means of payment.

To determine the shape of $\bar{i}(\kappa)$, we assume A2.1-A2.4 and observe the following:

Claim 2.1. We have $\bar{i}(\bar{\kappa}) = \sigma \frac{\theta}{1-\theta}$.

Proof: When $\kappa > \bar{\kappa}$, agents do not use the asset for trade and $a = 0$, so $q_1(m, 0, b) = q_2(m, 0, b) = q(m, 0, b)$ and (2.35) simplifies to $i \geq \sigma L(q(m, 0, b))$. At $\bar{i}(\bar{\kappa})$, agents are indifferent between carrying and not carrying money into the DM and assume $m = 0$. Substituting $L(0) = \frac{\theta}{1-\theta}$ into $\bar{i} = \sigma L(q(m, 0, b))$ at equality we get $\bar{i}(\bar{\kappa}) = \sigma \frac{\theta}{1-\theta}$. *End of proof.*

Claim 2.2. For $\kappa < \bar{\kappa}$, we have $\bar{i}(\kappa) \leq \bar{i}(\bar{\kappa})$, and $\bar{i}(\kappa) \to \bar{i}(\bar{\kappa})$ as $\kappa \to \bar{\kappa}$.

Proof: For $\kappa = \bar{\kappa}$, agents are indifferent between using the asset for trade and we assume $a = 0$. For $\kappa < \bar{\kappa}$, agents use the asset for trade and $a > 0$. For $a > 0$, we have $q_2(m, a, b) > q_2(m, 0, b)$ so $L(q_2(m, a, b)) < L(q_2(m, 0, b))$. Hence (2.35) at equality, for $m = 0$, we get $\bar{i}(\kappa) = \sigma(1-\alpha)\frac{\theta}{1-\theta} + \sigma\alpha L(q_2(m, a, b)) < \sigma(1-\alpha)\frac{\theta}{1-\theta} + \sigma\alpha L(q_2(m, 0, b)) < \sigma(1-\alpha)\frac{\theta}{1-\theta} + \sigma\alpha L(q_2(0,0, b)) = \sigma\frac{\theta}{1-\theta} = \bar{i}(\bar{\kappa})$.

End of proof.

Claim 2.3. For $\kappa \in [0, \bar{\kappa}]$, we have $\bar{i}(\kappa) = \sigma(1-\alpha)\frac{\theta}{1-\theta} + \frac{1}{\beta}\left[\frac{\psi}{\psi+\eta} + \kappa - \beta\right]$ is linear.

Proof: Solving (2.36) at equality we get $L\big(q_2(m,a,b)\big) = \frac{1}{\beta\sigma\alpha}\left[\frac{\psi}{\psi+\eta} + \kappa - \beta\right]$.

Substituting into (2.35) we get $\bar{\imath}(\kappa) = \sigma(1-\alpha)\frac{\theta}{1-\theta} + \frac{1}{\beta}\left[\frac{\psi}{\psi+\eta} + \kappa - \beta\right]$. From the expression, $\bar{\imath}(\kappa)$ is linear in κ by factor of $\frac{1}{\beta}$. The horizontal intercept is given by substituting $\kappa = 0$ and solving to get $\bar{\imath}(0) = \sigma(1-\alpha)\frac{\theta}{(1-\theta)} + \frac{1}{\beta}\left[\frac{\psi}{\psi+\eta} - \beta\right]$. *End of proof.*

Figure 2.2 shows the regions demarcated by $\bar{\imath}(\kappa)$.

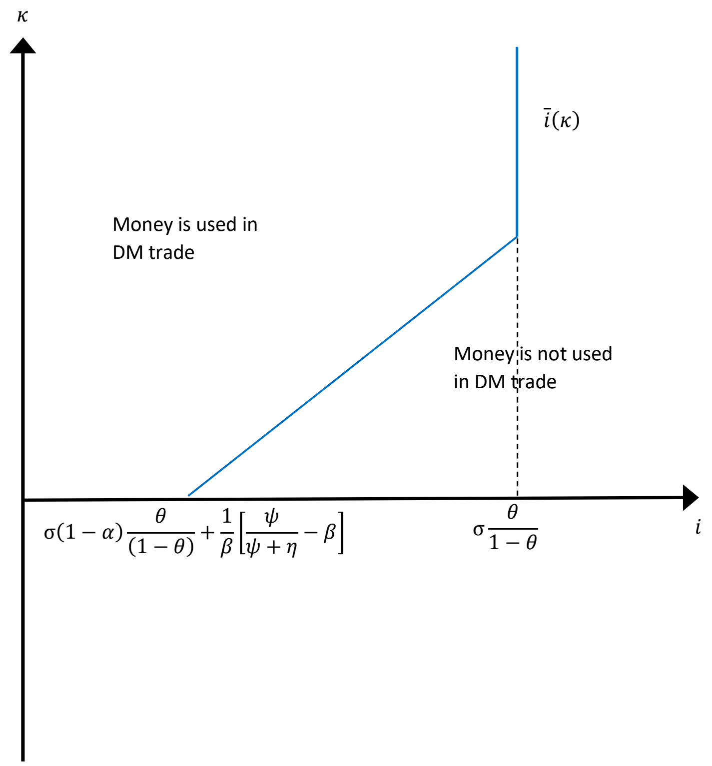


Figure 2.2: Cut-off for Usage of Money in DM trade

To determine the shape of $\bar{\kappa}(i)$, we observer the following:

Claim 2.4. We have $\bar{\kappa}(\bar{\imath}) = \beta\left[1 + \sigma\alpha\frac{\theta}{1-\theta}\right] - \frac{\psi}{\psi+\eta}$.

Proof: When $i > \bar{\imath}$, agents do not carry the money for trade and $m = 0$. At $\kappa = \bar{\kappa}$, $a = 0$ and (2.36) at equality gives $\bar{\kappa} = \beta\left[1 + \sigma\alpha\frac{\theta}{1-\theta}\right] - \frac{\psi}{\psi+\eta}$. *End of proof.*

Claim 2.5. For $i < \bar{\imath}$, we have $\bar{\kappa}(i) \le \bar{\kappa}(\bar{\imath})$, and $\bar{\kappa}(i) \to \bar{\kappa}(\bar{\imath})$ as $i \to \bar{\imath}$.

Proof: For $i = \bar{i}$, agents are indifferent between using money for trade and we assume $m = 0$ and for $i < \bar{i}$, $m > 0$. Note that for $a = 0$ we have $q_1(m, 0, b) = q_2(m, 0, b) = q(m, 0, b)$. For $m > 0$, we have $q(m, 0, b) > q(0,0, b)$ so $L(q(m, 0, b)) < L(q(0,0, b))$. Solving (2.35) at equality gives $L(q(m, 0, b)) = \frac{i}{\sigma}$. So $L(q(m, 0, b)) < L(q(0,0, b))$ gives $\frac{i}{\sigma} < \frac{\theta}{1-\theta}$. Substituting $L(q(m, 0, b)) = \frac{i}{\sigma}$ into (2.36) at equality gives $\bar{\kappa}(i) = \beta[1 + \alpha i] - \frac{\psi}{\psi+\eta} < \beta\left[1 + \sigma\alpha\frac{\theta}{1-\theta}\right] - \frac{\psi}{\psi+\eta} = \bar{\kappa}(\bar{i})$. So $\bar{\kappa}(i) \leq \bar{\kappa}(\bar{i})$. *End of proof.*

Claim 2.6. For $i \in [0, \bar{i}]$, we have $\bar{\kappa}(i) = \beta[1 + \alpha i] - \frac{\psi}{\psi+\eta}$ is linear.

Proof: From $\bar{\kappa}(i) = \beta[1 + \alpha i] - \frac{\psi}{\psi+\eta}$, $\bar{\kappa}(i)$ is linear in i by factor of $\beta\alpha$. The vertical intercept is given by substituting $i = 0$ and solving to get $\bar{\kappa}(0) = \beta - \frac{\psi}{\psi+\eta}$. Note that $\beta - \frac{\psi}{\psi+\eta} < 0$. The horizontal intercept is given by solving $\bar{\kappa}(i) = 0$ which gives $i = \frac{1}{\beta\alpha}\left[\frac{\psi}{\psi+\eta} - \beta\right]$. *End of proof.*

Figure 2.3 shows the regions demarcated by $\overline{\kappa}(i)$.

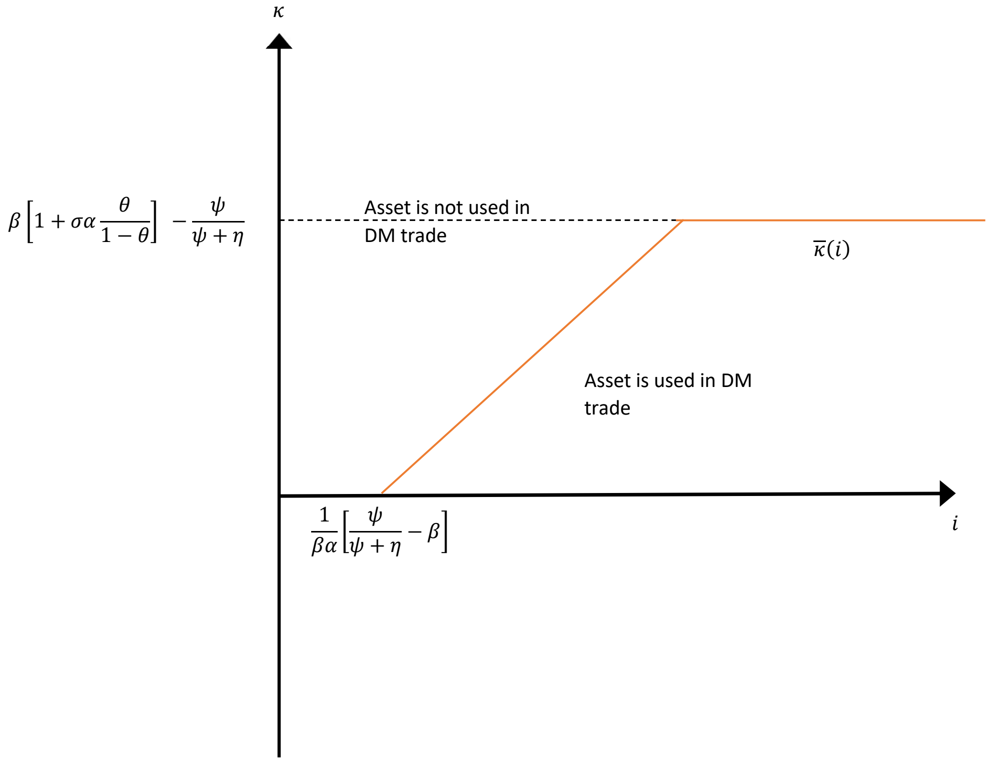


Figure 2.3: Cut-off for Adoption of Asset as Means-of-Payment in DM trade

Putting together $i(\kappa)$ and $\kappa(i)$, we get the following equilibrium regions depicted in Figure 2.4:

- Region I – Autarky/no trade in DM
- Region II – Only money is used as means-of-payment
- Region III – Only asset is used as means-of-payment
- Region IV – Both money and asset are used as means-of-payment

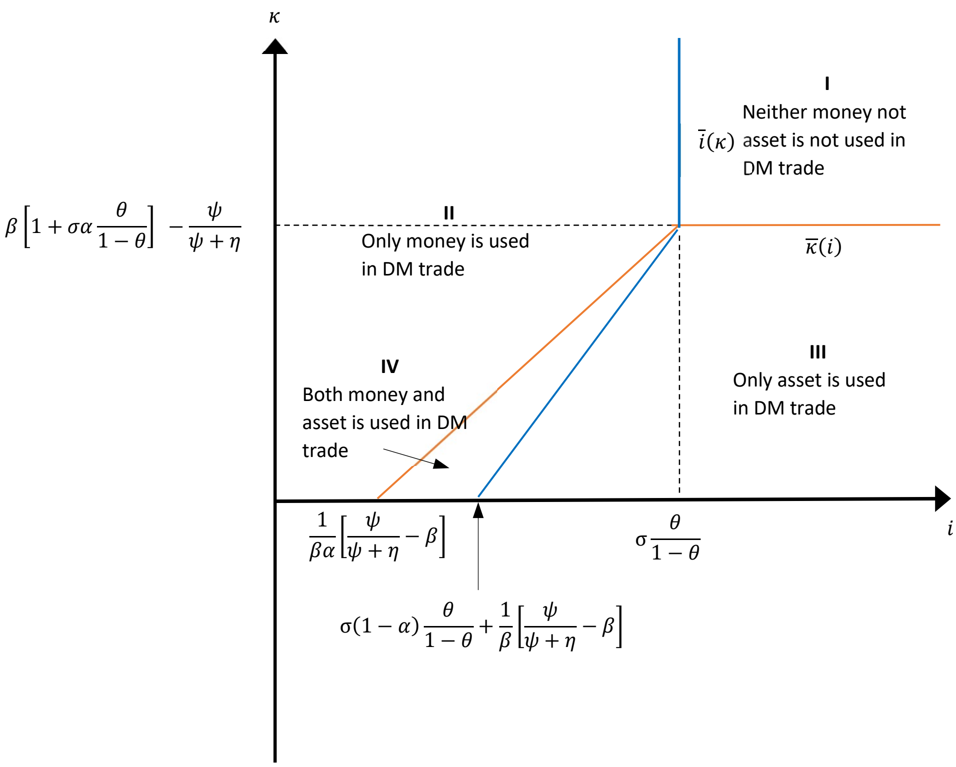


Figure 2.4: Cut-off Regions in DM trade

Note ψ is endogenous and depends on exogenous parameters such as κ and i.

2.4.2 *Numerical Analysis*

Throughout the rest of the sections, we assume the following functional forms for $u(q)$ and $c(q)$:

$$u(q) = A_u \frac{(q+\varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}}{1-\gamma}$$

$$c(q) = A_c \frac{q^{1+\xi}}{1+\xi}$$

For illustration we assign the following values to the following parameters and study how parameters the acceptance rate α and the buyer's bargaining power θ affect the equilibrium regions.

A_u	ε	γ	A_c	ξ	β	σ	α	θ	η

1.00	0.00	0.60	1.00	3.80	0.90	0.25	0.70	0.50	0.10

where we assume the asset is sufficient for the agent to obtain q^* so $\beta = \frac{\psi}{\psi+\eta}$.

We have the following regions as shown in Figure 2.5.

$$\bar{i}(\bar{\kappa}) = \sigma \frac{\theta}{1-\theta} = 0.25$$

$$\bar{\kappa}(\bar{i}) = \beta \left[1 + \sigma\alpha \frac{\theta}{1-\theta}\right] - \frac{\psi}{\psi+\eta} = 0.1575$$

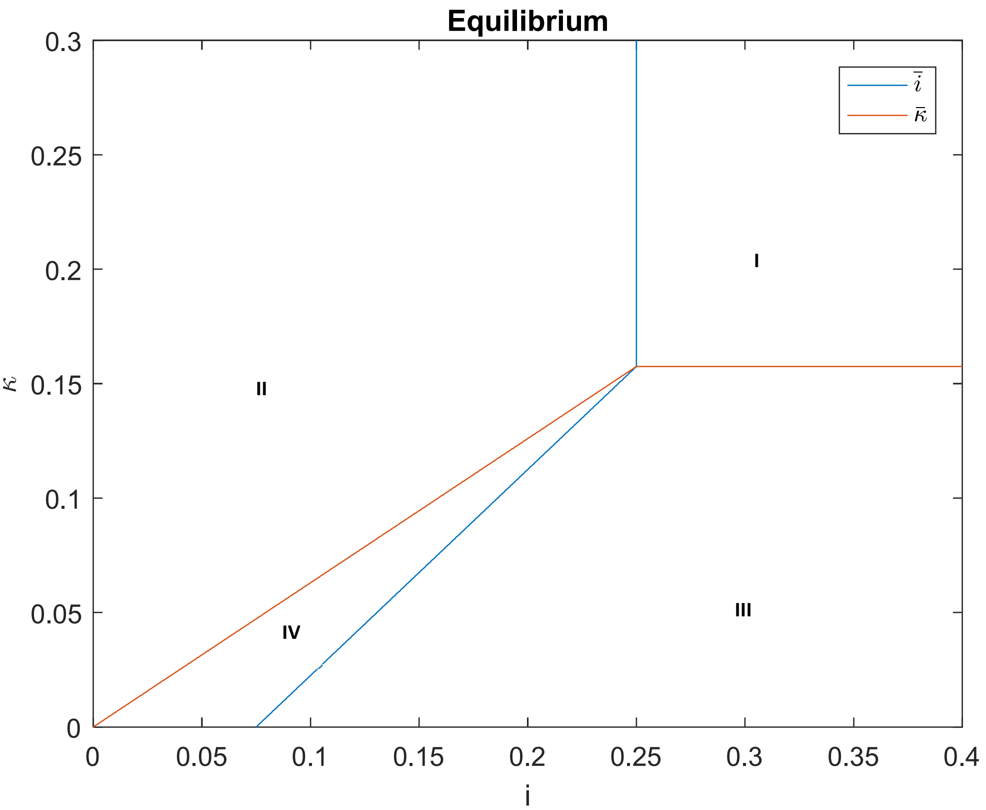


Figure 2.5: Numerical Example of Cut-off Region

2.4.3 Effect of Acceptance Rate α

Figure 2.6 illustrates how the equilibrium regions vary with varying values of α.

The values of $\bar{i}(\bar{\kappa})$ and $\bar{\kappa}(\bar{i})$ are computed below:

α	0.30	0.50	0.70	1.00
$\bar{i}(\bar{\kappa})$	0.25	0.25	0.25	0.25
$\bar{\kappa}(\bar{i})$	0.0675	0.1125	0.1575	0.2225

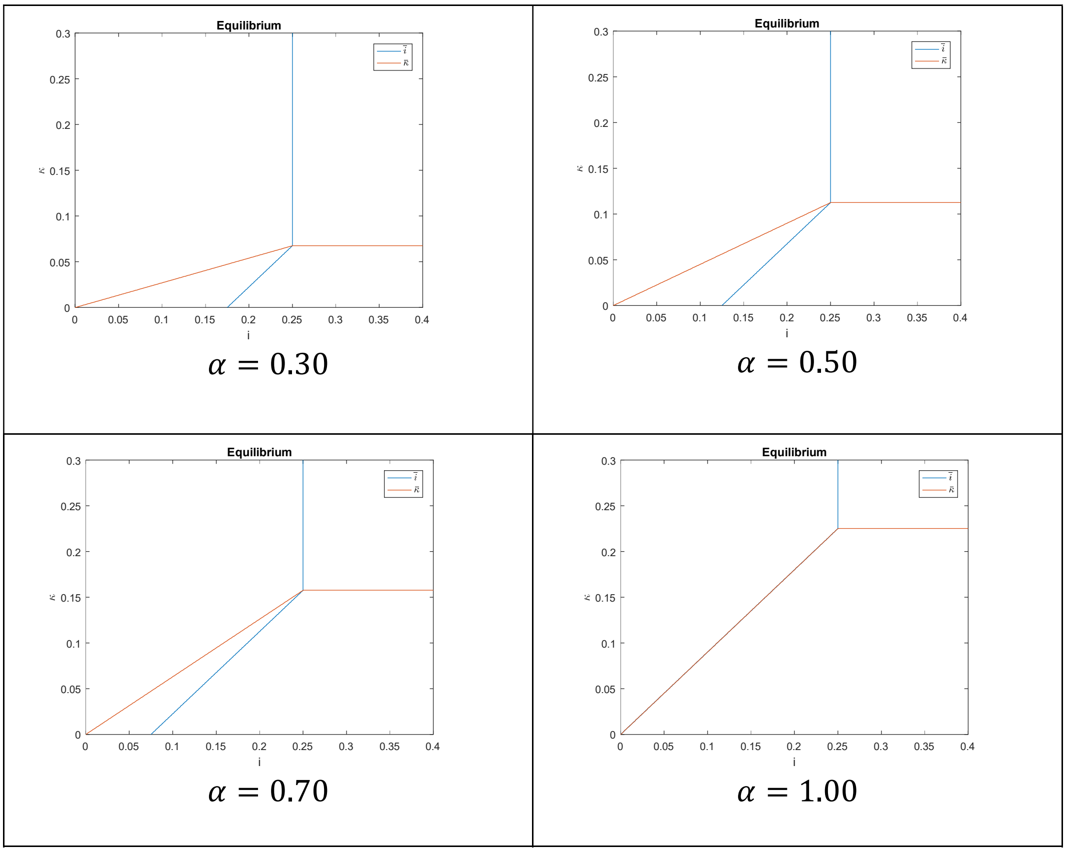


Figure 2.6: Equilibrium Regions when α changes

Proposition 2.2. As $\alpha \to 1$, region IV vanishes.

Proof: Note that as $\alpha \to 1$, we have $\sigma(1-\alpha)\frac{\theta}{1-\theta} + \frac{1}{\beta}\left[\frac{\psi}{\psi+\eta} - \beta\right] \to \frac{1}{\beta}\left[\frac{\psi}{\psi+\eta} - \beta\right]$ and $\frac{\psi}{\beta\alpha(\psi+\eta)} - \beta \to \frac{1}{\beta}\left[\frac{\psi}{\psi+\eta} - \beta\right]$, and region IV vanishes. *End of proof.*

Claim 2.7. $\bar{\imath}(\bar{\kappa})$ is independent of α.

Proof: $\frac{\partial \bar{\imath}(\bar{\kappa})}{\partial \alpha} = 0$. *End of proof.*

Claim 2.8. $\bar{\kappa}(\bar{\imath})$ is increasing in α.

Proof: $\frac{\partial \bar{\kappa}(\bar{\imath})}{\partial \alpha} = \sigma\frac{\theta}{1-\theta} > 0$. *End of proof.*

As $\alpha \to 1$, the asset becomes a perfect substitute for money as a means-of-payment, agents substitute money out for the asset. The cut-off $\bar{\kappa}$ increases as $\alpha \to 1$ meaning it will take more to discourage agents from using the asset as a means-of-payment. When $\alpha = 1$ and the asset is a perfect substitute for money, Region IV vanishes and agents use either money or the asset as means-of-payment depending on which side of the cut-off the equilibrium is in.

2.4.4 Effect of Bargaining Power θ

Figure 2.7 illustrates how the equilibrium regions vary with varying values of θ:

θ	0	0.25	0.50	0.75
$\bar{i}(\kappa)$	0	0.0833	0.25	0.75
$\bar{\kappa}(\bar{i})$	0	0.0525	0.1575	0.4725

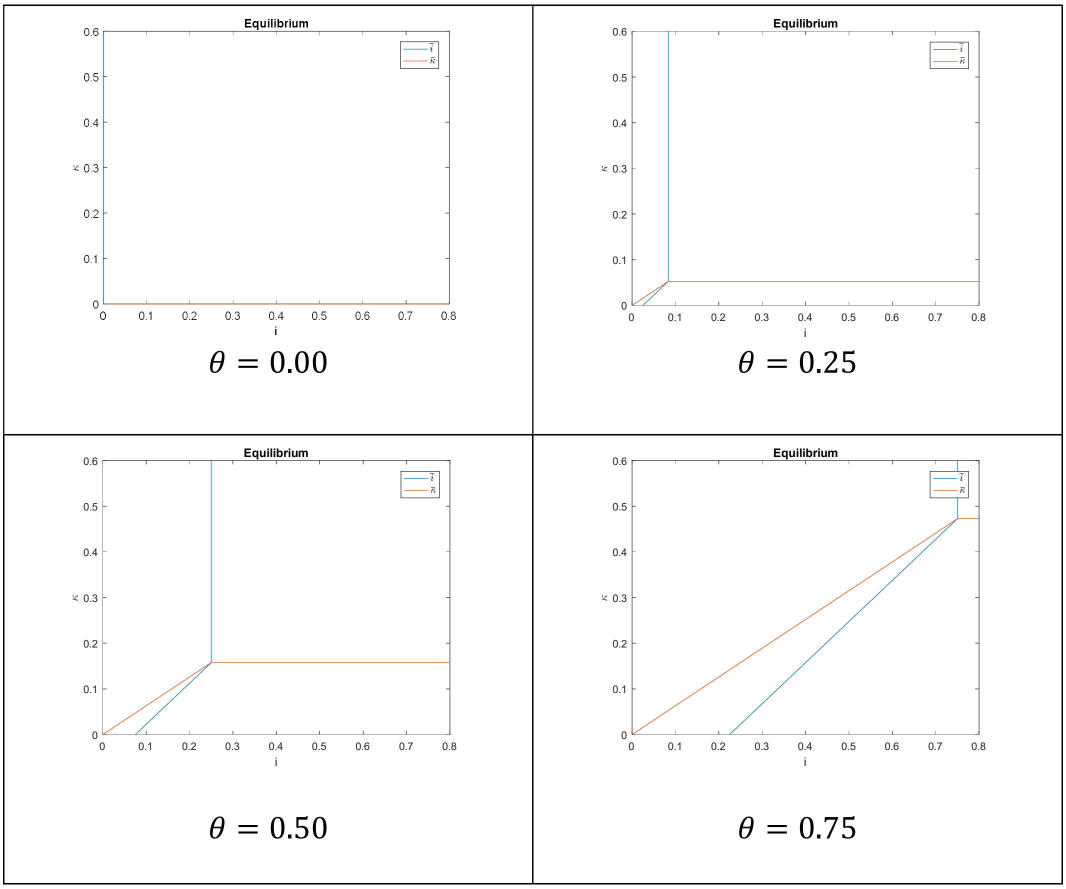


Figure 2.7: Equilibrium Regions when θ changes

When $\theta = 0$, the buyer has no bargaining power and is unable to extract any surplus from a meeting. $\bar{i}(\kappa) = 0$ and $\bar{\kappa}(\bar{i}) = 0$, and the only equilibrium is Region I where Autarky or no trade in the DM.

Claim 2.9. $\bar{i}(\kappa)$ is increasing in θ.

Proof: $\frac{\partial \bar{\iota}(\bar{\kappa})}{\partial \theta} = \frac{\sigma}{(1-\theta)^2} > 0$. *End of proof.*

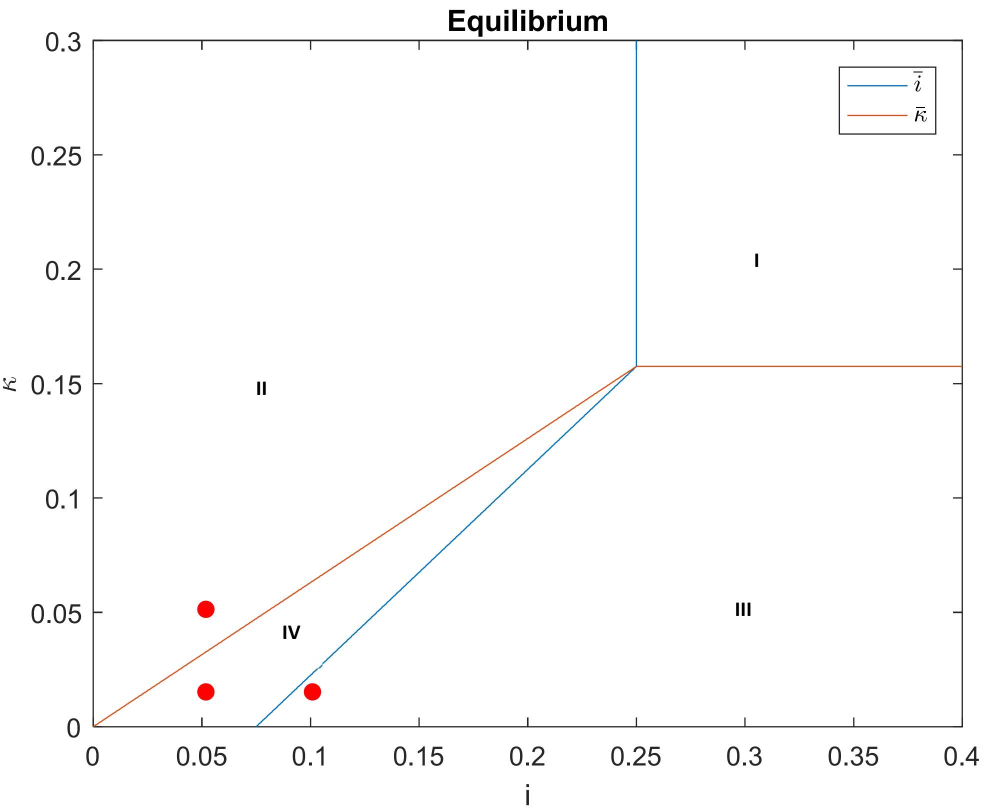
Claim 2.10. $\bar{\kappa}(\bar{\iota})$ is increasing in θ.

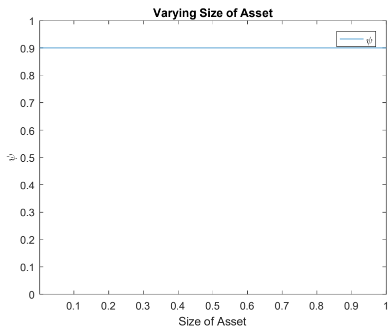
Proof: $\frac{\partial \bar{\kappa}(\bar{\iota})}{\partial \theta} = \frac{\sigma\alpha}{(1-\theta)^2} > 0$. *End of proof.*

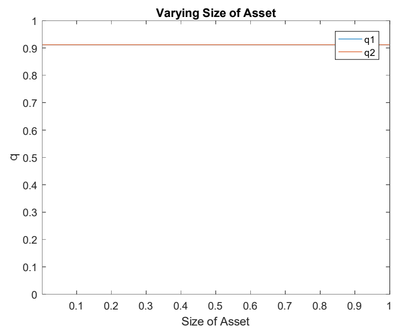
As the buyer's bargaining power θ increases, Regions II, II and IV expands as the buyer is more willing to trade as he can extract more from the trade surplus. The cut-offs $\bar{\iota}(\bar{\kappa})$ shifts right and $\bar{\kappa}(\bar{\iota})$ shifts up as buyers are willing to pay more costs to use money or the asset trade, making the coexistence of money and the alternative means of payment possible.

