6-2022

Essays on social choice and implementation theory

Paulo Daniel SALLES RAMOS
Singapore Management University, pauloramos.2018@phdecons.smu.edu.sg

Singapore Management University

PhD Dissertation

Essays on Social Choice and Implementation Theory

PAULO DANIEL SALLES RAMOS

Supervised by
Professor Shurojit Chatterji

2022

ESSAYS ON SOCIAL CHOICE AND IMPLEMENTATION THEORY

PAULO DANIEL SALLES RAMOS

A DISSERTATION

In

ECONOMICS

Presented to the Singapore Management University in Partial Fulfilment

of the Requirements for the Degree of Ph.D. in Economics

2022

____

Supervisor of Dissertation

____

Ph.D. in Economics, Programme Director

Essays on Social Choice and Implementation Theory

Paulo Daniel Salles Ramos

Submitted to School of Economics in Partial Fulfillment of
the Requirements for the Degree of Doctor of Philosophy in Economics

Dissertation Committee:

Shurojit Chatterji (Supervisor / Chair)
Professor of Economics
Singapore Management University

Takashi Kunimoto (Co-chair)
Associate Professor of Economics
Singapore Management University

Xue Jingyi
Associate Professor of Economics
Singapore Management University

Peng Liu
Assistant Professor of Economics
East China Normal University

Singapore Management University
2022

Essays on Social Choice and Implementation Theory

Paulo Daniel Salles Ramos

Abstract

We present here three essays. The first two are about Social Choice Theory and explore under what domain restrictions is possible to obtain a Social Choice Function that is Well Behaved as well as Monotonic, and what other characteristics can be inferred about such functions on that domain. The last essay is about Implementation Theory and explore how to obtain a more compelling form of implementation than pure Nash Equilibrium for finite environments using finite mechanisms.

Contents

Acknowledgements and Dedication

I would like to thank my family, friends and colleagues for the support on this journey, as well as my advisor and the other members of my committee for their valuable contributions to my work. It was a long journey, coming from Brazil to Singapore and changing my career from the private sector back to academia. There were many obstacles in my path, and I only managed to overcome them thanks to the help, kindness and support of several other people I met in my journey. I am truly thankful to all who helped me reach this goal.

Eu gostaria de dedicar essa tese à minha tia Cleonice, que me abrigou em sua casa há mais de dez anos atrás, no início da minha jornada acadêmica. Muito obrigado, Tia Conice.

Chapter 1

The Model

This Chapter presents the basic model and notations that will be used in Chapters 2 and 3. Chapter 4 deals with a separate model and contains its own framework and notations.

1.1 Basic Framework

Let $N = \{1, ..., n\}$ be the finite set of voters, with $n \geq 2$, and A be a finite set with $m \geq 3$ alternatives. Voters have *strict preference relations* over A. An individual voter's preference is denoted by P_i, and for any two distinct elements $x, y \in A$, the notation xP_iy reads as "x is strictly preferred to y according to the preference relation P_i". The set of all *admissible preference relations* is denoted by $\mathscr{D}$ and is called the **preference domain**. A **profile** $P = (P_1, ..., P_n)$ is a list of preference relations, one for each voter. For a given preference relation P_i, we say that an alternative a is s-th ranked in P_i if $|\{x \in A | aP_ix\}| = m - s$ for $s = 1, ..., m$, and we use the notation $r_s(P_i)$ to denote the $s-$th ranked alternative in P_i. In particular, we call the 1st ranked (or top-ranked) alternative for a given voter a **vote**, and the expression ”number of votes for an alternative” means the number of voters that had that alternative ranked at the top of their preferences in a given profile. We will use the notation $v(a,P)$ to denote the number of votes alternative a has at profile P.

A **Social Choice Function** (SCF) is a mapping $f : \mathscr{D}^n \to A$ that assigns for each profile P an alternative $f(P)$. In this paper, we are interested in SCFs that satisfy a set of properties: anonymity, unanimity and tops-only. An **unanimous** SCF has the property that $f(P) = a$ whenever $v(a,P) = n$. A SCF is said to be **anonymous** whenever $f(P) = f(P')$ where P' is any permutation [1] of the preferences in the profile P. Finally, a SCF has the **tops-only** property if $f(P) = f(P')$ whenever $r_1(P_i) = r_1(P'_i)$ for every voter $i = 1,...,n$; that is, the outcome of the SCF is completely determined by the top-ranked alternatives in each preference of the profile. Any SCF that satisfies these three properties is said to be a **Well Behaved SCF**.

Well Behaved SCFs possess an important property that will be extensively explored in later sections of this manuscript. The outcome of such SCFs can be entirely determined by the number of votes for each alternative in a profile, with profiles that have an identical distribution of votes across alternatives sharing the same outcome.

Property WB: Assume that f is a Well Behaved SCF. Then, if $\forall\, a \in A$, $v(a,P) = v(a,P')$, we must have $f(P) = f(P')$

[1] P' is said to be a permutation of P if and only if there is a bijection $h : N \to N$ such that for every $i = 1,...,n$, $P_i = P'_{h(i)}$.

We will restrict our attention to Well Behaved SCFs that satisfy one additional property, commonly called **Maskin Monotonicity (MM)**. In order to define Maskin Monotonicity we need to introduce an intermediary concept first. Let $P_i, P_i' \in \mathscr{D}$ be any two preference relations and $a \in A$ an arbitrary alternative. We then say that a **maintains its position** from P_i to P_i' if, for every alternative x, $aP_ix \Rightarrow aP_i'x$ holds, and we use the following notation $P_i \mapsto_a P_i'$ denote that a maintains its position from P_i to P_i'. We can extend this notion to profiles and say that a maintains its position from P to P' if for every voter $i = 1,...,n$, $P_i \mapsto_a P_i'$ holds, in which case, we use the notation $P \mapsto_a P'$. Then, we say that the SCF is **monotonic** (or, alternatively, that it satisfies MM) if $[P \mapsto_a P' \wedge f(P) = a] \Rightarrow f(P') = a$. In the appendix, however, we will provide another expression of this concept, one that takes advantage of our other assumptions and provides a more convenient condition to work with.

Related to the concept of MM, we have a few characteristic sets for both the domain and the SCF. First, define the set $\hat{A}$ as the subset of A^2 such that the two coordinates are different: $\hat{A} = \{(x,y) \in A^2 : x \neq y\}$. Then, call the set W_D^a the set of all pairs $(P_i, P_i') \in \mathscr{D}^2$ such that $P_i \mapsto_a P_i'$. Call the subset D^a of $\mathscr{D}$ the subset of all orderings P_i^a such that a is the top-ranked alternative in P_i^a.

Finally, define the set M_D^a as the set of all pairs $(b,c) \in \hat{A}$ such that there exists at least one $P_i^b \in D^b$ and one $P_i^c \in D^c$ so that the pair $(P_i^b, P_i^c) \in W_D^a$. In words, the pair (b,c) is in M_D^a if there is a way to change a vote from b to c while making alternative a maintain its position in this change. Conversely, a pair (b,c) is *not* in the set M_D^a if for every pair of preferences $P_i^b \in D^b$ and $P_i^c \in D^c$ there exists at least one alternative x such that aP_i^bx and xP_i^ca. We call this - an instance where $(b,c) \notin M_D^a$ - a **reversal** for alternative a.

Another concept related to Monotonicity is that of pivotal changes. A pair (b,c) is called a **pivotal change for** a if there is at least one profile P such that, by changing the vote of a single voter from b to c at that profile, the outcome of the SCF changes from a to something else. We can specify the set of all pivotal changes for an alternative as follows. First, for a social choice function f, define the set C_f as the subset of $A \times \mathscr{D}^n \times \hat{A} \times \mathscr{D}^n$ such that every element $[a, P, (b,c), P']$ satisfies the following properties:

1. $f(P) = a$ and $f(P') \neq a$;
2. $v(b,P) = v(b,P') + 1$;
3. $v(c,P) = v(c,P') - 1$;
4. $v(d,P) = v(d,P'), \forall d \neq b, c$

Then, fix an alternative a and define the set C_f^a as:

$$C_f^a = \{(b,c) \in \hat{A} | \exists P, P' \in \mathscr{D} : [a, P, (b,c), P'] \in C_f\}$$

C_f^a is the set of all pivotal changes that can occur when the outcome of the SCF is a. M_D^a and C_f^a are two sets that share some similarities. Firstly, they are both defined over the same space $\hat{A}^2$, that is, they are both sets of pairs of distinct alternatives. The first one, M_D^a, however, is delimited by the domain $\mathscr{D}$ alone and is related to possible ways in which alternative a maintains its position after a change of votes. The second, set C_f^a, depends crucially on f and describes changes of votes that cause a change in the outcome of the SCF from a to something else. As we shall see in the appendix, Monotonicity will require these two sets to be disjoint for every alternative a.

1.2 Graphs

Let $G = \langle A, \mathscr{E}^A \rangle$ denote an undirected graph where A is the set of nodes and $\mathscr{E}^A \subset A^2$ is the set of edges. We say that two nodes in a graph are **adjacent** if there is an edge on the graph connecting the two. If x is an arbitrary node of G, we use the notation $\alpha(x)$ to describe the set of adjacent nodes of x; that is, $y \in \alpha(x)$ if and only if $(x,y) \in \mathscr{E}^A$. A node in a graph is an **extreme node** if it is adjacent to only one other node.

In our theory, we will work primarily with **tree graphs**. We say that a graph is a tree if it is a connected acyclic graph. For this kind of graph, there are a few concepts, definitions and results that will be extensively used in the next sections. If the graph is a tree, and hence the path [2] connecting any two nodes is unique, we will employ the notation $\langle a,b \rangle$ to describe the set of nodes in the unique path connecting nodes a and b. Because of this, $\langle a,b \rangle = \langle b,a \rangle$ in this case. We also define a **maximal path** as any path that contains exactly two extreme nodes.

A second class of sets that are possible to be defined on a tree graph are the **spans** of a node, capturing the notion of the set of nodes in a tree graph that ”stem” from a given node from a certain direction. Let y,z be nodes on a tree graph G. We call the set $\xi(y,z)$ the **span of y from z**. It is the set of all nodes such that $x \in \xi(y,z) \Leftrightarrow y \in \langle x,z \rangle$. So the span of y from z is the set of nodes that includes y on their path to z.

Given a tree graph G and a set of nodes $B \subset G$, we can define the subgraph $G(B)$ as the unique connected induced subgraph that satisfies:

[2] A path is formally defined as a sequence of *distinct* nodes $(a_1, a_2, ..., a_k)$ such that for any $j = 1, ..., n-1$, the pair (a_j, a_{j+1}) constitutes an edge on the graph.

- The set of nodes in $G(B)$ contains B.
- Let $x,y \in B$. The graph $G(B)$ has an edge (x,y) only if (x,y) has an edge in G.
- $G(B)$ is connected.
- $y \in G(B)$ if and only if $y \in \langle x,z \rangle$ where $x,z \in B$.

In essence, a subgraph is formed by a collection of nodes B of G and all the paths that connect the nodes of B. In particular, any path $\langle a,b \rangle$ is a subgraph for $B = \{a,b\}$.

Another important concept that will be used in our work related to tree graphs is the concept of **projection of a node x in a subgraph** $G(B)$. Formally, given a subgraph $G(B) \subset G$ and a node $x \notin G(B)$, the projection of x on $G(B)$ is the unique node $\beta_x(B) \in G(B)$ such that for every node $y \in G(B)$ we have $\beta_x(B) \in \langle y,x \rangle$. For $x \in G(B)$ the projection of x on $G(B)$ is x itself.

A tree graph G where the set of nodes is equal to the set of alternatives of a Domain and a fixed node $t \in G$ together compose what we will call an **admissible pair**, denoted (G,t) for that Domain. Admissible pairs allow us to define a series of projections, which will be useful later on.

1.3 Richness Condition

We are interested in Domains that possess a minimal richness condition called MD-Connectedness. We first state the definition of MD-Connectedness for two alternatives a,b. We say these two alternatives are **MD-connected** in D if, for every alternative $c \neq a,b$ there are sequence of alternatives $\{x_j\}_{j=1}^{k}$ and $\{y_j\}_{j=1}^{l}$ such that $x_1 = y_l = a$, $y_1 = x_k = b$ and for every $j < k$ we have $(x_j, x_{j+1}) \in M_D^c$

and for every $j < l$ we have $(y_j, y_{j+1}) \in M_D^c$. We use the notation $a \approx b$ to denote that a and b are MD-connected.

Now consider a graph whose nodes are the elements of A. Two nodes in this graph constitute an edge if and only if they are MD-connected. Call this graph the **Connectivity Graph** of domain D. We can now define the MD-Connected property for D in terms of its Connectivity Graph.

Definition: *The domain D has the* ***MD-Connected*** *property if its Connectivity Graph is connected.*

We call to attention that this richness condition *does not require* for the Connectivity Graph to be a tree, only for it to be a connected graph. Some of our results rely on only a much weaker version of this richness condition, called Minimal Richness. We say that a domain is **minimally rich** when $\mathscr{D}^a$ is non-empty for all $a \in A$.

Remark 1: The MD-Connectedness condition was inspired by the strategyproofness literature. Many notions of connectivity between alternatives are used in that area. One in particular is a stronger version of MD-connectedness, **Strong Path-Connectedness**. In this condition, we say that two alternatives a, b are (strongly) connected if there are orderings $P_i, P_i' \in D$ such that a is ranked first and b in P_i, b is ranked first and a is ranked second in P_i' and every other alternative except for a and b is ranked exactly the same in both P_i and P_i'. The notion of connectedness for a domain from the connectedness between two alternatives is then constructed exactly in the same fashion as above; that is, we say that a domain is Strong Path-Connected if the Connectivity Graph (constructed using strong connectivity between alternatives, rather than MD-connectivity) is connected. We can check that if two alternatives are strongly connected, they will also be MD-connected, as every other alternative $c \neq a, b$ maintains its position when going from P_i to P_i' and from P_i' to P_i. However, this could be achieved by using other preference orderings, possibly involving lengthy chain of orderings rather than only two. As such, MD-connectedness is a much weaker restriction than Strong Path-Connectedness, particularly for domains that contain a reasonably large number of orderings. ■

Remark 2: Unfortunately, however, one of the main drawbacks of Strong Path-Connected domains remains with MD-Connectedness: this condition is incompatible with many forms of multidimensional domains. In particular, they do not work with top-separable multidimensional domains. We will try to illustrate the issue with these types of domains with an example. Suppose that a preference on computers could be decomposed into two components, software and hardware. There are many choices of software and many choices of hardware and alternatives are formed of exactly one element of the set of softwares and one element of the set of hardwares. If we think that, for each preference, the top element of a preference is composed of the ”best” elements in each component, and that somehow these characteristics are transferrable, then a bundle that has the best hardware (and some software) is always preferrable to a bundle that has the same software, but some other hardware.

The main issue that this causes with MD-Connected preferrences is that the restriction implied by top-separability will in turn imply many reversals. Imagine two alternatives, one given by (a, a) and another given by (a, b), and two preferences, $P_i^{(c,a)}$ which has (c,a) as his top-ranked alternative, and preference $P_i^{(c,b)}$, which has (c,b) as his top-ranked alternative. In preference $P_i^{(c,a)}$, alternative (a,a) is preferred to alternative (a,b), whereas in preference $P_i^{(c,b)}$, (a,b) is preferred to (a,a). As such, bundle (a,a) does not maintain its position when going from any preference with (c,a) on top to any preference with (c,b) on top; similarly, (a,b) does not maintain its position when going from any preference with (c,b) on top to any preference with (c,a) on top Hence, $[(c,a),(c,b)] \notin M_D^{(a,a)}$. While this is not in itself a problem, any top-separable domain has enough reversals like this to the point where the Connectivity Graph of the domain becomes no longer a connected graph. ■

1.4 Our additional condition

Lastly, we define the most important concept for our work before we state our results. We call it the Minimal Reversals Condition for a given MD-Connected domain.

The Minimum Reversals Condition: *Given a tree Graph G, denote by A_G^* the set of all alternatives that are not extreme nodes in G. We say that a domain satisfies the Minimum Reversals condition if there is an admissible pair (G,t) such that, for every $b \in A_G^*$, every pair $a,c \in \alpha(b)$, $a \neq c$ and every pair $x \in \xi(a,b)$, $y \in \xi(c,b)$ we have $t \notin \xi(c,b) \Rightarrow (y,x) \notin M_D^b$.*

The Minimum Reversals Condition is a holistic condition on the domain. It relates whether each alternative maintains its position through every possible change of preferences to its relative position on a tree graph G and a special node

t. As such, it cannot be fully summarized in terms of restrictions on individual preferences on the domain.

Remark 3: A given domain might satisfy the Minimum Reversals Condition for potentially several different admissible pairs. In fact, Single-Peaked Domains are an example of Domains that are compatible with several admissible pairs, as any node in the Connectivity Graph of a Single Peaked domain can play the role of t for the admissible pair. Thus, there would be as many admissible pairs compatible with the Minimum Reversals Condition for a Single Peaked domain as there are alternatives. ■

Remark 4: As a corollary of the first remark, a given domain might have many more pairs (x,y) that do not belong to the set M_D^b (i.e. more reversals) for some alternative b besides the ones specified by the Condition. As the name suggests, this is just a minimal condition. ■

1.4.1 Verifying the Minimum Reversals Condition

We show now an example on how the Condition can be verified, illustrating its functioning.

Let the diagram in figure 1.1 illustrate an admissible pair, where the node in green plays the role of alternative t for the admissible pair. Pick then an alternative and its corresponding node in the Graph to play the role of alternative b in the definition of the Condition. This alternative will divide the Graph into a number of subgraphs equal to the number of edges that node had. Each of these subgraphs corresponds to a span of the form $\xi(a,b)$, where $a \in \alpha(b)$. Figure 1.2 illustrates

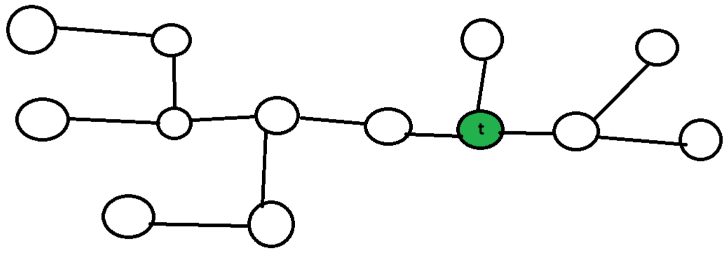

Figure 1.1: An admissible pair

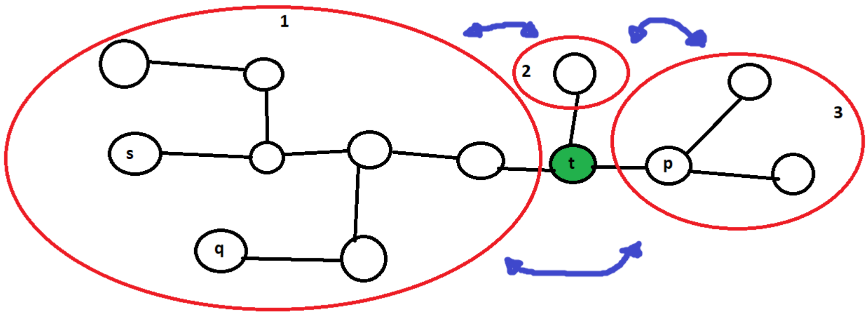

Figure 1.2: Reversals for t

this for the case where alternative t also plays the role of b, where the Graph is divided into three subgraphs. Pick now a pair of preferences, such that the top ranked alternative of each one comes from a different subgraph, for example, (p,q). It is evident that t itself is not a part of any of these subgraphs, and thus, by the Minimal Reversals Condition, alternative t cannot maintain its position when changing from either preference to the other; that is, $(p,q),(q,p) \notin M_D^t$. Note that the condition is silent about what happens between pairs in the same subgraph, like (q,s). For these cases, t can either maintain its position or not between preferences.

Now, let's consider a different case. Pick alternative x, as illustrated by Figure 1.3. This time, it divides the graph into two different subgraphs. Once more, we are concerned only with changes involving alternatives represented by nodes lying in different subgraphs, in this case, the single node in subgraph 1 (which we are also calling node 1, with some abuse in notation) and any of the nodes in subgraph 2, for example, s. We can easily see that node t is not in the same subgraph as node 1 and hence, for any preference whose top-ranked alternative lies in subgraph 2, say, s, we have $(1,s) \notin M_D^x$. Note, however, that the pair $(s,1)$ **can** belong to the set M_D^x, as t belongs to the subgraph 2, and so on this direction there are no implications made by the Minimal Reversals Condition. In other words, t can maintain its position when going from a preference whose top is s to a preference whose top is 1, but not the other way around.

We aggregate now both cases in a single example on Figure 1.4. In this case, t belongs to the subgraph 3. Thus, if we choose a pair of alternatives where one comes from subgraph 1, like x and the other comes from subgraph 2, like q we have that both $(x,q),(q,x) \notin M_D^y$. Similarly, a pair of preferences such that the

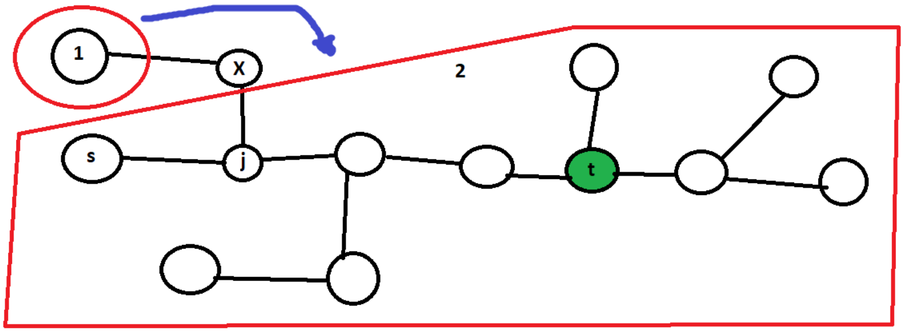

Figure 1.3: Reversals for x

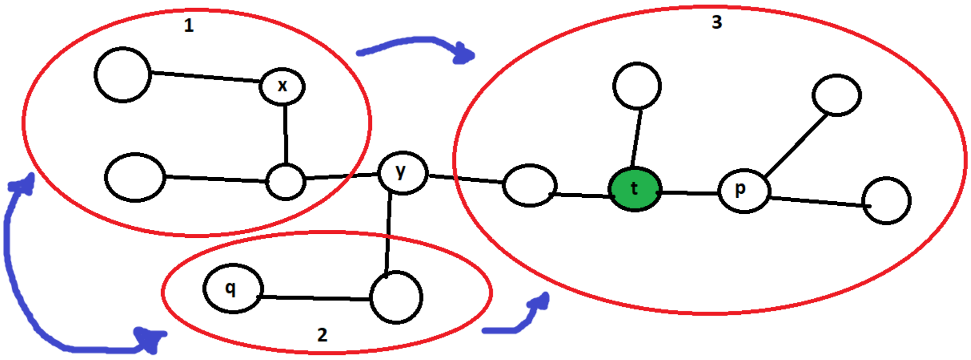

Figure 1.4: Reversals for y

top of the first one comes from either subgraph 1 or 2 and the top of the second one comes from subgraph 3 would also require alternative y to not maintain its position. For example, consider once more the pair (q,p). From that, we can readily infer that $(q,p) \notin M_D^y$. Similar arguments hold for the pair (x,p), for example. However, note that we cannot say anything about the pairs (p,x) or (p,q). As the first preference has a top-ranked alternative that is in the same subgraph as t, the Minimal Reversals Condition does not state anything about this case.

1.5 Single-Peaked Domains and its Generalizations

We will present here two generalizations of the class of Single-Peaked domains that will be used in this work, as well as an alternative (equivalent) formulation of Single-Peaked domains.

Starting with the most general to the most restrictive, we have:

Definition: *We say that a Domain $\mathbb{D}$ is a Weak Semi-Single-Peaked Domain if there is an admissible pair (G,t) such that, for all $P_i \in \mathbb{D}$ and every maximal path $\delta \in \mathbb{P}(G)$ with $r_1(P_i) \in \delta$ we have :*

- $[a_r, a_s \in \delta$ such that $a_r, a_s \in \langle r_1(P_i), \beta_t(\delta)\rangle$ and $a_r \in \langle r_1(P_i), a_s\rangle] \Rightarrow [a_r P_i a_s]$.
- $[a_r \in \delta \cap \alpha(\beta_t(\delta))$ and $\beta_t(\delta) \in \langle r_1(P_i), a_r\rangle] \Rightarrow [\beta_t(\delta) P_i a_r]$

Definition: *We say that a Domain $\mathbb{D}$ is a Semi-Single-Peaked Domain if there is an admissible pair*[3] *(G,t) such that, for all $P_i \in \mathbb{D}$ and every maximal path*

[3]The original definition of a Semi-Single-Peaked domain in Chatterji et al. (2013) uses a slightly different notion of an admissible pair, where the pair is defined as a tree graph and the set of projections of a specific node on all the maximal paths of the graph, rather than a graph and a

$\delta \in \mathbb{P}(G)$ with $r_1(P_i) \in \delta$ we have :

- $[a_r, a_s \in \delta$ such that $a_r, a_s \in \langle r_1(P_i), \beta_t(\delta)\rangle$ and $a_r \in \langle r_1(P_i), a_s\rangle] \Rightarrow [a_r P_i a_s]$.
- $[a_r \in \delta$ and $\beta_t(\delta) \in \langle r_1(P_i), a_r\rangle] \Rightarrow [\beta_t(\delta) P_i a_r]$

Definition: *We say that a Domain $\mathbb{D}$ is a Single-Peaked Domain (on a tree) if there is a tree graph G such that, for every $t \in A$, the domain $\mathscr{D}$ is a Weak Semi-Single-Peaked domain with respect to the admissible pair (G,t).*

Corollary: *For every preference P_i in a Single-Peaked domain and every pair of nodes $a_r, a_s \in A$ such that $a_r \in \langle r_1(P_i), a_s\rangle$, we must have $a_r P_i a_s$.*

The comparison of SSP domains and WSSP should be clear: the WSSP domains weaken the requirements for how the preferences behave on any path *after* $\beta_t(\delta)$ for that path. On SSP domains, those alternatives that are located away from the peak and after $\beta_t(\delta)$ must be ranked below $\beta_t(\delta)$. On the WSSP domains, the preferences between the peak and $\beta_t(\delta)$ must decrease similarly to what we see in SSP domains, but after $\beta_t(\delta)$ only *the alternative adjacent to* $\beta_t(\delta)$ must be ranked lower than it. For the case of Single-Peaked domains, since every node can be taken as a part of an admissible pair, we have that the preferences are always decreasing along the path from the peak of the preference to any other alternative. This is essentially an extension of Single-Peaked domains on linear orders - which are tree graphs with a single path - to more general structures. This particular formulation in terms of a Weak Semi-Single-Peaked domain simply makes the comparison with the other two domains more clear. Additionally, it

---

node. We adapted such notion to be compatible with the rest of our work.

will also make some properties of Single-Peaked domains more prominent on later sections.

1.6 Generalized Median Rules

We provide now a formal definition of a median function (with phantoms) on a tree, which will henceforth be called simply by ”median function” or ”median rule”. This definition has been adapted from the concept of median functions on a tree to serve as Social Choice Function.

Definition: A SCF f is a *median function on a tree* G if it satisfies all of the following properties:

- For each node $x \in G$, there is a number $f_x \geq 0$ associated with it. We call this number the *number of phantoms at node* x.
- $\sum_{x \in G} f_x = n - 1$.
- if $f(P) = x$, then $\sum_{z \in \xi(y,x)} v(z,P) + f_z < n$, for all $y \in \alpha(x)$.

We call the number f_x associated with a given node x the *number of phantoms* (or, equivalently, the *number of phantom voters*) on node x. For instance, if $f_x = 3$, we interpret this as meaning that the median function has 3 phantom voters casting votes for alternative x. Essentially, a median rule is a median of a set of $2n - 1$ nodes, $n - 1$ of which are fixed and n that are taken as an input to the function. As with any median function, the median point is chosen as the point that minimizes the sum of distances from itself to every one of the other $2n - 1$ nodes. As $2n - 1$ is always odd, the result not only exists, but it is also always uniquely determined. The last property from the list above is simply an equivalent (and, for our purposes, more convenient) way of stating this for a set with an odd number of nodes, as the only node that minimizes the sum of distances will be the one in which the sum of votes and phantoms on each of its spans adds to less than n [4]; see ORTEGA and KRISTON (2013) for more details. Lastly, an important property of median rules to take note is that, due to its construction, such rules are always Well Behaved, though not always Monotonic.

Examples We seek to illustrate now how median rules on a tree work, by showing how a particular rule selects the outcome of two different preference profiles. The rule has 10 voters and 9 phantoms. The black letters symbolize the alternative names, the notation in red $P = x$ represents that a given node has $x > 0$ phantom votes for that node (this is omitted for nodes with 0 phantoms), while the notation in green $V = y$ represents that a given node has $y > 0$ votes (again, we omit nodes with 0 votes). The figures below illustrate this same SCF, but different profiles of preferences.

We claim that in Example 1, the outcome of the SCF for that profile is alternative D. To verify that this is true, according to the last property of median rules, we then need to have that the sum of votes (both from phantoms and voters) in each of these sets must be less than 10: $\{E,F\}$, $\{K,I\}$, $\{A,B,C,G,H,J\}$. These three sets represent the three spans of nodes adjacent to node D. Indeed, the sum of votes in the set $\{E,F\}$ is 5, the sum of votes in set $\{K,I\}$ is 4, and the sum of votes in set $\{A,B,C,G,H,J\}$ is 9. The reader can verify that node D is the only node such that the span of adjacent nodes has a sum of votes less than 10,

[4] Essentially, if a particular span has n or more votes and phantoms, then it is always possible to minimize the sum of distances by moving towards that span, as this will decrease the distance to n nodes or more, but only increase the distance of $n-1$ nodes or less.

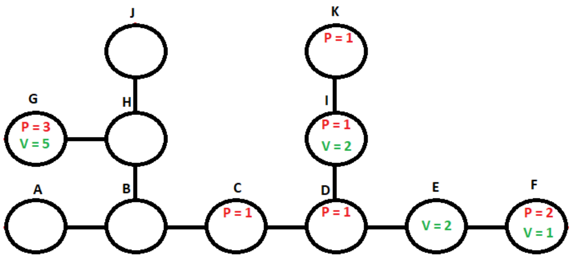

Figure 1.5: Example 1

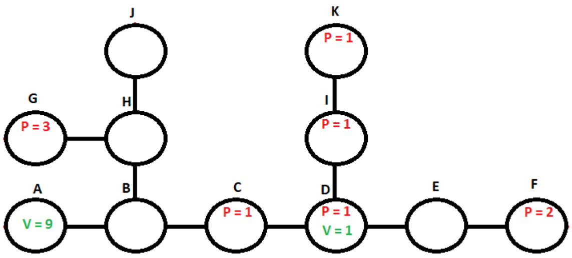

Figure 1.6: Example 2

so that the outcome is indeed unique. For instance, node C has two sets of spans: $\{A,B,G,H,J\}$ and $\{D,E,F,I,K\}$. In this second set, the sum of votes is exactly 10, and hence, C cannot be the outcome of the SCF at that profile, according to the last property of a median rule.

Example 2 represents the same SCF, as it can be seen from the placement of phantom votes remaining unchanged, but with a different profile, as seen from the change of the numbers in green. For this profile, the outcome of the SCF is alternative B. Once more, this node has three spans: $\{A\}$, $\{G,H,J\}$ and $\{C,D,E,F,I,K\}$. We can readily verify that in each of these three sets the sum of votes is below 10. Once more, this is the only node with such property. For instance, any node in the set $\{C,D,E,F,I,K\}$ will have a span that contains both the set $\{G,H,J\}$ and node A. As node A alone has 9 votes and node G has another 3, it is impossible for any alternative in the set $\{C,D,E,F,I,K\}$ to be the outcome of the SCF at that profile.

1.7 Eligible Thresholds

An important concept related to the Minimum Reversals Condition is the set of *eligible thresholds* for a MD-Connected domain that satisfies the Condition. Essentially, since a given domain might satisfy the MRC for multiple admissible pairs, the set of eligible thresholds give us an idea of how many admissible pairs are compatible with the MRC for a given domain. To make such a characterization, we fix the graph component of the admissible pair as the Connectivity Graph of the domain, and call the node component of such pair a *threshold*. The set of eligible thresholds is then the set of all nodes that can act as a threshold for the MRC along the Connectivity Graph of that domain. As it will be seen in later section, such set has important implications for the shape of the domain.

Definition Let $\mathscr{D}$ be an MD-Connected domain and the set $\tau_D \subset A$ be defined as the set of all alternatives t such that domain $\mathscr{D}$ satisfies the Minimal Reversals Condition for the admissible pair (G,t) formed by alternative t and the Connectivity graph of $\mathscr{D}$. We call the set τ_D the set of *eligible thresholds for* $\mathscr{D}$.

Examples: For all Single-Peaked domains, $\tau_D = A$, that is, every node is an eligible threshold. This is formally proved in Proposition 2 of Chapter 2. We check one more example of a domain and its associated set τ_D. Consider the domain below, composed of five alternatives, A,B,T,X,Y. We will identify the preferences on the Domain by numbers.

1	$A > B > Y > T > X$	2	$B > A > Y > T > X$
3	$B > T > A > Y > X$	4	$T > B > A > Y > X$
5	$T > X > A > Y > B$	6	$X > T > A > Y > B$
7	$X > Y > A > T > B$	8	$Y > X > A > T > B$

We can easily check that the Domain is Strong Path-Connected, (and thus, MD-Connected as well) and the Connectivity Graph is rather simple: $A \approx B \approx T \approx X \approx Y$. We also observe that $(t,a) \in M_D^b$ and $(t,y) \in M_D^x$. Now, we try to verify the Minimum Reversals Condition for each of the five possible admissible pairs involving G: $(G,a),(G,b),(G,t),(G,x)$ and (G,y). We verify quite easily that the Minimum Reversals Condition requires $(t,a) \notin M_D^b$ for both the admissible pairs (G,a) and (G,b), so neither a nor b belong to the set τ_D. Next, we also verify that the Minimum Reversals Condition requires $(t,y) \notin M_D^x$ for the both the admissible pairs (G,x) and (G,y), so neither x nor y belong to the set τ_D as well. The only remaining possible alternative is t, and indeed, the Minimum Reversals Condition holds for the admissible pair (G,t) [5]. Thus, for the domain above, $\tau_D = \{t\}$.

[5]This is verified in detail in section 2.4.2.

Chapter 2

Domains for Well Behaved Monotonic Social Choice Functions

We present here a set of necessary and sufficient conditions for an MD-Connected Domain to support a Well Behaved Monotonic Social Choice Function. We require the domain to have a minimal number of preferences in which a pair of alternatives flips their relation, and these reversals must occur in accordance to a tree graph. While this condition cannot be summarized by a set of restrictions on individual preferences, we provide two alternative characterizations that can, one that is necessary and another that is sufficient.

2.1 Introduction

In this paper we look into the problem of characterizing domains that allow us to define Social Choice Functions (SCF) that satisfy some desirable properties. Our main property of interest is Maskin Monotonicity (often abbreviated to

Monotonicity or just MM). This property has, first and foremost, a strong intuitive appeal. Muller and Satterthwaite (1977) introduced it as an axiom representing a desirable (if also somewhat intuitive) property in a SCF that casting more votes for an alternative in a ballot would never cause that alternative to be dropped once it has already been selected. Besides this intuitive appeal, there are also some methodological reasons to be interested in exploring under what conditions can Monotonic functions be obtained. In the paper that ended up naming the property, Maskin (1999) showed that Monotonicity is of fundamental importance to Nash Implementation. More recent work suggests that Monotonicity might play an important role in other implementation concepts as well, as seen on the work of Bergemann, Morris, and Tercieux (2011). Thus, this is a condition that has not only an appeal to common sense, but also is of importance to the literature.

