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Quantitative effects of two kinds of robots in a neo-classical growth model

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QUANTITATIVE EFFECTS OF TWO KINDS OF ROBOTS IN A NEO-CLASSICAL GROWTH MODEL

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Quantitative Effects Of Two Kinds Of Robots

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Quantitative Effects of Two Kinds of Robots in a Neo-Classical Growth Model

by
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2019

Declaration

I hereby declare that this PhD dissertation is my original work and it has been written by me in its entirety.
I have duly acknowledged all the sources of information which have been used in this dissertation.

This PhD dissertation has also not been submitted for any degree in any university previously.

VU HOANG PHUONG QUE

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Abstract

Advances in artificial intelligence are leading to many revolutions in robotics. How will the arrival of robots impact the growth of the economy, the workers' wage, consumption, and lifetime welfare? This dissertation attempts to answer this question by presenting a standard neoclassical growth model with two different kinds of robots, reflecting two ways that robots can transform the labor market. The first chapter introduces additive robots- a perfect substitution for human labor, while the second chapter employs multiplicative robots- a type of robots that augments human labor. The prevailing main result is that even in the case with no population growth and technical progress, the application of robots is enough to create a long term economic growth. Nevertheless, there is a difference in the behavior of real wage. The presence of additive robot solely makes wage jumps down and then stays constant throughout while utilization of multiplicative robots alone can increase productivity thus real wage increases fast over time.

In the last chapter, both types of robots are applied in the economy with a shrinking population, motivated by Japan. Under the perfect homogeneous labor market, there will be a shift of workers from jobs that can be substituted by additive robots to jobs that can be supported by multiplicative robots. This enables Japan to continue to enjoy the perpetual growth in real wage, consumption and wealth even after the labor market has finished its adjustment. However, as the interest rate would slowly decrease, proportionate to the decline of the population, there would be a point where it is no longer profitable to adopt robots although it would take a long time for the economy to face that issue.

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Dedication

This thesis is especially dedicated to my mom and dad who never give up on me, who encourage me to pursue my dream and who love me till the end. Without their endless love, forgiveness and support, I will never be able to overcome all the difficulties in order to finish this thesis as well as my Ph.D journey.

Chapter 1

Introduction

1.1 Motivation Factors

1.1.1 Invasion of Robots

The recent progress in technologies, especially those in automation machines, have been bringing forward many undeniable advantages for human production and living conditions. A new term has emerged throughout the information in news, discussion forums as well as research community...: Robots.

In general public understanding, robots are equal to industrial robots, especially those that are used in automation production processes. However, according to the International Federation of Robotics (IFR), a robot is "***automatically controlled, reprogrammable, multipurpose manipulator, programmable in three or more axes, which may be either fixed in place or mobile...***" (**IFR 2017**). IFR has reported constant increases in industrial robots and estimated that the trend will be continued at around 16% per year for the next few years.

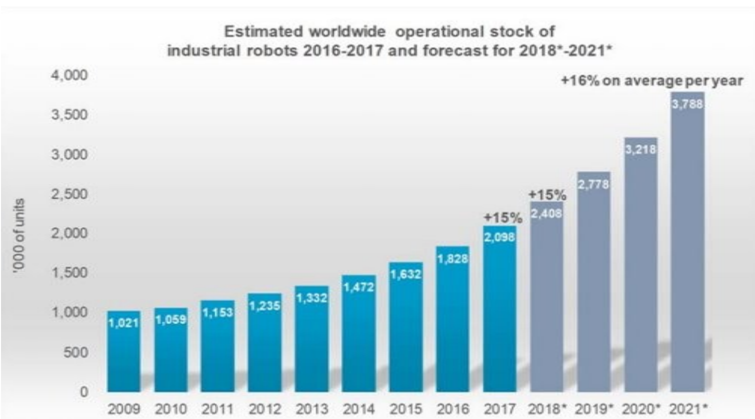


Figure 1.1: Industrial Robots (2009 - est.2021)[1]

However, the application of robots is no longer limited within the industrial sector. They are in fact expanding their invasion into the service sector more and more. The number of professional service robots grew 85% in 2016, 32% in 2017, and is expected to continue to rise at the rate of about 21% in the next 3 years, reaching 736,600 units by 2021[2].

On the other hand, human population growth shows a consistently decreasing trend in the last 30 years, currently staying at around only 1% per year worldwide.

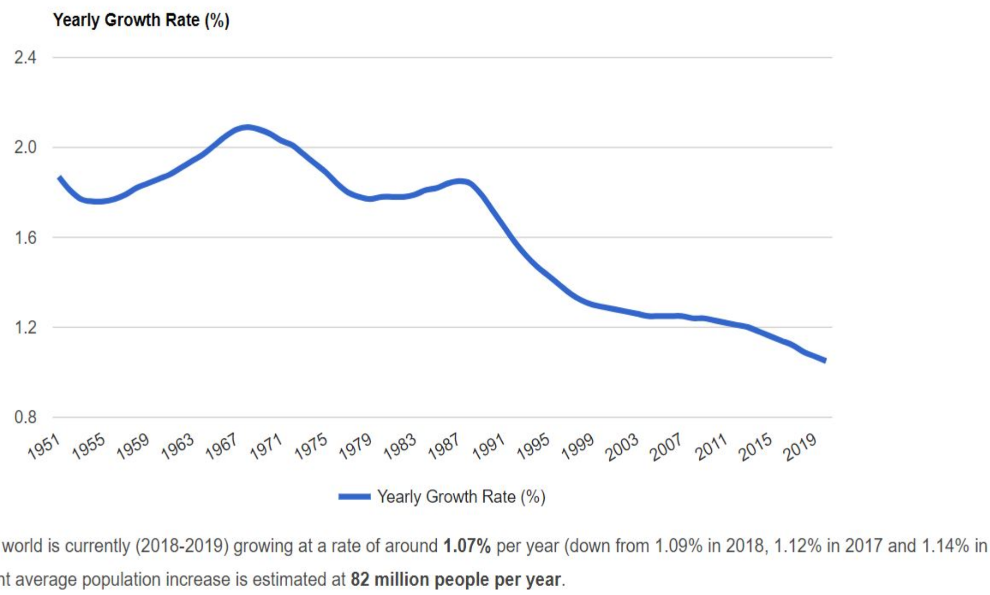


Figure 1.2: World Population Growth Rates 1950-2019[3]

There is no double that there are many benefits and convenience that these robots bring to our society. Nevertheless, are there purely opportunities or is there any drawback that we should know or prepare ahead? This big wave of robots invasion is hence posting huge questions to the research community. Economists and researchers are trying to analyze and predict the impacts that robots might bring. The most concern aspects are the effects on workers employment and the real wage.

1 https://ifr.org/downloads/press2018/Executive_Summary_WR_2018_Industrial_Robots.pdf

2 https://ifr.org/downloads/press2018/WR_Presentation_Industry_and_Service_Robots_rev_5_12_18.pdf

3 https://www.worldometers.info/world-population/

1.1.2 Jobs Replacement Apprehension

Early Predictions

Some have been suspecting that with rapid technology improvement, eventually, we will head toward a robotic economy where human workers are slowly and then totally replaced by robots and machines, creating mass unemployment. Such fears arose quite early in history, especially after any extended periods of high unemployment rates. During the Great Depression (1929-1939), John Maynard Keynes has foreseen the rapid growth of technological progress in the next 90 years which we are close to the end of it. However, he also noted that "***we are being afflicted with a new disease of which some readers may not have heard the name, but of which they will hear a great deal in the years to come - namely, technological unemployment***" **(Keynes 1930)**.

Two decades later, Wassily Leontief also predicted a similar problem when writing "***labor will become less and less important... more and more workers will be replaced by machines. I do not see that new industries can employ everybody who wants jobs***" **(Leontief 1952)**.

In the 1990s, just before the dot com bubble busted which lowered the unemployment rate to an all-time dip, Jeremy Rifkin had predicted that technology would produce the "End of Work" - **(Rifkin 1995)**.

Even though we have gone through three different stages of technology advances and (luckily?) these predictions have not become true as the industry evolution 4.0 is approaching. Instead, we saw a huge improvement in the productivity of humans and much efficient production all over the world. However, what set the current technology innovation wave apart from the previous three earlier phrases?

Recent Studies and Evidence

We are entering industrial 4.0 with recent rapid development in technology, especially with artificial intelligence, which led to a further surge of public interest in automation and robotics. Headlines about the application of new robots unitized in different aspects of the economy are everywhere and daily. However, driving that surge is again not only the fascination with prosperous changes that these technologies can bring forward but also the concern of the impact on employment that more and more jobs are now done by robots instead of human labor. A very recent title from Straits Times stated a fearful prediction:

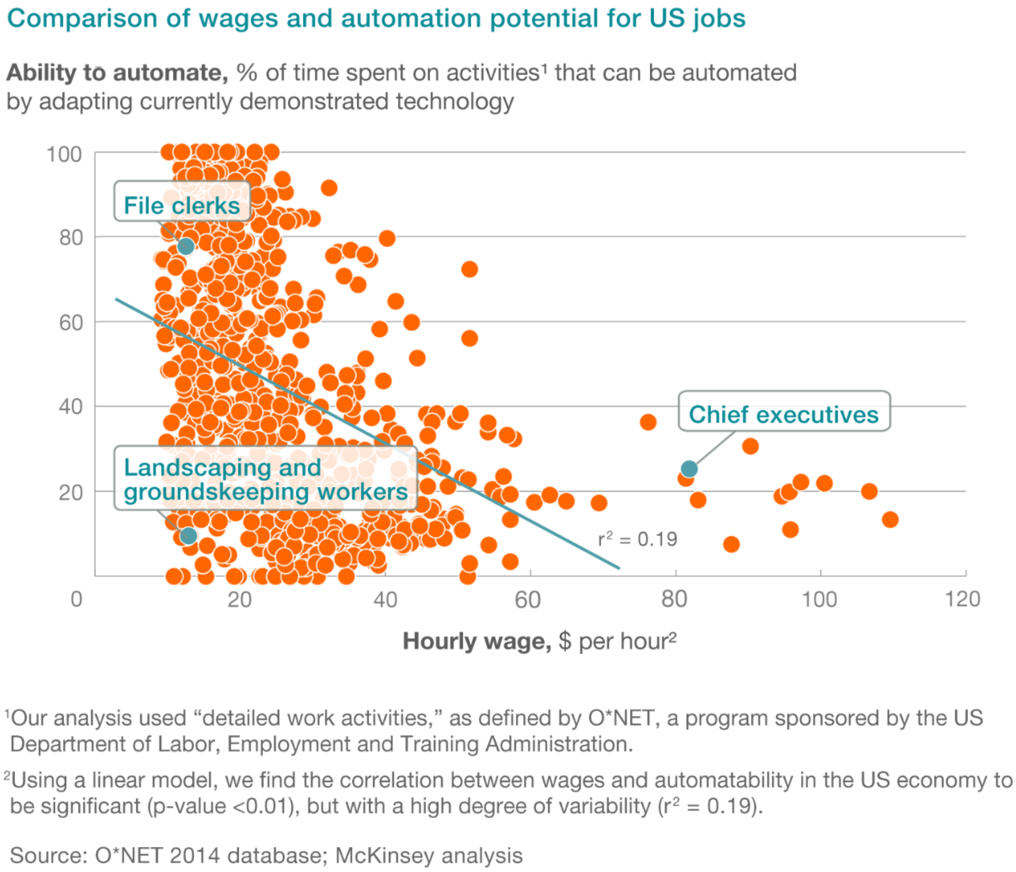
Robots to wipe out 20 million jobs around the world by 2030: Study

The authors emphasized that: "...***The implications are huge. We will see a significant boost to productivity and economic growth and some new types of jobs we cant even yet foresee***". They, however, noted that starting from the booming in the 2000s, some 1.7 million manufacturing jobs have already been lost to robots, including around 400,000 in Europe, 260,000 in the US, and 550,000 in China. The number is expected to be ten times in the next 10 years, and "...***business models will be disrupted or upturned and millions of existing workers will be displaced***[4]...".

Frey and Osborne, by using a Gaussian process classifier, have estimated the prob-

[4] https://www.straitstimes.com/tech/robots-to-wipe-out-20-million-jobs-around-the-world-by-2030-study

ability of computerization for 702 detailed occupations, and found out that about **47% of total US employment** is at risk. However, only **fewer than 5%** of occupations can be **entirely** automated although **about 60%** of occupations could have 30% or more of their activities substituted by machines (**Frey and Osborne 2017**).



There are two main conclusions which we can derive from their results. First, it is very difficult for a job to be **fully** automated with no human intervention. In other words, it is hard for workers to lose their job completely to robots. Second, opposite to normal belief, their results also showed that "it is no longer the case that only routine, modifiable activities are candidates for automation and that activities requiring tacit knowledge or experience that is difficult to translate into task specifications are immune to automation". For example, according to their data, it is true that routine tasks such as file clerks can be automated closed to 80% of their jobs cope. But many other lower-skilled jobs such as home health aides, landscapers, and maintenance workers, only a very small percentage of activities could be done by machines (less than 20%). Conversely, around 25% of a CEOs working time could be automated.

1.2 Recent Advances in Literature

1.2.1 Pessimistic vs. Optimistic Economic Growth

Technological Pessimists

Sachs and Kotlikoff painted a dark picture by building an over-lapping generation (OG) model with the labor market consist of the old skilled generation and a CES combination of robots and the young unskilled generation. Improvement in robots productivity thus reduces the demand for young unskilled workers and as the result depresses their wages. As being the only generation that saves, young workers will reduce their savings as well as the investment for robots and capital in the next period when they are old[5]. The lower investment will translate into a lower total output, which further reduces wages for the next young generation (on top of any reduction due to the advances in robots). Consequently, the economy will be sent into a spiral of perpetual contractions, with wages further goes down and down over time. Under the CES combination of robots and human labor in the workforce, the higher the elasticity of substitution between robots and human labor, the more severe the results are (**Sachs and Kotlikoff** 2012). DeCanio further supported Sachs and Kotlikoff by fitting different distributions to cross-sectional data on US productivity and found that when the elasticity of substitution between human and robotic labor is higher than 1.9 then technologies will cause a decline in aggregate wage (**DeCanio** 2016).

[5] Under log consumption utility function, savings do not depend on investment return rates

Berg et al. showed their pessimistic view even from the name of their paper "Should we fear the robot revolution (The correct answer is yes)". They focused on rather short-term effects[6] and in all scenarios of their models, wages would be decreased as robots productivity increases (**Berg, Buffie, and Zanna 2018**). While Freeman agreed with them in a way that advances in robots are bad for inequality, he also emphasized that all benefits will be redirected to increase the return for those owned technology advances only not for normal workers as their wages will be cut down (**Freeman 2015**).

Technological Optimists

On the other hand, some technology optimists have not denied the short-term disruption caused by advances in automation. Their focus, after all, is more on the medium and long-term, and they agreed that there is a trade-off between short-term suffering and long-term benefits.

Nevertheless, there is not much progress in theoretical models to support this path. The latest in the literature is a recent innovative paper by Acemoglu and Restrepo. Their model uses a task-based approach in which robots replace human labor in more and more tasks over time. The twisted point is that new tasks, where human labor has comparative advantages, have emerged at the same time. Hence, the model would reach a balanced growth path where the rate in which workers are replaced is the same as the rate that new tasks are invented. Although the short/medium results are not really clear in their paper, the long-run effect, after the economy reaches the balanced growth path, is that real wages will increase and labor share will be back to the original level (**Acemoglu and Restrepo 2018**).

[6] The short-term "defined" by the period when the rates of return for normal capital and robots are adjusted to match each other as they assumed that at the initial point the rate of return of robots would be much **higher** than that of normal capital.

Even in the paper by **Berg et al.** (2018), if the focus is shifted toward the longer term for the economy, their models do report the positive growth in wages in long-run[7]. Pointing out short-term adverse effects is important for suggesting mitigation policies[8] toward the long-run prosperity rather than is the denial of the hard path.

1.2.2 Robots, Productivity, and Employment

It is undeniable that technology improvement has been increasing human productivity tremendously over the past few decades. Robots are naturally another wave of technology advances, no double will further improve overall productivity. In their study which focused on robotic impacts across 17 countries, Graetz and Michaels concluded[9] that increases in robots densification contributed approximately 0.36% points to labor productivity growth, raising total factor productivity and lowering output prices across 17 countries studied **(Graetz and Michaels 2018)**.

In another recent research done by the Center for Economics and Business Research in 2017 to study the impact of automation with data from OECD countries from 1993 to 2016, it is found that one unit increased in the robot density is associated with 0.04% increases in labor productivity **(Economics and Research 2017)**.

Furthermore, there is a link between productivity and demand **(Graetz and Michaels 2018)**, that higher productivity if can translate into higher wages, will induce higher household demand. That supports the arguments in **(Bessen 2016)**. Bessen tried to explain why automation can increase employment in some industries while de-

[7] According to them it would take a few generations though.

[8] For example, (Freeman 2015) pointed out that ownership is the key to resolve the problem of inequality. The more workers are encouraged to own part of the technology that replaces them, the smaller the inequality gap they have to suffer.

[9] Based on a model of firm decisions to adobe robots as a Technology choice to replace some of the tasks from using labor to using robots in relatives with robots prices (with shreds of evidence that Robots prices have been decreasing over years). Their estimates also suggested that robots only reduce low-skilled workers employment share.

pleting labor in some others. He considered automation as labor augmenting technological changes, which make labors more productive. Although the productivity of robots defines the magnitude of the changes in that industrys employment, price elasticity of demand decides the sign of the changes. He argued that in some sectors, in the beginning when demand is still highly elastic, the demand can increase fast to catch up with the increase in the productivity of production and hence actually push up demand for labor workers.

On top of that, increased demand in one sector/industry will then have spilled over effect to other complementary sectors and eventually spread to the whole economy (**Zierahn, Gregory, and Arntz 2016**). As a consequence, robots have no or very little effect on total hours worked (**Graetz and Michaels 2018**).

1.2.3 Jobs Substitution vs. Jobs Re-allocation

In their The Second Machine Age book, (**Brynjolfsson and McAfee 2014**) one more time raised a similar concern with the predecessors that the rapid automation of jobs will make human labor redundant in the future. As quoted in (**Akst 2013**), economic historian Robert Heilbroner confidently stated that "***as machines continue to invade society, duplicating greater and greater numbers of social tasks, it is human labor itself - at least, as we now think of labor - that is gradually rendered redundant***. The fear is that human labor will find it more and more difficult to compete against robotics and artificial intelligence just like horses lose their jobs when cars were invented.

In many tasks or jobs, robots indeed have significantly higher productivity or is a safer choice for dangerous duties. However, focusing too much on the substitution effect shadows the fact that, in many cases, automation does not necessarily lead to job replacement but jobs re-allocation instead.

Economics follows comparative advantages rather than absolute advantages. Hence, if that is still true, human labor will always find jobs at the borders at which they have higher comparative strength than machines. Furthermore, with the emergence of new technology, we can see the born of many new occupations or tasks where technology is not yet able to follow. For example, now with live stream features, many lecturers and teachers can conduct their classes live online, and students can even watch the videos conveniently at their schedule. Behind the scene, however, that might need a lot of technical support labor that not used to or needed to do those tasks before. Another quite obvious example is that the more robots (rather than just machines) are used, the more labor is needed for producing, programming, and maintaining those robots themselves. **(Bessen 2016)** hence, concluded that: "***Automation might not cause mass unemployment, but it may well require workers to make disruptive transitions to new industries, requiring new skills and occupations***." **(Acemoglu and Restrepo 2018)** also added on to that direction by noting there would be "...***new tasks ranging from engineering and programming functions to those performed by audio-visual specialists, executive assistants, data administrators, and analysts, meeting planners and computer support specialists***". Their task-based approach indeed assumes that human has a comparative advantage in new and more complex tasks, while old tasks are gradually automated by robots. If the creation of new tasks is sufficient, then the labor share can even be back to a constant level over time.

1.2.4 Mixed Empirical Results on Wages

Employment is just one side of the labor calculus, wages are far more important to the humans well-being. Even if we agree that net employment does not change or even increase, there is nothing that can ensure that compensation for workers will increase at the same time. Empirically, we have very mixed results.

(**Acemoglu and Restrepo 2017**) analyzed the effects of the increase in industrial robots from 1990 to 2007 on the US labor market. The local impacts are obtained by regressing the changes in wages on robot usage at the industry-level and across industries. According to their estimates, one more robot per thousand workers **reduces wages by 0.42%**.

On the other hand, (**Graetz and Michaels 2018**) regressions reported a positive relationship between robot density and wages. Hence, using more robots actually can **increase wages** for workers in general. If the results by (**Acemoglu and Restrepo 2017**) are at industry-level, Graetz and Michael used data for 17 countries from 1993 to 2007. Hence, it might be that their conclusions are more general or at a more aggregated level.

1.3 Research Statement

1.3.1 Research Questions and Discussions

With the vast invasion of robots or any type of machine with artificial intelligence, the big question is that how the economy is affected especially in terms of economic growth in both the short and long term, as well as that how wages and workers lifetime welfare will be?

The huge public fear of robots replacing human labor has driven the model of the first type of robots where they are a perfect substitution of labor in production. Nevertheless, focus on this aspect of robots and forget the all-long-time effect that technology has been brought forward which boosting the humans productivity. Just like Autor commented in his paper (**Autor 2014**) that "...[people] ***tend to overstate the extent of machine substitutions for human labor and ignore the strong complementaries between automation and labor that increase productivity, raise earnings, and augment the demand for labor... Focusing only on what is lost***

misses a central economic mechanism by which automation affects the demand for labor: raising the value of tasks that workers supply uniquely". And so we have enough empirical support for the idea of the second type of robots introduced in Chapter 3, which helps to increase human labors productivity.

Furthermore, the mixed empirical results on wages also suggested that there might be more than one channel in which robots affect workers wages and we should expect opposite effects from different channels. It follows that only one way to model robots is not sufficient. Additionally, it can be argued that not like other subjects in economics where we have enough empirical data to support parameter values or regressors, effects from robots might not easily be concluded if they are built in very complex models. We need a simple but rich enough model to study the quantitative effects of these robots - a new type of capital[10].

At the same time, supported by the discussed results (Bessen 2016, Zierahn, Gregory, and Arntz 2016, Graetz and Michaels 2018, Autor 2015), total working hours or days is kept unchanged most of the time in models presented in this thesis. The changes, if there are, are coming from the supply side when workers decide that they do not need to work as much any more[11] or naturally is reduced by the shrinking population as in Chapter 4.

1.3.2 Directly Related Literature

The models I presented in this thesis are the direct extensions of what has been suggested by (**Hoon 2020**) which based on a simple Solow model without any population growth or technological progress[12]. There, robots can be thought of as a new type of capital (just like many other types of capital), helicopter-dropped into

[10] The general idea is that in most of the economic models, inputs that are not human labor will be lump into one category of capital including machines, factories, computers....

[11] When they do value leisure in the utility function

the economy, which differs from the conventional machines by the built-in artificial intelligence. He also suggested two different types of robots, one can substitute human labor and the other type can support or increase workers productivity.

The Solow Model and the Central Question of Growth Theory

The reason for no population or no technology progress assumptions is that in traditional growth economics, we can have perpetual growth only when there is an increase in either population or productivity factor. In the standard Solow model, entering production function is the effective labor $\boldsymbol{AL}$ where the productivity factor $\boldsymbol{A}$ grows exogenously at rate $\boldsymbol{g}$. With the constant saving rate, assuming no population growth, then $\boldsymbol{g}$ determines the growth rate of capital stock $\boldsymbol{K}$, capital per worker $\boldsymbol{K/L}$, and output per worker $\boldsymbol{Y/L}$. Since the saving rate is constant, consumption, as well as consumption per worker, also grows at the same rate of $\boldsymbol{g}$ at the balanced growth path.

The Solow model identifies two possibles sources for the variations in output per worker: accumulation of capital per worker and the effectiveness of labor $\boldsymbol{A}$. The interpretations of $\boldsymbol{A}$ are factors such as the education, skills of the labor force, the quality of the infrastructure... However, only the growth in the effectiveness of labor $\boldsymbol{A}$ can lead to permanent growth in output per worker. While the accumulation of capital does not account for the large differences in output per worker across either time or countries since such a huge difference will require vast differences in the rate of return (**Lucas** 1990) which we do not see.

Nevertheless, there was little to say about the effectiveness of labor since the Solow model assumed it to be exogenously improved. Our model with helicopter robots presented here, on the other hand, even without both the productivity factor and

[12] No technological progress means that rather than the changes persisted to robots, there are no advances in all other types of capital - which called conventional machines.

population growth, output per worker still grows perpetually, making consumption per worker grows steadily over time and thus sending the economy into infinite endogenous growth phrase. In other words, the usage of robots alone is enough to create infinite economic growth.

Most Resembled Work

Apart from the direct framework suggested by (**Hoon** 2020), (**Hanson** 2001) paper mostly shared similar ideas with my models. He constructed a simple exogenous growth model in which computers are not just complementing human labor but slowly become more productive and finally be able to substitute human. However, there are many differences. His Cobb Douglas production function includes three inputs rather than two:

$$Y = K^{\alpha} M^{\beta} L^{\gamma}$$

With this construction, he did not consider computer ($\boldsymbol{M}$) as a special type of capital which is moved to join the human labor force[13] as we did here. The improvement in productivity of computers making it more abundant and hence reduces its rate of return, making it more attractive than human labor[14] and hence reduce the labor demand from the production side. When the effect is still low, the computer is just a complement to labor and hence still can help to increase wages. However, when the effect is so dominant that the demand for labor is going down substantially and finally is replaced by the computer then wages will start to go down. Even though the final result is similar to mine in the way that the substitution effect reduces wages while the complementary effect increases wages, the mechanism is actually different[15].

[13] This is one of the most crucial basic assumptions for all the models presented in this thesis.

[14] He assumed that the marginal product of capital stays constant all the time.

[15] We will examine the detailed mechanism the following chapters.

The other difference is that the models presented here are more general equilibrium methods where both household and firms maximization problems are used while Hanson only considered the production side and studied the changes in total output which derive the changes in wages (his method is based on Slow-Swan model while ours is more Ramsey-Cass-Koopmans model). The lack of general equilibrium prohibited him from observing the changes in consumption, workers wealth, and welfare.

Another distinguished point is that his model is static, and the focus is on comparative statics which is different from my dynamic growth model. Hence, we can view it as one-period results while the price is adjusted. That is why his model cannot see the changes in the labor market even when he includes both types of robots (which defined as two types of computers in his model).

On a different view, that enables his work to complement my work in some ways. He differentiated the price of computers from the price of ordinary machines (the argument for a production function with 3 inputs) and assume that the price of computers falling much faster from an initially high value until the price of the computer matches with the price of normal capital. Hence, this can be viewed as the transition result before the market reaches no arbitrate condition where the returns on both types of capital (conventional and robotic) should be equal[16].

1.3.3 Contributions and Main Results

Main Contributions

The series of the model presented in this thesis is built directly upon what has been suggested by (**Hoon 2020**). However, Hoons models are of general benchmarks

[16] In all models here, I assume that the market will adjust immediately right after the introduction of robots

with no specific function forms; hence the initial results he got are general and purely theoretical. As a result, this dissertation adds contributions to his initial research in several ways:

- **Firstly**, I confirmed Hoons initial results in the general form by using specific forms of functions for both households and producers. I also further extend the model with the different economic environments such as flexible vs. inflexible labor market or shrinking population.

- **Secondly**, I introduce many new elements into the model. In his model, Hoon assumed that the comparative productivity of robots and humans is always 1:1 which does not affect the general results. However, parameterizing the productivity of robots allows me to see other things such as the effect of advances in robotics, or comparing different scenarios of productivity for two types of robots. The most prominent extension is in the last chapter when we consider both types of robots into the economy. If Hoon is stopped at a very general setup and hence general prediction, I have set up an extensive model with the specific structure of the labor market[17] which enables me to study the movement of human labor.

- **Thirdly**, only by using specific functional forms, I can identify different conditions for the models to work. For example, the profitability condition for robots resulting in a minimum productivity level for robots to be beneficial enough so that producers are willing to convert part of their capital from conventional machines to robots. Especially for Multiplicative Robots in Chapter 3, I identified the participation condition for the usage of robots in the economy. Even though Multiplicative Robots is productive enough but producers

[17] In (**Hoon 2020**), he fixed the structure of the labor market which prohibits him from observing the changes in labor.

still not able to use them unless they have a participation condition in which returns rate for production inputs are only fractions of marginal products of those inputs. This new markup element complicates the model (hence requires a numerical method to solve) but at the same time allows us to observe the trend of another economic behavior. Markup is not a new thing, especially if we have a monopoly market, however, the purpose of using markup, in this case, is different. It is a facility that allows the producer to alleviate the cost of financing robots to other input factors.

- **Fourthly**, by calibrating parameters with either standard literature values or specific values[18], I can further evaluate how strong the quantitative effects that predicted by theoretical models are. It also allows us to compare the effects of two types of robots which a general model can not do. However, one observation is that the results are very sensitive to the values of parameters. In other words, changing the values of parameters might change the comparison results. Except for parameters that use standard literature values, the remaining parameters have value based on my judgment. Since this topic of robots and artificial intelligence is still a new branch of economics, many new parameters do not have any preceding convenient values. Once we have enough data to back up or to generate more reliable values then the results would be more correct. However, the framework I presented here still can serve as the building block for any further quantitative evaluation.

- **Lastly**, the model is simulated in the long run to verify the existence of a balanced growth path where the economy converges to the new steady level. In the last chapter, the long-run consideration is even more important as we need to identify the feasibility period - the period in which robots are still

[18] With appropriate judgments

worth to use.

The focus of the paper by Hoon is mostly lying with the economic growth archived by using robots and the changes in workers wages. My extensions further enable us to observe many other economic variables such as the comparison between the stock of robots and conventional machines, market rates such as interest rate, markup rates, saving rates, and human shares ratio.

Main Results

This dissertation modelings two different kinds of robots, reflecting two ways that robots can transform the labor market, under a crucial assumption that capital is fully malleable or instantaneously convertible from conventional machines to robots and vice versa. In the first chapter, I introduce additive robots - a perfect substitution for human labor, while in the second chapter, we employ a multiplicative robot - a type of robot that helps to augment the human labor. In the third chapter, we investigate if the utilization of both types of robots can help to mitigate the situation with a shrinking population, motivated by the case of Japan.

The prevailing result is that even in the absence of population growth and technological progress, the application of helicopter robots (either type) into a simple aggregate neo-classical model itself is enough to create long-term economic growth, despite unfavorable outcomes in the process. The economy will be out of the initial steady-state and enter into perpetual growth. The interest rates under profitability conditions will reach a higher level. Workers can enjoy a higher stream of consumption despite the initial drops. Total wealth increases fast over time and hence the lifetime welfare of households also surges. The higher the robots productivity, the stronger the quantitative effects. There is also a minimum productivity level so that robots are worth using.

In the long run, both cases (assumed that start with the correct starting point in the first period after the introduction of robots), there always exists a balance growth path where all variables are converged to grow at the same rate.

Compared to Additive Robots, Multiplicative Robots create a better growth effect. An Additive Robot is assumed to be as productive as 33% of human labor while a Multiplicative Robot help to increase the whole work forces productivity only by 1%. Under these values, with Multiplicative Robots usage, there is a less drop in the first-period consumption (26% compared to 59% for Additive Robots) and a faster growth along the balanced growth path (10.5% compared to 5.6% for Additive Robots). As a result, workers will have a higher lifetime welfare increases (112% compared to 14.6% for Additive Robots). Notably, growth is faster with smaller saving rates (48% compared to 81% for Additive Robots) since not only non-human wealth but human wealth is increasing over time. That means the investment with Multiplicative Robots is more effective. However, since the qualitative results are extremely sensitive to the value of robot productivity, this is only true upon a specific set of parameters. If Additive Robots productivity increases so much and dominates Multiplicative Robots, the story will be reversed.

Besides, there is a difference in the behavior of real wages. Under the presence of additive robots solely, the workers wage jumps down and then stays constant throughout, while with multiplicative robots alone, after the initial drop, real wage increases fast over time, in response to the increased human labors productivity. The increases in real wage reduce overtime and converge to a constant level (10.5%) the same as other variables.

As suggested by empirical results, the two types of robots would create two opposite mechanisms on real wages. Hence, it would be interesting to see what happen when both kinds of robots are used in the economy, i.e both forces are possible on the workers wage. It is no surprise that the economy would be no longer in the initial steady-state but grows permanently since each type of robot will be enough to create that effect. Apart from that, there are three remarkable results.

First, under the homogeneous labor market, we would see a shift of workers from jobs that can be substituted by additive robots to jobs that can be helped by multiplicative robots, to enjoy the higher wages. This result does not depend on the robots productivity level. As long as the market is efficient, labor will shift toward jobs that are not able to be substituted by robots. The force to pull up wages will hence be stronger, making not only the interest rate but also real wage increases. Under a shrinking population, the shift will even be faster to compensate for the loss in the labor force.

Second, the economy can continue to enjoy the perpetual growth in real wages, consumption, and wealth even after the labor market has finished its adjustment. When there is no more labor doing jobs that are threatened to be done by robots, their wage is no longer to be pulled down by the force created by Additive Robots but only enjoy the higher return paid to higher productivity.

Nevertheless, thirdly, while in the case of a fixed population, the interest rate would converge to a constant level, under the shrinking population, the interest rate would slowly decrease, proportionate to the decline of the population. Eventually, there would be a point where it is no longer profitable to adopt robots although it would take a rather long time for the economy to face that issue. However, there are mitigating solutions to prolong that feasibility period.

Chapter 2

Additive Robots

2.1 Introduction

In this chapter, I introduce the first type of robots which can **perfectly** substitute for human tasks, in other words, a robot can do tasks exactly like a human. These type of robots is creating the most hyped fear that human's jobs are taken away by robots.