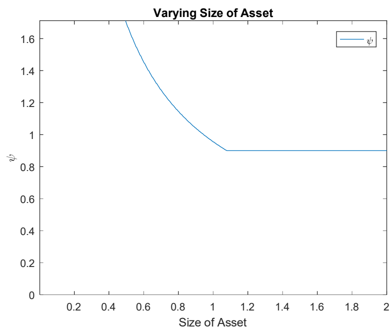
2.4.5 Effect of Asset Size A

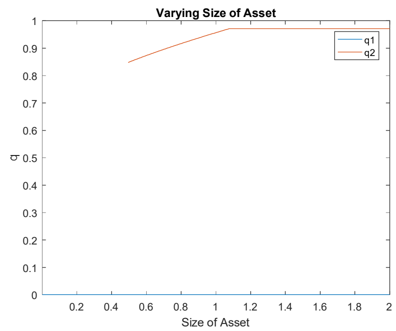
If the asset is in short supply, there is insufficient asset to relax liquidity constraints and the asset commands a liquidity premium and is priced above its fundamental value. On the other hand, if the asset is in excess supply, agents will only use just enough asst to obtain the optimal DM quantity and keep the rest as a store of value. Hence the asset will be priced at its fundamental value. This result is consistent with Lester et.al. (2012). Figure 2.8 illustrates the effect of varying asset size.



<table>
<tr><th>Region</th><th>Asset Price</th><th>DM trade</th></tr>
<tr><td>II</td><td>

$i = 0.05$
$\kappa = 0.05$</td><td>

$i = 0.05$
$\kappa = 0.05$</td></tr>
<tr><td>III</td><td>

$i = 0.10$
$\kappa = 0.01$</td><td>

$i = 0.10$
$\kappa = 0.01$</td></tr>
</table>

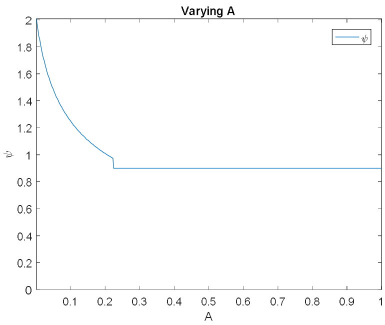
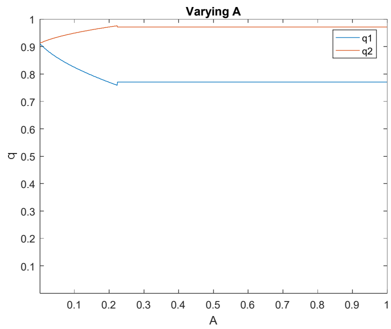
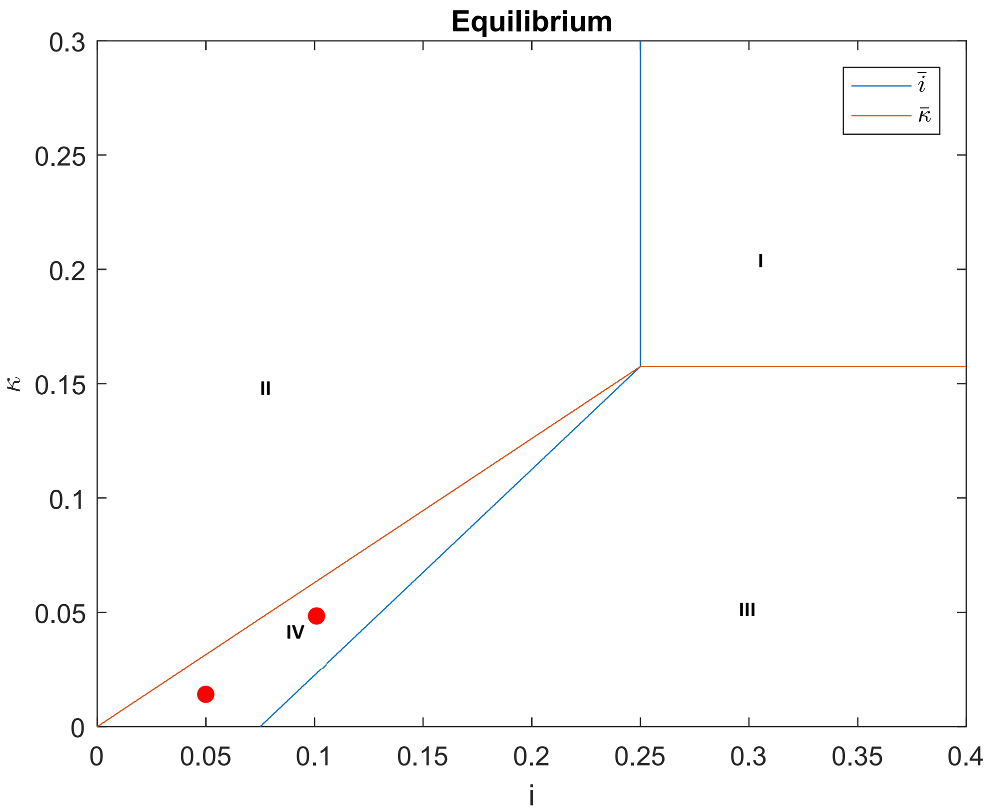
IV	 $i = 0.05$ $\kappa = 0.01$	 $i = 0.05$ $\kappa = 0.01$

Figure 2.8: Effect of Asset Size on Asset Price

In Region II, agents do not use the asset at all, the asset is priced at its fundamental value regardless of the size of the asset. In Region III, only the asset is used for trade in the DM. Hence the availability of asset for trade reduces the liquidity premium and price of the asset greatly. Note that because of the adoption cost, agents never obtain the optimal quantity trading only using the asset. In Region IV, at low values of A, there is insufficient asset for agents to acquire the quantity of DM good that they desire and there is a liquidity premium to the asset and the asset is priced above its fundamental value. As A, increases, the price of the asset falls towards its fundamental value. Note the discontinuity in the asset price and DM quantity traded is due to the adoption cost of using the asset as an alternative means of payment.

Staying in Region IV, as we move away from the origin agents use less of the asset as a means-of payment. Hence the cut-off size in which the liquidity price premium decreases. Figure 2.9 illustrates the cut-off liquidity price premium.



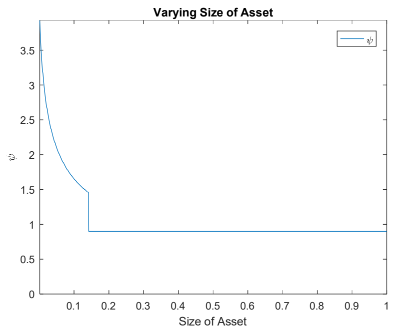
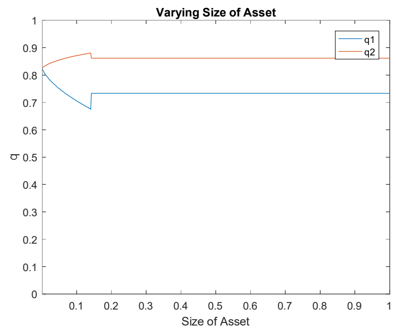
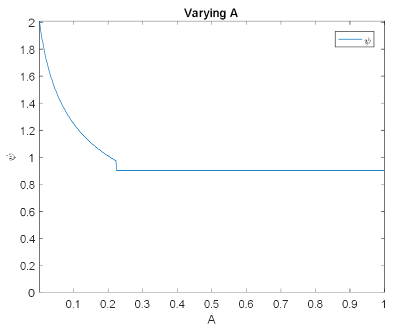
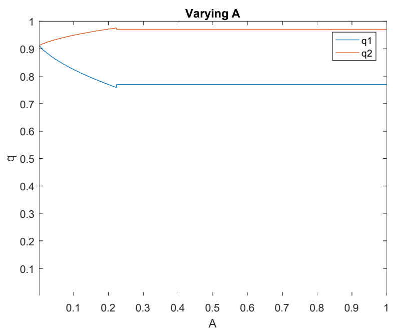
Region	Asset Price	DM trade
IV	 $i = 0.10$ $\kappa = 0.05$	 $i = 0.10$ $\kappa = 0.05$
IV	 $i = 0.05$ $\kappa = 0.01$	 $i = 0.05$ $\kappa = 0.01$

Figure 2.9: Cut-off Liquidity Price Premium

2.4.6 Comparative Statics

It is natural to ask how the quantity of asset used as a means-of-payment where $a = q_2 - q_1$ varies with the various parameters. We show this through comparative statics for the following regions:

- Regions I – Autarky

The endogenous variables are independent of the exogenous variables since $q_1 = 0$ and $q_2 = 0$.

- Region II – Only money is used

ϑ	i	α	κ
$\frac{dq_1}{d\vartheta}$	$-$	no effect	no effect
$\frac{dq_2}{d\vartheta}$	$-$	no effect	no effect
$\frac{da}{d\vartheta}$	no effect	no effect	no effect

Proof: Only (2.36) hold at equality with $q_1 = q_2 = q$.

Differentiating with respect to i and solving we get

$$\frac{\partial q_1}{\partial i} = \frac{\partial q_2}{\partial i} = \frac{1}{\sigma L'(q)} < 0$$

End of proof.

- Region III – Only asset is used

ϑ	i	α	κ
$\frac{dq_1}{d\vartheta}$	no effect	no effect	no effect
$\frac{dq_2}{d\vartheta}$	no effect	$+$	$-$
$\frac{da}{d\vartheta}$	no effect	$+$	$-$

Proof: Only (2.37) hold at equality

Differentiating with respect to i we get

$$\frac{\partial q_2}{\partial i} = 0$$

Differentiating with respect to α we get

$$\frac{\partial q_2}{\partial \alpha} = -\frac{\mathrm{L}(q_2)}{\alpha L'(q_2)} > 0$$

Differentiating with respect to κ we get

$$\frac{\partial q_2}{\partial \kappa} = \frac{1}{\sigma\alpha L'(q_2)} < 0$$

End of proof.

- Region IV – Both money and asset are used

ϑ	i	α	κ
$\frac{dq_1}{d\vartheta}$	−	0	+
$\frac{dq_2}{d\vartheta}$	−	+	−
$\frac{da}{d\vartheta}$	depends	+	−

Proof: Both (2.36) and (2.37) hold at equality

$$i = \sigma(1-\alpha)L\big(q_1(m,a,b)\big) + \sigma\alpha L\big(q_2(m,a,b)\big)$$

$$0 = \beta\sigma\alpha L\big(q_2(m,a,b)\big) - \kappa$$

Differentiating with respect to i and solving we get

$$\frac{\partial q_1}{\partial i} = \frac{1}{\sigma(1-\alpha)L'(q_1)} < 0$$

$$\frac{\partial q_2}{\partial i} = \frac{1}{\sigma\alpha L'(q_2)} < 0$$

Since $q_2 > q_1$, $L(q_1) > L(q_2)$ and $L'(q_2) < L'(q_1) < 0$. If $\alpha \geq 0.5$, then $\frac{\partial q_a}{\partial i} = \frac{\partial q_2}{\partial i} - \frac{\partial q_1}{\partial i} > 0$. However, if $\alpha < 0.5$, then the sign of $\frac{\partial q_a}{\partial i}$ is not fixed.

Differentiating with respect to α and solving we get

$$\frac{\partial q_1}{\partial \alpha} = 0$$

$$\frac{\partial q_2}{\partial \alpha} = -\frac{\mathrm{L}(q_2)}{\alpha L'(q_2)} > 0$$

$$\frac{\partial q_a}{\partial i} = \frac{\partial q_2}{\partial i} - \frac{\partial q_1}{\partial i} > 0$$

Differentiating with respect to κ and solving we get

$$\frac{\partial q_1}{\partial \kappa} = -\frac{1}{\sigma(1-\alpha)L'(q_1)} > 0$$

$$\frac{\partial q_2}{\partial \kappa} = \frac{1}{\sigma(1-\alpha)L'(q_2)} < 0$$

$$\frac{\partial q_a}{\partial \kappa} = \frac{\partial q_2}{\partial \kappa} - \frac{\partial q_1}{\partial \kappa} < 0$$

End of proof.

2.5 Welfare and Policy

In this section, we investigate the welfare which is given by the surplus extracted from DM trade.

Assume that there are a total of n_b buyers and n_s sellers in the economy. The probability σ_B of a buyer meeting a seller in the DM, and the probability σ_S of a seller meeting a buyer in the DM are

$$\sigma_B = \begin{cases} 1 & if\ n_b \geq n_s \\ \frac{n_B}{n_S} & if\ n_b < n_s \end{cases}$$

$$\sigma_S = \begin{cases} \frac{n_S}{n_B} & if\ n_b \geq n_s \\ 1 & if\ n_b < n_s \end{cases}$$

Let the buyer's surplus in a DM trade meeting be $\mathrm{S}^B = u(q) - p$, then the total surplus of a buyer is

$$\mathrm{S}^{Buyer} = \sigma_B[(1-\alpha)\mathrm{S}^B(q_1) + \alpha \mathrm{S}^B(q_2)] - \kappa(\psi + \eta)a$$

where the first term on the RHS says with probability σ_B the buyer meets a seller, and with probability $1-\alpha$ the seller does not accept the asset but only money as a means-of-payment and with probability α the seller accepts both the asset and money as a means-of-payment. The second term in the RHS is the adoption cost for loading the portfolio with a units of liquid asset to be used in a type 2 meeting.

Let the seller's surplus in a DM trade meeting be $\mathrm{S}^S = p - c(q)$, then the surplus of the j-th seller is

$$S_j^{Seller} = \sigma_S[\mathbb{I} \cdot [\mathrm{S}^S(q_2)] + (1 - \mathbb{I}) \cdot \mathrm{S}^S(q_1)] - \mathbb{I} \cdot I_j$$

where $\mathbb{I}$ is the indicator function indicating whether the seller had invested in the acceptance cost to accept the asset as a means-of-payment. The RHS says with probability σ_S the seller meets a buyer in the DM. If the seller had invested in the acceptance cost, he becomes a type 2 seller and extracts the surplus from a type 2 meeting. On the other hand, if the seller did not invest in the acceptance cost, he becomes a type 1 seller and extracts surplus from a type 1 meeting.

If the j-th seller had invested in the acceptance cost, any meeting between the seller and a buyer is a type 2 meeting and the seller's surplus simplifies to

$$S_{j,\mathbb{I}=1}^{Seller} = \sigma_S \mathrm{S}^S(q_2) - I_j$$

If the j-th seller did not invest in the acceptance cost, any meeting between the seller and a buyer is a type 1 meeting and the seller's surplus simplifies to

$$S_{j,\mathbb{I}=0}^{Seller} = \sigma_S \mathrm{S}^S(q_1)$$

Letting $\mathrm{S}(q) = u(q) - c(q)$, from the bargaining solution we have

$$\mathrm{S}^B(q) = \theta \mathrm{S}(q)$$

$$\mathrm{S}^S(q) = (1 - \theta)\mathrm{S}(q)$$

So

$$\mathrm{S}^{Buyer} = \sigma_B[\alpha\theta \mathrm{S}(q_2) + (1 - \alpha)\theta \mathrm{S}(q_1)] - \kappa(\psi + \eta)a \tag{2.38}$$

$$S_{j,\mathbb{I}=1}^{Seller} = \sigma_S(1 - \theta)\mathrm{S}(q_2) - I_j \tag{2.39}$$

$$S_{j,\mathbb{I}=0}^{Seller} = \sigma_S(1 - \theta)\mathrm{S}(q_1) \tag{2.40}$$

Total welfare is given by the sum of welfare of all sellers is

$$\mathrm{S}^{All\ sellers} = \sum_{i=1}^{j} S_{i,\mathbb{I}=1}^{Seller} + \sum_{i=j+1}^{n_S} S_{i,\mathbb{I}=0}^{Seller} \tag{2.41}$$

Let $\mu_I = \frac{1}{j}\sum_{i=1}^{j} I_i$ be the average acceptance cost of all the sellers who invested in the acceptance cost. Let $\mathcal{M}_B \in [0,1]$ be the measure of buyers and $\mathcal{M}_S \in [0,1]$ be the measure of sellers. Then $\sum_{i=1}^{j} S_{i,\mathbb{I}=1}^{Seller} = \mathcal{M}_S\alpha[\sigma_S \mathrm{S}^S(q_2) - \mu_I]$. Also note that $\sum_{i=j+1}^{n_S} S_{i,\mathbb{I}=0}^{Seller} = \mathcal{M}_S(1-\alpha)\sigma_S \mathrm{S}^S(q_1)$. Substituting into (2.41), together with (2.38), we simplify the total welfare of the economy $\mathcal{W} = \sum_{i=1}^{n_B} S_i^{Buyer} + \sum_{i=1}^{j} S_{i,\mathbb{I}=1}^{Seller} + \sum_{i=j+1}^{n_S} S_{i,\mathbb{I}=0}^{Seller}$ as

$$\mathcal{W} = \mathcal{M}_B[\sigma_B[\alpha\theta \mathrm{S}(q_2) + (1-\alpha)\theta \mathrm{S}(q_1)] - \kappa(\psi+\eta)a] + \mathcal{M}_S\big[\alpha[\sigma_S(1-\theta)\mathrm{S}(q_2) - \mu_I] + (1-\alpha)[\sigma_S(1-\theta)\mathrm{S}(q_1)]\big]$$

Simplifying gives

$$\mathcal{W} = [\mathcal{M}_B\sigma_B\theta + \mathcal{M}_S\sigma_S(1-\theta)][\alpha \mathrm{S}(q_2) + (1-\alpha)\mathrm{S}(q_1)] - \mathcal{M}_B\kappa(\psi+\eta)a - \mathcal{M}_S\alpha\mu_I$$

where the first term of the RHS is the surplus generated from DM trade, the second term is the adoption costs paid by buyers for using the asset as a means-of-payment and the last term is the acceptance costs paid by sellers who invested in the acceptance costs.

2.5.1 Endogenous Adoption

To endogenize the buyer's adoption of the asset as a means-of-payment, we look at the shape, uniqueness and concavity of the buyer's response, that is, the quantity of asset used as means-of-payment a in response to the fraction of sellers α who accept the asset as a means-of-payment.

Shape

Note that the buyer's optimal portfolio (m, a, b), $m(\alpha)$ and $a(\alpha)$ depends on α and are functions of α. Subsequently, $q_1(\alpha)$ and $q_2(\alpha)$ depends on $m(\alpha)$ and $a(\alpha)$ which are functions of α.

For $1 + i = \frac{\phi}{\beta\hat{\phi}}$ and $\gamma_a = \frac{\psi}{\hat{\psi}+\hat{\eta}}$ fixed, we assume $m > 0$ and $a > 0$ and interior solutions of the bargaining problem, i.e. (2.35) and (2.36) hold at equality. Let $\ddot{\alpha} > \dot{\alpha}$, then for κ fixed, from (2.36), we have $L\big(q_2(\ddot{\alpha})\big) < L\big(q_2(\dot{\alpha})\big)$ which implies that

$$q_2(\ddot{\alpha}) > q_2(\dot{\alpha})$$

However note that $\ddot{\alpha}L\big(q_2(\ddot{\alpha})\big) = \dot{\alpha}L\big(q_2(\dot{\alpha})\big)$ because (2.36) holds at equality for $a > 0$. From (2.35) at equality for $\ddot{\alpha} > \dot{\alpha}$, we must have $L\big(q_1(\ddot{\alpha})\big) > L\big(q_1(\dot{\alpha})\big)$ which implies

$$q_1(\ddot{\alpha}) < q_1(\dot{\alpha})$$

Since q_1 solely depends on m only, as more sellers accept the asset, buyers hold less money and substitute out money for the asset, that is

$$m(\ddot{\alpha}) < m(\dot{\alpha})$$

with $m(\ddot{\alpha}) < m(\dot{\alpha})$. It is obvious that for $q_2(\ddot{\alpha}) > q_2(\dot{\alpha})$, and we must have

$$a(\ddot{\alpha}) > a(\dot{\alpha})$$

So a is increasing in α.

<u>Uniqueness</u>

For uniqueness of a, we can re-arrange (2.36) and substitute into (2.35). This uniquely determines $q_1(\alpha)$ which in turn pins down the uniqueness of $q_2(\alpha)$.

Alternatively, one can prove uniqueness of q_1 and q_2 by expressing $m(a)$ as function of a and $a(m)$ as function of m, and then showing that they cross at a unique point.

Concavity

Note that $L'(q_2) = \frac{\theta[u''(q)c'(q) - u'(q)c''(q)]}{[(1-\theta)u'(q)+\theta c'(q)]^2} < 0$ and $L''(q_2) > 0$ for $q_2 \in [0, \bar{q}_2]$. For (2.36) at equality, for α small, we have $L(q_2(\alpha))$ large and a small increase in α reduces $L(q_2(\alpha))$ by the same amount. For $L(q_2(\alpha))$ large, $q_2(\alpha)$ is close to zero and a small increase in $q_2(\alpha)$ is able to achieve this. Similarly, for α large, we need a large amount of $q_2(\alpha)$ increase $L(q_2(\alpha))$. So $q_2(\alpha)$ is convex in α, or in other words $a(\alpha)$ is convex in α.

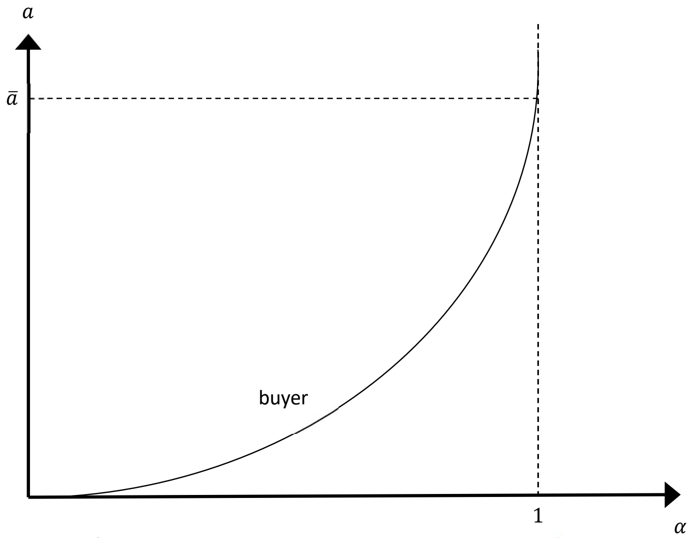


Figure 2.10: Buyer's Response Function

2.5.2 Endogenous Acceptance

Assume that there are n sellers in the economy. Let $I_j > 0$ be the investment cost to the seller j for accepting asset a as means-of-payment payable in the CM before the start of the DM. This could be in the form of a card-reader, or charges by banks for clearing checks. Assume that I_j is increasing in j.

In the DM, if a seller chooses not to pay the acceptance cost to accept asset a, he will only receive payment based on the money holdings of the buyer during the DM, which is $p_1(m) - c_1(m)$. On the other hand, if he chooses to pay the investment cost, he will be able to accept asset a and can potentially receive payments in both money and asset (if the buyer adopts the alternative asset), that is, $p_2(m,a) - c_2(m,a)$ in the DM.

The benefit for a seller accepting asset a as means-of-payment is

$$\varepsilon(m,a,b) = \sigma\left[\left(p_2(m,a,b) - c_2(m,a,b)\right) - \left(p_1(m,a,b) - c_1(m,a,b)\right)\right] \tag{2.42}$$

where the RHS says with probability σ the seller meets a buyer and if he accepts asset a, the entire buyer's wealth can be used as payment, while if he does not accept asset a, only the buyer's money holdings can be used as payment. Note that here, unlike in the case of the buyer where a buyer has a probability of either meeting a seller who accepts both means of payment or only money, the seller meets the homogeneous buyer with probability σ since buyers enter the DM with the same portfolio.

A seller j will only accept asset a if $I_j \leq \varepsilon(m,a,b)$. Re-order the measure of sellers in increasing values of I_j. We denote $\bar{j}$ as the highest j where $I_j \leq \varepsilon(m,a,b)$. The fraction of sellers $\frac{\bar{j}}{n}$ gives the probability of acceptance α. Thus we have

$$\alpha \begin{cases} = 0 & if\ \bar{j} = 0 \\ \in (0,1) & if\ 0 < \bar{j} < n \\ = 1 & if\ \bar{j} = n \end{cases} \tag{2.43}$$

We investigate the seller's optimal acceptance choice taking as given a, the asset used as means-of-payment in the portfolio of buyers entering the DM. Assuming that sellers are ordered in increasing acceptance cost, we have $\bar{j} = \sum_{j=1}^{n} \mathbb{I}_{I_j \leq \varepsilon(m,a,b)}$ which is the sum of all the sellers whose acceptance cost is equal to or less than the benefit for accepting asset a as means-of-payment, making it worthwhile for them to invest in the acceptance cost. As a increases, $\varepsilon(m, a, b)$ increases, meaning $\bar{j}$ increases and α increases.

The seller's maximization problem is

$$\max_{\mathbb{I}_{I\in\{0,1\}}} \left\{\mathbb{I}\left[\sigma\left(p_2(m,a,b) - c_2(m,a,b)\right) - I\right] + (1-\mathbb{I})\sigma\left(p_1(m,a,b) - c_1(m,a,b)\right)\right\}$$

Using $(1-\theta)S(q_1) = u\big(q_1(m,a,b)\big) - p_1(m,a,b)$ and $(1-\theta)S(q_2) = u\big(q_2(m,a,b)\big) - p_2(m,a,b)$, the above simplifies to

$$\sigma(1-\theta)S(q_1) + \max_{\mathbb{I}_{I\in\{0,1\}}} \{\mathbb{I}\sigma(1-\theta)[S(q_2) - S(q_1)] - I, 0\}$$

Let

$$\Psi = \mathbb{I}\cdot\sigma(1-\theta)[S(q_2) - S(q_1)] - I$$

Given q_2 is increasing in a, we have $S(q_2) - S(q_1)$ increasing in a and Ψ increasing in a.

$$\operatorname*{Max}_{\mathbb{I}_{I\in\{0,1\}}} \{\Psi, 0\}$$

The seller will choose $\mathbb{I} = 1$ if $\Psi \geq 0$. As a increases, the number of seller with $\Psi \geq 0$ increases and so α increases.

The shape and uniqueness of the seller's response function is straightforward. Figure 2.11 illustrates the seller's response given the quantity of a buyers use in the DM.

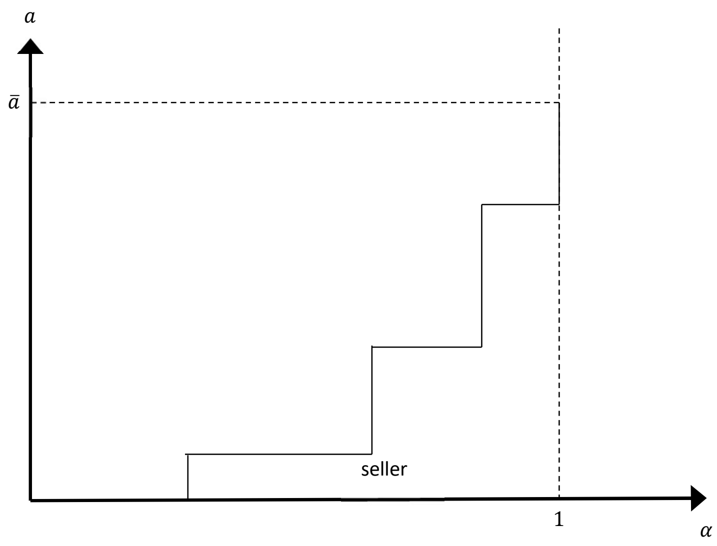


Figure 2.11: Seller's Response Function

Putting the buyer's and seller's response functions together, the model may permit multiple equilibria depending on the number of crossover points as shown in Figuer 2.12.

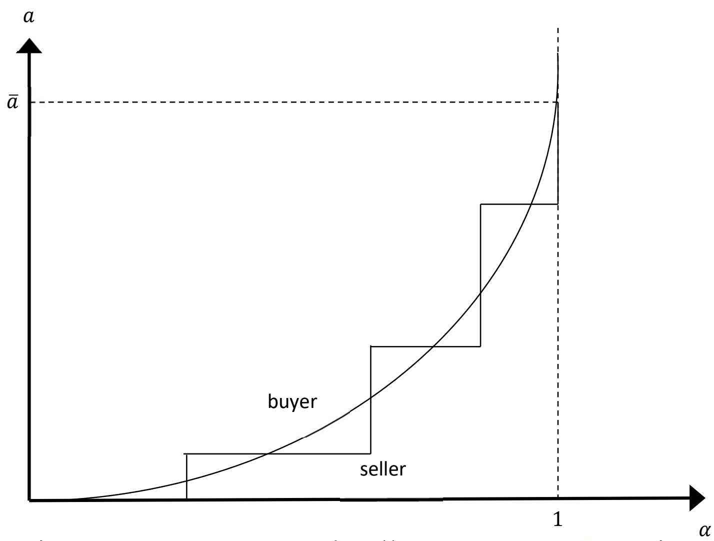


Figure 2.12: Buyer's and Seller's Response Functions

For $\alpha \in (0,1)$ to admit an equilibrium, since a and α are uniquely determined by the buyer's response function, from the seller's response function, for a (and m) given, we need

$$I_{\bar{j}-1} > \sigma\left[\left(p_2(m,a,b) - c_2(m,a,b)\right) - \left(p_1(m,a,b) - c_1(m,a,b)\right)\right]$$

$$I_{\bar{j}} \leq \sigma\left[\left(p_2(m,a,b) - c_2(m,a,b)\right) - \left(p_1(m,a,b) - c_1(m,a,b)\right)\right]$$

We demonstrate how the fraction of sellers accepting the asset as means-of-payment affect the stability of the equilibrium.

Scenario 1:

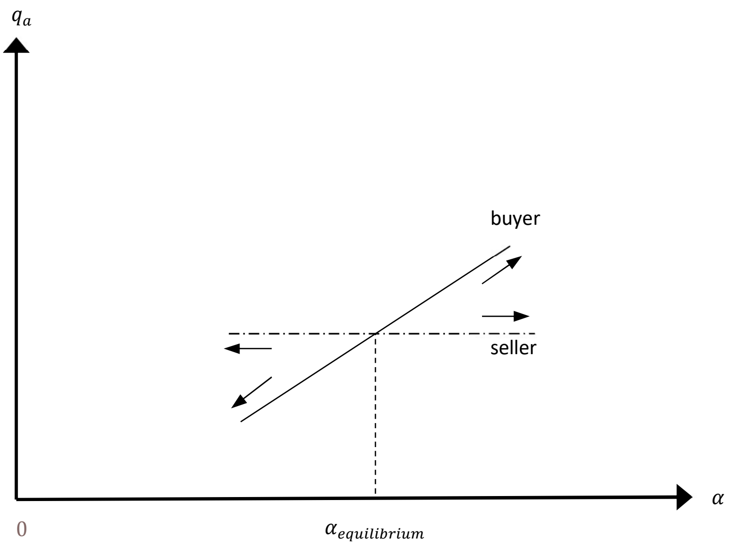


Figure 2.13: Unstable Equilibrium point

Off-equilibrium moves away from $\alpha_{equilibrium}$.