As known from the Muller-Satterthwaite Theorem (Muller and Satterthwaite (1977)), this assumptions is not without a cost, and it will demand that some preferences must be excluded from the universe of possible preference relations held by the agents (also known as the preference domain) [1]. In fact, the more restricted the domain is, the easier it gets to formulate SCFs that satisfy almost any properties of interest, to the point where it becomes trivial in an environment where agents can only have one preference. To ensure that the domains being considered in our analysis are useful, we impose a richness condition that ensures that the domains we are considering allow for a rich representation of different points of view held by agents in a given framework. The richness condition chosen is MD-Connectedness, a condition that is based on the notion of sets

[1]The Theorem shows that Monotonicity, along with Unanimity, implies Dictatorship in the Universal domain. Thus, if one wishes to avoid Dictatorship, but retain Monotonicity and Unanimity, domain restrictions become necessary.

in which a given alternative maintain its position. Loosely speaking, for each alternative x in the Domain, it is required that it exists a few pair of preferences, with specific alternatives on top, such that alternative x maintain its position between these two preferences. This condition is a generalization of the Strong Path-Connectedness presented in Chatterji, Sanver, and Sen (2013). It draws from notions of connection between alternatives that appeared in both Aswal, Chatterji, and Sen (2003), and Chatterji and Sen (2011).

A second axiom we will impose on the SCFs studied in this paper is that they must be Well Behaved. This axiom requires the functions to satisfy three distinct properties: Anonymity, Unanimity and Tops-Only. Once again, there are both normative as well as methodological reasons to impose this axiom. Unanimity has a very intuitive appeal for any rule that seeks to be used in collective decision making. Anonymity is another often invoked property of such contexts, with appeals to fairness and equity. In particular, Anonymity implies a stronger version of non-dictatorship. These first two assumptions are broadly adopted in many branches of the literature on Social Choice. Lastly, Tops-Onlyness is a property that greatly simplifies the informational requirements for the application of any decision rule, as the rule depends exclusively on the top ranked alternative of each agent. On a methodological level, these properties make the analysis simpler and the problem tractable. Anonymity and Tops-Onlyness both introduce sources of ”rigidity” on the SCF (understood as a measure of how much the output of the function is unaffected by a change in the inputs) that, in conjunction with Monotonicity, allows us to pin down what are the critical profiles where the SCF must change the output in order to satisfy Unanimity. As we will explain in greater detail below and in subsequent sections, this is of fundamental importance to our analysis.

The first fundamental step in our approach is to take advantage of the assumptions adopted in our setup and rewrite the expression of Monotonicity in a more convenient way that makes it explicit the way the SCF and the domain need to interact with one another for the condition to be upheld. We can summarize it as saying that the richer a domain is, the less responsive the SCF needs to be in order for it to satisfy Monotonicity. Our approach to the problem can then be described as trying to find the minimum level of responsiveness that is presumed from the existence of a Well Behaved SCF and, from them, deriving the restrictions that must be placed on the domain in order for the SCF to be Monotonic while still allowing for the domain to be MD-Connected.

The final set of restrictions, collectively called the Minimum Reversals Condition in this paper, is expressed using the same language for the alternative expression of Monotonicity. This highlights the importance of that step in our analysis. The Minimum Reversals Condition involves arranging the preferences of the domain according to a tree graph and then placing constraints on which alternatives are allowed to maintain their positions when the preferences change from one area of the graph to another.

As the Minimum Reversals Condition is a condition that is expressed in terms of the domain as whole, rather than something that can be checked from properties on the individual preferences, it can be hard to verify. We then provide a set of necessary (but not sufficient) conditions that is based on constraints over the individual preferences. We call the class of domains satisfying this condition Weak Semi-Single-Peaked domains, as the conditions found are a weaker version of the requirements on preferences in a Semi-Single-Peaked domain. We also provide a way to strengthen a Weak Semi-Single-Peaked domain such that we obtain a set of sufficient (but not necessary) conditions that is also easier to express than the Minimal Reversals Condition.

The rest of the paper is organized as follows: section 2 presents the framework we will adopt for our work, introducing the basic notations and definitions used. Section 3 contains the main results of our analysis, the set of necessary and sufficient conditions for a domain to admit a Well Behaved Monotonic SCF, as well as a necessary condition for individual preferences. Section 4 illustrates these ideas by providing examples, highlighting the differences between Semi-Single-Peaked domains, the domains that satisfy the Minimum Reversals Condition and the Weak Semi-Single Peaked domains. Section 5 concludes, with the proofs of the results left for the appendix.

2.1.1 Related Literature

This paper is situated in a broad literature that investigates what sort of domain restrictions can yield positive results in the face of the known Impossibility Theorems. While the possibilities of Single Peaked domains have been known even before 1950 (see Black (1948)), the earliest attempts of characterizing possible domain restrictions that ensure the existence of rules satisfying a set of properties date back to 1977, with the work of Kalai and Muller (1977).

Our paper focuses specifically on domains that admit Monotonic rules. Bochet and Storcken (2010) started by investigating the possibilities for a Pareto-Optimal, Monotonic and Anonymous SCF to exist by placing restrictions on a single agent's preferences; their work differs from ours as we impose identical restrictions on the preferences of each agent. These different restrictions yield considerably different results for the shape of the domain. Kutlu (2009) followed a different approach and examined the conditions under which the only Unanimous and Monotonic SCFs are the dictatorial ones, hence delimiting a set of necessary restrictions to be placed on domains if one expects to find non-dictatorial rules. Our work is able to provide necessary and sufficient conditions for the case of Well Behaved SCFs. Klaus and Bochet (2013) examined under what restrictions Monotonicity and Strategy-Proofness are equivalent, thus allowing the branch of the literature that explores Monotonic rules to borrow some results from the more extensive branch on Strategy-Proof rules. Our work takes this literature forward by providing positive results on a class of problems yet unexplored, while also developing a new approach that highlights how the properties of the SCF interact with the shape of the domain on which they are defined.

We adopt some of the methodology of the literature on Strategy-Proof SCFs. Chatterji, Sanver, and Sen (2013) examine a similar problem to ours, but with a stronger richness condition and swapping Monotonicity for Strategy-Proofness. While our sufficiency results are similar, the necessity part of our work differs substantially from theirs. Not only our richness condition is weaker, but on a restricted domain environment with strict preferences, Monotonicity is also a weaker condition than Strategy-Proofness, and thus, the characterization of necessary conditions becomes harder. [2]

[2] Other notable works on the branch of Strategy-Proof domains are Chatterji and Massó (2018), Aswal, Chatterji, and Sen (2003), Demange (1982) and Moulin (1980)

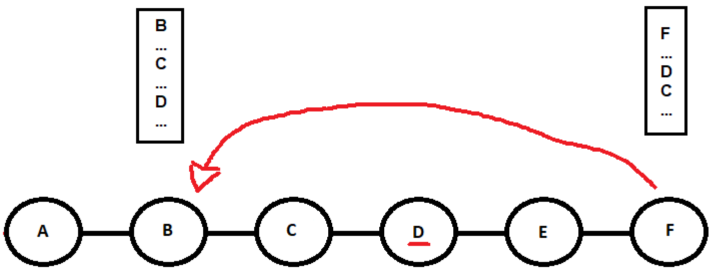

Figure 2.1: A property of Single-Peakedness.

2.2 A Preliminary Illustration

Before presenting the results, we would like to offer an heuristic example to help the reader understand the nature of our findings, as well as to relate the results to other more familiar concepts.

Consider a Single-Peaked domain over some arbitrary linear order, for instance, the linear order shown in Figure 2.1. One of its properties is that, for any d that is not an extreme node, and any f,b that lie on **opposite sides** of d, we will have $(f,b),(b,f) \notin M_D^d$. This property guarantees that any median voter SCF (which, by design, are Well Behaved functions) are Monotonic on a Single-Peaked domain. To see why, suppose that the outcome of the SCF at a given profile is d. Then, any profile involving a permutation of votes of the same side of d (say, changing a vote for a to a vote for b) will not alter the position of the median. The only way to change the outcome of the SCF from d to something else is to move votes across alternatives at different sides of d, like from f to b. But, as we saw, for any of these changes, alternative d does not maintain its position, and thus Monotonicity is always preserved.

However, as convenient as the property of Single-Peakedness is for guaranteeing that Monotonicity will hold for *any* median rule (which, by construction, are also always Well Behaved as well), this imposes heavy restrictions on the domains. Since we don't require that all Well Behaved rules be Monotonic, but rather, finding a domain that has a single Well Behaved rule that is also Monotonic is enough for our purpose, we wonder if it is possible to somehow relax the restrictions while still preserving Monotonicity for at least one median rule. If we can find such a relaxation that still preserves Monotonicity for one of these rules, we will have found a domain that is larger than a Single Peaked domain and in which we can still define a Monotonic, Well Behaved SCF, given by that median rule.

It turns out, this is possible if we pick a SCF that exhibits Veto Power in at least some of its sections. Consider the median rule for three players that has two phantoms, one at d and one at e, depicted in Figure 2.2. Alternative c can veto alternative b, as a single vote for c is enough to make it impossible for the outcome of the SCF to be b, even when all the other voters vote for b. This implies that whenever alternative b is selected as the outcome, there is no (non-phantom) voter voting for any alternative to the right of b. Since there will never be a voter voting for c when b is selected, we don't need to worry about a change of votes coming from that node - for instance, changing a vote from c to a - violating Monotonicity. In turn, this allows us to have $(c,a) \in M_D^b$ without violating Monotonicity for that function. For instance, we could have in the domain a pair of preferences like $P_i = a > b > c > d > e > f$ and $P_i' = c > a > d > e > f > b$, such that b maintain its position when going from preference P_i to preference P_i. For a different SCF - say, one where the phantoms are located at nodes a and f, so that there are no alternatives that can be vetoed - such pair of preferences could be a problem,

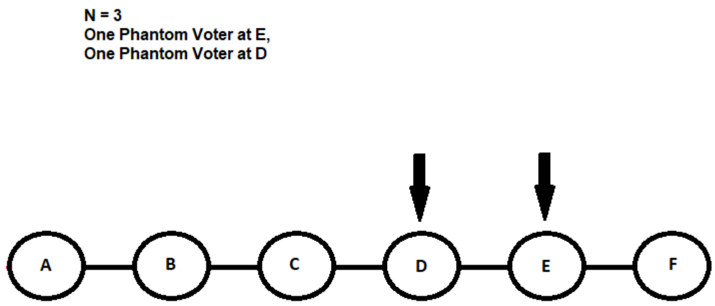

Figure 2.2: A Median Rule.

because it is possible to find a scenario where Monotonicity is violated using these preferences. For instance, if the phantoms are at nodes a and f, then we could pick a profile where player 1 has a preference with a on top, player 2 has a preference with b on top and player 3 has preference P_i'. The outcome of such profile would be b. However, if player 3 changes his preference from P_i' to P_i (while the other two players keep the same preferences), the outcome of this SCF would change from b to a, even though alternative b would maintain its position from one profile to the other. In the case where alternative b can be vetoed by alternative c, we know that such a scenario can never happen, since whenever b is selected as the outcome, there are no agents voting for c, and so a preference like P_i' does not cause any issues. We have successfully identified a form to relax the domain from Single-Peaked to something more general without losing Monotonicity for that particular SCF.

We call each of the relaxations in the form $(y,x) \in M_D^b$ (where x and y are two nodes lying on opposite sides of b) a *breach* [3]. As each breach represents a deviation from the constraints of Single-Peakedness, mapping the set of all

[3] In the proof section we will give an equivalent, but slightly different definition for a breach, one that it is more formal and more suited for our proofs, rather than for exposition.

possible breaches is equivalent to mapping the possibilities for a domain to deviate from a Single-Peaked one. Our proof follows this logic, first exploiting the fact that each breach is associated with an occurrence of Veto Power in a SCF to draw properties on how each breach must be placed on a graph. For instance, if alternative a can veto alternative b, then alternative b cannot veto alternative a. Similarly, we also find that if a can veto b and b can veto c, then a will also be able to veto c. As the number of alternatives is finite, this suggests that there must be at least one alternative that cannot be vetoed by any other alternative. In turn, characterizing the set of alternatives that cannot be vetoed by any other alternative is equivalent to characterizing the set of alternatives that **can** be vetoed, as the two sets are complements. All of these properties have counterparts as implications for the placement of breaches. For instance, we are able to conclude that there must exist (at least) one node from which we can derive the position of every breach in the domain, characterizing this set of breaches. Moreover, since we characterize the set of breaches by defining its complement, our characterization will also look like a set of statements of the form $(y,x) \notin M_D^b$, rather than $(y,x) \in M_D^b$

This insight is also further generalized and explored on Chapter 3. There, we show that, similarly to what Moulin have proved for Single-Peaked domains, the Social Choice Functions that are Well Behaved and Monotonic (and are defined on an MD-Connected domain) must also be median functions of some sort, exactly as it must be the case for Single-Peaked domains. However, since we are departing from Single-Peaked domains, there is an extra condition that is placed on these functions. The extra condition that these SCFs must satisfy is to exhibit Veto Power (which imply restrictions on the placement of phantoms) exactly in conformity with the segments where the domain deviates from a traditional

Single-Peaked one, as defined above.

2.3 Results

2.3.1 Necessary and Sufficient Conditions

We state now our main result.

Theorem 1: *If an MD-Connected domain admits a Well Behaved Monotonic SCF, then there must be an admissible pair* (G,t) *such that the domain satisfies the Minimum Reversals Condition for that admissible pair. Moreover, any domain that that satisfies the Minimum Reversals Condition for some admissible pair* (G,t) *admits a Well Behaved Monotonic SCF.*

We leave the proof to the appendix.

The Minimum Reversals Condition is then necessary and sufficient for an MD-Connected domain to admit a Well Behaved Monotonic function. While the proof will be presented later, we would like to call to attention that the sufficiency result relies on an identical version of the generalization of the median voter rule for a tree that was used on Chatterji, Sanver, and Sen (2013). Similarly to their result, MD-Connectedness is not needed for the sufficiency part of the result. The necessity part, while more convoluted, is also related to their work. As we will show next, the conditions for a domain to admit a Well Behaved Strategy Proof SCF imply the conditions for a domain to admit a Well Behaved Monotonic SCF within our richness condition. This is not a surprise, as Klaus and Bochet (2013) have shown that for domains with only strict preferences, Strategy Proofness implies MM.

We also want to emphasize a point made by Chatterji, Sanver, and Sen (2013) about the tree structure that emerges for the Connectivity graph. This was not imposed as a primitive to the model, but rather something that emerged endogenously. In principle, the Connectivity graph could assume the shape of any connected graph, like a complete graph.

2.3.2 Additional Characterization Results

The class of domains that satisfy the Minimum Reversals Condition has a complicated characterization, as it requires checking for each alternative the changes of preferences where that alternative maintains its position. It would be desirable to have a more convenient characterization of the domain, one that could be expressed in terms of restrictions on individual preferences, so that it could shed a light on how those preferences would have to behave. This is possible, if we strengthen our richness condition to Strong Path-Connectedness. When we do so, we are able to find a few properties that every individual preference in a domain that satisfies the Minimum Reversals Condition must exhibit. These properties are related to the Weak Semi-Single-Peaked domains that were presented earlier. We summarize these findings in the following Proposition:

Proposition 1: *If a Strong Path-Connected domain satisfies the Minimum Reversals Condition for an admissible pair* (G,t)*, then it is a Weak Semi-Single-Peaked domain.*

Remark 5: Not all Weak Semi-Single-Peaked domains will satisfy the Minimum Reversals Condition, even when they are Strong Path-Connected. This is illustrated on a further section through examples. As such, the class of Strong Path-Connected domains that are Weak Semi-Single-Peaked is larger than (and contains) the class of Strong Path-Connected domains that satisfy the Minimum Reversals Condition.

Remark 6: Proposition 2 requires Strong-Path Connectedness instead of MD-Connectedness. We need a stronger version of the richness condition than the one used in Theorem 2 because MD-Connectedness is a much weaker property, with few implications on the preferences of a domain, whereas Strong Path-Connectedness allows us to make inferences about the way certain alternatives must be ranked within each preference with a particular alternative on top. As such, it is possible to have a domain that satisfies the Minimum Reversals Condition, but is not a Weak Semi-Single-Peaked domain, if such domain is not rich enough to be Strong Path-Connected. Nonetheless, for domains that have enough variety in preferences to satisfy this stronger richness condition, this Proposition gives useful properties that individual preferences of such domains must exhibit. In turn, these properties allow us to state a few more results.

Proposition 2: *If a domain is Single-Peaked on the tree G, then it satisfies the Minimum Reversals Condition for all admissible pairs (G,t) that include G. Moreover, if the domain is Strong Path-Connected with Connectivity Graph G and satisfies the Minimum Reversals Condition for all admissible pairs that include G, then the domain is Single-Peaked.*

Proposition 3: *If a Strong Path-Connected domain satisfies the Minimum Reversals Condition for an admissible pair (G,t) and it is not a Single-Peaked domain, then any Well Behaved and Monotonic SCF defined on that domain violates No Veto Power*

Proposition 2 makes the connection between Single-Peaked domains and the flexibility in selecting an admissible pair to satisfy the Minimum Reversals Condition more evident. As we see here, the key difference between a Single-Peaked domain and a richer domain that still satisfies the Minimum Reversals Condition lies on having a smaller set of alternatives that can function as an admissible pair. Proposition 3, in turn, links this with the presence of Veto Power on the Well Behaved and Monotonic SCFs for that domain, as we have alluded on our preliminary illustration of the results. Lastly, we have an additional characterization result that is related to the previous two:

Proposition 4: *Let $\mathscr{D}$ be an MD-Connected domain that satisfies the Minimum Reversals Condition and $\tau_D \in A$ its associated set of eligible thresholds. If x,z are two distinct nodes such that $x,z \in \tau_D$, then for every $y \in \langle x,y \rangle$ we also have $y \in \tau_D$.*

The idea behind this last result draws on the relation of Veto Power, departures from the Single-Peaked domain and the shrinking of the set of eligible thresholds. As seen, every time that a domain deviates from a Single-Peaked domain, two things happen: i) we are able to find a set of alternatives that can be vetoed in all (Well Behaved and Monotonic) SCFs defined on that domain; ii) the set of eligible thresholds decreases in size. These two things are not unrelated: if an alternative can be vetoed by all SCFs on that domain, then that alternative cannot be an eligible threshold. As one set expands, the other set contracts. This proposition then essentially says that all the alternatives that can be vetoed by either x or z will also be vetoed by y, and so y will also be in the set of eligible thresholds. Hence, the set of eligible thresholds is ”convex” in the sense that if two distinct nodes, x and z are part of the set, then every node y in between them must also be a part of the set τ_D. This helps painting a clearer picture on how domains that satisfy the Minimal Reversals Condition must look like - something useful, given how elusive a characterization of such domains tends to be.

2.4 Examples

2.4.1 The Case of Single Peaked Domains

Our first example is a Single Peaked domain (though, for simplicity reasons, not *the* full Single-Peaked domain), to serve as a simple illustration of the ideas presented so far.

Let the set of alternatives be $\{a_k\}_{k=1}^4$. The preferences on the domain are shown in the table below, with the numbers on the first column being used to identify each preference:

1	$a_1 > a_2 > a_3 > a_4$	2	$a_2 > a_1 > a_3 > a_4$
3	$a_2 > a_3 > a_1 > a_4$	4	$a_3 > a_2 > a_1 > a_4$
5	$a_3 > a_4 > a_2 > a_1$	6	$a_4 > a_3 > a_2 > a_1$

This is an MD-Connected (and in particular a Strong Path-Connected) domain and a Single Peaked domain for the linear order $a_1 > a_2 > a_3 > a_4$. To verify the Minimum Reversals Condition, we need to specify first an admissible pair. For this, consider the graph given by the Connectivity Graph of this domain: $a_1 \approx a_2 \approx a_3 \approx a_4$, along with node a_1 as threshold. The Minimum Reversals Condition for this admissible pair then requires the following:

- Alternative a_2 cannot maintain its position when going from preferences 4, 5 or 6 to preference 1.
- Alternative a_3 cannot maintain its position when going from preference 6 to preferences 1, 2, or 3.

As it can be easily verified, these conditions are met. However, we could also use the admissible pair given by the same Connectivity Graph, but taking node a_3 instead. In this case, the Minimum Reversals Condition would then require the following:

- Alternative a_3 cannot maintain its position when going from preference 6 to preferences 1, 2 or 3, or when going from preferences 1, 2 or 3 to preference 6.
- Alternative a_2 cannot maintain its position when going from preference 1 to preferences 4, 5 or 6.

These conditions are also met. In fact, alternatives a_2 and a_4 could also be picked to create an admissible pair together with the Connectivity Graph that would also satisfy the Minimum Reversals Condition. We could even use a different graph, say, permuting the positions of a_2 and a_3 on the graph. This shows that some domains might be compatible with different thresholds for the same path. This is not a problem, and our Theorem requires only that there must exist at least one admissible pair such that the Minimum Reversals Condition is satisfied for that domain to be compatible with the existence of a Well Behaved Monotonic SCF. More generally, we have that any Single-Peaked domain satisfies the Minimal Reversals Condition, as seen in Proposition 2.

2.4.2 Weak SSP as a superset of SSP domains

We show now one example of a Weak Semi-Single-Peaked domain that is not a Semi-Single-Peaked domain. This example will also highlight a distinctive feature of this new class of domains, which is that we can find a SCF that satisfies MM without being strategy proof.

The Domain

The Domain is composed of five alternatives, A, B, T, X, Y. We will identify the preferences on the Domain by numbers.

1	$A > B > Y > T > X$	2	$B > A > Y > T > X$
3	$B > T > A > Y > X$	4	$T > B > A > Y > X$
5	$T > X > A > Y > B$	6	$X > T > A > Y > B$
7	$X > Y > A > T > B$	8	$Y > X > A > T > B$

We can easily check that the Domain is Strong Path-Connected, (and thus, MD-Connected as well) and the Connectivity Graph is rather simple: $A \approx B \approx T \approx X \approx Y$. However, this is not a semi-single-peaked Domain. We can see that by checking that there is no alternative that can act as a threshold for the domain. For a semi-single-peaked domain, there must be an admissible pair that uses the

Connectivity Graph of the domain [4], so we need only to check for thresholds on that graph, rather than checking all possible graphs.

- Alternative A cannot be the threshold, as preference 8 would violate the condition that alternatives decrease in ranking from the peak to the threshold, since $A > T$.
- Alternative B cannot be the threshold, as preference 5 would violate the condition that alternatives beyond the threshold must be ranked lower than the threshold.
- Alternative T cannot be the threshold, as preference 1 would violate the condition that alternatives beyond the threshold must be ranked lower than the threshold.
- Alternative X cannot be the threshold, as preference 1 would violate the condition that alternatives beyond the threshold must be ranked lower than the threshold.
- Alternative Y cannot be the threshold, as preference 1 would violate the condition that alternatives decrease in ranking from the peak to the threshold, since $Y > T$.

Nonetheless, we can verify the following in this domain:

- B does not maintain its position when going from preference 1 to preferences $4,5,6,7,8$, as $B > T$ in 1, but $T > B$ in $4,5,6,7,8$.
- X does not maintain its position when going from preference 8 to preferences $1,2,3,4,5$, as in 8 we have $X > T$ and $T > X$ in $1,2,3,4,5$

[4] This comes from the fact that if a domain is Semi-Single-Peaked, it admits a Well Behaved strategy-proof SCF, and, conversely, if it is a Strong Path-Connected domain that admits a Well Behaved and strategy-proof SCF, then there must be an admissible pair using the Connectivity Graph such that the domain is Semi-Single-Peaked with relation to that admissible pair.

- T does not maintain its position when going from 1, 2 or 3 to either 6, 7 or 8, as $T > X$ in the first three, but $X > T$ in the last three. Conversely, it also does not maintain its position when going from $6, 7, 8$ to $1, 2, 3$, as $T > B$ in $6, 7, 8$, but $B > T$ in $1, 2, 3$.

This is enough to check the Minimal Reversals Condition, using the Connectivity Graph and node T as an admissible pair. For this admissible pair, the Condition requires the following:

- $\{(A,T),(A,X),(A,Y)\} \notin M_D^B$
- $\{(Y,T),(Y,B),(Y,A)\} \notin M_D^X$
- $\{(A,X),(A,Y),(B,X),(B,Y),(X,A),(X,B),(Y,A),(Y,B)\} \notin M_D^T$

As we have just seen, these conditions are met for this domain. Thus, by Proposition 1, the domain is a Weak SSP domain, despite not being a SSP domain.

The SCF

The SCF is a simple one, with only two players and it takes the form of a median voter rule with a phantom voter at alternative T and the linear order of $A > B > T > X > Y$. Hence $f(P_i, P_j) = median(r_1(P_i), r_1(P_j), T)$. We can see quite easily that this function is Well Behaved.

Monotonic, but not Strategy-Proof

First we check that the function does not satisfy strategy-proofness. Indeed, look at the profile where the first player has preference 1 and the second has preference 8. In this scenario, the first player has an incentive to report having preference 8 instead, since the outcome when he reports truthfully is T, but the outcome of the misrepresentation is Y, which is preferred to T under his true preference. This is expected, as there can be no strategy-proof and Well Behaved rule on this domain, as it is Strong Path-Connected, but not Semi-Single-Peaked.

Nonetheless, this SCF does satisfy MM. To check it, we need to look at the pivotal scenarios.

- First, note that whenever the SCF changes because the alternative selected loses a vote, this does not violate MM. It is clear that changing a preference from one where the alternative was the top-ranked one to anything else it can't be the case that the alternative maintains its position, so the change is warranted.
- Hence, any changes when the outcome is either A or Y never violate MM, as these alternatives are only selected when they get both votes, so the only way to move away from them is by changing a vote for them to another alternative.
- For alternative B, it is possible to change the outcome of the SCF by changing a vote for A to either T, X or Y (while keeping the other vote in B). But as we saw earlier, B does not maintain it's position in these cases.
- A similar argument applies for the case where X is the outcome of the SCF. That can only happen when either X loses a vote or when a vote changes from Y to either A, B or T, and in none of this scenarios X maintain it's position.
- Finally, when the outcome of the SCF is T, it means that the profile is *not* one of the following: $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$, $(2,3)$, $(3,1)$, $(3,2)$, $(3,3)$, $(6,6)$, $(6,7)$, $(6,8)$, $(7,6)$, $(7,7)$, $(7,8)$, $(8,6)$, $(8,7)$, $(8,8)$. Then, for the SCF to change from T to another outcome, it must involve going from one of the profiles *not* listed above to a profile that was listed. All these

changes involve one of three scenarios: i) a change from a preference 4 or 5 to another preference, which imply a loss of votes for T, so a valid change; ii) changing a preference from $1,2,3$ to $6,7,8$, which as seen before imply that T also does not maintain its position; or iii) changing a preference from $6,7,8$ to $1,2,3$, which again, as seen before, also imply that T does not maintain its position.

Hence, the SCF satisfies MM on this Domain.

2.4.3 Weak SSP and the Minimum Reversals Condition

To illustrate the subtlety of the Minimum Reversals Condition, we will present two examples of Weak Semi-Single-Peaked domains, one that violates the Minimum Reversals Condition and one that satisfies it.

The first example is a domain composed of six alternatives, A, B, C, T, X, Y. Once more, we identify the preferences on the Domain by numbers:

1	$A>Y>B>C>T>X$	7	$T>C>X>B>A>Y$
2	$A>B>C>T>X>Y$	8	$T>X>C>B>A>Y$
3	$B>A>C>T>X>Y$	9	$X>T>C>B>A>Y$
4	$B>C>T>X>A>Y$	10	$X>Y>T>C>B>A$
5	$C>B>T>X>A>Y$	11	$Y>X>T>C>B>A$
6	$C>T>X>B>A>Y$	12	$Y>B>X>A>T>C$

We can check easily that this is a Strong Path-Connected (and thus, an MD-Connected) domain whose Connectivity Graph is given by $A \approx B \approx C \approx T \approx X \approx Y$. Moreover, using T as the threshold of the unique path of this graph, we can also easily verify this is a Weak Semi-Single-Peaked domain. Nonetheless, it *violates* the Minimum Reversals Condition since alternative B maintains its position when going from preference 1 to preference 12. As $A \in \xi(A,B), T \notin \xi(A,B)$ and $Y \in \xi(C,B)$ we have that $(A,Y) \in M_D^B$ violates the Minimum Reversals Condition.

Our second example involves eight alternatives, A, B, C, T, X, Y, O, P. The individual preferences of this domain are:

1	$A > Y > B > C > T > X > O > P$	9	$X > T > C > B > A > Y > O > P$
2	$A > B > C > T > X > Y > O > P$	10	$X > Y > T > C > B > A > O > P$
3	$B > A > C > T > X > Y > O > P$	11	$Y > X > T > C > B > A > O > P$
4	$B > C > T > X > A > Y > O > P$	12	$Y > P > B > X > A > T > C > O$
5	$C > B > T > X > A > Y > O > P$	13	$P > O > T > C > B > A > X > Y$
6	$C > T > X > B > A > Y > O > P$	14	$O > P > T > C > B > A > X > Y$
7	$T > C > X > B > A > Y > O > P$	15	$O > T > C > B > A > X > Y > P$
8	$T > X > C > B > A > Y > O > P$	16	$T > O > C > B > A > X > Y > P$

This is once more an example of Weak Semi-Single-Peaked domain. The Connectivity Graph of this domain is composed of three maximal paths: $A \approx B \approx C \approx T \approx X \approx Y$; $A \approx B \approx C \approx T \approx O \approx P$; and $O \approx P \approx T \approx X \approx Y$. Alternative T acts as the threshold in all three paths.

This domain not only satisfies the Minimum Reversals Condition, but it has an unusual connection with the last example: along the path $A \approx B \approx C \approx T \approx X \approx Y$ every alternative is ordered the same as in the last domain. In fact, if we erase alternatives O and P, the first 12 preferences of this new domain are identical to the 12 preferences of the previous one. However, here a move from preference 1 to preference 12 *does not violate the Minimum Reversals Condition*, and the reason for it is that alternative B beats alternative P on preference 1, but is beaten by it on preference 12. Thus, it was an alternative that is not on the same path that contains alternatives A, B and Y that made possible for that change of preferences to satisfy the Minimum Reversals Condition. This shows how it is necessary to look at the domain as whole, rather than at how individual preferences are restricted, even along a particular path.

Strengthening Weak Semi-Single-Peaked domains

As seen above, the restrictions implied by Weak Semi-Single-Peaked domains are also implied by the Minimal Reversals Condition, but the latter is stronger than the former, demanding more. A natural question is to ask if there is a way to strengthen the conditions of Weak SSP domains, so that they are now sufficient to guarantee that the Minimal Reversals Condition hold. Alternatively, we could also ask if we can somewhat relax the conditions on SSP domains - which do contain the Minimal Reversals - , so that we include more preferences while not losing the existence of a Well Behaved Monotonic SCF.

Indeed, it is possible to find an intermediary between both. Let $\mathbb{D}^{WSSP}_{(G,t)}$ and $\mathbb{D}^{SSP}_{(G,t)}$ denote the sets of Weak SSP domains compatible with the admissible pair (G,t) and the set of all the SSP domains compatible with the admissible* pair $(G,\beta_t(\delta))$, where $\beta_t(B\delta)$ is the function that assigns the projection of t onto every maximal path δ of G [5]. We can then state the following proposition:

[5] This is the original definition of admissible pair of Chatterji, Sanver, and Sen (2013), which we are here calling an admissible* pair. On their work, they define an admissible pair as a tree graph and the set of projections of a specific node on every maximal path of that graph. We opted

Proposition 5: *Take any* $\mathscr{D}^{SSP} \in \mathbb{D}^{SSP}_{(G,t)}$*, and any* $\mathscr{D}^{WSSP} \in \mathbb{D}^{WSSP}_{(G,t)}$*. Fix some* $p \in A, q \in \alpha(p)$ *and let* $D^p, D^q \subset \mathscr{D}^{WSSP}$ *be the subset of all orderings* P_i^p *and* P_i^q *such that p and q are the top-ranked alternatives in* P_i^p *and* P_i^q*, respectively, as defined in section 2. Then the domain* $\mathscr{D} = \mathscr{D}^{SSP} \cup D^p \cup D^q$ *satisfies the Minimal Reversals Condition.*

While we defer the proof to the appendix once more, we note that Proposition 5 corroborates our broader point on the Minimal Reversals Condition being a holistic condition that cannot be expressed solely on terms of preferences restrictions. Indeed, as shown by Proposition 5, *any* preference P_i^a that is compatible with a Weak SSP domain can be a part of a domain that satisfies the Condition, by choosing the appropriate set D^a and a suitable SSP domain to append it. The Condition is only violated when many of such preferences are found on the same domain.

2.5 Conclusion

In this paper we have attempted to characterize (rich) domains of preferences that admit Well Behaved and Monotonic social choice functions. These domains are shown to be related other variants of single-peaked domains. Unlike traditional variants of single-peaked domains, however, this new class of domains cannot be fully described by a set of restrictions on individual preferences. Instead, the characterization is more holistic and requires relating each preference to every other in the domain.

---

to define it based on the node itself, instead of its projections, and gave the original definition a slightly different name to avoid confusions.

To reach this characterization, we developed a new methodology. We translated the expression of Maskin Monotonicity to a property that links certain features of the domain to features of the social choice function. This allows us to work primarily with the social choice functions, which are objects easier to manipulate than preference domains. Once the properties of the social choice function are uncovered, we can translate them back as restrictions on the domain. This is different from the approach adopted by Chatterji, Sanver, and Sen (2013). Their method involves mapping the (even) N-agent case to a 2-agent case. With only two agents, Unanimity implies that in every profile where the two agents disagree there is a way to change the outcome of the SCF, by having a single agent to change its vote to agree with the other. In other words, with only two agents, every profile is either unanimous or pivotal (or both). This makes it much easier to check if the more classic definition of MM (as opposed to the alternative one we employed in our proofs) holds for each profile. In contrast, our approach is more transparent, as we work directly with the properties of the existent social choice functions, instead of relying on a two-player representation of them. We believe that our approach, when adapted to express strategy-proofness (instead of Monotonicity) as a set of joint restrictions on domains and SCFs, could also be applied to solve the unresolved case of an odd number of players on that paper.

We conjecture that there is a connection between the number of possible admissible pairs and the SCF. The more admissible pairs a domain has that satisfy the Minimal Reversals Condition, the greater is the number of SCFs that will be Well Behaved and monotonic on that domain. For example, for generalized median voter rules, we believe that the phantom voters can only be placed on alternatives such that there is an admissible pair where that alternative plays the role of t. This is expected, as more admissible pairs compatible with the domain means that there are many reversals on that domain, which in turn implies that there are less restrictions placed on the shape of the SCFs for that domain, via the relation between the M_D and C_f sets. Moreover, the Lemmas in the appendix reveal already some properties that the SCF must exhibit. For instance, *any* Well Behaved Monotone SCF defined on such a domain must behave similarly to a median rule in the sense that changing votes between alternatives that lie on the same "side" of the outcome (which, on a tree graph, it would correspond to alternatives lying on the same subgraph) does not change the outcome of the SCF.

While the necessity part of our paper deviated substantially from Chatterji, Sanver, and Sen (2013), the proof of sufficiency employed exactly the same SCF. The expression of that SCF is convenient for the properties we wanted to check on that section, but it could also be expressed as a particular version of a median voter rule on a tree. Once more, given how single-peakedness and median rules are related, this is unsurprising. In fact, all the additional properties of the SCFs that were implied by Lemmas 1, 2, 3 and 7 are properties shared by median rules on a tree.

Lastly, in this paper we also described a way to express the holistic restrictions implied by the Minimal Reversals Condition in terms of a restriction on individual preferences. We found that some of the restrictions implied by the Condition can be expressed by a set of of properties of each preference relation, naming the class of domains where these restrictions hold Weak Semi-Single-Peaked domains. However, the restrictions implied by these domains are not sufficient to enforce the Minimal Reversals Condition. We provided then a method to create domains that are compatible with the Condition that involves checking restrictions only on the individual preferences, by combining some preferences that came from a Weak Semi-Single-Peaked domain with preferences that satisfy the more stringent requirements of Semi-Single-Peaked domains.