It follows that if the economy adopts R^A units of robots, the "**effective labor**" would become:

$$L^A = \left(\Lambda_A R^A + H\right) \tag{2.1}$$

in which H is human labor and Λ_A is robots' productivity. One unit of robots is as productive as Λ_A units of human labor. That is why we name them "**Additive Robots**": a simple addition to the labor force. The labor market now includes both human labor and robotic labor.

It is worth to discuss the role and value of Λ_A. In general, people believe that Λ_A should be more than one, which means robots are much more productive than human labor. It might be true for many automatic resemble or package process. However, that might not be the case at an aggregate level. As mentioned in the first chapter, **(Frey and Osborne 2017)** have analyzed a total of **702** occupations and shown that only less than 5% of occupations can be entirely replaced by current technology. It is very hard for a robot to completely do a job without human intervention. We would still need to supervise it, maintain it, and still need human labor to communicate with other departments or connect between tasks. For example, robots are replacing surgeons to do surgical operations. These tasks are highly sophisticated but sometimes robots can do even better than human doctors. At least, they might not be influenced by emotion or environment. As long as they are programmed well they will just do what had been planned with almost perfect precision. Undoubtedly, we will still need doctors to be there to supervise the whole process, to troubleshoot if any unexpected situations occur. A robot might not be prepared for all possible scenarios. As a result, at the aggregate level, Λ_A should be less than one.

What should then be the value of Λ_A?

There is no such preceded value before in literature. Again as Frey and Osborne have shown there is a wide range for that value. Almost 80% of a file clerk can be done by machines, giving us the impression that most simple jobs can be replaced highly by robots. However, another simple job like sweeping the floor can only be automated by around 10%. That is because when there are obstacles or corners, the movement of robots is no longer smooth hence it will not be able to clean well or might take a longer time to clean. Especially when there are stairs and robots are not able to bend down or walk down thus unable to do the tasks or can not do tasks as efficient as a human. On the other hand, even more than 20% of a CEO's work can be automated by current technology.

However, their results also show that "...about 60% of occupations could have 30% or more of their activities substituted by machines..." **(Frey and Osborne 2017)**. That makes the minimum value for Λ_A should be around 0.18. For calibration purposes, I use the value of 0.2, slightly higher than the minimum value, and conservative enough to not create any unreasonable results.

Before introducing this type of robot into the economy, this chapter will layout the baseline standard aggregate one-sector model. Then I will present and discuss the results of model simulations after the adoption of robots. In the last past, I allow three different extensions. The first one is a variation with flexible labor supply, the second is allowing a transition to higher productivity for robots and lastly, we generalize the production technology to a CES function to make use of the elasticity of substitution.

2.2 The Model Setup

2.2.1 Standard Aggregate One-Sector Model

In this section, I will first describe the setup for the standard Solow-Swan one sector aggregate model and layout its important solutions. This serves as the baseline formulas before we introduce the robots into the economy.

- **The Representative Household**

  The setup follows exactly what has been described in (**Hoon 2020**). The representative household has infinite lives[1], has θ as the subjective time preference rate.

  At any period t, the agent works a **fixed** number of hours H, receives v_t as a real hourly wage and earn a real interest rate of r_t on his non-human wealth w_t^n. He will need to decide his consumption c_t in each period in order to maximize his lifetime utility:

$$U_t = \int_t^\infty \log c_\kappa \exp\big(-\theta(\kappa - t)\big) d\kappa \tag{2.2}$$

[1] Although this is a usual setup in the general equilibrium model, we can interpret this assumption as a continuous overlapping generation, in which in each period, the old generation will pass on the inheritance wealth to the young generation. This allows capital accumulation in the economy through many generations.

subject to a budget constraint or non-human wealth accumulation rule:

$$\dot{w}_t^n = \frac{dw_t^n}{dt} = r_t w_t^n + v_t H - c_t \tag{2.3}$$

and the transversality condition that prevents the household agent from going indefinitely into debt[2].

Denote the **human wealth** which is the total discounted present value of all future labor income streams as:

$$w_t^h = \int_t^\infty v_\kappa H \exp\left(\int_t^\kappa -r(\nu)d\nu\right) d\kappa \tag{2.4}$$

with the accumulation rule as following[3]:

$$\dot{w}_t^h = r_t w_t^h - v_t H$$

The **solution** to the agent's problem[4] is then given by the following consumption rule in each period t:

$$c_t = \theta\left(w_t^h + w_t^n\right) \tag{2.5}$$

with a Euler equation for the consumption growth equates the difference between the real interest rate of period t and the time preference rate.

$$\frac{\dot{c}_t}{c_t} = r_t - \theta \tag{2.6}$$

- **The Representative Firm**

  On the production side, we start with the simple Cobb Douglas production function. At each period t, the representative firm produces final goods using technology:

$$Y_t = F(K,L) = K_t^\alpha H^{1-\alpha} \quad \text{with } \alpha \in [0,1]$$

[2] Details can be found in **Appendix A.1.1**

[3] Proof in **Appendix A.1.1**

[4] Proof in **Appendix A.1.1**

Denote $y_t = \frac{Y_t}{H}$ as the output per day then $y_t = f(k_t) = k_t^{\alpha}$ with $k_t = \frac{K_t}{H}$ is the **capital intensity** - how much capital is spent per working day. These notations are the same with per labor variables since we have the representative household as the only one worker.

Thus, the profit function for the period t is:

$$\pi_t = y_t - r_t^K k_t - v_t$$

where r_t^K is return rate or rental rate on capital which is the sum of the real interest rate and depreciation rate δ:

$$r_t^K = r_t + \delta$$

First-order conditions of profit maximization problem give us the following equations for market rental rate and real wage[5]:

$$r_t^K = f'(k_t) = \alpha k_t^{\alpha-1} = r_t + \delta \tag{2.7}$$

$$v_t = f(k_t) - r_t^K k_t = k_t^{\alpha} - \alpha k_t^{\alpha-1} k_t = (1-\alpha)k_t^{\alpha} \tag{2.8}$$

Financial market clearing condition requires $w_t^n = K_t$ hence $k_t = \frac{K_t}{H} = \frac{w_t^n}{H}$ so that:

$$\dot{k}_t = \frac{\dot{w}_t^n}{H}$$

From the budget constraint(2.3), we have:

$$\frac{\dot{w}_t^n}{H} = r_t \frac{K_t}{H} + v_t - \frac{c_t}{H}$$

Using the real interest rate (2.7) and wage equations (2.8) we have the instantaneous flow rule for the capital intensity:

$$\dot{k}_t = f(k_t) - \frac{c_t}{H} - \delta k_t \tag{2.9}$$

Furthermore, substitute (2.7) into (2.6) we have the following rule for household's

[5] Proof in **Appendix A.1.2**. Remember that the labor market condition requires that $H_t = L_t$ so that $\frac{K_t}{L_t} = \frac{K_t}{H_t} = k_t$.

consumption growth rate at labor market clearing condition:

$$\frac{\dot{c}_t}{c_t} = \alpha k_t^{\alpha-1} - \delta - \theta \tag{2.10}$$

- **Initial Steady State**

  Before the introduction of robots, we assume that the economy is at an initial steady state, meaning that there are no changes in all variables. Hence, by letting $\dot{k}_t = \dot{c}_t = 0$ we have:

$$\alpha k_{ss}^{\alpha-1} - \delta - \theta = 0$$
$$f(k_{ss}) - \delta k_{ss} = \frac{c_{ss}}{H}$$

  Hence the following system of equations characterizes the steady state:

$$\begin{aligned} k_{ss} &= \left(\frac{\alpha}{\delta+\theta}\right)^{\frac{1}{1-\alpha}} \\ c_{ss} &= \left[\left(\frac{\alpha}{\theta+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{\alpha}{\theta+\delta}\right)^{\frac{1}{1-\alpha}}\right] H \\ r_{ss}^K &= f'(k_{ss}) = \alpha k_{ss}^{\alpha-1} = \delta + \theta \\ r_{ss} &= r_{ss}^K - \delta = \theta \\ v_{ss} &= (1-\alpha)k_{ss}^{\alpha} = (1-\alpha)\left(\frac{\alpha}{\delta+\theta}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned} \tag{2.11}$$

  And the lifetime welfare for the household at any period t is that:

$$U_t = \int_t^{\infty} \log c_{\kappa} e^{-\theta(\kappa-t)} dt = \frac{1}{\theta} \log c_{ss}$$

  since $c_{\kappa} = c_{ss}$ for all periods κ.

2.2.2 Main Assumptions and Implications

Let assume that there is a sudden invention in technology that makes robots available in the market, which can be called "helicopter robots". There are two crucial assumptions when we introduce these robots into the economy:

Assumption 2.1: Capital is fully malleable.

This assumption says that capital is instantaneously convertible from one type to the other. Why can we make such an assumption?

This is an aggregated model, in which there are only two types of input for production. Anything that is not labor would be included in an aggregated category named capital K. This figure includes buildings, factories, machines, computers, tables, chairs... and many more. In reality, in the production process, these components are changing every period. However, we do not distinguish among such different types of capital or take into account the time of converting from one type to the other, just only concern about the total aggregated number.

Now, with a new invention of technology, robots appear as **a new type of capital** while all previously used capital can be considered as "the conventional machines". We continue to assume that **it takes no time to convert one unit of the conventional machine to one unit of robots and vice versa.** As long as the aggregate number is what we need to use, this assumption is valid.

Under the no arbitrate condition for the capital market, all units of capital should bear the same rate of return. Hence, this assumption implies that **the return of conventional machines should be the same as the return on robots**:

$$r_t^K = r_t^{R^A} \quad \text{for all } t \tag{2.12}$$

where r_t^K is the return rate on conventional machine and $r_t^{R^A}$ is the return rate on Additive Robots at period t.

Assumption 2.2: Robots are a perfect substitution for human labor.

As explained in the introduction, Additive Robots perfectly perform some tasks of human labor. The parameter productivity Λ_A can be interpreted as the marginal rate of substitution between robotic labor and human labor. In other words, the producer is indifferent between one unit of Additive Robots and Λ_A units of human labor.

Hence, the no-arbitrage condition implies that **return on one unit of Additive Robots to equate Λ_A units of human labor**:

$$\Lambda_A v_t = r_t^{R^A} \quad \text{for all } t \tag{2.13}$$

where v_t is an hourly wage of representative household and $r_t^{R^A}$ is the return rate on Additive Robots at period t.

2.2.3 Economy with Additive Robots

The decision to utilize Additive Robots lies with the firm side. Hence we will start this section with the changes on the technology side and condition that makes Additive Robots feasible for the producer to use before move on to the consequences for market rates and household's side and end with model predictions in long run.

1. **The New Level of Effective Capital Intensity**

   In each period t, with R_t^A units of Additive Robots, the effective labor is as in (2.1), hence, the firm profit function will be:

$$\begin{aligned}
\pi_t^A &= F(K_t, \Lambda_A R_t^A + H) - r_t^K K_t - r_t^{R^A} R_t^A - v_t H \\
&= F(K_t, \Lambda_A R_t^A + H) - r_t^K K_t - \Lambda_A v_t R_t^A - v_t H
\end{aligned}$$

$$\Leftrightarrow \quad \frac{\pi_t^A}{\Lambda_A R_t^A + H} = F\left(\frac{K_t}{\Lambda_A R_t^A + H}, 1\right) - r_t^K \frac{K_t}{\Lambda_A R_t^A + H} - v_t$$

   Define

$$k_t^A = \frac{K_t}{\Lambda_A R_t^A + H}$$

   as the new "**effective capital intensity**" at period t which is the number of conventional machines K_t per unit of new "**effective labor**" L_t. It follows that the output per effective labor is $f(k_t^A) = F(k_t^A, 1) = \left(k_t^A\right)^\alpha$, and first order conditions of maximization profit problem similar to (2.7) and (2.8) give us marginal product of conventional machines and marginal product of labor:

$$r_t^K = f'(kA_t) = \alpha \left(k_t^A\right)^{\alpha-1} \tag{2.14}$$

$$v_t = f(k_t^A) - r_t^K k_t^A = f(k_t^A) - f'(k_t^A)k_t^A = (1-\alpha)\left(k_t^A\right)^{\alpha} \tag{2.15}$$

Hence, according to assumption 2.1, effective capital intensity turns out to be a **constant number:** $k_t^A = \gamma_A \quad \forall t$ such that:

$$f'(\gamma_A) = \Lambda_A \left[f(\gamma_A) - f'(\gamma_A)\gamma_A\right]$$

Use the above marginal products we have:

$$\alpha(\gamma_A)^{\alpha-1} = \Lambda_A(1-\alpha)(\gamma_A)^{\alpha}$$

$$\Leftrightarrow \quad \gamma_A = \frac{1}{\Lambda_A}\left(\frac{\alpha}{1-\alpha}\right) \tag{2.16}$$

It means that the new constant level of capital intensity depends on the robot's productivity and initial capital share α.

2. **Profitability Condition**

   In every period t, investors will only find it profitable to convert part of their assets (in the form of conventional machines) to robots if their return is higher than the return of capital in the initial steady state, i.e. $r^{R^A} \geq r_{ss}^K$ which means that $f'(\gamma_A) \geq f'(k_{ss})$ or

$$\gamma_A \leq k_{ss}$$

since $f(.)$ is a concave function.

With a higher return rate, demand for robots goes up, the market will adjust r^K up (due to comparatively less supply) while r^{R^A} is down until the two are matched with each other at $f'(\gamma^A)$. The reached equilibrium level should still be higher than the steady-state level of return on conventional machines $r_{ss}^K = f'(k_{ss})$. If the equilibrium level is lower than the steady-state level then robots will vanish since no investor would find it beneficial to invest in robots anymore and the economy moves back to the initial steady state level. Hence, the profitability condition indicates that the capital intensity will jump down to a lower level than the initial steady state. It can be explained that part of initial capital K_{ss} is converted to robots and becomes robotic labor, the new and lower level of "**conventional machines**" $(K_{ss} - R_t^A)$ is spread thinner to a more abundant labor force $L_t > H$.

In this series of chapters, I ignore the process of reaching the equilibrium rate of return since this is the general equilibrium model. The adjustment of two markets - conventional machines and robots - are partial equilibriums. We assume that it would take a relatively short time for the markets to adjust and reach equilibriums and just take the final rates for the aggregated model [6] .

The profitability condition is equivalent to:

$$\begin{aligned} \frac{1}{\Lambda_A}\frac{\alpha}{1-\alpha} &\leq \left(\frac{\alpha}{\delta+\theta}\right)^{\frac{1}{1-\alpha}} \\ \Leftrightarrow \qquad \frac{1}{\Lambda_A} &\leq \left(\frac{\alpha}{\delta+\theta}\right)^{\frac{1}{1-\alpha}}\frac{1-\alpha}{\alpha} \\ \Leftrightarrow \qquad \Lambda_A &\geq \frac{\alpha}{1-\alpha}\left(\frac{\delta+\theta}{\alpha}\right)^{\frac{1}{1-\alpha}} \end{aligned} \tag{2.17}$$

Let

$$\Lambda_A^* = \frac{\alpha}{1-\alpha}\left(\frac{\delta+\theta}{\alpha}\right)^{\frac{1}{1-\alpha}}$$

then Λ_A^* is the **minimum productivity level** of the robot such that it is profitable to convert conventional machine to robot.

3. **Constant Returns rate on conventional machines, Robots and Labor**

Since effective capital intensity k_t^A stays constant at Λ_A all the time as per (2.16),

[6] **(Berg, Buffie, and Zanna 2018)** did a supplement to my work in this dimension. They assumed that there would be convertible costs to exchange one unit of a conventional machine to one unit of robots and vice versa. Hence, their set of results can be considered very short term result - only the first period of my model, when the market still adjusting for the no-arbitrage condition.

from (2.12), (2.14) and (2.15), all return rates are also constant:

$$r^K = r^{R^A} = f'(\gamma_A) = \alpha\,(\gamma_A)^{\alpha-1} = \alpha\left(\frac{1}{\Lambda_A}\frac{\alpha}{1-\alpha}\right)^{\alpha-1}$$
$$v = \frac{1}{\Lambda_A}r^{R^A} = \frac{1}{\Lambda_A}\alpha\,(\gamma_A)^{\alpha-1} = \frac{1}{\Lambda_A}\alpha\left(\frac{1}{\Lambda_A}\frac{\alpha}{1-\alpha}\right)^{\alpha-1}$$

The market interest rate is hence also at the constant level:

$$r = r^K - \delta = f'(\gamma_A) - \delta$$

To see the changes of these rates from the initial steady stead level, we need to consider the profitability condition, under which $f'(\gamma_A) \geq f'(k_{ss})$. That leads to the return rate on conventional machines will jump up to a higher equilibrium level $r^K \geq r^K_{ss}$, and interest rate also increases to more than time preference rate:

$$r \geq r_{ss} \quad \Leftrightarrow \quad r = f'(\gamma_A) - \delta > \theta \tag{2.18}$$

How about the real wage? From (2.8), we can prove that real wage v is decreasing in variable k (since $\alpha < 1$ then $v'_k < 0$). The profitability condition implies $\gamma_A \leq k_{ss}$ therefor $v \leq v_{ss}$.

In summary, when Additive Robots is introduced into the economy, the returns on conventional machines, robots as well as real interest rate jumps to permanently higher levels. Nevertheless, the real wage is depressed to a lower level and stays there forever.

4. **Consumption: Initial drop but has positive growth rates consequently**

   At the first period, total wealth still stays at the initial steady state level $K_{ss} = K_0 + R^A_0$. On the other hand, human wealth is the total present discounted value of all future labor income streams. Household know that they can expect a higher interest rate coupling with a lower wage, hence human wealth w^h will be smaller compared to the level of the initial steady-state:

$$\begin{aligned} w_t^h &= \int_t^{\infty} vH \exp\left(\int_t^{\kappa} -r d\nu\right) d\kappa = vH \int_t^{\infty} \exp\left(-r \int_t^{\kappa} d\nu\right) d\kappa \\ &= vH \int_t^{\infty} \exp\left(-r(\kappa - t)\right) d\kappa = vH \frac{-1}{r} \left[exp(-\infty) - exp(0)\right] \\ &= \frac{vH}{r} < \frac{v_{ss}H}{r_{ss}} \end{aligned} \tag{2.19}$$

We know that, the solution of the household's maximization problem follow consumption rule (2.5). During the initial steady state the consumption is:

$$c_{ss} = \theta \left(K_{ss} + \frac{v_{ss}H}{r_{ss}}\right)$$

while in the first period immediately after the introduction of robots, consumption will drop to

$$c_0 = \theta \left(w_0^h + w_0^n\right) = \theta \left(K_{ss} + \frac{vH}{r}\right) < c_{ss}$$

However, the Euler equation (2.10) implies that consumption will later continue to grow at a same positives rate g_c permanently, which is the different between the new interest rate and time preference rate. And the economy moves out of the initial steady state:

$$\frac{\dot{c}_t}{c_t} = \alpha \left(\gamma_A\right)^{\alpha-1} - \delta - \theta \quad \Leftrightarrow \quad g_c = r^K - \delta - \theta = r - \theta > 0 \tag{2.20}$$

as a direct result of (2.18).

5. **Non-human wealth growth, stock of conventional machine vs. stock of robots**

Substitute the consumption function (2.5) and human wealth (2.4) into the budget constraint (2.3) we have:

$$\begin{aligned} \dot{w}_t^n &= r w_t^n + vH - \theta \left(w_t^n + \frac{vH}{r}\right) \\ &= (r - \theta) \left(w_t^n + \frac{vH}{r}\right) \end{aligned}$$

From (2.18), we have $\dot{w}_t^n > 0$. In other words, under the profitability condition, total wealth will keep growing positively over time.

On the other hand, non-human wealth is the sum of both conventional machines and robots: $w_t^n = K_t + R_t^A$, while the stock of conventional machine and robots are adjusted to achieve the constant effective capital intensity of γ_A. Therefore, we have the following expressions of conventional machines and Additive Robots:

$$\gamma_A = \frac{K_t}{\Lambda_A R_t^A + H} = \frac{w_t^n - R_t^A}{\Lambda_A R_t^A + H} = \frac{1}{\Lambda_A}\frac{\alpha}{1-\alpha}$$

$$\Leftrightarrow \quad R_t^A = (1-\alpha)w_t^n - \frac{\alpha H}{\Lambda_A} \tag{2.21}$$

$$\Leftrightarrow \quad K_t = w_t^n - R_t^A = \alpha w_t^n + \frac{\alpha H}{\Lambda_A} \tag{2.22}$$

which indicates that the stock of robots and conventional machine will grow along with non-human wealth:

$$\dot{R}_t^A = (1-\alpha)\dot{w}_t^n \qquad \text{and} \qquad \dot{K}_t = \alpha \dot{w}_t^n$$

It depends on the parameter values which one is higher in absolute terms, conventional machines or robots. But we can compare the growth rates by examining:

$$\frac{\dot{R}^A}{R^A} = \frac{(1-\alpha)\dot{w}^n}{(1-\alpha)w^n - \frac{\alpha H}{\Lambda_A}}$$
$$= \frac{(1-\alpha)\frac{\dot{w}^n}{w^n}}{(1-\alpha) - \frac{\alpha H}{\Lambda_A w^n}} = \frac{\dot{w}^n}{w^n}\left(\frac{(1-\alpha)}{(1-\alpha) - \frac{\alpha H}{\Lambda_A w^n}}\right) \tag{2.23}$$

As $\frac{\alpha H}{\Lambda_A w^n} > 0$ the element inside the big bracket is more than 1. Hence the stock of Additive Robots will grow even faster than the total nun-human wealth.

On the contrary, with the same method, we have the stock of conventional machines grows at a slower rate than total non-human wealth.

$$\frac{\dot{K}}{K} = \frac{\dot{w}^n}{w^n}\left(\frac{1-\alpha}{1-\alpha+\frac{\alpha H}{\Lambda_A w^n}}\right) < \frac{\dot{w}^n}{w^n} \tag{2.24}$$

6. **Convergence to a balanced growth path**

From (2.23) and (2.24), as wealth keeps accumulated over time ($w_t^n \xrightarrow{t\to\infty} \infty$), the element in the big bracket will converge to 1, indicating that conventional machines, Additive Robots and total non-human wealth should grow at a same rate in the long run, which indicates a new balanced growth path.

Furthermore, we can examine the behavioral of consumption by rewriting the accumulation rule of total wealth as following[7],[8]:

$$\begin{aligned}\dot{w}_t^n &= F\left(K_t, \Lambda_A R_t^A + H\right) - c_t - \delta w_t^n = \left(\Lambda_A R_t^A + H\right) f(\gamma_A) - c_t - \delta w_t^n \\ &= \Lambda_A(1-\alpha)\left(w_t^n + \frac{1}{\Lambda_A}H\right) f(\gamma_A) - c_t - \delta w_t^n\end{aligned}$$

Then the growth rate of non-human wealth would be:

$$\frac{\dot{w}_t^n}{w_t^n} = \Lambda_A(1-\alpha)\left(1 + \frac{1}{\Lambda_A}\frac{H}{w_t^n}\right) f(\gamma_A) - \frac{c_t}{w_t^n} - \delta \tag{2.25}$$

In the long run, when w_t^n keeps increasing and H is fixed, thus $\frac{H}{w_t^n} \xrightarrow{t\to\infty} 0$ and again use formula of effective capital intensity (2.16) we have:

$$\begin{aligned}\frac{\dot{w}_t^n}{w_t^n} &= \Lambda_A(1-\alpha) f(\gamma_A) - \frac{c_t}{w_t^n} - \delta \\ &= (1-\alpha)\frac{1}{\gamma_A}\frac{\alpha}{1-\alpha} f(\gamma_A) - \frac{c_t}{w_t^n} - \delta = \alpha(\gamma_A)^{\alpha-1} - \delta - \frac{c_t}{w_t^n} \\ &= \frac{\dot{c}_t}{c_t} + \theta - \frac{c_t}{w_t^n} \end{aligned} \tag{2.26}$$

Ultimately, there will be a the balance growth path where all variables move at the same rate g_c, making the ratio between consumption and non-human wealth stays constant.

$$\frac{\dot{w}_t^n}{w_t^n} = \frac{\dot{c}_t}{c_t} \quad \Leftrightarrow \quad \frac{c_t}{w_t^n} = \theta$$

7. **Lifetime Welfare Increases**

[7] From the expression of Additive Robots, we can achieve the effective labor formula:

$$\Lambda_A R_t^A = \Lambda_A(1-\alpha)(w_t^n - \gamma_A H) \quad \Leftrightarrow \quad \Lambda_A R_t^A + H = \Lambda_A(1-\alpha)\left(w_t^n + \frac{1}{\Lambda_A}H\right)$$

[8] From (2.3) (2.14) (2.15), it is easy to prove that:

$$r_t w_t^n + v_t H = (r_t^K - \delta) w_t^n + v_t H = r_t^K (K_t + R_t^A) + v_t H - \delta w_t^n = F(K_t, \Lambda_A R_t^A + H) - \delta w_t^n$$

since $F(K_t, \Lambda_A R_t^A + H) = r_t^K K_t + r_t^K R_t^A + v_t H$ in perfect competition condition.

At any period t after the introduction of Additive Robots, the lifetime welfare is[9]:

$$U_t = \frac{1}{\theta}\log(c_t) + \frac{g_c}{\theta^2} = \frac{1}{\theta}\log(c_t) + \frac{r-\theta}{\theta^2} \tag{2.27}$$

Compared to the level at the initial steady state $U_{ss} = \frac{1}{\theta}\log c_{ss}$, the first-period consumption is less thus the first element is less than U_{ss}. However, under the profitability condition, the second element is always positive. Hence whether the lifetime welfare in the first period is higher or lower than the steady stead level depends on our choices of parameters. In any case, we are sure that Lifetime welfare would increase over time as later consumption keeps growing.

2.3 Model Simulation Results and Discussions

2.3.1 Calibration Parameters

In the baseline model, there are only three parameters, and one constant need to be determined. The values of these parameters and constants are given in Table 2.1.

Parameters	Explanations	Values
α	Capital share	0.33
δ	Depreciation Rate (annually)	0.085
θ	Subjective Time preference	0.04
H	Fixed working hours (annually)	0.23*365

Table 2.1: Parameters' Values in Baseline Model

For the household maximization problem in (2.2), there is only one parameter which is the subjective time preference rate θ is chosen to match with the real interest rate of 4%.

For both parameters of the production side, capital share α and depreciation rate δ,

[9] For detail calculation, please refer to **Appendix A.1.1**

I use their standard values in literature.

The fixed number of working hours H is a portion of 365 days per year that workers have to work. The ratio is determined using the assumption that workers work 8 hours per day, 5 days per week and there are 52 weeks per year.

With this set of parameters, we have the minimum productivity for robots Λ_A^* is only 0.1157. That means robots only need to work effectively as 11.57% of human labor then it is already profitable for the producer to adopt them. As a starting point, as explained in the Introduction part [10], I choose robots productivity of 20%, so that $\Lambda_A = 0.2$, a very conservative value which can generate reasonable results. In the last exercise, I will examine the comparative statics in all the results in case we have a higher productivity level.

2.3.2 Results of the Baseline Model

- **Changes in Constant Variables of the Economy**

  Table (2.2) summarizes the changes in all constant variables of the economy after the introduction of robots. All of these changes are permanent, in other words, all these variables keep their values over time.

  As predicted, the capital intensity drops from the initial steady state level $k_{ss} = 4.26$ to $\gamma_A = 2.46$, a decrease of approximately 42%. That makes the return rates on conventional machines and hence the real interest rate all increase. The interest rate almost doubles from 4% to 9.5%. Unfortunately, there is a permanent drop in the real wage of 16.5%.

  However, supported by the perpetual increase in consumption of 5.54% every period, even the first-period lifetime welfare increases by 11.5% (and thereafter

[10] Derive from (Frey and Osborne 2017) the empirical robot productivity is around 0.18

will grow alongside with consumption).

Variables	**Initial Steady State Value**	**Values after robots**	**Percentage Changes**
Capital Intensity	4.258	2.462	- 42.2%
Real Interest Rate	0.040	0.095	138.5%
Return on conventional machines	0.125	0.18	44%
Real Wage	1.081	0.902	- 16.5%
First period Lifetime Welfare	122.988	137.019	11.5%
Consumption growth rate	0	5.54%	-

Table 2.2: Constant Variables under Baseline Model

- **Consumption and Total Wealth**

  As explained in (2.19), since wage is pressed down to a lower level, at the same time interest rate increases to a higher level (see table (2.2)), the expected human wealth (the present discounted value of all future wage income) becomes smaller compared with the initial wage. Hence, consumption in the first period drops to 60.06 from the initial steady state level of 136.99, a decrease of 55.7%.

  Nevertheless, under profitability conditions, we will have a positive endogenous perpetual growth for consumption as per (2.20). Consumption will grow at the rate of 5.54% for every period and bounce back to the initial steady state level after around 16 years. Since the initial drop is slightly more than half, 16 years is also the required time for consumption to double.

  Total wealth as the sum of both conventional machines and Additive Robots also grows over time. The starting value for total wealth at the first period is still K_{ss} which is used to allocate into either conventional machines or Additive Robots. We will see in detail how the number of robots and conventional machine change in the next part.

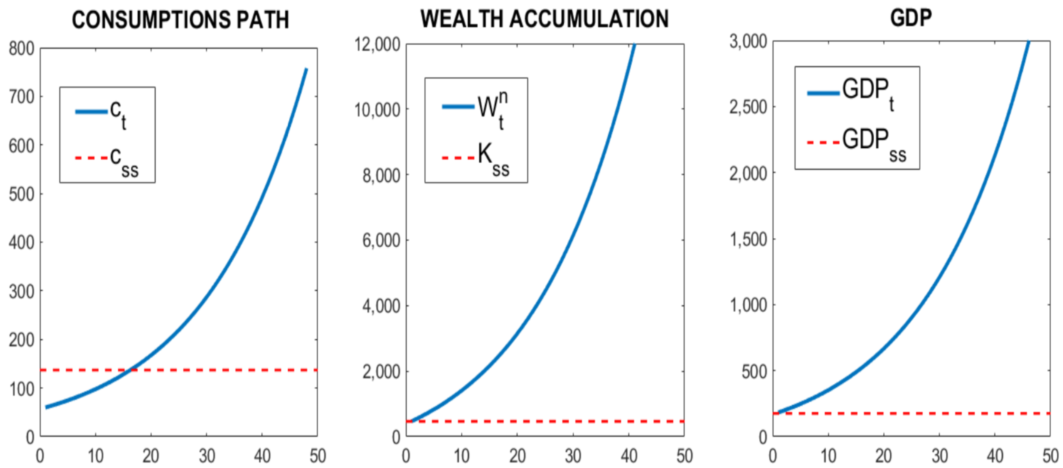


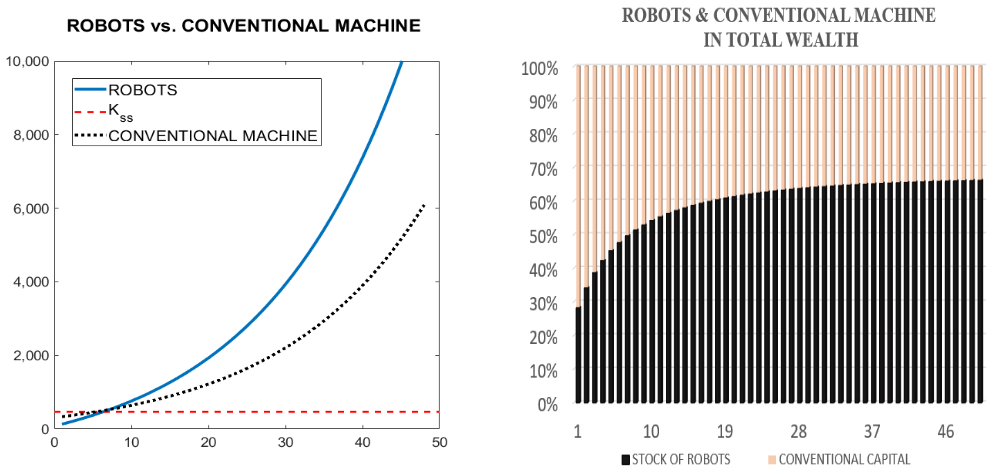
Figure 2.1: Consumption Path, Wealth Accumulation and GDP Growth with Additive Robots

- **Stock of Robots vs. Stock of Conventional Machine**

  Although effective capital intensity does not change over time, and we have an assumption that labor H is fixed, it does not indicate that conventional machines and robots (added to the effective labor) will not change. Their values are adjusted every period just to keep the ratio $k_t^A = \frac{K_t}{\Lambda_A R^A + H}$ constant at γ_A. In the first period, K_{ss} itself was divided into K_0 and R_0^A.

  From (2.23) and (2.24), we have that as long as wealth keeps accumulating positively, both robots and conventional machines also grow, however at different rates, which is shown in Figure 2.2a. Additive Robot stock grows faster than total wealth, which in turn increases faster than conventional machines. As a result, the number of robots is only lower than conventional machines for the first 7 years. After that, robots dominate in total wealth.

  This result can be further pictured in figure 2.2b. The fraction of wealth that is used to finance Additive Robots increases over time. In the long run, this ratio converges to a constant of $(1-\alpha)$.



(a) (b)

Figure 2.2: Stock of Additive Robots vs. Stock of Conventional Machines

- **Balanced Growth Path**

  The economy does not reach a new steady-state (where all the variables stay constant) although effective capital intensity stays constant already. Instead, it will fall into a perpetual endogenous growth as consumption increase positively all the time. However, in the long run, the economy does go to a balanced growth path where all the variables (given that they do change) will change at the same rate, making their ratios constant. This result presents in Figure (2.3).

  Over time, the growth rate of non-human wealth slows down and converges to the growth rate of consumption, making the consumption to wealth ratio also converge to the constant level at time preference rate θ as proofed in (2.26).