- If left of $\alpha_{equilibrium}$, buyers do not carry enough q_a, so sellers start to exit the market and α decreases.
  - If $\sigma_B = 1$, σ_S increases and seller's response shifts downwards. $\alpha_{equilibrium}$ shifts left but economy may continue to move away from $\alpha_{equilibrium}$.
  - If $\sigma_S = 1$, σ_B decreases and buyer's response shifts upwards. $\alpha_{equilibrium}$ shifts left but economy may continue to move away from $\alpha_{equilibrium}$.
- If right of $\alpha_{equilibrium}$, buyers carry more than enough q_a, so sellers start to enter the market and α increases.

- If $\sigma_B = 1$, σ_S decreases and seller's response shifts upwards. $\alpha_{equilibrium}$ shifts right but economy may continue to move away from $\alpha_{equilibrium}$.
- If $\sigma_S = 1$, σ_B increases and buyer's response shifts downwards. $\alpha_{equilibrium}$ shifts right but economy may continue to move away from $\alpha_{equilibrium}$.

Scenario 2:

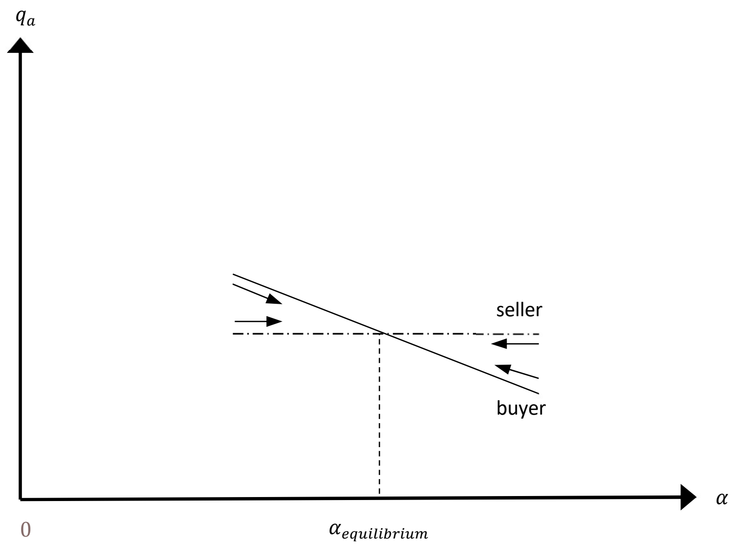


Figure 2.14: Stable Equilibrium point

Off-equilibrium moves towards from equilibrium α.

- If left of $\alpha_{equilibrium}$, buyers carry more than enough q_a, so sellers start to enter the market and α increases.
  - If $\sigma_B = 1$, σ_S decreases and seller's response shifts upwards. $\alpha_{equilibrium}$ shifts left and economy reaches $\alpha_{equilibrium}$ sooner.
  - If $\sigma_S = 1$, σ_B increases and buyer's response shifts downwards. $\alpha_{equilibrium}$ shifts left and economy reaches $\alpha_{equilibrium}$ sooner.

- If right of $\alpha_{equilibrium}$, buyers do not carry enough q_a, so sellers start to exit the market and α decreases.
  - If $\sigma_B = 1$, σ_S increases and seller's response shifts downwards. $\alpha_{equilibrium}$ shifts right and economy reaches $\alpha_{equilibrium}$ sooner.
  - If $\sigma_S = 1$, σ_B decreases and buyer's response shifts upwards. $\alpha_{equilibrium}$ shifts right and economy reaches $\alpha_{equilibrium}$ sooner.

Hence the equilibrium points in scenario 1 are unstable while the points in scenario 2 are stable. To illustrate the stability and feasibility of equilibrium points, we assume the baseline values for the following parameters

A_u	ε	γ	A_c	ξ	β	σ_B	σ_S	θ	η	κ	i
0.60	0.00	0.90	1.00	3.80	0.95	0.025	1.00	0.95	0.10	0.01	0.04

In addition, sellers assume acceptance cost function:

$$I_j = \begin{cases} 0 & if\ j = 1 \\ I_{j-1} + 0.0000050 & if\ 1 < j \leq 0.3n_S \\ I_{j-1} + 0.0000675 & if\ 0.3n_S < j \leq 0.6n_S \\ I_{j-1} + 0.0000500 & if\ 0.6n_S < j \leq 0.8n_S \\ I_{j-1} + 0.0020000 & if\ 0.8n_S < j \leq 0.9n_S \\ I_{j-1} + 0.0002500 & if\ j > 0.9n_S \end{cases}$$

Figure 2.5 shows the possible equilibrium points where an * indicates stable equilibrium points. We note that there are two potential stable equilibrium points, one near $\alpha = 0.4$ and the other near $\alpha = 0.8$.

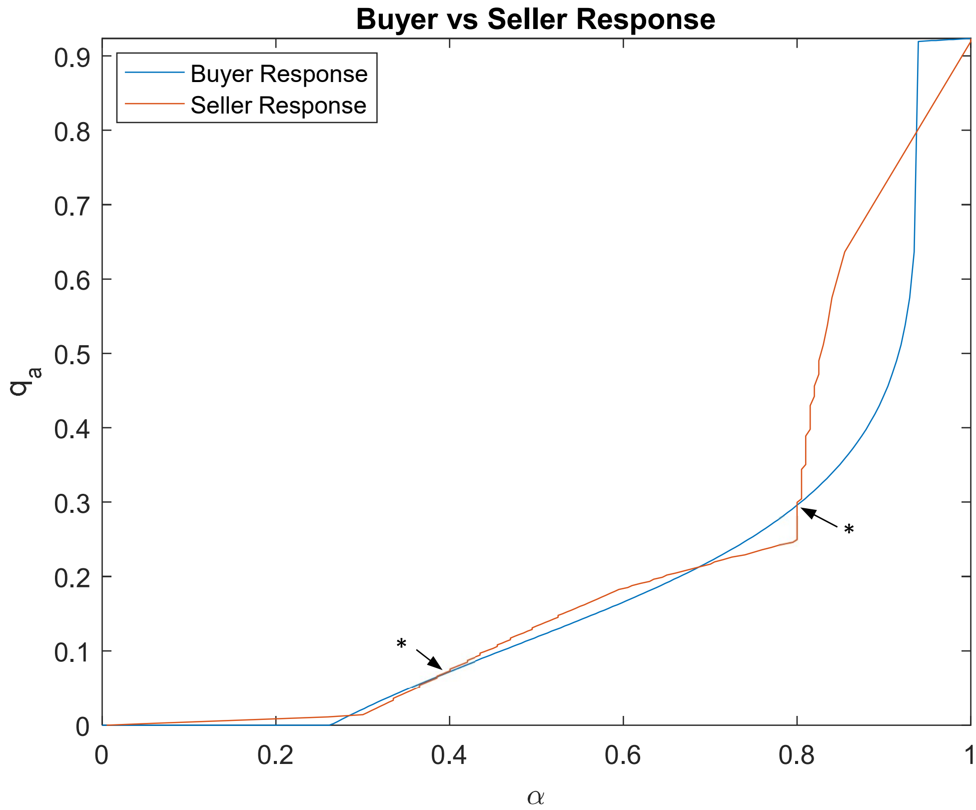


Figure 2.15: Equilibrium points of baseline parameters

Note that as the values of the parameters change, the buyer's response and the seller's response function shifts. Hence, we cannot vary the values of the parameters too much without losing equilibrium points. For example, as i increases, the buyer's response curve shifts upwards as depicted in Figure 2.16.

Figure 2.17 shows the change in equilibrium α as i changes. Because of the gentler slope of the seller's response at the equilibrium point near $\alpha = 0.40$, the equilibrium α shifts more as the buyer's response curve shifts. In contrast, because of the steep slope of the seller's response at the equilibrium point near $\alpha = 0.80$, there is little change to the equilibrium α because the buyer's response and seller's response functions cuts at nearly the same equilibrium α value.

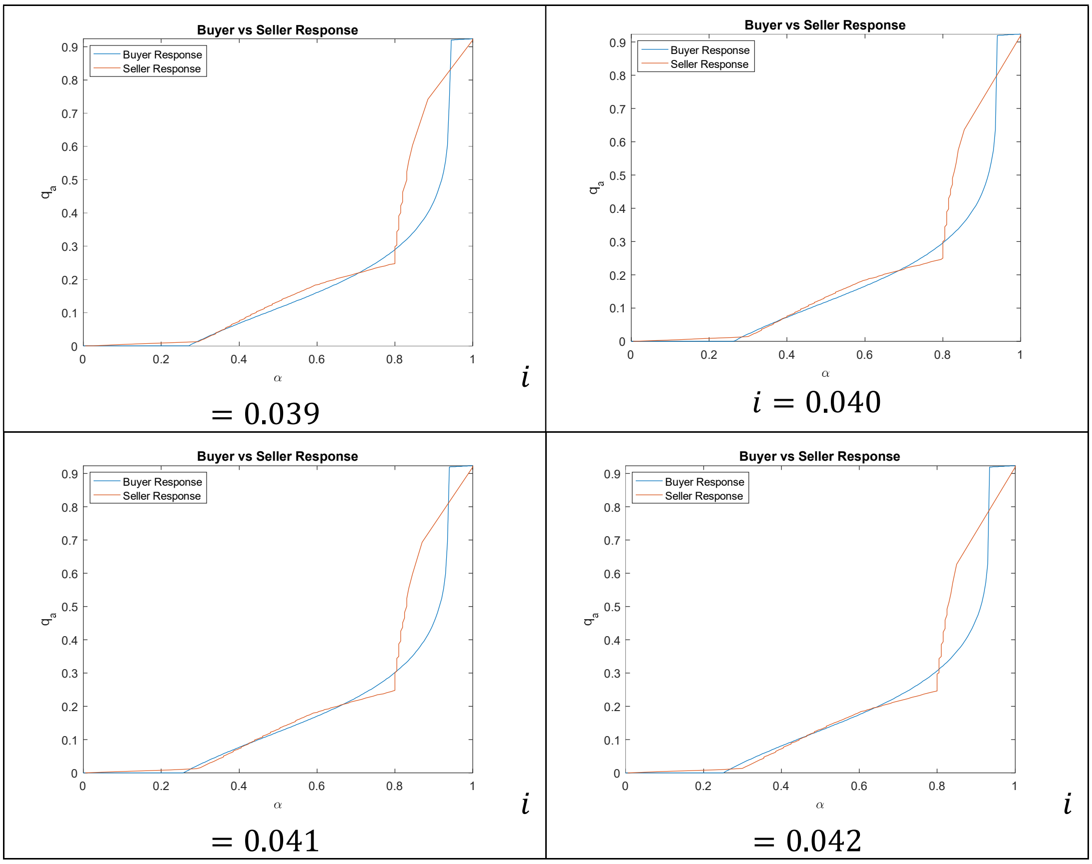


Figure 2.16: Change in buyer's response as i changes

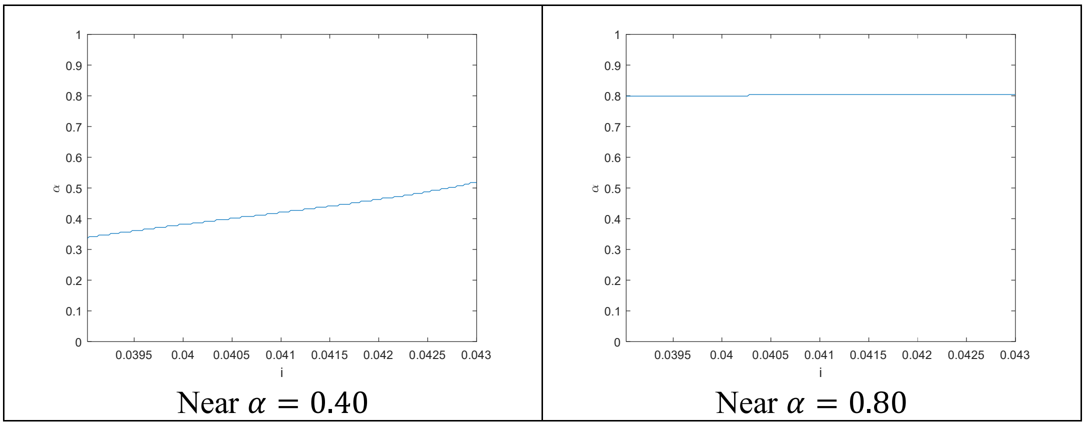


Figure 2.17: Change in equilibrium α as i changes

2.5.3 Feasibility of Equilibrium Points

To check the feasibility of equilibrium points, we need to check the buyer's welfare in an economy with both money and the asset as means-of-payment vis-à-vis an economy with only money as a means-of-payment, and also the marginal seller's welfare for accepting the asset as means-of-payment vis-à-vis not accepting the asset as means-of-payment.

Let $\sim$ denote quantities in an economy where only fiat currency is available. Substituting $\tilde{q} = q_1 = q_2$ into (2.35), the equilibrium condition becomes

$$i \geq \sigma L\big(\tilde{q}(\widetilde{m})\big)$$

with equality if $\widetilde{m} > 0$.

We investigate the effect of θ on the buyer's welfare in Figure 2.18.

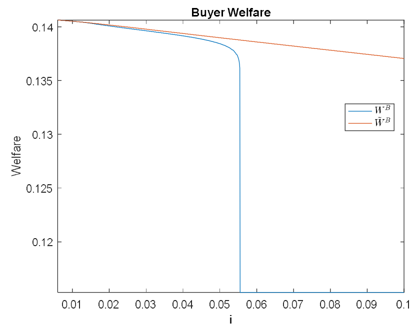
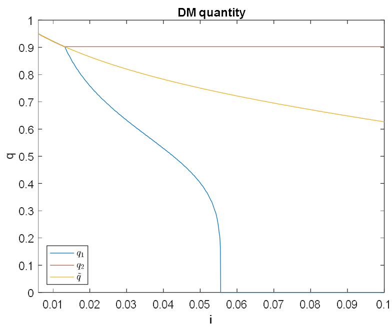
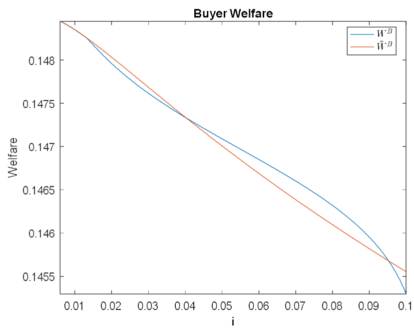
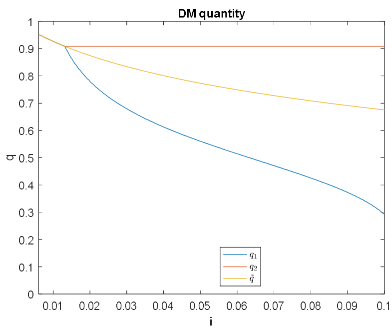
θ	Buyer's Welfare	DM Trade
0.90		
0.95		

Figure 2.18: Effect of θ in a pure fiat economy vis-à-vis economy with fiat and alternative means of payment

First, we observe that for low buyer's bargaining power θ, the buyer's welfare will not improve by introducing an alternative means-of-payment. Hence the buyer will not choose to adopt the asset as an alternative means-of-payment, i.e., an equilibrium where money and the asset as an alternative means-of-payment co-existing does not happen. At very high levels of inflation, the agent will substitute the asset for money completely as a means-of-payment instead. However, if the buyer's bargaining power θ is high enough, buyer's welfare is improved for high levels of inflation and the buyer is most likely to adopt the asset as an alternative means-of-payment.

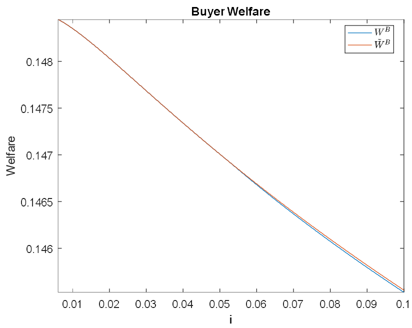
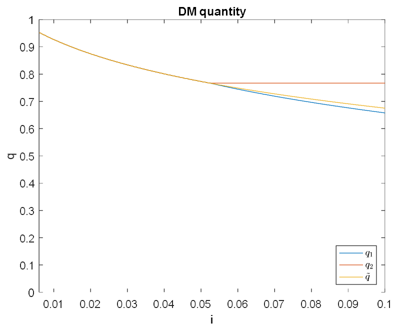
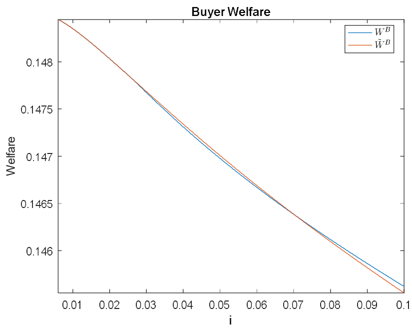
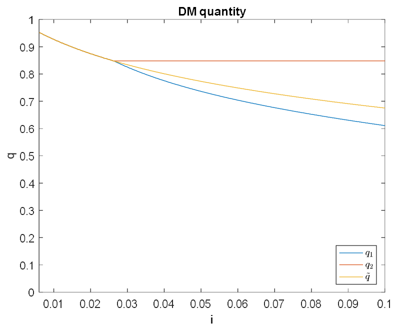
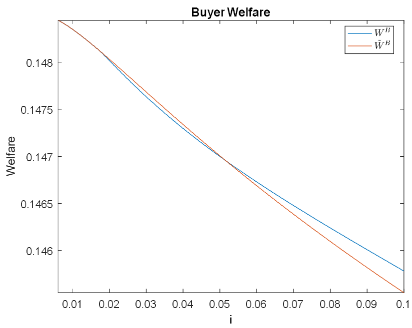
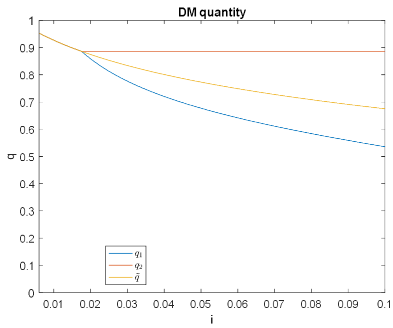
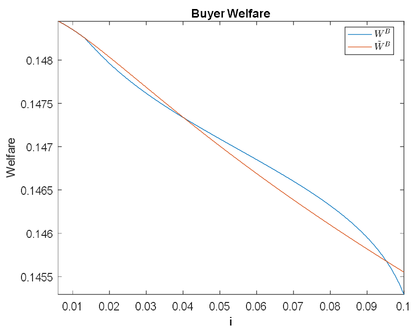
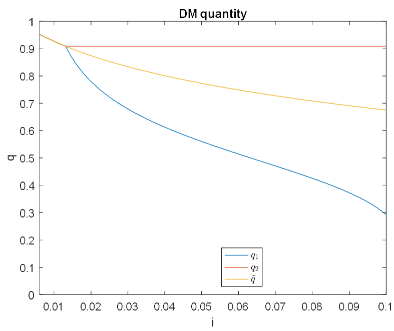
α	Buyer's Welfare	DM Trade
0.20		
0.40		
0.60		
0.80		

Figure 2.19: Effect of α in a pure fiat economy vis-à-vis economy with fiat and alternative means of payment

Assuming θ is high enough that the equilibrium where both money and the asset as an alternative means-of-payment co-exists, from Figure 2.19, we see that as more sellers accept the asset as an alternative means-of-payment, the feasible equilibrium region where both money and the asset as an alternative means-of-payment co-exists shifts to the left, i.e., it exists at lower levels of inflation.

Next, we investigate the incentive for sellers to invest in the acceptance cost. In Figure 2.20, S_0 denotes a seller who did not invest in the acceptance cost while S_1 denotes a seller who invested in the acceptance cost. For $\alpha = 0.40$, we see that the welfare of the seller who pays the investment cost and accepts the asset as a means-of-payment exceeds that of a seller who did not pay the investment cost and does not accept the asset as a means-of-payment for $i > 0.0294$. This means that any equilibrium point $i > 0.0294$ is feasible from the seller's standpoint. From the buyer's standpoint, equilibrium is only feasible for $i > 0.0703$ when the welfare of a buyer who adopts the asset as means-of-payment exceeds the welfare of a buyer in a pure fiat economy. Putting together, equilibrium points are only feasible for $i > 0.0703$.

Similarly, for $\alpha = 0.80$, equilibrium points after $i > 0.01963$ are feasible from the seller's standpoint and equilibrium points after $i > 0.04036$ are feasible from the buyer's standpoint, implying that equilibrium points are only feasible for $i > 0.04036$.

$\alpha = 0.40$	$\alpha = 0.80$

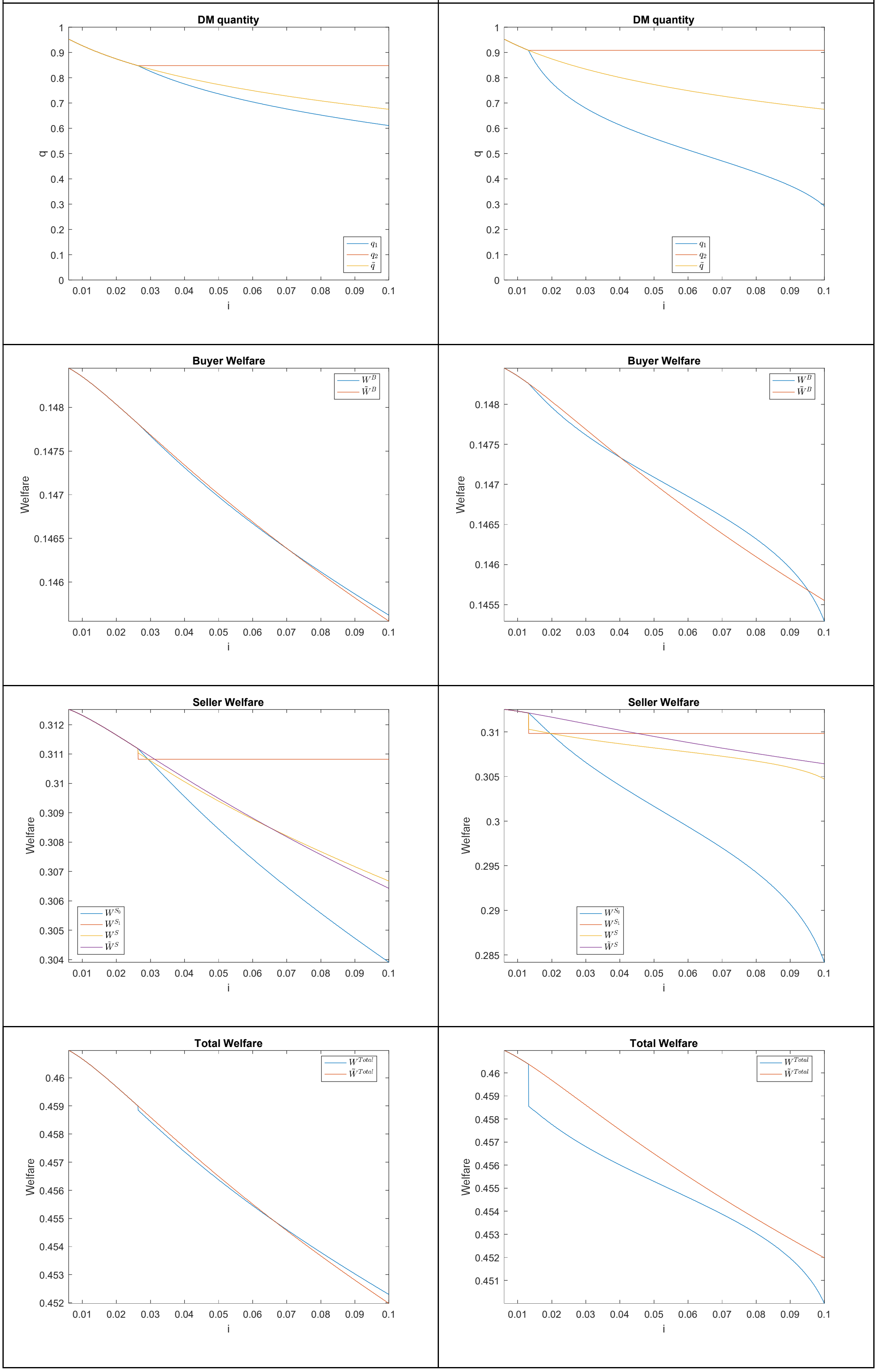


Figure 2.20: Effect of α on Buyer's, Seller's and Total Welfare

In terms of total welfare, similar to Chiu and Wong (2014), Lotz and Vasselin (2019), and Carli and Uras (2022), the introduction of an alternative means-of-payment may not always be welfare improving. For $\alpha = 0.40$, when i is low, the cost of holding money is not great so the total welfare of using the asset as an alternative means-of-payment as compared to a pure fiat economy is negative. This is due to the adoption and acceptance costs. At high $i > 0.06644$, the asset is adopted and accepted as an alternative means-of-payment, and the welfare gains are positive as compared to a pure fiat economy due to the high cost of holding money and the liquidity the asset brings to DM trade. For $\alpha = 0.80$, due to higher acceptance cost for the tail-end sellers from the step function of the cost, total welfare of adopting and accepting the asset as an alternative means-of-payment is always negative as compared to a pure fiat economy. This implies that encouraging widespread adoption of an alternative means-of-payment is not always welfare improving.

2.6 Conclusion

This chapter investigates what determines a buyer's decision to adopt an alternative means-of-payment and a seller's optimal strategy to accept the means-of-payment.

We derive the cut-off values for inflation and adoption costs, beyond which money and the asset are not used as means-of-payment respectively. Equilibrium conditions are sensitive to the seller's acceptance cost distribution. Multiple equilibria may exist for the right cost function chosen, provided i is not too high or not too low.

We find that welfare decreases as i increases. In addition, for money and the asset to co-exist as means of payment, the buyer's bargaining power must be high enough. In such equilibria, increasing the proportion of sellers who accept the asset as an alternative means of payment encourages the adoption of the asset at lower inflation rates.

In general, buyers will not choose to adopt the asset as an alternative means of payment unless inflation is high enough. Widespread adoption of an alternative means-of-payment is not always welfare improving due to higher acceptance cost for the tail-end sellers. Unless inflation is very high, having a smaller fraction of sellers accepting the alternative means-of-payment has higher welfare than a larger fraction of sellers accepting the alternative means-of-payment. Sellers who do not accept the alternative means-of-payment are better off because buyers carry more money when a smaller fraction of sellers accepts the alternative means-of-payment than when a larger fraction of sellers accepts the alternative means-of-payment.

2.7 References

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Chapter 3

Consumer Behaviour and Credit

3.1 Introduction

This chapter aims to study how agents with different profiles such as income level and consumption needs use credit to smooth consumption. Credit is useful when there is a cost to carrying money and agents want to consume more than their money wealth can obtain in the rare event of higher consumption needs. We model secured and unsecured credit and investigate how the various agents use the different types of debt to improve their liquid wealth when faced with liquidity constraints.

Our main contribution is the analysis of a wide variety of agents with different liquidity and consumption needs co-existing simultaneously in the Lagos and Wright (2005) framework. There are four types of agents – (1) low-income agents with high consumption needs, (2) high-income agents with high consumption needs, (3) low-income agents with low consumption needs, and (4) high-income agents with low consumption needs. Along with each agent's probability of accessing a financial market, this gives rise to a total of eight different agents. Here, everyday consumption such as grocery purchases is modelled as low consumption while big ticket items such as car purchases are modelled as high consumption. We then determine the conditions under which each type of agent will use credit to finance their consumption needs. Because of our variety of heterogeneous agents, wea are able to analyse individually the welfare loss of increasing inflation on each type of agent.

As inflation increases, the cost of money increases resulting in agents carrying less fiat currency and rely more on credit to finance their consumption needs. Low-income agents with high consumption needs are always the first to require credit while in most situations, high-income agents with low consumption needs never need credit. Calibrating our model to US data, we find that a half-percent increase in inflation leads to about 10% welfare loss. Low-income agents finance their big ticket items with secured credit while using unsecured credit for everyday purchases. High-income agents on the other hand also finance big ticket items with secured credit but almost never need credit for their everyday purchases.

The paper is organized as follows. Section 2 reviews literature related to alternative means of payment. Section 3 describes the model and environment, including the trading mechanisms and terms of trade. Section 4 describes the general equilibrium and investigates how the various equilibrium regions vary with parameters. Section 5 provides quantitative analysis with calibration, and Section 6 concludes.

3.2 Literature Review

We present a brief literature review in this chapter as extensive review of the literature is covered in Chapter 1.

Due to imperfect record keeping (or agent anonymity), the original LW model does not permit credit. In an attempt to incorporate credit, Telyukova and Wright (2008) introduced a third subperiod where a centralized market with no anonymity assumption operates and agents choose to use interest-bearing credit even when they have money due to idiosyncratic uncertainty about liquidity need.

Sanches and Williamson (2010) introduced the threat of theft with money holding for credit to coexist as competing media of exchange under different record-keeping technologies and ways of money injections.

Nosal and Rocheteau (2011) introduced endogeneous record-keeping but assumes that repayments are perfectly enforced. Lotz and Zhang (2015) extends the model by deriving an endogenous debt limit under limited commitment.

Bethune et. al. (2015) integrate the LW with a frictional Mortensen-Pissarides labour market to demonstrate the relationship between household unsecured debt, liquid assets, and aggregate unemployment. Households cannot commit and are heterogeneous in terms of their access to unsecured credit.

We aim to incorporate the features of existing literature in a general search model. One key feature of this chapter is the co-existence of secured and unsecured credit. Secured credit usually has lower interest rate then secured credit but there is a fixed cost associated to assess it.