Chapter 3

Characterizing Well Behaved Monotonic Social Choice Functions

3.1 Introduction

In this paper, we seek to characterize anonymous, unanimous, tops-only (henceforth referred collectively as Well Behaved) and Monotonic Social Choice Functions for domains that satisfy a particular richness condition. In a previous work (Ramos 2022), a complete characterization of the domains that can support the existence of a Social Choice Function (SCF) exhibiting the aforementioned properties was provided. This paper is a counterpart to that work, now providing a characterization for the (Well Behaved and Monotonic) SCFs that exist on such domains.

This problem is an expansion of the work done by Moulin (1980) on Single-Peaked domains. In that paper, the author looks at the problem of characterizing the SCFs that satisfy tops-onlyness, anonymity, efficiency and strategy-proofness on a Single-Peaked domain. Here, we are relaxing those hypothesis in a number of ways. On the side of the SCF, we are relaxing efficiency for unanimity and strategy-proofness for monotonicity [1]. On the side of the domain, we are relaxing the assumption of Single-Peakedness by instead assuming only that the domain satisfies a particular richness condition, as well as the necessary conditions for the existence of a Well Behaved and Monotonic SCF, as outlined in Chapter 2.

Our results are that the universe of possible SCFs satisfying our criteria does not expand as we relax these assumptions, but rather, it contracts. For Single-Peaked domains, any median rule retains the property of monotonicity, and they are the only Well Behaved rules to exhibit that property in such domains [2]. In contrast, for domains that are not Single-Peaked, only a few of those median rules still keep the property of Monotonicity. This exemplifies the nature of the trade-off between flexibility of the SCF and the richness of the domain encapsulated by Monotonicity, as seen in Chapter 2. Thus, by relaxing the assumption of Single-Peaked domains, we limit the number of Well Behaved rules that keep the property of being Monotonic. We show in this paper that the exact form which the median rules must take depends intrinsically on how Single-Peakedness was relaxed.

The rest of the paper is organized as follows: Section 2 outlines the model for our work, defining most concepts that will be used in our work, as well as the notation. Section 3 contains the main result of the paper, the characterization of the SCFs that have the desired properties. Section 4 concludes with some applications of those results. All proofs are left for the Appendix.

[1] See Klaus and Bochet (2013) for a demonstration that strategy-proofness implies monotonicity when preferences are strict

[2] This is a corolary of Proposition 2 of the previous chapter, as well as Theorem 2, proved in this chapter

3.2 A Preliminary Illustration

We want to illustrate the main ideas in this paper through an example. Consider the following domain, composed of six alternatives, a, b, c, d, e, f, and with individual preferences identified by numbers:

1	$a > b > e > c > d > f$	6	$d > c > e > f > b > a$
2	$b > a > e > c > d > f$	7	$d > e > c > b > a > f$
3	$b > c > f > d > a > e$	8	$e > d > c > b > a > f$
4	$c > b > f > d > a > e$	9	$d > f > c > b > e > a$
5	$c > d > e > f > b > a$	10	$f > d > c > b > e > a$

We can see easily that this is a Strong Path-Connected domain, with a forked Connectivity Graph that has three maximal paths: $a \approx b \approx c \approx d \approx e$, $a \approx b \approx c \approx d \approx f$ and $e \approx d \approx f$. Using this Connectivity Graph, we can also see that the domain is *not* a Single-Peaked domain, but it is a Weak Semi-Single-Peaked domain for either of the thresholds a, b, c. By following the steps outlined in Chapter 2, we can also verify that this domain satisfies the Minimum Reversals Condition, and the set of admissible thresholds is $\tau_D = \{a, b, c\}$. By Theorem 1, then, there is at least one Well Behaved and Monotonic SCF for this domain. We seek now to further uncover some more properties for such functions.

One way to uncover many features of the Well Behaved and Monotonic functions on that domain is to exploit the relationship between M_D and C_f sets implied by Monotonicity, as seen in Appendix A1. Since we need only the domain to characterize the M_D sets, this allows us to have at least a partial characterization on what pairs *cannot* be on each C_f set, which in turn, tells us sections where **any** of the (Monotonic and Well Behaved) SCFs for that domain cannot change its outcome. Thus, we once more look into the M_D sets for this domain, this time seeking which pairs do belong to those sets.

Strong Path-Connectedness gives us an easy start to this task. We know that if two alternatives, x and y, are strongly connected, then the pairs $(x,y),(y,x)$ belong to the set M_D^z for each other alternative z on that domain. Thus, for instance, on the set M_D^a we have the following pairs: (b,c), (c,b), (c,d), (d,c), (d,e), (e,d), (d,f), (f,d). This list is not exhaustive; there are more pairs that also belong to the M_D^a set, like (c,e). However, for the purposes of characterizing the pairs that *cannot* belong to the set C_f^a, this list is enough. The reason for this is as follows: (b,c) *not* being on the set C_f^a implies that, when the outcome of the SCF is a, no change of votes from alternative b to alternative c will ever alter the outcome of the SCF. Similarly, the pair (c,d) not being on that set means that when the outcome is a, we can also change votes from c to d without altering the outcome. But then, essentially, this means that we could change a vote from b to d without altering the outcome as well, if we first change it from b to c and then, since the outcome is still a, from c to d. Even if the pair (b,d) is not on the set M_D^a, it is enough that the pairs (b,c) and (c,d) are on that set to conclude that the pair (b,d) must not be on the set C_f^a. In other words, if the path between any two alternatives x,y does not contain alternative z, then $(x,y),(y,x) \notin C_f^z$, even if we don't have $(x,y),(y,x) \in M_D^z$. When this is applied to an extreme node, like a, we are able to conclude for any two alternatives x,y different from a, we have $(x,y),(y,x) \notin C_f^a$: this means that *no change of votes between any alternatives that are not a can change the outcome from a to anything else*; the only way the outcome can be changed from a to something else is if some voters change their votes from a to some other alternative.

This logic holds for any of the other extreme alternatives, like f and e: whenever these alternatives are selected, permutations of votes between other alternatives will not impact the outcome. We can also apply the same logic for interior alternatives, like c. Take any two alternatives that are on the same side of c, like a and b or f and e, but not b and d. Strong Path-Connectedness will imply that these pairs, like (e,f) and (f,e), do not belong to C^c_f. Hence, once more, no permutation between votes for these alternatives will change the outcome of the SCF from c to something else. This is a property that is shared by median rules with phantoms: the outcome of these rules is always given by the alternative that is in a certain position in a profile. Any permutation of votes does not change the outcome of these rules, unless the permutation also changes which alternative is in a certain position (like the median) for that profile.

However, not every median rule will be compatible with our domain. Once more, look at the set M^d_D. We check that besides the pairs implied by Strong Path-Connectedness, the following pairs belong to this set: (a,e), (b,e), (b,f) and (c,f). Besides the direct implications of this to the set C^d_f, we can apply the same reasoning outlined above to draw more conclusions about this set. For instance, even though (a,f) does not belong to the set M^d_D, this pair cannot belong to C^d_f either, since the pairs (a,b) and (b,f) belong to M^d_D. Hence, in this domain, the outcome of any Monotonic median rule cannot change from d when there is a change of votes from a, b or c to either e or f. This excludes a few possible median rules. For instance, let f be the median rule for two players that has one phantom at node d. Then, the outcome of the profile where the first voter has preference 1 and the second voter has preference 8, $f(1,8)$, is d, while $f(8,8) = e$. This would violate Monotonicity, as alternative d maintains its position when going from preference 1 to preference 8, and hence, such SCF is not compatible with the domain above. The key to understand which SCFs are compatible lies in Proposition 3, as well as on the ideas presented on section 2.2: since this domain is not Single-Peaked, we must have that every SCF (that is Well Behaved and Monotonic) on this domain must exhibit a particular form of Veto Power. Specifically, alternative d must be vetoed by alternatives a, b or c. Only when this happens we can be sure that any time that alternative d is the outcome there will be no voters able to change their votes from a, b or c to e or f (which could cause the outcome of the SCF to change from d to something else). To achieve this, we cannot have any phantoms to be placed on alternatives d, e or f, since this would prevent alternative d from being vetoed by alternatives a, b and c. The only nodes where phantoms can be placed end up being the nodes in the set τ_D: a, b and c. Hence, we can conclude that the SCFs that exhibit the desired properties on our domain must be median rules where the phantoms are only placed on the nodes of the set τ_D. As it will be seen in Theorem 2, these are not features exclusive to the domain we selected, but rather to all domains that are MD-Connected and satisfy the Minimum Reversals Condition.

3.3 Results

We state now our main result:

Theorem 2: *Let $\mathscr{D}$ be an MD-Connected domain that satisfies the Minimum Reversals Condition and $\tau_D \in A$ its associated set of eligible thresholds. Then, the following two statements are equivalent:*

- *The SCF f is Well Behaved and Monotonic on $\mathscr{D}$.*
- *The SCF f is a median function with phantoms on the Connectivity Graph of $\mathscr{D}$, and for all $x \notin \tau_D$ we have $f_x = 0$.*

The proof is left to the Appendix. This result establishes formally a link that was suggested in the previous Chapter between median rules and the domains that satisfy the Minimal Reversals Condition. It is also useful to contrast it with the pioneering results of Moulin (1980) for Single Peaked Domains. When the domain is Single-Peaked, any median rule will be Monotonic when defined on that domain, and similarly, any Well Behaved and Monotonic rule defined on such domain will be a median rule of some variety. As such, it is no surprise that when we relax some of the restrictions of Single-Peaked domains (as long as the domain is still MD-Connected), the only Well Behaved and Monotonic rules are still medians. However, not all median rules will be Monotonic on such domains. As we relax some of the restrictions, the only rules that still preserve that property are the ones where the phantom voters are placed in particular alternatives.

3.4 Applications

Some applications of the main result.

3.4.1 More Comprehensive Necessary Conditions for Strategy-Proofness

An important implication of these results is that, for domains with only strict preferences, as strategy-proofness implies Monotonicity (Klaus and Bochet (2013)), then for Semi-Single-Peaked domains that are also MD-Connected the only Well Behaved and Strategy-Proof SCF are also median rules following the same restrictions on the placement of phantom voters. Since MD-Connectedness is itself a more general restriction than Strong Path-Connectedness, this result complements the findings of Chatterji, Sanver, and Sen (2013) for Semi-Single-Peaked domains and strategy-proofness. In particular, it allows us to provide necessary conditions for strategy-proofness that work both for odd and even number of players.

Proposition 6: *Let $\mathscr{D}$ be a Strong Path-Connected domain, with Connectivity Graph given by G. If there is a $t \in A$ such that $\mathscr{D}$ satisfies the Minimum Reversals Condition for the admissible pair (G,t), but no $t \in A$ such that $\mathscr{D}$ is a Semi-Single-Peaked domain with respect to (G,t), then every median rule that is Monotonic in $\mathscr{D}$ is not Strategy-Proof.*

Corollary: *Semi-Single-Peakedness is a necessary condition for a Strong Path-Connected domain to exhibit any SCF that is Well Behaved and Strategy Proof.*

The proof of Proposition 6 is left for the Appendix. As for the Corollary, it comes from a series of other results. First, assume, by way of contradiction, that there exist a domain $\mathscr{D}$ that has only strict preferences, is Strong Path-Connected, not Semi-Single-Peaked and on which you can define a Well Behaved and strategy-proof rule f. First, because you have only strict preferences, due to Klaus and Bochet (2013), we know that strategy-proofness implies Monotonicity, so f is also Monotonic. Secondly, as Strong Path-Connectedness implies MD-Connectedness, we can apply Theorem 1 and conclude that $\mathscr{D}$ satisfies the Minimum Reversals Condition. Then, we can apply Theorem 2 and conclude that f is a median rule with phantoms. However, that contradicts Proposition 6, as there is no median rule with phantoms that is strategy-proof on a domain that is not Semi-Single-Peaked. Hence, Semi-Single-Peakedness is a necessary condition for strategy-proofness.

The notable feature of such result is that it no longer requires the number of players to be even, something that was necessary for the original result in Chatterji, Sanver, and Sen (2013). We are able to improve on their results thanks to the additional information pertaining the characterization of the SCFs on the domain. With this characterization, we are able to construct a particular profile where there is an opportunity for manipulation, regardless of the number of players.

3.4.2 Implications for Single-Peaked Domains

Given the relevance of Single-Peaked domains for the literature, we discuss now in detail the implications of Theorem 2 for such domains. First, we have that any median rule is Monotonic on a Single-Peaked domain. This is a direct consequence from the fact that the set of eligible thresholds for a Single-Peaked domain is equal to the entire set of alternatives, $\tau_D = A$, but such conclusion could also be reached from the classic result of Moulin (1980) on strategy-proofness and the result from Klaus and Bochet (2013) that strategy-proofness (under strict preferences) implies Monotonicity.

The novelty comes when we strengthen the assumptions on the domain. If in addition to Single-Peakedness we also assume that the domain is MD-Connected, then, by Theorem 2, median rules are the only kinds of Well Behaved rules that are also Monotonic on such domains, giving us a version of Moulin's result for Monotonicity instead of strategy-proofness. Moreover, if we further strengthen the richness condition to Strong Path-Connectedness, then, by Propositions 2 and 3, we have that Single-Peaked domains are the only domains where every median rule is Monotonic. Under that richness condition, any deviation from Single-Peakedness will result in Monotonicity requiring particular alternatives to be vetoed by other alternatives, which in turn will exclude certain median rules from the universe of Monotonic, Well Behaved SCFs for that domain.

3.5 Conclusion

In this paper, we characterized the set of Monotonic and Well Behaved Social Choice Functions on MD-Connected domains. These functions must all be median rules with phantoms. Moreover, the placement of the phantoms must coincide with the set of admissible thresholds of a domain that satisfies the Minimum Reversals Condition (and all domains that possess a function with such characteristics satisfy it). Lastly, we showed that any median rule with phantoms satisfying this condition will be Monotonic on Minimum Reversal domains. These results are achieved by exploiting the connections between breaches and veto power, as well as building up on many properties of the SCF uncovered in the previous chapter.

This result extend the findings of Moulin (1980) on strategy-proofness on Single-Peaked domains to also include Monotonicity. We also go one step further and show that Single-Peaked domains are the only class of domains where every median rule with phantoms is Monotonic, under the assumption of Strong Path-Connectedness. These results build upon the similarity between Monotonicity and strategy-proofness when only strict preferences are considered.

Lastly, we are also able to exploit this connection between the two aforementioned properties to obtain another result on the literature of strategy-proofness. As Strong Path-Connectedness implies MD-Connectedness and strategy-proofness implies Monotonicity, we are able to invoke the results of Theorem 2 to show that Semi-Single-Peakedness is a necessary condition for a domain to admit a Well Behaved and strategy-proof SCF, regardless of the number of agents. This expands the earlier results of Chatterji, Sanver, and Sen (2013), which covered only the case of an even number of agents.

Chapter 4

Compellingness in Nash Implementation

4.1 Introduction

This last Chapter presents a model that is independent from the framework developed in the previous three Chapters. It is a stand-alone essay on implementation theory, and it was co-authored with Shurojit Chatterji and Takashi Kunimoto.

The theory of implementation attempts to answer two questions. First, can one design a mechanism that successfully structures the interactions of agents in such a way that, in each state of the world, they always choose actions which result in the socially desirable outcomes for that state? Second, if agents possess information about the state and interact through a given mechanism, what properties do the resulting outcomes, viewed as a map from states to outcomes (and called social choice function (henceforth, SCFs), possess? In answering these, the consequences of a given mechanism are predicted through the application of game theoretic solution concepts.

In this paper we adopt Nash equilibrium as a solution concept, consider complete information environments with two agents, and ask if a SCF is implementable, i.e., when we can design a mechanism in which "every" Nash equilibrium outcome induces outcomes consistent with the SCF. Although the literature claims to care about all equilibria, it often ignores mixed strategy equilibria and only focuses on pure strategy equilibria. Jackson (1992) provides the most forceful argument for why the omission of mixed strategy equilibria brings about a serious consequence. In his Example 4, Jackson (1992) constructs a two-person environment and a SCF such that (i) there is a finite mechanism that pure Nash implements the SCF; and (ii) every finite pure Nash implementing mechanism always has a mixed strategy equilibrium that gives a lottery that is preferred by both agents to the outcome of the SCF. Thus, if we insist on using finite mechanisms, which is to be anticipated in an environment with finite number of alternatives and agents, we must question why agents would limit themselves to playing only pure strategies, particularly when there is a mixed strategy equilibrium that would be strictly preferred by both of them than any pure strategy equilibrium. We call such a mixed strategy equilibrium *compelling*.

We consider a two-person finite environment with respect to an SCF on which we impose Condition P+M, which delineates a set of conditions where it is always possible to construct a finite mechanism which pure Nash implements the SCF without compelling mixed strategy equilibria. We call such a notion of implementation *compelling implementation*. Importantly, compelling implementation might admit other mixed equilibria which result in outcomes that are not consistent with the those induced by the SCF. However, since such a mixed strategy equilibrium is not compelling, at least one agent will be worse off with any pure strategy equilibrium outcome. Hence, compelling implementation is considered a compromise between pure Nash implementation where only pure strategy equilibria considered and mixed Nash implementation where all mixed strategy equilibria are fully considered.

Our solution has several notable features: In addition to the information about the agents' ordinal strict preferences, what is required is the information regarding the smallest difference in cardinal utilities between any two distinct alternatives. We can think of such information as the smallest unit in which the agents' utilities are measured. As long as that unit of measure is positive, we can construct a mechanism that compellingly implement the SCF. In this sense, while our compelling implementation is not completely ordinal, it can be made ordinal as much as it can possibly be. Our mechanism is finite so that it does not use the *integer games* which are often considered a questionable devise in the literature.[1] The use of transfers can completely be dispensable, which exhibits a stark contrast with Chen, Kunimoto, Sun, and Xiong (2022) who establish mixed Nash implementation by a finite mechanism in environments with transfers and lotteries.

We organize the rest of the paper as follows: Section 2 presents the environment, notation, mechanism and solution concepts, as well as a small discussion on Maskin Monotonicity. Section 3 revisits Example 4 of Jackson (1992), which motivates our inquiry. Section 4 slightly modifies the environment in Section

[1] In the integer game, each agent announces some integer and the person who announces the highest integer gets to name his favorite outcome.

3 and presents an illustration of the main result. Section 5 contains the main result of the paper together with a specific family of mechanisms that can achieve compelling implementation under Condition P+M. Section 6 shows that part of Condition P+M is necessary for pure Nash Implementation and the other part of Condition P+M is indispensable for compelling implementation in the sense that our mechanism fails to achieve compelling implementation when the other part of Condition P+M are not satisfied. Section 7 compares our mechanism with the canonical mechanism of Moore and Repullo (1990), showing that there is a class of environments in which the canonical mechanism of Morre and Repullo (1990) admits a compelling mixed strategy equilibrium. Lastly, section 8 concludes.

4.2 Preliminaries

Throughout the paper, we consider an environment in which there are only two agents. Let Θ be the finite set of states. It is assumed that the underlying state $\theta \in \Theta$ is commonly certain among the agents. This is the complete information assumption. Let A denote the set of social alternatives, which are assumed to be independent of the information state. We shall assume that A is finite, and denote by $\Delta(A)$ the set of probability distributions over A. Associated with each state θ is a preference profile $\succeq^\theta = (\succeq_i^\theta)_{i \in N}$ where $\succeq_i^\theta$ is agent i's preference relation over A at θ. We write $a \succeq_i^\theta a'$ when agent i weakly prefers a to a' in state θ. We also write $a \succ_i^\theta a'$ if agent i strictly prefers a to a' in state θ and $a \sim_i^\theta a'$ if agent i is indifferent between a and a' in state θ. We can now define an *environment* as $\mathscr{E} = \left(\{1,2\}, A, \Theta, (\succeq_i^\theta)_{i \in \{1,2\}, \theta \in \Theta}\right)$, which is implicitly understood to be commonly certain among the agents. Throughout the paper, we assume that the environment

$\mathscr{E}$ admits strictly preferences only, that is, for any $i \in N,\ \theta \in \Theta$, and $a, a^{'} \in A$, it follows that either $a \succ_i^{\theta} a^{'}$ or $a^{'} \succ_i^{\theta} a$.

We assume that any preference relation $\succeq_i^{\theta}$ is representable by a von Neumann-Morgenstern utility function $u_i(\cdot, \theta) : \Delta(A) \to \mathbb{R}$. We say that $u_i(\cdot, \theta)$ is consistent with $\succeq_i^{\theta}$ if, for any $a, a^{'} \in A$, $u_i(a, \theta) \geq u_i(a^{'}, \theta) \Leftrightarrow a \succeq_i^{\theta} a^{'}$. We denote by $\mathscr{U}_i^{\theta}$ the set of all possible cardinal representations $u_i(\cdot, \theta)$ that is consistent with $\succeq_i^{\theta}$. We formally define $\mathscr{U}_i^{\theta}$ as follows:

$$\mathscr{U}_i^{\theta} = \left\{ u_i(\cdot, \theta) \in [0,1]^{|A|} \;\middle|\; \begin{array}{l} u_i(\cdot, \theta) \text{ is consistent with } \succeq_i^{\theta}; \min_{a \in A} u_i(a, \theta) = 0; \\ \text{and } \max_{a \in A} u_i(a, \theta) = 1 \end{array} \right\},$$

where $|A|$ denotes the cardinality of A. Let $\mathscr{U}^{\theta} \equiv \times_{i \in N} \mathscr{U}_i^{\theta}$ and $\mathscr{U} \equiv \times_{\theta \in \Theta} \mathscr{U}^{\theta}$. We denote any subset of $\mathscr{U}_i^{\theta}$ by $\hat{\mathscr{U}}_i^{\theta}$ and any subset of $\mathscr{U}^{\theta}$ by $\hat{\mathscr{U}}^{\theta}$, respectively

The planner's objective is specified by a *social choice function (henceforth, SCF)* $f : \Theta \to \Delta(A)$ a *social choice function*. Although many papers deal with multi-valued social choice correspondences in the literature of Nash implementation, we focus only on single-valued SCFs.

4.2.1 Mechanism and Solution Concepts

Let $\Gamma = ((M_i)_{i \in \{1,2\}}, g)$ be a two-person finite mechanism where M_i is a nonempty **finite** *set of messages* available to agent i; $g : M \to A$ (where $M \equiv \times_{i \in N} M_i$) is the *outcome function*. At each state $\theta \in \Theta$ and profile of representations $u \in \mathscr{U}$, the environment and the mechanism together constitute a *game with complete information* which we denote by $\Gamma(\theta, u)$. By $\Gamma(\theta)$ we mean the game in which the preference profile $(\succeq_i^{\theta})_{i \in N}$ is commonly certain among the agents so that any representation $u \in \mathscr{U}$ is admissible. Note that the restriction of M_i to a finite set rules out the use of integer games (See, for example, Maskin (1999)).

Let $\sigma_i \in \Delta(M_i)$ be a mixed *strategy* of agent i in the game $\Gamma(\theta, u)$. A strategy profile $\sigma = (\sigma_1, \sigma_2) \in \Delta(M_1) \times \Delta(M_2)$ is said to be a mixed-strategy *Nash equilibrium* of the game $\Gamma(\theta, u)$ if, for all agents $i \in \{1, 2\}$ and all messages $m_i \in \operatorname{supp}(\sigma_i)$ and $m_i' \in M_i$, we have

$$\sum_{m_j \in M_j} \sigma_j(m_j) u_i(g(m_i, m_j), \theta) \geq \sum_{m_j \in M_j} \sigma_j(m_j) u_i(g(m_i', m_j), \theta).$$

A *pure-strategy* Nash equilibrium is a mixed-strategy Nash equilibrium σ such that each agent i's mixed-strategy σ_i assigns probability one to some $m_i \in M_i$. Let $NE(\Gamma(\theta, u))$ denote the set of mixed-strategy Nash equilibria of the game $\Gamma(\theta, u)$ and $pureNE(\Gamma(\theta))$ denote the set of pure strategy Nash equilibria of the game $\Gamma(\theta)$. As far as we are only concerned with pure strategy equilibria, we only need ordinal preferences so that we can write $pureNE(\Gamma(\theta))$. We also define

$$NE(\Gamma(\theta)) = \bigcup_{u \in \mathscr{U}^\theta} NE(\Gamma(\theta, u))$$

as the set of all Nash equilibria of the class of games $\Gamma(\theta, u)$ across all possible representation $u \in \mathscr{U}^\theta$. Since it does not depend upon cardinal utilities, $NE(\Gamma(\theta))$ is defined only in terms of ordinal preferences. We introduce the notion of pure strategy Nash implementation.

Definition 1 *An SCF f is* ***pure Nash implementable*** *if there exists a finite mechanism $\Gamma = (M, g)$ such that for every state $\theta \in \Theta$, (i) $pureNE(\Gamma(\theta)) \neq \emptyset$; and (ii) $m \in pureNE(\Gamma(\theta)) \Rightarrow g(m) = f(\theta)$.*

For each $\theta \in \Theta$, define

$$\text{supp}(NE(\Gamma(\theta))) = \left\{ m \in M | \ \exists u \in \mathscr{U}^{\theta} \ \text{ such that } \ \exists \sigma \in NE(\Gamma(\theta, u)) \ \text{ with } \sigma(m) > 0 \right\}$$

as the set of message profiles which can be played with positive probability in a Nash equilibrium of the game $\Gamma(\theta, u)$ associated with some $u \in \mathscr{U}^{\theta}$. We next introduce a notion of mixed strategy Nash implementation.

Definition 2 *An SCF f is* ***mixed Nash implementable*** *if there exists a finite mechanism $\Gamma = (M, g)$ such that for every state $\theta \in \Theta$, (i) pure$NE(\Gamma(\theta)) \neq \emptyset$; and (ii) $m \in$ supp$(NE(\Gamma(\theta))) \Rightarrow g(m) = f(\theta)$.*

This definition is proposed by Maskin (1999) but it is different in that Maskin (1999) allow for infinite mechanisms. The notion of mixed Nash implementation is stronger than that of pure Nash implementation because the former guarantees that every message profile that can be played with positive probability in a Nash equilibrium results in the outcome specified by the SCF. Since it is extremely demanding to take care of all mixed strategy equilibria, we propose a notion of compellingness, which singles out the class of mixed equilibria on which we give a serious consideration.

Definition 3 *Fix $\theta \in \Theta$ and $u \in \mathscr{U}^{\theta}$. We say that σ is a* ***compelling*** *mixed strategy equilibrium of the game $\Gamma(\theta, u)$ if, for any $m \in$ pure$NE(\Gamma(\theta))$ and $i \in \{1, 2\}$,*

$$\sum_{\tilde{m} \in M} \sigma(\tilde{m}) u_i(g(\tilde{m}), \theta) \geq u_i(g(m), \theta),$$

with at least one strict inequality for some $i \in \{1, 2\}$.

For each $\theta \in \Theta$, we denote by $\hat{\mathscr{U}}^{\theta}$ an arbitrary subset of $\mathscr{U}^{\theta}$. We write $\hat{\mathscr{U}} \equiv \times_{\theta \in \Theta} \hat{\mathscr{U}}^{\theta}$. We now introduce what we call *compelling implementation* which takes $\hat{\mathscr{U}}$ as the set of admissible cardinal utilities explicitly. The basic tenet underlying our notion of Nash implementation is that we ignore mixed strategy equilibria which are "not" compelling, while we take compelling mixed strategy equilibria seriously.

Definition 4 *Let $\hat{\mathscr{U}} \subseteq \mathscr{U}$. An SCF f is* ***compellingly implementable (C-implementable) with respect to*** $\hat{\mathscr{U}}$ *if there exists a finite mechanism $\Gamma = (M, g)$ such that for every state $\theta \in \Theta$, (i) pure$NE(\Gamma(\theta)) \neq \emptyset$; (ii) $m \in$ pure$NE(\Gamma(\theta)) \Rightarrow g(m) = f(\theta)$; and (iii) for any $u \in \hat{\mathscr{U}}^{\theta}$, the game $\Gamma(\theta, u)$ has no compelling mixed strategy equilibria.*

Our notion of compelling implementation strengthens the definition of pure Nash implementation with the following additional requirement: there be no compelling mixed strategy equilibria within the class of games $\Gamma(\theta, u)$ across all representation $u \in \hat{\mathscr{U}}^{\theta}$. On the other hand, our notion of compelling implementation weakens the definition of mixed Nash implementation by allowing the following possibilities: (i) there might exist a state $\theta \in \Theta$ and a message profile $m \in \text{supp}(NE(\Gamma(\theta))$ such that $g(m) \neq f(\theta)$ and (ii) there might exist $\theta \in \Theta$, $u \in \mathscr{U}^{\theta} \backslash \hat{\mathscr{U}}^{\theta}$, and $\sigma \in NE(\Gamma(\theta, u))$ such that σ is compelling. The first possibility means that our implementing mechanism might admit a bad Nash equilibrium that is not compelling. The second possibility means that our implementing mechanism might admit a compelling equilibrium if we allow a possible representation u to be outside of $\hat{\mathscr{U}}$.

4.2.2 Maskin Monotonicity

We now restate the definition of Maskin monotonicity that Maskin (1999) proposes for Nash implementation.

Definition 5 *An SCF f satisfies **Maskin monotonicity** if, for every pair of states $\tilde{\theta}$ and θ with $f(\tilde{\theta}) \neq f(\theta)$, some agent $i \in \mathscr{I}$ and some allocation $a \in A$ exist such that*

$$f(\tilde{\theta}) \succeq_i^{\tilde{\theta}} a \text{ and } a \succ_i^{\theta} f(\tilde{\theta}). \tag{4.1}$$

To show that Maskin monotonicity a necessary condition for compelling implementation, suppose that the SCF f is C-implementable by a mechanism $\Gamma = (M, g)$. When $\tilde{\theta}$ is the true state, there exists a pure-strategy Nash equilibrium $m \in M$ in $\Gamma(\tilde{\theta})$ which induces $f(\tilde{\theta})$. If $f(\tilde{\theta}) \neq f(\theta)$ and θ is the true state, then m cannot be a Nash equilibrium, i.e., there exists some agent i who has a profitable deviation. Suppose that the deviation induces outcome a, i.e., agent i strictly prefers a to $f(\tilde{\theta})$ at state θ. Since m is a Nash equilibrium at state $\tilde{\theta}$, such a deviation cannot be profitable in state $\tilde{\theta}$; that is, agent i weakly prefers $f(\tilde{\theta})$ to a at state $\tilde{\theta}$. In other words, a belongs to agent i's lower contour set at $f(\tilde{\theta})$ of state $\tilde{\theta}$, whereas it belongs to the strict upper-contour set at $f(\tilde{\theta})$ of state θ. Therefore, Maskin monotonicity is a necessary condition for compelling implementation; in fact, it is a necessary condition even for pure Nash implementation.

4.3 Example 4 of Jackson (1992)

We start from describing Example 4 of Jackson (1992) which motivates our inquiry immensely. Consider the environment with two agents. Suppose that there are four alternatives $A = \{a, b, c, d\}$ and two states $\Theta = \{\theta, \theta'\}$. Suppose that agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \sim_1 d$ and agent 2 has the preference $a \succ_2^\theta b \succ_2^\theta d \succ_2^\theta c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \sim_2^{\theta'} d$ at state θ'. Consider the SCF f such that $f(\theta) = a$ and $f(\theta') = c$.

First, Jackson (1992) constructs a finite mechanism $\Gamma = (M, g)$ (described in the table below) that implements the SCF f in pure-strategy Nash equilibria:

$g(m)$		Agent 2		
		m_2^1	m_2^2	m_2^3
Agent 1	m_1^1	c	d	d
	m_1^2	d	a	b
	m_1^3	d	b	a

There are two pure strategy Nash equilibria, (m_1^2, m_2^2) and (m_1^3, m_2^3), in the game $\Gamma(\theta)$, both of which result in outcome a. In the game $\Gamma(\theta^{'})$, the unique pure-strategy Nash equilibrium is (m_1^1, m_2^1), which results in outcome c. Thus, the SCF f is implementable by the above finite mechanism in pure-strategy Nash equilibria. Due to the necessity of Maskin monotonicity for Nash implementation, we know that the SCF f satisfies Maskin monotonicity. However, in the game $\Gamma(\theta^{'})$, there is a mixed-strategy Nash equilibrium, where each agent i plays m_i^2

and m_i^3 with equal probability, which results in outcomes a and b, each with probability $1/2$. Both agents strictly prefer any outcome of the mixed-strategy equilibrium to the outcome of the pure-strategy equilibrium. Thus, according to our terminology, this mixed strategy Nash equilibrium is compelling. Note that there is a conflict of interests between the two agents over a and b in state $\theta^{'}$, i.e., while agent 1 prefers a to b, agent 2 prefers b to a. This conflict of interests allows us to have the unique pure strategy Nash equilibrium in the game $\Gamma(\theta^{'})$, which results in outcome c. At the same time, this logic for the uniqueness of the pure-strategy equilibrium is extremely dubious because outcomes a and b are strictly better for both agents than outcome c.

Jackson (1992) further shows that his argument applies to *any* finite implementing mechanism. That is, for any finite mechanism which implements the SCF f in pure-strategy Nash equilibria, there must also exist a compelling mixed-strategy Nash equilibrium such that at state θ', the equilibrium outcome differs from c with positive probability. Therefore, the SCF f is "not" C-implementable with respect to $\mathscr{U}$, which is the set of "all" representations. It thus follows that the identified compelling mixed strategy equilibrium persists independently of any cardinal representation.

4.4 Illustration of the Main Result

One crucial feature Jackson's Example 4 has is that its argument seems to rely heavily on the extreme inefficiency of the SCF, i.e., the SCF f assigns the common worst outcome in state $\theta^{'}$.[2] To investigate how robust Jackson's argument is, we

[2]Jackson (1992, p.770) is well aware of this point.

only make the following modification: both agents now strictly prefer c to d in state θ', i.e., $c \succ_i^{\theta'} d$ for each $i = 1, 2$. Recall that this modification is consistent with our setup, as the environment we consider in this paper only admits strict preferences.

We summarize the basic setup. Agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \succ_1 d$ and agent 2 has the preference $a \succ_2^{\theta} b \succ_2^{\theta} d \succ_2^{\theta} c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \succ_2^{\theta'} d$ at state θ'. Consider the same SCF f such that $f(\theta) = a$ and $f(\theta') = c$. This way the SCF never assigns the worst outcome for any agent in either state (a feature that will also be implied by our sufficient condition).

With this modification, we are able to construct a mechanism that not only implements the SCF in pure-strategy Nash equilibrium, while we guarantee that all mixed-strategy equilibria of the constructed mechanism give each agent the expected payoff arbitrarily close to that of d, which is worse than that of c, the outcome induced by the SCF f at state θ'. Hence, we essentially overturn the implication of Jackson's Example 4 by assuming that there is a uniform bound for the utility difference.

For each integer $k \geq 2$, we define $\Gamma^k = (M^k, g^k)$ as a mechanism with the following properties: (i) for each $i \in N$, $M_i^k = \{0, 1, \ldots, k\}$ and (ii) the outcome function $g^k : M^k \to A$ is given by the following rules: for each $m \in M^k$,

- If $m = (k, k)$, then $g^k(m) = c$;
- If there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, h)$, then $g^k(m) = a$;
- If there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, (h + 1 \bmod k))$, then $g^k(m) = b$; and

- Otherwise, $g^k(m) = d$.

We illustrate this mechanism as follows:

$g^k(m)$		Agent 2								
		k	$k-1$	$k-2$	$k-3$	$\cdots$	3	2	1	0
Agent 1	k	c	d	d	d	$\cdots$	d	d	d	d
	$k-1$	d	a	d	d	$\cdots$	d	d	d	b
	$k-2$	d	b	a	d	$\cdots$	d	d	d	d
	$k-3$	d	d	b	a	$\cdots$	d	d	d	d
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	3	d	d	d	d	$\cdots$	a	d	d	d
	2	d	d	d	d	$\cdots$	b	a	d	d
	1	d	d	d	d	$\cdots$	d	b	a	d
	0	d	d	d	d	$\cdots$	d	d	b	a

When $k = 2$, our mechanism is reduced to the one introduced by Jackson (1992) where we set $m_i^1 = 2; m_i^2 = 1$; and $m_i^3 = 0$ for each $i \in \{1, 2\}$.