  Figure 2.3 together with Figure 2.2b above show us the convergence of other ratios such as $\frac{R^A}{K}$ and $\frac{C}{Y}$. At the productivity level of only $\Lambda_A = 0.2$, the convergences take place quite fast, after around 50 years. In the later extension, we will be able to see if the changes in productivity can have any effect on the model convergence.

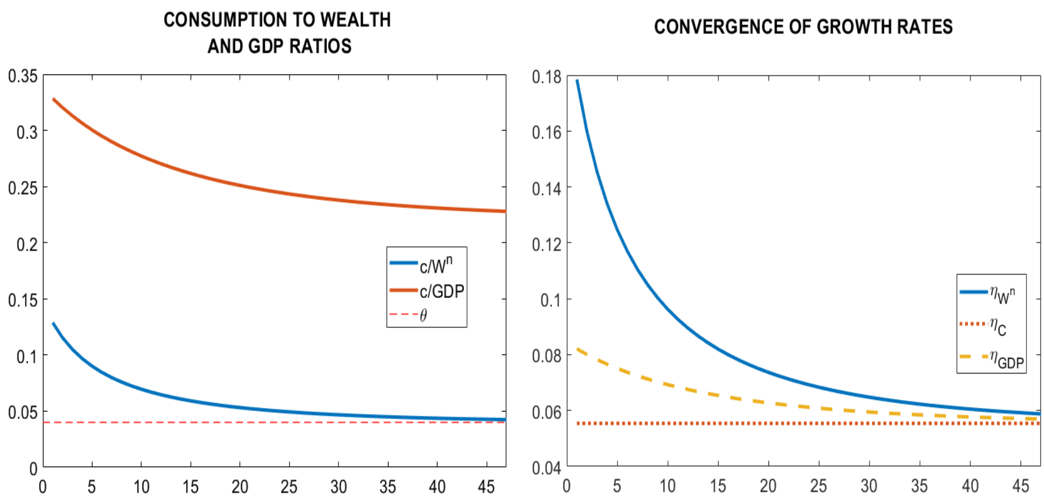


Figure 2.3: Balanced Growth Path Convergences

- **Human shares and total effective labor share**

  Since H is fixed and real wage v does not change over time, at every period total human (labor) incomes vH do not change. On the other hand, total GDP keeps increasing which has been shown in Figure 2.1. It follows that human share in national incomes $\frac{vH}{F(K,L)}$ converges to zero over time.

  However, if we include the incomes generated by owning robots to form one category called human and robotic incomes $vH + r_t^K R_t^A$, then the ratio of this income in national incomes always stays constant[11] at $(1-\alpha)$ which is the same with labor shares ratio of the initial steady state.

[11] This indicates that by any means if we can transfer some earning from robots to workers (who own labor only) in can help to soften the reduction in human labor incomes.

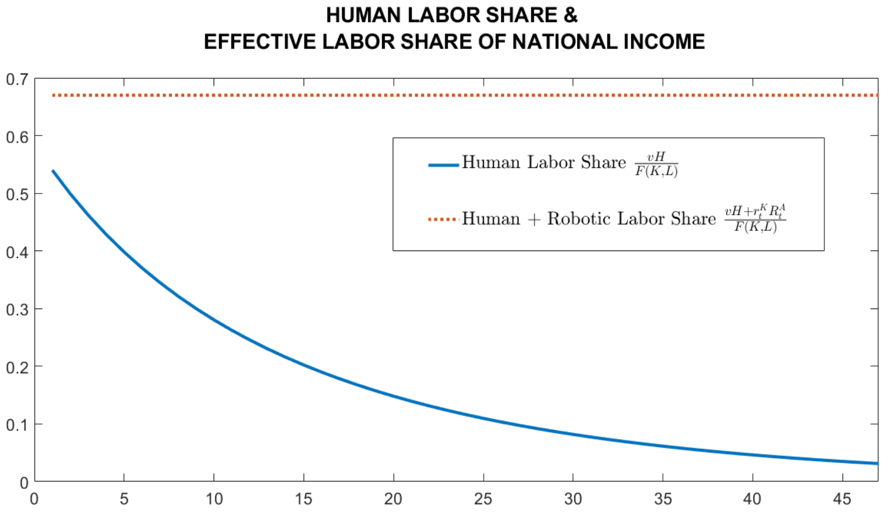


Figure 2.4: Effective Human and Robotic Labor Shares

2.4 Some Extensions from the Baseline Model

2.4.1 Flexible Labor Supply

- **Extension Description**

  In the baseline model, we assume that H stays constant throughout time. Although this might be a valid assumption in reality, for example, the number of working hours per day and the number of working days per year are fixed. However, this restriction does not allow us to see the effects (if any) on the labor market when robots are introduced.

  Hence, for the first extension, I include leisure into the utility function so that the number of working hours is now one of the results of the household's maximization problem. For simplification, we use a simple separable c and H utility function:

$$U(c,H) = logc_t + Blog(\tilde{H} - H_t)$$

  with $\tilde{H} - H_t$ being leisure time left.

  For a separable utility function, $U_{cH} = 0$ thus we have the Euler equation the same with a fixed H model as in (2.6)[12].

$$r = \theta - \frac{U_{cc}}{U_c}\dot{c} = \theta - \frac{\dot{c}}{c} \Leftrightarrow \frac{\dot{c}}{c} = r - \theta$$

The intratemporal condition of the optimization problem gives us the labor supply rules:

$$v_t = -\frac{U_H(c_t, H_t)}{U_c(c_t, H_t)} = \frac{Bc_t}{\tilde{H} - H_t} \tag{2.28}$$

This extension does not impose any change in the production side, all analysis on that side still holds except for the fact that now H in all the equations[13] does change over time satisfying (2.28). Hence, the economy still reaches a constant effective capital intensity level:

$$\gamma_A = \frac{1}{\Lambda_A}\frac{\alpha}{1-\alpha} = \frac{K_t - R_t^A}{\Lambda_A R_t^A + H_t} < k_{ss}$$

The difference is that for this flexible labor model, not only K, R^A change every period, now H also changes to achieve a constant ratio of γ_A every period.

It follows that all market conditions still stay the same throughout. Return rates on conventional machine and robots, as well as real wage, are all constant, making interest rate also constant:

$$r^K = r^{R^A} = f'(\gamma_A) \quad ; \quad r = r^K - \delta$$

$$v_t = f(\gamma_A) - f'(\gamma_A)\gamma_A = \Lambda_A f'(\gamma_A)$$

It indicates that from the Euler equation, consumption will still grow at a constant positive rate since the interest rate is higher than the previous steady-state level (which equals to θ).

At the same time from the intra-temporal condition (2.28), if v_t is fixed, while c_t increases over time, then H has to gradually decrease over time either to 0 or

[12] Refer to the Appendix A.2.1 for detailed proof

[13] Refer to the Appendix A.2.2 for firm's profit maximization problem

to the lowest positive level. The latter is more proper since although it might be reasonable to not work anymore as the earning from the Additive Robots is enough to finance household's consumption, in fact, we still need some human labor[14].

- **Numerical Solution for Initial Condition**

  With the decrease in H, we are not able to have a closed-form solution for non-human wealth as in (2.19), hence we will not be able to have a value of initial consumption right from available model conditions. Instead, we need to use a simple shooting algorithm to guess for this value.

  1. Guess c_0 and generate the time series $\{c_t\}_{t=1}^{T}$ using the calculated growth rate of consumption from (2.20).

  2. Calculate H_t using the intratemporal equation

     - If $\left(\bar{H}_{t+1} - \frac{B}{v}c_t\right) > 0$ then $H_{t+1} = \bar{H} - \frac{B}{v}c_t$

     - If $\left(\bar{H}_{t+1} - \frac{B}{v}c\right) < 0$ then $H_{t+1} = H_t$ (keeps the constant minimal positive labor)

  3. Calculate human wealth w_t^h using the simulated labor path with fixed wage v and use fixed interest rate r to time discount the whole series to obtain w_0^h.

  4. Recalculate the value of consumption in the first period $c_0' = \theta(K_{ss} + w_0^h)$ since in the first period $w_0^n = K_{ss}$

  5. Compare c_0' with initial guess. If not match then update[15] the guess until we have a match[16].

[14] Even in the case Additive Robots are completely able to replace human labor in performing tasks, we will still need workers to manage, maintain... robots.

[16] The difference is smaller than a very small threshold

- **The importance of Leisure**

  Under this extension, we have one more parameter B that needs to calibrate. The value of B shows the importance of leisure in the utility function. The higher the value of B, the stronger the household values leisure. In other words, based on (2.28), given the same wage and consumption preference, the higher the value of B, the lower the value of H, meaning household will accept to work less.

  To calibrate B, we would want to induce the same H_{ss} as in the baseline model. So that the initial steady state conditions in both cases are exactly compatible. Then we can see what will happen to the model if we choose a higher value of B. Hence, given the same value of baseline model H_{ss}, we will have the value of B as follow[17]:

$$B = \frac{(1-\alpha)\left(\frac{\tilde{H}}{H_{ss}} - 1\right)}{1 - \delta k_{ss}^{1-\alpha}}$$

- **Decreases in Labor Supply only after an initial jump**

  As we know and already predicted, employment will decrease over time to a very low level. However, looking at the simulated series as per Figure 2.5, there is another important observation. Initially, the household will work more (almost double of their initial steady-state level) then slowly reduce over time.

  In general, we might think that since the future stream of H is expected to be lower, the total present value of human wealth would be even lower in the elastic H case compared to the baseline model. This could make the initial drop in consumption be more than in the inelastic labor case. As can be seen from Table (2.3), this turns out to be not true. Now household can change their labor supply, and they know that the wage is going to be reduced to a lower level. To not have

[16] Updated rule $c_0'' = (1-\lambda)c_0 + \lambda c_0'$ with λ is very small to avoid outline values.

[17] Refer to Appendix A.2.3

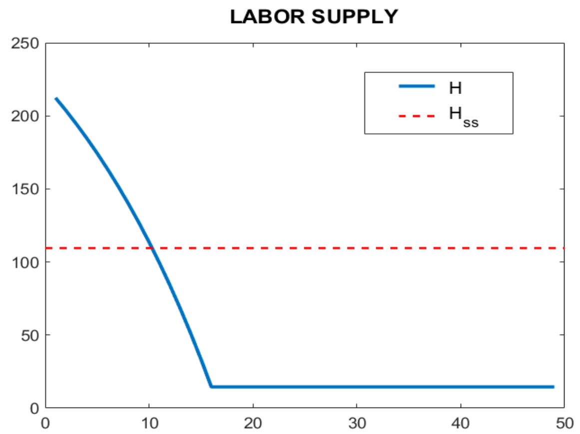


Figure 2.5: Changes in Labor Supply (number of working hours per year)

Note: Under $B = 2$ and $\Lambda_A = 0.2$ and without control for non-negative R^A

to compensate for very low consumption levels, they work more. Compared to the reduction of initial consumption in the inelastic labor case we even have a lighter reduction.

Initial Steady State	**Inflexible H - first period**	**Flexible H - first period**
136.99	60.06	65.07
Changes	-55.7%	-52.5%

Table 2.3: Initial Consumption after Robots: Flexible vs. Inflexible Labor Supply.

- **Higher Productivity or More Accumulated Wealth**

  Furthermore, a closer look into the base scenario with $B = 2$ and productivity $\Lambda_A = 0.2$ above reveals that we need to make further constraint on the model parameters to generate meaningful results. Since given that base scenario, in the first few periods, there are no Additive Robots are used.

  From (2.21), the expression for Additive Robots under flexible labor supply is such that:

$$R_t^A = (1-\alpha)(w_t^n - \gamma_A H_t)$$

  Hence, the number of working hours needs to satisfy the following condition so that the number of Additive Robots to be positive:

$$H_t < \frac{w_t^n}{\gamma_A} \tag{2.29}$$

For the scenario with a low value of B (household does not value leisure much), in the first few periods H_t will be too high to satisfy the above condition. After wealth is accumulated high enough, there are two consecutive mechanisms. The right-hand side becomes bigger, while the left-hand side becomes smaller and the condition is achieved. This can be explained that as the new wage is too low, at the same time household does not value leisure highly so they would need to work more to maximize their utility first before the economy can start to use extra wealth to finance Additive Robots.

For a higher value of B, in contrast, household values leisure more and does not want to work as much to maximize their utility since the higher leisure already generate higher utility for them. It would then take the household less time to accumulate enough wealth to start using robots.

There is another way of solving the issue of (2.29). Instead of waiting for wealth to accumulate over time, we need the capital intensity γ_A to have a lower value. If there is no change in the production function (i.e. α unchanged) then the only way is to have higher productivity of robots, meaning higher Λ_A.

The mechanism is that, with a higher productivity level, capital intensity is lower, making return rates of robots and conventional machines are higher, and wage goes down less. With a less reduced wage, the household does not need to increase their labor that much to maximize their utility, compared the case with lower productivity. That makes Additive Robots becomes possible more easily. Put it differently, robots with higher productivity earn higher returns, sufficiently compensate for the drop in human labor incomes, helping households accumulate more wealth. It follows that they can increase their consumption faster (now the difference between the interest rate and time preference rate is higher) and generate more utility [18].

- **Similar Long run Convergence**

  However, as long as H_t hits 0 or the minimum positive level, the model comes back to the inelastic case. Hence, it sure has the same convergence paths compared to scenario under fixed H, implying that $\frac{\dot{w}_n}{w_n} = \frac{\dot{c}}{c}$ with $\frac{c}{w_n} = \theta$. Given the condition that consumption equation (2.5) is satisfied.

  In conclusion, this extension shows us that when labor is flexible, the introduction of Additive Robots does create a decrease in employment over time. However, to finance their preferred consumption streams, the household would need to work more initially. The lesser they value leisure in their utility function, the more they will work initially or economy would require higher productivity of the robots to start using them and move out of the initial steady state.

2.4.2 Higher Productivity of Additive Robots

In the previous extension, we briefly discuss the need for higher productivity in case household does not value leisure much in their utility function.

However, all the results are assumed under helicopter robots, meaning robots are suddenly available in the market at a return rate that is profitable for firms to adopt them. The appearance should be more gradually. With technology improvement, robot's productivity increases slowly, up to the point that it becomes possible to adopt in production. The model then ignores all individual market partial equilibrium, and general equilibrium is immediately achieved in the first period.

In this variation, I assume that from the previous level, technology is further im-

[18] These rationales can be seen clearer under the next extension

proved, increasing productivity to a new level. However, this will not be applied abruptly into the market. Instead, we have a gradual increase in robots' productivity over a few periods until it reaches the new level, meaning:

$$\Lambda_{A(t)} = \Lambda_{A(2)} + \left(\Lambda_{A(1)} - \Lambda_{A(2)}\right)e^{nt} \tag{2.30}$$

with $n < 0$ is the improved speed.

Then the effective capital intensity is no longer constant but converges to the new steady level, as per:

$$\gamma_{A(t)} = \frac{\alpha}{1-\alpha}\frac{1}{\Lambda_{A(t)}} \tag{2.31}$$

As a result, all other variables such as return on conventional machines (and hence on robots), real interest rate and real wage also convergences together to their new respective balanced levels.

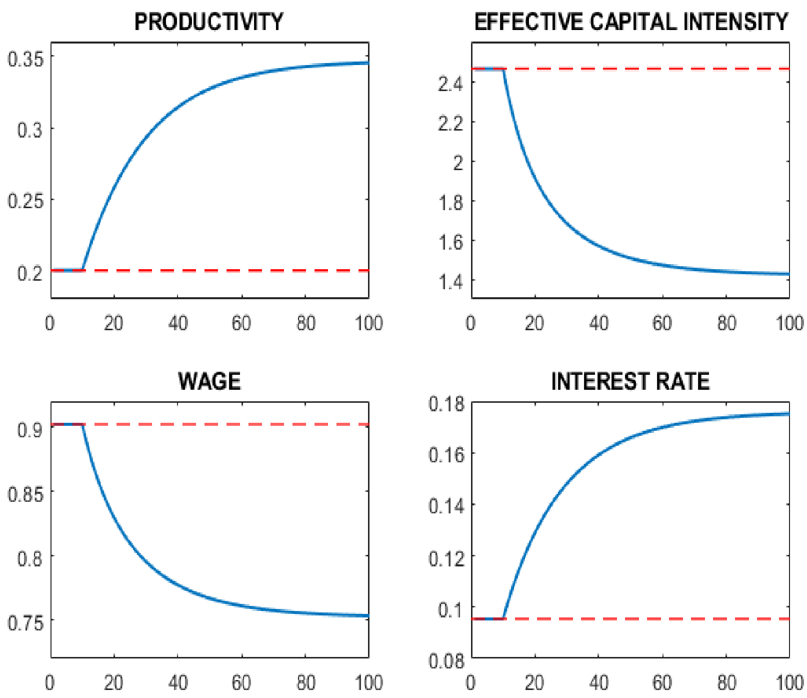


Figure 2.6: Convergences to New Balanced Levels under Productivity Improvement (Flexible Labor)

From Figure (2.6), as robots' productivity increases, all the results of the baseline model becomes stronger. The declined variables will fall even lower. The increased variables will continue to climb up higher. In summary we will have stronger quantitative effects under higher productivity as we can see from the below figure:

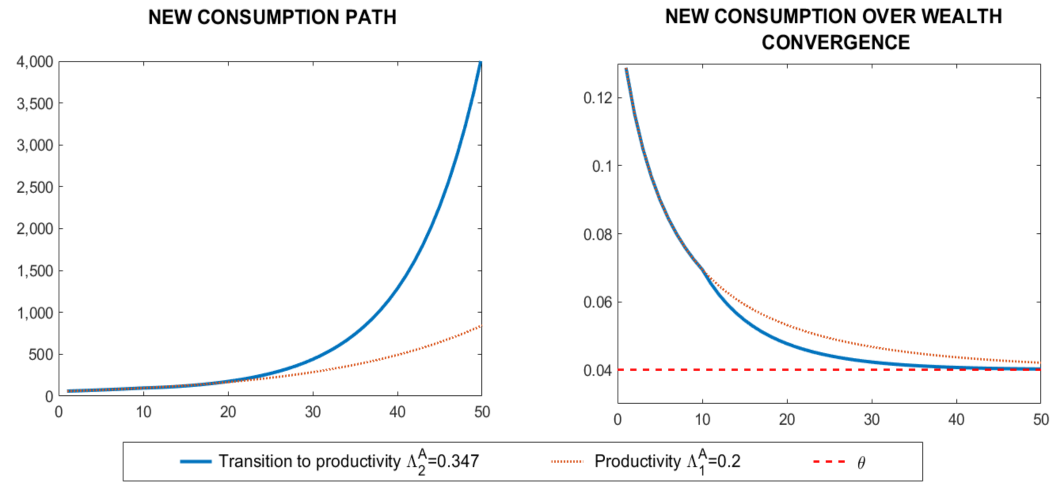


Figure 2.7: Higher Consumption Growth and Faster Convergence to the Balanced Growth Path (Flexible Labor)

We can expect a faster convergence to the balanced growth path as well as faster growth in variables that grow perpetually.

2.4.3 Extension with CES production function

In the last extension, we come back to the inflexible labor market but use a more general form of production function:

$$Y_t = \left[\alpha K_t^{-\rho} + (1-\alpha) H^{-\rho}\right]^{-1/\rho}$$

with $0 \leq \alpha \leq 1$ and $\rho \geq -1$

Then output per unit labor will be: $y_t = \left[\alpha k_t^{-\rho} + (1-\alpha)\right]^{-1/\rho} = f(k_t)$.

The analysis is similar to the baseline model. After the introduction of robots, the new effective capital intensity and interest rate now depend on values of the elasticity of substitution between capital and labor ρ.

$$\gamma_A = \left(\frac{1}{\Lambda_A}\frac{\alpha}{1-\alpha}\right)^{\frac{1}{1+\rho}}$$
$$r = \alpha\left(\frac{y}{\gamma_A}\right)^{1+\rho} = \alpha\left(\frac{1}{\Lambda_A}\frac{\alpha}{1-\alpha}\right)y^{1+\rho} - \delta \tag{2.32}$$

In Cobb Douglas production function, which satisfies the Inada conditions, when we have a reduction in capital intensity we would have a higher interest rate (and other return rates) and hence a faster consumption growth. On the other hand, with CES production function, a lower value of ρ would sure make interest rate increase in (2.32) but the change in capital intensity is undetermined.

In the general case, the changes in capital intensity depend on whether the value inside the bracket is more than or less than one. But interest rate does move the same direction with the elasticity of substitution as illustrated in Table 2.4, which means in any case, a lower value of ρ will create stronger quantitative effects for our model.

	ρ	γ_A	y	r^K
$\Lambda_A < \frac{\alpha}{1-\alpha}$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$
$\Lambda_A > \frac{\alpha}{1-\alpha}$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$

Table 2.4: Different Scenarios of Elasticity of Substitution

This extension can be applied for industry-specific cases, in which different industries can have different values of Elasticity of substitution. A lower elasticity of substitution means that it is easier for that industry to substitute capital by labor.

Since the type of robots in this chapter is additive to human labor with similar functionality, it follows that it will be easier to replace conventional machines with Additive Robots, creating stronger quantitative effects.

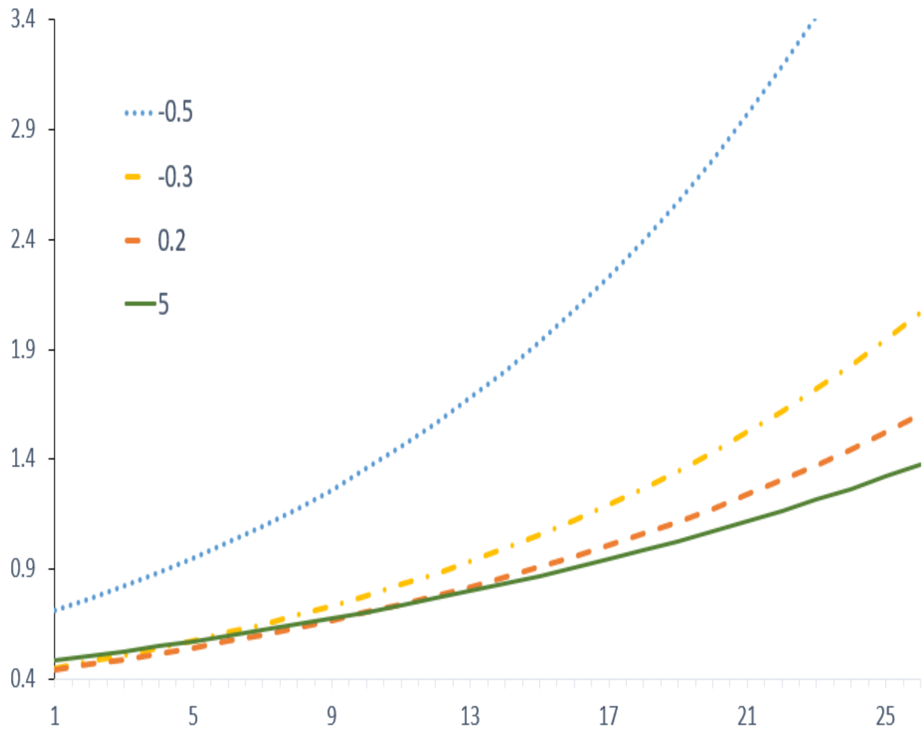


Figure 2.8: Consumption Growth ($\frac{c_t}{c_{ss}}$) with Different Values of Elasticity of Substitution.

2.5 Conclusion

In this chapter, robots are a direct substitution of labor however only with a fraction of a humans capability. Thus, Additive Robots become a "**direct extension**" of the labor force. Under a standard literature set of parameters and a Cobb Douglas production function, the introduction of robots into the initial steady state of the economy does make capital per labor becomes thinner. Therefore, the effective capital intensity drops to a permanently lower value, causing the real interest rate to increase 1.5 times, and the workers wage decreases to a 16.5% lower level. On the household's side, after the initial drop of a half, consumption increases at a constant rate of 5.54% perpetually, driving the lifetime welfare to increase by 11.5%. Total wealth including robots grows steadily over time, with the stack of robots grows at a faster rate compared to the conventional machine. In the long run, the simulation of the economy does confirm the existence of a balanced growth path, where the growth of wealth converges to the growth of consumption.

It is significant to notice that the economy will start to adopt robots with as low productivity as only 11.5% of a human. The higher the productivity of the robots, the higher the return on robots required for the capital owner to find it profitable to convert part of their capital from conventional machines to robots. Hence, all the quantitative results will be stronger by the increase in the robots productivity.

Under the elastic labor supply extension, the household works longer initially to compensate for the drop in wage and hence slightly mitigates the initial drop in consumption. Nevertheless, as wealth starts to accumulate to sustain the consumptions growth, the model does predict a decrease in employment over time.

Lastly, the exercise with different elasticities of substitution between capital and labor demonstrates that for industries in which it is easier to substitute capital by labor, all the effects will be quantitatively stronger.

Chapter 3

Multiplicative Robots

3.1 Introduction

In this chapter, I introduce the second type of robot, called Multiplication Robots, into the baseline model. What are the differences between Additive Robots and Multiplicative Robots?

Additive Robot is a perfect substitution for human's tasks. It can do exactly what human labor would do even though that might be only a fraction of the activities of their jobs. Multiplicative Robots, on the other hand, is not used to substitute human labor. Instead, this type of robot is used to support human labor in doing their jobs in, for example, a faster or more accurate way. The differences are not abilities or technology levels but rather functionalities depend on the nature of the jobs that use it.

Let us take the examples that I used in the Introduction part of Chapter 2. Instead of having a robot to do a fraction of Λ_A of a cleaner's job, we can give the worker better cleaning tools that help him complete the work more efficiently. In the case of the surgeon, Multiplicative Robots can help him in many different ways such as collecting data to have more accurate diagnoses or supporting him while he does the operation himself.

Clarification of this point paves the road for the last chapter where we consider both types of robots at the same time in the economy. In chapter 2 as well as this chapter 3, we study the effects under the single application of each type of robot without the appearance of the other type. In other words, in this chapter, when workers are given Multiplicative Robots, they do not have an option to use Additive Robots.

In the modeling language, unlike the Additives Robots which enter the labor force in an additive way $L^A = \Lambda_A R^A + H$, Multiplicative Robots help humans increase their productivity. Hence, the "**effective labor**" would become:

$$L^M = \left(1 + \Lambda_M R^M\right) H$$

Due to the way that Multiplicative Robots transform the labor force, the use of Multiplicative Robots requires more conditions than the simple Additive Robots which will need to be discussed first in this chapter before describing how the model reacts with the introduction of Multiplicative Robots. The identification of the first period immediately after the appearance of robots as well as the stabilization of the long-run convergence are not straightforward like in the case with Additive Robots, requiring a numerical method. Only after this stability condition is achieved then we can have the results to compare with what has been obtained with Additive Robots.

3.2 Market Descriptions and Assumptions

3.2.1 Main Assumptions and Implications

Analog to chapter 2, we have 2 critical assumptions for the model set up in this chapter. The first one is the same as the first assumption in chapter 2, while the other is different, showing the differences in the characteristic of two types of robots.

Assumption 3.1: Capital is fully malleable.

This assumption has been explained in the previous chapter when we introduce Additives Robots into the market. It works in the same way with Multiplicative Robots. Capital is instantaneously convertible between conventional machines and

Multiplicative Robots. Hence by no-arbitrage condition, we require the same return rates on both types of capital, conventional and robotic:

$$r_t^K = r_t^{R^M} \quad \forall \text{ t} \tag{3.1}$$

Assumption 3.2: Multiplicative Robots acts as a labor augmented factor.

This assumption assumes that this new type of robots can help to increase the productivity of human labor. It seems to be very intuitive since, for so many years, robots are mainly used to support humans in many areas. With the help of robots, human labor can perform tasks either faster or more accurately.

To model that characteristic of robots, we assume that robots act as a labor augmented factor with Λ_M as their productivity. Hence with R^M number of robots will help to increase **every** human labor's productivity by $\Lambda_M R^M$. That means the **effective labor** is now:

$$L^M = \left(1 + \Lambda_M R^M\right) H \tag{3.2}$$

We define the "marginal effective labor" with respect to humans or robots is an increase in effective labor if we have an additional unit of human labor or robots respectively.

$$MeL_H = \frac{\partial L^M}{\partial H} = \left(1 + \Lambda_M R^M\right)$$
$$MeL_R = \frac{\partial L^M}{\partial R^M} = \Lambda_M H$$

Constant Effective Capital Intensity

As the previous chapter, we define the "**effective capital intensity**" as $k^M = \frac{K}{L^M}$, and maintain the homogeneous of degree 1 production function: $F\left(K, L^M\right) = L^M f\left(k^M\right)$.

The respective Marginal Product of Conventional Machines (MPK) and Total Marginal

Product of Effective Labor ($MPeL$) are still as normal:

$$MPK = F_1\left(K, L^M\right) = \frac{\partial F(.)}{\partial K} = \frac{\partial\left(L^M f(k^M)\right)}{\partial(k^M L^M)} = f'\left(k^M\right)$$

$$\begin{aligned} MPeL = F_2\left(K, L^M\right) = \frac{\partial F(.)}{\partial L^M} &= \frac{\partial\left(L^M f(k^M)\right)}{\partial L^M} \\ &= f\left(k^M\right) + L^M f'\left(k^M\right)\frac{\partial k^M}{\partial L^M} \\ &= f\left(k^M\right) - k^M f'\left(k^M\right) = (1-\alpha) f\left(k^M\right) \end{aligned}$$

We can explain the mechanism as follow, each an extra unit of human labor will create an extra MeL_H units of effective labor whose each unit can create an extra $MPeL$ units of final products. Accordingly, we have the final Marginal Product of human labor (MPL) and Marginal Product of robots (MPR) as:

$$MPL = MeL_H \times MPeL = \left(1 + \Lambda_M R^M\right)(1-\alpha) f\left(k^M\right) \tag{3.3}$$

$$MPR = MeL_R \times MPeL = (\Lambda_M H)(1-\alpha) f\left(k^M\right) \tag{3.4}$$

Combine the two main assumptions, we reach a similar condition as in the previous chapter that to have the same return rate on both conventional machines and Multiplicative Robots, the effective capital intensity will have to stay at a constant level, i.e. $k_t^M = \gamma_M \ \ \forall t$, which satisfies [1]:

$$f'(\gamma_M) = (\Lambda_M H)(1-\alpha) f(\gamma_M) \tag{3.5}$$

With the standard Cobb Doughlas production function:

$$\alpha\left(\gamma_M^{\alpha-1}\right) = \Lambda_M H(1-\alpha)\gamma_A^{\alpha} \quad \Leftrightarrow \quad \gamma_M = \frac{1}{\Lambda_M H}\left(\frac{\alpha}{1-\alpha}\right) \tag{3.6}$$

which is constant under fixed H.

[1] The difference is that in Chapter 2, this result does not depend on the assumption on H. Hence, it still hold even in the extension with flexible H. While in this chapter, it is only correct under the assumption that labor supply H is fixed. However, the effect of changes in H will rather be incorporated into Chapter 4.

3.2.2 Participation Conditions - Markups

With the new effective labor (3.2), the firm' profit function will be changed to

$$\pi_t^M = F\left(K_t, \left(1+\Lambda_M R_t^M\right)H\right) - r_t^K K_t - r_t^{R^M} R_t^M - v_t H$$

and first order conditions of profit maximization problem require that each production inputs are paid (in terms of final product) according to their respective marginal products:

$$r^K = MPK \qquad r^{R^A} = MPR \qquad v = MPL$$

Hence the total cost incurred would be:

$$\begin{aligned} TC &= K \times MPK + H \times MPL + R^M \times MPR \\ &= KF_1 + H\left(1+\Lambda_M R^M\right)F_2 + R^M\left(\Lambda_M H\right)F_2 \end{aligned}$$

However, since production function is homogeneous of degree one, according to Euler Theorem, the following must be true:

$$Y = F\left(K, \left(1+\Lambda_M R^M\right)H\right) = KF_1 + \left(1+\Lambda_M R^M\right)HF_2$$

Compare the two expressions above, we see that the output Y is just enough to cover the cost for conventional machines and human labor. Therefore firms will always suffer from a negative profit if they choose to use the Multiplicative Robots.

This is a similar problem with **Solow Residuals** in the Solow model as well as the Ramsey model. Empirically, there are increases in the final output that is not accounted for by capital accumulation and/or increases in labor **(Solow 1957)**. In the Solow model, that was taken care of by the technology progress or the total factor productivity term A in $F(K,AL)$. However, A is exogenously given, without any extra cost incurred for producers. There is no incentive for the producer to invest in A or create the technological progress since the extra output created is used to increase the pay for labor $v = AF_2$ as they demand higher pay for higher productivity.

Similarly, in our model with Multiplicative Robots, the producer does not want to both employ costly Multiplicative Robots and increase the pay for human labor. The output is just **not enough to cover the cost if all input resources are paid according to their marginal products above**. Hence, the condition for Multiplicative Robots to be applied is that producers can pay lower than each inputs' marginal products. On other words, they pay the same [2] fraction $\frac{1}{\mu_t} < 1$ with $\mu_t > 1$ of marginal product to each input factors. The idea is similar to the monopolistic competition. Under perfect competition we would have the profit maximization condition is that all the firms will set price equals marginal cost: $P = MC$. However, monopolist with market power will be able to charge at a higher price: $\frac{P}{MC} = \mu > 1$ to have a positive profit.

That means we need to impose the followings to the first-order conditions:

$$r^K = \frac{1}{\mu}MPK \qquad r^{R^A} = \frac{1}{\mu}MPR \qquad v = \frac{1}{\mu}MPL \tag{3.7}$$

with $\mu > 1$.

Note that by applying the same fraction across all production inputs, it does not make any changes to the no-arbitrage condition between conventional machines and Multiplicative Robots in (3.5).