3.3 Model

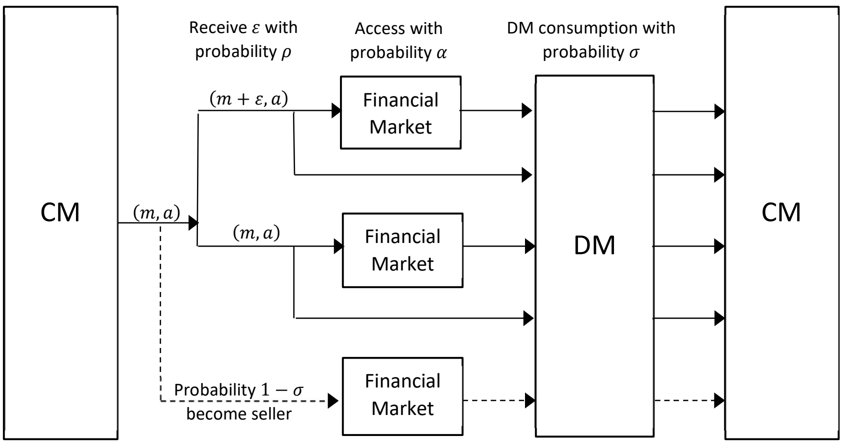


Figure 3.1: Timing of a period

As in the framework developed by Lagos and Wright (2005), time is discrete and divided into periods. Each period is further divided into two subperiods called day and night. The time horizon is infinite, and agents live forever. Agents apply a discount factor $\beta \in (0,1)$ across periods but not between subperiods. There is only one means-of-payment used for trade which is money m. However, agents can take on unsecured or secured debt to finance their consumption needs. There is an asset a which pays dividend η and can be pledged as collateral for secured debt.

Agents leave the night market with a probability σ of being a buyer in a day DM market and a probability $1 - \sigma$ of being a seller in the DM market.

During the day, agents have a probability ρ of receiving an endowment ε. This endowment income process can be thought of as a state-independent random model where all agents own Lucas trees, but some agents' trees bear fruits that period, while the trees of other agents do not bear fruit.

Upon realising their income, agents then get to access a credit market with probability α. The credit market can be thought of as a small open economy where the banks have access to funds in an international financial market at the risk-free interest rate. As there is cost (e.g. legal fees) to lending credit, these firms mark up the interest rate on credit to b_u for unsecured credit and b_a for secured credit. Due to the free entry of firms, all firms have zero profits and provide credit at the same interest rates. Also, as unsecured credit is more risky than secured credit, we assume that $b_u \geq b_a$. Sellers in the DM market always have full access to the credit market where they can deposit their money holdings and receive interest payments b_l. For simplicity, buyers can only borrow and may not deposit money.

After the chance of accessing the credit market and re-adjusting their portfolio, agents then meet bilaterally in a decentralized market (DM)**.** To rule out barter trade, only sellers can produce $q \in \mathbb{R}_+$ units of the DM good at cost $c(q)$ which only buyers want to consume with utility $\xi u(q)$ but cannot produce where $\xi \in \{\xi^l, \xi^h\}$ is shock to DM consumption utility. The optimal consumption and production q^* is given by $\xi u'(q^*) = c'(q^*)$. It is assumed that $u(0) = 0$, $u'(q) > 0$, $u'(0) = \infty$, $u''(q) < 0$, and $c(0) = 0$, $c'(q) > 0$, $c'(0) = 0$, $c''(q) > 0$. The

DM good is assumed to be perfectly divisible, perishable. Since the DM good is non-storable, it is not carried over to the night subperiod, bringing the quantity of DM good traded as $q \in [0, q^*]$.

At night, agents trade in a Walrasian centralized market (CM). Here, agents can choose to work h units of labour to produce a general CM good where for simplicity it is assumed that h is non-binding and it is normalised that 1 unit of labour produces 1 unit of the CM good. Agents can also choose to consume x units of the CM good at utility $U(x)$ by producing the good themselves or buying from an agent that produces the CM good. It is assumed that $U(0) = 0$, $U'(x) > 0$ and $U''(x) \leq 0$. It is also assumed that the CM good is perfectly divisible, perishable, and non-storable and may not be carried over to the day subperiod. In the CM, agents have to pay all debts owed or risk being excluded from trade forever. At the end of the CM subperiod, agents decide the quantity of money and assets they wish to bring into the next subperiod DM.

3.3.1 CM Value Function

An agent entering the CM with a portfolio of m units of money, a units of asset, d_u units of unsecured debt, d_a units of secured debt and l units of deposits has the following CM function:

$$
\begin{aligned}
W(m, a, d_u, d_a, l) \\
= \max_{x,h,\hat{m},\hat{a}} \{ & U(x) - h \\
& + \beta\sigma[\alpha\pi\rho J^h(\hat{m} + \hat{\varepsilon}, \hat{a}) + \alpha\pi(1-\rho)J^h(\hat{m}, \hat{a}) \\
& + \alpha(1-\pi)\rho J^l(\hat{m} + \hat{\varepsilon}, \hat{a}) + \alpha(1-\pi)(1-\rho)J^l(\hat{m}, \hat{a}) \\
& + (1-\alpha)\pi\rho V^h(\hat{m} + \hat{\varepsilon}, \hat{a}, 0,0) + (1-\alpha)\pi(1-\rho)V^h(\hat{m}, \hat{a}, 0,0) \\
& + (1-\alpha)(1-\pi)\rho V^l(\hat{m} + \hat{\varepsilon}, \hat{a}, 0,0) \\
& + (1-\alpha)(1-\pi)(1-\rho)V^l(\hat{m}, \hat{a}, 0,0)] \\
& + \beta(1-\sigma)\rho W(\hat{m} + \hat{\varepsilon}, \hat{a}, 0,0, l) \\
& + \beta(1-\sigma)(1-\rho)W(\hat{m}, \hat{a}, 0,0, l)\}
\end{aligned}
$$

s.t. $x + \phi\hat{m} + \psi\hat{a} = h + \phi m + (\psi + \eta)a - (1 + b_u)d_u - (1 + b_a)d_a + (1 + b_l)l + T$

where J is the credit market value function and V is the DM value function. ϕ is the price of money, ψ is the price of the asset, η is the dividend obtained from holding the asset, x the CM good, h labour worked, b_u is the interest payment on unsecured debt and b_a is the interest payment on secured debt. T are taxes for balancing endowments received at the beginning of the period.

Assuming there is a x^* such that $U(x^*) = x^*$, then above becomes

$$
\begin{aligned}
&W(m, a, d_u, d_a, l) \\
&\quad = \phi m + (\psi + \eta)a - (1 + b_u)d_u - (1 + b_a)d_a + (1 + b_l)l + T \\
&\quad + \max_{\widehat{m}, \hat{a}} \{-\phi \widehat{m} - \psi \hat{a} \\
&\quad + \beta \sigma [\alpha \pi \rho J^h(\widehat{m} + \hat{\varepsilon}, \hat{a}) + \alpha \pi (1 - \rho) J^h(\widehat{m}, \hat{a}) \\
&\quad + \alpha (1 - \pi) \rho J^l(\widehat{m} + \hat{\varepsilon}, \hat{a}) + \alpha (1 - \pi)(1 - \rho) J^l(\widehat{m}, \hat{a}) \\
&\quad + (1 - \alpha) \pi \rho V^h(\widehat{m} + \varepsilon, \hat{a}, 0,0) + (1 - \alpha) \pi (1 - \rho) V^h(\widehat{m}, \hat{a}, 0,0) \\
&\quad + (1 - \alpha)(1 - \pi) \rho V^l(\widehat{m} + \hat{\varepsilon}, \hat{a}, 0,0) \\
&\quad + (1 - \alpha)(1 - \pi)(1 - \rho) V^l(\widehat{m}, \hat{a}, 0,0)] \\
&\quad + \beta (1 - \sigma) W(0, \hat{a}, 0,0, \widehat{m})\}
\end{aligned} \tag{3.1}
$$

That is, with probability ρ the agent receives an endowment ε, with probability π the agent has high consumption needs in the DM and with probability α the agent gets to access the credit market. With probability σ the agent gets to consume in the DM and with probability $1 - \sigma$ the agent are sellers in the DM and goes straight to the period's CM. Note that since sellers have no need for money, they will deposit all their money to earn interest.

The FOCs with respect to $\widehat{m}$ are

$$
\begin{aligned}
\phi \geq \beta \Bigg[&\sigma \alpha \pi \rho \frac{\partial J^h(\widehat{m} + \hat{\varepsilon}, \hat{a})}{\partial \widehat{m}} + \sigma \alpha \pi (1 - \rho) \frac{\partial J^h(\widehat{m}, \hat{a})}{\partial \widehat{m}} \\
&+ \sigma \alpha (1 - \pi) \rho \frac{\partial J^l(\widehat{m} + \hat{\varepsilon}, \hat{a})}{\partial \widehat{m}} + \sigma \alpha (1 - \pi)(1 - \rho) \frac{\partial J^l(\widehat{m}, \hat{a})}{\partial \widehat{m}} \\
&+ \sigma (1 - \alpha) \pi \rho \frac{\partial V^h(\widehat{m} + \hat{\varepsilon}, \hat{a}, 0,0)}{\partial \widehat{m}} \\
&+ \sigma (1 - \alpha) \pi (1 - \rho) \frac{\partial V^h(\widehat{m}, \hat{a}, 0,0)}{\partial \widehat{m}} \\
&+ \sigma (1 - \alpha)(1 - \pi) \rho \frac{\partial V^l(\widehat{m} + \hat{\varepsilon}, \hat{a}, 0,0)}{\partial \widehat{m}} \\
&+ \sigma (1 - \alpha)(1 - \pi)(1 - \rho) \frac{\partial V^l(\widehat{m}, \hat{a}, 0,0)}{\partial \widehat{m}} \\
&+ \beta (1 - \sigma) \frac{\partial W(0, \hat{a}, 0,0, \widehat{m})}{\partial \widehat{m}} \Bigg]
\end{aligned}
$$

with equality if $\hat{m} > 0$.

FOCs with respect to $\hat{a}$ is

$$\psi \geq \beta \left[\sigma\alpha\pi\rho \frac{\partial J^h(\hat{m}+\hat{\varepsilon},\hat{a})}{\partial \hat{a}} + \sigma\alpha\pi(1-\rho)\frac{\partial J^h(\hat{m},\hat{a})}{\partial \hat{a}} \right.$$

$$+ \sigma\alpha(1-\pi)\rho \frac{\partial J^l(\hat{m}+\hat{\varepsilon},\hat{a})}{\partial \hat{a}} + \sigma\alpha(1-\pi)(1-\rho)\frac{\partial J^l(\hat{m},\hat{a})}{\partial \hat{a}}$$

$$+ \sigma(1-\alpha)\pi\rho \frac{\partial V^h(\hat{m}+\hat{\varepsilon},\hat{a},0,0)}{\partial \hat{a}}$$

$$+ \sigma(1-\alpha)\pi(1-\rho) \frac{\partial V^h(\hat{m},\hat{a},0,0)}{\partial \hat{a}}$$

$$+ \sigma(1-\alpha)(1-\pi)\rho \frac{\partial V^l(\hat{m}+\hat{\varepsilon},\hat{a},0,0)}{\partial \hat{a}}$$

$$+ \sigma(1-\alpha)(1-\pi)(1-\rho) \frac{\partial V^l(\hat{m},\hat{a},0,0)}{\partial \hat{a}}$$

$$\left. + \beta(1-\sigma) \frac{\partial W(0,\hat{a},0,0,\hat{m})}{\partial \hat{a}} \right]$$

with equality if $\hat{a} > 0$.

3.3.2 Credit Market Value Function

After receiving their endowment if any and learning of their DM consumption needs, the agent get to access a Credit Market with probability α where they can take on unsecured and secured credit to finance their DM consumption. To access secured credit, agents need to pay a fixed cost c and pledge their asset holdings as collateral.

$$J^h(m+\varepsilon,a) = \max_{d_u^{Hh},d_a^{Hh}} \left\{ V^h(m+\varepsilon+(d_u^{Hh}+d_a^{Hh})/\phi, a, d_u^{Hh}, d_a^{Hh}) - \mathbb{I}_{d_a^{Hh}>0} \cdot c \right\} \tag{3.2a}$$

s.t. $\quad d_u^{Hh} \leq D_u$ and $d_a^{Hh} \leq \kappa\psi a = D(a)$

$$J^l(m+\varepsilon,a) = \max_{d_u^{Hl},d_a^{Hl}} \left\{V^l(m+\varepsilon+(d_u^{Hl}+d_a^{Hl})/\phi, a, d_u^{Hl}, d_a^{Hl}) - \mathbb{I}_{d_a^{Hl}>0}\cdot c\right\}$$

(3.2b)

$$\text{s.t.} \quad d_u^{Hl} \le D_u \text{ and } d_a^{Hl} \le \kappa\psi a = D(a)$$

$$J^h(m,a) = \max_{d_u^{Lh},d_a^{Lh}} \left\{V^h(m+(d_u^{Lh}+d_a^{Lh})/\phi, a, d_u^{Lh}, d_a^{Lh}) - \mathbb{I}_{d_a^{Lh}>0}\cdot c\right\}$$

(3.2c)

$$\text{s.t.} \quad d_u^{Lh} \le D_u \text{ and } d_a^{Lh} \le \kappa\psi a = D(a)$$

$$J^l(m,a) = \max_{d_u^{Ll},d_a^{Ll}} \left\{V^l(m+(d_u^{Ll}+d_a^{Ll})/\phi, a, d_u^{Ll}, d_a^{Ll}) - \mathbb{I}_{d_a^{Ll}>0}\cdot c\right\}$$

(3.2d)

$$\text{s.t.} \quad d_u^{Ll} \le D_u \text{ and } d_a^{Ll} \le \kappa\psi a = D(a)$$

where D_u is the debt limit on unsecured debt and $D(a)$ is the debt limit on secured debt. Here it is assumed that due to transaction costs to liquidating seized assets in secured debt, the borrower can only get funds up to a fraction κ of the total assets pledged.

Let the subscript $\tilde{c}$ denotes the scenario where secured credit is used and the subscript $\tilde{n}$ denote the scenario where secured credit is not used which could be just money alone or money with unsecured credit. If the agent does not use secured credit, he will have

$$J_{\tilde{n}}(m,a) = V(m+d_a/\phi, a, 0, d_a)$$

$$= [\xi u(q_{\tilde{n}}) - \omega_{\tilde{n}}] + W\left(m+\varepsilon+\frac{d_a}{\phi}, a, 0, d_a, 0\right)$$

If he uses secured credit, he will have

$$J_{\tilde{c}}(m,a) = -c + V(m+(d_u+d_a)/\phi, a, d_u, d_a) = [\xi u(q_{\tilde{c}}) - \omega_{\tilde{c}}]$$

$$= -c + W\left(m+\varepsilon+\frac{(d_u+d_a)}{\phi}, a, d_u, d_a, 0\right)$$

Using $W\left(m+\varepsilon+\frac{(d_u+d_a)}{\phi},a,d_u,d_a,0\right)=W\left(m+\varepsilon+\frac{d_a}{\phi},a,0,d_a,0\right)+\phi\frac{d_a}{\phi}-(1+b_a)d_a=W\left(m+\varepsilon+\frac{d_a}{\phi},a,0,d_a,0\right)-b_a d_a$, and agents are will prefer using secured credit as long as $J_{\tilde{c}}(m,a)\geq J_{\tilde{n}}(m,a)$, we have

$$[\xi u(q_{\tilde{c}})-\omega_{\tilde{c}}]-c-(1+b_a)d_a\geq[\xi u(q_{\tilde{n}})-\omega_{\tilde{n}}]$$

Agents will borrow secured credit as long as the benefit from using secured credit exceeds the fixed cost c.

$$c\leq[\xi u(q_{\tilde{c}})-\omega_{\tilde{c}}]-[\xi u(q_{\tilde{n}})-\omega_{\tilde{n}}]-b_a d_a$$

3.3.3 DM Value Function

In the DM, in a match where a buyer meets a seller, the terms of trade are determined by Kalai (1977) proportional bargaining. Let p be the payment handed over to the seller for quantity q of the DM goods if an agreement is reached. The proportional solution is then given by solving

$$\max_{p,q}\{\xi u(q)-p\}$$

$$\text{s.t.}\quad \xi u(q)-p=\theta[\xi u(q)-c(q)]$$

where $\theta\in[0,1]$ is the buyer's bargaining power. Define the payment in a successful trade as

$$\omega(q)=(1-\theta)\xi u(q)+\theta c(q)$$

Let y^* denote the wealth required to acquire the optimal consumption, that is, $y^*=\omega(q^*)$ where q^* is the optimal production and consumption given by $\xi u'(q^*)=c'(q^*)$. Hence buyers will only want to consume up to the optimal quantity q^*, and we have the quantity of DM good traded $q\in[0,q^*]$.

Depending on the liquid wealth y of the buyer, if the buyer has sufficient liquid wealth, i.e. $y\geq y^*$, he pays only $p=y^*$, consumes $q=q^*$ units of the DM good and keeps the rest of his money and asset holdings. If the buyer has insufficient wealth, i.e. $y < y^*$, he exhausts all his wealth holdings and pays $p = y$ to consume $q < q^*$ units of the DM good where q solves $\omega(q) = y$. This shows that p and q are functions of the amount of liquid wealth the buyer can use in a meeting which is dependent on the composition of his portfolio. We note that $p \in [0, y^*]$.

DM terms of trade with payment ω for q units of DM goods are

$$q_c^{Hh} = \begin{cases} q^{h*} & if\ \phi(m+\varepsilon) + d_u^{Hh} + d_a^{Hh} \geq y^{h*} \\ \omega^{-1}(\phi(m+\varepsilon) + d_u^{Hh} + d_a^{Hh}) & if\ \phi(m+\varepsilon) + d_u^{Hh} + d_a^{Hh} < y^{h*} \end{cases}$$

$$\omega_c^{Hh} = \omega(q_c^{Hh}) = \begin{cases} y^{h*} & if\ \phi(m+\varepsilon) + d_u^{Hh} + d_a^{Hh} \geq y^{h*} \\ \phi(m+\varepsilon) + d_u^{Hh} + d_a^{Hh} & if\ \phi(m+\varepsilon) + d_u^{Hh} + d_a^{Hh} < y^{h*} \end{cases}$$

$$q_c^{Hl} = \begin{cases} q^{l*} & if\ \phi(m+\varepsilon) + d_u^{Hl} + d_a^{Hl} \geq y^{l*} \\ \omega^{-1}(\phi(m+\varepsilon) + d_u^{Hl} + d_a^{Hl}) & if\ \phi(m+\varepsilon) + d_u^{Hl} + d_a^{Hl} < y^{l*} \end{cases}$$

$$\omega_c^{Hl} = \omega(q_c^{Hl}) = \begin{cases} y^{l*} & if\ \phi(m+\varepsilon) + d_u^{Hl} + d_a^{Hl} \geq y^{l*} \\ \phi(m+\varepsilon) + d_u^{Hl} + d_a^{Hl} & if\ \phi(m+\varepsilon) + d_u^{Hl} + d_a^{Hl} < y^{l*} \end{cases}$$

$$q_n^{Hh} = \begin{cases} q^{h*} & if\ \phi(m+\varepsilon) \geq y^{h*} \\ \omega^{-1}\big(\phi(m+\varepsilon)\big) & if\ \phi(m+\varepsilon) < y^{h*} \end{cases}$$

$$\omega_n^{Hh} = \omega(q_n^{Hh}) = \begin{cases} y^{h*} & if\ \phi(m+\varepsilon) \geq y^{h*} \\ \phi(m+\varepsilon) & if\ \phi(m+\varepsilon) < y^{h*} \end{cases}$$

$$q_n^{Hl} = \begin{cases} q^{l*} & if\ \phi(m+\varepsilon) \geq y^{l*} \\ \omega^{-1}\big(\phi(m+\varepsilon)\big) & if\ \phi(m+\varepsilon) < y^{l*} \end{cases}$$

$$\omega_n^{Hl} = \omega(q_n^{Hl}) = \begin{cases} y^{l*} & if\ \phi(m+\varepsilon) \geq y^{l*} \\ \phi(m+\varepsilon) & if\ \phi(m+\varepsilon) < y^{l*} \end{cases}$$

$$q_c^{Lh} = \begin{cases} q^{h*} & if\ \phi(m+\varepsilon) + d_u^{Lh} + d_a^{Lh} \geq y^{h*} \\ \omega^{-1}(\phi(m+\varepsilon) + d_u^{Lh} + d_a^{Lh}) & if\ \phi(m+\varepsilon) + d_u^{Lh} + d_a^{Lh} < y^{h*} \end{cases}$$

$$\omega_c^{Lh} = \omega(q_c^{Lh}) = \begin{cases} y^{h*} & if\ \phi(m+\varepsilon) + d_u^{Lh} + d_a^{Lh} \geq y^{h*} \\ \phi(m+\varepsilon) + d_u^{Lh} + d_a^{Lh} & if\ \phi(m+\varepsilon) + d_u^{Lh} + d_a^{Lh} < y^{h*} \end{cases}$$

$$q_c^{Ll} = \begin{cases} q^{l*} & if\ \phi(m+\varepsilon) + d_u^{Ll} + d_a^{Ll} \geq y^{l*} \\ \omega^{-1}(\phi(m+\varepsilon) + d_u^{Ll} + d_a^{Ll}) & if\ \phi(m+\varepsilon) + d_u^{Ll} + d_a^{Ll} < y^{l*} \end{cases}$$

$$\omega_c^{Ll} = \omega(q_c^{Ll}) = \begin{cases} y^{l*} & if\ \phi(m+\varepsilon) + d_u^{Ll} + d_a^{Ll} \geq y^{l*} \\ \phi(m+\varepsilon) + d_u^{Ll} + d_a^{Ll} & if\ \phi(m+\varepsilon) + d_u^{Ll} + d_a^{Ll} < y^{l*} \end{cases}$$

$$q_n^{Lh} = \begin{cases} q^{h*} & if\ \phi(m+\varepsilon) \geq y^{h*} \\ \omega^{-1}\big(\phi(m+\varepsilon)\big) & if\ \phi(m+\varepsilon) < y^{h*} \end{cases}$$

$$\omega_n^{Lh} = \omega(q_n^{Lh}) = \begin{cases} y^{h*} & if\ \phi(m+\varepsilon) \geq y^{h*} \\ \phi(m+\varepsilon) & if\ \phi(m+\varepsilon) < y^{h*} \end{cases}$$

$$q_n^{Ll} = \begin{cases} q^{l*} & if\ \phi(m+\varepsilon) \geq y^{l*} \\ \omega^{-1}\big(\phi(m+\varepsilon)\big) & if\ \phi(m+\varepsilon) < y^{l*} \end{cases}$$

$$\omega_n^{Ll} = \omega(q_n^{Ll}) = \begin{cases} y^{l*} & if\ \phi(m+\varepsilon) \geq y^{l*} \\ \phi(m+\varepsilon) & if\ \phi(m+\varepsilon) < y^{l*} \end{cases}$$

The FOCs are

$$\frac{\partial q_c^{Hh}}{\partial m} = \begin{cases} 0 & if\ q_c^{Hh} \geq q^{h*} \\ \dfrac{\phi}{\omega_c^{Hh'}} & if\ q_c^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_c^{Hh}}{\partial m} = \begin{cases} 0 & if\ q_c^{Hh} \geq q^{h*} \\ \phi & if\ q_c^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial q_c^{Hh}}{\partial a} = \begin{cases} 0 & if\ q_c^{Hh} \geq q^{h*} \\ \dfrac{\kappa\psi}{\omega_c^{Hh'}} & if\ q_c^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_c^{Hh}}{\partial a} = \begin{cases} 0 & if\ q_c^{Hh} \geq q^{h*} \\ \psi + \eta & if\ q_c^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial q_c^{Hl}}{\partial m} = \begin{cases} 0 & if\ q_c^{Hl} \geq q^{l*} \\ \dfrac{\phi}{\omega_c^{Hl'}} & if\ q_c^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_c^{Hl}}{\partial m} = \begin{cases} 0 & if\ q_c^{Hl} \geq q^{l*} \\ \phi & if\ q_c^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial q_c^{Hl}}{\partial a} = \begin{cases} 0 & if\ q_c^{Hl} \geq q^{l*} \\ \dfrac{\kappa\psi}{\omega_c^{Hl'}} & if\ q_c^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_c^{Hl}}{\partial a} = \begin{cases} 0 & if\ q_c^{Hl} \geq q^{l*} \\ \psi + \eta & if\ q_c^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial q_n^{Hh}}{\partial m} = \begin{cases} 0 & if\ q_n^{Hh} \geq q^{h*} \\ \dfrac{\phi}{\omega_n^{Hh'}} & if\ q_n^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_n^{Hh}}{\partial m} = \begin{cases} 0 & if\ q_n^{Hh} \geq q^{h*} \\ \phi & if\ q_n^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial q_n^{Hh}}{\partial a} = \begin{cases} 0 & if\ q_n^{Hh} \geq q^{h*} \\ 0 & if\ q_n^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_n^{Hh}}{\partial a} = \begin{cases} 0 & if\ q_n^{Hh} \geq q^{h*} \\ 0 & if\ q_n^{Hh} < q^{h*} \end{cases}$$

$$\frac{\partial q_n^{Hl}}{\partial m} = \begin{cases} 0 & if\ q_n^{Hl} \geq q^{l*} \\ \dfrac{\phi}{\omega_n^{Hl'}} & if\ q_n^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_n^{Hl}}{\partial m} = \begin{cases} 0 & if\ q_n^{Hl} \geq q^{l*} \\ \phi & if\ q_n^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial q_n^{Hl}}{\partial a} = \begin{cases} 0 & if\ q_n^{Hl} \geq q^{l*} \\ 0 & if\ q_n^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_n^{Hl}}{\partial a} = \begin{cases} 0 & if\ q_n^{Hl} \geq q^{l*} \\ 0 & if\ q_n^{Hl} < q^{l*} \end{cases}$$

$$\frac{\partial q_c^{Lh}}{\partial m} = \begin{cases} 0 & if\ q_c^{Lh} \geq q^{h*} \\ \dfrac{\phi}{\omega_c^{Lh'}} & if\ q_c^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_c^{Lh}}{\partial m} = \begin{cases} 0 & if\ q_c^{Lh} \geq q^{h*} \\ \phi & if\ q_c^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial q_c^{Lh}}{\partial a} = \begin{cases} 0 & if\ q_c^{Lh} \geq q^{h*} \\ \dfrac{\kappa\psi}{\omega_c^{Lh'}} & if\ q_c^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_c^{Lh}}{\partial a} = \begin{cases} 0 & if\ q_c^{Lh} \geq q^{h*} \\ \psi + \eta & if\ q_c^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial q_c^{Ll}}{\partial m} = \begin{cases} 0 & if\ q_c^{Ll} \geq q^{l*} \\ \dfrac{\phi}{\omega_c^{Ll'}} & if\ q_c^{Ll} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_c^{Ll}}{\partial m} = \begin{cases} 0 & if\ q_c^{Ll} \geq q^* \\ \phi & if\ q_c^{Ll} < q^* \end{cases}$$

$$\frac{\partial q_c^{Ll}}{\partial a} = \begin{cases} 0 & if\ q_c^{Ll} \geq q^{l*} \\ \frac{\kappa\psi}{\omega_c^{Ll'}} & if\ q_c^{Ll} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_c^{Ll}}{\partial a} = \begin{cases} 0 & if\ q_c^{Ll} \geq q^{l*} \\ \psi + \eta & if\ q_c^{Ll} < q^{l*} \end{cases}$$

$$\frac{\partial q_n^{Lh}}{\partial m} = \begin{cases} 0 & if\ q_n^{Lh} \geq q^{h*} \\ \frac{\phi}{\omega_n^{Lh'}} & if\ q_n^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_n^{Lh}}{\partial m} = \begin{cases} 0 & if\ q_n^{Lh} \geq q^{h*} \\ \phi & if\ q_n^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial q_n^{Lh}}{\partial a} = \begin{cases} 0 & if\ q_n^{Lh} \geq q^{h*} \\ 0 & if\ q_n^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial \omega_n^{Lh}}{\partial a} = \begin{cases} 0 & if\ q_n^{Lh} \geq q^{h*} \\ 0 & if\ q_n^{Lh} < q^{h*} \end{cases}$$

$$\frac{\partial q_n^{Ll}}{\partial m} = \begin{cases} 0 & if\ q_n^{Ll} \geq q^{l*} \\ \frac{\phi}{\omega_n^{Ll'}} & if\ q_n^{Ll} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_n^{Ll}}{\partial m} = \begin{cases} 0 & if\ q_n^{Ll} \geq q^{l*} \\ \phi & if\ q_n^{Ll} < q^{l*} \end{cases}$$

$$\frac{\partial q_n^{Ll}}{\partial a} = \begin{cases} 0 & if\ q_n^{Ll} \geq q^{l*} \\ 0 & if\ q_n^{Ll} < q^{l*} \end{cases}$$

$$\frac{\partial \omega_n^{Ll}}{\partial a} = \begin{cases} 0 & if\ q_n^{Ll} \geq q^{l*} \\ 0 & if\ q_n^{Ll} < q^{l*} \end{cases}$$

The buyer's DM value functions are

$$V^h(m + \varepsilon + (d_u^{Hh} + d_a^{Hh})/\phi, a, d_u^{Hh}, d_a^{Hh})$$

$$= \xi^h u(q_c^{Hh}) - \omega_c^{Hh} + W\left(m + \varepsilon + \frac{(d_u^{Hh} + d_a^{Hh})}{\phi}, a, d_u^{Hh}, d_a^{Hh}, 0\right)$$

$$V^l(m+\varepsilon+(d_u^{Hl}+d_a^{Hl})/\phi, a, d_u^{Hl}, d_a^{Hl})$$

$$= \xi^l u(q_c^{Hl}) - \omega_c^{Hl} + W(m+\varepsilon+(d_u^{Hl}+d_a^{Hl})/\phi, a, d_u^{Hl}, d_a^{Hl}, 0)$$

$$V^h(m+\varepsilon, a, 0,0) = \xi^h u(q_n^{Hh}) - \omega_n^{Hh} + W(m+\varepsilon, a, 0,0,0)$$

$$V^l(m+\varepsilon, a, 0,0) = \xi^l u(q_n^{Hl}) - \omega_n^{Hl} + W(m+\varepsilon, a, 0,0,0)$$

$$V^h(m+(d_u^{Lh}+d_a^{Lh})/\phi, a, d_u^{Lh}, d_a^{Lh})$$

$$= \xi^h u(q_c^{Lh}) - \omega_c^{Lh} + W(m+(d_u+d_a)/\phi, a, d_u^{Lh}, d_a^{Lh}, 0)$$

$$V^l(m+(d_u^{Ll}+d_a^{Ll})/\phi, a, d_u^{Ll}, d_a^{Ll})$$

$$= \xi^l u(q_c^{Ll}) - \omega_c^{Ll} + W(m+(d_u^{Ll}+d_a^{Ll})/\phi, a, d_u^{Ll}, d_a^{Ll}, 0)$$

$$V^h(m, a, 0,0) = \xi^h u(q_n^{Lh}) - \omega_n^{Lh} + W(m, a, 0,0,0)$$

$$V^l(m, a, 0,0) = \xi^l u(q_n^{Ll}) - \omega_n^{Ll} + W(m, a, 0,0,0)$$

(3.3a) to (3.3h)

3.4 General Equilibrium

WLOG, we drop subscripts and superscripts for ease of notation. Define the liquidity premium as

$$L^h(q) = \frac{\xi^h u'(q)}{\omega'(q)} - 1 = \begin{cases} 0 & if\ q \geq q^* \\ \dfrac{\theta[\xi^h u'(q) - c'(q)]}{(1-\theta)\xi^h u'(q) + \theta c'(q)} & if\ q < q^* \end{cases}$$

$$L^l(q) = \frac{\xi^l u'(q)}{\omega'(q)} - 1 = \begin{cases} 0 & if\ q \geq q^* \\ \dfrac{\theta[\xi^l u'(q) - c'(q)]}{(1-\theta)\xi^l u'(q) + \theta c'(q)} & if\ q < q^* \end{cases}$$

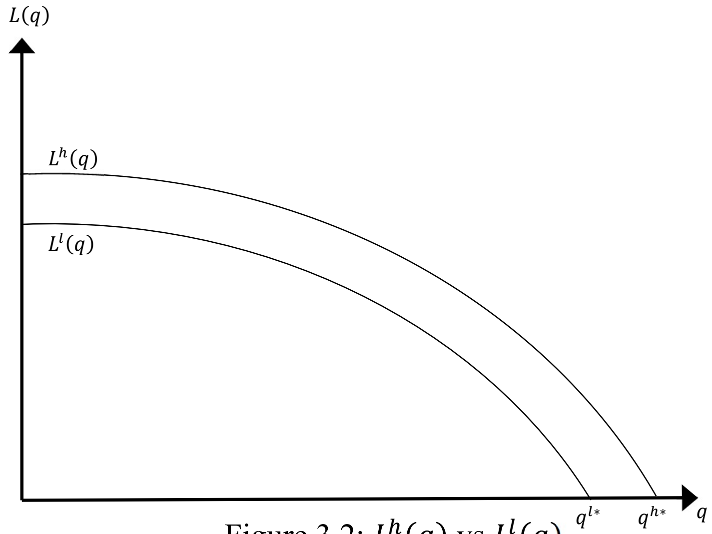


Figure 3.2: $L^h(q)$ vs $L^l(q)$

From assumptions of $u(q)$ and its curvature, we have, we have $L'(q) < 0$ and $L''(q) < 0$ for $q \in [0, q^*]$. This implies that for $q_2 > q_1$, we have $L(q_2) < L(q_1)$.