$g(m)$		Agent 2		
		2	1	0
Agent 1	2	c	d	d
	1	d	a	b
	0	d	b	a

For each $\theta \in \Theta$, $i \in \{1, 2\}$, and $\varepsilon > 0$, we define $\mathscr{U}_i^{\theta,\varepsilon}$ as a subset of $\mathscr{U}_i^{\theta}$ as follows:

$$\mathscr{U}_i^{\theta,\varepsilon} = \left\{ u_i \in \mathscr{U}_i^{\theta} \middle| \, |u_i(a,\theta) - u_i(a^{'},\theta)| \geq \varepsilon, \, \forall a \in A, \forall a^{'} \in A \backslash \{a\}, \, \forall \theta \in \Theta \right\}.$$

Let $\mathscr{U}^{\theta,\varepsilon} \equiv \times_{i \in N} \mathscr{U}_i^{\theta,\varepsilon}$ and $\mathscr{U}^{\varepsilon} \equiv \times_{\theta \in \Theta} \mathscr{U}^{\theta,\varepsilon}$. We observe that $\mathscr{U}^{\varepsilon}$ possesses the following monotonicity:

$$\varepsilon > \varepsilon^{'} > 0 \Rightarrow \mathscr{U}^{\varepsilon} \subsetneq \mathscr{U}^{\varepsilon^{'}} \subseteq \mathscr{U} \subseteq \mathscr{U}^{0}.$$

Loosely speaking, if we choose $\varepsilon > 0$ small enough, we can approximate $\mathscr{U}$ by $\mathscr{U}^{\varepsilon}$ to an arbitrary degree. We are now ready state the main result of this section.

Proposition 1 *For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ large enough such that the SCF f is C-implementable with respect to $\mathscr{U}^{\varepsilon}$ by the mechanism Γ^K.*

Proof: The proof is completed by a series of lemmas. For the moment, we fix k in the proof and we ignore the dependence of the mechanism on k. The following lemma is our key result characterizing the set of Nash equilibria of the mechanism Γ^k in state $\theta^{'}$.

Lemma 1 *The mechanism Γ^k implements the SCF in pure-strategy Nash Equilibrium.*

Proof: The message profile $(1,1)$ is a Nash equilibrium of the game $\Gamma^k(\theta)$, as it yields a which is their most preferred outcome for both agents so that no agent can find a profitable deviation. We claim that a is the unique Nash equilibrium outcome of the game $\Gamma^k(\theta)$. Let m be a message profile such that $g(m) \neq a$. We will show that m is "not" a Nash Equilibrium in the game $\Gamma^k(\theta)$:

- If $g(m) = b$, there exists an integer h with $0 \leq h \leq k-1$ such that $m = (h, (h+1 \text{ mod } k))$. Then agent 1 has an incentive to send a message $h+1 \text{ mod } k$ so that outcome a is induced.
- If $g(m) = c$, then $m = (k,k)$. Then, agent 2 has an incentive to send any message other than k so that outcome d is induced, as he strictly prefers outcome d to outcome c at state θ.
- If $g(m) = d$, then we have $m = (m_1, m_2)$ where $m_1 \neq m_2$. Then, agent 1 has an incentive to deviate from m_1 to m_2 so that outcome a is induced.

We next claim that (k,k) is a Nash Equilibrium of the game $\Gamma^k(\theta^{'})$ because any unilateral deviation from (k,k) yields d, which is inferior to c induced by (k,k) for both agents. Moreover, no other outcome can be a Nash Equilibrium in this game: every message profile $m = (m_1, m_2)$ where $m_2 < k$ and $g(m) \neq a$ has a profitable deviation for player 1 at $m_1' = m_2$, while every message profile $m = (m_1, m_2)$ where $m_1 < k$ and $g(m) \neq b$ has a profitable deviation for player 2 at $m_2' = m_1 + 1 \text{ mod } k$. Since $g(m) = a$ implies $m_1 < k$ and $g(m) = b$ implies $m_2 < k$, we have that there is no possible Nash Equilibrium with either $m_1 < k$ or $m_2 < k$. Thus, the only possible Nash Equilibrium in pure strategies for this game is (k,k).
■

Lemma 2 *For each $i \in \{1,2\}$, let $\sigma_i = (\sigma_i(0), \sigma_i(1), ..., \sigma_i(k))$ denote agent i's strategy and for each $x \in \{0,1,\ldots,k\}$, let $\sigma_i(x)$ denote the probability that agent i chooses x. If $\sigma = (\sigma_1, \sigma_2)$ is a Nash equilibrium in the game $\Gamma^k(\theta^{'})$, then, for each $i \in \{1,2\}$, there is a number $p^i \in [0,1]$ such that $\sigma_i(x) = p^i/k$ for each $x \in \{0,\ldots,k-1\}$. Moreover, $p^1 = 0$ if and only if $p^2 = 0$.*

Proof: Recall that we set $u_i(d;\theta^{'}) = 0$ for each $u_i \in \mathscr{U}_i^{\theta^{'}}$ and $i \in \{1,2\}$. Let σ be a Nash equilibrium of the game $\Gamma^k(\theta^{'})$. If $\sigma_i(k) = 1$ for each $i \in \{1,2\}$, such p^i in the lemma is guaranteed to exist by setting $p^i = 0$. Thus, we assume that there exists $i \in \{1,2\}$ for whom $\sigma_i(k) < 1$. We divide the proof into a series of steps, whose proofs will be found in the appendix:

Step 1a: If there exists $x \in \{0,\ldots,k-1\}$ such that $\sigma_1(x) > 0$, then $\sigma_2(x) > 0$.

Step 1b: If there exists $x \in \{1,\ldots,k-1\}$ such that $\sigma_2(x) > 0$, then $\sigma_1(x-1) > 0$. Moreover, if $\sigma_2(0) > 0$, then $\sigma_1(k-1) > 0$.

Step 1c: If there exist $i \in \{1,2\}$ and $x^{'} \in \{0,\ldots,k-1\}$ for whom $\sigma_i(x^{'}) > 0$, then $\sigma_1(x) > 0$ and $\sigma_2(x) > 0$ for all $x \in \{0,\ldots,k-1\}$.

Step 2: If there exist $i \in \{1,2\}$ and $x,x^{'} \in \{0,\ldots,k-1\}$ such that $\sigma_i(x) > 0$ and $\sigma_i(x^{'}) > 0$, then $\sigma_i(x) = \sigma_i(x^{'})$.

It follows from both Steps 2 and 1c that $\sigma_i(x) = \sigma_i(x^{'})$ for every $x,x^{'} \in \{0,\ldots,k-1\}$ and $i \in \{1,2\}$. Thus, we can set $p^i = \sum_{x=1}^{k} \sigma_i(x)$. Since we assume $\sigma_i(k) < 1$ for each $i \in \{1,2\}$, we have $p^i > 0$. This completes the proof of Lemma 2. ■

As we can easily see in the proof of Lemma 1, there are no (compelling) mixed strategy Nash equilibria of the game $\Gamma^k(\theta)$ because, in state θ, the unique Nash equilibrium outcome is a, which is the best outcome for both agents. It thus remains to prove that there exists no compelling mixed strategy equilibria in the game $\Gamma^k(\theta^{'})$.

If $k \geq 3$, we let σ^k be a nontrivial mixed-strategy Nash equilibrium in the game $\Gamma^k(\theta^{'})$. Then, the resulting outcome distribution induced by σ^k is given by

$$g \circ \sigma^k = \begin{cases} c & \text{w.p. } (1-p^1)(1-p^2) \\ a & \text{w.p. } (p^1 p^2)/k \\ b & \text{w.p. } (p^1 p^2)/k \\ d & \text{w.p. } ((k-2p^1 p^2)/k) - ((1-p^1)(1-p^2)), \end{cases}$$

where $p^1, p^2 \in (0,1]$ and $p^i = \sum_{x=1}^{k} \sigma_i(x)$ for each $i \in \{1,2\}$. Recall the following pieces of notations:

$$\begin{aligned} \mathscr{U}_1^{\theta^{'}} &= \left\{ u_1(\cdot;\theta^{'}) \in [0,1]^A \middle| \, 1 = u_1(a;\theta^{'}) > u_1(b;\theta^{'}) > u_1(c;\theta^{'}) > u_1(d;\theta^{'}) = 0 \right\}; \\ \mathscr{U}_2^{\theta^{'}} &= \left\{ u_2(\cdot;\theta^{'}) \in [0,1]^A \middle| \, 1 = u_2(b;\theta^{'}) > u_2(a;\theta^{'}) > u_2(c;\theta^{'}) > u_2(d;\theta^{'}) = 0 \right\}. \end{aligned}$$

Let $\mathscr{U}^{\theta^{'}} \equiv \mathscr{U}_1^{\theta^{'}} \times \mathscr{U}_2^{\theta^{'}}$. For each $\varepsilon \in (0,1)$, we have

$$\begin{aligned} \mathscr{U}_1^{\theta^{'},\varepsilon} &= \left\{ u_1(\cdot;\theta^{'}) \in \mathscr{U}_1^{\theta^{'}} \,\middle|\, u_1(c;\theta^{'}) \geq \varepsilon \right\}; \\ \mathscr{U}_2^{\theta^{'},\varepsilon} &= \left\{ u_2(\cdot;\theta^{'}) \in U_2^{\theta^{'}} \,\middle|\, u_2(c;\theta^{'}) \geq \varepsilon \right\}. \end{aligned}$$

Similarly, let $\mathscr{U}^{\theta^{'},\varepsilon} \equiv \mathscr{U}_1^{\theta^{'},\varepsilon} \times \mathscr{U}_2^{\theta^{'},\varepsilon}$.

By the lemma below, we show that for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ large enough so that, for any $u \in \mathscr{U}^{\theta^{'},\varepsilon}$, the game $\Gamma^K(\theta^{'}, u)$ has no compelling mixed strategy equilibria.

Lemma 3 *For each $\varepsilon > 0$, there exists an integer $K \in \mathbb{N}$ large enough so that for any $k \geq K$, $i \in \{1,2\}$, and $(u_1(\cdot,\theta^{'}),u_2(\cdot;\theta^{'})) \in \mathscr{U}^{\theta^{'},\varepsilon}$,*

$$U_i(\sigma^k;\theta^{'}) \leq u_i(c;\theta^{'}),$$

where $U_i(\sigma^k;\theta^{'}) = \sum_{x=0}^{k} \sigma_1^k(x) \sum_{x^{'}=0}^{k} \sigma_2^k(x^{'}) u_i(g(x,x^{'});\theta^{'})$.

Proof: Fix $\varepsilon > 0$ and $i \in \{1,2\}$. We compute

$$U_i(\sigma^k;\theta^{'}) \quad = \quad \frac{p^1 p^2}{k}[u_i(a;\theta^{'}) + u_i(b;\theta^{'})] + (1-p^1)(1-p^2)u_i(c;\theta^{'}).$$

For each $(p^1,p^2) \in [0,1]^2$, we define

$$k(p^1,p^2) = \frac{u_i(a;\theta') + u_i(b;\theta')}{u_i(c;\theta')} \left[\frac{1}{p^1} + \frac{1}{p^2} - 1\right]^{-1}.$$

In the rest of the proof, we make use of the following properties of $k(p^1,p^2)$:

- $k(\cdot,\cdot)$ is strictly increasing in both arguments over $[0,1]^2$.
- $k(p_h^1,p_h^2)$ converges to zero no matter how the sequence $\{(p_h^1,p_h^2)\}_{h=1}^{\infty}$ approaches $(0,0)$. Thus, $k(0,0) \equiv \lim_{(p^1,p^2)\to(0,0)} k(p^1,p^2) = 0$.
- $k(1,1) = [u_i(a;\theta') + u_i(b;\theta')]/u_i(c;\theta') = \max_{(p^1,p^2)\in[0,1]^2} k(p^1,p^2)$.
- We can conveniently rewrite $k(p^1,p^2)$ as

$$k(p^1,p^2) = \frac{u_i(a;\theta') + u_i(b;\theta')}{u_i(c;\theta')} \frac{p^1 p^2}{[1-(1-p^1)(1-p^2)]}.$$

We set $K = \min\{k \in \mathbb{N} | k \geq 2/\varepsilon\}$. As $2/\varepsilon \geq [u_i(a;\theta') + u_i(b;\theta')]/u_i(c;\theta')$ for any $u_i(\cdot;\theta') \in \mathscr{U}_i^{\theta'}[\varepsilon]$, we have that $K \geq k(1,1)$. Due to the strict monotonicity of $k(p^1,p^2)$ with respect to p^1 and p^2, we have that $K \geq k(p^1,p^2)$ for any $(p^1,p^2) \in [0,1]^2$. Hence, for any $k \geq K$:

$$\begin{aligned}
U_i(\sigma^k;\theta') &= \frac{p^1p^2}{k}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&\leq \frac{p^1p^2}{k(p^1,p^2)}[u_i(a;\theta') + u_i(b;\theta')] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&\quad (\because\ k \geq K \geq k(p^1,p^2)\ \forall (p^1,p^2) \in [0,1]^2) \\
&= u_i(c;\theta')[1-(1-p^1)(1-p^2)] + (1-p^1)(1-p^2)u_i(c;\theta') \\
&= u_i(c;\theta').
\end{aligned}$$

This completes the proof of Lemma 3. ■

Combining Lemmas 2, 1, and 3 together, we complete the proof of Proposition 1. ■

4.5 The Main Result

We generalize Proposition 1 of the previous section. First, we introduce a notion of *acceptability*.

Definition 6 *Given two subsets of alternatives $\mathscr{A}, \mathscr{B} \subseteq A$, we say that alternative x is $(\mathscr{A},\mathscr{B})$-**acceptable** at state θ if $x \in \mathscr{A} \cup \mathscr{B}$ and the following two conditions hold:*

- *There is no alternative $a \in \mathscr{A}$ such that $a \succ_1^\theta x$.*
- *There is no alternative $b \in \mathscr{B}$ such that $b \succ_2^\theta x$.*

The property that x is $(\mathscr{A}, \mathscr{B})$-acceptable at state θ guarantees that $\mathscr{A}$ is contained in agent 1's lower contour set at x in state θ and $\mathscr{B}$ is contained agent 2's lower contour set at x in state θ. In the rest of the argument below, we write $\Theta = \{\theta_0, \theta_1, \ldots, \theta_J\}$ where $J = |\Theta| - 1$. So, we are now ready to introduce the key condition for our characterization.

Definition 7 *The environment $\mathscr{E} = \left(\{1,2\}, A, \Theta, (\succeq_i^\theta)_{i \in \{1,2\}, \theta \in \Theta}\right)$ satisfies **Condition P+M** with respect to the SCF f if there exist a function $z : \{0, \ldots, J\} \times \{0, \ldots, J\} \to A$ with image Z, and two collections of subsets $\{\mathscr{A}_j\}_{j=0}^J, \{\mathscr{B}_j\}_{j=0}^J \subseteq A$ such that*

1. *$z(j_1, j_2) \in \mathscr{A}_{j_2} \cap \mathscr{B}_{j_1}$ for all $(j_1, j_2) \in \{0, \ldots, J\} \times \{0, \ldots, J\}$;*
2. *For each state $\theta \in \Theta$ and each pair $(j_1, j_2) \in \{0, \ldots, J\} \times \{0, \ldots, J\}$, if $f(\theta) \neq z(j_1, j_2)$, there exists either $a_{(j_1, j_2)} \in \mathscr{A}_{j_2}$ such that $a_{(j_1, j_2)} \succ_1^\theta z(j_1, j_2)$ or $b_{(j_1, j_2)} \in \mathscr{B}_{j_1}$ such that $b_{(j_1, j_2)} \succ_2^\theta z(j_1, j_2)$;*
3. *For every $j \in \{0, \ldots, J\}$, $f(\theta_j)$ is $(\mathscr{A}_j, \mathscr{B}_j)$-acceptable at state θ_j;*
4. *For every $\theta \in \Theta$ and every $j \in \{0, \ldots, J\}$, if there exists $x \in A$ such that x is $(\mathscr{A}_j, \mathscr{B}_j)$-acceptable at θ, then $x = f(\theta)$,*
5. *For each $\theta \in \Theta$, if $f(\theta) \in Z$, there exists no $x \in \bigcup_{j=0}^J \mathscr{A}_j \cup \mathscr{B}_j$ such that $x \succ_i^\theta f(\theta)$ for all $i \in \{1,2\}$.*
6. *For each $\theta \in \Theta$, if $f(\theta) \notin Z$, then $f(\theta) \succ_i^\theta z$ for all $i \in \{1,2\}$ and $z \in Z$.*

In what follows, Properties 1, 2, 3, and 4 in Condition P+M are collectively called *Condition P* and Properties 5 and 6 in Condition P+M are collectively called *Condition M*, respectively. By "Condition P," we mean the property concerning "pure" Nash implementation and by "Condition M," we mean the property concerning "mixed" Nash implementation.

We recall the following notation. For each $\varepsilon > 0$, $\theta \in \Theta$, and $i \in \{1,2\}$, we have

$$\mathscr{U}_i^{\theta,\varepsilon} = \left\{ u_i \in \mathscr{U}_i^{\theta} \middle| \, |u_i(a,\theta) - u_i(a^{'},\theta)| \geq \varepsilon, \; \forall a \in A, \forall a^{'} \in A \backslash \{a\}, \; \forall \theta \in \Theta \right\}$$

We let $\mathscr{U}^{\theta,\varepsilon} = \mathscr{U}_1^{\theta,\varepsilon} \times \mathscr{U}_2^{\theta,\varepsilon}$ and $\mathscr{U}^{\varepsilon} = \times_{\theta \in \Theta} \mathscr{U}^{\theta,\varepsilon}$.

Theorem 1 *Let f be an SCF. Suppose that the finite two-person environment $\mathscr{E} = \left(\{1,2\}, A, \Theta, (\succeq_i^{\theta})_{i \in \{1,2\}, \theta \in \Theta}\right)$ satisfies Condition P+M with respect to the SCF f. Then, for any $\varepsilon > 0$, the SCF f is C-implementable with respect to $\mathscr{U}^{\varepsilon}$.*

Proof: Suppose that the finite two-person environment $\mathscr{E} = \left(\{1,2\}, A, \Theta, (\succeq_i^{\theta})_{i \in \{1,2\}, \theta \in \Theta}\right)$ satisfies Condition P+M with respect to the SCF f. For each integer $k \geq 2$, we construct a mechanism $\Gamma^k = (M^k, g^k)$ as follows: For each $i \in \{1,2\}$, we set $M_i^k = \{0,1,\ldots,(J+1)k-1\} \times A$, i.e., each message $m_i = (o_i, x_i)$ agent i sends to the mechanism is composed of a pair of an integer which lies between 0 and $(J+1)k-1$ and an alternative in A.

Define the function $n^k : \{0,\ldots,(J+1)k-1\} \to \{0,\ldots,J\}$ as follows: for each $o_i \in \{0,\ldots,(J+1)k-1\}$,

$$n^k(o_i) = \max\{n \in \mathbb{N} | \; n \times k \leq o_i\}.$$

In words, we first compute o_i/k, then round the computed number down to the nearest integer, and finally set the obtained integer as $n^k(o_i)$. For example, if $o_i = 13$ and $k = 5$, we have $n^5(13) = 2$.

To define the outcome function below, we introduce the following permutation $\pi^k : \{0,\dots,(J+1)k-1\} \to \{0,\dots,(J+1)k-1\}$: for each $\tilde{o} \in \{0,\dots,(J+1)k-1\}$,

$$\pi^k(\tilde{k}) = \begin{cases} n^k(\tilde{o})k & \text{if } \tilde{o} = (n^k(\tilde{o})+1)k-1, \\ \tilde{o}+1 & \text{otherwise.} \end{cases}$$

We can interpret π^k as a series of $J+1$ cycles which moves $\tilde{o}$ to $\tilde{o}+1$ for all $\tilde{o} \in \{n^k(\tilde{o})k,\dots,(n^k(\tilde{o})+1)k-2\}$, while moves $(n^k(\tilde{o})+1)k-1$ to $n^k(\tilde{o})k$. The outcome function $g^k : M^k \to A$ is dictated by the following three rules: for each $m \in M^k$ where $m = (m_1,m_2) = ((o_1,x_1),(o_2,x_2))$,

Rule 1: If $o_1 = o_2$ and $x_1 \in \mathscr{A}_{n^k(o_1)}$, then $g^k(m) = x_1$.

Rule 2: If $o_2 = \pi^k(o_1)$ [3] and $x_2 \in \mathscr{B}_{n^k(o_2)}$, then $g^k(m) = x_2$.

Rule 3: For all other cases, $g^k(m) = z(n^k(o_1),n^k(o_2))$.

We describe how the mechanism Γ^k can be played as follows: first, each agent selects a number between 0 and J (represented in the mechanism by the values of $n^k(o_1)$ for agent 1 and $n^k(o_2)$ for agent 2). If $n^k(o_1) \neq n^k(o_2)$, the outcome is given by $z(n^k(o_1),n^k(o_2))$. If $n^k(o_1) = n^k(o_2)$, then they proceed to play a particular form of modulo game: each announces a second number between 0 and $k-1$. If they both select the same number, which implies $o_1 = o_2$, agent 1 wins and he can select any outcome from the set $\mathscr{A}_{n^k(o_1)}$. If agent 2 picks a number o_2 that is higher than the number o_1 picked by agent 1 exactly by one unit (or picks 0, in case agent 1 picks $k-1$) but still $n^k(o_1) = n^k(o_2)$, then agent 2 wins

[3] We note that this implies both $o_1 \neq o_2$ and $n^k(o_1) = n^k(o_2)$, as these properties will be exploited later.

$g(m_1,m_2)$			Agent 2																
			$n^k(o_2) = 0$				$n^k(o_2) = 1$				$n^k(o_2) = 2$					$n^k(o_2) = J$			
			0	1	...	k-1	k	k+1	...	2k-1	2k	2k+1	...	...	...	Jk	Jk+1	...	J(k+1)-1
Agent 1	$n^k(o_1) = 0$	0	a(0)	b(0)	...	z(0,0)	z(0,1)				z(0,2)				...	z(0,J)			
		1	z(0,0)	a(0)	...	z(0,0)													
		...	...	...	...	...													
		k-1	b(0)	z(0,0)	...	a(0)													
	$n^k(o_1) = 1$	k	z(1,0)				a(1)	b(1)	...	z(1,1)	z(1,2)				...	z(1,J)			
		k+1					z(1,1)	a(1)	...	z(1,1)									
		...					...	...	...	...									
		2k-1					b(1)	z(1,1)	...	a(1)									
	$n^k(o_1) = 2$	2k	z(2,0)				z(2,1)				a(2)	b(2)	...	z(2,2)	...	z(2,J)			
		2k+1									z(2,2)	a(2)	...	z(2,2)					
		...									...	...	...	...					
		...									b(2)	z(2,2)	...	a(2)					
		...	...				...				...				...	...	...	...	...
	$n^k(o_1) = J$	Jk	z(J,0)				z(J,1)				z(J,2)				...	a(J)	b(J)	...	z(J,J)
		Jk+1														z(J,J)	a(J)	...	z(J,J)
		...														...	...	...	...
		J(k+1)-1														b(J)	z(J,J)	...	a(J)

Figure 4.1: Table for the game

and can pick any outcome from the set $\mathscr{B}_{n^k(o_2)}$. In any other case, the outcome is, once again, given by $z(n^k(o_1), n^k(o_1))$. This is illustrated in Figure 1, which depicts only the integer part of the messages. To represent the alternatives, we use function $a(j)$, which is equal to x_1 if $x_1 \in \mathscr{A}_j$ (and equal to $z(j,j)$ otherwise) and function $b(j)$, which is equal to x_2 if $x_2 \in \mathscr{B}_j$ (also equal to $z(j,j)$ otherwise).

We complete the rest of the proof in a series of lemmas.

Lemma 4 *If Condition P is satisfied,* Γ^k *pure Nash implements* f.

Proof of Lemma 4: Fix $j \in \{0,\ldots,J\}$. By Property 3 in Condition P, $f(\theta_j)$ is $(\mathscr{A}_j, \mathscr{B}_j)$-acceptable at state θ_j. This implies that $f(\theta_j) \in \mathscr{A}_j \cup \mathscr{B}_j$. Assume first that $f(\theta_j) \in \mathscr{A}_j$. We then define $m = (m_1, m_2) = ((jk, f(\theta_j)), (jk, z(j,j)))$. By construction, m induces Rule 1 in the mechanism Γ^k so that we have $g(m) = f(\theta_j)$. We consider $m_1^{'}$ as an arbitrary deviation strategy of agent 1 and argue that $m_1^{'}$ never be a better reply than m_1 against m_2. If $(m_1^{'}, m_2)$ induces Rule 1, by Property 3 of Condition P, $m_1^{'}$ is not a profitable deviation. If $(m_1^{'}, m_2)$ induces Rule 2, then we have $g(m_1^{'}, m_2) = z(j,j)$, which, by Property 1, is also a part of $\mathscr{A}_j$ and thus, by Property 3, not a profitable deviation either. If $(m_1^{'}, m_2)$ induces Rule 3, there exists $j_1 \in \{0,\ldots,J\}$ such that $g(m_1^{'},m_2) = z(j_1,j)$. By Property 1, $z(j_1,j) \in \mathscr{A}_j$ and thus by Property 3 in Condition P, $m_1^{'}$ is not a profitable deviation.

We next consider $m_2^{'}$ as an arbitrary deviation strategy of agent 2 and argue that $m_2^{'}$ never be a better reply than m_2 against m_1. If $(m_1,m_2^{'})$ induces Rule 1, we have $g(m_1,m_2^{'}) = g(m_1,m_2)$ so that $m_2^{'}$ is not a profitable deviation. If $(m_1,m_2^{'})$ induces Rule 2, then $g(m_1,m_2^{'}) \in \mathscr{B}_j$ and it follows from Property 3 of Condition P that $m_2^{'}$ is not a profitable deviation, since $f(\theta_j)$ being $(\mathscr{A}_j,\mathscr{B}_j)$-acceptable at that state means there can be no element in $\mathscr{B}_j$ that is preferred to $f(\theta_j)$ by agent 2. Finally, if $(m_1,m_2^{'})$ induces Rule 3, there exists $(j,j_2) \in \{0,\ldots,J\} \times \{0,\ldots,J\}$ such that $g(m_1,m_2^{'}) = z(j,j_2)$. It follows from Property 1 that $z(j,j_2) \in \mathscr{B}_j$ and from Property 3 of Condition P that $m_2^{'}$ is not a profitable deviation. Thus, $(m_1,m_2) = ((jk,f(\theta_j),(jk,z(j,j))$ is a Nash equilibrium of the game $\Gamma^k(\theta_j)$ in this case.

Finally, consider the scenario when $f(\theta_j) \notin \mathscr{A}_j$, which, by the definition of acceptability, must imply that $f(\theta_j) \in \mathscr{B}_j$. Then, we define $m = (m_1,m_2) = ((jk,z(j,j)),(jk+1,f(\theta_j)))$. This message induces Rule 2 and results in $f(\theta_j)$ as the outcome, as desired. To check that there are no profitable deviations, consider first m_1' as an arbitrary deviation by agent 1. If (m_1',m_2) induces Rule 1, by Property 3 this is not a profitable deviation. If it induces Rule 2, the outcome is unchanged, so again, no benefit for the agent. Finally, if it induces Rule 3, then there is a (j_1,j) such that $g(m_1',m_2) = z(j_1,j)$, but as $z(j_1,j) \in \mathscr{A}_j$ according to Property 1, we can once more invoke Property 3 to conclude that this is not a profitable deviation either.

We move on to deviation strategies attempted by agent 2, denoting by m_2' an arbitrary deviation. As above, Property 3 ensures that any deviation that induces

Rule 2 cannot be profitable. If (m_1, m_2') induces Rule 3 instead, then we can find a (j, j_2) such that $g(m_1', m_2) = z(j, j_2)$, but as $z(j, j_2) \in \mathscr{B}_j$ according to Property 1, we can once more invoke Property 3 to conclude that this is not a profitable deviation either. Lastly, if it induces Rule 1, the outcome must be $z(j, j)$ and then by Properties 1 and 3 this cannot be a profitable deviation either.

Fix $\theta \in \Theta$. We shall show that $m \in pureNE(\Gamma^k(\theta))$ implies $g(m) = f(\theta)$. We assume by way of contradiction that there exists $m \in pureNE(\Gamma^k(\theta))$ such that $g(m) \neq f(\theta)$. We write $m = (m_1, m_2) = ((o_1, x_1), (o_2, x_2))$. We complete the proof by considering the following separate cases.

Case 1: *m* induces Rule 1

Assume that m induces Rule 1. Then, we have $g(m) = x_1$. Since $x_1 \neq f(\theta)$ from our hypothesis, Property 4 of Condition P implies that for every $j = \{0, \ldots, J\}$, x_1 is not $(\mathscr{A}_j, \mathscr{B}_j)$-acceptable. As $x_1 \in \mathscr{A}_{n^k(o_2)}$, this implies that there must exist either $a \in \mathscr{A}_{n^k(o_2)}$ such that $a \succ_1^\theta x_1$ or $b \in \mathscr{B}_{n^k(o_2)}$ such that $b \succ_2^\theta x_1$. Assume it is the former. Then $m_1' = (o_1, a)$ is a profitable deviation for agent 1. If it is the latter, then $m_2' = (\pi^k(o_1), b)$ is a profitable deviation for agent 2.

Case 2: *m* induces Rule 2

Assume that m induces Rule 2. Then, we have $g(m) = x_2$. Since $x_2 \neq f(\theta)$ from our hypothesis, Property 4 of Condition P implies that for every $j = \{0, \ldots, J\}$, x_1 is not $(\mathscr{A}_j, \mathscr{B}_j)$-acceptable. As $x_2 \in \mathscr{B}_{n^k(o_1)}$, this implies that there must exist either $a \in \mathscr{A}_{n^k(o_2)}$ such that $a \succ_1^\theta x_1$ or $b \in \mathscr{B}_{n^k(o_2)}$ such that $b \succ_2^\theta x_1$. Assume it is the former. Then $m_1' = (o_2, a)$ is a profitable deviation for agent 1. If it is the latter, then $m_2' = (o_2, b)$ is a profitable deviation for agent 2.

Case 3: m induces Rule 3

If m induces Rule 3, we have $g(m) = z(j_1, j_2)$ where $(j_1, j_2) = (n^k(o_1), n^k(o_2))$. By assumption, $z(j_1, j_2)) \neq f(\theta)$, then we can invoke Property 2 and find either $a_{(j_1,j_2)} \in \mathscr{A}_{j_2}$ such that $a_{(j_1,j_2)} \succ_1^\theta z(j_1, j_2)$ or $b_{(j_1,j_2)} \in \mathscr{B}_{j_1}$ such that $b_{(j_1,j_2)} \succ_2^\theta z(j_1, j_2)$. Assume it is the former; then agent 1 has a profitable deviation strategy in $m_1' = (o_2, a_{(j_1,j_2)})$. If it is the latter, than agent 2 has a profitable deviation strategy in $m_1' = (\pi^K(o_1), b_{(j_1,j_2)})$. ■

Next, we show that if $f(\theta) \in Z$, there are no compelling mixed strategy equilibria in the game.

Lemma 5 *Assume that Condition P+M holds. If $f(\theta) \in Z$, then, for any $u \in \mathscr{U}^\theta$, the game $\Gamma^k(\theta, u)$ has no compelling mixed strategy equilibria.*

Proof of Lemma 5: Fix $u \in \mathscr{U}^\theta$. Suppose by way of contradiction that there is a compelling mixed strategy Nash equilibrium σ of the game $\Gamma^k(\theta, u)$. As $\Gamma^k = (M^k, g^k)$ pure Nash implements the SCF f under Condition P by Lemma 4, we have that for each $i \in \{1, 2\}$,

$$\sum_{\tilde{m} \in M} \sigma(\tilde{m}) u_i(g^k(\tilde{m}), \theta) \geq u_i(f(\theta), \theta),$$

with at least one strict inequality for some $i \in \{1, 2\}$. This implies that there exist $i \in \{1, 2\}$ and $m \in \mathrm{supp}(\sigma)$ such that

$$u_i(g^k(m), \theta) > u_i(f(\theta), \theta).$$

We write $m = (m_1, m_2) = ((o_1, x_1), (o_2, x_2))$. If m induces Rule 1, we have $g^k(m) = x_1$ and $x_1 \in \mathscr{A}_{n^k(o_1)}$. This contradicts Property 5 of Condition M. If m induces Rule 2, we have $g^k(m) = x_2$ and $x_2 \in \mathscr{B}_{n^k(o_2)}$. This also contradicts Property 5 of Condition M. If m induces Rule 3, we have $g^k(m) = z(n^k(o_1), n^k(o_2))$. By Property 1 of Condition P, we have $z(n^k(o_1), n^k(o_2)) \in \mathscr{A}_{n^k(o_2)}$. This contradicts Property 5 of Condition M. ■

It remains now for us to show that when $f(\theta) \notin Z$, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for any $u \in \mathscr{U}^{\varepsilon}$, the game $\Gamma^K(\theta, u)$ has no compelling mixed strategy equilibria. The proof of this case requires us to take a series of steps.

For each $j \in \{0, \ldots, J\}$ and $\theta \in \Theta$, let a_j^θ be the best outcome for player 1 within $\mathscr{A}_j$ at state θ, and b_j^θ the best outcome for player 2 within $\mathscr{B}_j$ at state θ, respectively. In the rest of the proof, we fix $\theta \in \Theta$ throughout.

Lemma 6 *Consider the mechanism $\Gamma^k = (M^k, g^k)$. For each message $m_1 = (o_1, x_1) \in M_1^k$, we can define the following message $m_1^*(m_1) = (o_1, a^\theta_{n^k(o_1)}) \in M_1^k$ (possibly $m_1^*(m_1) = m_1$) such that $g^k(m_1^*, m_2) \succeq_1^\theta g^k(m_1, m_2)$ for each $m_2 \in M_2^k$. Moreover, if $g^k(m_1^*, m_2) \neq g^k(m_1, m_2)$ for some $m_2 \in M_2^k$, then $g^k(m_1^*(m_1), m_2) \succ_1^\theta g^k(m_1, m_2)$. Similarly, for each message $m_2 = (o_2, x_2) \in M_2^k$, we can define the following message $m_2^*(m_2) = (o_2, b^\theta_{n^k(o_2)}) \in M_2^k$ (possibly $m_2^*(m_2) = m_2$) such that $g^k(m_1, m_2^*(m_2)) \succeq_2^\theta g^k(m_1, m_2)$ for each $m_1 \in M_1^k$. Moreover, if $g^k(m_1, m_2^*(m_2)) \neq g^k(m_1, m_2)$ for some $m_1 \in M_1^k$, then $g^k(m_1, m_2^*(m_2)) \succ_2^\theta g^k(m_1, m_2)$.*

Proof of Lemma 6: Let $m_1 = (o_1, x_1)$ denote player 1's a generic message in the mechanism Γ^k. We define the following partition over M_2^k given m_1:

$$\begin{aligned} M_2^1(m_1) &= \left\{m_2 \in M_2^k | \ (m_1, m_2) \text{ induces Rule 1}\right\}, \\ M_2^2(m_1) &= \left\{m_2 \in M_2^k | \ (m_1, m_2) \text{ induces Rule 2}\right\}, \\ M_2^3(m_1) &= \left\{m_2 \in M_2^k | \ (m_1, m_2) \text{ induces Rule 3}\right\}. \end{aligned}$$

By construction, we have $M_2^1(m_1) \cup M_2^2(m_1) \cup M_2^3(m_1) = M_2^k$. Define $m_1^*(m_1) = (o_1, a^{\theta}_{n^k(o_1)})$. When either Rule 2 or Rule 3 is induced, player 1's announcement about alternatives is irrelevant. So, by construction of $m_1^*(m_1)$, we obtain the following property: for any $m_2 \in M_2^2(m_1) \cup M_2^3(m_1)$,

$$g(m_1, m_2) = g(m_1^*(m_1), m_2) \Rightarrow g(m_1^*(m_1), m_2) \sim_1^{\theta} g(m_1, m_2).$$

When (m_1, m_2) induces Rule 1, by its construction, $(m_1^*(m_1), m_2)$ also induces Rule 1. Under Rule 1, we know that player 1's announcement about alternatives solely dictates the outcome. Once again, by construction of m_1^*, along with the fact that we have strict preferences, we obtain the following property: for any $m_2 \in M_2^1(m_1)$,

$$g(m_1, m_2) \neq g(m_1^*, m_2) \Rightarrow g(m_1^*, m_2) \succ_1^{\theta} g(m_1, m_2).$$

This completes the argument for player 1.