3.2.3 Overhead Labor Cost

As discussed in the previous part, for Multiplicative Robots to be applicable, we need to impose markups which means a depress in the payments to all production inputs which is similar to monopolistic condition.

However, I would not want to impose the monopolistic condition in the model

[2] Markups for each production's inputs do not need to be the same. However, since markup rates are not the focus of this thesis, for simplification, I assume that they are the same.

as well as just to have intervention so that after the introduction of Multiplicative Robots, producers suddenly have market power and can pay less than the marginal products. Instead, We would make the changes in the market condition even from the original initial steady state so that the existence of markups rate μ_t is expected oven before the application of Robots. After robots, producers just need to adjust the value of markup.

That is not impossible. In literature, there are already few methods to do so, for example, to impose a fixed cost into the production function. So that right from the beginning the producers need to pay at a lower level than the marginal products to finance the fixed cost.

Another method is to assume that there is an overhead labor cost that requires producers to cover besides the cost of the inputs. And this is the way I choose to apply to the model. In the next subsection, I will introduce the notion of overhead labor as well as derive the implied markups' formula.

3.2.4 Overhead Labor

By nature, overhead or overhead expense refers to ongoing expenses of operating a business that can't be linked to creating or producing a product or service. Hence overhead labor can be understood as part of working hours that are wasted and are not used to produce the final products. This can include time spend for breaks, socialization, or even unfocused time (such as in the early morning).

If all the working days provided by the workers are H then a fraction of it: $\bar{h} = \frac{\bar{H}}{H}$ is overhead labor fraction with $\bar{H}$ is the overhead labor. Therefore, the "**actual productive working hours**" is only:

$$H - \bar{H} = H\left(1 - \bar{h}\right)$$

We can assume that $\bar{h}$ does not change, meaning that you always have the same fraction of working time without producing any product. As we assume H is fixed, overhead labor $\bar{H}$ is also fixed. This is to ensure the same behavior in the labor market.

Overhead labor expenses and Markups

This part will explain why overhead issue requires the application of markups and how can we derive the relationship between $\bar{h}$ and markups μ.

The shortage arises when the firm still needs to pay for the wasted overhead labor time while not produce any final output. According to profit maximization, it follows that the normal total cost is $TC = F_1K + F_2H$. However, outputs produced is just enough to cover the cost of "non-overhead" labor as per Euler theorem:

$$Y = F_1K + F_2H\left(1-\bar{h}\right) < F_1K + F_2H = TC$$

That is why firms will be able to pay only a fraction of each marginal product. Here I emphasize again that we use the same fraction across all inputs. Thus, the real total cost becomes:

$$TC = \left(\frac{1}{\mu}F_1\right)K + \left(\frac{1}{\mu}F_2\right)H$$

As the product market is perfectly competitive, the zero-profit condition needs to hold or $Y = TC$. Hence,

$$\begin{aligned}
\mu Y = &\quad F_1K + F_2H(1-\bar{H}) + F_2\bar{H} = Y + F_2\bar{H} \\
\Leftrightarrow &\quad (\mu-1)H(1-\bar{H})f(k) = \Big(f(k) - kf'(k)\Big)H\bar{h} \\
\Leftrightarrow &\quad \mu = \left(\frac{f(k)-kf'(k)}{f(k)}\right)\frac{\bar{h}}{1-\bar{h}} + 1
\end{aligned}$$

Given the Cobb Douglas production function, we have the formula for the markup ratio which can be used for the initial steady state.

$$\mu = (1-\alpha)\left(\frac{\bar{h}}{1-\bar{h}}\right) + 1 \tag{3.8}$$

3.2.5 The Purpose of Using Markups

Through the above calculation, we see the role of markups in this case. With overhead labor cost, by right the productivity of human labor is lower. In other words, instead of being paid F_2, workers should be paid only $(1-\bar{h})F_2$. However, for some reason, producers are not able to do so. For example, the producers only can observe the cost incurred but not be able to identify the source. It can be included in a fixed cost, which is not able to decide which factor should bear it. Hence, by using the same markups across all inputs, we distribute that fixed cost to all production inputs. More precisely, **markups redistribute the overhead labor cost from workers to capital owners** instead of letting the workers bear all of it.

As a result, we do not use $\bar{h}$ to adjust the value of the marginal product of labor alone. However, it appears in the expression for markups and we use markups to adjust both MPK and MPL.

3.2.6 Additives Robots Not Require Markups

Unlike Multiplicative Robots that can increase the productivity of labor workers, Additives Robots are just simple additions (extensions) into the labor force. They do take part in the production and create extra final outputs. However, as productivity is unchanged, the wage for human labor does not change [3]. As a consequence, the extra final outputs created can be used to finance the cost of Additives Robots. Although at the end of the day, it would be returned to the robots' owner, ultimately, the representative household.

By the Euler theorem for homogeneous of degree one function, we have output is

[3] This refers to the changes just because of the increase/decrease in productivity, not to say that there is no change due to the introduction of robots.

the same with the total cost in the model with Additives Robots.

$$Y = F\left(K, \Lambda_A R^A + H\right) = F_1 K + F_2\left(\Lambda_A R^A + H\right)$$
$$= F_1 K + (F_2 \Lambda_A) R^A + F_2 H = TC$$

If we impose the overhead labor cost in the model with Additives Robots, that will be the only factor contributes to the need for markups as well as affects the markups' value. Therefore, the expression for markups in (3.8) is applicable for both the initial steady-state and after the introduction of robots.

There are only two changes that need to be made to model under Additive Robots. Firstly, we need to use the markups in (3.8) to readjust all the return rates on conventional machines, robots, and human labor [4]. Secondly, the effective labor with Additive Robots in (2.1) needs to be adjusted to:

$$L_A = \Lambda^A R^A + H(1 - \bar{h})$$

For the rest of this chapter (as well as the next chapter), in all comparisons between two scenarios, the economy with either Additives Robots or Multiplicative Robots, the model with Additives Robots is adjusted (as compared to what have been presented in Chapter 2) with overhead labor cost. As a result, both models will have the same initial steady-state which will be express in the next section.

3.3 The Model Setup

3.3.1 Initial Steady State

We maintain the same simple setup used in the framework model in Chapter 2 with representative household and representative producer. Notably, the introduction of overhead labor cost and hence the use of markups will change the market conditions

[4] Hence, interest rate and consumption growth are both scaled down by markups accordingly.

that market participants need to accept, which are the real interest rate and real wage:

$$r_t = r_t^K - \delta = \frac{1}{\mu_t} f'(k_t) - \delta$$
$$v_t = \frac{1}{\mu_t}\left(f(k_t) - k_t f'(k_t)\right) = \frac{1}{\mu_t}(1-\alpha) f(k)$$

The household still has to solve the same maximization problem given the above market conditions. Hence the growth rate of consumption as per Euler Equation in (2.6) needs to be modified as follow:

$$\frac{\dot{c}_t}{c_t} = r_t - \theta = \frac{1}{\mu_t} f'(k_t) - \delta - \theta \tag{3.9}$$

However, as now only $H(1-\bar{h})$ is effective in the production, the capital intensity is now:

$$k = \frac{K}{H\left(1-\bar{h}\right)}$$

and growth rate of k as in (2.9) will be calculated as follow:

$$\dot{K} = Y - \delta K - c = H\left(1-\bar{h}\right) f(k) - \delta K - c$$
$$\Leftrightarrow \quad \dot{k} = \frac{\dot{K}}{H\left(1-\bar{h}\right)} = f(k) - \delta k - \frac{c}{H\left(1-\bar{H}\right)}$$

Hence,we have the followings to solve for the initial steady state:

$$\frac{1}{\mu_{ss}} \alpha k_{ss}^{\alpha-1} - \delta - \theta = 0$$
$$k_{ss}^{\alpha} - \delta k_{ss} = \frac{c_{ss}}{H\left(1-\bar{h}\right)}$$

As a result, the following system of equations [5] characterizes the initial steady state:

$$k_{ss} = \left(\frac{\alpha}{\mu_{ss}(\delta+\theta)}\right)^{\frac{1}{1-\alpha}} \qquad K_{ss} = k_{ss} H\left(1-\bar{h}\right)$$
$$y_{ss} = k_{ss}^{\alpha} \qquad Y_{ss} = H\left(1-\bar{h}\right) y_{ss} \qquad c_{ss} = Y_{ss} - \delta K_{ss}$$
$$v_{ss} = \frac{1}{\mu_{ss}}\left(f(k_{ss}) - k_{ss} f'(k_{ss})\right) = \frac{(1-\alpha)}{\mu_{ss}} f(k_{ss}) = \frac{(1-\alpha)}{\mu_{ss}}\left(\frac{\alpha}{\mu_{ss}(\delta+\theta)}\right)^{\frac{\alpha}{1-\alpha}}$$

with μ_{ss} follows (3.8).

[5] Compared to the system of equations in Chapter 2, only those have H will need to be adjusted to $H\left(1-\bar{h}\right)$

And the Lifetime Welfare for household at any period t is still:

$$U_t = u(c_{ss}) \int_t^{\infty} e^{-\theta t} dt = u(c_{ss}) \frac{1}{-\theta} \left[e^{-\theta t} \right]_0^{\infty} = \frac{1}{\theta} u(c_{ss}) = \frac{1}{\theta} \log c_{ss}$$

3.3.2 Introduction of Multiplicative Robots into the Economy

Since now only a fraction $(1-\bar{h})$ of labor participate in production function ($\bar{h}$ is wasted), we have the followings:

Effective Labor in (3.2):

$$L_M = \left(1 + \Lambda_M R^M\right) H \left(1 - \bar{h}\right) \tag{3.10}$$

Effective Capital Intensity:

$$k_t^M = \frac{K_t}{\left(1 + \Lambda_M R^M\right) H \left(1 - \bar{h}\right)} = \frac{w_t^n - R_t^M}{\left(1 + \Lambda_M R^M\right) H \left(1 - \bar{h}\right)} \tag{3.11}$$

Total Output:

$$Y_t = F\left(K_t, \left(1 + \Lambda_M R^M\right) H \left(1 - \bar{h}\right)\right) = \left(1 + \Lambda_M R_t^M\right) H \left(1 - \bar{h}\right) f\left(k_t^M\right)$$

which satisfies the Euler theorem for homogeneous of degree 1 function:

$$Y_t = F_1 K_t + F_2 \left(1 + \Lambda_M R_t^M\right) H \left(1 - \bar{h}\right) \tag{3.12}$$

And first-order conditions of profit maximization problem as per (3.7) become:

$$r_t^K = \frac{1}{\mu_t} f'\left(k_t^M\right) \tag{3.13}$$

$$v_t = \frac{1}{\mu_t} \left(1 + \Lambda_M R_t^M\right) \left[(1-\alpha) f\left(k_t^M\right) \right] \tag{3.14}$$

$$r_t^{R^M} = \frac{1}{\mu_t} \Lambda_M H \left[(1-\alpha) f\left(k_t^M\right) \right] \tag{3.15}$$

We need two preconditions to solve the model. One is the value of effective capital intensity k_t which we already have under the implications of two main assumptions in (3.6) [6]. The second is the markup rate. We need to recalculate the markup

[6] As noted, the application of the same markups for all production inputs does not change the expression for the no-arbitrage condition. Also, the use of overhead labor cost does not affect the marginal

rates after the introduction of Multiplicative Robots as explained before since Multiplicative Robots are also a source for Solow residual (besides overhead labor in our model).

Before going into the solutions, we derive some expressions of effective capital intensity that are useful for later manipulations.

From (3.5):

$$\gamma_M = \frac{1}{\Lambda_M H}\frac{\alpha}{1-\alpha} \quad \Leftrightarrow \quad \left(1-\gamma_M\Lambda_M H\right) = \frac{1}{1-\alpha}$$

Then, apply into (3.6) we have:

$$\Lambda_M H f\left(\gamma_M\right) = f'\left(\gamma_M\right)\frac{1}{1-\alpha} = f'\left(\gamma_M\right)\left(1-\gamma_M\Lambda_M H\right) \tag{3.16}$$

- **Markups**

  Since in this setup, the economy is closed without any trade or borrowing or lending from outside, the national income and national output need to be equal. In the same manner with what we did for the initial steady-state, we obtain the markup rate for each period:

$$\begin{aligned}
& Y_t && = TC = \tfrac{1}{\mu_t}\left(F_1K_t + F_2\Lambda_M H R_t^M + F_2\left(1+\Lambda_M R_t^M\right)H\right)\\
\Leftrightarrow\quad & \mu_t Y_t && = \left[F_1K_t + F_2\left(1+\Lambda_M R_t^M\right)H\left(1-\bar{h}\right)\right] + F_2\left(1+\Lambda_M R_t^M\right)H\bar{h} + F_2\Lambda_M H R_t^M\\
& && = Y_t + F_2\left(1+\Lambda_M R_t^M\right)H\bar{h} + F_2\Lambda_M H R_t^M\\
\Leftrightarrow\quad & \left(\mu_t-1\right)Y_t && = HF_2\left[\left(1+\Lambda_M R_t^M\right)\bar{h} + \Lambda_M R_t^M\right]\\
\Leftrightarrow\quad & \mu_t - 1 && = \frac{H\left(f(\gamma_M)-\gamma_M f'(\gamma_M)\right)\left[(1+\Lambda_M R_t^M)\bar{h}+\Lambda_M R_t^M\right]}{(1+\Lambda_M R_t^M)H\left(1-\bar{h}\right)f(\gamma_M)}\\
& && = \left(\frac{f(\gamma_M)-\gamma_M f'(\gamma_M)}{f(\gamma_M)}\right)\left(\frac{(1+\Lambda_M R_t^M)\bar{h}+\Lambda_M R_t^M}{(1+\Lambda_M R_t^M)\left(1-\bar{h}\right)}\right)\\
& && = \left(\frac{f(\gamma_M)-\gamma_M f'(\gamma_M)}{f(\gamma_M)}\right)\left(\frac{\bar{h}}{1-\bar{h}} + \frac{\Lambda_M R_t^M}{(1+\Lambda_M R_t^M)\left(1-\bar{h}\right)}\right)
\end{aligned}$$

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product expression hence the formula for γ_M is still the same.

Hence [7] , we have

$$\mu_t - 1 = \frac{\bar{h}(1-\alpha)}{1-\bar{h}} + \frac{(1-\alpha)}{(1-\bar{h})}\left(\frac{\Lambda_M R_t^M}{1+\Lambda_M R_t^M}\right)$$

$$\mu_t = \left(\mu_{ss} + \frac{1-\alpha}{(1-\bar{h})}\right) - \frac{1-\alpha}{(1-\bar{h})\left(1+\Lambda_M R_t^M\right)} \tag{3.17}$$

There are **three observations** we can derive from the above formula of markup ratio:

- **Firstly**, (3.17) can be rewrite as:

$$\mu_t = \mu_{ss} + \frac{1-\alpha}{1-\bar{h}}\left(1 - \frac{1}{1+\Lambda_M R_t^M}\right) \tag{3.18}$$

  Since the second component is always positive, in the first period after the introduction of robots $\mu_0 > \mu_{ss}$. In general we have $\mu_t > \mu_{ss}\ \forall t \geq 0$. That means the markup rates after the introduction of Multiplicative Robots are always more than the markup of the initial steady-state μ_{ss}. This is because although the Multiplicative Robots can help to make non-overhead labor becomes more productive i.e produce more final outputs, at the same time this non-overhead labor is paid a higher wage. Hence, there is a pressure on producers to set aside investment to invest in robots still there, pushing them to charge a higher price over the marginal cost.

- **Secondly**, as R_t^M increases, $\frac{1}{1+\Lambda_M R_t^M}$ decreases thus μ_t increases over time. Put it differently, as the stock of robots grows, there are more and more investment is needed to finance them.

- However, **thirdly** we see that as $R_t^M \xrightarrow{t\to\infty} \infty$, $\frac{1}{1+\Lambda_M R_t^M} \to 0$ then in the long run

[7] With Cobb Douglas function we have:

$$\frac{f(k_t) - k_t f'(k_t)}{f(k)} = \frac{k_t^\alpha - \alpha k_t k_t^{\alpha-1}}{k_t^\alpha} = 1-\alpha$$

μ_t will converges to a constant level μ^*.

$$\mu^* = \mu_{ss} + \frac{1-\alpha}{(1-\bar{h})} = (1-\alpha)\frac{\bar{h}}{1-\bar{h}} + 1 + \frac{1-\alpha}{(1-\bar{h})}$$
$$= \frac{1-\alpha}{(1-\bar{h})}(1+\bar{h}) + 1 \tag{3.19}$$

Hence from (3.17), we have:

$$\mu_t - \mu^* = \frac{1-\alpha}{(1-\bar{h})\left(1+\Lambda_M R_t^M\right)} \tag{3.20}$$

In summary, after the introduction of Multiplicative Robots, the markup jumps to a higher level then keeps increasing over time, and finally converges to the upper boundary μ^*.

- **Return rates on Conventional Machine, Multiplicative Robots, and Real interest rate**

  In the model with Additives Robots, when the effective capital intensity stays constant at γ_A then all of these rates are constant[8]. On the contrary, due to the use of markup rates which changes over time, these rates are no longer constant with Multiplicative Robots. From (3.13) we have return rates and market real interest rate as follow:

$$r_t^K = r_t^{R^M} = \frac{1}{\mu_t} f'(\gamma_M) \quad \text{and} \quad r_t = r_t^K - \delta \tag{3.21}$$

  As have shown in the previous part, when the stock of robots increases, μ_t increases over time, making the return rates and interest rate decrease over time. As t goes to infinity, $\mu_t \to \mu^*$. It follows that in the long run, return rates on conventional machines and Multiplicative Robots would also converge.

$$r_t^K = r_t^{R^M} \xrightarrow{t\to\infty} \frac{1}{\mu^*} f'(\gamma_M)$$
$$r_t \xrightarrow{t\to\infty} \frac{1}{\mu^*} f'(\gamma_M) - \delta$$

[8] Even when we adjust the model with overhead labor cost, the markup is unchanged throughout time and hence only proportionately devalues these rates.

- **Profitability Condition**

With the above predictions of market rates, we can answer the next question, regarding the profitability condition.

For Additive Robots: $\gamma_A < k_{ss}$ is enough to ensure the profitability condition which is the requirement that:

$$r^{R^A} = f'(\gamma_A) > r^K_{ss} = f'(k_{ss}) = \theta + \delta \quad \forall t$$

In the case of the Multiplicative Robots, due to the participation condition μ_t, investors will only find it profitable to convert part of their assets (in the form of the conventional machine) to robots if $r_t^{R^M} \geq r^K_{ss}$ which means:

$$r_t^{R^M} = \frac{1}{\mu_t} f'(\gamma_M) > \frac{1}{\mu_{ss}} f'(k_{ss}) = r^K_{ss} = \theta + \delta \tag{3.22}$$

It is quite clear that the effective capital intensity γ_M is less than the steady state capital intensity k_{ss}, as the numerator (conventional machines) decreases while denominator (labor) increases. Since γ_M is constant, we can just use the first period formula to demonstrate:

$$\gamma_M = \frac{K_{ss} - R_0^M}{\left(1 + \Lambda_M R_0^M\right) H(1-\bar{h})} < \frac{K_{ss}}{H(1-\bar{h})} = k_{ss}$$

Hence, $f'(\gamma_M) > f'(k_{ss})$, similar to Additives Robots case. Nevertheless, in the case of Multiplicative Robots, we have the markups that: $\frac{1}{\mu_t} < \frac{1}{\mu_{ss}}$. That is why (3.22) can not be achieved automatically.

Besides, the profitability condition needs to be held for all periods. And in the previous part we know that μ_t increases overtime and converges to μ^*. Hence, $\frac{1}{\mu_t}$ is the minimum value of $r_t^{R^M}$ for all t. So in order to achieve (3.22) for all period t, we require the following constraints on the model's parameters:

$$\begin{aligned}
\tfrac{1}{\mu^*}\alpha(\gamma_M^{\alpha-1}) &> \alpha\tfrac{1}{\mu^*}(k_{ss}^{\alpha-1}) = \delta+\theta \\
\Leftrightarrow \quad \gamma_M &< \left(\tfrac{\mu^*(\theta+\delta)}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
\Leftrightarrow \quad \tfrac{1}{\Lambda_M H}\tfrac{\alpha}{1-\alpha} &< \left(\tfrac{\mu^*(\theta+\delta)}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
\Leftrightarrow \quad \Lambda_M &> \tfrac{\alpha}{(1-\alpha)H}\left(\tfrac{\mu^*(\theta+\delta)}{\alpha}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

The choice of Multiplicative Robots' productivity need to be higher than the minimum productivity level Λ_M^*

$$\Lambda_M > \Lambda_M^* = \frac{\alpha}{(1-\alpha)}\left(\frac{(\theta+\delta)}{\alpha}\right)^{\frac{1}{1-\alpha}}\frac{(\mu^*)^{\frac{1}{1-\alpha}}}{H} \tag{3.23}$$

with μ^* can be taken from (3.19).

Compared to the Additives Robots case, the minimum required productivity should be lower. Multiplicative Robots help increase workers' productivity hence each unit of Multiplicative Robots will have effects on the whole labor force H, rather than just a single add up into the labor force like Additives Robots. Consequently, if given the same R units of robots (we do not take into account the issue with markups since it is not applicable in Additives Robots), we have:

$$\Lambda_A R + H = (1+\Lambda_M R)H \Leftrightarrow \Lambda_M = \frac{1}{H}\Lambda_A$$

- **Stock of Multiplicative Robots and Conventional Machine**

  From the formula for effective capital intensity (3.6), we have:

$$\gamma_M = \frac{w_t^n - R_t^M}{\left(1+\Lambda_M R_t^M\right)H(1-\bar{h})} = \frac{1}{\Lambda_M H}\left(\frac{\alpha}{1-\alpha}\right)$$

  With some manipulations

$$\begin{aligned}
\alpha\left(1+\Lambda_M R_t^M\right)(1-\bar{h}) &= \Lambda_M(1-\alpha)\left(w_t^n - R_t^M\right) \\
\Leftrightarrow \quad \alpha(1-\bar{h})\Lambda_M R_t^M + \alpha(1-\bar{h}) &= (1-\alpha)\Lambda_M w_t^n - (1-\alpha)\Lambda_M R_t^M \\
\Leftrightarrow \quad \Lambda_M R^M\Big(\alpha(1-\bar{h}) + (1-\alpha)\Big) &= (1-\alpha)\Lambda_M w_t^n - \alpha(1-\bar{h}) \\
\Leftrightarrow \quad \left(1+\Lambda_M R_t^M\right)(1-\alpha\bar{h}) &= (1-\alpha)\Lambda_M w_t^n + 1 - \alpha \\
&= (1-\alpha)\left(1+\Lambda_M w_t^n\right)
\end{aligned}$$

We have the following results:

$$\left(1+\Lambda_M R_t^M\right) = \frac{1-\alpha}{1-\alpha\bar{h}}\left(1+\Lambda_M w_t^n\right) \tag{3.24}$$

$$R_t^M = \frac{(1-\alpha)}{(1-\alpha\bar{h})}w_t^n - \frac{\alpha(1-\bar{h})}{(1-\alpha\bar{h})\Lambda_M} \tag{3.25}$$

$$K_t = w_t^n - R_t^M = \frac{\alpha(1-\bar{h})}{1-\alpha\bar{h}}w_t^n + \frac{\alpha(1-\bar{h})}{(1-\alpha\bar{h})\Lambda_M} \tag{3.26}$$

A closer look into equation (3.25) reveals that the profitability condition (3.23) should be enough to make sure that R^M has a positive value[9]. In order to compare the changes speed of conventional machines and Multiplicative Robots, we need to compare the fraction of the increases in non-human wealth that is used to invest in Multiplicative Robots versus the fraction invested in conventional machines:

$$\frac{(1-\alpha)}{(1-\alpha\bar{h})} > \frac{\alpha(1-\bar{h})}{1-\alpha\bar{h}} \quad \Leftrightarrow \quad 1-\alpha > \alpha(1-\bar{h}) \quad \Leftrightarrow \quad \alpha < \frac{1}{2-\bar{h}}$$

If the above inequality holds, then the stock of Multiplicative Robots will grow faster than the stock of conventional machines as total wealth is accumulated.

In fact, with $\frac{1}{2-\bar{h}} > \frac{1}{2}$, that should be the case for most of the frequently used value of α as standard literature value for α is only one third.

- **Real Wage Increases under Profitability Condition**

  We now turn to real wage. From the first-order condition for real wage in profit maximization problem (3.14) we have:

$$v_t = \frac{1}{\mu_t}\left(1+\Lambda_M R_t^M\right)\left[f(\gamma_M) - \gamma_M f'(\gamma_M)\right]$$
$$= \frac{1}{\mu_t}\frac{(1-\alpha)^2}{(1-\alpha\bar{h})}(1+\Lambda_M w_t^n)f(\gamma_M) \tag{3.27}$$

  As noted, as R_t^M increases μ_t increases along, making $\frac{1}{\mu_t}$ decreases overtime. Then how real wage change will depend on whether the increase in robots is faster or the increase in markups rate is faster before the convergence of markups.

[9] $R^M > 0 \Leftrightarrow \Lambda_M > \frac{\alpha(1-\bar{h})}{(1-\alpha)K_{ss}}$ which is same with profitability condition (3.23).

After μ_t converges to a constant level μ^*, v_t will grow at a proportional rate of Multiplicative Robots.

However, we can also see that:

$$\begin{aligned}
\frac{\partial v_t}{\partial R_t^M} &= \frac{(1-\alpha)^2}{(1-\alpha\bar{h})} f(\gamma_M)\left[\Lambda_M \frac{1}{\mu_t} + \left(1+\Lambda_M R_t^M\right)\frac{-1}{(\mu_t)^2}\frac{\partial \mu_t}{\partial R_t^M}\right] \\
&= \frac{(1-\alpha)^2 f(\gamma_M)}{(1-\alpha\bar{h})\mu_t}\left[\Lambda_M - \left(1+\Lambda_M R_t^M\right)\frac{1}{\mu_t}\frac{1-\alpha}{(1-\bar{h})}\frac{\Lambda_M}{\left(1+\Lambda_M R_t^M\right)^2}\right] \\
&= \frac{\Lambda_M (1-\alpha)^2 f(\gamma_M)}{(1-\alpha\bar{h})\mu_t}\left[1 - \frac{1-\alpha}{(1-\bar{h})}\frac{1}{\mu_t\left(1+\Lambda_M R_t^M\right)}\right] > 0
\end{aligned}$$

The inequality hold since both μ_t and $\left(1+\Lambda_M R_t^M\right)$ is more than 1, thus $\frac{1}{\mu_t(1+\Lambda_M R_t^M)} < 1$. At the same time, most of the case $\bar{h} < \alpha$ as we should not have that much wasted overhead labor, thus $\frac{1-\alpha}{1-\bar{h}} < 1$.

Hence in the normal case, real wage will rise if the stock of Multiplicative robots grows overtime even before the convergence of markups. Unless we have very high overhead labor cost that $\frac{1-\alpha}{1-\bar{h}}$ dominates $\frac{1}{\mu_t(1+\Lambda_M R_t^M)}$.

For the initial period, real wage would be:

$$v_0 = \frac{1}{\mu_0}\frac{(1-\alpha)^2}{(1-\alpha\bar{h})}\left(1+\Lambda_M K_{ss}\right)(\gamma_M)^{\alpha}$$

while at the initial steady state

$$v_{ss} = \frac{1}{\mu_{SS}}(1-\alpha)(k_{ss})^{\alpha}$$

then we have:

$$\begin{aligned}
\frac{v_0}{v_{ss}} &= \frac{\mu_{ss}}{\mu_0}\frac{(1-\alpha)}{(1-\alpha\bar{h})}\left(1+\Lambda_M K_{ss}\right)\left(\frac{\gamma_M}{k_{ss}}\right)^{\alpha} \\
&= \left(\frac{\mu_{ss}}{\mu_0}\left(\frac{\gamma_M}{k_{ss}}\right)^{\alpha-1}\right)\frac{(1-\alpha)}{(1-\alpha\bar{h})}\left(1+\Lambda_M K_{ss}\right)\left(\frac{\gamma_M}{k_{ss}}\right) \\
&= \left(\frac{\frac{1}{\mu_0}f'(\gamma_M)}{\frac{1}{\mu_{ss}}f'(k_{ss})}\right)\frac{(1-\alpha)}{(1-\alpha\bar{h})}\left(\frac{\gamma_M}{k_{ss}} + \frac{\Lambda_M(k_{ss}H)\gamma_M}{k_{ss}}\right) \\
&= \left(\frac{\frac{1}{\mu_0}f'(\gamma_M)}{\frac{1}{\mu_{ss}}f'(k_{ss})}\right)\left(\frac{(1-\alpha)\gamma_M}{(1-\alpha\bar{h})k_{ss}} + \alpha\right)
\end{aligned}$$

From the profitability condition in the first period:

$$\frac{1}{\mu_0} f'(\gamma_M) > \frac{1}{\mu_{ss}} f'(k_{ss})$$

thus the first element is greater than 1. However, the second component is the weighted average between γ_M and k_{ss}. And since $\gamma_M < k_{ss}$ this weighted average should be less than k_{ss}. As a result, the second component is always less than 1.

That means in the first period, it is **undetermined** if the real wage increases or decreases from the initial steady-state level. That depends on the parameters of the model. The only thing can be determined is that wage will increase as the stock of Multiplicative Robots grows.

This result is different from the model under Additives Robots, in which from the initial steady state, real wage will definitely go down and then stay constant forever. Multiplicative Robots bring extra productivity for human labor thus create wage effects as wage continues to grow as long as Multiplicative Robots are used.

3.3.3 Long run Balanced Growth Path

We have the following to complete the model:

$$w_t^n = K_t + R_t^M$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\mu_t} f'(\gamma_M) - \delta - \theta = r_t^K - \delta - \theta \tag{3.28}$$

$$\dot{w}_t^n = \left[\Lambda_M R_t^M + 1\right] H(1-\bar{h}) f(\gamma_M) - \delta w_t^n - c_t \tag{3.29}$$

where the first one means that total wealth is used to finance either conventional machines or Multiplicative Robots. The second equation is the consumption rule for households that the growth of consumption is the difference between the market interest rate and time preference rate with a market interest rate is return on either conventional machines or Multiplicative Robots minus off the depreciation rate.

The third equation is the wealth accumulation rule or budget constraint for a household rewrite in the form of general equilibrium (which means take into account the production side). The first element is the total market output

$$F\left(K_t, \left[\Lambda_M R_t^M + 1\right] H(1-\bar{h})\right) = \left[\Lambda_M R_t^M + 1\right] H(1-\bar{h}) f(\gamma_M)$$

Then take out the necessary cover for depreciation of current wealth of household δw_t^n (or also the total investment capital of firm) and take out the household consumption c_t, what is left is the increase (accumulated amount) for wealth (or total capital investment) of the next period.

- **Reducing but Positive Consumption growth**

  From (3.28), there are two conclusions can be made regarding the growth rate of consumption. **Firstly**, since the effective capital intensity stays constant $k_t = \gamma_M$ while μ_t increases over time (as the stock of robots continues to grow), the growth rate of consumption will reduce over time. Until the time when μ_t converges to the constant μ^* then consumption changes with the same rate:

$$\eta_C^* = \frac{1}{\mu^*} f'(\gamma_M) - \delta - \theta > 0$$

  Secondly, as long as the profitability condition (3.22) still holds even when μ_t has been converged, we always have $r_t^K - \delta > r_{ss}$, that means (3.28) is always positive.

  In summary, consumption **always** grow at a positive rate, but the growth rate reduces over time and converges to the constant η_c^*.

- **Non-human wealth growth**

  From (3.29), the growth rate of non-human wealth is:

$$\dot{w}_t^n = \frac{(1-\alpha)}{(1-\alpha\bar{h})}(1+\Lambda_M w_t^n) H(1-\bar{h}) f(\gamma_M) - c_t - \delta w_t^n$$
$$\Leftrightarrow \frac{\dot{w}_t^n}{w_t^n} = \frac{(1-\alpha)}{(1-\alpha\bar{h})}\left(\frac{1}{w_t^n} + \Lambda_M\right) H(1-\bar{h}) f(\gamma_M) - \frac{c_t}{w_t^n} - \delta$$

  In the long run, as H is fixed while w_t^n keeps increasing, the fraction $\frac{1}{w_t^n} \to 0$.

Thus from (3.16) we have

$$\frac{\dot{w}_t^n}{w_t^n} \to \frac{(1-\alpha)}{(1-\alpha\bar{h})}\Lambda_M H(1-\bar{h})f(\gamma_M) - \frac{c_t}{w_t^n} - \delta$$
$$= \frac{1-\bar{h}}{(1-\alpha\bar{h})}f'(\gamma_M) - \frac{c_t}{w_t^n} - \delta$$

It is not straightforward to derive the same expression to observe the convergence of $\frac{c}{w}$ ratio as in (2.26) since the growth rate of consumption is no longer fixed (due to markups and overhead labor cost). Theoretically, when we have the convergence of $\frac{\dot{w}_t^n}{w_t^n}$ then based on the above expression, the ratio $\frac{c_t}{w_t^n}$ should also converge to a constant.

- **Convergence of consumption to wealth ratio**

  To identify the convergence of the consumption to wealth ratio, we use the growth rate of wealth from household budget constraint:

$$\frac{\dot{w}^n}{w^n} = \frac{rw^n + vH - c}{w^n} = r + \frac{vH}{w^n} - \frac{c}{w^n}$$
$$= \frac{\dot{c}}{c} + \theta + \frac{vH}{w^n} - \frac{c}{w^n}$$

  From the above expression, we see that if consumption and non-human wealth grow at the same rate then $\frac{c}{w^n} = \frac{vH}{w^n} + \theta$.