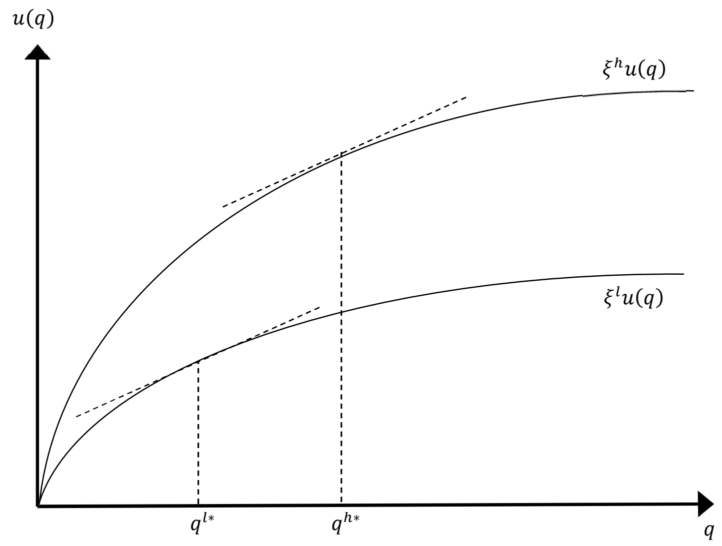


Figure 3.3: $\xi^h u(q)$ vs $\xi^l u(q)$

Let q_c denote the DM quantity consumed with access to credit market and q_n without access to credit market. Let $q_u = \frac{d_u}{p}$ denote the additional DM quantity consumed with unsecured credit and $q_a = \frac{d_a}{p}$ with secured credit. Then $q_c = q_n + q_u + q_a$. Note that $q_c \geq q_n$ so $L(q_c) \leq L(q_n)$.

Next to solve the debt decision, we substitute (3.3a)-(3.3d) into (3.2a)-(3.2d) to get

$$J^h(m+\varepsilon, a) = \max_{d_u^{Hh}, d_a^{Hh}} \left[\xi^h u(q_c^{Hh}) - \omega_c^{Hh} + W\left(m + \varepsilon + \frac{(d_u^{Hh} + d_a^{Hh})}{\phi}, a, d_u^{Hh}, d_a^{Hh}, 0\right) - \mathbb{I}_{d_a^{Hh}>0} \cdot c\right]$$

$$J^l(m+\varepsilon, a) = \max_{d_u^{Hl}, d_a^{Hl}} \left[\xi^l u(q_c^{Hl}) - \omega_c^{Hl} + W(m + \varepsilon + (d_u^{Hl} + d_a^{Hl})/\phi, a, d_u^{Hl}, d_a^{Hl}, 0) - \mathbb{I}_{d_a^{Hl}>0} \cdot c\right]$$

$$J^h(m, a) = \max_{d_u^{Lh}, d_a^{Lh}} \left[\xi^h u(q_c^{Lh}) - \omega_c^{Lh} + W(m + (d_u + d_a)/\phi, a, d_u^{Lh}, d_a^{Lh}, 0) - \mathbb{I}_{d_a^{Lh}>0} \cdot c\right]$$

$$J^l(m, a) = \max_{d_u^{Ll}, d_a^{Ll}} \left[\xi^l u(q_c^{Ll}) - \omega_c^{Ll} + W(m + (d_u^{Ll} + d_a^{Ll})/\phi, a, d_u^{Ll}, d_a^{Ll}, 0) - \mathbb{I}_{d_a^{Ll}>0} \cdot c\right]$$

WLOG, we show the solution for q_c^{Hh}. Solving the rest of the DM quantities are similar. Substituting $q_c^{Hh} = q_n^{Hh} + q_u^{Hh} + q_a^{Hh}$, we get

$$J^h(m+\varepsilon, a) = \max_{d_u^{Hh}, d_a^{Hh}} \left[\xi^h u(q_c^{Hh}) - \omega_c^{Hh} + W(m + \varepsilon + (d_u^{Hh} + d_a^{Hh})/\phi, a, d_u^{Hh}, d_a^{Hh}, 0) - \mathbb{I}_{d_a^{Hh}>0} \cdot c\right]$$

FOC with respect to d_u^{Hh} gives

$$\frac{\partial J^h}{\partial d_u^{Hh}} = \frac{\partial}{\partial d_u^{Hh}} \left[\xi^h u(q_c^{Hh}) - \omega_c^{Hh} + W(m + \varepsilon + (d_u^{Hh} + d_a^{Hh})/\phi, a, d_u^{Hh}, d_a^{Hh}, 0) - \mathbb{I}_{d_a^{Hh}>0} \cdot c\right]$$

$$\frac{\partial J^h}{\partial d_u^{Hh}} = \xi^h \frac{\partial u(q_c^{Hh})}{\partial q_c^{Hh}} \cdot \frac{\partial q_c^{Hh}}{\partial d_u^{Hh}} - \frac{\partial \omega_c^{Hh}}{\partial q_c^{Hh}} \cdot \frac{\partial q_c^{Hh}}{\partial d_u^{Hh}} + \frac{\partial}{\partial d_u^{Hh}} \phi \left[\frac{m + \varepsilon + (d_u^{Hh} + d_a^{Hh})}{\phi}\right] - \frac{\partial}{\partial d_u^{Hh}} [(1 + b_u) d_u^{Hh}]$$

$$\frac{\partial J^h}{\partial d_u^{Hh}} = \xi^h u'(q_c^{Hh}) \cdot \frac{1}{\omega'(q_c^{Hh})} - \omega'(q_c^{Hh}) \cdot \frac{1}{\omega'(q_c^{Hh})} + 1 - (1 + b_u)$$

$$\frac{\partial J^h}{\partial d_u^{Hh}} = L^h(q_c^{Hh}) - b_u$$

Maximizing d_u^{Hh}, we get

$$b_u \geq L^h(q_c^{Hh})$$

with equality if $d_u^{Hh} > 0$. This says that agents will take on additional unsecured debt up till the level $b_u = L^h(q_c^{Hh})$.

This also implicitly implies that agents will only take on additional unsecured debt if $b_u < L^h(q_n^{Hh})$ if unsecured debt is cheaper than secured debt or $b_u < L^h(q_n^{Hh} + q_a^{Hh})$ if secured debt is cheaper than unsecured debt.

For secured debt, we can derive similar condition

$$b_a \geq L^h(q_c^{Hh})$$

with equality if $d_a^{Hh} > 0$ and agents will take on additional secured debt up till the level $b_a = L^h(q_c^{Hh})$.

However, in addition to the conditions for unsecured debt, agents will only take on secured debt if the additional utility from taking on the debt exceeds the fixed cost to access the debt.

If $b_u > b_a$, the fixed cost condition is

$$c \leq \xi^h[u(q_c^{Hh}) - u(q_n^{Hh})] - \omega_c^{Hh} - \omega_n^{Hh} - b_a d_a$$

If $b_u < b_a$, the fixed cost condition is

$$c \leq \xi^h[u(q_c^{Hh}) - u(q_n^{Hh} + q_u^{Hh})] - \omega_c^{Hh} - \omega_{n+u}^{Hh} - b_a d_a$$

Next for optimal portfolio decision for m and a, we work backwards by differentiating 3a-3h with respect to m and a. In general,

$$\frac{\partial V}{\partial m} = L(q)\frac{\partial q}{\partial m} + \frac{\partial W(m, a, d_u, d_a, 0)}{\partial m}$$

$$\frac{\partial V}{\partial a} = L(q)\frac{\partial q}{\partial a} + \frac{\partial W(m, a, d_u, d_a, 0)}{\partial a}$$

FOC for q_n

$$\frac{\partial q_n}{\partial m} = \begin{cases} 0 & if\ q_n \geq q^* \\ \dfrac{\phi}{\omega'(q_n)} & if\ q_n < q^* \end{cases}$$

$$\frac{\partial \omega(q_n)}{\partial m} = \begin{cases} 0 & if\ q_n \geq q^* \\ \phi & if\ q_n < q^* \end{cases}$$

$$\frac{\partial q_n}{\partial a} = \begin{cases} 0 & if\ q_n \geq q^* \\ 0 & if\ q_n < q^* \end{cases}$$

$$\frac{\partial \omega(q_n)}{\partial a} = \begin{cases} 0 & if\ q_n \geq q^* \\ 0 & if\ q_n < q^* \end{cases}$$

where we dropped the superscripts for brevity.

FOC for q_c

$$\frac{\partial q_c}{\partial m} = \begin{cases} 0 & if\ q_c \geq q^* \\ \dfrac{\phi}{\omega'(q_c)} & if\ q_c < q^* \end{cases}$$

$$\frac{\partial \omega(q_c)}{\partial m} = \begin{cases} 0 & if\ q_c \geq q^* \\ \phi & if\ q_c < q^* \end{cases}$$

$$\frac{\partial q_c}{\partial a} = \begin{cases} 0 & if\ q_c \geq q^* \\ \dfrac{\kappa\psi}{\omega'(q_c)} & if\ q_c < q^* \end{cases}$$

$$\frac{\partial \omega(q_c)}{\partial a} = \begin{cases} 0 & if\ q_c \geq q^* \\ \psi + \eta & if\ q_c < q^* \end{cases}$$

where we dropped the superscripts for brevity.

Putting everything together, we get

$$\begin{aligned} \phi \geq \hat{\phi}\beta[1 &+ \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl}) \\ &+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) \\ &+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl}) \\ &+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})] \end{aligned}$$

Let $1 + i = \frac{\phi}{\beta\hat{\phi}}$. Then above becomes

$$
\begin{aligned}
i \geq\; & \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl}) \\
& + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) \\
& + \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl}) \\
& + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})
\end{aligned}
\tag{3.4}
$$

Similarly, for the asset

$$
\begin{aligned}
\psi \geq \left(\hat{\psi} + \hat{\eta}\right)\beta[1 & + \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl}) \\
& + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) \\
& + \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl}) \\
& + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})]
\end{aligned}
\tag{3.5}
$$

Here, if the stock of asset is not limiting, then the asset is priced fundamentally where $\psi = \left(\hat{\psi} + \hat{\eta}\right)\beta$ or $\underline{\psi} = \frac{\beta\eta}{1-\beta}$. For simplicity, we assume that the asset is always in abundance so that it is priced fundamentally.

Refer to the appendix for alternative method of solving the equilibrium by the Lagrangian method.

3.4.1 Equilibrium Regions

WLOG, agents take as given exogenous parameters such as the inflation rate i, and borrowing costs b_u and b_a along with the fixed cost for using secured credit c.

To solve for the general equilibrium we need to work our way upwards, by first assuming that agents regardless of types carry sufficient money such that they do not require credit to obtain q^{l*}.

Next we slowly reduce the money holding until the first agent that needs to use credit is encountered while the rest of the agents still do not need credit. This first agent is the low-income high-consumption need type. We work out the debt conditions for this scenario and compare if the exogenous debt parameters meet such a scenario.

If the exogenous debt parameters do not meet the above scenario, we then continue by reducing the money holdings even further until the second agent who requires credit is encountered, checking the debt conditions for a match. And so on.

Region	q_n^{Hl}	q_n^{Hh}	q_n^{Ll}	q_n^{Lh}
A	q^{l*}	q^{h*}	q^{l*}	q^{h*}
B	q^{l*}	q^{h*}	q^{l*}	ϕm
C1	q^{l*}	$\phi(m+\varepsilon)$	q^{l*}	ϕm
C2	q^{l*}	q^{h*}	ϕm	ϕm
D	q^{l*}	$\phi(m+\varepsilon)$	ϕm	ϕm
E	$\phi(m+\varepsilon)$	$\phi(m+\varepsilon)$	ϕm	ϕm

Region A

Agents carry sufficient money and do not require credit to obtain q^{l*}. Here $q_n^H \geq q^{l*}$ and $q_n^H \geq q^{h*}$, and $q_n^L \geq q^{l*}$ and $q_n^L \geq q^{h*}$ so we have

$$q_c^{Hl} = q_n^{Hl} = q^{l*}$$

$$q_c^{Hh} = q_n^{Hh} = q^{h*}$$

$$q_c^{Ll} = q_n^{Ll} = q^{l*}$$

$$q_c^{Lh} = q_n^{Lh} = q^{h*}$$

So (3.4) gives

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$
$$i = \sigma\alpha\pi\rho L^h(q^{h*}) + \sigma\alpha\pi(1-\rho)L^h(q^{h*}) + \sigma\alpha(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q^{l*}) + \sigma(1-\alpha)\pi\rho L^h(q^{h*})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q^{h*}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q^{l*})$$
$$i = 0 \tag{3.6A}$$

This says that this scenario occurs only if $i = 0$ regardless of what the debt onditions are

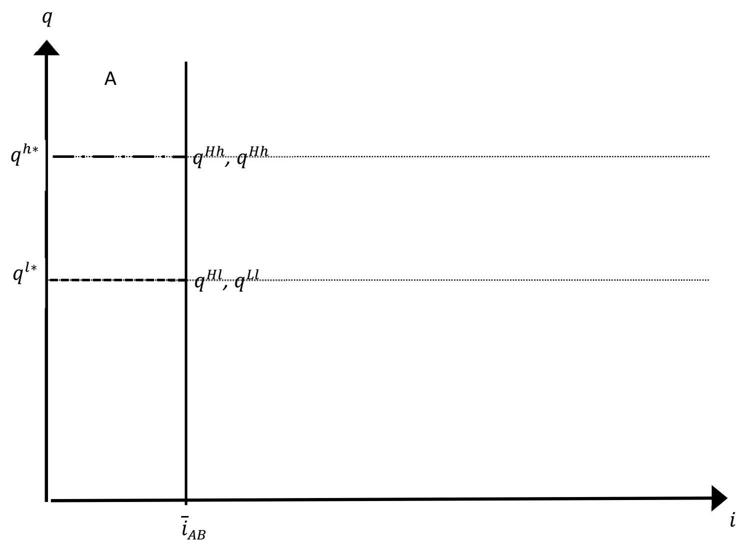


Figure 3.4: Region A

Region B

Since $i > 0$ generally, agents carry less money than what is optimal. As i increases from 0, we have

$$q_c^{Hl} = q_n^{Hl} = q^{l*}$$

$$q_c^{Hh} = q_n^{Hh} = q^{h*}$$

$$q_c^{Ll} = q_n^{Ll} = q^{l*}$$

$$q_n^{Lh} < q^{h*}$$

So (3.4) gives

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$i = \sigma\alpha\pi\rho L^h(q^{h*}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q^{l*}) + \sigma(1-\alpha)\pi\rho L^h(q^{h*})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q^{l*})$$

$$i = \sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})] \quad (3.6B)$$

The above scenario (agents carry less and less money) continues until the second agent requires who requires credit is encountered.

For continuity, the boundary $\bar{i}_{AB}$ is defined when (A)=(B), and the conditions are

$$0 = \sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})]$$

$$0 = \sigma\pi(1-\rho)[\alpha L^h(q^{h*}) + (1-\alpha)L^h(q^{h*})]$$

which says that at $\bar{i}_{AB}$, low-income high consumption agents must carry enough money to get q^{h*}.

Next we check the debt conditions.

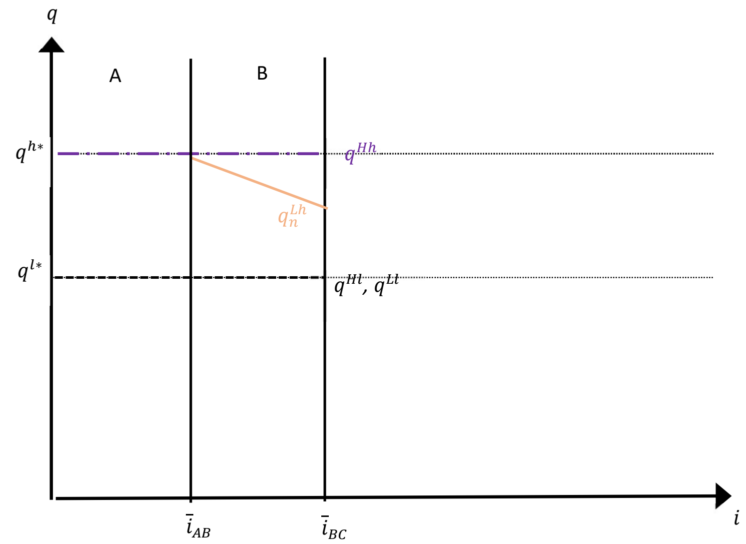


Figure 3.5: Region B

Low-income high consumption need agents will borrow until the limit

$$b_u = L(q_c)$$

$$b_a = L(q_c)$$

which means that if the debt limit and fixed costs are not binding, they will borrow to consume q^{h*} only if $b_u = 0$, $b_a = 0$, or both. Else they will borrow up to a quantity less than q^{h*} indicated by the horizontal line just below q^{h*}.

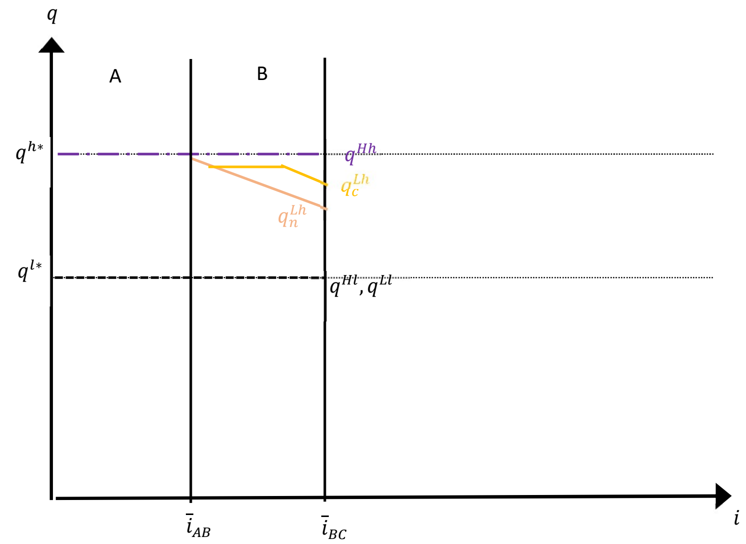


Figure 3.6: Region B with low money holdings

Region C

As i increases further, agents carry less and less money. The next agent to require credit to achieve q^* depends on the size of the endowment ε and the levels of q^{h*} and q^{l*}.

Assume the worst-case scenario where low-income agents carry just enough money to consume q^{l*}. Then high-income agents can consume $q^{l*} + \varepsilon$. If $q^{h*} > q^{l*} + \varepsilon$, then high-income agents do not have enough to consume q^{h*}. If $q^{h*} \leq q^{l*} + \varepsilon$, then high-income agents have enough to consume q^{h*}, and as we move a little more where low-income agents do not carry enough money to consume q^{l*}, they are the next agents who need credit while high-income agents still can consume q^{h*} without credit.

Proposition 3.1: If $q^{h*} - q^{l*} > \varepsilon$, Region C1 holds. If $q^{h*} - q^{l*} \leq \varepsilon$, Region C2 holds.

Proof. See derivation of q^{h*} for C1 and q^{l*} for C2 below. *End of proof.*

If $q^{h*} - q^{l*} > \varepsilon$, the second agent requiring credit are high-income high consumption agents.

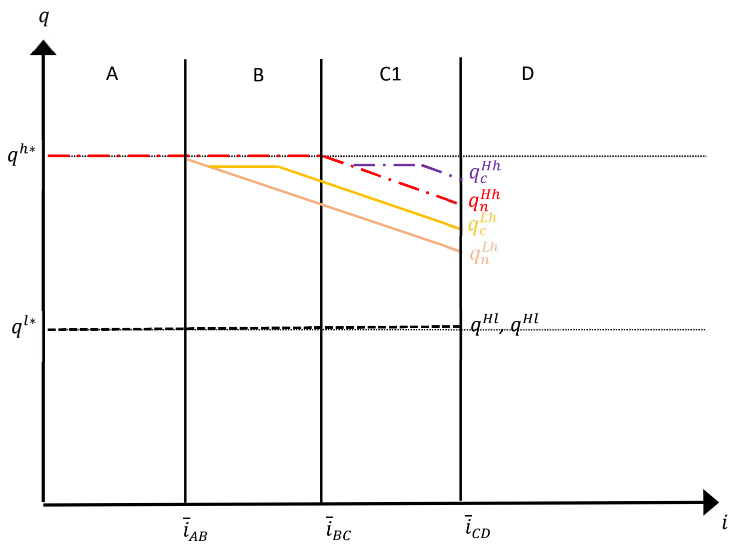


Figure 3.7: Region C1

$$q_c^{Hl} = q_n^{Hl} = q^{l*}$$

$$q_n^{Hh} < q^{h*}$$

$$q_c^{Ll} = q_n^{Ll} = q^{l*}$$

$$q_n^{Lh} < q^{h*}$$

So (3.4) gives

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q^{l*}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q^{l*})$$

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})]$$

(3.6C1)

For continuity, the boundary $\bar{i}_{BC}$ is defined when (B)=(C1), and the conditions are

$$\sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})]$$
$$= \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})]$$
$$0 = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$

which says that at $\bar{i}_{BC}$, high-income high consumption agents must carry enough money to get q^{h*}.

If $q^{h*} - q^{l*} \leq \varepsilon$, the second agent requiring credit are low-income low consumption agents.

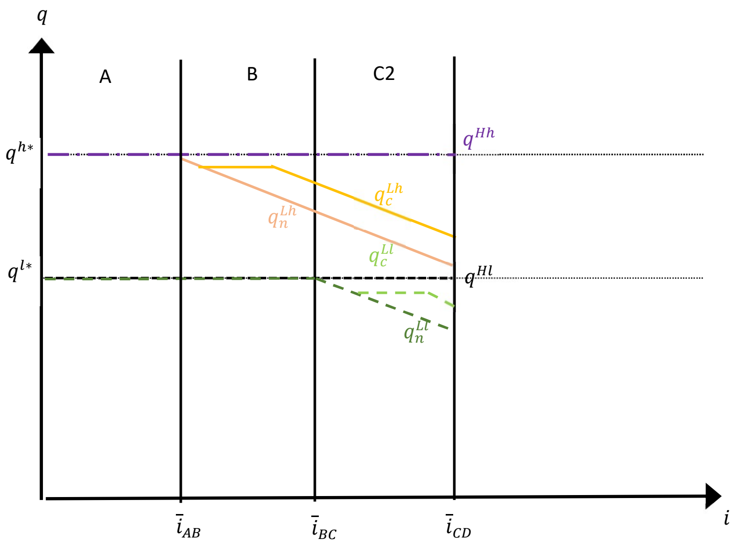


Figure 3.8: Region C2

$$q_c^{Hl} = q_n^{Hl} = q^{l*}$$

$$q_c^{Hh} = q_n^{Hh} = q^{h*}$$

$$q_n^{Ll} < q^{l*}$$

$$q_n^{Lh} < q^{h*}$$

So (3.4) gives

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl}) + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) + \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$i = \sigma\alpha\pi\rho L^h(q^{h*}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q^{l*}) + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q^{h*}) + \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q^{l*}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$i = \sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})] + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

(3.6C2)

The above scenario (agents carry less and less money) continues until the second agent requires who requires credit is encountered.

For continuity, the boundary $\bar{i}_{BC}$ is defined when (3.6B) equals (3.6C2), and the conditions are

$$\sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})]$$
$$= \sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})]$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$
$$0 = \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

which says that at $\bar{i}_{AB}$, low-income low consumption agents must carry enough money to get q^{h*}.

Region D

As i increases further, agents carry even less money. In this region, three agents require credit to achieve q^* which are high-income high consumption agents, low-income high consumption agents and low-income low consumption agents.

$$q_c^{Hl} = q_n^{Hl} = q^{l*}$$
$$q_n^{Hh} < q^{h*}$$
$$q_n^{Ll} < q^{l*}$$
$$q_n^{Lh} < q^{h*}$$

So (3.4) gives

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q^{l*})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll})$$
$$+ \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) + \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

(3.6D)

If we come here from Region C1, then for continuity (3.6C1) equals (3.6D)

$$\sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) + \sigma\pi(1-\rho)[\alpha L^h(q_c^{Lh}) + (1-\alpha)L^h(q_n^{Lh})]$$
$$= \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$
$$0 = \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

which says that at $\bar{i}_{CD}$, low-income low consumption agents carry enough money to get q^{h*}.

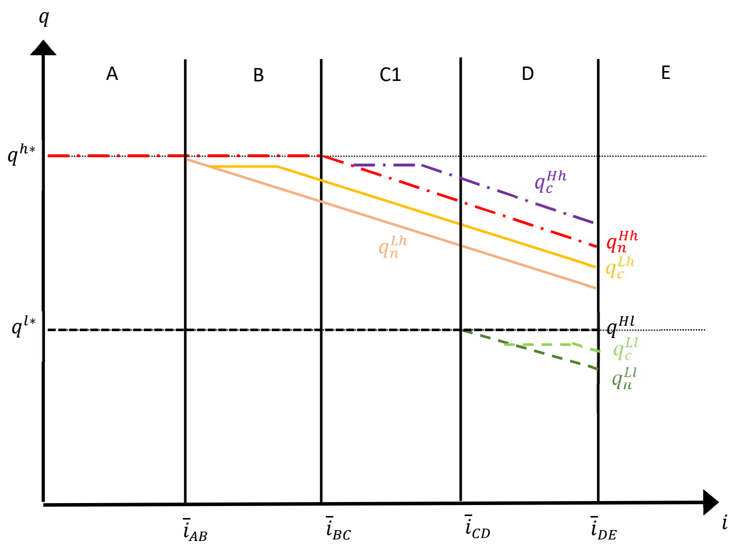


Figure 3.9: Region D from C1

If we come here from Region C2, then for continuity (3.6C2) equals (3.6D)

$$\sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$= \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh})$$

$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$

$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$

$$0 = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh})$$

which says that at $\bar{i}_{CD}$, high-income high consumption agents must carry enough money to get q^{h*}.

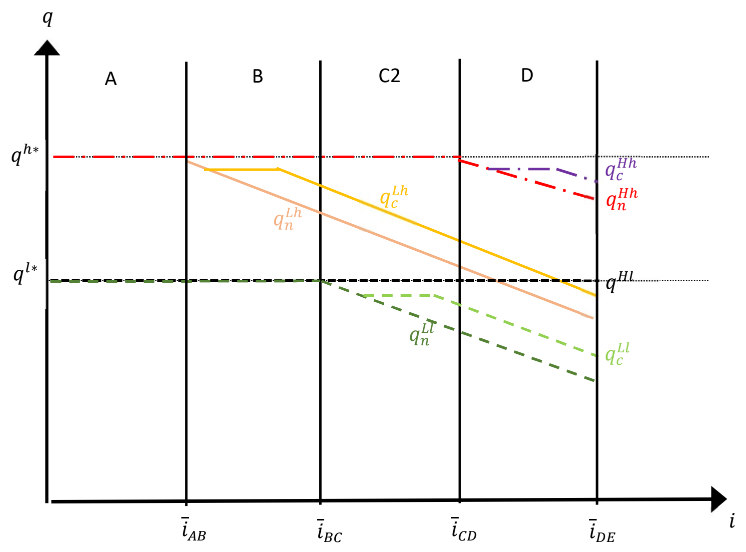


Figure 3.10: Region D from C2

Region E

As i increases further, agents carry even less money. In this region, all agents do not carry enough money to consume q^* and would rely on credit.

$$q_n^{Hl} < q^{l*}$$

$$q_n^{Hh} < q^{h*}$$

$$q_n^{Ll} < q^{l*}$$

$$q_n^{Lh} < q^{h*}$$

So (3.4) gives

$$i = \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl}) + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) + \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl}) + \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll}) \tag{3.6E}$$

For continuity, the boundary $\bar{i}_{DE}$ is defined when (D)=(E), and the conditions are

$$\sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll})$$
$$+ \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh}) + \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$
$$= \sigma\alpha\pi\rho L^h(q_c^{Hh}) + \sigma\alpha\pi(1-\rho)L^h(q_c^{Lh}) + \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl})$$
$$+ \sigma\alpha(1-\pi)(1-\rho)L^l(q_c^{Ll}) + \sigma(1-\alpha)\pi\rho L^h(q_n^{Hh})$$
$$+ \sigma(1-\alpha)\pi(1-\rho)L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl})$$
$$+ \sigma(1-\alpha)(1-\pi)(1-\rho)L^l(q_n^{Ll})$$
$$0 = \sigma\alpha(1-\pi)\rho L^l(q_c^{Hl}) + \sigma(1-\alpha)(1-\pi)\rho L^l(q_n^{Hl})$$

which says that at $\bar{i}_{DE}$, high-income low consumption agents must carry enough money to get q^{h*}.