Let $m_2 = (o_2, x_2)$ be a generic message agent 2 sends to the mechanism Γ^k.

We define the following partition over M_1^k given m_2:

$$\begin{aligned} M_1^1(m_2) &= \left\{ m_1 \in M_1^k | \ (m_1, m_2) \text{ induces Rule 1} \right\}, \\ M_1^2(m_2) &= \left\{ m_1 \in M_1^k | \ (m_1, m_2) \text{ induces Rule 2} \right\}, \\ M_1^3(m_2) &= \left\{ m_1 \in M_1^k | \ (m_1, m_2) \text{ induces Rule 3} \right\}. \end{aligned}$$

By construction, we have $M_1^1(m_2) \cup M_1^2(m_2) \cup M_1^3(m_2) = M_1^k$. Define $m_2^*(m_2) = (o_2, b^{\theta}_{n^k(o_2)})$. When either Rule 1 or Rule 3 is induced, player 2's announcement about alternatives is irrelevant. So, by construction of m_2^*, we obtain the following property: for any $m_1 \in M_1^1(m_2) \cup M_1^3(m_2)$,

$$g(m_1, m_2^*(m_2)) = g(m_1, m_2) \Rightarrow g(m_1, m_2^*(m_2)) \sim_2^{\theta} g(m_1, m_2).$$

When (m_1, m_2) induces Rule 2, by its construction, $(m_1, m_2^*(m_2))$ also induces Rule 2. Under Rule 2, we know that player 2's announcement about alternatives solely dictates the outcome. Once again, by construction of $m_2^*(m_2)$ along with the fact that we have strict preferences, we obtain the following property: for any $m_1 \in M_1^2(m_2)$,

$$g(m_1, m_2^*(m_2)) \neq g(m_1, m_2) \Rightarrow g(m_1, m_2^*(m_2)) \succ_2^{\theta} g(m_1, m_2).$$

This completes the argument for player 2. ■

We introduce the following notation in the mechanism $\Gamma = (M, g)$: for any agent $i \in \{1, 2\}$ and mixed strategy $\sigma_i \in \Delta(M_i)$, we can define another mixed strategy $\sigma_i^*[\sigma_i]$ as follows: for each $m_i \in M_i$,

$$\sigma_i^*[\sigma_i](m_i) = \sum_{\tilde{m}_i : m_i = m_i^*(\tilde{m}_i)} \sigma_i(\tilde{m}_i).$$

Then, we establish the following lemma.

Lemma 7 *Fix $u \in \mathscr{U}^\theta$ and $\sigma \in NE(\Gamma(\theta,u))$. Then, $\sigma^*[\sigma] \in NE(\Gamma(\theta,u))$. Moreover, if σ is compelling in the game $\Gamma(\theta,u)$, $\sigma^*[\sigma]$ is also compelling in the same game $\Gamma(\theta,u)$.*

Proof of Lemma 7: This follows directly from Lemma 6. ■

Define

$$NE^*(\Gamma(\theta,u)) \equiv \bigcup_{\sigma \in NE(\Gamma(\theta,u))} \{\sigma^*[\sigma]\}.$$

By Lemma 7, we have $NE^*(\Gamma(\theta,u)) \subseteq NE(\Gamma(\theta,u))$. The contrapositive form of Lemma 7 says that if σ^* is not a compelling mixed strategy equilibrium, σ is also not a compelling mixed strategy equilibrium. This implies that there is no loss of generality to focus on $NE^*(\Gamma(\theta,u))$, as far as we are concerned with the nonexistence of compelling mixed strategy equilibria in the game $\Gamma(\theta,u)$. If we only focus on $NE^*(\Gamma(\theta,u))$, we can only focus on mixed strategies where the players randomize only on the integers the choose (the first component of the message), with the alternative (second component) being always the most preferred alternative from their choice set associated.

With this specific structure of mixed strategies, we introduce the following notation. Let $m_1(o_1) = (o_1, a^\theta_{n^k(o_1)})$ for each $o_1 \in \{0,\ldots,(J+1)k-1\}$ and $m_2(o_2) = (o_2, b^\theta_{n^k(o_2)})$ for each $o_2 \in \{0,\ldots,(J+1)k-1\}$. Then, for each $i \in \{1,2\}$, denote by σ_i the strategy that assigns probability $\sigma_i(o_i)$ to message $m_i(o_i)$, with $\sum_{o_i=0}^{(J+1)k-1} \sigma_i(o_i) = 1$.

Lemma 8 *Suppose that Condition P+M holds and $f(\theta) \notin Z$. Let $u \in \mathscr{U}^\theta$ and $\sigma = (\sigma_1, \sigma_2) \in NE^*(\Gamma^k(\theta, u))$ be a compelling mixed strategy equilibrium. Then, for each $i \in \{1,2\}$ and $j \in \{0, \dots, J-1\}$ such that $a_j^\theta \succ_1^\theta f(\theta)$ or $b_j^\theta \succ_2^\theta f(\theta)$, there is a number $p_j^i \in [0,1]$ such that $\sigma_i(x) = p_j^i/k$ for each x such that $n^k(x) = j$.*

Proof of Lemma 8: For the sake of notation, we denote $u_i^a = u_i(a_j^\theta; \theta)$, $u_i^b = u_i(b_j^\theta; \theta)$, $u_i^z = u_i(z(j,j); \theta)$, and $\sigma_i^j = \sum_{x:n^k(x)=j} \sigma_i(x)$ for $i = 1,2$. Lastly, for each $i \in \{1,2\}$, define sets S_i^{max} and S_i^{min} as

$$S_i^{max} = \arg \max_{n^k(\tilde{x})=n^k(x)} \sigma_i(\tilde{x}) \text{ and } S_i^{min} = \arg \min_{n^k(\tilde{x})=n^k(x)} \sigma_i(\tilde{x})$$

We further divide the proof into a series of substeps:

Step 8a: If $S_2^{max} \neq S_2^{min}$, then $\sigma_1(x) = 0$ for each $x \in S_2^{min}$.

Proof of Step 8a:

Assume $S_2^{max} \neq S_2^{min}$ and take $x \in S_2^{max}$, $x' \in S_2^{min}$. We want to show that message $\sigma_1(x')$ is dominated by message $\sigma_1(x)$, and thus is never sent with positive probability in an equilibrium. There are two cases to consider: either $u_1^a > u_1^b > u_1^z$ or $u_1^a > u_1^z > u_1^b$. We start with the first case, where $u_1^b > u_1^z$.

The expected payoff for agent 1 of sending integer x' against σ_2 in the game

$\Gamma^k(\theta)$ is given by

$$U_1(m(x^{'}),\sigma_2;\theta) = \sigma_2(x^{'})u_1^a + \sigma_2(\pi^k(x^{'}))u_1^b + (\sigma_2^j - \sigma_2(x^{'}) - \sigma_2(\pi^k(x^{'})))u_1^z + \hat{z}_1^{\sigma^2}$$

On the other hand, The expected payoff for agent 1 of sending message x against σ_2 in the game $\Gamma^k(\theta)$ is given by

$$U_1(m(x),\sigma_2;\theta) = \sigma_2(x)u_1^a + \sigma_2(\pi^k(x))u_1^b + (\sigma_2^j - \sigma_2(x) - \sigma_2(\pi^k(x)))u_1^z + \hat{z}_1^{\sigma^2}$$

Taking the difference between the two, we compute

$$\begin{aligned}
& U_1(x,\sigma_2;\theta) - U_1(x^{'},\sigma_2;\theta) \\
= \ & [\sigma_2(x) - \sigma_2(x^{'})]u_1^a + [\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]u_1^b + [\sigma_2(x) - \sigma_2(x^{'}) + \sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]U_1^z \\
= \ & [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^z) + [\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))](u_1^b - u_1^z) \\
\geq \ & [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^z) - [\sigma_2(x) - \sigma_2(x^{'})](u_1^b - u_1^z) \\
& (\because [\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))] \geq -[\sigma_2(x) - \sigma_2(x^{'})], u_1^b > u_1^z) \\
= \ & [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^b) \\
> \ & 0.
\end{aligned}$$

This implies that a message with integer x is a strictly better response for agent 1 against σ_2 than $x^{'}$ in the game $\Gamma^k(\theta)$. Thus, message $\sigma(x')$ is never sent with positive probability in an equilibrium if $u_1^b > u_1^z$.

We move to the next case, where $u_1^z > u_1^b$. Take again $x' \in S_2^{min}$. To choose x, we use now the following procedure: take first x_1 such that $\pi^k(x_1) = x'$. If $x_1 \notin S_2^{min}$, then $x = x_1$. If $x_1 \in S_2^{min}$, then take x_2 such that $\pi^k(x_2) = x_1$ and check if $x_2 \in S_2^{min}$. If it is not in the set, then $x = x_2$, else we iterate the argument. As $S_2^{min} \neq S_2^{max}$, eventually we will find an x such that $x \notin S_2^{min}$, but $\pi^k(x) \in S_2^{min}$. We then compute the difference in expected payoffs between these two messages:

$$
\begin{aligned}
& U_1(x, \sigma_2; \theta) - U_1(x^{'}, \sigma_2; \theta) \\
= \quad & [\sigma_2(x) - \sigma_2(x^{'})]u_1^a + [\sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]u_1^b + [\sigma_2(x) - \sigma_2(x^{'}) + \sigma_2(\pi^k(x)) - \sigma_2(\pi^k(x'))]U_1^z \\
= \quad & [\sigma_2(x) - \sigma_2(x^{'})](u_1^a - u_1^z) + [\sigma_2(\pi^k(x')) - \sigma_2(\pi^k(x))](u_1^z - u_1^b) \\
> \quad & 0. \\
& (\because [x', \pi^k(x) \in S_2^{min};\ x \notin S_2^{min}], u_1^a > u_1^z > u_1^b)
\end{aligned}
$$

This shows that message $\sigma(x')$ is also never sent with positive probability in an equilibrium if $u_1^z > u_1^b$. ■

Step 8b: If $S_1^{max} \neq S_1^{min}$, then $\sigma_2(\pi^k(x)) = 0$ for each $x \in S_1^{min}$.

Proof of Step 8b: Assume $S_1^{max} \neq S_1^{min}$ and take $x \in S_1^{max}$, $x' \in S_1^{min}$. We want to show that message $\sigma_2(\pi^k(x'))$ is dominated by message $\sigma_1(\pi^k(x))$, and thus is never sent with positive probability in an equilibrium. There are two cases to consider: either $u_2^b > u_2^a > u_2^z$ or $u_2^b > u_2^z > u_2^a$. We start with the first case, where $u_2^a > u_1^z$.

The expected payoff for agent 2 of sending message $\pi^k(x^{'})$ against σ_1 in the game $\Gamma^k(\theta)$ is given by

$$U_2(\sigma_1, m(\pi^k(x^{'})); \theta) = \sigma_1(\pi^k(x^{'}))u_2^a + \sigma_1(x^{'})u_2^b + (\sigma_1^j - \sigma_1(\pi^k(x^{'})) - \sigma_1(x^{'}))u_2^z + \hat{z}_2^{\sigma^1}$$

On the other hand, the expected payoff for agent 2 of sending message $\pi^k(x)$ against σ_1 in the game $\Gamma^k(\theta)$ is given by

$$U_2(\sigma_1, m(\pi^k(x)); \theta) = \sigma_1(\pi^k(x))u_2^a + \sigma_1(x)u_2^b + (\sigma_1^j - \sigma_1(\pi^k(x)) - \sigma_1(x))u_1^z + \hat{z}_2^{\sigma^1}$$

Taking the difference between the two, we compute

$$\begin{aligned}
& U_2(\sigma_1, m(\pi^k(x)); \theta) - U_2(\sigma_1, \pi^k(x^{'}); \theta) \\
= \; & [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'}))]u_2^a + [\sigma_1(x) - \sigma_1(x')]u_2^b + [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'})) + \sigma_1(x) - \sigma_1(x')]u_2^z \\
= \; & [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'}))](u_2^a - u_2^z) + [\sigma_1(x) - \sigma_1(x')](u_2^b - u_2^z) \\
\geq \; & [\sigma_1(x) - \sigma_1(x^{'})](u_2^b - u_2^z) - [\sigma_1(x) - \sigma_1(x^{'})](u_2^a - u_2^z) \\
& (\because [\sigma_1(\pi^k(x)) \; - \sigma_1(\pi^k(x'))] \geq -[\sigma_1(x) - \sigma_1(x^{'})], u_2^a > u_2^z) \\
= \; & [\sigma_1(x) - \sigma_1(x^{'})](u_2^b - u_2^a) \\
> \; & 0.
\end{aligned}$$

This implies that message $\sigma(\pi^k(x))$ is a strictly better response for agent 2 against σ_1 than $\sigma(\pi^k(x^{'}))$ in the game $\Gamma^k(\theta^{'})$. Thus, message $\sigma(\pi^k(x^{'}))$ is never sent with positive probability in an equilibrium if $u_2^a > u_2^z$.

We move to the next case, where $u_2^z > u_2^a$. Take again $x' \in S_1^{min}$ and choose x such that $x \notin S_1^{min}$, but $\pi^k(x) \in S_1^{min}$. Since $S_1^{min} \neq S_1^{max}$, such x must exist. We then compute the difference in expected payoffs between these two messages:

$$
\begin{aligned}
& U_2(\sigma_1, m(\pi^k(x)); \theta) - U_2(\sigma_1, \pi^k(x^{'}); \theta) \\
= \; & [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'}))]u_2^a + [\sigma_1(x) - \sigma_1(x')]u_2^b + [\sigma_1(\pi^k(x)) - \sigma_1(\pi^k(x^{'})) + \sigma_1(x) - \sigma_1(x')]u_2^z \\
= \; & [\sigma_1(\pi^k(x^{'})) - \sigma_1(\pi^k(x))](u_2^z - u_2^a) + [\sigma_1(x) - \sigma_1(x')](u_2^b - u_2^z) \\
> \; & 0. \\
& (\because [x', \pi^k(x) \in S_1^{min};\ x \notin S_1^{min}], u_2^b > u_2^z > u_2^a)
\end{aligned}
$$

This shows that message $\sigma(\pi^k(x'))$ is also never sent with positive probability in an equilibrium if $u_2^z > u_2^a$. ■

Step 8c: $\sigma_1(x) = 0$ for every x such that $n^k(x) = j$ if and only if $\sigma_2(x) = 0$ for every x such that $n^k(x) = j$.

Proof of Step 8c: Assume that $\sigma_1(x) = 0$ for every x such that $n^k(x) = j$. Then, the expected payoff $\sigma_2(x)$ for any x such that $n^k(x) = j$ is the same and equal to

$$
U_2(\sigma_1, \sigma_2(x); \theta) = \sum_{l \in \{0,\ldots,J\}/j} \sum_{y: n^k(y) = l} \sigma_2(y) u_1(z(j, n^k(y)), \theta)
$$

.

From Property 6, we know that $f(\theta) \succ_2^\theta z$ for any $z \in Z$, and thus, we must have $u(f(\theta)) > U_2(\sigma_1, \sigma_2(x); \theta)$. However, since every message that receives positive probability in a strategy that is a part of a mixed-strategy equilibrium must offer the same expected payoff as other messages of that strategy, having $\sigma_2(x)$

be sent with positive probability implies that σ is not a compelling equilibrium, contradicting our earlier assumption. Thus, $\sigma_1(x) = 0$ for every x such that $n^k(x) = j$ only if $\sigma_2(x) = 0$ as well. A similar argument shows that the reverse must hold as well.

Step 8d: For each $i = \{1,2\}$, $S_i^{max} = S_i^{min}$.

Proof of Step 8d: Assume by way of contradiction that $S_2^{max} \neq S_2^{min}$. We can then apply Step 8a to conclude that $\sigma_1(x) = 0$ for each $x \in S_2^{min}$. As $0 = \min \sigma_1(\tilde{x})$, this in turn implies that $S_2^{min} \subset S_1^{min}$. Moreover, $S_2^{max} \neq S_2^{min}$ implies that $\sigma_2(x) > 0$ for some x such that $n^k(x) = j$, which, by Step 8c, implies that $\sigma_1(x) > 0$ for some x in the same range and thus, $S_1^{max} \neq S_1^{min}$ as well. Then we apply Step 8b to show that $\sigma_2(\pi^k(x)) = 0$ for each $x \in S_1^{min}$. Since $0 = \min \sigma_2(\tilde{x})$ and for no value of x we have $\pi^k(x) = x$, we must either have $x \in S_2^{min}$ but not in S_1^{min}, contradicting $S_2^{min} \subset S_1^{min}$, or we must have $\sigma_1(x) = 0$ for every x such that $n^k(x) = j$, which is also a contradiction. Thus, we cannot have $S_2^{max} \neq S_2^{min}$.

If, on the other hand, we have $S_1^{max} \neq S_1^{min}$, then we start by noting that this implies $\sigma_1(x) > 0$ for some x such that $n^k(x) = j$. Then, we apply Step 8c to conclude that $\sigma_2(x) > 0$ for some x in the same range. Next, we apply Step 8b to show that $\sigma_2(\pi^k(x)) = 0$ for each $x \in S_1^{min}$. Since $0 = \min \sigma_2(\tilde{x})$, we must have $S_2^{max} \neq S_2^{min}$. Then we can apply the same arguments as above and complete the proof of this step. ■

It follows from Step 8.d that $\sigma_i(x) = \sigma(x')$ for every x, x' with $n^k(x) = n^k(x') = j$ and $i \in \{1,2\}$. Thus we can set $p_j^i = \sum_{x=jk}^{(j+1)k-1} \sigma_i(x)$. This completes the proof of Lemma 8. ■

Lemma 8 needs to assume that there exists $j \in \{0,\ldots,J-1\}$ such that $a_j^\theta \succ_1^\theta f(\theta)$ or $b_j^\theta \succ_2^\theta f(\theta)$ to characterize the structure of compelling mixed strategy equilibria. If such a condition is not satisfied, we do not know the structure of compelling equilibria by Lemma 8. Therefore, the next lemma guarantees that the premise for Lemma 8 is nonvacous.

Lemma 9 *Suppose that Condition P+M holds and $f(\theta) \notin Z$. Let $u \in \mathscr{U}^\theta$ and $\sigma = (\sigma_1, \sigma_2) \in NE^*(\Gamma^k(\theta,u))$ be a compelling mixed strategy equilibrium. Then, there exists a $j \in \{0,\ldots,J-1\}$ such that $a_j^\theta \succ_1^\theta f(\theta)$ or $b_j^\theta \succ_2^\theta f(\theta)$.*

Proof of Lemma 9: Fix $u \in \mathscr{U}^\theta$ and $\sigma \in NE^*(\Gamma^k(\theta,u))$. Assume that σ is a compelling mixed strategy equilibrium of the game $\Gamma^k(\theta,u)$. As we know from Lemma 5 that $\Gamma^k = (M^k, g^k)$ pure Nash implements the SCF under Condition P, we have that for each $i \in \{1,2\}$,

$$\sum_{\tilde{m} \in M} \sigma(\tilde{m}) u_i(g^k(\tilde{m}),\theta) \geq u_i(f(\theta),\theta),$$

with at least one strict inequality for some $i \in \{1,2\}$. This implies that there exist $i \in \{1,2\}$ and $m \in \text{supp}(\sigma)$ such that

$$u_i(g^k(m),\theta) > u_i(f(\theta),\theta).$$

We can write $m = (m_1,m_2) = ((o_1, a^\theta_{n^k(o_1)}),(o_2, b^\theta_{n^k(o_2)}))$. By Properties 5 and 6 of Condition M, m induces either Rule 1 or Rule 2 so that there must exist a $j \in \{0,\ldots,J-1\}$ such that $g^k(m) \in \mathscr{A}_j \cup \mathscr{B}_j$. More specifically, if m induces Rule 1, then $g^k(m) = a_j^\theta \in \mathscr{A}_j$ where $j = n^k(o_1)$, while if m induces Rule 2, then $g^k(m) = b_j^\theta \in \mathscr{B}_j$ where $j = n^k(o_2)$. Thus, for some i, we have either $a_j^\theta \succ_i^\theta$

$f(\theta)$ or $b_j^\theta \succ_i^\theta f(\theta)$. This implies four possible different scenarios, with two immediately completing our proof. It remains to show that the last two scenarios, where $a_j^\theta \succ_2^\theta f(\theta)$ or $b_j^\theta \succ_1^\theta f(\theta)$, will also result in either $a_j^\theta \succ_1^\theta f(\theta)$ or $b_j^\theta \succ_2^\theta f(\theta)$.

Assume first that $a_j^\theta \succ_2^\theta f(\theta)$ and $f(\theta) \succ_2^\theta b_j^\theta$ both hold. This implies that $a_j^\theta \neq f(\theta)$ and a_j^θ is $(\mathscr{A}_j, \mathscr{B}_j)$-acceptable at state θ, which would contradict Property 4 of Condition P. Hence, $a_j^\theta \succ_2^\theta f(\theta)$ implies $b_j^\theta \succ_2^\theta f(\theta)$ A similar argument holds to show that $b_j^\theta \succ_1^\theta f(\theta)$ implies $a_j^\theta \succ_1^\theta f(\theta)$. Thus, we have found a j such that $a_j^\theta \succ_1^\theta f(\theta)$ or $b_j^\theta \succ_2^\theta f(\theta)$, completing the proof. ■

Lemma 10 *Suppose that Condition P+M holds and $f(\theta) \notin Z$. Then, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that there are no compelling mixed strategy equilibria of the game $\Gamma^K(\theta, u)$ for all $u \in \mathscr{U}^\varepsilon$.*

Proof of Lemma 10: We prove this by contradiction. That is, there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there exist $u \in \mathscr{U}^\varepsilon$ and $\sigma^k \in NE^*(\Gamma^k(\theta, u))$ for which σ^k is a compelling mixed strategy equilibrium of the game $\Gamma^k(\theta, u)$. We fix k large enough so that by our hypothesis, we can fix $u \in \mathscr{U}^\varepsilon$ and a compelling mixed strategy equilibrium $\sigma^k \in NE^*(\Gamma^k(\theta, u))$. When we determine the exact size of k later, we guarantee that such k potentially depends on ε but not on u. Since σ^k is compelling in the game $\Gamma^k(\theta, u)$ and the mechanism Γ^k pure Nash implements the SCF f under Condition P by Lemma 4, we have that, for each $i \in \{1, 2\}$,

$$\sum_{\tilde{m} \in M} \sigma^k(\tilde{m}) u_i(g^k(\tilde{m}), \theta) \geq u_i(f(\theta), \theta),$$

with at least one strict inequality for some $i \in \{1, 2\}$. This implies that there exist

$i \in \{1,2\}$ and $m \in \text{supp}(\sigma^k)$ such that

$$u_i(g^k(m),\theta) > u_i(f(\theta),\theta).$$

Fix such $i \in \{1,2\}$. We introduce the following partition over $\text{supp}(\sigma^k)$: $\{\{M^+\},\{M^0\},\{M^-\}\} = \text{supp}(\sigma^k)$ such that

$$\begin{aligned} M^+ &= \{m \in M^k |\ u_i(g^k(m),\theta) > u_i(f(\theta),\theta)\}, \\ M^0 &= \{m \in M^k |\ u_i(g^k(m),\theta) = u_i(f(\theta),\theta)\}, \\ M^- &= \{m \in M^k |\ u_i(g^k(m),\theta) < u_i(f(\theta),\theta)\}. \end{aligned}$$

By construction, we have $M^+ \neq \emptyset$. Using the characterization of compelling mixed strategy equilibria in $NE^*(\Gamma^k(\theta,u))$ obtained by Lemmas 8 and 9, σ^k induces Rule 3 with positive probability. By Property 6 of Condition M, we also have $M^- \neq \emptyset$. Define the following notation:

$$\begin{aligned} u_+ &\equiv \max_{m\in M^+} u_i(g(m),\theta), \\ u_- &\equiv \max_{m\in M^-} u_i(g(m),\theta). \end{aligned}$$

By construction, we have $u_+ > u_-$ and $u_i(f(\theta),\theta) > u_-$. We now define

$$K \equiv \min\left\{k \in \mathbb{N} \middle|\ k > \frac{2(1-\varepsilon)}{\varepsilon}\right\}.$$

We fix $k = K$. Since $\sigma^k \in NE^*(\Gamma^k(\theta,u))$, by the definition of $NE^*(\Gamma^k(\theta,u))$, no agents randomize over alternatives. Since we assume $f(\theta) \notin Z$, by Property 6 of Condition M, we have that $m \in M^+$ only if m induces either Rule 1 or Rule 2.

Furthermore, using the characterization of compelling mixed strategy equilibria in $NE^*(\Gamma^k(\theta,u))$ by Lemmas 8 and 9 and the construction of the mechanism Γ^k, we conclude that the probability that σ^k induces messages in M^+ is "at most" $2/k$. Moreover, by the construction of the mechanism Γ^k, we have that $m \in M^-$ if m induces Rule 3. Once again as $f(\theta) \notin Z$, by Property 6 of Condition M and the construction of the mechanism Γ^k, we conclude that the probability that σ^k induces messages in M^- is "at least" $1-2/k$. Then,

$$
\begin{aligned}
U_i(\sigma^k,\theta) &= \sum_{m\in M^+}\sigma^k(m)u_i(g^k(m),\theta)+\sum_{m\in M^0}\sigma^k(m)u_i(g^k(m),\theta)+\sum_{m\in M^-}\sigma^k(m)u_i(g^k(m),\theta)\\
&\leq u_+\times\frac{2}{k}+u_-\times\left(1-\frac{2}{k}\right)\\
&\quad \Bigg(\because M^+\neq\emptyset,\ M^-\neq\emptyset,\ \sum_{m\in M^+\cup M^0}\sigma^k(m)u_i(g^k(m),\theta)\leq u_+\times(2/k)\ \text{ and}\\
&\quad \sum_{m\in M^-}\sigma^k(m)u_i(g^k(m),\theta)\leq u_-\times(1-2/k)\Bigg)\\
&= \frac{2}{k}(u_+-u_-)+u_-,
\end{aligned}
$$

which we define as $h(k)$. Since $u\in\mathscr{U}^{\varepsilon}$, we have

$$\frac{2(u_+-u_-)}{u_i(f(\theta),\theta)-u_-}\leq\frac{2(1-\varepsilon)}{\varepsilon}<K.$$

As $h(k)$ is strictly decreasing in k, we have

$$h(K)<h\left(\frac{2(1-\varepsilon)}{\varepsilon}\right)\leq h\left(\frac{2(u_+-u_-)}{u_i(f(\theta),\theta)-u_-}\right)=u_i(f(\theta),\theta).$$

Therefore, when $k = K$, we have

$$U_i(\sigma^K, \theta) \leq h(K) < u_i(f(\theta), \theta).$$

This contradicts the hypothesis that σ^K is a compelling mixed strategy equilibrium of the game $\Gamma^K(\theta, u)$. ■

The proof of Theorem 1 is completed as follows. By Lemma 4, the mechanism Γ^k pure Nash implements the SCF f. When $f(\theta) \in Z$, by Lemma 5, for any $u \in \mathscr{U}$, the game $\Gamma^k(\theta, u)$ has no compelling mixed strategy equilibria. When $f(\theta) \notin Z$, by Lemma 10, there exists $K \in \mathbb{N}$ large enough so that for any $u \in \mathscr{U}$, the game $\Gamma^K(\theta, u)$ has no compelling mixed strategy equilibria. Thus, f is C-implementable with respect to $\mathscr{U}$ by the mechanism Γ^k. ■

4.6 Indispensability of Condition P+M

While our result deals only with sufficient - rather than necessary - conditions, one can inquire if it is possible to make the same argument with weaker assumptions. In other words, if the collection of subsets $\{\mathscr{A}_j\}_{j=0}^J, \{\mathscr{B}_j\}_{j=0}^J$ and the function $z : \{0, \ldots, J\} \times \{0, \ldots, J\} \to A$ does not satisfy Condition P+M, will any mechanism in the class of mechanisms constructed from these elements as outlined in Theorem 1 still achieve compelling implementation? We seek to show below that if any of the properties in Condition P+M is missing, then no mechanism in the class Γ^k will C-implement the SCF f for the environment $\mathscr{E}$. In this sense, we say that all of the properties in Condition P+M are *indispensable* for our argument.

4.6.1 Condition P is Necessary for pure Nash implementation

Lemma 11 *Suppose that the SCF f is pure Nash implementable. Then, there exist two collections of subsets of alternatives $\{\mathscr{A}_j\}_{j=0}^J, \{\mathscr{B}_j\}_{j=0}^J \subseteq A$ and a function $z : \{0,\dots,J\} \times \{0,\dots,J\} \to A$ such that Condition P holds.*

Proof: Let $J = |\Theta| - 1$, the number of possible states of the world, minus one, and let $j : \Theta \to \{0,\dots,J\}$ be a bijection that indexes each of the possible states of the world. For example, if $\Theta = \{\theta_t\}_{t=0}^J$, then $j(\theta_t) = t$. Let $\Gamma = (M, g)$ be a mechanism that pure Nash implements f. For each $\theta \in \Theta$, we define $m^\theta = (m_1^\theta, m_2^\theta)$ as a pure strategy Nash equilibrium of the game $\Gamma(\theta)$. The existence of m^θ is guaranteed by by our hypothesis that f is pure Nash implementable by the mechanism Γ. Then we can define sets $\mathscr{A}_{j(\theta)}, \mathscr{B}_{j(\theta)}$ as follows:

$$\mathscr{A}_{j(\theta)} = \{a \in A \mid \exists\, m_1 \in M_1 \text{ such that } g(m_1, m_2^\theta) = a\},$$

and

$$\mathscr{B}_{j(\theta)} = \{a \in A \mid \exists\, m_2 \in M_2 \text{ such that } g(m_1^\theta, m_2) = a\}.$$

Let j^{-1} be the inverse function of j, which is a bijective mapping from Θ to $\{0,\dots,J\}$. For each $(j_1, j_2) \in \{0,\dots,J\} \times \{0,\dots,J\}$, we define $z(j_1, j_2) = g(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$. We can see clearly from the definition of the sets $\mathscr{A}_{j(\theta)}, \mathscr{B}_{j(\theta)}$ above that point 1 (of condition P+N) is satisfied.

We claim now that message m^θ is a pure strategy Nash Equilibrium of $\Gamma(\theta)$ if and only if it is $(\mathscr{A}_{j(\theta)}, \mathscr{B}_{j(\theta)})$-acceptable at state θ. Indeed, if m^θ is a Nash Equilibrium, then there is no other message (m_i, m_{-i}^θ) such that $g(m_i, m_{-i}^\theta) \succ_i^\theta$

$g(m^\theta)$, which due to the way that the sets were constructed, implies that $g(m^\theta)$ is $(\mathscr{A}_{j(\theta)}, \mathscr{B}_{j(\theta)})$-acceptable. Similarly, if m^θ is not a Nash Equilibrium of $\Gamma(\theta)$ in pure strategies, then there must be some message m' and some player i such that $g(m') \succ_i^\theta g(m^\theta)$, which once again, due to the way the sets were constructed, implies that $g(m^\theta)$ is not a Nash Equilibrium of the game. Having established that, the proof for points 3 and 4 being satisfied follows from the definition of implementation in pure strategies: for each state θ, there is a message m^θ such that m^θ is a NE of the game and $g(m^\theta) = f(\theta)$. Additionally, if m is a Nash Equilibrium in pure strategies, then $g(m) = f(\theta)$.

Finally, for point 2, if for some state θ there exists a pair (j_1, j_2) such that $z(j_1, j_2) \succ_1^\theta a$ for each $a \in \mathscr{A}_{j_2}$ and $z(j_1, j_2) \succ_2^\theta b$ for each $b \in \mathscr{B}_{j_1}$, then $(m_1^{j^{-1}(j_1)}, m_2^{j^{-1}(j_2)})$ must be a Nash Equilibria of the implementing mechanism, and thus, by the definition of function z above, $f(\theta) \in Z$. Hence, if $f(\theta) \notin Z$, there must be for each pair $(j_1, j_2) \in \{0, \ldots, J\}^2$ either an $a_{(j_1,j_2)} \in \mathscr{A}_{j_2}$ such that $a_{(j_1,j_2)} \succ_1^\theta z(j_1, j_2)$ or a $b_{(j_1,j_2)} \in \mathscr{B}_{j_1}$ such that $b_{(j_1,j_2)} \succ_2^\theta z$, which are the requirements of point 2. ■

The lemma above shows that Condition P is a necessary condition for pure Nash implementation when there are two players. Indeed, they are exact counterparts of the conditions identified by Moore and Repullo (1990) for pure Nash implementation for the case of two players. It follows from their work that these conditions are also sufficient (which is already verified in Lemma 4).

4.6.2 Indispensability of Condition M

We show now that if the environment does not satisfy either of the properties in Condition M it is possible to find a compelling equilibrium in our class of mechanisms, regardless of the value of k is chosen. In this way, both properties in Condition M are indispensable for our C-implementation.

We start with Property 6. Indeed, Jackson's Example 4 illustrates an environment where all properties of Condition P+M are satisfied, with the exception of Property 6. Properties 1 to 4 are satisfied since, as seen, they are necessary for pure Nash implementation. We can verify that they imply $J = 1$, $z(j_1, j_2) = d \; \forall (j_1, j_2) \in \{0,1\}^2$, $\mathscr{A}_0 = \mathscr{B}_0 = \{c,d\}$, $\mathscr{A}_1 = \mathscr{B}_1 = \{a,b,d\}$. In this case, Property 5 is vacuously satisfied, as there $f(\tilde{\theta}) \neq d$ for $\tilde{\theta} \in \{\theta, \theta'\}$. As such, properties 1 to 5 are satisfied, and yet, it is known that no finite mechanism can C-implement the SCF in this environment. Thus, all of the mechanisms in the Γ^k class of mechanism will have a compelling mixed-equilibrium at θ'.

For the indispensability of Property 5, consider the same environment as Section 4. However, rather than defining the function z as it would be implicitly defined by the mechanism of that section - that is, a constant function $z(j_1, j_2) = d$, we instead define it incorrectly in a way to violate Property 5 alone. Formally, take $J = 1$, $\mathscr{A}_0 = \mathscr{B}_0 = \{c,d\}$, $\mathscr{A}_1 = \mathscr{B}_1 = \{a,b,d\}$, and define function $z(j_1, j_2)$ as $z(1,1) = c$, $z(1,0) = z(0,1) = z(0,0) = d$. We can check easily that properties 1 to 4, as well as Property 6 are still satisfied. Nonetheless, as Property 5 fails, every mechanism of the class Γ^k created by these sets and the function $z(j_1, j_2)$ in the manner described by Theorem 1 will have a compelling mixed-strategy equilibrium at state θ'. Such equilibrium occurs when both agents randomize uniformly over the integers in the interval $\{k, \dots, 2k-1\}$, with agent 1 always selecting a as the second component of his messages and agent 2 always selecting b as his second component of his messages. In this mixed equilibrium, outcome d is realized with probability zero and both a and b are realized with positive probabilities, thus, the expected utility of the outcome is always strictly above the utility of the socially optimal outcome for both agents.

4.7 Comparison with the Canonical Mechanism of Moore and Repullo (1990)

We compare the mechanism introduced in this paper and the natural finite version of the canonical mechanism developed by Moore and Repullo (1990). We will show that when applied to the setting of our modified version of Example 4 of Jackson (1992), the finite version of the canonical mechanism still admit a compelling mixed strategy equilibrium. In contrast, we know that this problem ceases to exist if we use our mechanism.

Suppose that agent 1 has the state-independent preference $a \succ_1 b \succ_1 c \succ_1 d$ and agent 2 has the preference $a \succ_2^\theta b \succ_2^\theta d \succ_2^\theta c$ at state θ and preference $b \succ_2^{\theta'} a \succ_2^{\theta'} c \succ_2^{\theta'} d$ at state θ'. Consider the SCF f such that $f(\theta) = a$ and $f(\theta') = c$.