  In the previous chapter, with Additive Robots, we easily have $\frac{vH}{w^n} \to 0$ since v is fixed and H is either fixed or decreases, at the same time w^n goes to infinity. And hence, consumption to wealth ratio will converge to θ. This result can also be proved directly from the consumption equation of household maximization problem:

$$c = \theta(w^n + w^h) \Rightarrow \frac{c}{w^n} = \theta + \theta\left(\frac{w^h}{w^n}\right)$$

  We have v, H and interest rate r are all fixed and human wealth w^h is fixed at $\frac{vH}{r}$ as proved in (2.19). Thus when wealth increase to infinity the second component goes to zero. It follows that we have $\frac{c}{w^n} \to \theta$.

However, with Multiplicative Robots, one of the main results is that wages will increase as the stock of robots increases (together with wealth accumulation). At the same time interest rate $r = \frac{1}{\mu} f'(\gamma_M) - \theta$ goes down as μ increases. Can we still achieve the same convergence here?

Using the wage equation from (3.27) we see that:

$$\frac{vH}{w^n} = \frac{(1-\alpha)^2(1+\Lambda_M w^n)f(\gamma_M)H}{(1-\alpha\bar{h})\mu w^n} = \frac{(1-\alpha)^2}{(1-\alpha\bar{h})} f(\gamma_M)\frac{1}{\mu}\left(\frac{1}{w^n} + \Lambda_M\right)$$
$$\longrightarrow \frac{(1-\alpha)^2}{(1-\alpha\bar{h})}\Big(\Lambda_M f(\gamma_M)H\Big)\frac{1}{\mu^*} = \frac{(1-\alpha)}{(1-\alpha\bar{h})}\frac{f'(\gamma_M)}{\mu^*}$$

which indicates that $\frac{vH}{w^n}$ does not go to zero but a positive constant as wealth goes to infinity. Hence, consumption to wealth ratio will converge to a higher level than Additives Robots case:

$$\frac{c}{w^n} \longrightarrow \theta + \frac{(1-\alpha)}{(1-\alpha\bar{h})}\frac{f'(\gamma_M)}{\mu^*} \tag{3.30}$$

3.4 Numerical Methods for Initial Condition

Why is the value of first-period consumption important? If this value is too high, meaning households consume too much compared to their utility maximization problem's solution, then the accumulation of non-human wealth is lower than the optimal level. In the long run, if consumption keeps increasing according to their optimal growth rate, there will be a point that non-human wealth will start to decrease toward zero.

The reverse is true. Household starts with too low consumption, i.e too high wealth accumulation will make the increase of non-human wealth too fast compared to the optimal level. Hence, instead of $\frac{c}{w^n}$ converges to a constant, it will go down all the way to zero as w^n increase too fast compared to consumption.

3.4.1 Problems with Simple Guessing Method

It seems that we can find the initial condition (consumption) by using the simple shooting algorithm as in the case of flexible H in the previous chapter. From the expression for markups after the introduction of robots (3.17), μ_t only depends on the value of Multiplicative Robots R_t^M and (3.24) shows the relationship between robots R_t^M and total wealth w_t^n. In the first period, wealth still stays at the initial steady-state level. Hence, markup rate in the first period is specified as:

$$\mu_0 = \mu_{ss} + \frac{1-\alpha}{1-\bar{h}} \left(1 - \frac{1-\alpha\bar{h}}{(1-\alpha)(1+\Lambda_M K_{ss})} \right) \tag{3.31}$$

Hence, we can implement the following steps for the fixed point iterations:

1. Given μ_0, calculate other market condition for the first period v_0 and r_0
2. Guess c_0
   - For the second period:
     - Calculate non-human wealth next period $w_1^n - K_{ss} = r_0 K_{ss} + v_0 H - c_0$
     - Use μ_0, γ_M and w_1^n to infer the next period market conditions: μ_1, v_1, r_1 from (3.21),(3.27),(3.31).
     - Use μ_1 and r_1 to update consumption for the next period: $\frac{\dot{c}_1}{c_1} = \frac{1}{\mu_1} r_1 - \theta$
   - Repeat the above 3 steps, simulate for T number of periods.
3. Use the series of wage $(v_t)_{t=1}^T$ and interest rate $(v_t)_{t=1}^T$ to calculate human wealth at $t = 0$.
4. Calculate $c_0' = \theta \left(w_0^h + K_{ss} \right)$ and compare with the guess. Update the guess until we have a match.

However, there is an issue with the above iteration. First period human wealth expression as in (2.4) is:

$$w_0^h = \int_0^\infty v_\kappa H \exp\left(\int_0^\kappa -r(\nu)d\nu\right) d\kappa \tag{3.32}$$

As time goes to infinity, the discounted factor does make the series inside the integral goes down to zero, meaning the further the human income is, the less important its contribution to the human wealth is. However, with Multiplicative Robots, wage increases to infinity in the long run. Thus, no matter how good our guess is and how long the simulation is, there are still some errors in the value of the initial guess of c_0.

This is different from the previous chapter. For the case where we have inflexible labor, i.e. fixed H, We can even derive the exact expression for human wealth as in (2.19). Or even under the extension of flexible labor H, as we simulate the economy long enough so that H reaches a constant then from there onward we can again calculate the value of non-human wealth. Hence, there was no error in updating the value for initial consumption in iterations. With Multiplicative Robots, all market conditions are changed as time goes hence the guessing technique can not preserve the convergence of the model. In other words, the model does not stable.

3.4.2 Convergence Stability Condition

As explained above, the simple technique to guess for the initial consumption does not work, hence, we need to find another technique to pin down the initial condition. Theoretically, we know that the ratio $\frac{c}{w^n}$ will converge to a ratio in the long run.

The idea for this part is that instead of finding the initial value of consumption c_0, we will find the initial value of that ratio $q_0 = \frac{c_0}{K_{ss}}$. Since the non-human wealth in the first period is still at K_{ss}, from there we can derive the first-period consumption. The first step is to examine the dynamics of c/w to see if its changes depend on any other variables.

Consumption over wealth ratio

Let denote $q = \frac{c}{w^n}$ or $c = qw^n$. Use wage equation (3.27) we have:

$$\begin{aligned}
\frac{\dot{q}}{q} = \frac{\dot{c}}{c} - \frac{\dot{w^n}}{w^n} &= (r-\theta) - \frac{rw^n + vH - c}{w^n} = -\frac{vH}{w^n} + q - \theta \\
&= q - \theta - \frac{(1-\alpha)^2(1+\Lambda_M w_t^n) f(\gamma_M) H}{(1-\alpha\bar{h})\mu_t w^n} \\
&= q - \theta - \frac{(1-\alpha)^2(1+\Lambda_M w_t^n)}{(1-\alpha\bar{h})\mu_t w^n} \frac{f'(\gamma_M)}{\Lambda_M(1-\alpha)} \\
&= q - \theta - \frac{(1-\alpha) f'(\gamma_M)}{(1-\alpha\bar{h})\mu_t} \left(\frac{1}{1 - \frac{1}{1+\Lambda_M w_t^n}} \right) \\
&= q - \theta - \frac{(1-\alpha) f'(\gamma_M)}{(1-\alpha\bar{h})\mu_t} \left(\frac{1}{1 - (\mu^* - \mu_t)\frac{(1-\bar{h})}{(1-\alpha\bar{h})}} \right)
\end{aligned}$$

Note that in the second line we use relationship in (3.16) and in the third line, both (3.20) with (3.24) are used.

There are two things can be noted from the above:

- Firstly, the change in q only depends on the change in the markup ratio.
- Secondly, as $\mu_t \to \mu^*$ then along the balanced growth path, q also converges to a constant value:

  $$q^* = \theta + \frac{(1-\alpha) f'(\gamma_M)}{(1-\alpha\bar{h})\mu^*}$$

  which is consistent with what we noted before in (3.30).

As a result, to have the convergence in q we need to find the rate of change in markup rate over time.

Growth rate of mark-up ratio

From the expression of markups in (3.17), use the relationship between Multiplicative Robots and non-human wealth in (3.24) we have:

$$
\begin{aligned}
\mu_t &= \mu^* - \frac{1-\alpha}{(1-\bar{h})\frac{(1-\alpha)(1+\Lambda_M w^n)}{(1-\alpha\bar{h})}} = \mu^* - \frac{(1-\alpha\bar{h})}{(1-\bar{h})(1+\Lambda_M w^n)} \\
\Leftrightarrow \quad \dot{\mu}_t &= \frac{\Lambda_M \dot{w}_t^n}{(1+\Lambda_M w_t^n)^2}\frac{1-\alpha\bar{h}}{(1-\bar{h})} = \frac{\Lambda_M \dot{w}_t^n}{(1+\Lambda_M w_t^n)}\frac{1-\alpha\bar{h}}{(1-\bar{h})}\frac{(\mu^*-\mu_t)(1-\bar{h})}{(1-\alpha\bar{h})} \\
&= (\mu^*-\mu_t)\frac{\Lambda_M \dot{w}_t^n}{(1+\Lambda_M w_t^n)} \qquad (3.33)
\end{aligned}
$$

Substitute the expression for non-human wealth accumulation and $c = qw^n$ we have:

$$
\begin{aligned}
\dot{\mu}_t &= \Lambda_M(\mu^*-\mu_t)\left[\frac{\frac{1-\alpha}{(1-\alpha\bar{h})}(1+\Lambda_M w_t^n)H(1-\bar{h})f(\gamma_M) - \delta w_t^n - q_t w_t^n}{(1+\Lambda_M w_t^n)}\right] \\
&= \Lambda_M(\mu^*-\mu_t)\left[(1-\bar{h})\frac{1-\alpha}{(1-\alpha\bar{h})}Hf(\gamma_M) - \frac{(\delta+q_t)w_t^n}{(1+\Lambda_M w_t^n)}\right] \\
&= (\mu^*-\mu_t)\left[(1-\bar{h})\frac{1-\alpha}{(1-\alpha\bar{h})}\Lambda_M Hf(\gamma_M) - (\delta+q_t)\left(1-\frac{1}{1+\Lambda_M w_t^n}\right)\right] \\
&= (\mu^*-\mu_t)\left[(1-\bar{h})\frac{1-\alpha}{(1-\alpha\bar{h})}\frac{f'(\gamma_M)}{1-\alpha} - (\delta+q_t)\left(1-(\mu^*-\mu_t)\frac{(1-\bar{h})}{1-\alpha\bar{h}}\right)\right] \\
&= (\mu^*-\mu_t)\left[\frac{1-\bar{h}}{1-\alpha\bar{h}}f'(\gamma_M) + (\delta+q_t)\left(\mu^*\frac{(1-\bar{h})}{1-\alpha\bar{h}} - 1\right) - (\delta+q_t)\frac{(1-\bar{h})}{1-\alpha\bar{h}}\mu_t\right]
\end{aligned}
$$

Along the balanced growth path, the consumption growth rate is positive. The non-human wealth growth rate will converge to the same rate as consumption, so the second component is always positive. As a result along the balanced growth path, markups are constant at μ^*.

Linearization around balanced level

- ***For markup ratio***: Let denote $\dot{\mu}_t = \psi(\mu_t, q_t)$. We have the following:

$$\psi(\mu^*, q^*) = 0$$

$$\left.\frac{\partial \psi}{\partial \mu_t}\right|_{\mu^*,q^*} = \left[-\left(\frac{1-\bar{h}}{1-\alpha\bar{h}} f'(\gamma_M) - (\delta + q_t)\left(1 - (\mu^* - \mu_t)\frac{(1-\bar{h})}{1-\alpha\bar{h}}\right)\right) + \right.$$
$$\left.\left. + (\mu^* - \mu_t)(\delta + q_t)\frac{(1-\bar{h})}{1-\alpha\bar{h}}\right]\right|_{\mu^*,q^*}$$
$$= (q^* + \delta) - \frac{1-\bar{h}}{1-\alpha\bar{h}} f'(\gamma_M)$$

$$\left.\frac{\partial \psi}{\partial q_t}\right|_{\mu^*,q^*} = (\mu^* - \mu_t)\left.\left(1 - (\mu^* - \mu_t)\frac{(1-\bar{h})}{1-\alpha\bar{h}}\right)\right|_{\mu^*,q^*} = 0$$

Then by Taylor expansion for two variables we have:

$$\psi(\mu_t, q_t) \approx \psi(\mu^*, q^*) + (\mu_t - \mu^*)\frac{\partial \psi}{\partial \mu_t}(\mu^*, q^*) + (q_t - q^*)\frac{\partial \psi}{\partial q_t}(\mu^*, q^*)$$
$$= 0 + (\mu_t - \mu^*)\left((q^* + \delta) - \frac{1-\bar{h}}{1-\alpha\bar{h}} f'(\gamma_M)\right) + 0$$
$$= (\mu_t - \mu^*)\left((q^* + \delta) - \frac{1-\bar{h}}{1-\alpha\bar{h}} f'(\gamma_M)\right)$$

Let $A = (q^* + \delta) - \frac{1-\bar{h}}{1-\alpha\bar{h}} f'(\gamma_M)$, since μ_t increases over time (i.e. $\dot{\mu}_t > 0$ and $\mu_t < \mu^*$) then we need $A < 0$.

- ***For*** c/w ***ratio:*** Denote $\Phi(\mu_t, q_t) = \dot{q}_t$, we have:

$$\Phi(\mu_t, q_t) = q_t^2 - \left[\theta + \frac{(1-\alpha)f'(\gamma_M)}{(1-\alpha\bar{h})\mu_t}\left(\frac{1}{1 - (\mu^* - \mu_t)\frac{(1-\bar{h})}{(1-\alpha\bar{h})}}\right)\right] q_t$$

$$\Phi(\mu^*, q^*) = 0$$

$$\left.\frac{\partial \Phi}{\partial q_t}\right|_{\mu^*,q^*} = \left.\left[2q_t - \theta - \frac{(1-\alpha)f'(\gamma_M)}{(1-\alpha\bar{h})\mu_t}\left(\frac{1}{1 - (\mu^* - \mu_t)\frac{(1-\bar{h})}{(1-\alpha\bar{h})}}\right)\right]\right|_{\mu^*,q^*}$$
$$= 2q^* - \left(\theta + \frac{(1-\alpha)f'(\gamma_M)}{(1-\alpha\bar{h})\mu^*}\right) = q^*$$

$$\left.\frac{\partial \Phi}{\partial \mu_t}\right|_{\mu^*,q^*} = \left[-q_t \frac{(1-\alpha)}{(1-\alpha\bar{h})} f'(\gamma_M) \left(\frac{-1}{(\mu_t)^2} \frac{1}{1-(\mu^*-\mu_t)\frac{(1-\bar{h})}{(1-\alpha\bar{h})}} \right.\right.$$

$$\left.\left.\left. + \frac{1}{\mu_t} \frac{-1}{\left(1-(\mu^*-\mu_t)\frac{(1-\bar{h})}{(1-\alpha\bar{h})}\right)^2} \right) \right] \right|_{\mu^*,q^*}$$

$$= q^* \frac{(1-\alpha) f'(\gamma_M)}{(1-\alpha\bar{h})(\mu^*)^2}(1+\mu^*)$$

Then by Taylor expansion, we have:

$$\Phi(\mu_t, q_t) \approx \Phi(\mu^*, q^*) + (q_t - q^*)\frac{\partial \Phi}{\partial q_t}(\mu^*, q^*) + (\mu_t - \mu^*)\frac{\partial \Phi}{\partial \mu_t}(\mu^*, q^*)$$

$$= (q_t - q^*)(q^*) + (\mu_t - \mu^*) q^* \frac{(1-\alpha) f'(\gamma_M)}{(1-\alpha\bar{h})(\mu^*)^2}(1+\mu^*)$$

Now we have a system of two differential equations:

$$\begin{cases} \dot{\mu}_t &= A(\mu_t - \mu^*) & (1) \\ \dot{q}_t &= q^*(q_t - q^*) + B(\mu_t - \mu^*) & (2) \end{cases}$$

with $A = (q^* + \delta) - \frac{1-\bar{h}}{1-\alpha\bar{h}} f'(\gamma_M)$ and $B = (1-\alpha) f'(\gamma_M) \frac{q^*}{(1-\alpha\bar{h})(\mu^*)^2}(1+\mu^*)$

The solution for the first equation is:

$$\mu_t - \mu^* = (\mu_0 - \mu^*) e^{At}$$

Plug this result into equation (2) we have a first-order linear differential equation for q_t:

$$\dot{q}_t = q^* q_t + \left(B(\mu_0 - \mu^*) e^{At} - (q^*)^2 \right)$$

The solution for this equation is:

$$\begin{aligned}
q_t &= q_0 e^{q^*t} + e^{q^*t}\int_0^t e^{-q^*s}\left(B(\mu_0-\mu^*)e^{As}-(q^*)^2\right)ds \\
&= q_0 e^{q^*t} + e^{q^*t}\left[B(\mu_0-\mu^*)\int_0^t e^{(A-q^*)s}ds-(q^*)^2\int_0^t e^{-q^*s}ds\right] \\
&= q_0 e^{q^*t} + e^{q^*t}\left[B(\mu_0-\mu^*)\frac{e^{(A-q^*)t}-1}{A-q^*}-(q^*)^2\frac{e^{-q^*t}-1}{-q^*}\right] \\
&= e^{q^*t}\left[q_0-\frac{B(\mu_0-\mu^*)}{A-q^*}+\frac{(q^*)^2}{-q^*}\right]+\frac{B(\mu_0-\mu^*)}{A-q^*}e^{At}-\frac{(q^*)^2}{-q^*} \\
&= e^{q^*t}\left[q_0-q^*-\frac{B(\mu_0-\mu^*)}{A-q^*}\right]+\frac{B(\mu_0-\mu^*)}{A-q^*}e^{At}+q^* \qquad (3.34)
\end{aligned}$$

Conditions for stability convergence

As we noted before, for μ_t to be able to converge to μ^* then we need $A<0$, thus the second component will converge to 0 as $t\to\infty$.

At the same time we have $e^{q^*}>1$ since $q^*>0$. Hence, in order to ensure the convergence of q_t to q^* the coefficient of e^{q^*t} needs to be zero as $e^{q^*}>1$. It follows that:

$$q_0 = q^* + \frac{B(\mu_0-\mu^*)}{A-q^*} \qquad (3.35)$$

Profitability Condition Revisit

The right chosen value for the initial q_0 will make sure that q converges over time. The stability of the whole model, however, depends on the initial condition of consumption which needs to satisfy (3.35):

$$c_0 = K_{ss}\left(q^*+\frac{B(\mu_0-\mu^*)}{A-q^*}\right)$$

However, there is still one more condition we have noted above but haven check. That is for markups convergence we need $A<0$. In this part, we will prove that this condition is taken care of by the profitability condition, i.e Multiplicative Robots have productivity higher than the minimum required level.

Remember the profitability condition in (3.22) is such that:

$$\frac{1}{\mu_t}f'(\gamma_M) > \frac{1}{\mu^*}f'(\gamma_M) > \frac{1}{\mu_{ss}}f'(k_{ss}) = \theta + \delta$$

Now for the stability of the model, we need $A < 0$, which mean:

$$\begin{aligned}
& q^* + \delta - \frac{1-\bar{h}}{1-\alpha\bar{h}}f'(\gamma_M) < 0 \\
\Leftrightarrow \quad & \theta + \frac{(1-\alpha)f'(\gamma_M)}{(1-\alpha\bar{h})\mu^*} + \delta - \frac{1-\bar{h}}{1-\alpha\bar{h}}f'(\gamma_M) < 0 \\
\Leftrightarrow \quad & \frac{f'(\gamma_M)}{(1-\alpha\bar{h})\mu^*}\left((1-\bar{h})\mu^* - (1-\alpha)\right) > \delta + \theta \\
\Leftrightarrow \quad & \frac{f'(\gamma_M)}{(1-\alpha\bar{h})\mu^*}(1-\alpha\bar{h}) > \delta + \theta
\end{aligned}$$

which is the same with profitability condition [10] .

Furthermore, we can show that $A < 0$ is also implied by the condition that wealth grows positively $\dot{w}_t^n > 0$.

$$\begin{aligned}
\dot{w}_t^n = \frac{(1-\alpha)}{(1-\alpha\bar{h})}(1+\Lambda_M w_t^n)H(1-\bar{h})f(\gamma_M) - (\delta+q_t)w_t^n > 0 \\
\frac{(1-\alpha)}{(1-\alpha\bar{h})}(1+\Lambda_M w_t^n)\frac{f'(\gamma_M)(1-\bar{h})}{\Lambda_M(1-\alpha)} > (\delta+q_t)w_t^n \\
\frac{1-\bar{h}}{1-\alpha\bar{h}}f'(\gamma_M) > (\delta+q_t)\frac{\Lambda_M w_t^n}{(1+\Lambda_M w_t^n)}
\end{aligned}$$

As $t \to \infty$, $\frac{\Lambda_M w_t^n}{(1+\Lambda_M w_t^n)} \to 1$ and $q \to q^*$ then the above inequality is equivalent to $A < 0$.

In summary, as long as the profitability condition holds, i.e. $\Lambda_M > \Lambda_M^*$ then we will have a positive non-human wealth growth as well as a stability convergence for markup rate and consumption.

[10] From formula for μ^* (3.19) we have:

$$\begin{aligned}
(1-\bar{h})\mu^* - (1-\alpha) &= (1-\bar{h})\left(\frac{(1-\alpha)(1+\bar{h})}{1-\bar{h}} + 1\right) - (1-\alpha) \\
&= (1-\alpha)\bar{h} + 1 - \bar{h} = 1 - \alpha\bar{h}
\end{aligned}$$

3.5 Model Simulation and Results

3.5.1 Parameters Calibration

Apart from all the same parameters as used in the previous chapter as per Table 2.1, we have two more variables that need to determine the results: the robots' productivity Λ_M and the overhead labor cost $\bar{h}$.

Multiplicative Robots' Productivity

The minimum productivity for Multiplicative Robots is $\Lambda_M^* = 0.004$, meaning as long as Multiplicative Robots can help to increase at least 0.4% of human labor's productivity, it is profitable to use them.

By construction, the application of the Multiplication Robots will help to increase the productivity of the whole workforce, not just a single worker which is a very strong effect. Compared to the Additives Robots case, the effective capital intensity as in (3.6) is already very low due to the fraction $\frac{1}{H}$. Hence we could expect stronger quantitative effects compared to the previous chapter. Furthermore, a higher value of Λ_M, as in the previous chapter will generate even higher quantitative effects. Therefore, we chose a very conservative rate of 1% which can help to generate reasonable results.

For comparison purposes, we will use Additives Robots at the productivity level of $\Lambda_A = 0.26$ slightly higher than in the previous chapter to account for the reduction due to the use of markup and overhead labor cost.

Calibrating value for overhead labor

As we assume the same $\bar{h}$ is hold even after the introduction of Multiplicative Robots. We need to calibrate the value for it. Note the following from the formula of markup ratio:

$$\begin{aligned} \frac{\bar{h}}{1-\bar{h}} = \frac{\mu_t - 1}{1-\alpha} &= \frac{1}{1-\bar{h}} + 1 \\ \Leftrightarrow \quad \bar{h} &= \frac{\mu_t - 1}{\mu_t - \alpha} \end{aligned} \tag{3.36}$$

Hence, we can infer the value of the overhead labor ratio $\bar{h}$ if we have the value of markup μ_{ss}.

Robert E. Hall recently reported the latest mean markup ratio of all industries is 1.31, grew from 1.12 in 1988 to 1.38 in 2015 (**Hall 1988**, **Hall 2018**). However, it is only the markup for labor cost while we are using μ as a **gross markup ratio**. Hence in terms of per labor we have,

$$\begin{aligned} &\frac{1}{\mu_G}(rk+v) = rk + \frac{1}{\mu_H}v \Leftrightarrow \frac{1}{\mu_G} f(k) = kf'(k) + \frac{1}{\mu_H}\big(f(k) - kf'(k)\big) \\ \Leftrightarrow \quad &\left(\frac{1}{\mu_G} - \frac{1}{\mu_H}\right) f(k) = kf'(k)\left(1 - \frac{1}{\mu_H}\right) \\ \Leftrightarrow \quad &\frac{1}{\mu_G} = \frac{kf'(k)}{f(k)}\left(1 - \frac{1}{\mu_H}\right) + \frac{1}{\mu_H} = \alpha + (1-\alpha)\frac{1}{\mu_H} \end{aligned} \tag{3.37}$$

Using the value suggested by **(Hall 2018)** we have $\mu_H = 1.31$ hence, the gross markup rate is $\mu_G = 1.18$.

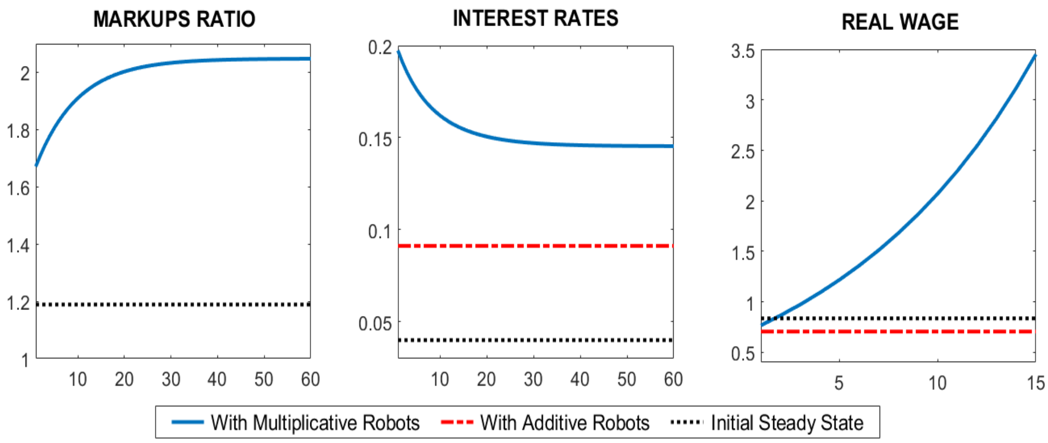
Substitute the value into (3.36), we obtain the value for overhead labor cost ratio of

$$\bar{h} = 0.2195$$

That means every unit of human labor put into the production only 79.1% effectively produce final products.

3.5.2 Stabilized Results in Comparison with Additives Robots

Market Rates



Note that for the markups under Additives Robots, it does not change even after the use of robots.

Figure 3.1: Market Rates after Multiplicative Robots

The Figure 3.1 shows the dynamics of market rates. Although the markup ratio in the initial steady-state stays at the low level of only $\mu_{ss} = 1.18$ [11] , after the introduction of Multiplicative Robots, markups rate jumps to $\mu_0 = 1.67$, keeps increasing over time, and then converges to value $\mu^* = 2.05$ after around 76 years. Intuitively, as we have more and more Multiplicative Robots, the higher the value of **Solow Residuals**, which requires the deeper depression of marginal products. While there is no such thing for Additives Robots, so the markups still stay the same with the initial level.

For the interest rate, even with so low productivity of Multiplicative Robots of 1%, the effective capital intensity was depressed lower compared to the Additives Robots of 26% productivity, making the interest rate increase to a much higher rate of around 19.73%, then reduce overtime to a constant level of 14.53% while interest

[11] As that the value we use to calibrate the value for overhead labor cost ratio $\bar{h}$.

rate only jump from steady stead level of 4% to 9.59% [12] with Additives Robots.

However, the main difference stays with the human labor wage. While wage reduces by 16.64% [13] from the initial steady-state with Additives Robots and then stays constant, it does not do the same under Multiplicative Robots.

From the right figure of Figure (3.1) we see that it is not only that the initial drop in wage is less for Multiplicative Robots (8.21%), but also as time goes by, with the help of Multiplicative Robots, human labor's productivity increases, wage increases very fast over time, surpass the initial steady-state level after only 3 years.

Consumption and Total Wealth

Similarly, the initial drop in consumption with Multiplicative Robots is less than the case with Additives Robots. From the initial consumption of $c_{ss} = 78.69$, the consumption drops by 58.09% to $c_0^A = 32.98$ in the model with Additives Robots while it drops by 26.59% to $c_0^M = 57.77$ in the model with Multiplicative Robots. We know that optimal consumption is θ fraction of the sum of total wealth and human wealth. Even though total wealth in the first period is still the same with initial steady-state total capital, human wealth depends on the household prediction. With Multiplicative Robots, although wages fall initially it grows fast afterward. However, interest rate also increases very high (Figure(3.1). In other words, further higher human income streams are discounted at a higher rate. Therefore, these high human incomes become less important to the household. That's why they still reduced their initial consumption. Nevertheless, the magnitude is less than under Additives Robots since Additives Robots depress the wage forever coupling with a higher interest rate compared to the initial steady state. After that, while

[12] The differences compared to the previous chapter is due to the use of overhead labor cost to adjust the results under Additives Robots.

[13] See footnote (12)

consumption grows at a constant rate of only 5.59% [14] , with Multiplication Robots, consumption grows at a much faster rate from nearly 20% converges to the constant level of 10.53% per period (which can be seen from Figure (3.4). This is the result of higher interest rates streams under Multiplicative Robots compared to Additives Robots case. The reasons why such a high consumption plan can be financed is because the growth in GDP as well as the speed of wealth accumulation are both very high, which can be seen in the Figure (3.2).

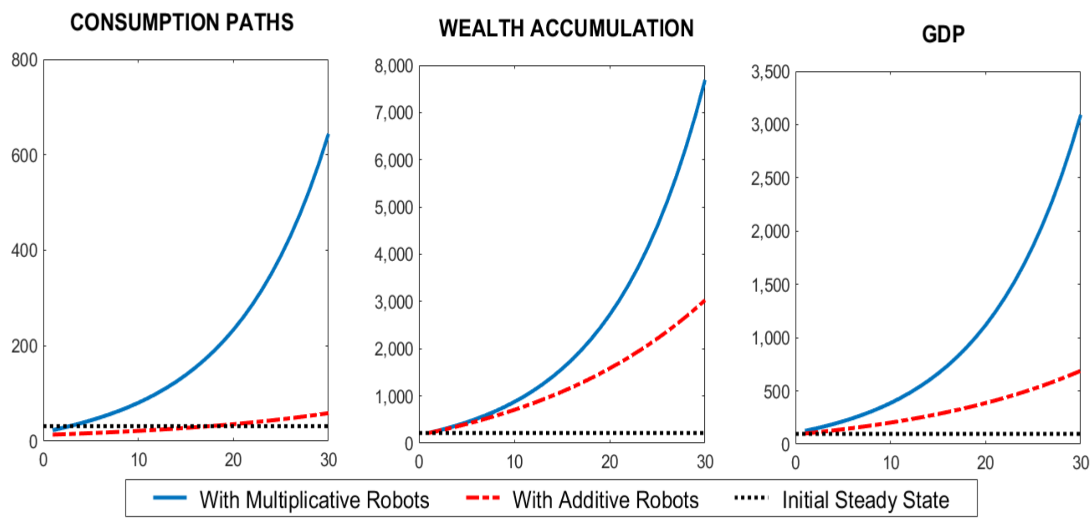


Figure 3.2: Consumption Path, Wealth Accumulation and GDP Growth

Stock of Robots vs. Conventional Machines

From Figure (3.3), we can see that as time goes by the gap between the stock of robots and the stock of conventional machines is widened in both types of robots. Multiplicative Robots even outnumber conventional machines right from the beginning. In terms of growth rates, robots even increase faster than non-human wealth. Although the Additives Robots does increase faster in the initial periods, the rate of its' change decreases faster making it is less than 10% after 12 years and converge

[14] See footnote (12)

to the same level with consumption growth of 5.59% [15] . While the multiplicative grows slower in the first place, the rate of change reduces slowly and converge to 10.53% after 77 years [16] . The economy will employ even more Multiplicative Robots than Additives Robots given our choice of robots' productivity. There is one more point worth noting that although during the first 10 years, Multiplicative Robots grows at lower rates, it generates much higher growth rates in non-human wealth compared to Additives Robots. This is because Multiplicative Robots are not the simple addition into the workforce but instead help to increase the productivity of the whole workforce which helps to generate much more output. This explains the fast-growing of the total wealth and GDP that we have seen in Figure (3.2).