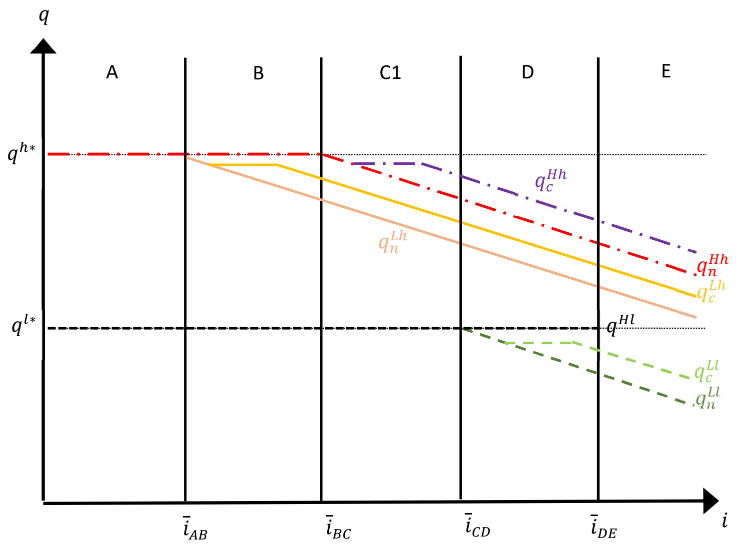


Figure 3.11: Region E from C1

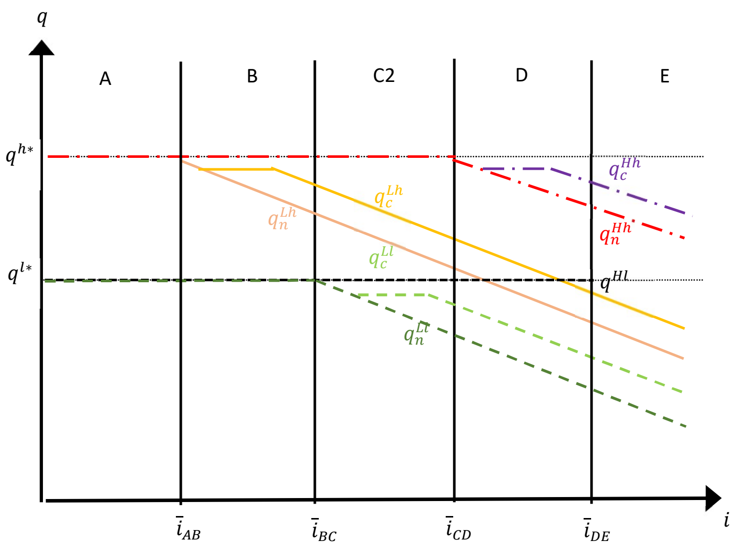


Figure 3.12: Region E from C2

3.4.2 Endogenous Choice of Debt

Zooming in to when the agent starts to require credit, agents' choice of debt is characterised by the following:

- [Region I] At the beginning, because the marginal benefit may not exceed the cost of debt, agents may not take on debt and so $q_c = q_n$.
- [Region II] When agents start to take on debt, if unconstrained by debt limits, they will take on debt until $b = L(q_c)$.
- [Region III] As agents take on even more debt, they become constrained by debt limits and borrow to consume q_c, but q_c is now $b < L(q_c)$.

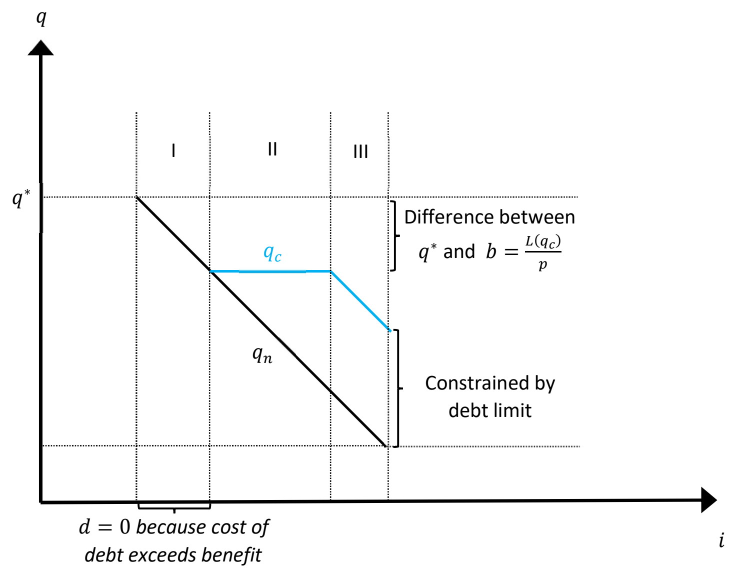


Figure 3.13: Regions I to III

<u>*Region I*</u>

In this region, agents do not find it optimal to increase consumption with debt. Hence both unsecured and credit must be unattractive:

- For unsecured debt, the cost of debt exceeds the marginal benefit of any additional consumption with credit

$$b_u > L(q_n)$$

- For secured credit, either the cost of debt exceeds the marginal benefit of any additional consumption with credit in which the fixed cost does not matter

$$b_a > L(q_n)$$

$$c \text{ does not matter}$$

  or the cost of debt does not exceed the marginal benefit of any additional consumption with credit but the fixed cost is expensive

$$b_a \leq L(q_n)$$

$$c > \xi[u(q_{\tilde{c}}) - u(q_{\tilde{n}})] - [\omega_{\tilde{c}} - \omega_{\tilde{n}}]$$

Region II

In this region, agents find it optimal to increase consumption with debt. Hence either unsecured credit, secured credit or both must be attractive:

Case	$b_u \leq L(q_n)$	$b_a \leq L(q_n)$ $c \leq \xi[u(q_{\tilde{c}}) - u(q_{\tilde{n}})] - [\omega_{\tilde{c}} - \omega_{\tilde{n}}]$
i	Yes	No
ii	No	Yes
iii	Yes	Yes

- [Case i]

  Agents will use unsecured credit only to increase consumption up till the point

$$b_u = L(q_c)$$

  where $q_c = q_n + q_u$. The agent is unconstrained by debt limit

$$d_u < D_u$$

  Note that since by assumption $b_u > b_a$ which implies $b_a \leq b_u \leq \frac{L(q_n)}{p}$, in this case, we must have $c > \xi[u(q_{\tilde{c}}) - u(q_{\tilde{n}})] - [\omega_{\tilde{c}} - \omega_{\tilde{n}}]$ in order for agents to not want to use secured credit.

- [Case ii]

  Agents will use secured credit only to increase consumption up till the point

$$b_a = L(q_c)$$

  where $q_c = q_n + q_a$. The fixed cost to access secured credit is less than the benefit of increased consumption

$$c \leq \xi[u(q_c) - u(q_n)] - [\omega_c - \omega_n]$$

  and the agent is unconstrained by debt limit

$$d_a < \kappa\psi a$$

- [Case iii]

  Agents will use mixture of unsecured and secured credit to increase consumption. Here, one debt constrain may bind but not both.

  Since by assumption $b_u > b_a$, we start with agents taking on secured debt first.

  Case iii-a

  Agents use only secured debt.

$$b_a = L(q_a + q_n)$$

$$c \leq \xi[u(q_a + q_n) - u(q_n)] - [\omega_{a+n} - \omega_n]$$

$$d_a < \kappa\psi a$$

$$b_u > L(q_a + q_n)$$

  Case iii-b

  Agents use secured debt first, but reaches debt limit and supplement with unsecured debt.

$$b_a = L(q_a + q_n)$$

$$c \leq \xi[u(q_a + q_n) - u(q_n)] - [\omega_{a+n} - \omega_n]$$

$$d_a = \kappa\psi a$$

$$b_u = L(q_u + q_a + q_n)$$

$$d_u < D_u$$

  Case iii-c

  Agents use only unsecured debt. Conditions are similar to Case i.

Region III

In this region, agents find it optimal to increase consumption with debt but are constrained by debt limits.

[Case i]

Agents use only secured debt.

$$b_a = L(q_a + q_n)$$

$$c \leq \xi[u(q_a + q_n) - u(q_n)] - [\omega_{a+n} - \omega_n]$$

$$d_a = \kappa\psi a$$

$$b_u > L(q_a + q_n)$$

[Case ii]

Agents use secured debt first, but reaches debt limit and supplement with unsecured debt.

$$b_a = L(q_a + q_n)$$

$$c \leq \xi[u(q_a + q_n) - u(q_n)] - [\omega_{a+n} - \omega_n]$$

$$d_a = \kappa\psi a$$

$$b_u = L(q_u + q_a + q_n)$$

$$d_u = D_u$$

[Case iii]

Agents use only unsecured debt.

$$b_a \leq L(q_n)$$

$$c > \xi[u(q_a + q_n) - u(q_n)] - [\omega_{a+n} - \omega_n]$$

$$b_u = L(q_u + q_n)$$

$$d_u = D_u$$

Note that because of the curvature of $u(q)$, if agents do not find it worthwhile to take on secured credit first because of the fixed cost, they will find it worthwhile to take on unsecured credit first than take on secured credit because the marginal benefit at a lower level of q is greater than the marginal benefit for same debt at higher q.

Let $G(c)$ be the q_c required for fixed cost c, where

$$G^{-1}(c) = \xi[u(q_c) - u(q_n)] - [\omega_c - \omega_n]$$

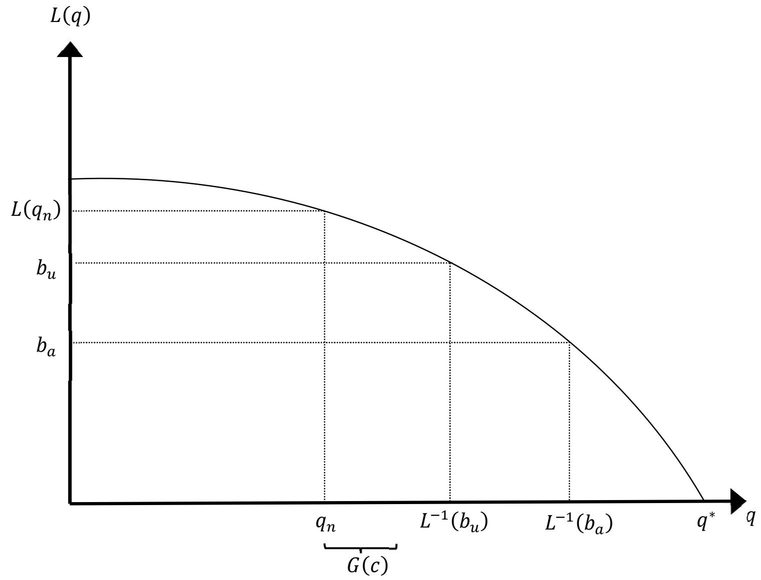


Figure 3.14: Curvature of $L(q)$

That is, if agents find it worthwhile to use secured credit, then there will be a "jump" $G(c)$ in the amount traded.

3.4.3 Comparative Statics

If only money is used

ϑ	i	b_u	b_a	α	ρ	π
$\frac{dm}{d\vartheta}$	$-$	no effect	no effect	no effect	$-$	$+$
$\frac{dd_u}{d\vartheta}$	no effect	no effect	no effect	no effect	no effect	no effect
$\frac{dd_a}{d\vartheta}$	no effect	no effect	no effect	no effect	no effect	no effect

Proof: Differentiating with respect to i, we get

$$1 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial i} + \sigma(1-\rho)\pi L^{h\prime}(q^{Lh})\frac{\partial q^{Lh}}{\partial i} + \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial i} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial i}$$

WLOG, consider Region 1a where $q^H = \phi m + \varepsilon$ and $q^L = \phi m$, which simplifies to $q^H = q^L + \varepsilon$. The above various q can then be replaced in terms of q^L. Since $L'(q) < 0$, this gives $\frac{\partial q^L}{\partial i} < 0$. From $\frac{dm}{di} = \frac{\partial q}{\partial i} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{di} < 0$. Regions 1b to 1e is similar but easier since one or more of the q is q^* and differentiating with respect to a constant gives zero and the respective $\frac{\partial q}{\partial i}$ can be removed from the equation.

Differentiating individual q with respect to b_u and b_a gives $\frac{\partial q}{\partial b} = 0$.

Differentiating with α, we get

$$0 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial \alpha} + \sigma(1-\rho)\pi L^{h\prime}(q^{Lh})\frac{\partial q^{Lh}}{\partial \alpha} + \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial \alpha} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \alpha}$$

which gives us $\frac{\partial q}{\partial \alpha} = 0$.

Differentiating with respect to ρ, we get

$$0 = \sigma\pi L^{h}(q^{Hh}) + \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial \rho} - \sigma\pi L^{h}(q^{Lh}) + \sigma(1-\rho)\pi L^{h\prime}(q^{Lh})\frac{\partial q^{Lh}}{\partial \rho} + \sigma(1-\pi)L^{l}(q^{Hl}) + \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial \rho} - \sigma(1-\pi)L^{l}(q^{Ll}) + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \rho}$$

Again WLOG, consider Region 1a where $q^H = q^L + \varepsilon$. Re-expressing the above in terms of q^L where $\frac{\partial q^H}{\partial \rho} = \frac{\partial q^L}{\partial \rho}$, we get

$$0 = \sigma\pi L^h(q^L + \varepsilon) + \sigma\rho\pi L^{h\prime}(q^L + \varepsilon)\frac{\partial q^L}{\partial \rho} - \sigma\pi L^h(q^L) + \sigma(1-\rho)\pi L^{h\prime}(q^L)\frac{\partial q^L}{\partial \rho}$$

$$+ \sigma(1-\pi)L^l(q^L + \varepsilon) + \sigma\rho(1-\pi)L^{l\prime}(q^L + \varepsilon)\frac{\partial q^L}{\partial \rho}$$

$$- \sigma(1-\pi)L^l(q^L) + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^L)\frac{\partial q^L}{\partial \rho}$$

Since $L(q) \geq 0$ and $L'(q) < 0$, along with $L(q^L + \varepsilon) < L(q^L)$, re-arranging the above gives

$$-\left[\sigma\rho\pi L^{h'}(q^L + \varepsilon) + \sigma(1-\rho)\pi L^{h'}(q^L) + \sigma\rho(1-\pi)L^{l'}(q^L + \varepsilon)\right.$$

$$\left. + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^L)\right]\frac{\partial q^L}{\partial \rho}$$

$$= \sigma\pi L^h(q^L + \varepsilon) - \sigma\pi L^h(q^L) + \sigma(1-\pi)L^l(q^L + \varepsilon)$$

$$- \sigma(1-\pi)L^l(q^L)$$

which gives us $\frac{\partial q^L}{\partial \rho} < 0$. From $\frac{dm}{d\rho} = \frac{\partial q}{\partial \rho} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{d\rho} < 0$.

Differentiating with respect to π, we get

$$1 = \sigma\rho L^h(q^{Hh}) + \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial \pi} + \sigma(1-\rho)L^h(q^{Lh})$$

$$+ \sigma(1-\rho)\pi L^{h'}(q^{Lh})\frac{\partial q^{Lh}}{\partial \pi} - \sigma\rho L^l(q^{Hl})$$

$$+ \sigma\rho(1-\pi)L^{l'}(q^{Hl})\frac{\partial q^{Hl}}{\partial \pi} - \sigma(1-\rho)L^l(q^{Ll})$$

$$+ \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \pi}$$

Again WLOG, consider Region 1a where $q^H = q^L + \varepsilon$ with $L^h(q) > L^l(q)$. Re-expressing the above in terms of q^L where $\frac{\partial q^H}{\partial \pi} = \frac{\partial q^L}{\partial \pi}$, we get

$$-\left[\sigma\rho\pi L^{h'}(q^L+\varepsilon)+\sigma(1-\rho)\pi L^{h'}(q^L)+\sigma\rho(1-\pi)L^{l'}(q^L+\varepsilon)\right.$$

$$\left.+\sigma(1-\rho)(1-\pi)L^{l\prime}(q^L)\right]\frac{\partial q^L}{\partial \pi}$$

$$=\sigma\rho L^h(q^L+\varepsilon)-\sigma\pi L^l(q^L+\varepsilon)+\sigma(1-\rho)L^h(q^L)$$

$$-\sigma(1-\rho)L^l(q^L)$$

which gives us $\frac{\partial q^L}{\partial \pi}>0$. From $\frac{dm}{d\pi}=\frac{\partial q}{\partial \pi}/\frac{\partial q}{\partial m}$, we get $\frac{dm}{d\pi}>0$.

End of proof.

If both money and unsecured debt are used

ϑ	i	b_u	b_a	α	ρ	π
$\frac{dm}{d\vartheta}$	−	+	no effect	−	−	+
$\frac{dd_u}{d\vartheta}$	+	−	no effect	+	−	+
$\frac{dd_a}{d\vartheta}$	no effect	no effect	no effect	no effect	no effect	no effect

Proof: Differentiating with respect to i, we get

$$1=\sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial i}+\sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q_c^{Lh}}{\partial i}$$

$$+\sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q_n^{Lh}}{\partial i}+\sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial i}$$

$$+\sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial i}$$

WLOG, consider Region 2-1ia where $q^{Ll}=q_n^{Lh}=\phi m$ and express the various q in terms of q^{Ll} or q_n^{Lh}. Since $L'(q)<0$, this gives $\frac{\partial q^{Ll}}{\partial i}<0$ or $q_n^{Lh}<0$. From $\frac{dm}{di}=\frac{\partial q}{\partial i}/\frac{\partial q}{\partial m}$, we get $\frac{dm}{di}<0$. Regions 1b to 1e is similar but easier since one or more of the q is q^* and differentiating with respect to a constant gives zero and the respective $\frac{\partial q}{\partial i}$ can be removed from the equation.

Since $b_u = L^h(q_c^{Hh})$ and differentiating with respect to b_u gives $1 = \frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} \cdot \frac{\partial q_c^{Lh}}{\partial b_u}$. Because $\frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} < 0$ we have $\frac{\partial q_c^{Lh}}{\partial b_u} < 0$. From $\frac{dm}{db_u} = \frac{\partial q}{\partial b_u} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{db_u} < 0$.

Differentiating individual q with respect to b_a gives $\frac{\partial q}{\partial b_a} = 0$.

Differentiating with respect to α, we get

$$0 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial \alpha} + \sigma(1-\rho)\pi L^h(q_c^{Lh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q_c^{Lh}}{\partial \alpha}$$

$$- \sigma(1-\rho)\pi L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q_n^{Lh}}{\partial \alpha}$$

$$+ \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial \alpha} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \alpha}$$

Re-expressing the above in terms of q^{Ll} or q_n^{Lh}, we get

$$0 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Ll}}{\partial \alpha} + \sigma(1-\rho)\pi L^h(q_c^{Lh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q^{Ll}}{\partial \alpha}$$

$$- \sigma(1-\rho)\pi L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q^{Ll}}{\partial \alpha}$$

$$+ \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Ll}}{\partial \alpha} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \alpha}$$

Since $L(q) \geq 0$ and $L'(q) < 0$, along with $L(q_c^{Lh}) < L(q_n^{Lh})$, re-arranging the above gives

$$-[\sigma\rho\pi L^{h\prime}(q^{Hh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})$$

$$+ \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh}) + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})]\frac{\partial q^L}{\partial \rho}$$

$$= \sigma(1-\rho)\pi L^h(q_c^{Lh}) - \sigma(1-\rho)\pi L^h(q_n^{Lh})$$

which gives us $\frac{\partial q^L}{\partial \alpha} < 0$. From $\frac{dm}{d\alpha} = \frac{\partial q}{\partial \alpha} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{d\alpha} < 0$.

End of proof.

If both money and secured debt are used

ϑ	i	b_u	b_a	α	ρ	π
$\frac{dm}{d\vartheta}$	−	no effect	+	−	−	+
$\frac{dd_u}{d\vartheta}$	no effect	no effect	no effect	no effect	no effect	no effect
$\frac{dd_a}{d\vartheta}$	+	no effect	−	+	−	+

Proof: Differentiating with respect to i, we get

$$1 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial i} + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q_c^{Lh}}{\partial i}$$

$$+ \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q_n^{Lh}}{\partial i} + \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial i}$$

$$+ \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial i}$$

WLOG, consider Region 2-1ia where $q^{Ll} = q_n^{Lh} = \phi m$ and express the various q in terms of q^{Ll} or q_n^{Lh}. Since $L'(q) < 0$, this gives $\frac{\partial q^{Ll}}{\partial i} < 0$ or $q_n^{Lh} < 0$. From $\frac{dm}{di} = \frac{\partial q}{\partial i} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{di} < 0$. Regions 1b to 1e is similar but easier since one or more of the q is q^* and differentiating with respect to a constant gives zero and the respective $\frac{\partial q}{\partial i}$ can be removed from the equation.

Differentiating individual q with respect to b_u gives $\frac{\partial q}{\partial b_u} = 0$.

Since $b_a = L^h(q_c^{Hh})$ and differentiating with respect to b_a gives $1 = \frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} \cdot \frac{\partial q_c^{Lh}}{\partial b_a}$. Because $\frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} < 0$ we have $\frac{\partial q_c^{Lh}}{\partial b_a} < 0$. From $\frac{dm}{db_a} = \frac{\partial q}{\partial b_a} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{db_a} < 0$.

Differentiating with respect to α, we get

$$0 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial \alpha} + \sigma(1-\rho)\pi L^h(q_c^{Lh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q_c^{Lh}}{\partial \alpha}$$

$$- \sigma(1-\rho)\pi L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q_n^{Lh}}{\partial \alpha}$$

$$+ \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial \alpha} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \alpha}$$

Re-expressing the above in terms of q^{Ll} or q_n^{Lh}, we get

$$0 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Ll}}{\partial \alpha} + \sigma(1-\rho)\pi L^h(q_c^{Lh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q^{Ll}}{\partial \alpha}$$

$$- \sigma(1-\rho)\pi L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q^{Ll}}{\partial \alpha}$$

$$+ \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Ll}}{\partial \alpha} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \alpha}$$

Since $L(q) \geq 0$ and $L'(q) < 0$, along with $L(q_c^{Lh}) < L(q_n^{Lh})$, re-arranging the above gives

$$-[\sigma\rho\pi L^{h\prime}(q^{Hh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})$$

$$+ \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh}) + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})]\frac{\partial q^L}{\partial \rho}$$

$$= \sigma(1-\rho)\pi L^h(q_c^{Lh}) - \sigma(1-\rho)\pi L^h(q_n^{Lh})$$

which gives us $\frac{\partial q^L}{\partial \alpha} < 0$. From $\frac{dm}{d\alpha} = \frac{\partial q}{\partial \alpha} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{d\alpha} < 0$.

End of proof.

If money, unsecured and secured debt are used

ϑ	i	b_u	b_a	α	ρ	π
$\frac{dm}{d\vartheta}$	$-$	$+$	$+$	$-$	$-$	$+$
$\frac{dd_u}{d\vartheta}$	$+$	$-$	no effect	$+$	$-$	$+$
$\frac{dd_a}{d\vartheta}$	$+$	no effect	$-$	$+$	$-$	$+$

Proof: Differentiating with respect to i, we get

$$1 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial i} + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q_c^{Lh}}{\partial i}$$

$$+ \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q_n^{Lh}}{\partial i} + \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial i}$$

$$+ \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial i}$$

WLOG, consider Region 2-1ia where $q^{Ll} = q_n^{Lh} = \phi m$ and express the various q in terms of q^{Ll} or q_n^{Lh}. Since $L'(q) < 0$, this gives $\frac{\partial q^{Ll}}{\partial i} < 0$ or $q_n^{Lh} < 0$. From $\frac{dm}{di} = \frac{\partial q}{\partial i} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{di} < 0$. Regions 1b to 1e is similar but easier since one or more of the q is q^* and differentiating with respect to a constant gives zero and the respective $\frac{\partial q}{\partial i}$ can be removed from the equation.

Since $b_u = L^h(q_c^{Hh})$ and differentiating with respect to b_u gives $1 = \frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} \cdot \frac{\partial q_c^{Lh}}{\partial b_u}$. Because $\frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} < 0$ we have $\frac{\partial q_c^{Lh}}{\partial b_u} < 0$. From $\frac{dm}{db_u} = \frac{\partial q}{\partial b_u} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{db_u} < 0$.

Since $b_a = L^h(q_c^{Hh})$ and differentiating with respect to b_a gives $1 = \frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} \cdot \frac{\partial q_c^{Lh}}{\partial b_a}$. Because $\frac{\partial L^h(q_c^{Hh})}{\partial q_c^{Lh}} < 0$ we have $\frac{\partial q_c^{Lh}}{\partial b_a} < 0$. From $\frac{dm}{db_a} = \frac{\partial q}{\partial b_a} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{db_a} < 0$.

Differentiating with respect to α, we get

$$0 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Hh}}{\partial \alpha} + \sigma(1-\rho)\pi L^h(q_c^{Lh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q_c^{Lh}}{\partial \alpha}$$

$$- \sigma(1-\rho)\pi L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q_n^{Lh}}{\partial \alpha}$$

$$+ \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Hl}}{\partial \alpha} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \alpha}$$

Re-expressing the above in terms of q^{Ll} or q_n^{Lh}, we get

$$0 = \sigma\rho\pi L^{h\prime}(q^{Hh})\frac{\partial q^{Ll}}{\partial \alpha} + \sigma(1-\rho)\pi L^h(q_c^{Lh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh})\frac{\partial q^{Ll}}{\partial \alpha}$$

$$- \sigma(1-\rho)\pi L^h(q_n^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})\frac{\partial q^{Ll}}{\partial \alpha}$$

$$+ \sigma\rho(1-\pi)L^{l\prime}(q^{Hl})\frac{\partial q^{Ll}}{\partial \alpha} + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})\frac{\partial q^{Ll}}{\partial \alpha}$$

Since $L(q) \geq 0$ and $L'(q) < 0$, along with $L(q_c^{Lh}) < L(q_n^{Lh})$, re-arranging the above gives

$$-[\sigma\rho\pi L^{h\prime}(q^{Hh}) + \sigma\alpha(1-\rho)\pi L^{h\prime}(q_c^{Lh}) + \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh})$$

$$+ \sigma(1-\alpha)(1-\rho)\pi L^{h\prime}(q_n^{Lh}) + \sigma(1-\rho)(1-\pi)L^{l\prime}(q^{Ll})]\frac{\partial q^L}{\partial \rho}$$

$$= \sigma(1-\rho)\pi L^h(q_c^{Lh}) - \sigma(1-\rho)\pi L^h(q_n^{Lh})$$

which gives us $\frac{\partial q^L}{\partial \alpha} < 0$. From $\frac{dm}{d\alpha} = \frac{\partial q}{\partial \alpha} / \frac{\partial q}{\partial m}$, we get $\frac{dm}{d\alpha} < 0$.

End of proof.

3.4.4 Endogenous Debt Rates

Assuming a reserved ratio is imposed on banks. Based on the reserve ratio R, banks can only loan out $1-R$ of their deposits. The total deposits from sellers is $(1-\sigma)l$ while the total loan to buyers is the sum of proportion of each type of buyers or $\sigma\alpha\pi\rho[d_u^{Hh}+d_a^{Hh}]+\sigma\alpha\pi(1-\rho)[d_u^{Lh}+d_a^{Lh}]+\sigma\alpha(1-\pi)\rho[d_u^{Hl}+d_a^{Hl}]+$ $\sigma\alpha(1-\pi)(1-\rho)[d_u^{Ll}+d_a^{Ll}]$. Hence, we must have

$$\sigma\alpha\pi\rho[d_u^{Hh}+d_a^{Hh}]+\sigma\alpha\pi(1-\rho)[d_u^{Lh}+d_a^{Lh}]+\sigma\alpha(1-\pi)\rho[d_u^{Hl}+d_a^{Hl}] + \sigma\alpha(1-\pi)(1-\rho)[d_u^{Ll}+d_a^{Ll}] \leq (1-R)(1-\sigma)l$$

which we assume is non-binding. Note here it is the total sum of debt issues, not the expected sum of debt the banks expect to get back.

Banks optimize profit on debt issued:

$$\begin{aligned}\max_{d_u,d_a}\Big\{&(1-n)\sigma\alpha\pi\rho\left[(1+b_u)d_u^{Hh}+(1+b_a)d_a^{Hh}+c\cdot\mathbb{I}_{d_a^{Hh}>0}\right]\\ &+(1-n)\sigma\alpha\pi(1-\rho)\left[(1+b_u)d_u^{Lh}+(1+b_a)d_a^{Lh}+c\cdot\mathbb{I}_{d_a^{Lh}>0}\right]\\ &+(1-n)\sigma\alpha(1-\pi)\rho\left[(1+b_u)d_u^{Hl}+(1+b_a)d_a^{Hl}+c\cdot\mathbb{I}_{d_a^{Hl}>0}\right]\\ &+(1-n)\sigma\alpha(1-\pi)(1-\rho)\left[(1+b_u)d_u^{Ll}+(1+b_a)d_a^{Ll}+c\cdot\mathbb{I}_{d_a^{Ll}>0}\right]+n\sigma\alpha\pi\rho\kappa\psi a^{Hh}+n\sigma\alpha\pi(1-\rho)\kappa\psi a^{Lh}\\ &+n\sigma\alpha(1-\pi)\rho\kappa\psi a^{Hl}+n\sigma\alpha(1-\pi)(1-\rho)\kappa\psi a^{Ll}\\ &-(1-\sigma)(1+b_l)l\Big\}\end{aligned}$$

where banks are only able to recover $1-n$ fraction of the loans. For the fraction n loans that are forfeited, only $\kappa\psi a$ are recovered from the secured loans. We assume perfect competition, so banks have zero profits and all have the same rates.