The natural finite adaptation of the canonical mechanism that pure Nash implements an SCF is obtained by replacing the integer game with a modulo game instead. The modulo game is regarded as a finite version of the integer game in which agents announce integers from a finite set. The agent who matches the modulo of the sum of the integers gets to name an allocation.

We formally define this canonical mechanism adapted for our finite setting (and the fact that we are using a function, rather than a correspondence) as follows: $\Gamma^{MR} = (M, g)$ is a mechanism composed of a message space $M_i = \{(\theta_i, b_i, n_i) \in \{\theta, \theta'\} \times \{a, b, c, d\} \times \{0, 1, 2\}\}$ identical for both agents, and an outcome function $g : M \to \{a, b, c, d\}$ defined by the following rules:

- If $\theta_1 = \theta_2 = \theta$, then $g(m) = f(\theta)$.
- If $\theta_1 \neq \theta_2$ and either $n_1 = 0$ or $n_2 = 0$, then $g(s) = d$.
- If $\theta_1 \neq \theta_2$ and $n_1 = n_2 \in \{1, 2\}$, then $g(m) = b_2$.
- If $\theta_1 \neq \theta_2$ and $n_1 \neq n_2$, with $n_1, n_2 > 0$, then $g(m) = b_1$.

This mechanism is similar to the one presented in the section of Example 4. In particular, it still features the same mixed strategy equilibrium at state θ', where player 1 randomizes uniformly between messages $(\theta, 1, a)$ and $(\theta, 2, a)$, while player 2 randomizes uniformly between messages $(\theta', 1, b)$ and $(\theta', 2, b)$. This yields an outcome that dominates the outcome of the SCF for state θ'. However, the mechanism proposed at section 5 does not suffer from this problem.

In particular, we can verify that Condition P+M is satisfied for this setting quite easily. Take $J = 1$, $z(j_1, j_2) = d\ \forall (j_1, j_2) \in \{0, 1\}^2$, $\mathscr{A}_0 = \mathscr{B}_0 = \{c, d\}$, $\mathscr{A}_1 = \mathscr{B}_1 = \{a, b, d\}$. Property 1 of the Condition is verified as $d \in \mathscr{A}_0, \mathscr{A}_1, \mathscr{B}_0, \mathscr{B}_1$. Property 5 of P+M is vacuously satisfied, as there is no $\theta'' \in \Theta$ such that $f(\theta) = d$. Property 2 is verified as $a \in \mathscr{A}_0$ and $c \in \mathscr{A}_1$. Property 3 is verified as a is $(\mathscr{A}_1, \mathscr{B}_1)$-acceptable in state θ and c is $(\mathscr{A}_0, \mathscr{B}_0)$-acceptable in state θ'; moreover, these are the only alternatives that are $(\mathscr{A}_j, \mathscr{B}_j)$-acceptable for some $j = 0, 1$ in each state. This verifies Property 4 as well. Lastly, Property 6 is verified as both players prefer a to d in state θ and c to d in state θ'. Thus, all properties in Condition P+M hold.

With Condition P+M satisfied, we can invoke Theorem 1 and have that there is some k such that all the mixed strategies in the mechanism Γ^k do not dominate $f(\theta'')$ for any $\theta'' \in \Theta$. This sheds some light on the importance of selecting the implementing mechanism carefully, particularly once one starts thinking about potentially disruptive mixed equilibria. As seen in this example, even canonical mechanisms can fail to withstand this plausability check posed by Jackson. While in his original example such problem was unavoidable, in many contexts this problem can be circumvented with a more careful selection of the implementing mechanism, as seen in this section.

4.8 Conclusion

This paper presented a concept of compelling implementation, by which we strengthen the requirement of pure-strategy Nash implementation by taking care of what we call compelling mixed strategy equilibria, but ignoring other mixed strategy equilibria. We call a mixed strategy equilibrium compelling if its outcome Pareto dominates any pure strategy equilibrium.

The main contribution of this paper is to provide Condition P+M under which compelling implementation is possible by finite mechanisms in environments with two agents. We also show that Condition P+M is indispensable for our result. Our implementing mechanism has desirable properties: transfers are not needed at all; only finite mechanisms are used; integer games are not invoked; and agents' risk attitudes do not matter.

We conclude this paper with possible extensions. First, we assume throughout the paper that agents have strict preferences over alternatives. A slight relaxation of this assumption can be made, as we only rely on the best alternative for each agent in each choice set $\mathscr{A}_j$ and $\mathscr{B}_j$ to be unique. However, since the SCF determines the choice of both collection of sets, this places domain restrictions which depend upon the chosen SCF. Thus, indifferences must be allowed on a case by case basis.

Second, our analysis covers only the case of two agents. It is well known in the literature of implementation that the analysis for two agents is very different from that for three or more agents. The main reason for our focus on two-agent environment comes from the following technical difficulty: A crucial step we exploit by Lemmas 6 and 7 to characterize the structure of compelling mixed equilibria is to argue that there is no loss of generality to focus on mixed strategies where agents only randomize over integers in our mechanism. Extending these results to the case of more than two agents is challenging because we then have no a priori clear way of obtaining a simple characterization of compelling equilibria such as Lemmas 6 and 7. Furthermore, we also do not know how to establish the counterpart of Lemma 8. Despite all the difficulties outlined, this is an important open question we pursue in our future research.

Bibliography

Aswal, N., Chatterji, S., Sen, A., 2003. Dictatorial domains. Economic Theory 22.

Barberà, S., 2011. Chapter twenty-five – strategyproof social choice.

Bergemann, D., Morris, S., Tercieux, O., 2011. Rationalizable implementation. Journal of Economic Theory 146.

Black, D., 1948. The decisions of a committee using a special majority. Econometrica 16.

Bochet, O., Storcken, T., 2010. Maximal domains for maskin monotone pareto optimal and anonymous choice rules.

Chatterji, S., Massó, J., 2018. On strategy-proofness and the salience of single-peakedness. International Economic Review 59.

Chatterji, S., Sanver, R., Sen, A., 2013. On domains that admit well-behaved strategy-proof social choice functions. Journal of Economic Theory 148.

Chatterji, S., Sen, A., 2011. Tops-only domains. Economic Theory 46.

Demange, G., 1982. Single-peaked orders on a tree. Mathematical Social Sciences 3.

Kalai, E., Muller, E., 1977. Characterization of domains admitting nondictatorial social welfare functions and nonmanipulable voting procedures. Journal of Economic Theory 16.

Klaus, B., Bochet, O., 2013. The relation between monotonicity and strategy-proofness. Social Choice and Welfare 40.

Kutlu, L., 2009. A dictatorial domain for monotone social choice functions. Economics Letters 105.

Maskin, E., 1999. Nash equilibrium and welfare optimality. Review of Economic Studies 66.

Moulin, H., 1980. On strategy-proofness and single peakedness. Public Choice 35, 437–455.

Muller, E., Satterthwaite, M. A., 1977. The equivalence of strong positive association and strategy-proofness. Journal of Economic Theory 14.

Nehring, K., Puppe, C., 2007. The structure of strategy-proof social choice - part i: General characterization and possibility results on median spaces. Journal of Economic Theory 135.

ORTEGA, O., KRISTON, G., 2013. The median function on trees. Discrete Mathematics, Algorithms and Applications 05, 1350033.

Saijo, T., 1988. Strategy space reduction in maskin's theorem: Sufficient conditions for nash implementation. Econometrica 56.

Appendix A

Appendix A covers all of the proofs in Chapter 2. Section A.1 covers the cornerstone of the analysis, an alternative way to express Monotonicity that takes advantage of our other assumptions. Section A.2 gives a simple sketch of the necessity part of the proof, with section A.2.2 explaining many of the key properties and ideas that will be exploited in the necessity part of the proof of Theorem 1. Section A.3 presents the complete proofs of the results in Chapter 2, with Theorem 1 being broken down into several Lemmas.

A.1 An Alternative Expression of Maskin's Monotonicity

Our problem can be understood, in a broad sense, as an analogous to a constrained maximization problem, as we are seeking to find the largest domains that can still sustain a SCF that is Monotonic and Well Behaved. This is a complex problem to solve. Fortunately, there is an alternative formulation of this problem that is more tractable. In this alternative formulation, the problem is expressed in terms of the SCF, which not only is an object that is easier to work with, but also is now compatible with the way the restrictions of the problem are expressed. To make this alternative formulation, we first need to express the Monotonicity condition in a way that is more convenient for us.

We can think of MM as a rule that broadly says "if certain inputs are provided to the SCF and certain outputs are obtained as a result, this rule is violated". Thus, if the rule is to *not* be violated, then we can either exclude the problematic inputs, so that they never happen, or change the outputs associated with those inputs, so that they are now acceptable. While this logic stands on its own, regardless of other assumptions, when we pair it with the properties of Well Behaved functions, we can make a precise formulation of it by using the M_D and C_f sets:

Claim: *A Well Behaved Social Choice Function f satisfies MM for the domain D if and only if for every alternative $a \in A$, the intersection $C_f^a \cap M_D^a$ is empty.*

Proof: Assume that there are alternatives a, b, c such that $(b,c) \in C_f^a \cap M_D^a$. As $(b,c) \in C_f^a$, that means that there are profiles P, P' such that a is the alternative selected by the SCF at P, but not the alternative selected at P', and the votes between the two profiles differ only by a single voter (assume that it is the first voter; by Anonymity, this is without loss of generality) flipping its vote from b to c. Because f satisfies Anonymity and Tops-Only, we can specify that $P = (P_{1b}, P_2, P_3, \ldots, P_n)$ and $P' = (P_{1c}, P_2, P_3, \ldots, P_n)$, where P_{1b} and P_{1c} are preferences with b and c ranked first, respectively. Now, because the pair (b,c) belongs to M_D^a, we can also find preferences $\hat{P}_{1b}$ and $\hat{P}_{1c}$ where b and c are also ranked first, respectively, but such that $\hat{P}_{1b} \mapsto_a \hat{P}_{1c}$. Then, we can specify the profiles $\hat{P} = (\hat{P}_{1b}, P_2, P_3, \ldots, P_n)$ and $\hat{P}' = (\hat{P}_{1c}, P_2, P_3, \ldots, P_n)$. By Tops-Only of f, we must have $f(\hat{P}) = a$ and $f(\hat{P}') \neq a$, which violates MM as $\hat{P} \mapsto_a \hat{P}'$. Thus, whenever

$C_f^a \cap M_D^a \neq \emptyset$, MM is violated.

For the suffiency part, assume that MM is violated, that is, there are two profiles $P^0 = (P_1, P_2, P_3, \ldots, P_n)$ and $P^n = (P_1', P_2', P_3', \ldots, P_n')$ such that $f(P^0) = a$, $f(P^n) \neq a$ and $P^0 \mapsto_a P^n$. Let profile P^1 be defined as $(P_1', P_2, P_3, \ldots, P_n)$. If $P^0 \mapsto_a P^n$, then we must have $P^0 \mapsto_a P^1$. Assume that $f(P^1) \neq a$. Then, because f satisfies Tops-Only, we must have that the top ranked alternatives in P_1 and P_1' are different, call them b and c, respectively. We then have that $P^0 \mapsto_a P^1 \Rightarrow P_1 \mapsto_a P_1' \Rightarrow (b,c) \in M_D^a$. Similarly, as P^0 and P^1 differ in a single vote, we will have that (b,c) also belongs to C_f^a, as it is a pivotal change when the outcome of the SCF is a. This imples that $(b,c) \in C_f^a \cap M_D^a$. If $f(P^1) = a$, we proceed to the next voter, constructing profile P^2 in a similar fashion as $P^2 = (P_1', P_2', P_3, \ldots, P_n)$. Because $P^0 \mapsto_a P^n$, we will have that $P^1 \mapsto_a P^2$ and, more generally, between any two profiles P^{j-1}, P^j constructed in this way we will have $P^{j-1} \mapsto_a P^j$. Similarly, because $f(P^n) \neq a$, we know that there will be an index $j = 1, \ldots, n$ that $f(P^{j-1}) = a$ and $f(P^j) \neq a$. Then we can apply the same reasoning illustrated above to that case. Thus, whenever MM is violated, $C_f^a \cap M_D^a \neq \emptyset$. $\star$

This alternative definition allows us to translate some of the properties of the SCF into properties for the domain of the function and vice-versa. As functions are easier to analyze, this is extensively used in our proofs.

A.2 Sketch of Proof

A.2.1 General Overview

To prove the necessity part, we need to establish three things: i) find a tree graph G whose set of nodes matches the set of alternatives for the domain in question; ii) among the nodes of this graph, pick one node t to form an admissible pair with G; iii) verify that the Minimal Reversals Condition holds for this choice of admissible pair. Hence, our proof for this is also divided in three blocks. The first block, composed by Lemmas 1 to 6, deals with proving that the Connectivity Graph of the domain is a tree graph. We do so by proving that any cycle in the connectivity graph implies a contradiction for the SCF. The second block, composed of Lemmas 7 to 12, deals with proving the existence of a special node (called the threshold) in the Connectivity Graph with some useful properties, which will be our choice for t. This is achieved by showing that all the breaches (a concept that will be presented later on) must be oriented in such a way that implies a common origin point. Finally, the third block, composed of Lemmas 13 to 15, proves that the Minimal Reversals Condition is indeed verified when taking the Connectivity Graph and the threshold as an admissible pair. We do this by exploiting the properties of the threshold found.

A.2.2 An Illustration

We provide now a small illustration of the most important properties exhibited by the domains and SCFs that we are studying. They are derived from the Lemmas in the necessity part and understanding them provides a good overview of the

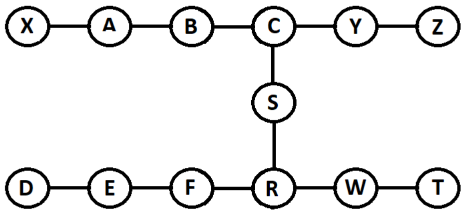

Figure A.1: A Connectivity Graph

reasoning behind that proof. This illustration will be based off a domain with the following Connectivity Graph:

- Take any two alternatives, say x and z. Since the graph is connected, there must be at least one path from x to z. Pick now a third alternative, say, s. Our first property is the following: whenever the outcome of the SCF is s for some profile P, if there is a path connecting x and z that **does not** include s, then we can take all of the votes for alternative x at profile P and give them to alternative z (creating a new profile, P') *without changing the outcome* of the SCF; i.e, at profile P', the outcome must still be s. We could also do the reverse and transfer all the votes from z to x, the direction is not important. All that matters is that there is a path between the two alternatives that does not pass through the outcome. So, for instance, if the outcome were c instead we would not be able to do this procedure between z and x - but would still be able to transfer these votes between x and a, for example. When we can move votes freely between two alternatives without altering the outcome of the SCF, we say that x and z *share votes* under s. This property is similar to what happens in generalized median rules, and it comes from Lemmas 1, 2 and 5. This is the essence of our approach, and it translates properties from the *domain* (like the Connectivity Graph) into implications about the (Well Behaved and Monotonic) *SCFs* on that domain.

- Conversely, alternatives that lie on opposite sides of a particular node, like x and z relative to c, must **not** share votes under c. One way transfer, like from z to x under c, but not the other way around, is still allowed, but we cannot have both x transferring votes to z and z transferring votes to x under c at the same time. This is explained in Lemmas 4 and 5.
- The Connectivity Graph must be a tree. This comes essentially from the two properties above. If a cycle is allowed - say, alternatives y and b are both connected to s - then we could find a path from x to z without passing through c. By the first property, x and z would share votes, but by the second they would not. The only way to keep consistency is by having no cycles in the graph.
- The possibilities of vote transferring can be ”added” together to form longer paths. For instance, if y can transfer votes to b under c, then essentially z can transfer votes to x under c: from z to y, then from y to b, then from b to x. Similarly, if it just so happens that x can transfer votes to d under s (something that is possible, even if not shown in the Connectivity Graph), then as a consequence r will be able to transfer votes to c under s, by going from r to d, then from d to x, then from x to c.
- Take a pair of adjacent alternatives, like w and r. If alternative w can transfer votes under r to some other alternative on the opposite side of r, like s, then we say that (r, w) forms a breach.
- A domain might not exhibit any breaches. If such is the case, then any node can be picked as a threshold node.

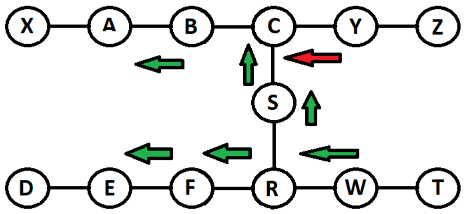

Figure A.2: Several breaches represented by arrows.

- If a domain does exhibit at least one breach, say, (r, w), then we can draw an arrow representing that breach, starting from the second node of the pair towards the first node (for example, the arrow would start at w and have the arrowhead at r). If there are any more breaches in a domain, no two arrows must be pointing towards the same node. Figure 8 illustrates this. All the breaches represented by the arrows in green are compatible with one another, but the breach (c, y), represented by the red arrow, is not compatible with the rest, at it points towards the arrow representing breach (c, s). This is explained in Lemma 8.
- As all breaches must follow the same orientation and the number of nodes in the graph is finite, there must be at least one node serving as an origin point for all breaches. In our example (ignoring the breach in red), that would be node t. This is explained in Lemmas 7, 8, 9, 10, 11 and 12. We call this origin point of the breaches the threshold node.

The last step of the proof is essentially combining all of the properties above to identify which alternatives cannot share votes and translate this set of restrictions for the SCF back into a set of restrictions for the domain. For instance, assume that we have breaches as depicted in Figure 8 (ignoring the one in red). It follows that it is not possible to transfer votes from x to z under c, since this movement would imply the existence of a breach (c,b) and this breach goes against the other green breaches, similarly to what happened with the breach (c,s) represented by the red arrow. This restriction for the SCF, in turn, implies that alternative c must not maintain its position when going from any preference with x on top to any preference with z on top (i.e., some reversals must occur between these preferences).

A.3 Complete proofs

Our starting point is a simple well known fact related to MM: changing a vote from any other alternative to alternative a makes a maintain its position and as such, if a was the social choice before, a must still be the social choice after receiving more votes, by MM. Then, by using the notation developed in the Model section, we have:

Fact 1: *Let f satisfy MM. Fix any alternative a. In any minimally rich domain, $(b,a) \notin C_f^a \; \forall b \neq a$.*

Assume henceforth that f is a Well Behaved SCF satisfying MM.

Lemma 1: *If $f(P) = a$ and $(b,c) \notin C_f^a$, then $f(P') = a$ for all profiles P' satisfying:*

1. $v(b,P') = v(b,P) - k$, $k \leq v(b,P)$;
2. $v(c,P') = v(c,P) + k$
3. $v(d,P') = v(d,P)$, $\forall d \neq b,c$

Proof: When $k = 1$, the result is immediate from the definition of the set C_f^a: as (b,c) is not a part of this set, then $f(P) = f(P') = a$. For k greater than one, we can proceed as follows. First, rearrange profile P such that the first k voters all vote b in P. This is allowed by Anonymity. Then, define profile P^1 as being identical to profile P, with the only difference that the first voter in P^1 votes for c instead of b. By the argument above, $f(P^1) = a$. Now construct profile P^2 in a similar fashion, with it being identical to profile P, except that now the first two voters vote for c instead of b. By the same logic, we must have $f(P^2) = a$. Proceed with this construction, until you reach P^k. We must have that $f(P^k) = a$. We also have that the total of votes in P^k is exactly the same as in P', for every alternative. Then we can use Anonymity once again to rearrange the voters in P^k to the configuration in P' and have that $f(P') = a$. $\star$

Definition: We say that **alternative** b **transfers votes to alternative** c **under** a whenever there is a sequence of alternatives $\{x_j\}_{j=1}^k$ such that $x_1 = b$, $x_k = c$ and for any $j < k$, we have $(x_j, x_{j+1}) \in M_D^a$. We also say that **alternatives** b **and** c **share votes under** a whenever b transfers votes to c and c transfers votes to b under a.

An important remark about the two definitions above is that the sequences implied by those definitions need *not* to form a path in the Connectivity Graph.

Lemma 2: *Let $\lambda \in A$ be a subset of alternatives such that any two alternatives in λ share votes under a. Then if $f(P) = a$ we have that $f(P') = a$ for all profiles P' satisfying:*

- $\sum_{j\in\lambda} v(j,P') = \sum_{j\in\lambda} v(j,P)$

- $v(d,P') = v(d,P)$, $\forall d \notin \lambda$

Proof: We proceed by induction on the number of alternatives in λ. First, we will show that when there are only two alternatives, the result holds. Call these two alternatives in λ b and c. We assume that b and c share votes under a, which translates into the existance of two sequences: the first is a sequence of alternatives such that the first element is b, the last element is c, and every pair of sucessive elements in the sequence, (x_j, x_{j+1}) is in the set M_D^a; the second is a sequence with the same properties as the first, except that the first element is c and the last element is b.

Since under our assumptions when a pair (x_j, x_{j+1}) belongs to M_D^a then the same pair does not belong to C_f^a, the conditions for Lemma 1 are satisfied and we can apply it multiple times. Let $k = v(b,P') - v(b,P)$ and assume first that k is negative, i.e., the initial profile P has more votes for b then the desired profile P'. Use Anonymity to rearrange profile P by making the first k voters all vote for b. From the assumption that b shares votes with c, there is an alternative x_2 such that $(b,x_2) \in M_D^a$. By MM of f, then $(b,x_2) \notin C_f^a$. Then we can apply Lemma 1 to change those k votes from b to x_2 without changing the outcome of the SCF. We now can repeat this process again with alternatives $x_3, \ldots, c$ within the sequence implied by the assumption that b and c share votes, until those first k voters all vote for c. By repeatedly applying Lemma 1, the outcome of the SCF at this new profile must still be a. Finally, by Anonymity, we can then rearrange back the order of the voters to match P'. For the case where k is negative, we do the inverse procedure, starting with the first k voters all voting for c and applying Lemma 1 along the alternatives in the sequence implied by the assumption that b and c share votes, until we have that the first k voters are all voting for b.

That shows that the result is valid for λ with only two alternatives. Now assume that this is true for any subset of A containing l alternatives or less, and let λ contain $l+1$ alternatives. Take λ' as an arbitrary subset of λ with l alternatives, and let $b \in \lambda'$ and $c \in \lambda - \lambda'$. Define $k = v(c,P') - v(c,P)$ and assume first that k is negative. Then, because b and c share votes, we can apply the case for 2 alternatives to create a profile P^1 that is identical to P, except that k voters that had voted for c in P now vote for b in P^1 and we still have $f(P^1) = a$. Now, we have that $v(c,P^1) = v(c,P')$ and hence $\sum_{j\in\lambda'} v(j,P^1) = \sum_{j\in\lambda'} v(j,P')$, $v(d,P^1) = v(d,P')$, $\forall d \notin \lambda'$ and we can now apply the induction hypothesis to go from P^1 to P' without changing the outcome of the SCF. For the case where k is positive, note first that $\sum_{j\in\lambda} v(j,P') = \sum_{j\in\lambda} v(j,P)$ and $k = v(c,P') - v(c,P)$ positive implies $\sum_{j\in\lambda'} v(j,P) = \sum_{j\in\lambda'} v(j,P') + k$. Now use the induction hypothesis to create a profile P^1 such that $v(b,P^1) = v(b,P') + k$ and the votes of every alternative other than b and c in P^1 matches the votes for those alternatives in P'. Then we apply again the case for two alternatives to change the votes of k voters that voted for b at P^1 to votes for c, arriving at P'. $\star$

Lemma 3: *Let $\lambda \in A$ denote a subset of alternatives, and let $\mathscr{P}(\lambda)$ denote the set of all profiles such that $P \in \mathscr{P}(\lambda)$ if and only if $v(a,P) = 0$ for all $a \notin \lambda$. Let also alternative $x \in \lambda$ be such that for every $p,q \in \lambda$, $p,q \neq x$, we have that p and q share votes under x. Then there is a number $v^*_x(\lambda)$ such that for any profile $P \in \mathscr{P}(\lambda)$, $f(P) = x \Leftrightarrow v(x,P) \geq v^*_x(\lambda)$.*

Proof: Let the set $\Phi_x \subset \mathbb{N}$ be defined as $v_x \in \Phi_x \Leftrightarrow \exists P' \in \mathscr{P}(\lambda) : v(x,P') = v_x$ and $f(P') = x$. By Unanimity, the set Φ_x is not empty and because there is a finite number of voters, there is a minimal element to this set, call it $v_x^*(\lambda)$.

Assume first that P is a profile in λ with $v(x,P) = v_x^*(\lambda)$. By the definition of λ, we have that $\sum_{y\in\lambda-\{x\}} v(y,P) = n - v(x,P) = n - v_x^*(\lambda)$ - that is, the sum of votes for every alternative other than x is equal to the total number of votes, minus the votes that x received in this profile, $v_x^*(\lambda)$. From the definition of $v_x^*(\lambda)$ there is a profile P' such that $\sum_{y\in\lambda-\{x\}} v(y,P') = n - v_x^*(\lambda)$, that is, a profile P' where x has the same number of votes than in P and thus, the number of votes for every alternative *other than* x is also the same as in P. Hence, if $v(x,P) = v_x^*(\lambda)$, $\sum_{y\in\lambda-\{x\}} v(y,P') = \sum_{y\in\lambda-\{x\}} v(y,P)$ and because $f(P') = x$ we can then apply Lemma 2 to have that $f(P) = x$ in this case.

The case where $v(x,P) > v_x^*(\lambda)$ comes then from Fact 1 and Lemma 1. Take the profile P' implied by $v_x^*(\lambda)$, and select any number of alternatives in λ such that their total number of votes is at least equal to $v(x,P) - v_x^*(\lambda)$ (which should exist, since by the definition of $P' \in \mathscr{P}(\lambda)$, no other alternatives receive any votes). By Fact 1, for any alternative a, $(a,x) \in C_f^x$, and then, by Lemma 1, we can convert the votes from each of those alternatives into votes for x without altering the outcome of the SCF. Call this new profile P''. Then we will have that $\sum_{y\in\lambda-\{x\}} v(y,P'') = \sum_{y\in\lambda-\{x\}} v(y,P)$ and $f(P'') = x$, so we can proceed as above and apply Lemma 2 to have that $f(P) = x$.

This proves the sufficiency part of the result. For the necessity part, it comes from the definition of $v_x^*(\lambda)$ as the minimal element of the set Φ_x. $\star$

Lemma 4: Let f be a Well Behaved SCF satisfying MM. Let also a, b, and c be alternatives in A such that:

- b and c share votes under a.
- b and a share votes under c
- if $|A| > 3$, then a, b, and c share votes under any other alternative $d \neq a, b, c$.

Then alternatives a and c cannot share votes under b.

Proof: We divide the proof into a series of steps. Define the set $\lambda = \{a, b, c\}$ and the set $\mathscr{P}(\lambda)$ again as the set of profiles P such that $v(d, P) = 0$ for any $d \notin \lambda$ (for a domain with only three alternatives, this distinction is not meaningful and $\mathscr{P}(\lambda)$ is the set of all possible profiles). Then, we can proceed to the first step.

Step 1: For any $P \in \mathscr{P}(\lambda)$, $f(P) \in \lambda$. This is trivial for the case where there are only three alternatives, so we will check the case where there are four or more. We will make a proof by contradiction, so assume the above statement is false. Then we can find a profile $P \in \mathscr{P}(\lambda)$ such that $f(P) = d \neq a, b, c$. However, the sum of votes for alternatives a, b, c in $\mathscr{P}(\lambda)$ is equal to n, and the three alternatives share votes under d. Then we can transfer all the votes to, say, alternative a. By Lemma 2, the SCF in this unanimous profile should not change from d, but this violates Unanimity. Hence, the outcome of the SCF for any profile in $\mathscr{P}(\lambda)$ must be one of the three alternatives a, b, c.

Step 2: There is a number $v_a^*(\lambda)$ such that for any $P \in \mathscr{P}(\lambda)$, $f(P) = a$ if and only if $v(a, P) \geq v_a^*(\lambda)$. A number $v_c^*(\lambda)$ also exists for alternative c. This is just an application of Lemma 3, as we assume that b and c share votes under a, and b and a share votes under c.

Step 3: Assuming that a and c share votes under b implies a contradiction. If a and c share votes under b, we have that $v_a^*(\lambda)+v_c^*(\lambda)=n+1$. To see this, consider the case where $v_a^*(\lambda)+v_c^*(\lambda)<n+1$. Then for the profile P^* where $v(a,P^*)\geq v_a^*(\lambda)$, $v(c,P^*)\geq v_c^*(\lambda)$ and $v(d,P^*)=0$, $\forall d\neq a,c$, we have that the SCF must select both a and c, which is a contradiction. For the case where $v_a^*(\lambda)+v_c^*(\lambda)>n+1$, we can construct a profile P^* in $\mathscr{P}(\lambda)$ such that $v(a,P^*)=v_a^*(\lambda)-1$, $v(c,P^*)=n-v(a,P^*)$ and the SCF cannot select either a or c, so it must select b as the outcome. However, this together with the assumption that a and c share votes under b allows us to transfer all the votes from c to a by using Lemma 2, which will imply by Unanimity that the outcome of the SCF is also a as well as b, another contradiction. So we must have $v_a^*(\lambda)+v_c^*(\lambda)=n+1$.

Then consider profile $P\in\mathscr{P}(\lambda)$ where $v(a,P)=v_a^*(\lambda)-1$, $v(c,P)=v_c^*(\lambda)-1$ and $v(b,P)=1$. By Step 2, the outcome of the SCF at P cannot be either a or c. By Step 1, it also cannot be any other alternative outside a,b,c, and hence, we must have $f(P)=b$. Without loss of generality, assume that $v_a^*(\lambda)\geq v_c^*(\lambda)$ (so that $v(a,P)\geq 1$). If a and c share votes under b, then we can transfer one vote from a to c without changing the outcome of the SCF. Call this new profile P'. So by Lemma 2 it must be the case that $f(P')$ is still equal to b. However, $v(c,P')=v_c^*(\lambda)$, which by Step 2 also implies that $f(P')=c$. Hence, we cannot have that a and c can share votes under b. $\star$

Lemma 5: *Assume that $\mathscr{D}$ is a MD-Connected domain with Connectivity Graph G and there is a set of alternatives $\{a_j\}_{j=1}^k$ such that they form a path in the Connectivity Graph $\langle a_1,a_k\rangle$ Then the following two properties hold:*

1. *a_1 and a_k share votes under any alternatives that are not a part of the path*

$\langle a_1, a_k \rangle$.

2. *For any j, $1 < j < k$, we have that alternatives a_{j-1} and a_{j+1} cannot share votes under a_j.*

Proof: We start with the first statement. It comes immediately from the definitions of MD-Connectedness and vote sharing that if two alternatives are MD-connected, they share votes under every other alternative. Thus, if alternative b is not a part of path $\langle a_1, a_k \rangle$, each alternative in this path shares votes with the alternatives that are adjacent to it under b. If we append each of the sequences implied by this fact, we can form two sequences, one that transfers votes from a_1 to a_k under b and another that transfers votes a_k to a_1 under b, thus showing that the two alternatives share votes under b.

For the second part, just use the result above to check the conditions for Lemma 4. Under any alternative other than a_{j-1}, a_j or a_{j+1}, the three alternatives share votes, as we saw above. Under a_{j+1}, a_j and a_{j-1} share votes (just take a sub segment of the original path that does not include a_{j+1}). The same happens for a_j and a_{j+1} under a_{j-1}. Then, by Lemma 4, the other two alternatives cannot share votes under a_j. ⋆

Lemma 6: *There can be no cycles in the Connectivity Graph of a domain that admits a Well Behaved Social Choice Function satisfying MM.*

Proof: Assume that there is a cycle between the alternatives in the set $\{a_j\}_{j=1}^{k}$. Since they form a cycle, between any two alternatives in the set, there are two entirely distinct paths connecting them, such that there are no nodes in common between these two paths, other than the starting and ending nodes. Hence, for any two alternatives in the set, and a third distinct alternative, it is possible to find a path between the first two that does not contain the third. Then, we can take any three arbitrary alternatives, say a_1, a_2 and a_3 and by the first statement of Lemma 5 we will have that a_2 share votes with a_3 under a_1, a_2 share votes with a_1 under a_3 and that a_1 and a_3 share votes under a_2. This then creates a contradiction, according to Lemma 4. $\star$

Definition: Given a Domain D that is MD-Connected and admits a Well Behaved SCF satisfying MM (and hence, whose Connectivity Graph is a tree) and a triple of adjacent nodes (a,b,c) with $a,c \in \alpha(b),\ a \neq c$ we will say that the pair (b,c) is a **breach** if c transfer votes to a under b. Additionally, whenever we refer to the *span of a breach* (b,c), we are referring to the span $\xi(b,c)$.

Finally, we will employ the following notation in the arguments below: $P(a_1 = v_1, a_2 = v_2, \ldots, a_k = v_k)$ describes the profile where alternatives in the set $\{a_j\}_{j=1}^{k}$ get votes equal to $\{v_j\}_{j=1}^{k}$, with $\sum_{j=1}^{k} v_j = n$ (which implies that any alternative outside of $\{a_j\}_{j=1}^{k}$ gets zero votes). In other words, $v(a_j, P(a_1 = v_1, a_2 = v_2, \ldots, a_k = v_k)) = v_j$.

Lemma 7 *Let $\delta = \langle a_1, a_m \rangle$ be a maximal path in G, and $(a_j, a_{j+1}) \in \delta$ be a breach such that $a_1 \in \xi(a_j, a_{j+1})$. Then $f(P(a_1 = n-1, a_{j+1} = 1)) = a_{j+1}$. Moreover, if $a_i \in \xi(a_j, a_{j+1}) \cap \delta$, we have that $f(P(a_1 = n-1, a_i = 1)) = a_i$.*

Proof: Let $\hat{a} \in \alpha(a_j)$ be the node such that a_{j+1} transfers votes to $\hat{a}$ under a_j, implied by the definition of a breach. Importantly, we do not assume that $\hat{a}$ belongs to δ. Start with $\lambda = \{\hat{a}, a_j, a_{j+1}\}$ and $\mathscr{P}(\lambda)$ as defined previously. We now proceed in a series of steps.

Step 1: for any P in $\mathscr{P}(\lambda)$, $f(P) \in \lambda$. This is similar to Step 1 of Lemma 4. If $f(P) = d \notin \lambda$, by the first statement of Lemma 5 $\hat{a}, a_j$ and a_{j+1} all share votes under d and since $P \in \mathscr{P}(\lambda)$ we can use Lemma 3 to imply that $f(P(a_j = n)) = d$, which is a contradiction with the Unanimity assumption of f.

Now, apply Lemma 3 to alternative $\hat{a}$ and to alternative a_{j+1}, defining the numbers $v^*_{\hat{a}}(\lambda) = v_1$ and $v^*_{a_{j+1}}(\lambda) = v_2$.

Step 2: Show that $v_1 + v_2 = n + 1$. This is also similar to Step 3 of Lemma 4. If $v_1 + v_2 < n + 1$, the SCF must have two outcomes at profile $P(\hat{a} = v_1, a_{j+1} = n - v_1)$, a contradiction. If, on the other hand $v_1 + v_2 > n + 1$, we will have that $f(P(\hat{a} = v_1 - 1, a_{j+1} = n + 1 - v_1)) = a_j$ (this comes from Step 1 and $v_2 > n + 1 - v_1$) but then we can use the assumption that (a_j, a_{j+1}) is a breach and apply Lemma 2 to have that $f(P(\hat{a} = n)) = a_j$, another contradiction. Hence, we must have that $v_1 + v_2 = n + 1$.

Step 3: Show that $v_2 = 1$. First notice that by the definition of v_2 from Lemma 3, it is impossible to have $v_2 = 0$, as this would imply a violation of Unanimity. Then, as $v_2 > 0$, we have that $f(P(\hat{a} = v_1 - 1, a_j = 1, a_{j+1} = v_2 - 1)) = a_j$. From here we can use the assumption that (a_j, a_{j+1}) is a breach to apply Lemma 2 and get that $f(P(\hat{a} = n - 1, a_j = 1)) = a_j$, which then implies that $v_1 = n$, and in turn, that $v_2 = 1$, as we claimed. Note that this implies $f(P(a_j = n - 1, a_{j+1} = 1)) = a_{j+1}$.