[15] Same with the convergence of consumption path, which is the proof of the balanced growth path in the next part

[16] See footnote (15)

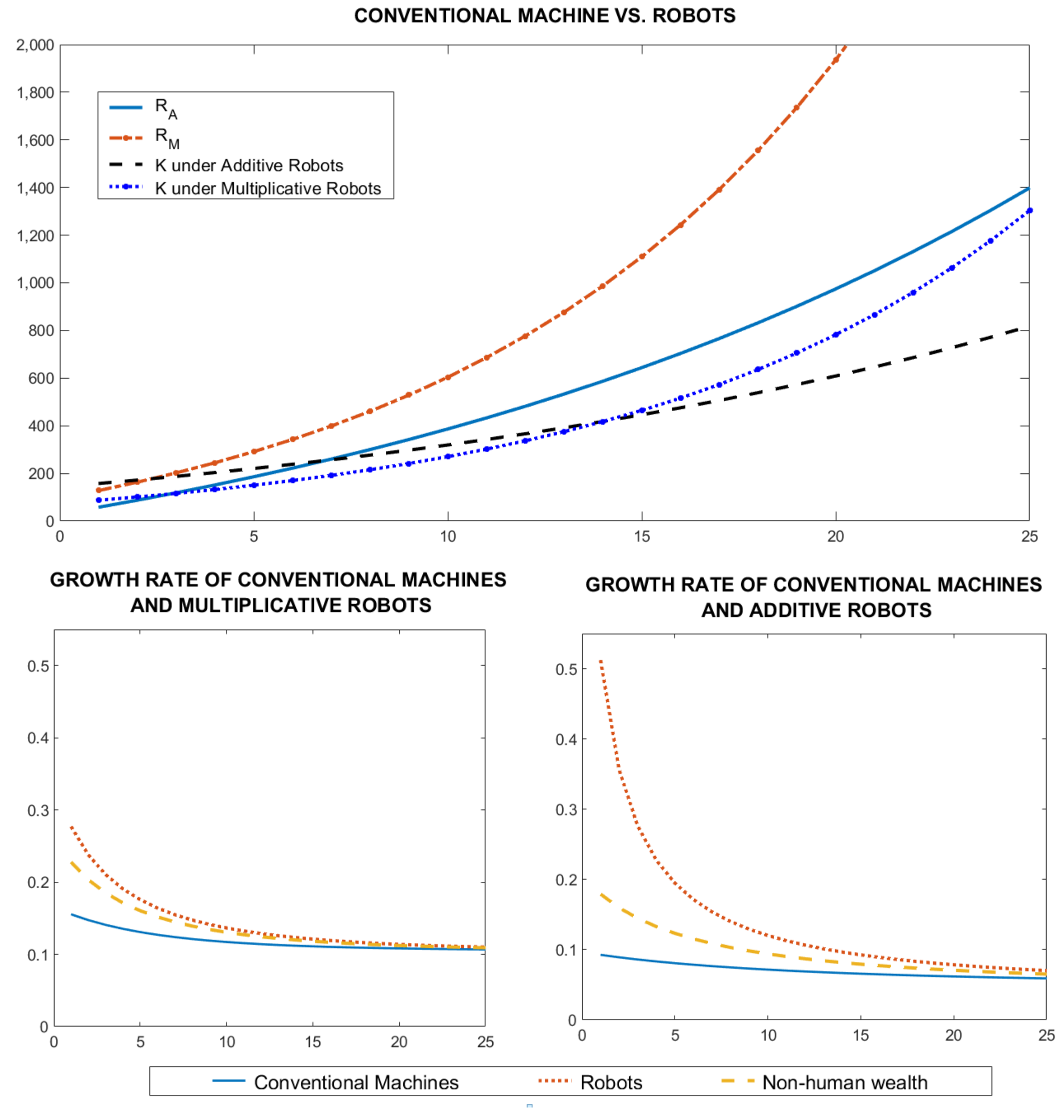


Figure 3.3: Robots vs. Conventional Machines

Balanced Growth Path

In the long run, for both cases, the economy converges to balanced growth paths where all the variables grow at the same rates. We already see that some proof of these in the previous parts, where over time, consumption growth rates are the same with the growth rate in stock of robots and non-human wealth, which can be visualized in (Figure (3.4)), making their ratios constant. As per Figure (3.5), consumption over wealth ratio while converges to the time preference rate θ with

Additives Robots, it converges to a much higher level (q^* is always θ plus a positive number).

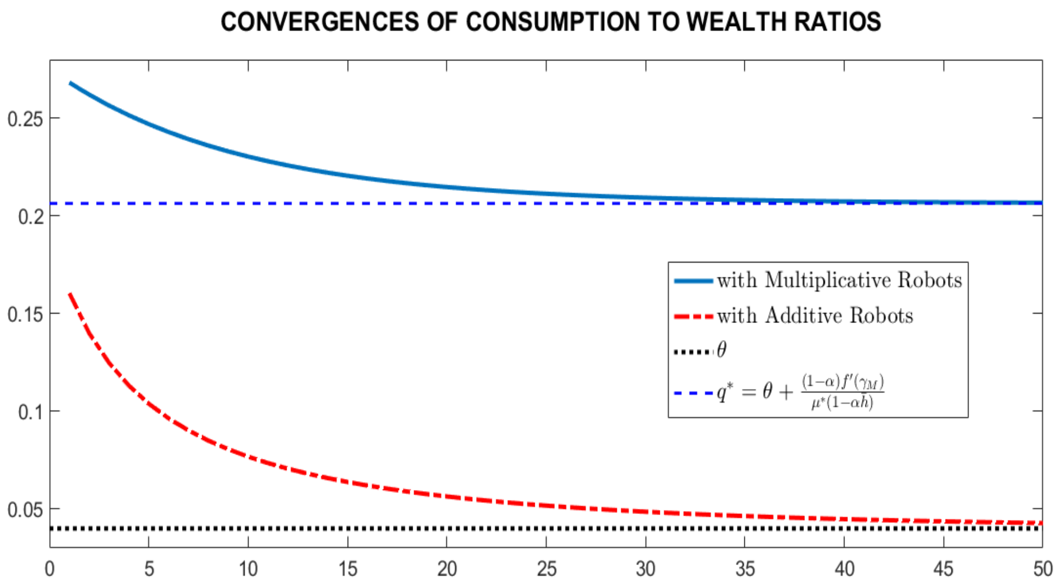


Figure 3.4: Growth rates of Consumption and Total Wealth

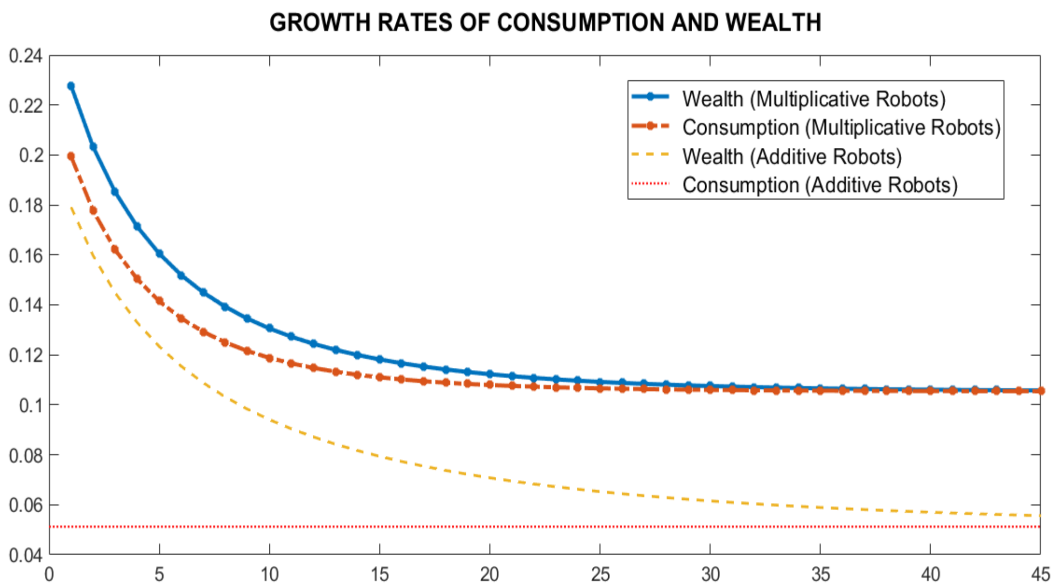


Figure 3.5: Convergence of Consumption over Wealth Ratio

The higher consumption stream under Multiplicative Robots can also be seen through the savings rate convergences. Over time, the household saves more and more under Additives Robots, while they spend more and more under Multiplicative Robots.

From the below figure, after reaching the balanced growth path, the household saves only 47.95% of their incomes under Multiplicative Robots, while they save 81.4% under Additives Robots.

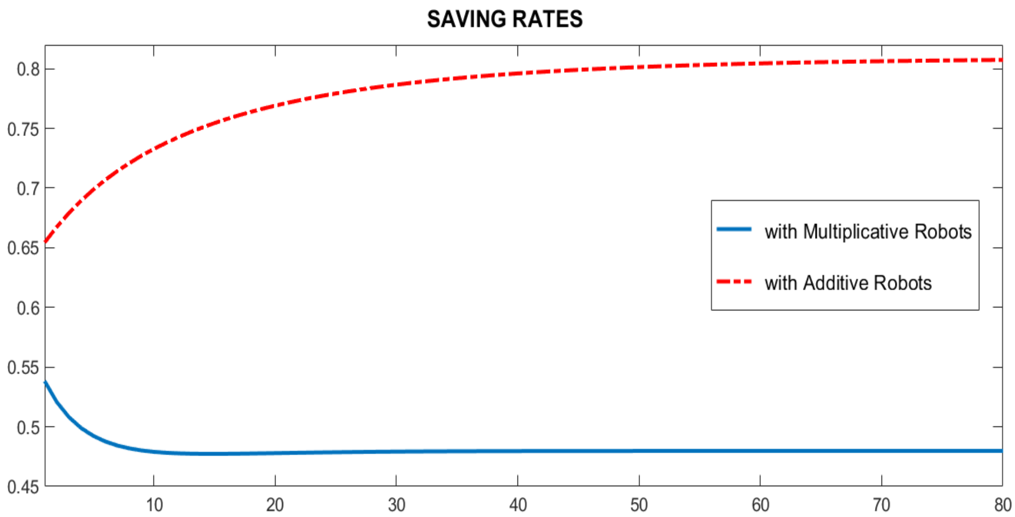


Figure 3.6: Convergence to Constant Saving Rates

This can be explained that under Multiplicative Robots, household human wealth (labor income) accumulates over time since they can expect higher and higher wages. On the other hand, under Additives Robots, wage stays constant, their human wealth is constant. Therefore, the only way for a household to accumulate more wealth is to save more (meaning they reinvest more into the capital and robots). This can be seen through Figure (3.7) of human income shares in the national income. As H stays constant, the wage is also constant while output grows as the stock of robots grows, the human shares in national incomes reduces toward zero in the long run for the model with Additives Robots. While in the model with Multiplicative Robots, these shares converge to a constant positive ratio (0.419).

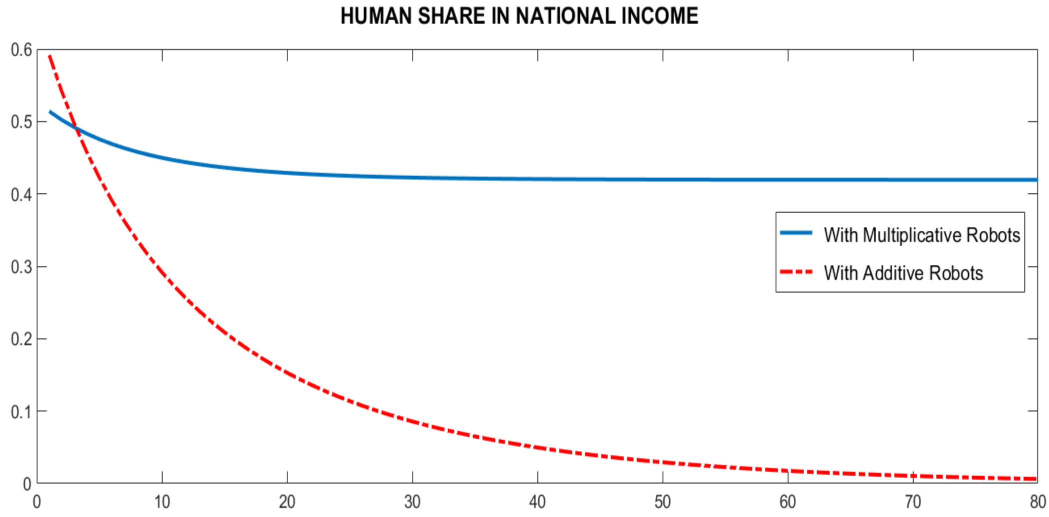


Figure 3.7: Human Share in National Incomes

3.6 Conclusion

In this chapter, we have seen that for producers to utilize Multiplicative Robots, there is not only profitability condition but also participation conditions. While profitability condition requires that the new rate of return (either on conventional machines or robots) is higher than the rate in the initial steady-state, making interest rate jumps to a very high level of 19.7% from the initial value of 4%. However, that is not enough for producers to decide whether to use robots. That is because of the way Multiplicative Robots affect real wages. Human labor, under the support of robots, is now having much higher productivity and will demand much higher wages. Unlike the Additive Robots case that real wage reduced permanently by 16.6%, not only the initial drop is less under Multiplicative Robots (8.21%) but it increases fast after that, surpassing the initial steady-state level only after 3 years.

The extra output created by higher efficient labor is hence used to pay higher wages for labor. There is no resource left to finance the cost of Multiplicative Robots [17].

[17] This is due to the use of a constant return to scale production function.

The use of markup is then necessary, to redistribute resources from conventional capital and labor to Multiplicative Robots. Producers use the same markup rate to adjust down the marginal products for conventional machines and labor to partially pay returns on robots. The more robots are used, the higher the markup ratios before it converges to a constant value along the balanced growth path.

Besides, to have markups even from the initial steady-state, I introduce the overhead labor cost into the model. The way overhead labor cost requires markups is similar but in the opposite way. Overhead labor reduces effective labor, but producers still have to pay full labor costs, causing not enough output produced. Hence, rates of return are only a fraction of marginal products.

Unlike the model with Additive Robots where we have closed-form solutions for the first-period variables. To have a stable convergence in the longterm, we need to use a numerical method for the model with Multiplicative Robots. This is because with wages keep increasing infinitely, there is no way to estimate the human wealth at any time. No matter how good is our estimation there will always be an error in obtaining the true value for first-period value. Theoretically, we know that markup ratios and consumption over wealth ratio need to converge. Hence, I need to use the linearization of these variables to derive the movement of other variables.

Compared to Additive Robots, Multiplicative Robots create much better economic growth. Not only first-period consumption drops only 26.6% (vs. 58.1% under Additive Robots), but also later, consumption, total wealth, and total output grow at a much faster rate which converges to 10.53% along the balanced growth path (vs. 5.59% under Additive Robots).

For both cases, the stock of robots increases even faster than total wealth. Additive Robots increase very fast in the first periods, from more than 50% compared to around 30% only of Multiplicative Robots. However, total wealth increases faster with Multiplicative Robots, making it a more effective investment. Over time, the growth rate of Additive Robots reduces fast, being less than 10% in just 12 years, while it takes around 77 years for Multiplicative Robots' growth rates to convert to 10.53% in the balanced growth path.

Another worth mentioning point is that Multiplicative Robots is a more effective investment even though the savings rate from the household is much lower than what they do under Additive Robots. Real wage reduces and stays constant over time under Additive Robots, while workers can expect higher and higher wages with the support of Multiplicative Robots. Hence, not just total wealth but human wealth is also accumulated over time. Therefore the only way for households under Additive Robots to invest more (or maintain their investment) is to save more.

Chapter 4

Two Kinds of Robots with Diminishing Population

4.1 Additive Robots vs. Multiplicative Robots

In this last chapter, I include both kinds of robots at the same time in the economy to study the combined net effect of using both types of robots in the economy. Before that let do a summary of the main results that we obtained in the previous two chapters.

Looking at table 4.1, both kinds of robots - satisfied profitability condition - when used would create economic growth. We will have perpetual growth in consumption, in total wealth (and hence lifetime welfare). However, the main difference lies with the real wage. The two types of robots have two different mechanisms and thus create two opposite forces on human labor's real wage.

Additive Robots are just a simple extension of the labor force which is similar to an increase in population. Hence, in the Production Inputs Frontier, the new market equilibrium point is achieved just by moving along (down) the curve, resulting in an increase in the interest rate and a decline in wage. Multiplicative Robots, on the other hand, works like technology progress factor, increasing the productivity of human labor even without any population growth. Hence, the application of Multiplicative Robots shifts the curve outward. In the short run, it might still result in a higher interest rate and lower wage. But with the constant shift of the frontier outward, there will be a point where wage starts to increase higher than the initial

Variables	Multiplicative Robots	Additive Robots
Markups	Increases over time and then converges to a constant level	Does not change (from the initial steady state level)
Interest Rates	• Jumps to a higher level $r \geq r_{ss}$ (under the profitability condition) • Later, decreases and converges (due to the behavior of markups) but still higher than initial level.	• Jumps to a **permanently** higher level $r \geq r_{ss}$ (under the profitability condition)
Real Wage	• Might drop down in the first period • Rapidly grows as the stock of robots increases, pushing up the human's productivity	• Depressed to a **permanently** lower level
Consumption	• Drops in the first period (requires numerical method - use the convergence of consumption to wealth ratio - to identify) • Later, continue grows perpetually with **a positive but decreasing rate** • Growth rate converges to a constant level (same with wealth accumulation growth rate) during balanced growth path.	• Drops initially (due to lower expected human wealth) • Grows at a **constant** rate (same with wealth accumulation growth rate.

Table 4.1: Comparison of Main Results under Additive Robots vs. Multiplicative Robots

equilibrium point, making both wage and interest rate increase over time.

Hence, it would be a very interesting question to see which force is stronger and under which conditions we can ensure that wages will not fall.

Another point for this chapter, motivated by the case of Japan, a country is suffering from negative population growth. Japan is not the only country experiencing a shrinking population but is one of the largest declines[1]

[1] https://www.ft.com/content/29d594fa-5cf2-11e9-9dde-7aedca0a081a

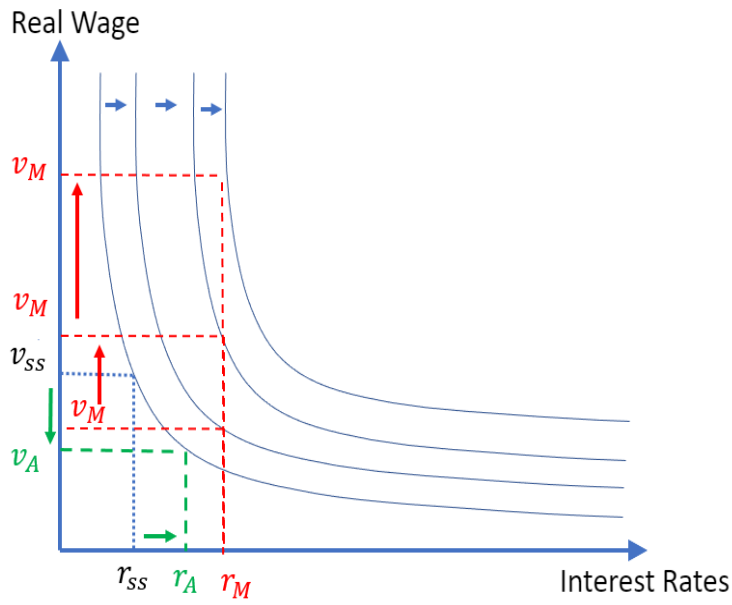


Figure 4.1: Production Factor Frontier

The change in the interest rate in both cases are due to the behavior of markups. In this graph, we abstract it away from those effects. Hence, the application of robots will result in an immediate jump in the interest rate only.

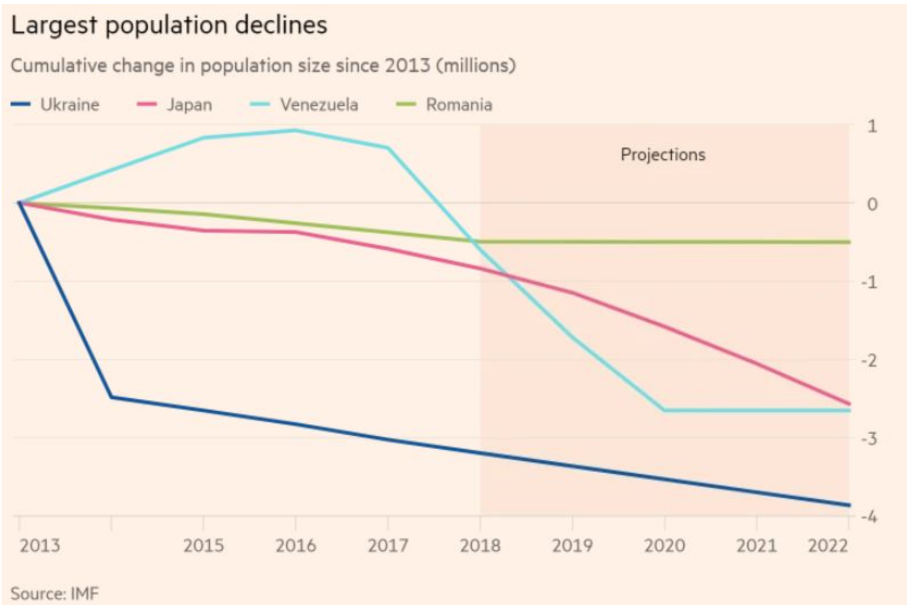


Figure 4.2: World Largest Population Declines

Despite that, Japan still enjoys economic growth, a high living standard, and very low interest rates. At the same time, Japan is also one of the countries with the highest robot application. By considering the scenario when the population is shrinking, we can see if my model of both types of robots can explain the case for Japan that with the introduction of robots can help send the economy into a spiral of perpetual growth, increases of wages and low interest rates.

4.2 Model Pre-Setup Conditions

4.2.1 Effective Labor Market

The first step in explaining the model is the re-categorization of the labor market, which enables the use of both kinds of robots. Additives Robots and Multiplicative Robots transform the human labor market in two different ways. As a result, for both of them being used, we need to differentiate two different types of labor or two types of jobs.

Labor type A:

This type of labor (which I refer to as worker type A) does jobs (which I refer to as type A jobs) that are repetitive and easy to be done or substituted by Additive Robots. Denote the Effective Labor type A as L^A then it is the combination of worker type A and Additive Robots. We follow the usual setup used in Chapter 2:

$$L_t^A = \Lambda_A R_t^A + H_t^A \tag{4.1}$$

in which R^A is the number of Additive Robots, Λ_A is Additive Robots' productivity compared to worker type A, and H^A is the number of worker type A.

Labor type M:

This type of labor (which I refer to as Worker type M) does jobs (which I refer to as type M jobs) that are not be able to fulfill do by robots. Hence, Multiplicative Robots can help to increase worker type M's productivity only but can not be able to replace them. Denote the Effective Labor type M as L^M then we follow the setup used in chapter 3:

$$L_t^M = \left(1+\Lambda_M R_t^M\right) H_t^M \tag{4.2}$$

in which R^M is the number of Multiplicative Robots, Λ_M is Additive Robots compared to worker type M, and H^M is the number of Worker type M.

Then, the **Total Effective Labor Market** is the function $G\left(L^A, L^M\right)$ with $G(.)$ is homogeneous of degree 1 such that:

$$L = G\left(L^A, L^M\right) = \left(L^A\right)^{\rho}\left(L^M\right)^{1-\rho} \tag{4.3}$$

Denote **Effective Labor Ratio** as $\phi_t = \frac{L^A}{L^M}$ then by constant return to scale properties, we have:

$$G\left(L^A, L^M\right) = L^M g(\phi) \quad \text{with } g(\phi) = \phi^{\rho} \tag{4.4}$$

Marginal Effective Labor of Worker (*MeLW*)- Marginal Effective Labor of Robots (*MeLR*) measure how many units of Effective Labor L we can achieve with every one extra unit of each type of labor and each type of robots respectively.

$$\begin{aligned} MeLW_A &= \frac{\partial G\left(L^A, L^M\right)}{\partial H^A} = \frac{\partial\left(L^M g(\phi)\right)}{\partial H^A} = L^M g'(\phi)\frac{\partial \phi}{\partial H^A} \\ &= L^M g'(\phi)\frac{1}{L^M}\frac{\partial L^A}{\partial H^A} = g'(\phi) \end{aligned}$$

$$\begin{aligned} MeLW_M &= \frac{\partial G\left(L^A, L^M\right)}{\partial H^M} = \frac{\partial\left(L^M g(\phi)\right)}{\partial H^M} = \frac{\partial L^M}{\partial H^M} g(\phi) + L^M g'(\phi)\frac{\partial \phi}{\partial H^M} \\ &= \left(1+\Lambda_M R^M\right) g(\phi) + L^M g'(\phi) L^A \frac{-1}{(L^M)^2}\left(1+\Lambda_M R^M\right) \\ &= \left(1+\Lambda_M R^M\right)\left(g(\phi) - \phi g'(\phi)\right) \\ &= \left(1+\Lambda_M R^M\right)(1-\rho) g(\phi) \end{aligned}$$

$$\begin{aligned} MeLR_A &= \frac{\partial G\left(L^A, L^M\right)}{\partial R^A} = \frac{\partial \left(L^M g(\phi)\right)}{\partial R^A} = L^M g'(\phi) \frac{\partial \phi}{\partial R^A} \\ &= L^M g'(\phi) \frac{1}{L^M} \frac{\partial L^A}{\partial R^A} = \Lambda_A g'(\phi) \end{aligned}$$

$$\begin{aligned} MeLR_M &= \frac{\partial G\left(L^A, L^M\right)}{\partial R^M} = \frac{\partial \left(L^M g(\phi)\right)}{\partial R^M} = \frac{\partial L^M}{\partial R^M} g(\phi) + L^M g'(\phi) \frac{\partial \phi}{\partial R^M} \\ &= \Lambda_M H^M g(\phi) + L^M g'(\phi) \frac{L^A}{H^M} \frac{-\Lambda_M}{\left(1 + \Lambda_M R^M\right)^2} = \Lambda_M H^M \left(g(\phi) - \phi g'(\phi)\right) \\ &= \Lambda_M H^M (1 - \rho) g(\phi) \end{aligned}$$

Lastly, we have:

$$H^A + H^M = H$$

4.2.2 Production Function and Total Marginal Products

We continue with the simple Cobb Douglas production function $F(K,L) = K^{\alpha} L^{1-\alpha}$. The only difference is now L is the Total Effective Labor function: $L = G\left(L^A, L^M\right)$.

Denote the **Effective Capital Intensity** as normal $k = \frac{K}{L}$ then:

$$F(K,L) = Lf(k) \text{ with } f(k) = k^{\alpha}$$

Hence, we have the following usual results:

Marginal Product of Conventional Machines

$$MPK = \frac{\partial F(K,L)}{\partial K} = f'(k) \tag{4.5}$$

Marginal Product of Total Effective Labor

$$MPL = \frac{\partial F(K,L)}{\partial L} = f'(k) - kf'(k) = (1-\alpha) f(k) \tag{4.6}$$

While MPK is straightforward, MPL needs some further explanation. MPL measures how many final products are produced if there is one more extra unit of **effective labor**. Total Effective Labor can be changed by the changes of four components: worker type A, worker type B, Additive Robots, and Multiplicative Robots.

Take Additive Robots as an example, the mechanism is that: one extra unit of Additive Robots will create $MeLR_A$ units of Effective Labor. In turn, each unit of this extra effective labor will create MPL units of the final product. Hence, the **Total Marginal Product of Additive Robots** should be:

$$MPR_A = MPL \times MeLR_A = \Lambda_A(1-\alpha)f(k)g'(\phi) \tag{4.7}$$

With the same logic we have the followings:

Total Marginal Product of Multiplicative Robots

$$MPR_M = MPL \times MeLR_M = \Lambda_M H^M(1-\alpha)f(k)(1-\rho)g(\phi) \tag{4.8}$$

Total Marginal Product of Worker type A

$$MPL_A = MPL \times MeLW_A = (1-\alpha)f(k)g'(\phi) \tag{4.9}$$

Total Marginal Product of Worker type M

$$MPL_M = MPL \times MeLW_M = \left(1+\Lambda_M R^M\right)(1-\alpha)f(k)(1-\rho)g(\phi) \tag{4.10}$$

4.2.3 Overhead Labor and Return Rates

As explained in the chapter 3, whenever we have Multiplicative Robots in the model, we encounter the issue with **"Solow residuals"** as outputs need to cover for both cost of Multiplicative Robots and increased wage for higher productivity human labor.

Hence, we need to use the markup rate to scale down the payments to all production inputs compared to their respective marginal product. In other words, return rates for all inputs are only the (same) fraction of marginal product.

We have the following system of equations of return rates:

$$\begin{aligned} r^K &= \frac{1}{\mu} MPK = \frac{1}{\mu} f'(k) \\ r^{R^A} &= \frac{1}{\mu} MPR_A = \frac{1}{\mu} \Lambda_A (1-\alpha) f(k) g'(\phi) \\ r^{R^M} &= \frac{1}{\mu} MPR_M = \frac{1}{\mu} \Lambda_M H^M (1-\alpha) f(k)(1-\rho) g(\phi) \\ v^A &= \frac{1}{\mu} MPL_A = \frac{1}{\mu} (1-\alpha) f(k) g'(\phi) \\ v^M &= \frac{1}{\mu} MPL_M = \frac{1}{\mu} \left(1+\Lambda_M R^M\right)(1-\alpha) f(k)(1-\rho) g(\phi) \end{aligned} \tag{4.11}$$

Again, we do not want markups to become abrupt after the introduction of robots. We would impost "**overhead labor**" even from the initial steady state. So that even in the initial state there exist markups.

Although all the marginal products above are not affected in formulas. Their changes are through the changes of effective labor ratio ϕ and effective capital intensity k since now only a fraction $(1-\bar{h})$ of labor is participating in the production (while $\bar{h}$ portion of time is wasted). Thus we have:

Effective Labor Ratio

$$\phi = \frac{L^A}{L^M} = \frac{\Lambda_A R^A + H^A(1-\bar{h})}{\left(1+\Lambda_M R^M\right) H^M (1-\bar{h})} \tag{4.12}$$

Total Effective Labor

$$\begin{aligned} L = G\left(L^A, L^M\right) &= \left(1+\Lambda_M R^M\right) H^M (1-\bar{h}) g(\phi) \\ &= g'(\phi) L^A + \left(g(\phi) - \phi g'(\phi)\right) L^M \\ &= \Lambda_A g'(\phi) R^A + g'(\phi) H^A (1-\bar{h}) + (1-\rho) g(\phi) \left(1+\Lambda_M R^M\right) H^M (1-\bar{h}) \end{aligned} \tag{4.13}$$

Effective Capital Intensity

$$k = \frac{K}{G\left(L^A, L^M\right)} = \frac{K}{\left(1+\Lambda_M R^M\right) H^M (1-\bar{h}) g(\phi)} \tag{4.14}$$

and **Total Output**

$$\begin{aligned} Y = F(K,L) = Lf(k) &= \left(1+\Lambda_M R^M\right) H^M (1-\bar{h}) g(\phi) f(k) \\ &= f'(k)K + (1-\alpha) f(k) \Bigg(g'(\phi) H^A (1-\bar{h}) + \Lambda_A g'(\phi) R^A + ... \\ &\qquad ... + (1-\rho) g(\phi) \left(1+\Lambda_M R^M\right) H^M (1-\bar{h}) \Bigg) \end{aligned}$$

4.3 Model Solving

4.3.1 Initial Steady State

Total Labor and Labor Multiplier

We assume that in the initial steady-state the effective labor takes a similar form.

$$L = \left(H^A\right)^{\rho} \left(H^M\right)^{1-\rho} \tag{4.15}$$

which is different from both of the previous chapters. Hence, we need to recalculate all the initial steady-state values.

For the baseline model, we consider the model under fixed H, before allowing H to be exogenously exponentially decline over time, meaning $H_t = H$ $\forall$t or $\frac{\dot{H}}{H} = 0$.

Denote capital intensity $k = \frac{K}{L}$, the profit maximization problem is

$$\max_{(K,H^A,H^M)} Lf(k) - r^K K - v^A H^A - v^M H^M$$

$$\text{such that: } L = \left(H^A\right)^{\rho} \left(H^M\right)^{1-\rho}$$

First order conditions:

$$
\begin{aligned}
r^K &= f'(k) \\
v^A &= \frac{\partial L}{\partial H^A} f(k) + L f'(k) \frac{\partial k}{\partial L} \frac{\partial L}{\partial H^A} \\
&= \frac{\partial L}{\partial H^A} \Big(f(k) - k f'(k) \Big) = \rho \left(\frac{H^M}{H^A} \right)^{1-\rho} \Big(f(k) - k f'(k) \Big) \\
v^M &= \frac{\partial L}{\partial H^M} f(k) + L f'(k) \frac{\partial k}{\partial L} \frac{\partial L}{\partial H^M} \\
&= \frac{\partial L}{\partial H^M} \Big(f(k) - k f'(k) \Big) = (1-\rho) \left(\frac{H^M}{H^A} \right)^{-\rho} \Big(f(k) - k f'(k) \Big)
\end{aligned}
$$

We assume that the labor market is perfectly mobile i.e. labor is freely movable between two types of jobs [2] . Hence, no arbitrage condition requires:

$$
v^A = v^M \quad \Leftrightarrow \quad \frac{H^A}{H^M} = \frac{\rho}{1-\rho} \quad \Leftrightarrow \quad H^A = \rho H \quad \text{and} \quad H^M = (1-\rho)H
$$

Then the total labor will be:

$$
L = H(\rho)^{\rho}(1-\rho)^{1-\rho} = (1-\rho)H \left(\frac{\rho}{1-\rho} \right)^{\rho}
$$

Denote the $\eta_L = \frac{L}{H} = \rho^{\rho}(1-\rho)^{1-\rho}$ as labor multiplier, which is a constant, then we have Total Effective Labor and real wage:

$$
\begin{aligned}
L &= \eta_L H \\
v &= \eta_L \Big(f(k) - k f'(k) \Big)
\end{aligned}
\tag{4.16}
$$

which are both constant since effective capital intensity k should be at the steady-state level.

Since η_L is a constant, the growth rate of total capital L will be the same with population growth, which is correct in the steady state:

$$
\frac{\dot{L}}{L} - \frac{\dot{H}}{H} = \frac{\dot{L}}{L} - n = 0
$$

[2] Will be discussed in details in the next section

Markup Rates

As explained in the chapter 3, we continue to use the overhead labor cost even in the initial steady state, meaning only $(1-\bar{h})L$ effectively produce final goods. Hence, the capital intensity is:

$$k = \frac{K}{(1-\bar{h})L} = \frac{K}{(1-\bar{h})\eta_L H}$$

And total output is:

$$Y = L(1-\bar{h})f(k) = \eta_L H(1-\bar{h})f(k)$$

At the same time, we know that we need to use the markups rate to adjust all the marginal products to achieve zero profit. Hence, we can infer the markup value as following [3] :

$$\begin{aligned}
& TC = \frac{1}{\mu}\Big(r^K K + vH\Big) = Y \\
\Leftrightarrow \quad & \mu Y = f'(k)K + \eta_L\Big(f(k) - kf'(k)\Big)H \\
& \quad = f'(k)K + \Big(f(k) - kf'(k)\Big)L(1-\bar{h}) + \Big(f(k) - kf'(k)\Big)L\bar{h} \\
& \quad = Y + \Big(f(k) - kf'(k)\Big)L\bar{h}
\end{aligned}$$

$$\Leftrightarrow \quad \mu - 1 = \frac{\Big(f(k) - kf'(k)\Big)L\bar{h}}{L(1-\bar{h})f(k)} \quad \Leftrightarrow \quad \mu = \frac{(1-\alpha)\bar{h}}{(1-\bar{h})} + 1 \tag{4.17}$$

the same expression that we got in the chapter 3.