Hence

$$0 = (1-n)\sigma\alpha\pi\rho\left[(1+b_u)d_u^{Hh} + (1+b_a)d_a^{Hh} + c\cdot\mathbb{I}_{d_a^{Hh}>0}\right]$$

$$+ (1-n)\sigma\alpha\pi(1-\rho)\left[(1+b_u)d_u^{Lh} + (1+b_a)d_a^{Lh} + c\cdot\mathbb{I}_{d_a^{Lh}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)\rho\left[(1+b_u)d_u^{Hl} + (1+b_a)d_a^{Hl} + c\cdot\mathbb{I}_{d_a^{Hl}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)(1-\rho)\left[(1+b_u)d_u^{Ll} + (1+b_a)d_a^{Ll} + c\right.$$

$$\left.\cdot\,\mathbb{I}_{d_a^{Ll}>0}\right] + n\sigma\alpha\pi\rho\kappa\psi a^{Hh} + n\sigma\alpha\pi(1-\rho)\kappa\psi a^{Lh}$$

$$+ n\sigma\alpha(1-\pi)\rho\kappa\psi a^{Hl} + n\sigma\alpha(1-\pi)(1-\rho)\kappa\psi a^{Ll}$$

$$- (1-\sigma)(1+b_l)l$$

Re-arranging, we get

$$(1-\sigma)(1+b_l)l$$

$$= (1-n)\sigma\alpha\pi\rho\left[(1+b_u)d_u^{Hh} + (1+b_a)d_a^{Hh} + c\cdot\mathbb{I}_{d_a^{Hh}>0}\right]$$

$$+ (1-n)\sigma\alpha\pi(1-\rho)\left[(1+b_u)d_u^{Lh} + (1+b_a)d_a^{Lh} + c\cdot\mathbb{I}_{d_a^{Lh}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)\rho\left[(1+b_u)d_u^{Hl} + (1+b_a)d_a^{Hl} + c\cdot\mathbb{I}_{d_a^{Hl}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)(1-\rho)\left[(1+b_u)d_u^{Ll} + (1+b_a)d_a^{Ll} + c\right.$$

$$\left.\cdot\,\mathbb{I}_{d_a^{Ll}>0}\right] + n\sigma\alpha\pi\rho\kappa\psi a^{Hh} + n\sigma\alpha\pi(1-\rho)\kappa\psi a^{Lh}$$

$$+ n\sigma\alpha(1-\pi)\rho\kappa\psi a^{Hl} + n\sigma\alpha(1-\pi)(1-\rho)\kappa\psi a^{Ll}$$

which says that the income from debt must equal the cost of deposits.

Assuming banks loan out the maximum, substituting the reserve ratio requirement at equality into the above equation, we get

$$\frac{\sigma\alpha\pi\rho[d_u^{Hh}+d_a^{Hh}]+\sigma\alpha\pi(1-\rho)[d_u^{Lh}+d_a^{Lh}]+\sigma\alpha(1-\pi)\rho[d_u^{Hl}+d_a^{Hl}]+\sigma\alpha(1-\pi)(1-\rho)[d_u^{Ll}+d_a^{Ll}]}{1-R}(1+b_l)$$

$$= (1-n)\sigma\alpha\pi\rho\left[(1+b_u)d_u^{Hh}+(1+b_a)d_a^{Hh}+c\cdot\mathbb{I}_{d_a^{Hh}>0}\right]$$

$$+ (1-n)\sigma\alpha\pi(1-\rho)\left[(1+b_u)d_u^{Lh}+(1+b_a)d_a^{Lh}+c\cdot\mathbb{I}_{d_a^{Lh}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)\rho\left[(1+b_u)d_u^{Hl}+(1+b_a)d_a^{Hl}+c\cdot\mathbb{I}_{d_a^{Hl}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)(1-\rho)\left[(1+b_u)d_u^{Ll}+(1+b_a)d_a^{Ll}+c\cdot\mathbb{I}_{d_a^{Ll}>0}\right]+n\sigma\alpha\pi\rho\kappa\psi a^{Hh}$$

$$+ n\sigma\alpha\pi(1-\rho)\kappa\psi a^{Lh}+n\sigma\alpha(1-\pi)\rho\kappa\psi a^{Hl}+n\sigma\alpha(1-\pi)(1-\rho)\kappa\psi a^{Ll}$$

In real terms, $\psi a^{ij}=(1+b_a)d_a^{ij}$ where $i\in\{L,H\}$ and $j\in\{l,h\}$, we further simplify the above to

$$\frac{\sigma\alpha\pi\rho[d_u^{Hh}+d_a^{Hh}]+\sigma\alpha\pi(1-\rho)[d_u^{Lh}+d_a^{Lh}]+\sigma\alpha(1-\pi)\rho[d_u^{Hl}+d_a^{Hl}]+\sigma\alpha(1-\pi)(1-\rho)[d_u^{Ll}+d_a^{Ll}]}{1-R}(1+b_l)$$

$$= (1-n)\sigma\alpha\pi\rho\left[(1+b_u)d_u^{Hh}+(1+b_a)d_a^{Hh}+c\cdot\mathbb{I}_{d_a^{Hh}>0}\right]$$

$$+ (1-n)\sigma\alpha\pi(1-\rho)\left[(1+b_u)d_u^{Lh}+(1+b_a)d_a^{Lh}+c\cdot\mathbb{I}_{d_a^{Lh}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)\rho\left[(1+b_u)d_u^{Hl}+(1+b_a)d_a^{Hl}+c\cdot\mathbb{I}_{d_a^{Hl}>0}\right]$$

$$+ (1-n)\sigma\alpha(1-\pi)(1-\rho)\left[(1+b_u)d_u^{Ll}+(1+b_a)d_a^{Ll}+c\cdot\mathbb{I}_{d_a^{Ll}>0}\right]+n\sigma\alpha\pi\rho\kappa(1+b_a)d_a^{Hh}$$

$$+ n\sigma\alpha\pi(1-\rho)\kappa(1+b_a)d_a^{Lh}+n\sigma\alpha(1-\pi)\rho\kappa(1+b_a)d_a^{Hl}+n\sigma\alpha(1-\pi)(1-\rho)\kappa(1+b_a)d_a^{Ll}$$

The first-order condition with respect to d_u's give

$$(1-n)b_u=\frac{b_l}{(1-R)}$$

$$b_u=\frac{b_l}{(1-R)(1-n)}$$

which gives $b_u>b_l$ for $0<R<1$ and $0<n<1$.

The first-order condition with respect to d_a's give

$$(1-n+n\kappa)b_a=\frac{b_l}{(1-R)}$$

$$b_a = \frac{b_l}{(1-R)[1-(1-\kappa)n]}$$

which gives $b_a > b_l$ for $0 < R < 1$, $0 < n < 1$ and $0 < \kappa < 1$.

For $0 < n < 1$ and $0 < \kappa < 1$, we have

$$b_u > b_a$$

and b_a is the discounted rate of b_u because $n\kappa$ units of assets from the total loans can be recovered.

We show the comparative statics of the effects of the various parameters on the endogenous debt rates:

ϑ	b_l	R	n	κ
$\frac{db_u}{d\vartheta}$	$+$	$+$	$+$	no effect
$\frac{db_a}{d\vartheta}$	$+$	$+$	$+$	$-$

Substituting $b_u = \frac{b_l}{(1-R)(1-n)}$ and $b_a = \frac{b_l}{(1-R)[1-(1-\kappa)n]}$ into the FOC for m, we solve the equilibrium.

3.5 Quantitative Analysis

We assume the following functional forms for $u(q)$ and $c(q)$:

$$u(q) = A_u \frac{(q+\varepsilon_u)^{1-\gamma} - \varepsilon_u^{1-\gamma}}{1-\gamma}$$

$$c(q) = A_c \frac{q^{1+\xi_c}}{1+\xi_c}$$

Para.	Description	Value	Source
β	Discount Factor	0.95	Venkateswaran and Wright (2014), β = 0.95 Zhang (2014), β = 0.966
θ	Buyer's bargaining power	0.50	Egalitarian bargaining rule for Kalai bargaining

σ	Probability of DM consumption	0.20	Set directly. For reference He et. al (2015), $\sigma = 0.25$
α	Probability of access to credit market	0.84	2021 US Adult Access to Credit Card from Forbes[1]
π	Probability of high DM consumption	0.20	Set directly
ξ^l	Low DM consumption utility shock	0.10	Set directly
ξ^h	High DM consumption utility shock	1.00	Set directly
ρ	Probability of receiving endowment	0.77	Hall and Kudlyak (2019) 15-month average of being employed: Employed to employed = 0.954 Unemployed to employed = 0.586
ε	Endowment	0.50	Set directly
A_u	Coefficient of buyer utility function	2.18	He et. al (2015)
ε_u	Coefficient in buyer utility function	0.00	He et. al (2015)
γ	Exponent in buyer utility function	0.16	He et. al (2015)
A_c	Coefficient of seller cost function	1.00	He et. al (2015)
ξ_c	Exponent in seller cost function	3.80	He et. al (2015)
i	Inflation	0.0155	2015-2019 Average inflation rate from Board of U.S. Bureau of Labor Statistics[2]
b_u	Interest payable on unsecured debt	0.1004	2015-2019 Average interest on personal loans from Board of Governors of the Federal Reserve System[3]
b_a	Interest payable on secured debt	0.0562	2015-2019 Average interest on car loans from Board of Governors of the Federal Reserve System[4]

[1] See https://www.forbes.com/advisor/credit-cards/credit-card-statistics/#:~:text=84%25%20of%20U.S.%20adults%20had%20a%20credit%20card%20in%202021

[2] See https://www.rateinflation.com/inflation-rate/usa-historical-inflation-rate which extracted data from Bureau of Labor Statistics.

[3] See https://www.federalreserve.gov/releases/g19/hist/cc_hist_tc_levels.html

[4] See https://www.federalreserve.gov/releases/g19/hist/cc_hist_tc_levels.html

b_l	Interest payable on savings deposit	0.00072	2015-2019 Average interest on savings deposits from Federal Deposit Insurance Corporation[5]
n	Rate of default on secured debt	0.0398	2015-2019 Average US Auto Loans Delinquency from Federal Reserve Bank of New York[6]
R	Reserve Ratio	0.0529	Set directly to match model
κ	Fraction of asset that can be recovered in open market during default	0.77	Pennington-Cross (2006) with foreclosure data from: Shilling, Benjamin and Sirmans (1990), 76% Forgey, Rutherford and VanBuskirk (1994), 77% Hardin and Wolverton (1996), 75%
c	Fixed cost to seller for loans	0.0116	Set directly to match model

Let welfare $\mathcal{W}$ be given by the total surplus in all DM meetings.

$$
\begin{aligned}
\mathcal{W} = \theta\Big[&\sigma\alpha\rho\pi[\xi^l u(q^{Hlc}) - \omega(q^{Hlc})] + \sigma\alpha(1-\rho)\pi[\xi^l u(q^{Llc}) - \omega(q^{Llc})] \\
&+ \sigma\alpha\rho(1-\pi)[\xi^h u(q^{Hhc}) - \omega(q^{Hhc})] \\
&+ \sigma\alpha(1-\rho)(1-\pi)[\xi^h u(q^{Lhc}) - \omega(q^{Lhc})] \\
&+ \sigma(1-\alpha)\rho\pi[\xi^l u(q^{Hln}) - \omega(q^{Hln})] \\
&+ \sigma(1-\alpha)(1-\rho)\pi[\xi^l u(q^{Lln}) - \omega(q^{Lln})] \\
&+ \sigma(1-\alpha)\rho(1-\pi)[\xi^h u(q^{Hhn}) - \omega(q^{Hhn})] \\
&+ \sigma(1-\alpha)(1-\rho)(1-\pi)[\xi^h u(q^{Lhn}) - \omega(q^{Lhn})]\Big] + \sigma\alpha\rho\pi c \\
&\cdot \mathbb{I}_{\tilde{d}_a^{Hlc}>0} + \sigma\alpha(1-\rho)\pi c \cdot \mathbb{I}_{\tilde{d}_a^{Llc}>0} + \sigma\alpha\rho(1-\pi)c \cdot \mathbb{I}_{\tilde{d}_a^{Hhc}>0} \\
&+ \sigma\alpha(1-\rho)(1-\pi)c \cdot \mathbb{I}_{\tilde{d}_a^{Lhc}>0} \\
&+ (1-\theta)\Big[\sigma\alpha\rho\pi[\omega(q^{Hlc}) - c(q^{Hlc})] \\
&+ \sigma\alpha(1-\rho)\pi[\omega(q^{Llc}) - c(q^{Llc})] \\
&+ \sigma\alpha\rho(1-\pi)[\omega(q^{Hhc}) - c(q^{Hhc})] \\
&+ \sigma\alpha(1-\rho)(1-\pi)[\omega(q^{Lhc}) - c(q^{Lhc})] \\
&+ \sigma\alpha(1-\alpha)\rho\pi[\omega(q^{Hln}) - \omega(q^{Hln})] \\
&+ \sigma\alpha(1-\alpha)(1-\rho)\pi[\omega(q^{Lln}) - c(q^{Lln})] \\
&+ \sigma\alpha(1-\alpha)\rho(1-\pi)[\omega(q^{Hhn}) - c(q^{Hhn})] \\
&+ \sigma\alpha(1-\alpha)(1-\rho)(1-\pi)[\omega(q^{Lhn}) - c(q^{Lhn})]\Big]
\end{aligned}
$$

[5] See https://www.thebalancemoney.com/savings-account-interest-rate-history-6742139 which extracted data from FDIC.

[6] See https://ycharts.com/indicators/us_auto_loans_delinquent_by_90_days#:~:text=US%20Auto%20Loans%20Delinquent%20by%2090%20or%20More%20Days%20is,long%20term%20average%20of%203.46%25 which extracted data from Federal Reserve Bank of New York.

which is the sum of the buyer's surplus minus fixed costs if credit is used plus the sum of the seller's surplus.

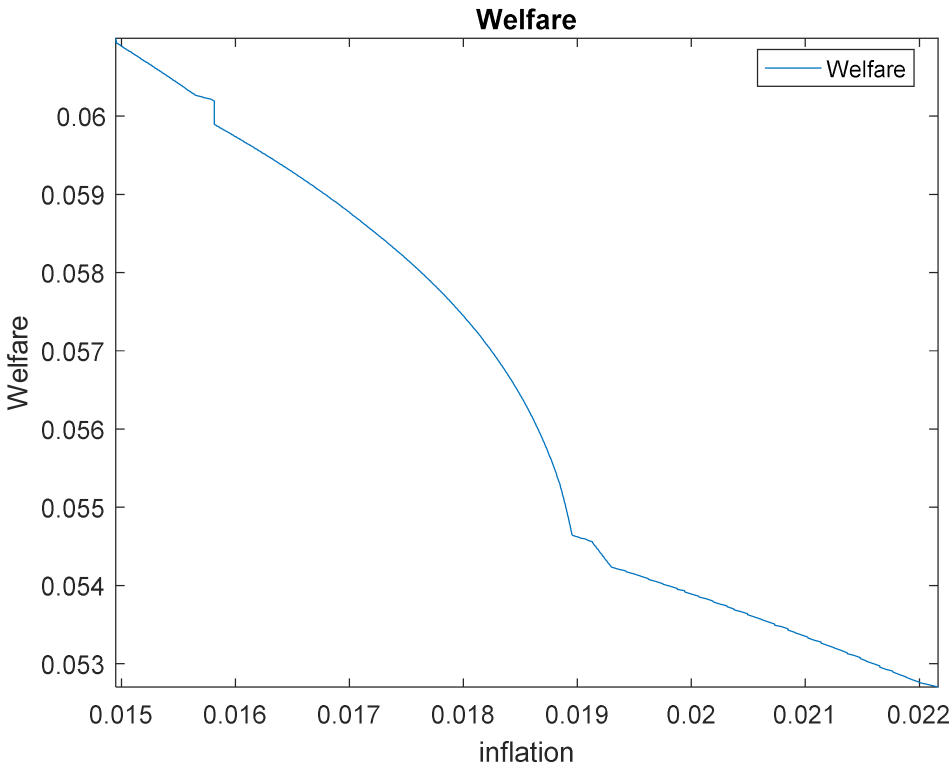
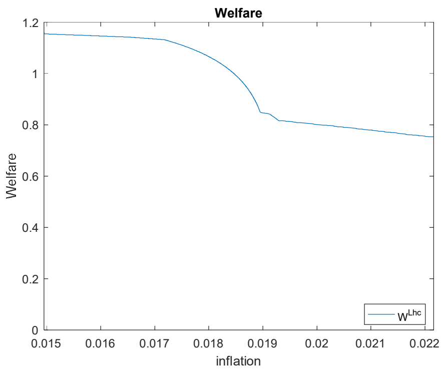
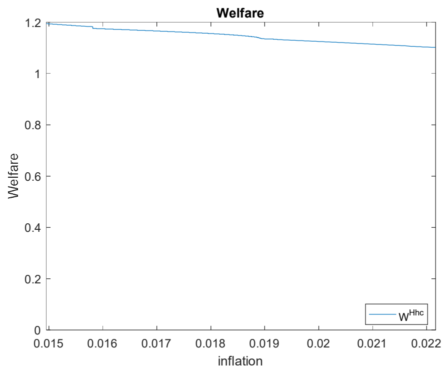


Figure 3.15: Effect of Inflation on Total Welfare

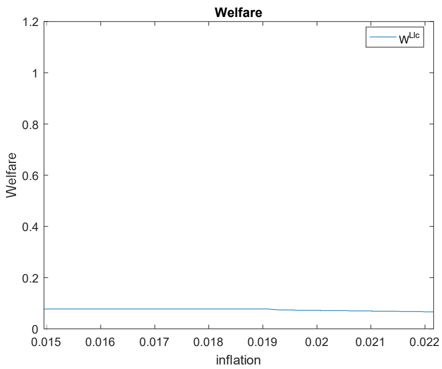
Welfare decreases as inflation increases because agents carry less money and rely on credit to finance consumption needs due to the cost of carrying money. Note that the welfare loss is not uniform. For a slight increase from equilibrium, total welfare loss is about 1.67% for every 0.1% increase in inflation. However, as inflation increase by about 0.3%, the welfare loss increases to 3.63% per 0.1% increase in inflation as both low-income and high-income agents with high consumption needs also become liquidity constrained. Thereafter, the welfare loss decreases again as low-income agents with low consumption needs become the next to be liquidity constrained. But because their liquidity needs are low, the welfare loss is about 1.88% per 0.1% increase in inflation.



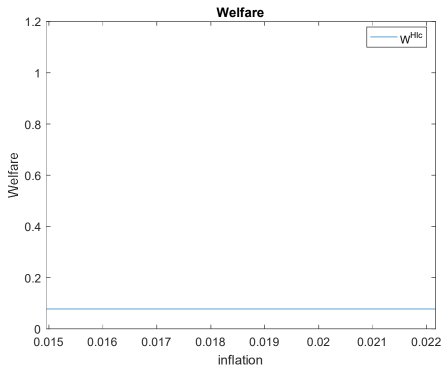
Low-income high consumption with access to credit



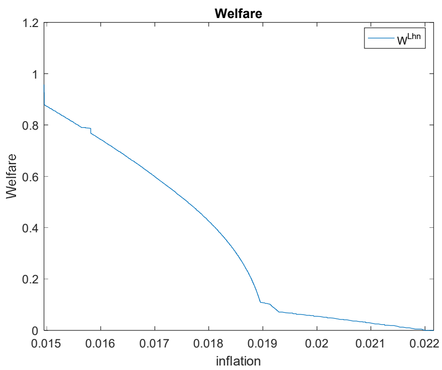
High-income high consumption with access to credit



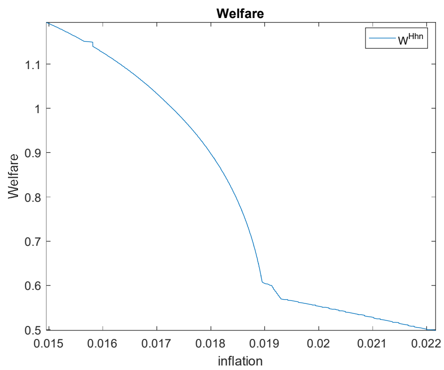
Low-income low consumption with access to credit



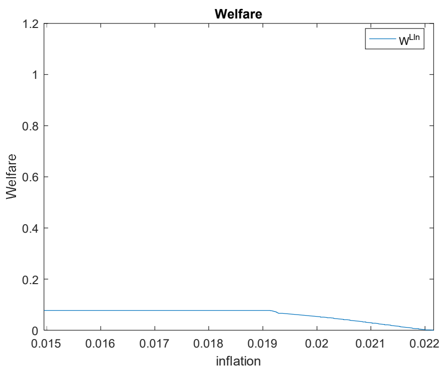
High-income low consumption with access to credit



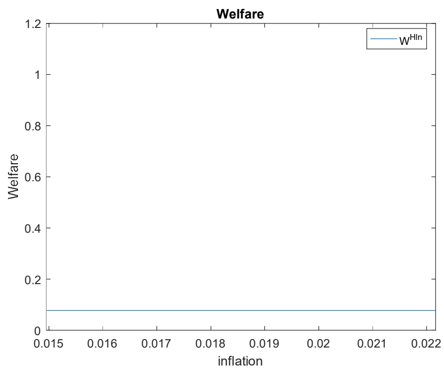
Low-income high consumption without access to credit



High-income high consumption without access to credit



Low-income low consumption without access to credit



High-income low consumption without access to credit

Figure 3.16: Effect of Inflation on individual Welfare

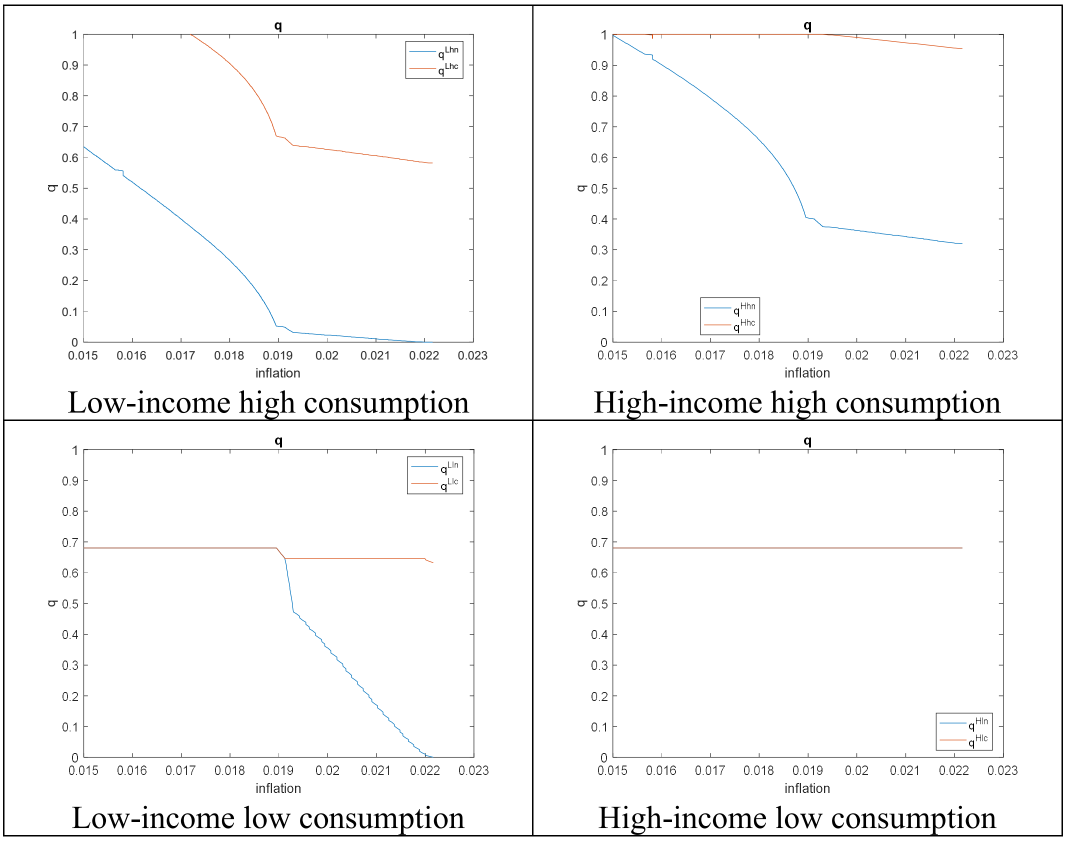


Figure 3.17: Effect of Inflation on DM consumption

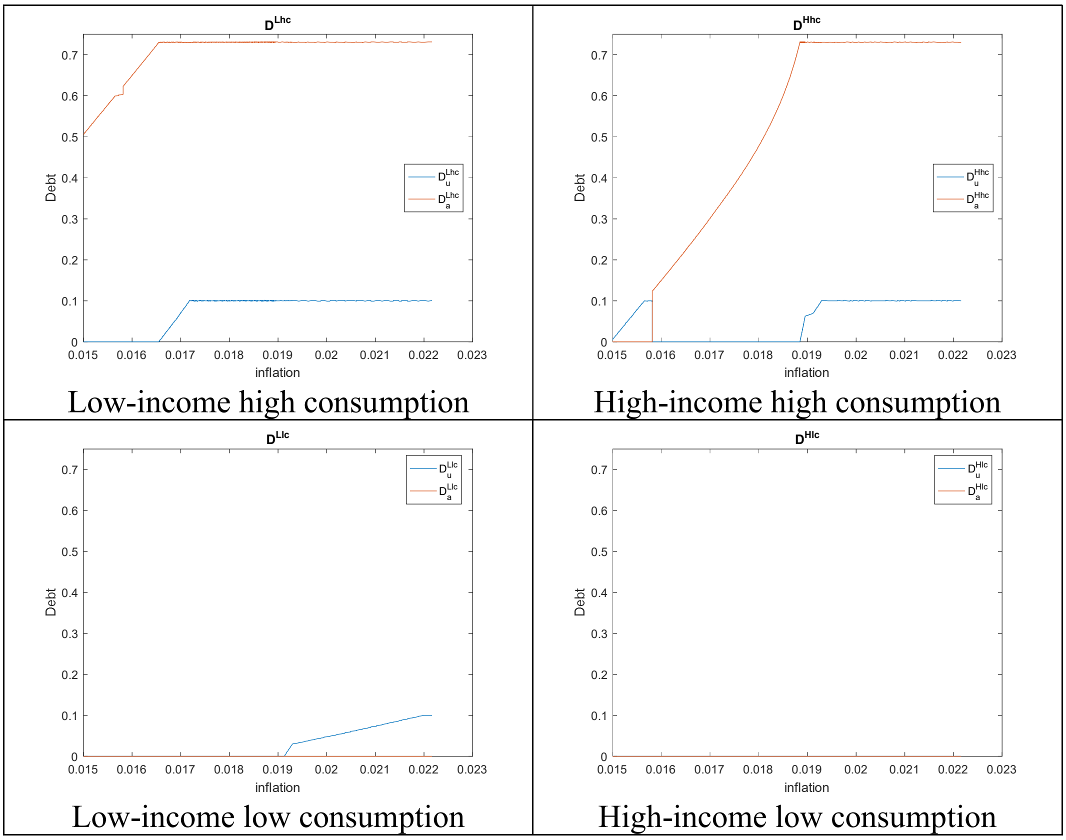


Figure 3.18: Effect of Inflation on debt

We study how the fixed cost c for taking on loans affects welfare. We investigate in the proximity of the equilibrium point by taking $i = 0.0155$ in the above calibrated model, we have

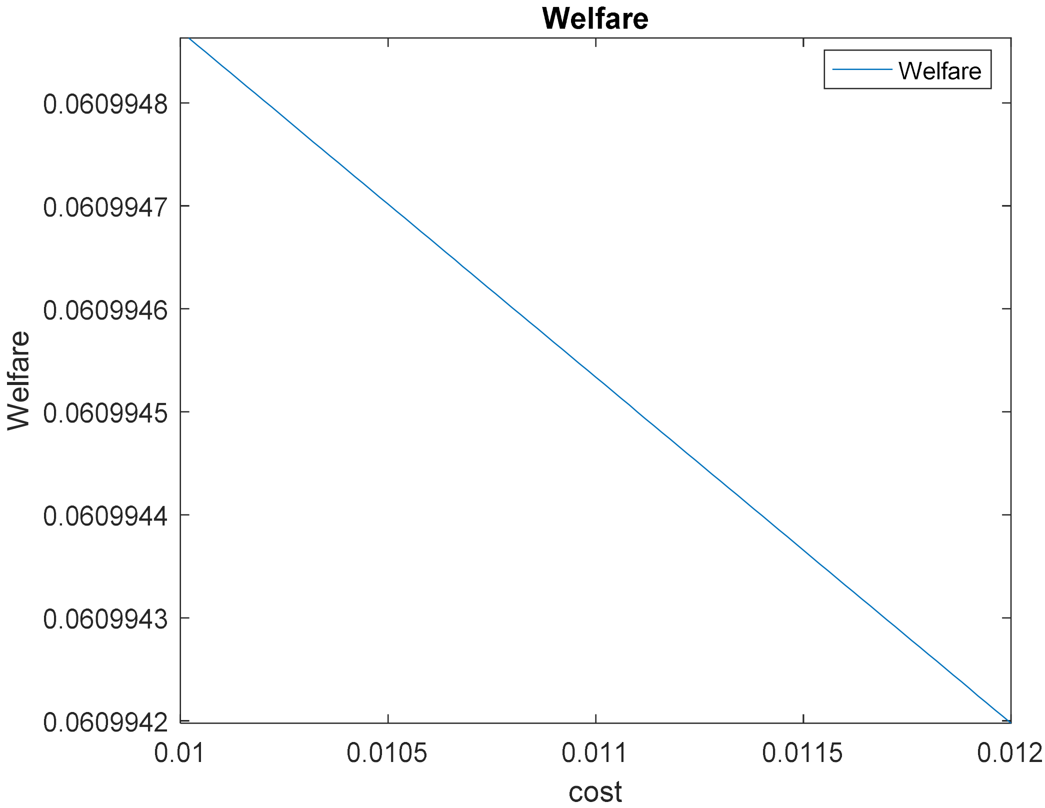


Figure 3.19: Effect of fixed cost on Welfare

Welfare decreases as the fixed cost c increases as agents pay more to take on debt. A 10% increase in c results in about 0.0005% welfare loss. Due to liquidity needs, agents still require and use secured credit (which relaxes liquidity constraint greatly) even though the fixed cost increased. This is because agents would still like to borrow to the debt limit to try and consume the optimal quantity.