Step 4: Show that $f(P(a_1 = n - 1, a_{j+1} = 1)) = a_{j+1}$. As $a_1 \in \xi(a_j, a_{j+1}) \Rightarrow a_{j+1} \notin \langle a_1, a_j \rangle$, we can apply Lemma 5 to have that a_1 and a_j share votes under a_{j+1}. Then we can apply Lemma 2 to have that $f(P(a_j = n - 1, a_{j+1} = 1)) = a_{j+1} \Rightarrow f(P(a_1 = n - 1, a_{j+1} = 1)) = a_{j+1}$

Step 5: Show that for all $a_i \in \xi(a_j, a_{j+1}) \cap \delta$, $f(P(a_1 = n-1, a_i = 1)) = a_i$. Assume that $f(P(a_1 = n-1, a_i = 1)) = x$. By the same reasoning of Step 1, we must have that $x \in \langle a_1, a_i \rangle$, as if it is not, we can apply the first statement of Lemma 5 and Lemma 2 to get a contradiction at $f(P(a_1 = n)) = x$. Then, assuming that $x \neq a_i$, we have that $x \notin \langle a_i, a_{j+1} \rangle$, which allow us to use Lemma 5 once more to have that a_i and a_{j+1} share votes under x and then Lemma 2 to have that $f(P(a_1 = n-1, a_{j+1} = 1)) = x$, which contradicts the earlier result that $f(P(a_1 = n-1, a_{j+1} = 1)) = a_{j+1}$. Therefore, $f(P(a_1 = n-1, a_i = 1)) = a_i$. This proves the statement. $\star$

Lemma 8 at first has a somewhat abstract formulation, but its implication is easily understood via drawings. It says that if we take all breaches (y, z) and for each breach we draw an arrow, starting the arrow at the last component of the ordered pair, z, and placing the arrowhead at the first component, y, then there will be no two arrows pointing at one another. In the language of the Lemma, (a_i, a_{i-1}), in this order, could not be a breach because the arrow drawn in this fashion would point in the opposite direction from the one drawn at breach (a_j, a_{j+1}). Hence, in a sense, all breaches must "point" away from a common origin point, rather than go against one another. This common origin point will be formally proved in Lemma 12.

Lemma 8: *Let $\delta = \langle a_1, a_m \rangle$ be a maximal path in G, (a_j, a_{j+1}) be a breach with $a_j, a_{j+1} \in \delta$ and (a_i, a_{i-1}) be a pair of adjacent nodes with $a_{i-1}, a_i \in \delta \cap \xi(a_j, a_{j+1})$. If $a_{j+1} \in \xi(a_i, a_{i-1})$, then (a_i, a_{i-1}) cannot be a breach.*

Proof: Assume it is not the case. Let (a_i, a_{i-1}) be a breach and $a_{j+1} \in \xi(a_i, a_{i-1})$. Assume without loss of generality that $a_1 \in \xi(a_j, a_{j+1})$, which implies $a_m \notin \xi(a_j, a_{j+1})$, since $a_j, a_{j+1} \in \delta$. Then $a_{i-1}, a_i \in \delta \cap \xi(a_j, a_{j+1})$ implies that $a_{i-1}, a_i \in \langle a_1, a_{j+1} \rangle$. Finally, $a_{i-1}, a_i \in \langle a_1, a_{j+1} \rangle$ together with $a_{j+1} \in \xi(a_i, a_{i-1})$ implies that $a_m \in \xi(a_i, a_{i-1})$. So, we have a clear picture of the placement of the nodes on the path δ: $a_1, ..., a_{i-1}, a_i, ..., a_j, a_j{+}1, ..., a_m$.

By the first part of Lemma 7, we have that $f(P(a_1 = n-1, a_{j+1} = 1)) = a_{j+1}$. By the first part of Lemma 5, a_1 share votes under a_{j+1} with any alternative in $\langle a_1, a_j \rangle$ hence we can use Lemma 2 to have $f(P(a_i = n-1, a_{j+1} = 1)) = a_{j+1}$. By Fact 1, we have that $(a_i, a_{j+1}) \notin C_f^{a_{j+1}}$ and by Lemma 1, we have that $f(P(a_i = 1, a_{j+1} = n-1)) = a_{j+1}$. Applying Lemma 7 for the breach (a_i, a_{i-1}), we have that $f(P(a_m = n-1, a_i = 1)) = a_i$. As $a_i \notin \langle a_{j+1}, a_m \rangle$, by the first part of Lemma 5 a_m and a_{j+1} share votes under a_i and hence by Lemma 2 we have that $f(P(a_{j+1} = n-1, a_i = 1)) = a_i$, which then contradicts our earlier conclusion that $f(P(a_i = 1, a_{j+1} = n-1)) = a_{j+1}$. Hence, (a_i, a_{i-1}) cannot be a breach $\star$.

Lemma 9: *Let (b, c) be a breach. Then, if there is a node d such that (c, d) is also a breach, we must have $d \neq b$.*

Proof: This proof follows an argument very similar to the one of Lemma 8. Assume it is not the case, that is, both (b, c) and (c, b) are breaches. Let a_1, a_m be a pair of extreme nodes, with $a_1 \in \xi(b, c)$ and $a_m \in \xi(c, b)$. By Lemma 7 on the breach (b, c), we must have $f(P(a_1 = n-1, c = 1)) = c$. Applying Lemmas 5 and 2, we have $f(P(b = n-1, c = 1)) = c$. Applying Fact 1 and Lemma 1, we further have $f(P(b = 1, c = n-1)) = c$. Now apply Lemma 7 once more, using breach (c,b) this time to have $f(P(a_m = n-1, b=1)) = b$. We can once more use Lemmas 5 and 2 to get $f(P(c=n-1,b=1)) = b$, which contradicts our earlier conclusion. Thus, we cannot have (b,c) and (c,b) to be both breaches. $\star$

Lemma 10: *Let (b,c) and (y,z) be breaches in G. Then,*

1. $y \in \langle c,z\rangle \Rightarrow b \notin \langle c,z\rangle$
2. $\xi(b,c) \subset \xi(y,z) \Leftrightarrow c \in \xi(y,z)$.
3. $\xi(b,c) \subset \xi(y,z) \Rightarrow z \notin \xi(b,c)$

Proof: We start by proving the first statement.

Assume that $b,y \in \langle c,z\rangle$, but $\langle b,y\rangle \neq \langle c,z\rangle$ (which must be true, since by Lemma 9 we cannot have $(b,c) = (z,y)$). Then we can make a maximal path that includes b,c,y and z by taking any extreme nodes $p \in \xi(c,b)$ and $q \in \xi(z,y)$. Hence, the path $\langle p,q\rangle$ will contain these nodes in the following order: $p,(\ldots),c,b,(\ldots),y,z,(\ldots),q$. Moreover, we then have that $b,c \in \xi(y,z)$ and $y,z \in \xi(b,c)$, which, by Lemma 8, implies that either (b,c) or (y,z) is not a breach, a contradiction to the assumption that both are breaches.

Now we move to the second statement.

First, we have that $\langle c,z\rangle \subset \langle b,z\rangle \cup \{c\}$, as $c \in \alpha(b)$. Thus, $y \in \langle c,z\rangle \Leftrightarrow y \in \langle b,z\rangle$. Then, $c \in \xi(y,z) \Leftrightarrow b \in \xi(y,z)$. As $b \in \xi(b,c)$, we have that $\xi(b,c) \subset \xi(y,z) \Rightarrow b \in \xi(y,z) \Rightarrow c \in \xi(y,z)$. This proves the first part.

For the second part, by the first statement already proven, $c \in \xi(y,z) \Rightarrow y \in \langle c,z\rangle \Rightarrow b \notin \langle c,z\rangle \Rightarrow z \notin \xi(b,c)$, and, as y is adjacent to z, we also have that $y \notin \xi(b,c)$. Hence, we have that for any $a \in \xi(b,c)$, we must have $x,y \notin \langle a,c\rangle$. Moreover, we must have $b \in \langle a,z\rangle$, as $b \notin \langle a,z\rangle \Rightarrow b \notin \langle a,c\rangle \subset \langle a,z\rangle \cup \langle z,c\rangle \Rightarrow a \notin$

$\xi(b,c)$. But then, since c is adjacent to b, the path $\langle a,z\rangle$ must also contain c and thus can be split into $\langle a,c\rangle$ and $\langle c,z\rangle$. Since $y \in \langle c,z\rangle$, we have that $y \in \langle a,z\rangle \Rightarrow a \in \xi(y,z)$.

Now, for the third statement, by the second part of this Lemma we have $\xi(b,c) \subset \xi(y,z) \Rightarrow c \in \xi(y,z) \Rightarrow y \in \langle c,z\rangle$. Then, by the first part of this Lemma, $y \in \langle c,z\rangle \Rightarrow b \notin \langle c,z\rangle \Rightarrow z \notin \xi(b,c)$. $\star$.

Lemma 11: *Let $(b,c),(j,k),(y,z)$ be breaches on G, with $c \in \xi(j,k)$ and $k \in \xi(y,z)$. Then $c \in \xi(y,z)$.*

Proof: By the second statement of Lemma 10, $k \in \xi(y,z) \Rightarrow \xi(j,k) \subset \xi(y,z) \Rightarrow c \in \xi(y,z)$. $\star$.

We prove the next Lemma by showing that any sequence of breaches $\{(b_k,c_k)\}_{k=0}^{T}$ in G such that for every $k > 0$ we have that c_{k-1} is in the span of (b_k,c_k) must have a finite $T < \infty$. As every of such sequences is finite, there is a last element that is not in the span of any other breach. Indeed, by Lemmas 10 and 11, we have a semblance of transitivity among the spans of breaches that allow us to make such claim when the number of alternatives is finite.

Lemma 12: *For any subtree $G(B)$ containing at least one breach there is a breach $(y,z) \in G(B)$ such that y is not in the span of any other breach in $G(B)$.*

Proof: Let $\{(b_k,c_k)\}_{k=0}^{T}$ be a sequence of breaches in G such that for every $k > 0$ we have that $\xi(b_{k-1},c_{k-1}) \subset \xi((b_k,c_k))$. Start with $\xi(b_0,c_0)^C$. As the set of nodes of G is equal to the set of alternatives, A, which is finite, any subtree of G will also have finite nodes, and thus, $\xi(b_0, c_0)^C$ must also be a finite set. Next, we have that for any $k > 0$ by the third statement of Lemma 10, $c_k \in \xi(b_{k-1}, c_{k-1})^C$, and by the second statement of that Lemma, $c_{k-1} \in \xi(b_k, c_k)$. Recall that for any span $\xi(b_{k-1}, c_{k-1})$, we have $c_{k-1} \notin \xi(b_{k-1}, c_{k-1})$, so $\xi(b_k, c_k) \not\subset \xi(b_{k-1}, c_{k-1})$. That, together with Lemma 11, which allow us to state that $\xi(b_0, c_0) \subset \xi(b_1, c_1) \subset ...\xi(b_{k-1}, c_{k-1}) \subset \xi(b_k, c_k)$ let us conclude that the sets $\xi(b_k, c_k)$ are expanding as k increases, incorporating at least one new element in each new set. Conversely, the sets $\xi(b_k, c_k)^C$ are shrinking, becoming smaller and smaller with each interaction. As they are all subsets of a finite set, the process itself is finite, meaning that for some k, there is no $c_{k+1} \in \xi(b_k, c_k)^C$. Then any sequence must be finite, with $T < m < \infty$. Let the last element of the largest of such sequences be a breach (y, z) such that z is not in the span of any other breach in this largest sequence. The last step of our analysis is to argue that z cannot be in the span of any other breach. Indeed, as we assumed that (y, z) is the last element of the largest of such sequences, if z were to be in the span of any other breach, then by the second statement of Lemma 10 we would have that the span of (y, z) is also contained in the span of this new breach, which contradicts the assumption that (y, z) was the last element of the largest sequence. Hence, z must not be in the span of any other breach. $\star$.

Corollary I: *For any maximal path $\delta \in G$ containing at least one breach there is a breach $(q, r) \in \delta$ such that r is not in the span of any other breach in δ. This is achieved by setting $G(B) = \delta$. Moreover, if G has any breaches, there is a breach $(y, z) \in G$ such that z is not in the span of any other breach in G. This is achieved by setting $G(B) = G$.*

In light of Lemma 12, we introduce now a new definition. We say that a node t on graph G is a **threshold node** if there are no breaches $(b,c) \in G$ such that $t \in \xi(b,c)$.

Lemma 13 *For any $x \in G$ and $y \in \alpha(x)$, $\xi(x,y)^C = \xi(y,x)$.*

Proof: First, we show that $\xi(y,x) \cap \xi(x,y) = \emptyset$. Without loss of generality, start with a node $a \in \xi(x,y)$. Then, $a \in \xi(x,y) \Rightarrow x \in \langle a,y \rangle \Rightarrow y \notin \langle a,x \rangle \Rightarrow a \notin \xi(y,x)$. The argument is symmetrical for the case where $a \in \xi(y,x)$.

Now, we show that $\xi(x,y) \cup \xi(y,x) = G$. First, because $x \in \alpha(y)$, for any node a we have $\langle a,x \rangle \subset \langle a,y \rangle \cup \{x\}$. Without loss of generality, start with $a \notin \xi(x,y)$. Then $x \notin \langle a,y \rangle$, which, by the above argument, implies $\langle a,x \rangle = \langle a,y \rangle \cup \{x\} \Rightarrow y \in \langle a,x \rangle \Rightarrow a \in \xi(y,x)$. Thus, $a \notin \xi(x,y) \Rightarrow a \in \xi(y,x)$. A symmetrical argument shows that $a \notin \xi(y,x) \Rightarrow a \in \xi(x,y)$, which establishes $\xi(x,y) \cup \xi(y,x) = G$, completing the proof. $\star$

Lemma 14: *Let $b \in G$ be an arbitrary, non-extreme node on G with a and c being two distinct nodes adjacent to b. If there are alternatives $x \in \xi(a,b)$ and $y \in \xi(c,b)$ such that y transfers votes to x under b, then (b,c) is a breach.*

Proof: We start first by proving the following claim: any two nodes p,q such that either $p,q \in \xi(a,b)$ or $p,q \in \xi(c,b)$ then p and q share votes under b. This can be verified as follows: without loss of generality, assume that they are both in $\xi(a,b)$. Then from the definition of $p \in \xi(a,b)$ we have $a \in \langle p,b \rangle \Rightarrow b \notin \langle p,a \rangle$ and similarly $b \notin \langle q,a \rangle$. Hence, $b \notin \langle p,q \rangle \subset \langle p,a \rangle \cup \langle a,q \rangle$. Then, by the first part of Lemma 5, p and q share votes.

We now assume that y transfer votes to x. By the above proposition, we have that, under b, a and x share votes, as do c and y. By the definitions of vote sharing and vote transference, we have that there are three sequences $\{f_j\}_{j=1}^{p}$, $\{g_j\}_{j=0}^{q}$ and $\{h_j\}_{j=0}^{r}$ such that $f_1 = c$, $f_p = g_0 = y$, $g_q = h_0 = x$ and $h_r = a$ and for any two sucessive members in either sequence we have $(f_j, f_{j+1}), (g_j, g_{j+1}), (h_j, h_{j+1}) \in M_D^b$. We can then construct a single sequence $\{w_j\}_{j=1}^{p+q+r}$ such that, for $j \in [1, p]$, $w_j = f_j$, for $j \in [0, q]$, $w_{p+j} = g_j$ and finally, for $j \in [0, r]$, $h_{p+q+j} = z_j$. This last sequence reads as $\{c, f_1, (...), y, g_1, (...), x, h_1, (...), a\}$. Then, there is a sequence starting from c and ending at a such that for any two sucessive members of this sequence we have $(w_j, w_{j+1}) \in M_D^b$. It follows that c transfers votes to a under b and hence (b, c) is a breach. $\star$

Lemma 15: *Let b be an arbitrary node in G with two distinct adjacent nodes, a, c. Let $x \in \xi(a, b)$ and $y \in \xi(c, b)$ and t be a threshold for G. If $t \notin \xi(c, b)$, then $(y, x) \notin M_D^b$*

Proof: Assume $t \notin \xi(c, b)$ and $(y, x) \in M_D^b$. By Lemma 14, $(y, x) \in M_D^b$ implies that (b, c) is a breach. And by Lemma 13, $t \notin \xi(c, b)$ implies $t \in \xi(b, c)$. But that contradicts the assumption that t is a threshold. $\star$

This concludes the necessity part of the proof. The next Lemma deals with the sufficiency part.

Lemma 16: *It is always possible to define a Well-Behaved SCF satisfying MM in a domain that satisfies the Minimum Reversals Condition.*

Proof: To show the desired result, we will first construct a particular Social Choice Function. Then, we will argue that this SCF satisfies MM. We will use the expression of MM outlined in section A.1 of the appendix to verify if MM holds for that SCF. Thus, we need to only find the pivotal changes of this SCF for each alternative (that is, the pairs in the set C_f^a for each $a \in A$) and verify if that pair is also in the set M_D^a. If the intersection of the two sets is empty, we know that MM holds. To verify that this intersection is always empty, we will invoke the Minimal Reversals Condition to show that whenever we have $(b,c) \in C_f^a$ this implies $(b,c) \notin M_D^a$ as well.

First let (G,t) denote an admissible pair for which the domain satisfies the Minimal Reversals Condition. For any profile $P \in D^n$, let $\{r_1(P)\}$ denote the set of all first ranked alternatives in the profile P, i.e. $\{r_1(P)\} = \{a_j \in A | r_1(P_i)$ for some $i \in N\}$, and $G(\{r_1(P)\})$ the subgraph containing this set. Then, we can define the SCF $f : D^n \to A$ as follows:

$$f(P) = \beta_t(G(\{r_1(P)\}))$$

This is the same SCF used in Chatterji, Sanver, and Sen (2013) and it follows from the construction of f that it is anonymous, unanimous and tops-only. We will now show that f also satisfies MM. We proceed now by analyzing in what ways the outcome of f can be changed by a single voter, and showing that in all the cases that the outcome of f can be changed, the changes don't violate MM. Before going in depth about this changes, we would like to bring to attention the fact that the only way this SCF can change its outcome at any given profile is if a voter changes its vote for some other alternative that lies on an *opposite* side of the outcome, akin to a median rule (this SCF is indeed identical to generalized median rule). As we will see, this allows us to invoke the Minimal Reversals Condition, which also deals with changes that involve pairs of alternatives lying at different sides of a focal alternative in the connectivity graph. We now explore these ideas in a more formal way.

Fix a profile P. Assume first that $f(P) = t$. Then we have one of two scenarios. Either P has some voters voting for t, or there are voters i, j such that $t \in \langle r_1(P_i), r_1(P_j) \rangle$. Assume the first case holds. As $f(P') = t$ for any P' where $v(t, P') > 0$, we have that the pivotal changes associated with this profile are of the kind (t, a), for some other $a \in A$, that is, changes where one voter changes his vote from t to some other alternative. These kinds of change never violate MM.

Assume now that the second case holds. We want to look at the pivotal changes involving player i, as defined above (since the choice of which is player i on that pair is arbitrary, this is without loss of generality). The only pivotal changes possible are those such that $t \notin \langle r_1(P_i'), r_1(P_j) \rangle$. Call now a, c the adjacent nodes of t such that $r_1(P_i'), r_1(P_j) \in \xi(a, t)$ and $r_1(P_i) \in \xi(c, t)$. These nodes always exist, as we can simply take $r_1(P_i)$ and $r_1(P_j)$ (or $r_1(P_i')$) if they happen to be adjacent to t. Moreover, they are necessarily distinct, since $t \in \langle r_1(P_i), r_1(P_j) \rangle$. We can check easily from the definition of the span of a node that $t \notin \xi(c, t)$. Thus, by the Minimal Reversals Condition [1], $(r_1(P_i), r_1(P_i')) \notin M_D^t$ and it follows that MM is not violated for any of these pivotal changes. This case is depicted by Figure 9. Node t plays the role of threshold, node i plays the role of both $r_1(P_i)$ and c, node a plays the role of a, node i' plays the role of $r_1(P_i')$, and node j plays the role of

[1] We have here that node t is playing the role of b as well, and that $(r_1(P_i), r_1(P_i'))$ play the role of (y, x), with nodes a and c playing the exact same roles as in the statement of the Condition.

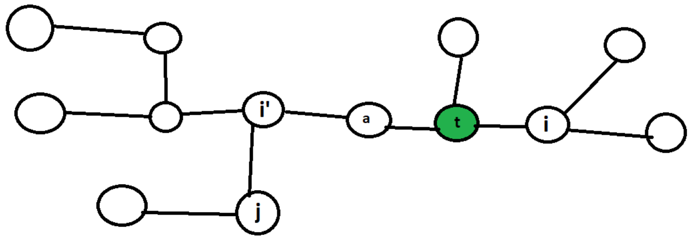

Figure A.3: An example for Lemma 16

$r_1(P_j)$. If we have only two players, then function f functions exactly as a median rule, taking the median (on a tree) between $t, r_1(P_i)$ and $r_1(P_j)$.

Now assume that $f(P) \neq t$, which then implies $f(P) = \beta_t(G(\{r_1(P)\}))$. Pivotal changes including individuals such that $r_1(P_i) = f(P)$ are ignored since these changes never violate MM. Thus, we will consider pivotal changes where an individual i with $r_1(P_i) \neq f(P)$ changes its vote from $r_1(P_i)$ to some other $r_1(P_i')$. In particular, as $r_1(P_i) \neq f(P)$, we can find a node c adjacent to $f(P)$ such that $r_1(P_i) \in \xi(c, f(P))$. We depict one of such scenario in Figure 10. There, node t plays again the role of threshold, B is the outcome of the SCF, $f(P)$, nodes i, i' and j play the role of $r_1(P_i)$,$r_1(P_i')$ and $r_1(P_j)$ (for some j), respectively, and finally nodes a and c will play the roles of alternatives a and c, mentioned below and used for the Minimal Reversals Condition. The nodes in blue represent the set $\xi(c, f(P))$. Once more, if we have only two players, then function f functions exactly as a median rule, taking the median (on a tree) between $t, r_1(P_i)$ and $r_1(P_j)$.

We now make two claims:

Claim 1: $t \notin \xi(c, f(P))$.

To see this, assume, by the way of contradiction, that both $t, r_1(P_i) \in$

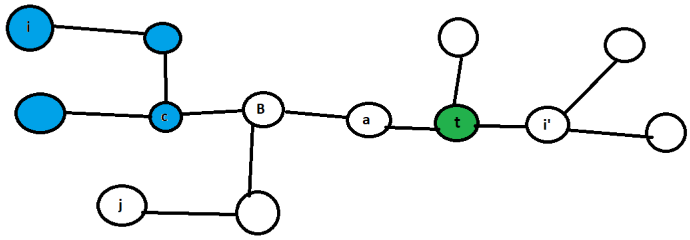

Figure A.4: Another example for Lemma 16

$\xi(c, f(P))$. Then we have that $f(P) \notin \langle c,t \rangle$ [2] as well as $f(P) \notin \langle c, r_1(P_i) \rangle$ (by the same reasoning). We can then use both of these results to conclude that $f(P) \notin \langle t, r_1(P_i) \rangle$ [3]. It follows then that $f(P) \neq \beta_t(G(\{r_1(P)\}))]$, as by definition, the projection of t must be a part of the path from every node of $G(\{r_1(P)\})$ to t. This is a contradiction, so $t \notin \xi(c, f(P))$ must be true.

Claim 2: if $f(P_i', P_{-i}) \neq f(P)$ then $r_1(P_i') \notin \xi(c, f(P))$.

This can be verified easily from a diagram or from the knowledge that f mimics a general median rule on a tree, but we can also formally check it in two steps. The first step is to show that if $f(P_i', P_{-i}) \neq f(P)$, then $f(P) \notin \langle r_1(P_i'), t \rangle$. This comes from the fact that both $f(P) = \beta_t(G(\{r_1(P)\}))$ and $f(P_i', P_{-i}) = \beta_t(G(\{r_1(P_i', P_{-i})\}))$. Thus, if $f(P_i', P_{-i}) \neq f(P)$, the two projections must be different. As $f(P) = \beta_t(G(\{r_1(P)\}))$ implies already that $f(P) \in \langle x, t \rangle$ for every $x \in \{r_1(P_{-i})\}$ and the only difference between $\{r_1(P_i', P_{-i})\}$ and $\{r_1(P)\}$ is $r_1(P_i')$, we must have $f(P) \notin \langle r_1(P_i'), t \rangle$, else we would have $\beta_t(G(\{r_1(P)\})) = \beta_t(G(\{r_1(P_i', P_{-i})\}))$.

[2] $t \in \xi(c, f(P))$ implies that c is in the middle of the path between t and $f(P)$, which further implies that $f(P)$ is outside the path between t and c.

[3] This follows from the fact that $\langle t, r_1(P_i) \rangle \subset \langle t, c \rangle \cup \langle c, r_1(P_i) \rangle$.

The second step is to show that if $r_1(P_i') \in \xi(c, f(P))$ then $f(P) \in \langle r_1(P_i'), t\rangle$. This can be verified from $[t \notin \xi(c, f(P)) \ \wedge \ r_1(P_i') \in \xi(c, f(P))] \Rightarrow \langle r_1(P_i'), t\rangle = \langle r_1(P_i'), f(P)\rangle \cup \langle f(P), t\rangle \Rightarrow f(P) \in \langle r_1(P_i'), t\rangle$. Together, these steps show that if $f(P_i', P_{-i}) \neq f(P)$ then $r_1(P_i') \notin \xi(c, f(P))$.

Thus, from the two claims above, we can invoke the Minimal Reversals Condition[4] and state that if $f(P_i', P_{-i}) \neq f(P)$, then $(r_1(P_i), r_1(P_i')) \notin M_D^{f(P)}$ and thus MM is not violated . We conclude that MM holds for every change in the outcome of the SCF constructed. $\star$

Proof of Proposition 1 We separate this proof into two Lemmas. Lemma 17 establishes a relation between the projection of the threshold node and breaches containing it, while Lemma 18 completes the remaining steps of the proof.

Lemma 17: *Let t be a threshold for G, δ be a maximal path not containing t, $(b,c) \in \delta$ a pair of adjacent nodes and $\beta_t(\delta)$ the projection of t onto δ. Then $\beta_t(\delta) \in \xi(b,c) \Rightarrow t \in \xi(b,c)$.*

Proof: First, fix $\delta = \langle p, q\rangle$ and without loss of generality, assume that $p \notin \xi(b,c)$, as well as $\beta_t(\delta) \in \xi(b,c)$. Then by the definition of $\xi(b,c)$ we have that $b \in \langle \beta_t(\delta), c\rangle$ and as b is adjacent to c and on the path δ, we have that $b, c \in \langle p, \beta_t(\delta)\rangle$ (which should read as $p, (...), c, b, (...), \beta_t(\delta)$). Next, consider the path $\langle p, t\rangle$. As $p \in \delta$ and $\beta_t(\delta)$ is the projection of t onto δ, this implies that $\beta_t(\delta) \in \langle p, t\rangle$. But then $\langle p, t\rangle = \langle p, \beta_t(\delta)\rangle \cup \langle \beta_t(\delta), t\rangle$ and thus $b, c \in \langle p, t\rangle$. Since $p \notin \xi(b,c) \Rightarrow b \notin \langle p, c\rangle$ this implies $b \in \langle c, t\rangle \Rightarrow t \in \xi(b,c)$. $\star$.

[4] To use the Condition here, node $f(P)$ plays the role of b, $(r_1(P_i), r_1(P_i'))$ play the role of (y,x), c plays the same role as in the statement of the Condition, and the node adjacent to $f(P)$ in the path $\langle r_1(P_i'), f(P)\rangle$ plays the role of a.

By our Theorem, we have that there is an alternative t in A such that together with the Connectivity Graph G of the domain they form an admissible pair. Thus, all that remains is to prove the statement below:

Lemma 18: *Let D be a Strong Path-connected domain admitting a Well Behaved SCF satisfying MM. Then, for all $P_i \in D$ and for every path δ such that $r_1(P_i) \in \delta$ and $\beta_t(\delta)$ is a threshold node of δ we have:*

- $[a_r, a_s \in \delta$ such that $a_r, a_s \in \langle r_1(P_i), \beta_t(\delta)\rangle$ and $a_r \in \langle r_1(P_i), a_s\rangle] \Rightarrow [a_r P_i a_s]$.
- $[a_r \in \delta,\ \beta_t(\delta) \approx a_r$ and $a_r \notin \langle r_1(P_i), \beta_t(\delta)\rangle] \Rightarrow [\beta_t(\delta) P_i a_r]$

Denote by a_1, a_m the extreme nodes of δ, such that $\delta = \langle a_1, a_m\rangle$, as conventioned so far. We start now the proof of the first statement by assuming its conditions, that is, $a_r, a_s \in \delta$ such that $a_r, a_s \in \langle r_1(P_i), \beta_t(\delta)\rangle$ and $a_r \in \langle r_1(P_i), a_s\rangle$. Without loss of generality, assume that $r_1(P_i) \in \langle a_1, \beta_t(\delta)\rangle$. It can then be easily verified that the alternatives in δ are arranged as follows: $a_1, ..., r_1(P_i), ..., a_r, ..., a_s, ..., \beta_t(\delta), ..., a_m$. Now, index the alternatives in $\langle a_s, a_r\rangle$ as $\{b_j\}_{j=0}^{k}$ with $b_0 = a_s$, $b_k = a_r$ and for every $j < k$, b_{j+1} is adjacent to b_j and to its left, so $\langle a_r, a_s\rangle = a_r, b_{k-1}, ..., b_1, a_s$. We will now verify the conditions of Lemma 15, as follows: let b_1 play the role of alternative b in Lemma 15, $r_1(P_i)$ play the role of y, b_2 play the role of c and a_s play the role of a and x. We have that $\beta_t(\delta) \in \xi(a_s, b_1)$, which, by Lemma 17, implies that $t \in \xi(a_s, b_1)$ and, by Lemma 13, $t \notin \xi(b_1, a_s)$. Thus, all the conditions for Lemma 15 are verified and we have $(r_1(P_i), a_s) \notin M_D^{b_1}$. Hence, there must be at least one alternative ranked below b_1 in P_i that is not ranked below b_1 in every preference that has a_s on top. However, since the domain is Strong Path-Connected and $b_1 \approx a_s$, we know that there is a preference with a_s on top where b_1 is ranked second. Therefore, every other alternative except for a_s itself is ranked below b_1 in this preference and thus, we must have $b_1 P_i a_s$. Now move to b_2. Once more, we can invoke Lemmas 17,13 and 15 in an almost identical fashion to conclude that $(r_1(P_i), b_1) \notin M_D^{b_2}$, which, by the same argument, should imply that $b_2 P_i b_1$ and thus $b_2 P_i a_s$. We can repeat these steps for every b_j in the sequence and conclude that $a_r P_i a_s$. The arguments for the case where $r_1(P_i) \in \langle \beta_t(\delta), a_m \rangle$ are mirrored for the case presented. This completes the proof for the first property. The proof for the second property follows the same idea: by Lemmas 17, 13 and 15, $(r_1(P_i), a_r) \notin M_D^{\beta_t(\delta)}$, so there must be at least one alternative ranked below $\beta_t(\delta)$ in P_i that is not ranked below $\beta_t(\delta)$ in every preference that has a_r on top. By the argument above, $\beta_t(\delta) \approx a_r$ implies that this alternative must be a_r and hence, $\beta_t(\delta) P_i a_r$, concluding the proof.

$\star$

Proof of Proposition 2: We start with the first part, that if a domain is Single-Peaked on the tree G, then it satisfies the Minimum Reversals Condition for all admissible pairs (G,t) that include G. As such, assume that we have an MD-Connected domain that is Single-Peaked, and G denotes its Connectivity Graph.

Take any three distinct alternatives x, y, b with the property that $b \in \langle x, y \rangle$, and let P_i^x denote an arbitrary preference with $r_1(P_i^x) = x$ and P_i^y its equivalent for an arbitrary preference with y on top. Then the property from Single-Peakedness implies that $b P_i^x y$ and $b P_i^y x$. This, together with the fact that $x P_i^x b$ and $y P_i^y b$ implies that $(x,y), (y,x) \notin M_D^b$. Since x, y and b were picked arbitrarily, we have that for any choice of t, the domain will satisfy the Minimum Reversals Condition.

The second part of the statement comes from Proposition 1: if the domain is Strong Path-Connected with Connectivity Graph G and satisfies the Minimum

Reversals Condition for a given admissible pair (G,t), then the domain is a Weak Semi-Single-Peaked domain for that admissible pair. Thus, if the domain satisfies the Minimum Reversals Condition for every admissible pair (G,t), this makes it a Weak Semi-Single-Peaked domain with respect to all these admissible pairs, which is the definition of a Single-Peaked domain on the graph G. $\star$

Proof of Proposition 3: From Proposition 2, if a domain is Strong Path-Connected, satisfies the Minimum Reversals Condition of an admissible pair (G,t) and is not a Single-Peaked domain, then there must be at least one node $z \neq t$ such that the domain does **not** satisfy the Minimum Reversals Condition for the admissible pair (G,z). Given the definition of a threshold node, this must mean that there exists a breach (y,x) such that $z \in \xi(y,x)$. Then, by Lemma 7, we can show that any Well Behaved and Monotonic SCF defined on that domain must violate No Veto Power. $\star$

Proof of Proposition 4: Let $\mathscr{D}$ be an MD-Connected domain that satisfies the Minimum Reversals Condition and $\tau_D \in A$ its associated set of eligible thresholds, with x,z being two distinct nodes such that $x,z \in \tau_D$. Assume now that we have a node y such that $y \in \langle x,z \rangle$, and, furthermore, assume that $y \neq x,z$ (else the proof becomes trivial). Since G is a tree, we must have that, for any node $c \in G$, either $\langle x,c \rangle = \langle x,y \rangle \cup \langle y,c \rangle$ or $\langle z,c \rangle = \langle z,y \rangle \cup \langle y,c \rangle$. Thus, any node $b \in \langle y,c \rangle$ will also belong to either $\langle x,c \rangle$ or $\langle z,c \rangle$. But since both $x,z \in \tau_D$, this means that there is no breach (b,c) such that $b \in \langle x,c \rangle$ or $b \in \langle z,c \rangle$. Hence, for every breach (b,c), we cannot have $b \in \langle y,c \rangle$. This means that y is also not in the span of any breach, and thus, $y \in \tau_D$ as well. $\star$

Proof of Proposition 5: We want to verify that the domain $\mathscr{D} = \mathscr{D}^{SSP} \cup D^p \cup D^q$ satisfies the Minimal Reversals Condition. First, note that, since SSP domains have stronger restrictions on the preferences of the domain than Weak SSP domains, whenever we can say that aP_ib for all P_i in some $\mathscr{D}^{WSSP} \in \mathbb{D}^{WSSP}_{(G,t)}$, it will also be true that aP_ib for all P_i in $\mathscr{D}^{SSP} \in \mathbb{D}^{SSP}_{(G,t)}$, that is, all preferences of a SSP domain that has a matching admissible* pair. Thus, if there is no pair $P_i \in \mathscr{D}^{WSSP}, P_i' \in \mathscr{D}^{SSP}$ such that $P_i \mapsto_a P_i'$, then there is no pair $P_i', P_i'' \in \mathscr{D}^{SSP}$ such that $P_i'' \mapsto_a P_i'$.

Assume now, by the way of contradiction, that $\mathscr{D}$ does not satisfy the Minimal Reversals Condition. Then there must be a pair of preferences P_i^x, P_i^y, with $r_1(P_i^x) = x, r_1(P_i^y) = y$ and a triple of adjacent nodes a,b,c such that $x \in \xi(a,b),\ y \in \xi(c,b),\ t \notin \xi(c,b)$ and $P_i^y \mapsto_b P_i^x$.

Next, we claim that if $P_i^y \in \mathscr{D}^{WSSP}$, then $P_i^x \in \mathscr{D}^{SSP}$, as $x \in \xi(a,b),\ y \in \xi(c,b)$ implies that x and y cannot be adjacent, and, by assumption, p and q are adjacent and if $r_1(P_i^x) \neq p,q$, then $P_i^x \notin \mathscr{D}^{WSSP}$. Similarly, if $P_i^x \in \mathscr{D}^{WSSP}$, then $P_i^y \in \mathscr{D}^{SSP}$, by the same argument.