Steady State Levels

The growth rate of consumption per working day:

$$\tilde{c} = \frac{c}{H} \quad \Leftrightarrow \quad \frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{c}}{c} - \frac{\dot{H}}{H} = \frac{1}{\mu}f'(k) - \delta - \theta - n \tag{4.18}$$

[3] where we use Euler theorem that

$$Y = F(K, L(1-\bar{h}) = f'(k)K + \big(f(k) - kf'(k)\big)L(1-\bar{h})$$

While Effective Capital Intensity k has growth rate of:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

On the other hand, from capital accumulation rules of production side we have:

$$\begin{aligned}
\frac{\dot{K}}{K} &= \frac{Y - \delta K - c}{K} = \frac{L(1-\bar{h})f(k) - \delta K - \tilde{c}H}{K} \\
&= \frac{L(1-\bar{h})f(k)}{kL(1-\bar{h})} - \frac{\tilde{c}H}{kH\eta_L(1-\bar{h})} - \delta = \frac{f(k)}{k} - \frac{\tilde{c}}{\eta_L(1-\bar{h})k} - \delta \\
\Leftrightarrow \quad \dot{k} &= f(k) - \frac{\tilde{c}}{\eta_L(1-\bar{h})} - (\delta + n)k
\end{aligned} \tag{4.19}$$

At the steady state we have $\frac{\dot{\tilde{c}}}{\tilde{c}} = \dot{k} = 0$. Hence the followings characterize the initial steady state:

$$\begin{aligned}
k_{ss} &= \left(\frac{\mu_{ss}(\delta + \theta + n)}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
\tilde{c}_{ss} &= (1-\bar{h})\eta_L\Big(f(k_{ss}) - (\delta + n)k_{ss}\Big) \\
r^K_{ss} &= \frac{1}{\mu_{ss}} f'(k_{ss}) = \delta + \theta + n \\
r_{ss} &= \theta + n \\
v_{ss} &= \frac{1}{\mu_{ss}} \eta_L (1-\alpha) f(k_{ss})
\end{aligned}$$

From the above expression, we see that all the variables will only stay constant when there is no change in population which translated into no changes in H. When H does change, although the capital intensity is constant, aggregate variables such as total wealth and consumption change at the same rate with population, keeping only the per working day (per H) variables are constant.

$$\begin{aligned}
K_t &= k_{ss}\eta_L(1-\bar{h})H_t \\
c_t &= H_t \tilde{c}_{ss}
\end{aligned}$$

4.3.2 Model Main Assumptions

Assumption 4.1: Labors can freely move between two types of jobs.

We can think that there are two types of jobs: type A and type M. Another alternative set up is that a representative worker will have two types of tasks for his daily work, one is type A tasks in which he can let Additive Robots replace him (depends on the robots' productivity Λ_A), other is type M tasks in which he has to do it but can be helped by Multiplicative Robots (with productivity Λ_M). Depends on the market equilibrium conditions (daily wages paid to each type of job), he will decide how much time he wants to spend for type A jobs: H^A and how much time he does type M jobs: H^M. Hence by the no-arbitrage condition, we require the following for all periods:

$$v^A = v^M \quad \Leftrightarrow \quad \frac{1}{\mu_t} MPL_A = \frac{1}{\mu_t} MPL_M$$

From (4.9) and (4.10) we have:

$$\begin{aligned}
(1-\alpha) f(k) g'(\phi) &= \left(1+\Lambda_M R^M\right)(1-\alpha) f(k)(1-\rho) g(\phi) \\
\Leftrightarrow \quad g'(\phi) &= \left(1+\Lambda_M R^M\right)(1-\rho) g(\phi) \qquad (4.20) \\
\Leftrightarrow \quad \rho(\phi)^{\rho-1} &= \left(1+\Lambda_M R^M\right)(1-\rho)(\phi)^{\rho} \\
\Leftrightarrow \quad \phi &= \left(\frac{\rho}{1-\rho}\right) \frac{1}{\left(1+\Lambda_M R^M\right)} \qquad (4.21)
\end{aligned}$$

Note that the above results only hold when we have perfect mobility between two types of jobs. Later, when we consider a different scenario, they might not behold.

Assumption 4.2 :Capital are perfectly malleable.

This assumption is a combination of Assumption (2.1) and Assumption (3.1) in two previous chapters. Capital is freely converted among conventional machines, Additive Robots, and Multiplicative Robots.

There are three implications under this assumption. Firstly, as long as we have both types of robots, the returns that investors earned from both of them need to be the same.

$$r^{R^M} = r^{R^A} \Leftrightarrow \frac{1}{\mu_t} MPR_A = \frac{1}{\mu_t} MPR_M$$

Substitute (4.7) and (4.8), we have the expression for capital ratio:

$$\begin{aligned} \Lambda_A (1-\alpha) f(k) g'(\phi) &= \Lambda_M H^M (1-\alpha) f(k)(1-\rho) g(\phi) \\ \Lambda_A g'(\phi) &= \Lambda_M H^M (1-\rho) g(\phi) \\ \phi &= \frac{\Lambda_A}{\Lambda_M} \left(\frac{\rho}{1-\rho} \right) \frac{1}{H^M} \end{aligned} \tag{4.22}$$

which will behold whenever we have both types of robots used in the economy.

The second implication requires that the rate of return for conventional machines needs to equate the rate of return for each type of robots. Use the formula for MPK and (4.7), the first equality give us effective capital intensity in terms of labor ratio:

$$\begin{aligned} r^K = r^{R^A} \quad &\Leftrightarrow \quad f'(k) = \Lambda_A (1-\alpha) f(k) g'(\phi) \\ &\Leftrightarrow \quad k = \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{1}{\rho \Lambda_A} \right) (\phi)^{1-\rho} \end{aligned} \tag{4.23}$$

Combine the formula for MPK and (4.8), we have another expression for the effective capital intensity in terms of labor ratio:

$$\begin{aligned} r^K = r^{R^M} \quad &\Leftrightarrow \quad f'(k) = \Lambda_M H^M (1-\alpha) f(k)(1-\rho) g(\phi) \\ &\Leftrightarrow \quad k = \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{1}{(1-\rho)\Lambda_M H^M} \right) (\phi)^{-\rho} \end{aligned} \tag{4.24}$$

Note that (4.23) and (4.24) are linked by (4.22). In the scenario when both types of robots are used, either (4.23) or (4.24) and (4.22) are enough. However, in scenario when only one type of robots is used, we need to use (4.22) and respective condition for that type of robots.

Assumption 4.3 : Additive Robots are perfect substitution for human labor.

This is the assumption we used in the second chapter. As long as Additive Robots is used, the no-arbitrage condition requires that the return rate of an Additive Robots is $\frac{1}{\Lambda_A}$ human labor do type A jobs.

By construction, it is achieved automatically:

$$v^A = (1-\alpha)f(k)g'(\phi)$$
$$r^{R^A} = \Lambda_A(1-\alpha)f(k)g'(\phi)$$

However, if both previous assumptions hold, we have wages for both types of workers are the same, and returns on both types of robots are also the same. Hence, the return on Multiplicative Robots also need to satisfy the following condition:

$$r^{R^M} = \Lambda_A v^M$$
$$\Lambda_M H^M(1-\alpha)f(k)(1-\rho)g(\phi) = \Lambda_A\left(1+\Lambda_M R^M\right)(1-\alpha)f(k)(1-\rho)g(\phi)$$
$$\Lambda_M H^M = \Lambda_A\left(1+\Lambda_M R^M\right) \tag{4.25}$$

4.3.3 Labor Market Adjustment

For convenience, in every period, we denote the fraction of human labors who do type M job is φ_t, meaning:

$$H_t^M = \varphi_t H \quad \Leftrightarrow \quad H_t^A = (1-\varphi_t)H$$

In this framework model, we focus on the scenario where we have both types of robots with perfect mobility for human labor. In other words, all assumptions in the previous section are held which can be used to identify the value and changes of φ_t.

Stock of Additive Robots

From effective labor ratio in (4.12) and (4.20), we have:

$$\begin{aligned}\phi &= \frac{\Lambda_A R^A + (1-\varphi)H(1-\bar{h})}{\left(1+\Lambda_M R^M\right)\varphi H(1-\bar{h})} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{1+\Lambda_M R^M}\\ \Leftrightarrow \quad & \Lambda_A R^A + (1-\varphi)H(1-\bar{h}) = \frac{\rho}{1-\rho}\varphi H(1-\bar{h})\\ \Leftrightarrow \quad & R^A = \frac{H}{\Lambda_A}(1-\bar{h})\left(\varphi\left(\frac{\rho}{(1-\rho)}+1\right)-1\right) = \frac{H}{\Lambda_A}(1-\bar{h})\left(\frac{\varphi}{(1-\rho)}-1\right)\end{aligned} \tag{4.26}$$

Stock of Conventional Machines

From (4.23), effective capital intensity in (4.14) and effective labor ratio (4.20) we have:

$$\begin{aligned}\frac{K}{\left(1+\Lambda_M R^M\right)\varphi H(1-\bar{h})g(\phi)} &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\rho\Lambda_A}\right)(\phi)^{1-\rho}\\ \Leftrightarrow \quad \frac{K}{\left(1+\Lambda_M R^M\right)\varphi H(1-\bar{h})} &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\rho\Lambda_A}\right)\phi\\ &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\rho\Lambda_A}\right)\left(\frac{\rho}{1-\rho}\right)\frac{1}{1+\Lambda_M R^M}\\ \Leftrightarrow \quad K = \varphi H\left(\frac{\alpha}{1-\alpha}\right)\frac{1-\bar{h}}{\Lambda_A(1-\rho)}\end{aligned} \tag{4.27}$$

Stock of Multiplicative Robots

Directly from (4.25) we have stock of Multiplicative Robots formula:

$$R^M = \left(\frac{\Lambda_M}{\Lambda_A}\varphi H - 1\right)\frac{1}{\Lambda_M} = \frac{1}{\Lambda_A}\varphi H - \frac{1}{\Lambda_M} \tag{4.28}$$

Total non-human wealth

$$\begin{aligned}
w^n &= \varphi H\left(\frac{1}{\Lambda_A}+\left(\frac{\alpha}{1-\alpha}\frac{1-\bar{h}}{\Lambda_A(1-\rho)}\right)+\frac{1}{\Lambda_A}\frac{(1-\bar{h}}{(1-\rho)}\right)-H\frac{1}{\Lambda_A}(1-\bar{h})-\frac{1}{\Lambda_M}\\
&=\frac{1}{\Lambda_A}\varphi H\left(1+\frac{1-\bar{h}}{1-\rho}\left(\frac{\alpha}{1-\alpha}+1\right)\right)-H\frac{1}{\Lambda_A}(1-\bar{h})-\frac{1}{\Lambda_M}\\
\Leftrightarrow 1+\Lambda_M w^n &= \frac{\Lambda_M}{\Lambda_A}\left(\varphi H\left(\frac{1-\bar{h}}{(1-\rho)(1-\alpha)}+1\right)-H(1-\bar{h})\right)\\
&= H\frac{\Lambda_M}{\Lambda_A}\left(\varphi\left(\frac{1-\bar{h}}{(1-\rho)(1-\alpha)}+1\right)-(1-\bar{h})\right) \qquad (4.29)
\end{aligned}$$

In the initial period, total wealth is still at K_{ss}, (4.29) results in the first period value φ_0. For consequent periods, wealth can be identified using the instantaneous flow of wealth $\dot{w}^n$ which helps us to calculate the value of φ_t in each period.

The important result we can observe from the above expressions is that φ_t needs to increase over time as wealth accumulates.

$$\dot{w} > 0 \quad \Leftrightarrow \quad \dot{\varphi} > 0 \qquad (4.30)$$

This is true for both cases of fixed H and diminishing H. When H decreases overtime, φ_t will need to increase even faster to keep the right-hand side still growing[4].

4.3.4 Markups and Market Rates during Labor Market Adjustment

Markups

In the previous section, we have obtained the value of φ_t, meaning we have the labor market allocation in each period. It follows that we can have the value of effective labor ratio ϕ_t in (4.22) and effective capital intensity k_t in (4.23) (by using with the value of R_t^M in (4.28)). To derive the interest rate and other return rates, we first need the expression for markups to adjust the marginal products system of equation

[4] Just for discussion if the economy is being able to keep population growth as normal, i.e. H increases over time, then there would be less pressure on the labor market adjustment to keep wealth grow.

(4.11).

We can derive the markup rate in the same manner in the chapter for Multiplicative robots. Below, I present an alternative way. Although they give the same result, this method shows the characteristic of markups. As all the marginal products are scaled down by the same markup, so is the total cost. And that lowered total cost needs to equate to actual total output under perfect competition. From another point of view, we need to scale up the actual total output to have the normal total cost (in the case we do not need to use markups) which is μY_t. In other words, the difference between that normal total cost with the actual output is due to the cost that needs to cover the overhead labor and to finance Multiplicative Robots if no markup is used.

Note that wages for both types of labor are same, and is equal to a Λ_A fraction of return on Multiplicative Robots which in turn equals to returns on conventional machines. Then we have:

$$\mu Y - Y = MPL \times H\bar{h} + MPR_M R^M = \frac{1}{\Lambda_A} f'(k) H\bar{h} + f'(k) R^M = f'(k)\left(\frac{1}{\Lambda_A} H\bar{h} + R^M\right)$$

$$\begin{aligned}
\mu - 1 &= \frac{f'(k)}{Y}\left(\frac{1}{\Lambda_A} H\bar{h} + R^M\right) = \frac{(1-\alpha) f(k) \Lambda_M H^M (1-\rho) g(\phi)}{(1+\Lambda_M R^M) H^M (1-\bar{h}) g(\phi) f(k)}\left(\frac{1}{\Lambda_A} H\bar{h} + R^M\right) \\
&= \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})} \frac{\Lambda_M}{1+\Lambda_M R^M}\left(\frac{1}{\Lambda_A} H\bar{h} + R^M\right) \\
&= \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}\left[1 - \frac{1}{1+\Lambda_M R^M}\left(1 - \frac{\Lambda_M H\bar{h}}{\Lambda_A}\right)\right]
\end{aligned}$$

Since we have all assumptions hold, we then can apply (4.25):

$$\begin{aligned}
\mu - 1 &= \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}\left[1 + \frac{\Lambda_A}{\Lambda_M \varphi H}\left(\frac{\Lambda_M H\bar{h}}{\Lambda_A} - 1\right)\right] \\
\Leftrightarrow \quad \mu &= \frac{1}{\varphi} \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}\left[\bar{h} - \frac{\Lambda_A}{\Lambda_M H}\right] + \left(\frac{(1-\alpha)(1-\rho)}{(1-\bar{h})} + 1\right) \qquad (4.31)
\end{aligned}$$

When H is fixed, the markup will only depend on the fraction φ of the human labor workforce that does type M jobs. However, the movement of markup depends on the sign of $\left(\frac{\Lambda_M H\bar{h} - \Lambda_A}{\Lambda_M H}\right)$. If it is negative then markups increase with the increase in φ and vice versa. The first component $\Lambda_M H\bar{h}$ is the increase in the overhead labor cost of the whole human workforce due to the increase in productivity under the support of Multiplicative Robots. In other words, although Multiplicative Robots help the human workforce H increases its productivity by $\Lambda_M H$, a $\bar{h}$ fraction of that increase is still wasted[5]. The higher this new overhead labor cost is, the higher the markup value is.

On the other hand, using a unit of Additive Robots does not encounter any extra overhead labor cost. As a result, using Additive Robots reduces the value of markup. The difference between these two forces determines the movement of markups[6].

Real Wage

Under two main assumptions, we know that both types of robots and conventional machines have the same return rate which equates $\frac{1}{\mu_t} f'(k_t)$ with k_t is the effective capital intensity of period t. At the same time wages for two types of jobs are also same and linked to robots and machine return rates by multiplier Λ_A. It follows that:

$$
\begin{aligned}
v = \frac{1}{\mu \Lambda_A} f'(k) \quad &\Leftrightarrow \quad \ln v = -\ln \mu + \ln\left(\frac{\alpha}{\Lambda_A}\right) + (\alpha - 1)\ln k \\
&\Leftrightarrow \quad d\ln v = -d\ln \mu - (1-\alpha) d\ln k \\
&\Leftrightarrow \quad \frac{\dot{v}}{v} = -\frac{\dot{\mu}}{\mu} - (1-\alpha)\frac{\dot{k}}{k}
\end{aligned}
$$

As μ is most likely increase over time, We need to examine what happens to the effective capital intensity k. From (4.23), we have:

$$
\begin{aligned}
&\ln k = \ln\left(\frac{\alpha}{1-\alpha}\frac{1}{\rho \Lambda_A}\right) + (1-\rho)\ln \phi \\
\Leftrightarrow \quad & d\ln k = (1-\rho) d\ln \phi \quad \Leftrightarrow \quad \frac{\dot{k}}{k} = (1-\rho)\frac{\dot{\phi}}{\phi}
\end{aligned}
\tag{4.32}
$$

[5] Under the assumption that human labor does not change their working pattern, i.e $\bar{h}$ is fixed.

[6] With our choice of parameter values, it is indeed negative. However, it is can be positive. In general, the condition for it to be positive is when we have a quite high level of overhead labor cost ratio.

Which in turns, depends on the rate of change for labor ratio ϕ. From (4.22), we have:

$$\ln\phi = \ln\left(\frac{\Lambda_A}{\Lambda_M}\frac{\rho}{1-\rho}\right) - \ln\varphi - \ln H$$

$$\Leftrightarrow \quad d\ln\phi = -d\ln\varphi - d\ln H \quad \Leftrightarrow \quad \frac{\dot{\phi}}{\phi} = -\frac{\dot{\varphi}}{\varphi} - \frac{\dot{H}}{H}$$

Hence, the rate of change for effective capital intensity is:

$$\frac{\dot{k}}{k} = -(1-\rho)\left(\frac{\dot{\varphi}}{\varphi} + \frac{\dot{H}}{H}\right) \tag{4.33}$$

Since $\rho - 1 < 0$, effective capital intensity moves opposite of labor market allocation φ and population. Previous analysis indicates that φ would increase. Thus, the effective capital intensity would reduce over time. In the scenario that $\frac{\dot{H}}{H} < 0$, the decrease of k will be slower.

The final rate of change of real wage is:

$$\frac{\dot{v}}{v} = -\frac{\dot{\mu}}{\mu} + (1-\alpha)(1-\rho)\left(\frac{\dot{\varphi}}{\varphi} + \frac{\dot{H}}{H}\right) \tag{4.34}$$

When H is fixed, as the labor market is adjusting, the increase of φ results in the increase in wage. However, since wage is adjusted using μ to finance overhead cost and Multiplicative Robots, the increase in μ will slow down the wage expansion.

When H is decreasing, it offsets partially the effect of labor market adjustment and also slows down wage expansion. If we ignore the effect from markups [7] , real wage will only not increase under the condition that:

$$\frac{\dot{\varphi}}{\varphi} + \frac{\dot{H}}{H} < 0 \quad \Leftrightarrow \quad \frac{\dot{\varphi}}{\varphi} < -n$$

The above condition says that wage will decline only when the rate at which the labor force shifting from type A to type M jobs is slower than the rate of decrease in population. That means the movement of wages depends on the effectiveness of

[7] In either case when the real wage is increasing or decreasing, markups just mitigate the changes of wages by diverting partially to return rates of other inputs.

the labor market.

Interest Rate

Since $r = r^K - \delta$, to see the change of interest rate, we need to examine the expression for the return rate of conventional machines:

$$\begin{aligned} r^K &= \frac{1}{\mu_t} f'(k_t) = \frac{1}{\mu_t} \alpha k^{\alpha-1} \\ \Leftrightarrow \quad \frac{\dot{r}^K}{r^K} &= -\frac{\dot{\mu}}{\mu} + (\alpha-1)\frac{\dot{k}}{k} = -\frac{\dot{\mu}}{\mu} + (1-\alpha)(1-\rho)\left(\frac{\dot{\varphi}}{\varphi} + \frac{\dot{H}}{H}\right) \end{aligned} \tag{4.35}$$

which is exactly same with (4.34). It is not surprising since return rate are linked to wage by no arbitrage condition for Additive Robots.

$$r = r^K - \delta = \Lambda_A v - \delta$$

Thus, the movement of all return rates are exactly same with changes of real wage. It follows that interest rate should also increase over time during the period of labor market adjustment.

In summary, during the adjustment phase of the labor market, real wage will most likely increase although the increase of the markup rates will mitigate the movement of wage. Since during this phase, we still have type A labor, all the market return rates will be binding to the wage of labor type A through Λ_A. That is why all the market rates will move parallel to the real wage. The final result will still depend on the effectiveness of the labor market (how quickly the labor is shifting from type A to type M) to win over the effect from the markup rates. We also have shown that if the labor market shifting rate is even slower than the decrease in the population, it is highly that real wage (and thus all other rates) will decrease.

4.3.5 Long run after Labor Market Adjustment

Recalculation after φ_t reaches its upper bound?

So far, all the above analyses hold during the process φ increases or the process that the labor market is adjusting. The next question we would like to ask is what happens if all the labor force finishes moving. It does not simply mean that we just impose the value $\varphi = 1$ in all the above analysis. Because when all the human labor force has moved, there would be no human worker who does type A jobs $H_A = 0$. That also means the no-arbitrage condition which makes real wages for two types of jobs equate in (4.20) is no longer valid, as well as comparable returns between robots and human (who do type A jobs). That is why we are not able to generate the relationship between R^M and φH (now is H) as in (4.25) while (4.22) and (4.23) both still hold, only instead of H^M is full H.

Total Wealth, Conventional Machines and Robots

As there is no more human labor doing type A job, labor ratio and and effective labor: will be:

$$\phi = \frac{\Lambda_A R^A}{\left(1+\Lambda_M R^M\right) H(1-\bar{h})} \qquad L = \left(1+\Lambda_M R^M\right) H(1-\bar{h}) g(\phi)$$

With these new relationships, we derive the new expression for conventional machines and two types of robots. From the expression of effective capital intensity and result in and (4.22) effective labor we have the expression of conventional machines.

$$k = \frac{K}{L} = \frac{K}{(1+\Lambda_M R^M)\,H(1-\bar{h})g(\phi)} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\rho\Lambda_A}\right)\phi^{1-\rho}$$

$$K = \left(1+\Lambda_M R^M\right)H(1-\bar{h})\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\rho\Lambda_A}\right)\phi$$

$$= \left(1+\Lambda_M R^M\right)H(1-\bar{h})\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{\rho\Lambda_A}\right)\frac{\rho\Lambda_A}{(1-\rho)\Lambda_M H}$$

$$= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{(1-\bar{h})}{\Lambda_M(1-\rho)}\right)\left(1+\Lambda_M R^M\right) \tag{4.36}$$

From new expression for labor ratio and (4.22), we have formula of Additive Robots

$$\frac{\Lambda_A R^A}{(1+\Lambda_M R^M)\,H(1-\bar{h})} = \frac{\rho\Lambda_A}{(1-\rho)\Lambda_M H}$$

$$R^A = (1-\bar{h})\left(\frac{\rho}{1-\rho}\right)\left(\frac{(1+\Lambda_M R^M)}{\Lambda_M}\right) \tag{4.37}$$

Then total non-human wealth would be

$$w^n = K + R^A + R^M$$

$$= \left(1+\Lambda_M R^M\right)\frac{1}{\Lambda_M}\left[\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{(1-\bar{h})}{(1-\rho)}\right) + (1-\bar{h})\left(\frac{\rho}{1-\rho}\right)\right] + R^M$$

$$\Lambda_M w^n = (1-\bar{h})\left(1+\Lambda_M R^M\right)\left[\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{(1-\rho)}\right) + \left(\frac{\rho}{1-\rho}\right)\right] + \Lambda_M R^M$$

$$1+\Lambda_M w^n = \left(1+\Lambda_M R^M\right)\left[(1-\bar{h})\left(\frac{\alpha}{(1-\alpha)}\frac{1}{(1-\rho)} + \frac{\rho}{1-\rho} + 1\right) + \bar{h}\right]$$

$$= \left(\frac{1-\bar{h}}{(1-\alpha)(1-\rho)} + \bar{h}\right)\left(1+\Lambda_M R^M\right) \tag{4.38}$$

We use the above expression to rewrite formulas for conventional machine and robots in terms of non-human wealth

$$\begin{aligned} R^A &= \frac{(1-\bar{h})\rho}{\Lambda_M(1-\rho)} \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}(1+\Lambda_M w^n) \\ &= \frac{(1-\alpha)\rho}{\Lambda_M}(1+\Lambda_M w^n) = (1-\alpha)\rho w^n + \frac{(1-\alpha)\rho}{\Lambda_M} \qquad (4.39) \\ R^M &= \left(\frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}(1+\Lambda_M w^n) - 1\right)\frac{1}{\Lambda_M} \\ &= \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})} w^n + \left(\frac{(1-\alpha)(1-\rho)}{(1-\bar{h})} - 1\right)\frac{1}{\Lambda_M} \qquad (4.40) \\ K &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{(1-\bar{h})}{\Lambda_M(1-\rho)}\right)\frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}(1+\Lambda_M w^n) \\ &= \frac{\alpha}{\Lambda_M}(1+\Lambda_M w^n) \qquad (4.41) \end{aligned}$$

Hence, we can see that as long as non-human wealth continues to accumulate, stock of conventional machines and robots are both increasing.

Markup Rate

To identify markups, we first need to have effective output and total cost:

$$\begin{aligned} L &= \Lambda_A R^A g'(\phi) + \left(1+\Lambda_M R^M\right) H(1-\bar{h})(1-\rho)g(\phi) \\ Y &= f'(k)K + (1-\alpha)f(k)L \\ &= f'(k)K + (1-\alpha)f(k)\left(\Lambda_A R^A g'(\phi) + \left(1+\Lambda_M R^M\right) H(1-\bar{h})(1-\rho)g(\phi)\right) \\ &= f'(k)K + f'(k)R^A + (1-\alpha)f(k)\left(1+\Lambda_M R^M\right) H(1-\bar{h})(1-\rho)g(\phi) \\ TC &= \frac{1}{\mu}\left[f'(k)(K+R^A+R^M) + (1-\alpha)f(k)(1-\rho)g(\phi)\left(1+\Lambda_M R^M\right) H\right] \end{aligned}$$

Zero-profit condition will give us the expression for markups:

$$\mu Y = Y + f'(k)R^M + (1-\alpha)f(k)(1-\rho)g(\phi)\left(1+\Lambda_M R^M\right)H\bar{h}$$

$$\mu - 1 = \frac{\bar{h}}{1-\bar{h}}(1-\alpha)(1-\rho) + \frac{\alpha k^{\alpha-1}R^M}{L^M g(\phi)k^\alpha} = \frac{\bar{h}}{1-\bar{h}}(1-\alpha)(1-\rho) + \frac{\alpha R^M}{L^M k g(\phi)}$$

$$= \frac{\bar{h}}{1-\bar{h}}(1-\alpha)(1-\rho) + \frac{\alpha R^M}{L^M \frac{\alpha}{1-\alpha}\frac{1}{\rho\Lambda_A}\phi} = \frac{\bar{h}}{1-\bar{h}}(1-\alpha)(1-\rho) + \frac{R^M(1-\alpha)\rho\Lambda_A}{L^A}$$

$$= \frac{\bar{h}}{1-\bar{h}}(1-\alpha)(1-\rho) + \frac{R^M(1-\alpha)\rho\Lambda_A}{\Lambda_A R^A} = \frac{\bar{h}}{1-\bar{h}}(1-\alpha)(1-\rho) + \frac{R^M(1-\alpha)\rho}{R^A}$$

$$= \frac{\bar{h}}{1-\bar{h}}(1-\alpha)(1-\rho) + \frac{R^M(1-\alpha)\rho}{(1-\bar{h})\left(\frac{\rho}{1-\rho}\right)\left(\frac{(1+\Lambda_M R^M)}{\Lambda_M}\right)}$$

$$= \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}\left(\bar{h} + \frac{\Lambda_M R^M}{1+\Lambda_M R^M}\right)$$

$$\mu = \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}\left(\bar{h} + 1 - \frac{1}{1+\Lambda_M R^M}\right) + 1$$

$$= \frac{(1-\alpha)(1-\rho)}{(1-\bar{h})}\left(\bar{h} + 1 - \frac{1-\bar{h}}{(1-\alpha)(1-\rho)(1+\Lambda_M w^n)}\right) + 1$$

$$= (1-\alpha)(1-\rho)\frac{\bar{h}+1}{(1-\bar{h})} + 1 - \frac{1}{(1+\Lambda_M w^n)} \tag{4.42}$$

Hence, **as non-human wealth increases, markup also increases and converges to the constant level**:

$$\mu^* = (1-\alpha)(1-\rho)\frac{\bar{h}+1}{(1-\bar{h})} + 1 \tag{4.43}$$

Interest Rate

Interest rate from (4.33) we have:

$$r = r^K - \delta = \frac{1}{\mu}f'(k) - \delta = \frac{1}{\mu}\alpha k^{\alpha-1} - \delta$$

$$\frac{\dot{r}}{r} = (\alpha-1)\frac{\dot{k}}{k} - \frac{\dot{\mu}}{\mu}$$

$$= (1-\alpha)(1-\rho)\left(\frac{\dot{\varphi}}{\varphi} + \frac{\dot{H}}{H}\right) - \frac{\dot{\mu}}{\mu} \tag{4.44}$$

Under **fixed** H, after labor market adjustment $\frac{\dot{\varphi}}{\varphi} = 0$, the movement of interest rate totally depends on markup. In the previous part, we know that markup increases overtime before converting to μ^*. As a result, **interest rate would decrease and**

then converge to a constant level when markups converge in the long run after labor market adjustment.

Under **diminishing** H, before markup converges, the decrease of interest rate is more than the scenario with the fixed H due to the contribution of decline in H. However, we can expect that even after markup has converged, the interest rate **continues to decrease over time** as the population shrinking since $\frac{\dot{H}}{H} < 0$.

Real wage

The difference between before and after labor market adjustment is that now real wage is no longer linked to interest rate through the no-arbitrage condition for Additive Robots. The real wage has its normal form based on worker's productivity.

$$\begin{aligned} v &= \frac{1}{\mu}\left(1+\Lambda_M R^M\right)\left(f(k)-f'(k)\right) \\ &= \frac{1}{\mu}\left(1+\Lambda_M R^M\right)(1-\alpha)k^{\alpha} \end{aligned} \tag{4.45}$$

Under **fixed** H, from (4.33) effective capital intensity will be fixed. Hence, wages continue to increase as human productivity keeps increase with the support from Multiplicative Robots. Although this increase will be offset slightly through the increase of markup (before the markup gets converged).

Furthermore, under the **diminishing** population, not just productivity improvement increases wage, but effective capital intensity k now increases also further making real wage increase. The effect of markup might be even more negligible in this case.

4.4 Main Results

4.4.1 Parameters Calibration

Apart from all the parameters' values that we have used in the previous two chapters, there are two new parameters in this chapter. Firstly is the parameter ρ: labor market allocation in 4.3. ρ is the percentage of jobs that can be substituted by Additive Robots and $(1-\rho)$ is the percentage of jobs that can only be supported by Multiplicative Robots. There is no preceded value of such a parameter in literature. Again, I took the hint from (**Frey and Osborne 2017**), suggesting that approximately 60% of occupations can be automated by technology. Hence, $\rho = 0.6$.

The second parameter is the decline growth rate of H when we consider the case of the population shrinking. For the last 5 years, Japan has been experiencing population shrinking[8]. The average population growth in the last 5 years for Japan is -0.2%. Hence,

$$n_H = -0.002$$

4.4.2 Endogenous Growth

As expected, the application of robots helps to move the economy out of the initial steady-state and enter endogenous growth.

[8] https://www.worldometers.info/world-population/japan-population/

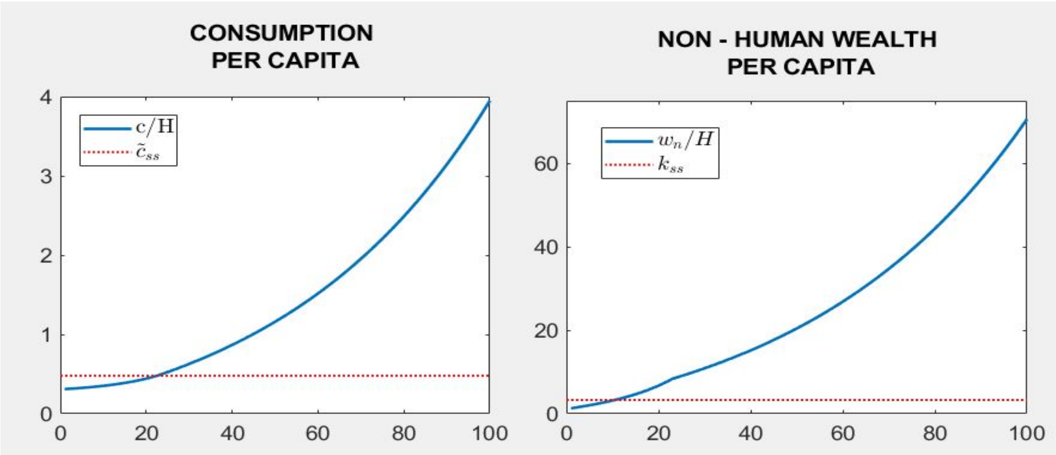


Figure 4.3: Consumption and Wealth per Capita

In the initial steady-state, all the aggregate variables change at the same speed as population growth (through the change in H). Hence, they are either constant or decreasing at the same rate with H. When H is diminishing then they are decreasing, but the per capita ratios (represented by the ratios over H) still stay constant. As a result, to see the overall growth, regardless of the change in H, I plot the graphs with per capita ratios instead of aggregate variables.