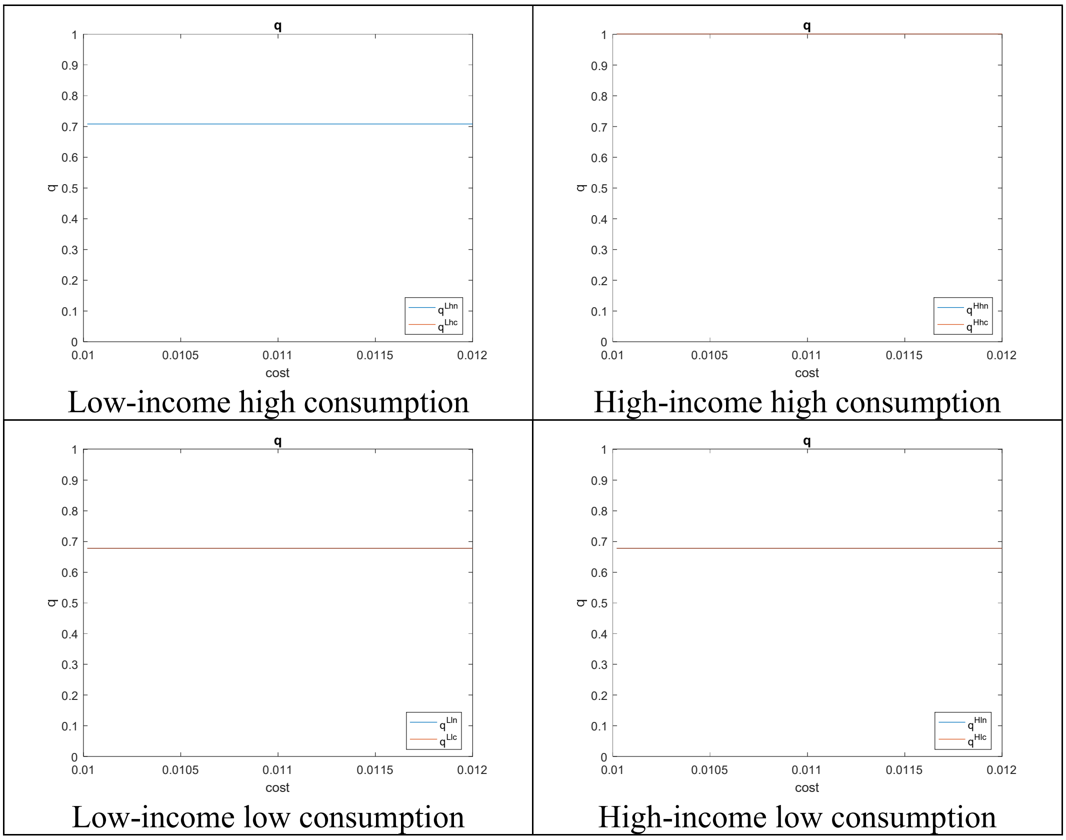


Figure 3.20: Effect of fixed cost on DM consumption

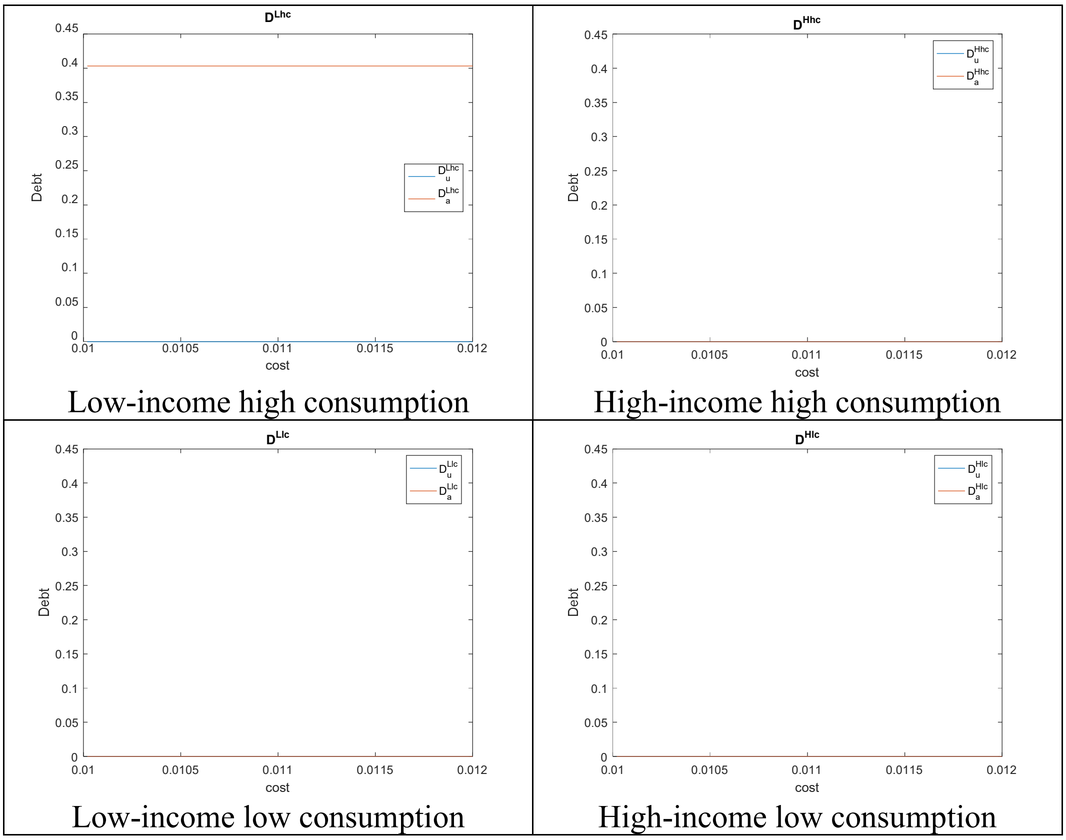


Figure 3.21: Effect of fixed cost on debt

3.6 Conclusion

In this chapter we investigated the co-existence of two types of credit – secured and unsecured, along with fiat currency, in an environment with four types of agents – (1) low-income agents with high consumption needs, (2) high-income agents with high consumption needs, (3) low-income agents with low consumption needs, and (4) high-income agents with low consumption needs.

Low-income agents with high consumption needs are always the first with liquidity needs. Next whether high-income agents with high consumption needs or low-income agents with low consumption needs are the next to face liquidity constraints depending on the size of the utility shock to the endowment received. High-income agents with low consumption needs are always the last to face liquidity constraints and, in most cases, almost never need credit.

As inflation increases, the cost of money increases resulting in agents carrying less fiat currency and rely more on credit to finance their consumption needs. Low-income agents with high consumption needs are always the first to require credit while in most situations, high-income agents with low consumption needs never need credit.

Welfare decreases as inflation increases because agents carry less money and rely on credit to finance consumption needs because agents have insufficient liquidity to obtain the optimal DM quantity of goods. Increase in the fixed cost for taking on secured loans have little impact on welfare loss because agents borrow to the debt limit to try and consume the optimal quantity.

Calibrating our model to US data, for a slight increase from equilibrium, total welfare loss is about 1.67% for every 0.1% increase in inflation. However, as inflation increase by about 0.3%, the welfare loss increases as both low-income and high-income agents with high consumption needs also become liquidity constrained. The welfare loss now becomes about 3.63% per 0.1% increase in inflation. Thereafter, the welfare loss decreases again as low-income agents with low consumption needs become the next to be liquidity constrained. But because their liquidity needs are low, the welfare loss is about 1.88% per 0.1% increase in inflation.

3.7 References

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Appendix

Solving the optimal portfolio choice by the Lagrangian method:

$$
\begin{aligned}
\mathcal{L} = -\phi\hat{m} - \psi\hat{a} + \beta\rho\varepsilon + (1-\sigma)b_l l - \beta\sigma\alpha\rho\pi b_u \tilde{d}_u^{Hlc} - \beta\sigma\alpha\rho\pi b_a \tilde{d}_a^{Hlc} \\
& - \beta\sigma\alpha(1-\rho)\pi b_u \tilde{d}_u^{Llc} - \beta\sigma\alpha(1-\rho)\pi b_a \tilde{d}_a^{Llc} \\
& - \beta\sigma\alpha\rho(1-\pi) b_u \tilde{d}_u^{Hhc} - \beta\sigma\alpha\rho(1-\pi) b_a \tilde{d}_a^{Hhc} \\
& - \beta\sigma\alpha(1-\rho)(1-\pi) b_u \tilde{d}_u^{Lhc} - \beta\sigma\alpha(1-\rho)(1-\pi) b_a \tilde{d}_a^{Lhc} \\
& + \beta W(\hat{m}, \hat{a}, 0,0,0) \\
& + \beta\{\sigma\alpha\rho\pi[\xi^l u(q^{Hlc}) - \omega(q^{Hlc})] \\
& + \sigma\alpha(1-\rho)\pi[\xi^l u(q^{Llc}) - \omega(q^{Llc})] \\
& + \sigma\alpha\rho(1-\pi)[\xi^h u(q^{Hhc}) - \omega(q^{Hhc})] \\
& + \sigma\alpha(1-\rho)(1-\pi)[\xi^h u(q^{Lhc}) - \omega(q^{Lhc})]\} \\
& + \beta\{\sigma(1-\alpha)\rho\pi[\xi^l u(q^{Hln}) - \omega(q^{Hln})] \\
& + \sigma(1-\alpha)(1-\rho)\pi[\xi^l u(q^{Lln}) - \omega(q^{Lln})] \\
& + \sigma(1-\alpha)\rho(1-\pi)[\xi^h u(q^{Hhn}) - \omega(q^{Hhn})] \\
& + \sigma(1-\alpha)(1-\rho)(1-\pi)[\xi^h u(q^{Lhn}) - \omega(q^{Lhn})]\} \\
& + \lambda_u^{Hlc}\sigma\alpha\rho\pi\left[D_u - \tilde{d}_u^{Hlc}\right] + \lambda_u^{Llc}\sigma\alpha(1-\rho)\pi\left[D_u - \tilde{d}_u^{Llc}\right] \\
& + \lambda_u^{Hhc}\sigma\alpha\rho(1-\pi)\left[D_u - \tilde{d}_u^{Hhc}\right] \\
& + \lambda_u^{Lhc}\sigma\alpha(1-\rho)(1-\pi)\left[D_u - \tilde{d}_u^{Lhc}\right] + \lambda_a^{Hlc}\sigma\alpha\rho\pi\left[D(a) - \tilde{d}_a^{Hlc}\right] \\
& + \lambda_a^{Llc}\sigma\alpha(1-\rho)\pi\left[D(a) - \tilde{d}_a^{Llc}\right] \\
& + \lambda_a^{Hhc}\sigma\alpha\rho(1-\pi)\left[D(a) - \tilde{d}_a^{Hhc}\right] \\
& + \lambda_a^{Lhc}\sigma\alpha(1-\rho)(1-\pi)\left[D(a) - \tilde{d}_a^{Lhc}\right] \\
& + \lambda_q^{Hlc}\sigma\alpha\rho\pi\left[\hat{\phi}\hat{m} + \hat{\varepsilon} + \tilde{d}^{Hlc} - \omega(q^{Hlc})\right] \\
& + \lambda_q^{Llc}\sigma\alpha(1-\rho)\pi\left[\hat{\phi}\hat{m} + \tilde{d}^{Llc} - \omega(q^{Llc})\right] \\
& + \lambda_q^{Hhc}\sigma\alpha\rho(1-\pi)\left[\hat{\phi}\hat{m} + \hat{\varepsilon} + \tilde{d}^{Hhc} - \omega(q^{Hhc})\right] \\
& + \lambda_q^{Lhc}\sigma\alpha(1-\rho)(1-\pi)\left[\hat{\phi}\hat{m} + \tilde{d}^{Lhc} - \omega(q^{Lhc})\right] \\
& + \lambda_q^{Hln}\sigma(1-\alpha)\rho\pi\left[\hat{\phi}\hat{m} + \hat{\varepsilon} - \omega(q^{Hln})\right] \\
& + \lambda_q^{Lln}\sigma(1-\alpha)(1-\rho)\pi\left[\hat{\phi}\hat{m} - \omega(q^{Lln})\right] \\
& + \lambda_q^{Hhn}\sigma(1-\alpha)\rho(1-\pi)\left[\hat{\phi}\hat{m} + \hat{\varepsilon} - \omega(q^{Hhn})\right] \\
& + \lambda_q^{Lhn}\sigma(1-\alpha)(1-\rho)(1-\pi)\left[\hat{\phi}\hat{m} - \omega(q^{Lhn})\right]
\end{aligned}
$$

The First-order Conditions are

$$\frac{\partial \mathcal{L}}{\partial \hat{m}} = -\phi + \beta \frac{\partial W(\hat{m}, \hat{a}, 0,0,0)}{\partial \hat{m}} + \lambda_q^{Hlc} \sigma \alpha \rho \pi \hat{\phi} + \lambda_q^{Llc} \sigma \alpha (1-\rho) \pi \hat{\phi}$$

$$+ \lambda_q^{Hhc} \sigma \alpha \rho (1-\pi) \hat{\phi} + \lambda_q^{Lhc} \sigma \alpha (1-\rho)(1-\pi) \hat{\phi}$$

$$+ \lambda_q^{Hln} \sigma (1-\alpha) \rho \pi \hat{\phi} + \lambda_q^{Lln} \sigma (1-\alpha)(1-\rho) \pi \hat{\phi}$$

$$+ \lambda_q^{Hhn} \sigma (1-\alpha) \rho (1-\pi) \hat{\phi} + \lambda_q^{Lhn} (1-\alpha)(1-\rho)(1-\pi) \hat{\phi}$$

$$= -\phi + \hat{\phi}\beta + \lambda_q^{Hlc} \sigma \alpha \rho \pi \hat{\phi} + \lambda_q^{Llc} \sigma \alpha (1-\rho) \pi \hat{\phi} + \lambda_q^{Hhc} \sigma \alpha \rho (1-\pi) \hat{\phi} + \lambda_q^{Lhc} \sigma \alpha (1-\rho)(1-\pi) \hat{\phi} + \lambda_q^{Hln} \sigma (1-\alpha) \rho \pi \hat{\phi} + \lambda_q^{Lln} \sigma (1-\alpha)(1-\rho) \pi \hat{\phi} + \lambda_q^{Hhn} \sigma (1-\alpha) \rho (1-\pi) \hat{\phi} + \lambda_q^{Lhn} (1-\alpha)(1-\rho)(1-\pi) \hat{\phi}$$

$$\geq 0$$

with equality if $\hat{m} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \hat{a}} = -\psi + \beta \frac{\partial W(\hat{m}, \hat{a}, 0,0,l)}{\partial \hat{a}} + \lambda_a^{Hl} \theta \rho \pi \left[\frac{\partial D(a)}{\partial \hat{a}}\right] + \lambda_a^{Ll} \sigma \alpha (1-\rho) \pi \left[\frac{\partial D(a)}{\partial \hat{a}}\right]$$

$$+ \lambda_a^{Hh} \sigma \alpha \rho (1-\pi) \left[\frac{\partial D(a)}{\partial \hat{a}}\right] + \lambda_a^{Lh} \sigma \alpha (1-\rho)(1-\pi) \left[\frac{\partial D(a)}{\partial \hat{a}}\right]$$

$$= -\psi + (\hat{\psi} + \hat{\eta})\beta + \lambda_a^{Hlc} \sigma \alpha \rho \pi \kappa (\hat{\psi} + \hat{\eta}) + \lambda_a^{Llc} \sigma \alpha (1-\rho) \pi \kappa (\hat{\psi} + \hat{\eta}) + \lambda_a^{Hhc} \sigma \alpha \rho (1-\pi) \kappa (\hat{\psi} + \hat{\eta}) + \lambda_a^{Lhc} \sigma \alpha (1-\rho)(1-\pi) \kappa (\hat{\psi} + \hat{\eta})$$

$$\geq 0$$

with equality if $\hat{a} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_u^{Hlc}} = -\beta \sigma \alpha \rho \pi \hat{b}_u - \lambda_u^{Hlc} \sigma \alpha \rho \pi + \lambda_q^{Hlc} \sigma \alpha \rho \pi$$

$$\geq 0$$

with equality if $\tilde{d}_u^{Hlc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_u^{Llc}} = -\beta \sigma \alpha (1-\rho) \pi \hat{b}_u - \lambda_u^{Llc} \sigma \alpha (1-\rho) \pi + \lambda_q^{Llc} \sigma \alpha (1-\rho) \pi$$

$$\geq 0$$

with equality if $\tilde{d}_u^{Llc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_u^{Hhc}} = -\beta\sigma\alpha\rho(1-\pi)\hat{b}_u - \lambda_u^{Hhc}\sigma\alpha\rho(1-\pi) + \lambda_q^{Hhc}\sigma\alpha\rho(1-\pi)$$

$$\geq 0$$

with equality if $\tilde{d}_u^{Hhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_u^{Lhc}} = -\beta\sigma\alpha(1-\rho)(1-\pi)\hat{b}_u - \lambda_u^{Lhc}\sigma\alpha(1-\rho)(1-\pi)$$

$$+ \lambda_q^{Lhc}\sigma\alpha(1-\rho)(1-\pi)$$

$$\geq 0$$

with equality if $\tilde{d}_u^{Lhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_a^{Hlc}} = -\beta\sigma\alpha\rho\pi\hat{b}_a - \lambda_a^{Hlc}\sigma\alpha\rho\pi + \lambda_q^{Hlc}\sigma\alpha\rho\pi$$

$$\geq 0$$

with equality if $\tilde{d}_a^{Hlc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_a^{Llc}} = -\beta\sigma\alpha(1-\rho)\pi\hat{b}_a - \lambda_a^{Llc}\sigma\alpha(1-\rho)\pi + \lambda_q^{Llc}\sigma\alpha(1-\rho)\pi$$

$$\geq 0$$

with equality if $\tilde{d}_a^{Llc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_a^{Hhc}} = -\beta\sigma\alpha\rho(1-\pi)\hat{b}_a - \lambda_a^{Hhc}\sigma\alpha\rho(1-\pi) + \lambda_q^{Hhc}\sigma\alpha\rho(1-\pi)$$

$$\geq 0$$

with equality if $\tilde{d}_a^{Hhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \tilde{d}_a^{Lhc}} = -\beta\sigma\alpha(1-\rho)(1-\pi)\hat{b}_a - \lambda_a^{Lhc}\sigma\alpha(1-\rho)(1-\pi)$$

$$+ \lambda_q^{Lhc}\sigma\alpha(1-\rho)(1-\pi)$$

$$\geq 0$$

with equality if $\tilde{d}_a^{Lhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Hlc}} = \beta\sigma\alpha\rho\pi\left[\xi^{l}\frac{\partial u(q^{Hlc})}{\partial q^{Hlc}} - \frac{\partial \omega(q^{Hlc})}{\partial q^{Hlc}}\right] - \lambda_q^{Hlc}\sigma\alpha\rho\pi\left[\frac{\partial \omega(q^{Hlc})}{\partial q^{Hlc}}\right]$$

$$= \beta\sigma\alpha\rho\pi[\xi^{l}u'(q^{Hlc}) - \omega'(q^{Hlc})] - \lambda_q^{Hlc}\sigma\alpha\rho\pi\omega'(q^{Hlc})$$

$$\geq 0$$

with equality if $q^{Hlc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Llc}} = \beta\sigma\alpha(1-\rho)\pi\left[\xi^{l}\frac{\partial u(q^{Llc})}{\partial q^{Llc}} - \frac{\partial \omega(q^{Llc})}{\partial q^{Llc}}\right] - \lambda_q^{Llc}\sigma\alpha(1-\rho)\pi\left[\frac{\partial \omega(q^{Llc})}{\partial q^{Llc}}\right]$$

$$= \beta\sigma\alpha(1-\rho)\pi[\xi^{l}u'(q^{Llc}) - \omega'(q^{Llc})] - \lambda_q^{Llc}\sigma\alpha(1-\rho)\pi\omega'(q^{Llc})$$

$$\geq 0$$

with equality if $q^{Llc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Hhc}} = \beta\sigma\alpha\rho(1-\pi)\left[\xi^{h}\frac{\partial u(q^{Hhc})}{\partial q^{Hhc}} - \frac{\partial \omega(q^{Hhc})}{\partial q^{Hhc}}\right]$$

$$- \lambda_q^{Hhc}\sigma\alpha\rho(1-\pi)\left[\frac{\partial \omega(q^{Hhc})}{\partial q^{Hhc}}\right]$$

$$= \beta\sigma\alpha\rho(1-\pi)[\xi^{h}u'(q^{Hhc}) - \omega'(q^{Hhc})] - \lambda_q^{Hhc}\sigma\alpha\rho(1-\pi)\omega'(q^{Hhc})$$

$$\geq 0$$

with equality if $q^{Hhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Lhc}} = \beta\sigma\alpha(1-\rho)(1-\pi)\left[\xi^{h}\frac{\partial u(q^{Lhc})}{\partial q^{Lhc}} - \frac{\partial \omega(q^{Lhc})}{\partial q^{Lhc}}\right]$$

$$- \lambda_q^{Lhc}\sigma\alpha(1-\rho)(1-\pi)\left[\frac{\partial \omega(q^{Lhc})}{\partial q^{Lhc}}\right]$$

$$= \beta\sigma\alpha(1-\rho)(1-\pi)[\xi^{h}u'(q^{Lhc}) - \omega'(q^{Lhc})] - \lambda_q^{Lhc}\sigma\alpha(1-\rho)(1-\pi)\omega'(q^{Lhc})$$

$$\geq 0$$

with equality if $q^{Lhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Hln}} = \beta\sigma(1-\alpha)\rho\pi\left[\xi^{l}\frac{\partial u(q^{Hln})}{\partial q^{Hln}} - \frac{\partial \omega(q^{Hln})}{\partial q^{Hln}}\right]$$

$$- \lambda_q^{Hln}\sigma(1-\alpha)\rho\pi\left[\frac{\partial \omega(q^{Hln})}{\partial q^{Hln}}\right]$$

$$= \beta\sigma(1-\alpha)\rho\pi\sigma[\xi^{l}u'(q^{Hln}) - \omega'(q^{Hln})] - \lambda_q^{Hln}\sigma(1-\alpha)\rho\pi\sigma\omega'(q^{Hln})$$

$$\geq 0$$

with equality if $q^{Hln} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Lln}} = \beta\sigma(1-\alpha)(1-\rho)\pi\left[\xi^{l}\frac{\partial u(q^{Lln})}{\partial q^{Lln}} - \frac{\partial \omega(q^{Lln})}{\partial q^{Lln}}\right]$$

$$- \lambda_q^{Lln}\sigma(1-\alpha)(1-\rho)\pi\left[\frac{\partial \omega(q^{Lln})}{\partial q^{Lln}}\right]$$

$$= \beta\sigma(1-\alpha)(1-\rho)\pi[\xi^{l}u'(q^{Lln}) - \omega'(q^{Lln})] - \lambda_q^{Lln}\sigma(1-\alpha)(1-$$

$$\rho)\pi\omega'(q^{Lln})$$

$$\geq 0$$

with equality if $q^{Lln} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Hhn}} = \beta\sigma(1-\alpha)\rho(1-\pi)\left[\xi^{h}\frac{\partial u(q^{Hhn})}{\partial q^{Hhn}} - \frac{\partial \omega(q^{Hhn})}{\partial q^{Hhn}}\right]$$

$$- \lambda_q^{Hhn}\sigma(1-\alpha)\rho(1-\pi)\left[\frac{\partial \omega(q^{Hhn})}{\partial q^{Hhn}}\right]$$

$$= \beta\sigma(1-\alpha)\rho(1-\pi)[\xi^{h}u'(q^{Hhn}) - \omega'(q^{Hhn})] - \lambda_q^{Hhn}\sigma(1-\alpha)\rho(1-$$

$$\pi)\omega'(q^{Hhn})$$

$$\geq 0$$

with equality if $q^{Hhn} > 0$.

$$\frac{\partial \mathcal{L}}{\partial q^{Lhn}} = \beta\sigma(1-\alpha)(1-\rho)(1-\pi)\left[\xi^{h}\frac{\partial u(q^{Lhn})}{\partial q^{Lhn}} - \frac{\partial \omega(q^{Lhn})}{\partial q^{Lhn}}\right]$$

$$- \lambda_q^{Lhn}\sigma(1-\alpha)(1-\rho)(1-\pi)\left[\frac{\partial \omega(q^{Lhn})}{\partial q^{Lhn}}\right]$$

$= \beta\sigma(1-\alpha)(1-\rho)(1-\pi)[\xi^h u'(q^{Lhn}) - \omega'(q^{Lhn})] - \lambda_q^{Lhn}\sigma(1-$

$\alpha)(1-\rho)(1-\pi)\omega'(q^{Lhn})$

≥ 0

with equality if $q^{Lhn} > 0$.

$$\frac{\partial\mathcal{L}}{\partial\lambda_u^{Hlc}} = \sigma\alpha\rho\pi\left[D_u - \tilde{d}_u^{Hlc}\right]$$

≥ 0

with equality if $\lambda_u^{Hlc} > 0$.

$$\frac{\partial\mathcal{L}}{\partial\lambda_u^{Llc}} = \sigma\alpha(1-\rho)\pi\left[D_u - \tilde{d}_u^{Llc}\right]$$

≥ 0

with equality if $\lambda_u^{Llc} > 0$.

$$\frac{\partial\mathcal{L}}{\partial\lambda_u^{Hhc}} = \sigma\alpha\rho(1-\pi)\left[D_u - \tilde{d}_u^{Hhc}\right]$$

≥ 0

with equality if $\lambda_u^{Hhc} > 0$.

$$\frac{\partial\mathcal{L}}{\partial\lambda_u^{Lhc}} = \sigma\alpha(1-\rho)(1-\pi)\left[D_u - \tilde{d}_u^{Lhc}\right]$$

≥ 0

with equality if $\lambda_u^{Lhc} > 0$.

$$\frac{\partial\mathcal{L}}{\partial\lambda_a^{Hlc}} = \sigma\alpha\rho\pi\left[D(a) - \tilde{d}_a^{Hlc}\right]$$

≥ 0

with equality if $\lambda_a^{Hlc} > 0$.

$$\frac{\partial\mathcal{L}}{\partial\lambda_a^{Llc}} = \sigma\alpha(1-\rho)\pi\left[D(a) - \tilde{d}_a^{Llc}\right]$$

≥ 0

with equality if $\lambda_a^{Llc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_a^{Hhc}} = \sigma\alpha\rho(1-\pi)\left[D(a) - \tilde{d}_a^{Hhc}\right]$$

$$\geq 0$$

with equality if $\lambda_a^{Hhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_a^{Lhc}} = \sigma\alpha(1-\rho)(1-\pi)\left[D(a) - \tilde{d}_a^{Lhc}\right]$$

$$\geq 0$$

with equality if $\lambda_a^{Lhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Hlc}} = \sigma\alpha\rho\pi\left[\hat{\phi}\widehat{m} + \varepsilon + \tilde{d}^{Hlc} - \omega(q^{Hlc})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Hlc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Llc}} = \sigma\alpha(1-\rho)\pi\left[\hat{\phi}\widehat{m} + \tilde{d}^{Llc} - \omega(q^{Llc})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Llc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Hhc}} = \sigma\alpha\rho(1-\pi)\left[\hat{\phi}\widehat{m} + \varepsilon + \tilde{d}^{Hhc} - \omega(q^{Hhc})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Hhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Lhc}} = \sigma\alpha(1-\rho)(1-\pi)\left[\hat{\phi}\widehat{m} + \tilde{d}^{Lhc} - \omega(q^{Lhc})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Lhc} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Hln}} = \sigma(1-\alpha)\rho\pi\left[\hat{\phi}\widehat{m} + \varepsilon - \omega(q^{Hln})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Hln} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Lln}} = \sigma(1-\alpha)(1-\rho)\pi\left[\hat{\phi}\hat{m} - \omega(q^{Lln})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Lln} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Hhn}} = \sigma(1-\alpha)\rho(1-\pi)\left[\hat{\phi}\hat{m} + \varepsilon - \omega(q^{Hhn})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Hhn} > 0$.

$$\frac{\partial \mathcal{L}}{\partial \lambda_q^{Lhn}} = \sigma(1-\alpha)(1-\rho)(1-\pi)\left[\hat{\phi}\hat{m} - \omega(q^{Lhn})\right]$$

$$\geq 0$$

with equality if $\lambda_q^{Lhn} > 0$.

where:

λ_u is a measure of how much unsecured debt the agent is willing to take on to finance his consumption. If the agent does not need to exhaust his debt entitlement or borrow to the debt limit D, then the constraint is slack and $\lambda_u = 0$. If the agent does not have enough money holdings and need to borrow until the debt limit, then the constraint is binding and $\lambda_u > 0$.

λ_a is a measure of how much secured debt the agent is willing to take on to finance his consumption. If the agent does not need to exhaust his debt entitlement or borrow to the debt limit D, then the constraint is slack and $\lambda_a = 0$. If the agent does not have enough money holdings and need to borrow until the debt limit, then the constraint is binding and $\lambda_a > 0$.

λ_q is a measure of how far off the agent is from consuming optimal q^*. For example, for a high-income agent, if the agent has sufficient wealth to consume q^*, then the payment $\omega(q)$ is less than his liquid wealth $\hat{\phi}\hat{m} + \hat{\varepsilon}$. So the constraint is slack and $\lambda_q = 0$. Vice versa if the agent does not have enough liquid wealth, the constraint is binding and $\lambda_q > 0$.