Another consequence of $x \in \xi(a,b),\ y \in \xi(c,b)$ is that $b \in \langle x,y \rangle$. Let then δ be a maximal path in G containing $\langle x,y \rangle$, and $\beta_t(\delta)$ be the projection of the threshold on path δ. There are exactly two possibilities for the relative disposition of nodes $x,y,a,b,c,\beta_t(\delta)$ on the path δ, as, per Lemma 17, $t \notin \xi(c,b)$ implies $\beta_t(\delta) \notin \xi(c,b)$. The first is that we have $\beta_t(\delta),...,x,...,a,b,c,...,y$, and the second is that we have $x,...,\beta_t(\delta),...,a,b,c,...,y$.

Assume first that we have $\beta_t(\delta),...,x,...,a,b,c,...,y$, and, additionally, that $P_i^y \in \mathscr{D}^{WSSP}$, which, as shown above, implies that $P_i^x \in \mathscr{D}^{SSP}$. By the properties of Weak SSP domains, we have $bP_i^y x$, while by the definition of P_i^x we also have

$xP_i^x b$. Thus, b does not maintain its position in this case. Now assume that $P_i^x \in \mathscr{D}^{WSSP}$ and $P_i^y \in \mathscr{D}^{SSP}$. Since $bP_i^y x$ held for $P_i^y \in \mathscr{D}^{WSSP}$, as discussed above, this should still hold for $P_i^y \in \mathscr{D}^{SSP}$. Similarly, $xP_i^x b$ also holds in this scenario, and we can reach the same conclusion as before. Lastly, the case where both $P_i^x, P_i^y \in \mathscr{D}^{SSP}$, as discussed above, is handled by the first case. Thus, for all possible properties of P_i^x and P_i^y, we have that alternative b does not maintain its position here.

Next, assume that we have now $x, ..., \beta_t(\delta), ..., a, b, c, ..., y$, and, additionally, that $P_i^y \in \mathscr{D}^{WSSP}$, which, once more, implies that $P_i^x \in \mathscr{D}^{SSP}$. By the properties of Weak SSP domains, we have that $bP_i^y \beta_t(\delta)$. Similarly, by the properties of SSP domains, we have that $\beta_t(\delta) P_i^x b$. So, b does not maintain its position in this scenario. As argued above, this conclusion also holds for the case where both $P_i^x, P_i^y \in \mathscr{D}^{SSP}$. Now, consider the last possible scenario, where $P_i^x \in \mathscr{D}^{WSSP}$. Then, by the properties of SSP domains, $bP_i^y x$, while by the definition of P_i^x, we have $xP_i^x b$. This implies that b does not maintain its position in all possible scenarios.

Hence, assuming that $\mathscr{D}$ does not satisfy the Minimal Reversals Condition imply a contradiction, concluding the proof. $\star$

Appendix B

Appendix B covers the proofs for Chapter 3. We start with Lemmas 19 to 25 covering the necessity part, Lemma 26 proves the relation between veto power and eligible thresholds, Lemma 27 covers the sufficiency part of the result and finally we have the proof of Proposition 6.

Lemma 19: For any $x \in G$ and $y \in \alpha(x)$, $\xi(x,y)$ is a connected subgraph.

Proof: For any $a \in \xi(x,y)$, we have that $\langle a,x \rangle \subset \xi(x,y)$, since $b \in \langle a,x \rangle \cap \xi(x,y)^C \Rightarrow x \notin \langle b,y \rangle$ (by definition of $\xi(x,y)$) and $x \notin \langle a,b \rangle$ (by uniqueness of $\langle a,x \rangle$). But that implies that $x \notin \langle a,y \rangle \subset \langle a,b \rangle \cup \langle b,y \rangle$, which contradicts $a \in \xi(x,y)$.

Now take any pair of alternatives $a,b \in \xi(x,y)$. By the above statement, $\langle a,x \rangle, \langle b,x \rangle \subset \xi(x,y)$. Hence, $\langle a,b \rangle \subset \langle a,x \rangle \cup \langle x,b \rangle \subset \xi(x,y)$, which shows that indeed for any two alternatives in $\xi(x,y)$, the path between them is also in $\xi(x,y)$, i.e., $\xi(x,y)$ is a connected subgraph. $\star$

Lemma 20: Given a set of alternatives λ, let $v(\lambda,P) = \sum_{k \in \lambda} v(k,P)$. For any $x \in A$ and $y \in \alpha(x)$, given f, there is a number $v^*_f(\xi(y,x))$ such that $f(P) \in \xi(y,x) \Leftrightarrow$

$v(\xi(y,x),P) \geq v^*_f(\xi(y,x))$.

Proof: Let the set $\Phi_{yx} \subset \mathbb{N}$ be defined as $v_{yx} \in \Phi_x \Leftrightarrow \exists P' : v(\xi(y,x),P') = v_{yx}$ and $f(P') \in \xi(y,x)$. By Unanimity, the set Φ_{yx} is not empty and because there is a finite number of voters, there is a minimal element to this set, call it $v^*_f(\xi(y,x))$.

Fix an arbitrary profile P with $v(\xi(y,x),P) = v^*_f(\xi(y,x))$, and let P' be one of the profiles implied by the definition of $v^*_f(\xi(y,x))$, such that $f(P') \in \xi(y,x)$. We want to show that $f(P)$ is also inside $\xi(y,x)$. To do so, we will first construct an intermediate profile, P^* that yields the same outcome as P'. First, since the sum of votes of alternatives in $\xi(y,x)$ is the same both in P and P', this means that the sum of the alternatives *outside of* $\xi(y,x)$ is also the same in both profiles. Then let the profile P^* be such that $v(a,P^*) = v(a,P')$ for any $a \in \xi(y,x)$ and $v(b,P^*) = v(b,P)$ for any $b \notin \xi(y,x)$. Next, by Lemmas 13 and 19, all the alternatives that are not part of $\xi(y,x)$ are connected, and then by (the first part of) Lemma 5, these alternatives share votes under any alternative in $\xi(y,x)$. We can then apply Lemma 2 to have that $f(P^*) = f(P')$, as we wanted. To establish that $f(P) \in \xi(y,x)$ and complete the proof, we will show that whenever this is *not* true, we must have $f(P) = f(P^*)$, contradicting the fact that $f(P^*)$ is also inside $\xi(y,x)$. This can be verified as follows: by Lemma 19, $\xi(y,x)$ is a connected set, then by Lemma 5 the alternatives in $\xi(y,x)$ share votes under any alternative outside of $\xi(y,x)$ (such as $f(P)$). Finally, since the sum of votes for alternatives in the set $\xi(y,x)$ is equal across P and P^*, we can apply Lemma 2 to have that $f(P^*) = f(P)$, which contradicts the earlier finding that $f(P^*) \in \xi(y,x)$. Thus, we must have $f(P) \in \xi(y,x)$.

The case where $v(\xi(y,x),P) \geq v^*_f(\xi(y,x))$ follows a similar argument used for this step in Lemma 3. Since we already have a profile that selects an alternative in $\xi(y,x)$ with $v_f^*(\xi(y,x))$ votes among those alternatives, if we have even more votes we can simply use Fact 1 and Lemma 1 to pass these extra votes to alternative $f(P')$ without altering the outcome. Thus, we can establish a profile $f(P'')$ with $f(P'') = f(P')$ and $v(\xi(y,x),P'') = v(\xi(y,x),P)$, then follow the steps above to prove that $f(P) = f(P'')$.

This establishes sufficiency part of the result. The necessity part comes from the definition of $v_f^*(\xi(y,x))$ as the minimal element of the set Φ_{yx}. $\star$

Lemma 21: Let $\Xi = \{\xi \mid \exists\, x \in G, y \in \alpha(x) : \xi = \xi(x,y)\}$. Then, for any $\xi \in \Xi$, $v_f^*(\xi) + v_f^*(\xi^C) = n+1$.

Proof: This proof is similar to what is shown in Step 3 of Lemma 4. Assume first that $v_f^*(\xi) + v_f^*(\xi^C) > n+1$. Then we can find a profile P with $v(\xi,P) < v_f^*(\xi)$ and $v(\xi^C,P) < v_f^*(\xi^C)$, such that $f(P)$ is empty. Similarly, if $v_f^*(\xi) + v_f^*(\xi^C) < n+1$ then we can find a profile P where $v(\xi,P) > v_f^*(\xi)$ and $v(\xi^X,P) > v_f^*(\xi^C)$ and $f(P)$ has two values. Neither scenario is allowed for a function, and thus the sum $v_f^*(\xi) + v_f^*(\xi^C)$ must be equal to $n+1$. $\star$

Definition: For a given $\xi \in \Xi$ (as defined above), let the number f_ξ (*the number of phantoms in* ξ *for SCF* f) be given by $f_\xi = n - v_f^*(\xi)$.

Lemma 22: For any $\xi \in \Xi$, $f_\xi + f_{\xi^C} = n-1$.

Proof: This is a direct consequence of Lemma 21: $f_\xi + f_{\xi^C} = n - v_f^*(\xi) + n - v_f^*(\xi^C) = 2n - (v_f^*(\xi) + v_f^*(\xi^C)) = 2n - (n+1) = n-1$.

Definition: For any node $x \in G$, let f_x (*the number of phantoms at node x for SCF f*) be given by $f_x = n - 1 - \sum_{y \in \alpha(x)} f_{\xi(y,x)}$

Lemma 23: The following statements are true:

- $f_\xi = \sum_{j \in \xi} f_j$.
- $\sum_{x \in G} f_x = n - 1$.

Proof: We will start with the first point, and do a proof by induction on the number of elements of the set ξ.

Assume the set ξ contains a single element, and call it x. As $\xi \in \Xi \Rightarrow \xi = \xi(x,y)$ for some $y \in \alpha(x)$, we must have that x is an extreme node. This comes from the fact that any other node adjacent to x would belong to $\xi(x,y)$. Then $f_\xi = f_x = n - 1 - f_{\xi(y,x)}$. By Lemma 22, $n - 1 - f_{\xi(y,x)} = f_{\xi(x,y)} = f_\xi$, proving the result.

Now assume that we have established that $f_{\xi^*} = \sum_{j \in \xi^*} f_j$ for any set ξ^* with k elements or less. Consider now the case where ξ has $k+1$ elements. Once more, because $\xi \in \Xi$,we have that there exists a pair of nodes (y,x) that are adjacent to one another such that the set ξ is of the form $\xi(y,x)$. This set can then be partioned into $\xi(y,x) = y \cup \xi(z_1,y) \cup ... \cup \xi(z_j,y)$, where the set $\{z_i\}_{i=1}^{j} = \alpha(y) \cap \xi(y,x)$ is the set of nodes adjacents to y, excluding x (which we know is not empty, since ξ has at least two elements). Since each $\xi(z_i,y)$ is a subset of ξ, they all have k elements or less; moreover, they are all disjoint $\xi(z_p,y) \cap \xi(z_q,y) = \emptyset \, \forall p,q \in [1,j]$ and y is not a member of any of them.

Using the definition f_y, we have then $f_y = n - 1 - f_{\xi(y,x)} - \sum_{i=1}^{j} f_{\xi(z_i,y)}$. Using Lemma 13 and 22, we have $f_y = f_\xi - \sum_{i=1}^{j} f_{\xi(z_i,y)}$. Rearranging, we get $f_\xi =$

$f_y + \sum_{i=1}^{j} f_{\xi(z_i,y)}$. Using the induction hypothesis, we reach $f_\xi = f_y + \sum_{j\in\xi-y} f_j = f_\xi = \sum_{j\in\xi} f_j$, concluding the proof.

The second statement now comes immediately from the first statement of this Lemma and Lemma 22. $\star$.

Definition: A SCF f is a *generalized median function on a tree G* if it satisfies all of the following properties:

- For each node $x \in G$, there is a number $f_x > 0$ associated with it (called the number of phantoms at node x).
- $\sum_{x\in G} f_x = n-1$.
- if $f(P) = x$, then $\sum_{z\in\xi(y,x)} v(z,P) + f_z < n$, for all $y \in \alpha(x)$.

Lemma 24: Any SCF f defined on a MRC domain that is Well Behaved and Monotonic is a generalized median function, with the tree corresponding to the graph G used in the MRC.

Proof: The previous Lemmas proved already the first two properties. To verify the last one, assume that $f(P) = x$ and for some $y \in \alpha(x)$ we have $\sum_{z\in\xi(y,x)} v(z,P) + f_z \geq n$. Using Lemma 23, rewrite this expression as $v(\xi(y,x),P) + f_{\xi(y,x)} \geq n$. Rearrange it into $v(\xi(y,x),P) \geq n - f_{\xi(y,x)}$. Now use the definition of $f_{\xi(y,x)}$ to get $v(\xi(y,x),P) \geq v_f^*(\xi(y,x))$. From the definition of $v_f^*(\xi(y,x))$, this implies $f(P) \in \xi(y,x)$, which contradicts $f(P) = x \notin \xi(y,x)$. Hence, we cannot have $\sum_{z\in\xi(y,x)} v(z,P) + f_z \geq n$. $\star$

Definition Let the set $\tau_D \subset A$ be defined as the set of all alternatives t such that there is a graph G for which (G,t) is an admissible pair and domain D satisfies the Minimal Reversals Condition for that admissible pair. We call the set τ_D the set of *eligible thresholds for D*.

Lemma 25 Let f be a generalized median function that is Monotonic on domain D, which satisfies the MRC. Then, for all $x \notin \tau_D$ we must have $f_x = 0$.

Proof: Assume that $x \notin \tau_D$. Then, there is a breach (a_j, a_{j+1}) such that $x \in \xi(a_j, a_{j+1})$. Let $a_1 \in \xi(a_j, a_{j+1})$ be any extreme node in $\xi(a_j, a_{j+1})$, and assume now that $f_x > 0$. By Lemma 7, we must have $f(P(a_1 = n-1, a_{j+1} = 1)) = a_{j+1}$, but that contradicts the third property of a generalized median function on a tree, as both $a_1, x \in \xi(a_j, a_{j+1})$ and thus $v(a_1, P(a_1 = n-1, a_{j+1} = 1)) + f_x \geq n$. Hence, f_x must be equal to 0. $\star$

Lemma 26 Let (b,c) be a breach in $\mathscr{D}$ and $x \in \xi(b,c)$ be a node on the span of that breach. Then there exists another alternative $y \neq x$ such that $f(P(x = n-1, y = 1)) \neq x$.

Proof: Let $y \in \xi(c,b)$. By Lemma 25, for all $z \in \xi(b,c)$ we have $f_z = 0$. Thus, we must have that $f_{\xi(c,b)} = n-1$. As $f_{\xi(c,b)} + v(y, P(x = n-1, y = 1)) = n$ we have $f(P(x = n-1, y = 1)) \in \xi(c,b)$. The conclusion then follows from $\xi(c,b) = \xi(b,c)^C$. $\star$

Corollary: If an alternative cannot be vetoed, then that alternative must be in the set of eligible thresholds for D, τ_D.

Lemma 27: *If f is a generalized median function on the tree G and $\mathscr{D}$ is a domain that satisfies the Minimum Reversals Condition for every admissible pair*

(G,t) such that $f_t > 0$, then f is Monotonic on $\mathscr{D}$.

Proof: We prove this Lemma by showing that, for every profile P such that the outcome $f(P)$ can be changed by a single voter changing his vote, that change does not violate Monotonicity, as defined in Claim 1.

Thus, assume that we have a profile P and a profile P' such that $r_1(P_1) \neq r_1(P_1')$, $P_i = P_i'$ for every $i > 1$, and $f(P) \neq f(P')$. Furthermore, to make notation easier, call $r_1(P_1) = y$ and $r_1(P_1') = x$. So, at profile P, the first voter votes for y and the outcome is $f(P)$, but if he changes his vote to x (and all other voters keep their votes the same), the outcome changes to $f(P')$.

First, we take care of a simple case, where $y = f(P)$. Clearly, $f(P)$ does not maintain its position in this change of preferences, and thus, Monotonicity is not violated here. Hence, assume from now on that $f(P) \neq y$.

Now, since f is a generalized median rule on G, there must be nodes $c, a \in \alpha(f(P))$, $a \neq c$ such that $x \in \xi(a, f(P))$ and $y \in \xi(c, f(P))$. Furthermore, since $\sum_{z \in \xi(c,f(P))} f_z = n-1$ would imply $\sum_{z \in \xi(c,f(P))} v(z,P) + f_z \geq n$ (contradicting the assumption that f is a generalized median function) we must have at least one node t such that $f_t > 0$ and $t \notin \xi(c, f(P))$. By assumption, $f_t > 0$ implies that the domain $\mathscr{D}$ satisfies the Minimum Reversals Condition for the admissible pair (G,t). In turn, $t \notin \xi(c, f(P))$ implies that $(y,x) \notin M_D^{f(P)}$. Hence, $f(P)$ must not maintain its position when going from any preference with y on top to any preference with x on top. Thus, f is Monotonic on domain $\mathscr{D}$. $\star$.

Proof of Proposition 6: We proceed in a series of steps.

Step 1: If $\mathscr{D}$ is not a SSP domain for any (G,t), then for every $t \in \tau_D$ we must *have a preference P_i and a node q, $q \neq r_1(P_i)$ with the following properties:*

- *If δ is a maximal path containing both $r_1(P_i)$ and q, then, $\beta_t(\delta) \in \langle r_1(P_i), q\rangle$.*
- *$q \notin \alpha(\beta_t(\delta))$.*
- *$qP_i\beta_t(\delta)$.*

Since $\mathscr{D}$ is a Strong Path-Connected domain and satisfy the MRC for the admissible pair (G,t), according to Proposition 1 of Ramos (2022), $\mathscr{D}$ is also a Weak Semi-Single-Peaked domain with regard to (G,t). Comparing the properties of a WSSP and a SSP domain, it must be the case that if $\mathscr{D}$ is not a SSP domain for (G,t), there must be some preference P_i and some node q such that all three statements above are true.

Step 2: Let δ be an arbitrary maximal path, and x,y,b,c nodes with the following properties: $x,y \in \delta$, $c \notin \delta$, $b \in \alpha(c)$ and $x \in \xi(b,c)$. Then $y \in \xi(b,c)$.

We decompose $\langle x,c\rangle$ as $\langle x,\beta_c(\delta)\rangle \cup \langle \beta_c(\delta),c\rangle$. Hence, $x \in \xi(b,c)$ implies $b \in \langle x,\beta_c(\delta)\rangle$ or $b \in \langle \beta_c(\delta),c\rangle$. Assume it is the first case. Then, as $b \in \alpha(c)$ we must have $b = \beta_c(\delta)$, which implies $b \in \langle y,c\rangle$ and thus $y \in \xi(b,c)$, as desired. Consider now the second case, $b \in \langle \beta_c(\delta),c\rangle$. Then, as we can also decompose $\langle y,c\rangle$ into $\langle y,\beta_c(\delta)\rangle \cup \langle \beta_c(\delta),c\rangle$, $b \in \langle \beta_c(\delta),c\rangle$ also implies that $b \in \langle y,c\rangle$ and thus $y \in \xi(b,c)$, completing the proof.

Fix a generalized median function f on G that is Monotonic on $\mathscr{D}$, and pick one node t such that $f_t > 0$. From Theorem 2, we must have $t \in \tau_D$. From Step 1, we must have a preference P_i and a node q with the three properties described on that Step. Take now a profile in which the first voter has preference P_i and all other voters have preferences with q on top and call it P.

Step 3: Show that $f(P) \in \delta$, where δ is a maximal path containing both q and $r_1(P_i)$.

Assume that $f(P)$ is not on δ, and let b be any node adjacent to $f(P)$ such that $q \in \xi(b, f(P))$. Then, by Step 2 above, we also have $r_1(P_i) \in \xi(b, f(P))$. But as the sum of votes between q and $r_1(P_i)$ is n, this violates one of the properties of the generalized median functions on a tree, preventing that node from being the outcome of f at profile P. Thus, we must have have $f(P) \in \delta$.

Step 4: Show that $f(P) \neq q$.

Assume that $f(P) = q$. Then, there must be some node t' such that $f_{t'} > 0$ and $q \in \langle \beta_{t'}(\delta), r_1(P_i) \rangle$, else q cannot be the outcome of f at profile P. As $\beta_t(\delta) \in \langle q, r_1(P_i) \rangle$, this also implies that $\beta_t(\delta) \in \langle \beta_{t'}(\delta), r_1(P_i) \rangle$. By the same arguments used in Step 1, we must have that $t' \in \tau_D$ and thus, as D is a WSSP domain that is also compatible with the admissible pair (G, t'), we must have $\beta_t(\delta) P_i q$, which contradicts the earlier assumption that $q P_i \beta_t(\delta)$. Thus, we must have $f(P) \neq q$.

Step 5: Show that $f(P) \in \langle \beta_t(\delta), q \rangle / q$.

Let δ_0, δ_1 be the extreme nodes (endpoints) of the maximal path δ. We can decompose δ into $\langle \delta_0, q \rangle \cup \langle q, \beta_t(\delta) \rangle \cup \langle \beta_t(\delta), \delta_1 \rangle$. By Step 3, $f(P)$ must be inside one of these three segments. Assume first $f(P) \in \langle \delta_0, q \rangle$. As we know $f(P) \neq q$ from Step 4, this implies we can find some node $b \in \alpha(f(P))$ such that both $q, r_1(P_i) \in \xi(b, f(P))$. However, as seen on Step 3, this contradicts one of the properties of generalized median functions on a tree. So this case is ruled out. Now consider the case where $f(P) \in \langle \beta_t(\delta), \delta_1 \rangle$, with $f(P) \neq \beta_t(\delta)$. Then, we can find another $b \in \alpha(f(P))$ such that $q, \beta_t(\delta) \in \xi(b, f(P))$, which by Lemma 17, implies $q,t \in \xi(b,f(P))$. Since $f_t > 0$, this again violates one of the properties of generalized median functions on a tree. Thus, we must have $f(P) \in \langle \beta_t(\delta),q\rangle/q$.

Step 6: Show that there is a node t' such that $f_{t'} > 0$ and $f(P) = \beta_{t'}(\delta)$.

If $f(P) = \beta_t(\delta)$, our proof is done, hence, assume $f(P) \neq \beta_t(\delta)$. By Step 5 and the properties of the outcome of a generalized median on a tree, this means that there must be another node t_1 such that $f_{t_1} > 0$ and $\beta_{t_1}(\delta) \in \langle q,\beta_t(\delta)\rangle$. As this is true for every other node between $\beta_t(\delta)$ and $\beta_{t_1}(\delta)$, none of them can be $f(P)$ either. We then check if $f(P) = \beta_{t_1}(\delta)$. If it is, our proof again is complete. If it is not, then once more there must exist another node t_2 such that $f_{t_2} > 0$ and $\beta_{t_2}(\delta) \in \langle q,\beta_{t_1}(\delta)$. Again, as this will be true for every node between $\beta_{t_1}(\delta)$ and $\beta_{t_2}(\delta)$, none of them can be $f(P)$ either. So we check if $f(P) = \beta_{t_2}(\delta)$. If it is our proof again is complete. If not, we iterate this argument. As it can be seen, the only suitable candidates for $f(P)$ in the interval $\langle \beta_t(\delta),q\rangle/q$ are nodes which are projections of a node t' with $f_{t'} > 0$, and since this process must be finite, there will be eventually a node that matches the desired criteria.

Step 7: Show that $qP_i f(P)$.

First, $f(P) \in \langle \beta_t(\delta),q\rangle/q$ and $\beta_t(\delta) \in \langle q,r_1(P_i)\rangle$ together imply that $\beta_t(\delta) \in \langle f(P),r_1(P_i)\rangle$. Let now t' be the node implied by Step 6. Next, since $f_{t'} > 0$, we have, by Theorem 2, that $t' \in \tau_D$ and thus D is a WSSP domain with respect to the admissible pair (G,t'). Then, $\beta_t(\delta) \in \langle f(P),r_1(P_i)\rangle$, along with $f(P) = \beta_{t'}(\delta)$ implies $\beta_t(\delta)P_i f(P)$. This, together with the assumption that $qP_i\beta_t(\delta)$ implies that $qP_i f(P)$, as desired.

Step 8: Show that if f is a generalized median function on G that is Monotonic on $\mathscr{D}$, then f is not Strategy-Proof.

Fix a generalized median function f on G that is Monotonic on $\mathscr{D}$, and pick one node t such that $f_t > 0$. From Theorem 2, we must have $t \in \tau_D$. From Step 1, we must have a preference P_i and a node q with the three properties described on that Step. Take now a profile in which the first voter has preference P_i and all other voters have preferences with q on top and call it P. If all agents report their true preferences at P, the outcome if $f(P)$. However, if instead of announcing his true preference, the first voter announces that q is his top-ranked alternative as well (while all the other voters still announce their true preferences), the outcome changes to q, as this is now an unanimous profile. By Step 7, $qP_if(P)$, and thus, this is a profitable manipulation for the first voter, showing that Strategy-Proofness is violated for this profile. $\star$

Appendix C

Appendix C covers the proof of Lemma 2 in Chapter 4. These steps are similar to those involved in Lemma 8, except that since Lemma 2 deals with one particular environment, the structure of the proofs is slightly less abstract. **Step 1a**: If there exists $x \in \{0,\dots,k-1\}$ such that $\sigma_1(x) > 0$, then $\sigma_2(x) > 0$.

Proof of Step 1a: Assume by way of contradiction that there exists an integer $x \in \{0,\dots,k-1\}$ such that $\sigma_1(x) > 0$ and $\sigma_2(x) = 0$. Then, there are two possibilities: either there exists $x' \in \{0,\dots,k-1\}\backslash\{x\}$ such that $\sigma_2(x') > 0$ or $\sigma_2(k) = 1$.

In the first case, let $x' \in \arg\max_{x'' \in \{0,\dots,k-1\}\backslash\{x\}} \sigma_2(x'')$. The expected payoff for agent 1 when sending message x is

$$U_1(x,\sigma_2;\theta') = \begin{cases} \sigma_2(x+1)u_1(b;\theta') & \text{if } x < k-1 \\ \sigma_2(0)u_1(b;\theta') & \text{if } x = k-1, \end{cases}$$

where we take into account that $u_i(d;\theta') = 0$. On the other hand, the expected payoff for agent 1 when sending message $x^{'}$ is given by

$$U_1(x^{'},\sigma_2;\theta^{'}) = \begin{cases} \sigma_2(x^{'})u_1(a;\theta^{'}) + \sigma_2(x^{'}+1)u_1(b;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_2(x^{'})u_1(a;\theta^{'}) + \sigma_2(0)u_1(b;\theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

As $u_1(a,\theta^{'}) > u_1(b,\theta^{'})$ and $\sigma_2(x^{'}) \geq \sigma_2(x+1)$, sending message $x^{'}$ is strictly better for agent 1 than sending x against σ_2, thus contradicting the hypothesis that message x is played with positive probability in the Nash equilibrium σ.

Consider the second possibility where agent 2 sends k with probability 1. Then, agent 1's expected payoff of sending message x is $U_1(x,\sigma_2;\theta^{'}) = 0$, while agent 1's expected payoff of sending message k is $U_1(\sigma_2;k;\theta^{'}) = u_1(c,\theta^{'}) > 0$, contradicting the hypothesis that message x is played with positive probability in the Nash equilibrium σ. ■

Proof of Step 1b: Assume by way of contradiction that there exists $x \in \{0,\ldots,k-1\}$ such that $\sigma_2(x) > 0$ and $\sigma_1(x-1) = 0$ if $x \geq 1$ and $\sigma_1(k-1) = 0$ if $x = 0$. Then we decompose our argument into the following two cases: (i) there exists $x^{'} \in \{0,\ldots,k-1\}$ such that $\sigma_1(x^{'}) > 0$ or (ii) $\sigma_1(k) = 1$.

We first consider Case (i). We assume without loss of generality that $x^{'} \in \arg\max_{x'' \in \{0,\ldots,k-1\}} \sigma_1(x^{''})$. Agent 2's expected payoff of sending message x against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1,x;\theta^{'}) = \sigma_1(x)u_2(a;\theta'),$$

while agent 2's expected payoff of sending message ($x^{'}+1$ mod k) against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x^{'}+1 \bmod k; \theta^{'}) = \begin{cases} \sigma_1(x^{'})u_2(b;\theta^{'}) + \sigma_1(x^{'}+1)u_2(a;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_1(x^{'})u_2(b;\theta^{'}) + \sigma_1(0)u_2(a;\theta^{'}) & \text{if } x^{'} = k-1, \end{cases}$$

where we take into account that $u_2(d;\theta^{'}) = 0$. Since $u_2(b;\theta^{'}) > u_2(a;\theta^{'}) > 0$, due to the way $x^{'}$ is defined, we have $U_2(\sigma_1, x^{'}+1 \bmod k; \theta^{'}) > U_2(\sigma_1, x; \theta^{'})$, which contradicts the hypothesis that message x is sent with positive probability in the Nash equilibrium σ.

We next consider Case (ii). Agent 2's expected payoff of sending message x against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x; \theta^{'}) = 0,$$

where we take into account that $u_2(d;\theta^{'}) = 0$. On the contrary, agent 2's expected payoff of sending message k against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, k; \theta^{'}) = u_2(c;\theta^{'}).$$

Since $u_2(c;\theta^{'}) > u_2(d;\theta^{'}) = 0$, we have $U_2(\sigma_1, k;\theta^{'}) > U_2(\sigma_1, x;\theta^{'})$, contradicting the hypothesis that message x is sent with positive probability in the Nash equilibrium σ in the game $\Gamma(\theta^{'})$. ■

Proof of Step 1c: Assume first that $i = 1$; that is, there exists $x^{'} \in \{0, \ldots, k-1\}$ such that $\sigma_1(x^{'}) > 0$. By Step 1a, we first have that $\sigma_2(x^{'}) > 0$. Second, by Step 1b, $\sigma_2(x^{'}) > 0$ implies $\sigma_1(x^{'}-1) > 0$ if $x^{'} \geq 1$ and $\sigma_1(k) > 0$ if $x^{'} = 0$. Third, using Step 1a once again, we conclude that $\sigma_2(x^{'}-1)>0$ if $x^{'}\geq 1$ and $\sigma_2(k)>0$ if $x^{'}=0$. Finally, iterating this argument, we are able to conclude that $\sigma_1(x)>0$ and $\sigma_2(x)>0$ for all $x\in\{0,\ldots,k-1\}$.

The case where $i=2$ is analogous to the previous one, only that we start the loop by applying Step 1b first, before Step 1a. This completes the proof of Step 1c. ■

Proof of Step 2: Assume by way of contradiction that there exist $i\in N$ and $x,x^{'}\in\{0,\ldots,k-1\}$ such that $\sigma_i(x)>\sigma_i(x^{'})>0$. By Step 1c, we know that $\sigma_i(\tilde{x})>0$ for all $\tilde{x}\in\{0,\ldots,k-1\}$. Then, we can choose x and $x^{'}$ satisfying the following property:

$$x\in\arg\max_{\tilde{x}\in\{0,\ldots,k-1\}}\sigma_i(\tilde{x})\ \text{ and }\ x^{'}\in\arg\min_{\tilde{x}\in\{0,\ldots,k-1\}}\sigma_i(\tilde{x}).$$

By Step 1c, we also know that $\sigma_j(\tilde{x})>0$ for each $\tilde{x}\in\{0,\ldots,k-1\}$, where $j\in\{1,2\}\backslash\{i\}$.

Assume that $i=2$. The expected payoff for agent 1 of sending message $x^{'}$ against σ_2 in the game $\Gamma(\theta^{'})$ is given by

$$U_1(x^{'},\sigma_2;\theta^{'})=\begin{cases}\sigma_2(x^{'})u_1(a;\theta^{'})+\sigma_2(x^{'}+1)u_1(b;\theta^{'}) & \text{if } x^{'}<k-1\\ \sigma_2(x^{'})u_1(a;\theta^{'})+\sigma_2(0)u_1(b;\theta^{'}) & \text{if } x^{'}=k-1\end{cases}$$

On the other hand, The expected payoff for agent 1 of sending message x against

σ_2 in the game $\Gamma(\theta^{'})$ is given by

$$U_1(x,\sigma_2;\theta^{'}) = \begin{cases} \sigma_2(x)u_1(a;\theta^{'}) + \sigma_2(x+1)u_1(b;\theta^{'}) & \text{if } x < k-1 \\ \sigma_2(x)u_1(a;\theta^{'}) + \sigma_2(0)u_1(b;\theta^{'}) & \text{if } x = k-1. \end{cases}$$

We compute

$$\begin{aligned} & U_1(x,\sigma_2;\theta^{'}) - U_1(x^{'},\sigma_2;\theta^{'}) \\ = \ & [\sigma_2(x) - \sigma_2(x^{'})]u_1(a;\theta^{'}) + [\sigma_2(x+1 \text{ mod } k) - \sigma_2(x'+1 \text{ mod } k)]u_1(b;\theta^{'}) \\ \geq \ & [\sigma_2(x) - \sigma_2(x^{'})]u_1(a;\theta^{'}) - [\sigma_2(x) - \sigma_2(x^{'})]u_1(b;\theta^{'}) \\ & (\because [\sigma_2(x+1 \text{ mod } k) - \sigma_2(x'+1 \text{ mod } k)] \geq -[\sigma_2(x) - \sigma_2(x^{'})], u_1(b;\theta^{'}) > 0) \\ = \ & [\sigma_2(x) - \sigma_2(x^{'})](u_1(a;\theta^{'}) - u_1(b;\theta^{'}) \\ > \ & 0. \end{aligned}$$

This implies that message x is a strictly better response for agent 1 against σ_2 than $x^{'}$ in the game $\Gamma(\theta^{'})$, contradicting the hypothesis that $\sigma_1(x^{'}) > 0$.

We next assume $i = 1$. The expected payoff for agent 2 of sending message $x^{'}+1$ against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1,x^{'}+1;\theta^{'}) = \begin{cases} \sigma_1(x^{'}+1)u_2(a;\theta^{'}) + \sigma_1(x^{'})u_1(b;\theta^{'}) & \text{if } x^{'} < k-1 \\ \sigma_1(0)u_2(a;\theta^{'}) + \sigma_1(x')u_1(b;\theta^{'}) & \text{if } x^{'} = k-1 \end{cases}$$

On the other hand, The expected payoff for agent 2 of sending message $x+1$

against σ_1 in the game $\Gamma(\theta^{'})$ is given by

$$U_2(\sigma_1, x+1; \theta^{'}) = \begin{cases} \sigma_1(x+1)u_1(a;\theta^{'}) + \sigma_1(x)u_2(b;\theta^{'}) & \text{if } x < k-1 \\ \sigma_1(0)u_1(a;\theta^{'}) + \sigma_1(x)u_2(b;\theta^{'}) & \text{if } x = k-1. \end{cases}$$

We compute

$$\begin{aligned}
& U_2(\sigma_1, x+1; \theta^{'}) - U_2(\sigma_1, x^{'}+1; \theta^{'}) \\
= \; & [\sigma_1(x+1) - \sigma_1(x^{'}+1)]u_2(a;\theta^{'}) + [\sigma_1(x) - \sigma_1(x')]u_2(b;\theta^{'}) \\
\geq \; & [\sigma_1(x+1) - \sigma_1(x^{'}+1)]u_2(b;\theta^{'}) - [\sigma_1(x) - \sigma_1(x^{'})]u_2(a;\theta^{'}) \\
& (\because [\sigma_1(x+1 \bmod k) - \sigma_1(x'+1) \bmod k)] \geq -[\sigma_1(x) - \sigma_1(x^{'})], u_2(a;\theta^{'}) > 0) \\
= \; & [\sigma_1(x) - \sigma_1(x^{'})](u_2(b;\theta^{'}) - u_2(a;\theta^{'})) \\
> \; & 0.
\end{aligned}$$

This implies that message $x+1$ is a strictly better response for agent 2 against σ_2 than $x^{'}+1$ in the game $\Gamma(\theta^{'})$, contradicting the hypothesis that $\sigma_2(x^{'}+1) > 0$. This completes the proof of Step 2. ■