From the previous two chapters, we already saw that the use of both types of robots will result in a reduction in the first period. Hence from the left-hand side of Figure (4.3), it is no surprise that consumption per capita will drop in the first period, then perpetually increase and surpass the initial level after around 22 years. Consumption growth rate increases from 1.1% to almost 4% then reduces slowly and converges to 2% after 270 years.

Non-human wealth increases very fast initially. The growth rate reduces from almost 13% and also converges to 2% just like consumption in the long run.

4.4.3 Labor Market Adjustment

As explained in (4.30), and we see in the previous part that non-human wealth does increase over time, hence labor will adjust over time. Starting from the optimal

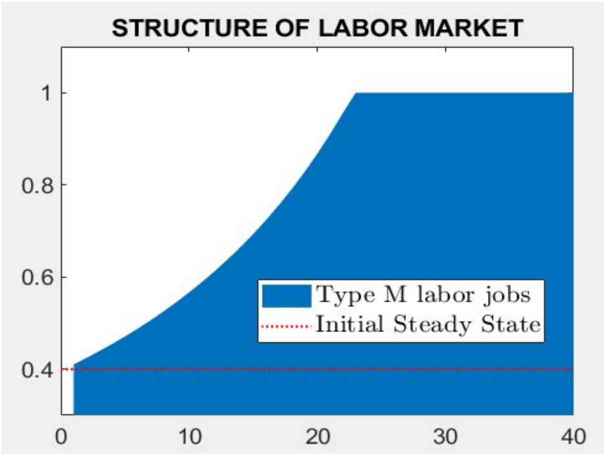


Figure 4.4: Labor Market Adjustment

allocation under the initial steady state, workers will slowly move to do type M jobs. Even though we assume that the labor market is perfectly mobile, meaning that wage should be equal between two types of jobs $v_A = v_M$, there can be two explanations why workers still move to do type M jobs:

- **Firstly**, there are more and more Additive Robots are used while there is no change in the structure of the jobs market (i.e. ρ does not change), there would be less and fewer jobs left for human labor. Hence, they need to move to other types of jobs where robots can not replace them.

- **Secondly**, the prospect that the support of Multiplicative Robots increases workers' productivity in type M jobs can create the expectation for a higher wage in the future.

The free movement creates the relative shortage of human labor supply for type A jobs and relatively excess human labor supply for type M jobs. As a result, wages for type A jobs are pulled up while wages for type M jobs are pulled down until the two are matched in the market equilibrium conditions.

4.4.4 Market Rates - Wages and Interest Rates

Interest Rate Increases Initially then Decreases and Converges

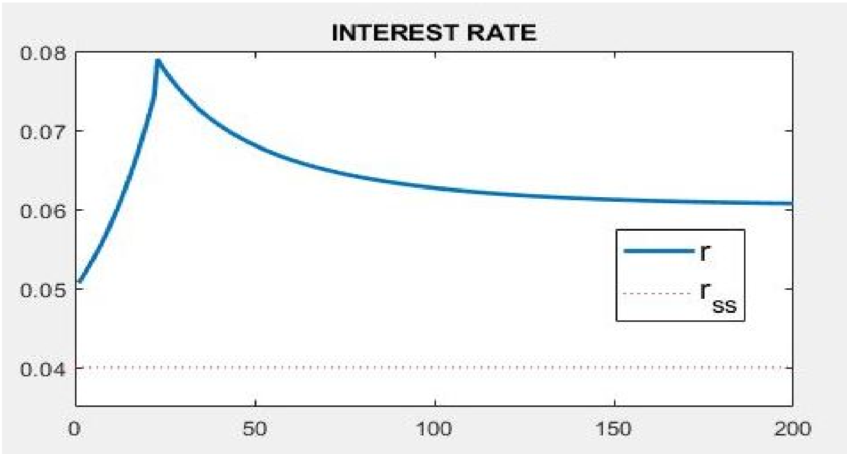


Figure 4.5: Interest Rates

From the interest rate of 4% in the initial steady state, the interest rate will immediately jump up to 5.2% then continues to increase as the labor market adjusting. This is expected since in the economy with each type of robot the increases in interest rate are observable. The effect of Multiplicative Robots causes a higher increase in the interest rate. As the labor market shift toward type M labor effectively, the interest rate would increase further. Only after the labor market has finished adjusting, if we do not have the effect of the markup, the interest rate should stay constant at that high level. But due to more and more Multiplicative Robots are used, the markup rate increases making interest rates fall. And only when markup has been converged, the interest rate also converges to a constant value of 6.1%. This value must be higher than the initial steady-state interest rate otherwise robots do not make any profit for the producer. The long-run interest rate is still higher than it is in the first period of 5.2%. This is because the stock of Multiplicative Robots later dominates the Additive Robots, putting a positive effect on the interest rate.

Real Wage Increases Throughout

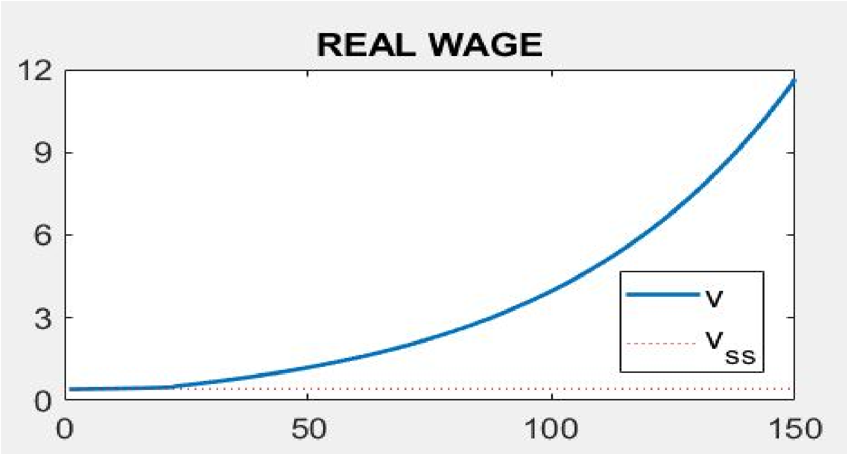


Figure 4.6: Wage gaps in Labor Market with and without Adjustment

Different from interest rate, which increases then decreases and converges, real wage increases through out over time. Although initially, as expected in economy with each type of robots, there is an initial drop in real wage:

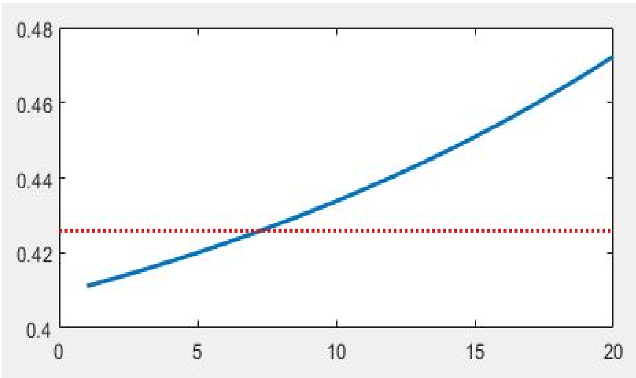


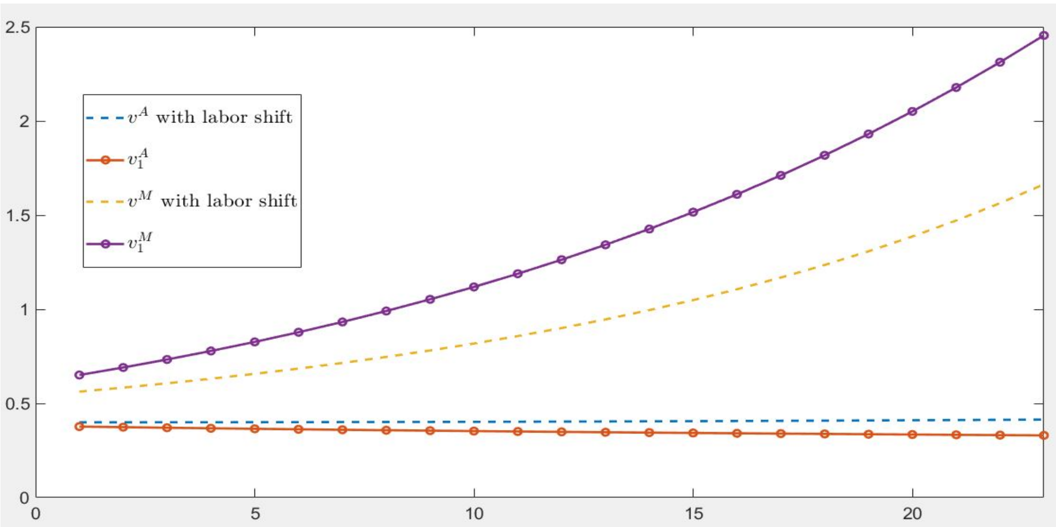
Figure 4.7: Initial Drop in Wage

It takes real wages only around 8 years to bounce back to the initial level. This is the result ensured by the assumption of a perfectly mobile labor market. As long as the labor market is effective, the movement of human labor will create an increase in the real wage. Even though, there are pulldown force from Additive Robots as well and markup rates. After the labor market finishes adjustment, the increase in wage is ensured by the use of Multiplicative Robots, which is used to boost human labor productivity. When productivity increases, wages will increase.

4.4.5 Real Wage and Inequality

However, in reality, the labor market is not perfectly mobile. Changing jobs requires training or cost and should not be a short term action. Sometimes it is even impossible for a worker in type A jobs to move to type M jobs. For example, investing cost and time to train a cleaner to be a professor might just be the necessary condition for a successful transition.

However, with more and more number of Additive Robots, there is a constant force pull down the wage for type A workers. While the bigger the number of Multiplicative Robots, the higher the wage for type M workers. The policy implication is hence such that even there is some movement (if any) is good enough to close the wage gap between two types of jobs. In Figure (4.8), the lines with marks show v_A



Wages under no labor shift are assumed that the allocation in the labor market stays at the initial level (i.e. 40% in type A and 60% in type M. On the other case, labor shift is kept at 3% every period.)

Figure 4.8: Wage gaps in Labor Market with and without Adjustment

and v_M if the movement is not allowed in the labor market. The allocation of workers between type A and type M jobs stays at the initial steady-state level. Starting from the initial gap, the wage gaps between two types of jobs get bigger and bigger over time.

For the dashed lines, although the gaps still get bigger over time, it is lesser compared to the total no movement scenario. The encouraged movement will help to lessen the reduction in wage for type A jobs while depressing the increase in wage for type M jobs.

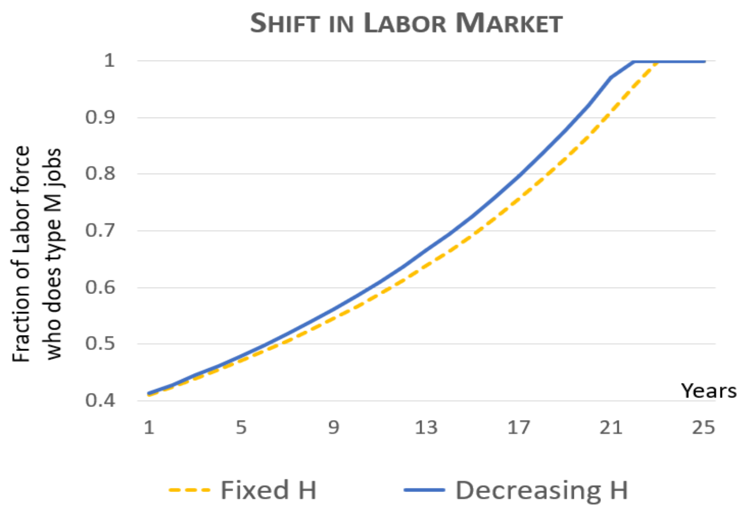
It suggests that the government can help to close the gaps by encouraging as much as possible the labor force to move from type A to type M jobs such as providing free and frequent training, upgrading skills courses. It might not be as perfect as the perfect mobile market, it still helps on certain levels. The faster the movement the better the gaps would be closed.

4.4.6 Fixed vs. Diminishing Population

For this part, we come back with a perfect labor market, where workers are free to move between two types of jobs. It is still worth to mention that the result would be only one wage for both types of jobs. Nevertheless, after the labor market has finished its adjustment, there still be only one wage for type M jobs as there are no more human workers doing type A jobs.

Faster Labor Market Adjustment

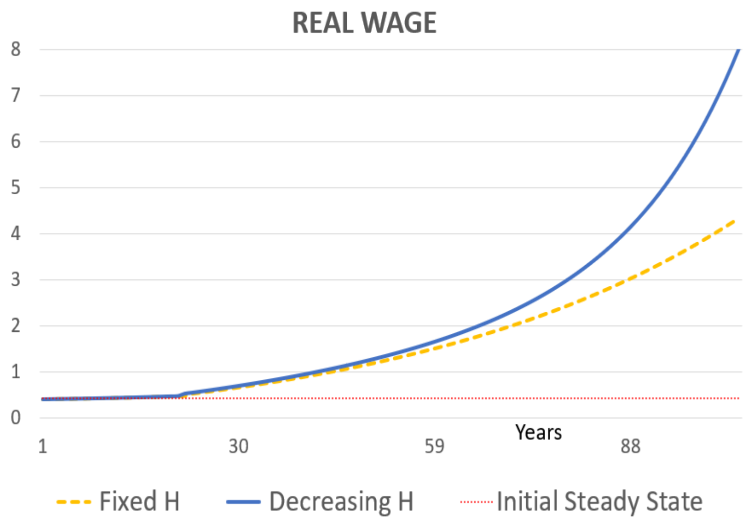
With a very conservative shrinking rate of 0.01%, it takes the labor market around 3 years less to complete the adjustment. When the population gets smaller, wealth accumulation in the absolute term is bigger (only the per capita ratios same with fixed H scenario), facilitate a faster increase in φ. Higher wealth enables faster increases in the number of robots (both Additive and Multiplicative Robots), making more and more type A workers who have jobs replaced need to move to type M jobs.



H is decreasing 0.01% every periods

Figure 4.9: Labor Market Adjustment under Fixed vs. Diminishing Population

Real Wage

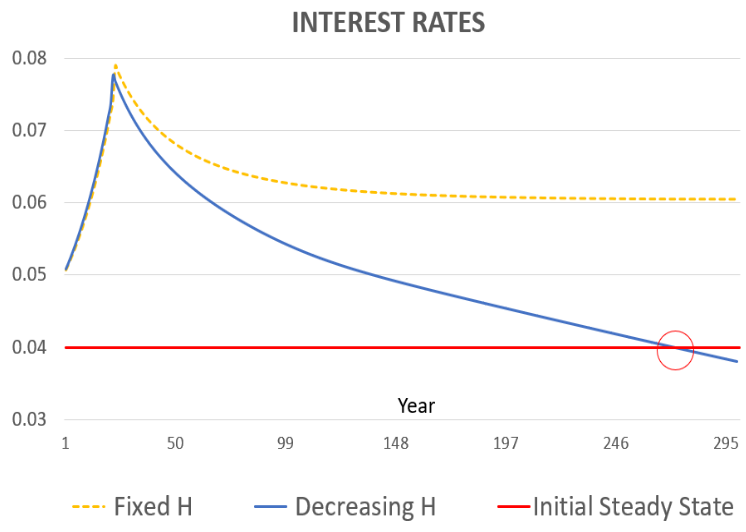


H is decreasing 0.01% every periods

Figure 4.10: Real Wage under Fixed vs. Diminishing Population

Under both scenarios, real wage will always increase either during the labor market adjusting as per (4.34) or as per (4.45). However, when the population is decreasing, wage increases even faster. That is because the human labor supply becomes relatively less, making effective labor L is less. It follows that effective capital intensity of $k = \frac{K}{L}$ is higher compared to the fixed H scenario. As a result, from (4.45), we have that wage will increase faster. The shortage of human labor will pull up the wage for workers.

Interest Rates



H is decreasing 0.01% every periods

Figure 4.11: Interest Rates under Fixed vs. Diminishing Population

We know that during the process of labor market adjustment, the interest rate will increase as shown in (4.44) in both scenarios, although the speed is less under diminishing H. The difference is after the adjustment. Under fixed H the decrease in interest rate is due to the use of markups. Hence when markups converge to the constant μ^* then interest rate also converges to a constant level of around 6% which is higher than the initial interest rate of 4%. Hence, profitability condition always holds, meaning robots is always a benefit to use.

However, under the case where H decreases over time, also from (4.44), the interest rate will keep falling. As a result, there will be a point where it crosses the initial interest rate of 4%. That means after that point, it is no longer profitable to use robots, however, it would be into a quite far future, around 300 years, based on our choice of parameters.

This can again be explained through the changes of effective capital intensity that keeps falling when H is decreasing, making the interest rates fall.

4.5 Regime Switching and Feasibility Period

4.5.1 Additive Robots Participation Condition

From the expression for Additive Robots (4.26), we see that the stock of Additive Robots will be positive, if and only if:

$$\frac{\varphi}{1-\rho} - 1 > 0 \quad \Leftrightarrow \quad \varphi > 1 - \rho \tag{4.46}$$

From (4.29), a **too low** φ can only be the result of a **too low** wealth. As φ increases and decreases together with wealth overtime, the only two concerns will be at the initial points and when wealth starts to decrease (if any). If initial wealth K_{ss} is not enough, then the condition (4.46) might not hold. As a result, the economy has to take more time to accumulate wealth until it can start to utilize Additive Robots. The reverse is true when wealth starts to decrease. There will be a point when the condition is violated and Additive Robots are no-longer usable.

4.5.2 Multiplicative Robots Participation Condition

Given the expression for Multiplicative Robots (4.28), stock of Multiplicative Robots will be positive if and only if:

$$\varphi > \frac{\Lambda_A}{\Lambda_M H} \tag{4.47}$$

There are two cases that the condition (4.47) does not hold. When H is fixed or in the periods where H is still high, it will only be violated when we have too low φ which again, similar to the condition for Additive Robots. Hence, even if the technology is available, the economy still has to wait until it accumulates enough wealth.

As wealth increase, φ increase, it will be easier for (4.47) to holds. However, under the scenario where H declines, the right-hand side would increase over time, and the condition might be violated. The human labor workforce is too small that even with the help of Multiplicative Robots, it can not compete with the productivity from a unit of Additive Robots.

Since $\varphi \leq 1$, this condition also indicates that, unless the increased productivity of all human labor who do type M jobs is higher than the productivity of an Additive Robots, then Multiplicative robots is not worth to use. As H is exogenously given (either fixed or decreased), and even if φ already increases up to its limit, i.e all workforce has already been working in type M jobs. We can only help $\frac{\Lambda_A}{\Lambda_M H} < 1$, by decreasing the ratio of $\frac{\Lambda_A}{\Lambda_M}$. The productivity of Multiplicative Robots needs to increase at a higher speed than the productivity of Additive Robots.

$$\frac{\dot{\Lambda}_M}{\Lambda_M} > \frac{\dot{\Lambda}_A}{\Lambda_A} \tag{4.48}$$

4.5.3 Prolonging the Feasibility Period

As discussed above, it depends on the parameter choices that make economy switching among Additive Robots alone, Multiplicative Robots alone, and both types of Robots. Even though our choices of parameters ensure the application of both types of robots (since it is the focus of our questions), there is still a point worth noticing. Under the result part, we see that in the case of a diminishing population, the interest rate will keep decreasing to the point it crosses the initial interest rate of the steady-state, making robots no-longer profit enough to use. Then how can we prolong that feasibility period? It is back to the fundamental reason why interest rate declines are due to the increase in capital intensity k which again due to the increase in the effective labor ratio ϕ as per (4.32). Hence to prolong the feasibility period, we need to slow down the increase of labor ratio which has the following expression from (4.22):

$$\phi = \frac{\Lambda_A}{\Lambda_M} \left(\frac{\rho}{1-\rho} \right) \frac{1}{H^M}$$

Hence, to control the increase of effective labor ratio, we need to:

- Firstly, it is very clear that the more the population decreases, the more ϕ increases. Hence we need to control to maintain population growth or at lease no change. It can be done by encouraging policies to increase the birth rate or through immigration.

- Secondly, control the increase in the productivity of Additive Robots. During the labor market adjustment process, this will even help to reduce the force on the labor market to adjust since Additive Robots is not so productive and replace human labor slower.

- And lastly, increase the productivity of Multiplicative Robots to further support the shrinking group of human labor. As long as the increase in productivity of Multiplicative Robots is more than the increase in productivity of Additive Robots is enough to ensure the decrease in ϕ rather than increase.

There is another method that is not straightforward from the expression is through the adjustment of ρ since in our model we assume that ρ is a constant. A decline in ρ can slow down the increase in ϕ. That suggests the creation of more jobs that can only be supported by Multiplicative Robots rather than focus on jobs that can be replaced by Additive Robots.

4.6 Conclusion

In this chapter, I incorporated both types of robots into one aggregate models and structured the labor market as a Cobb Douglas combination of two types of jobs: type A jobs are those that can be automated by Additive Robots and type M jobs are those that can only be supported by Multiplicative Robots. Furthermore, after the labor market has finished adjustment, there would be no more human labor that does type A jobs. The real wage is thus only payable to human labor do type M jobs, who have higher and higher productivity under the support of Multiplicative Robots. As a result, the real wage will continue to increase over time.

Under the assumption that the labor market is perfectly mobile, along with the wealth accumulation process, there is a transition of labor from doing type A jobs to doing type M jobs. If the labor market is so effective that the transition is unconditional and smooth, both interest rates and real wages will effectively increase. In other words, the pull-up force from Multiplicative Robots is so attractive and stronger than the pull-down force from Additive Robots.

Even under the case where the labor market is not perfect, meaning there exist two different wages for two types of jobs, as long as there is a small transfer of labor from type A to type M, it is good enough to narrow the wage gap. It might be the case that changing jobs require costly training or upgrading skill. As a result, the government can help to mitigate the inequality by policies that encourage such job transfer such as free training or education loans.

Apart from real wage, the main focus, the model does generate endogenous growth. This is no surprise since each type of robots can create economic growth, the effect should be stronger when both types are used.

Under the extension where the population is shrinking, the labor market even has to adjust faster to compensate for the shortage in labor supply. The relative shortfall of human labor makes the marginal product of labor increase more than under the case of a fixed population. Consequently, the real wage will also increase faster. However, as the labor force becomes smaller, capital is spread thinner, reducing effective capital intensity. That makes interest rate start to fall after the labor market's adjustment (since now there is no longer the effective force - the movement of labor - to keep the interest rate increase). If the time is long enough, the interest rate might even fall below the original interest rate in the initial steady-state. Upon that point, the profitability condition is violated, making robots is no longer beneficial to apply. Although a long time easily means hundreds of years, there might be a good idea to have precautions to prolong that feasible period.

Appendix A

Quantitative Appendix

A.1 Baseline Model

A.1.1 Solving the consumer maximization problem

The representative agent maximizes

$$\int_t^\infty \log c_\kappa e^{\left(-\theta(\kappa-t)\right)} d\kappa$$

subject to

$$\frac{dw_t^n}{dt} = r_t w_t^n + v_t H - c_t$$

and the transversality condition that prevent agents from going indefinitely into debt.

$$\lim_{\kappa\to\infty} \left\{ w_\kappa^n . e^{-\int_t^\kappa r(\nu)d\nu} \right\} = 0$$

Present Value Halmitonian function:

$$H(c,\mu) = U(c)e^{-\theta(\kappa-t)} + \mu\left[vH + rw^n - c\right]$$

First Order Condition

$$\begin{cases} U_c e^{-\theta(\kappa-t)} & = \mu \quad (1) \\ \dot{\mu} & = -r\mu \quad (2) \end{cases}$$

Transversality condition: From (2) the evolution of the shadow price is

$$\mu_{t+\kappa} = \mu_t . e^{\left[-\int_t^\kappa r(\nu)d\nu\right]}$$

substitute into the transversality:

$$\lim_{\kappa\to\infty} [w_\kappa^n . \mu_{t+\kappa} . \mu_t] = 0$$

, with $\mu_t = U_c > 0$ hence the transversality condition becomes:

$$\lim_{\kappa\to\infty}\left\{w^N_\kappa.\mu_{t+\kappa}\right\} = 0$$

The Euler Equation: Differentiate (1) with respect to time to get $\dot{\mu}$ and then substitute into (2), we get:

$$r = -\frac{\dot{\mu}}{\mu} = -\frac{U_{cc}\dot{c}e^{-\theta t} - \theta e^{-\theta t}U_c}{U_c e^{-\theta t}} = -\frac{U_{cc}}{U_c}\dot{c} + \theta = \theta - \left[\frac{U_{cc}.c}{U_c}\right]\frac{\dot{c}}{c}$$

Given the choice of utility function, we have:

$$r = \theta + \frac{\dot{c}}{c}$$

The consumption function

Define the average interest rate between time t and time κ as following:

$$\bar{r}(\kappa) = \frac{1}{\kappa - t}\int_t^\kappa r(\nu)d\nu$$

then the present discount factor to convert an unit of income in period κ to an unit of income at time t will be: $e^{-\bar{r}(\kappa)(\kappa-t)}$.

Solve the flow budget constrain as a first order differential equation in w^n to get an intertemporal budget constrain for any period $T > t$:

$$w^n_T.e^{-\bar{r}(T)T} + \int_t^T c_\kappa e^{-\bar{r}(\kappa)\kappa}d\kappa = w^n_t + \int_t^T v_\kappa H e^{-\bar{r}(\kappa)\kappa}d\kappa$$

As $T \to \infty$ the first element of the left hand side vanish as per transversality condition.

Let denote human wealth at time t - the sum of discounted future wage income as such: $w^h_t = \int_t^\infty v_\kappa H e^{-\bar{r}(\kappa)\kappa}d\kappa$ then we have:

$$\int_t^\infty c_\kappa e^{-\bar{r}(\kappa)\kappa}d\kappa = w^n_t + w^h_t$$

At the same time from the Euler Equation integrate between time t and κ and solve the first order differential equation in c we have:

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= r_t - \theta \\
\Leftrightarrow \quad \int_t^{\kappa} \frac{\dot{c}_\nu}{c_\nu} d\nu &= \int_t^{\kappa} (r_\nu - \theta) d\nu \\
\Leftrightarrow \quad \int_t^{\kappa} d(\ln c_\nu) &= \int_t^{\kappa} r_\nu d\nu - \theta \int_t^{\kappa} d\nu \\
\Leftrightarrow \quad \ln \frac{c_\kappa}{c_t} &= (\bar{r}(\kappa) - \theta)(\kappa - t) \\
\Leftrightarrow \quad c_\kappa &= c_t.exp((\bar{r}(\kappa) - \theta)(\kappa - t))
\end{aligned}$$

Plugin back into the intertemporal budget constrain:

$$c_t.e^{\left(\bar{r}(\kappa)-\theta\right)(\kappa-t)} e^{-\bar{r}(\kappa)\kappa} d\kappa = w_t^n + w_t^h$$

Use the formula for $\bar{r}(\kappa)$, and let $u(\kappa) = \bar{r}(\kappa)t - \theta(\kappa - t)$ the left hand side would be:

$$\begin{aligned}
LHS &= c_t \int_t^{\infty} e^{[(\bar{r}(\kappa)-\theta)(\kappa-t)-\bar{r}(\kappa)\kappa]} d\kappa \\
&= c_t \int_t^{\infty} e^{[\bar{r}(\kappa)t-\theta(\kappa-t)]} d\kappa \\
&= c_t \left[\frac{1}{u'(\kappa)} e^{u(\kappa)} \right]_{\kappa=t}^{\infty}
\end{aligned}$$

When $\kappa = \infty$, $\bar{r}(\kappa) = 0$ as $\frac{1}{\kappa - t} \longrightarrow 0$ and $u(\kappa) \longrightarrow -\infty$ hence $e^{u(\kappa)} \longrightarrow 0$

When $\kappa = t$ then $\bar{r}(\kappa) = 0$ since $\int_t^t r(\nu)d\nu = 0$ and $u(\kappa) = 0$ hence $e^{u(\kappa)} = 1$.

Also we have: $u'(\kappa)|_{\kappa=t} = [\bar{r}'(\kappa)t - \theta]_{\kappa=t}$ with

$$\bar{r}'(\kappa)|_{\kappa=t} = \frac{d\bar{r}(\kappa)}{d\kappa}|_{\kappa=t} = \left[(-1)(\kappa - t)^{-2} \int_t^{\kappa} r(\nu)d\nu + \frac{1}{\kappa - t} \frac{d}{d\kappa} \int_t^{\kappa} r(\nu)d\nu \right]_{\kappa=t} = 0$$

Hence, $LHS = c_t(0 - \frac{1}{-\theta}) = \frac{1}{\theta} c_t$

The solution to agent's problem is then given by:

$$c_t = \theta \left(w_t^h + w_t^n \right)$$

where human wealth is given by

$$w_t^h = \int_t^{\infty} \nu_\kappa H e^{\left(\int_t^{\kappa} -r(\nu)d\nu\right)} d\kappa$$

Human wealth instantaneous flow

Use Leibniz's Rule to find $\dot{w}_t^h$

$$\frac{dw_t^h}{dt} = \frac{d}{dt}\int_t^\infty v_\kappa H e^{\left(\int_t^\kappa -r(\nu)d\nu\right)} d\kappa$$
$$= \int_t^\infty v_\kappa H e^{\left(\int_t^\kappa -r(\nu)d\nu\right)} \left(\frac{d}{dt}\int_t^\kappa -r(\nu)d\nu\right) d\kappa +$$
$$v_\kappa H e^{\left(\int_t^\kappa -r(\nu)d\nu\right)}_{\kappa=\infty} \frac{d(\infty)}{dt} - v_\kappa H \exp\left(\int_t^\kappa -r(\nu)d\nu\right)_{\kappa=t} \frac{dt}{dt}$$
$$= \int_t^\infty v_\kappa H \exp\left(\int_t^\kappa -r(\nu)d\nu\right)\left(\frac{d}{dt}\int_t^\kappa -r(\nu)d\nu\right) d\kappa + 0 - v_t H$$

Use Leibniz's one more time we have:

$$\frac{d}{dt}\int_t^\kappa -r(\nu)d\nu = 0 + (-r(\nu)|_{\nu=\kappa})\frac{d\kappa}{dt} - (-r(\nu))|_{\nu=t})\frac{dt}{dt} = r_t$$

Hence,

$$\dot{w}_t^h = \left[r_t \int_t^\infty v_\kappa H \exp\left(\int_t^\kappa -r(\nu)d\nu\right) d\kappa\right] - v_t H = r_t w_t^h - v_t H$$

Lifetime Welfare after Introduction of Robots The new level of lifetime welfare would be

$$U = \int_t^\infty log(c_\kappa)e^{-\theta(\kappa-t)}d\kappa = \int_t^\infty log[c_t exp(\eta_c(\kappa-t))]e^{-\theta(\kappa-t)}d\kappa$$
$$= \int_t^\infty [log(c_t) + \eta_c(\kappa-t)]\, e^{-\theta(\kappa-t)}d\kappa$$
$$= \int_t^\infty log(c_t)e^{-\theta(\kappa-t)}d\kappa + \int_t^\infty \eta_c(\kappa-t)e^{-\theta(\kappa-t)}d\kappa$$
$$= \frac{1}{-\theta}log(c_t)[e^{-\theta(\kappa-t)}]_t^\infty + \eta_c\int_t^\infty (\kappa-t)e^{-\theta(\kappa-t)}d\kappa$$
$$= \frac{1}{\theta}log(c_t) + \eta_c\int_t^\infty (\kappa-t)e^{-\theta(\kappa-t)}d\kappa$$

Integration by part for the second integral:

$$\int_t^\infty (\kappa-t)e^{-\theta(\kappa-t)d\kappa}d\kappa = \int_0^\infty \mu e^{-\theta\mu}d\mu$$
$$= \left[\mu\left(\frac{1}{-\theta}e^{-\theta\mu}\right)\right]_{\mu=0}^\infty - \int_0^\infty \frac{1}{-\theta}e^{-\theta\mu}d\mu$$
$$= \frac{1}{\theta}\left[\frac{1}{-\theta}e^{-\theta\mu}\right]_{\mu=0}^\infty = \frac{1}{(\theta)^2}$$

Hence, the lifetime utility is

$$U = \frac{1}{\theta}\log(c_t) + \frac{\eta_c}{(\theta)^2}$$

A.1.2 Solving the firm profit maximization problem

Each period t, the representative firm will have to decide the level of capital used K_t and amount of labor employed L_t in order to maximize the profit:

$$\max_{K_t, L_t} \left(F(K_t, H_t) - r_t^K K_t - v_t L_t\right)$$

with $F(K_t, H_t) = K_t^{\alpha} L_t^{1-\alpha}$.

The firs-order conditions:

$$\alpha K_t^{\alpha-1} L_t^{1-\alpha} - r_t^K = 0$$
$$(1-\alpha) K_t^{\alpha} L_t^{-\alpha} - v_t^K = 0$$

In term of per labor variables we have:

$$r_t^K = \alpha \left(\frac{K_t}{L_t}\right)^{\alpha-1}$$
$$v_t^K = (1-\alpha) \left(\frac{K_t}{L_t}\right)^{\alpha}$$

And labor market clearing condition for general equilibrium model requires that labor demand to meet the labor supply from the representative household:

$$L_t = H_t$$

A.2 Baseline Model with Flexible Labor Supply

A.2.1 Consumer's Maximization Problem

Present Value Halmitonian function:

$$H(c, H, \mu) = U(C, H) e^{-\theta t} + \mu \left[vH + ra - c\right]$$

First Order Condition

$$\begin{cases} U_C e^{-\theta t} &= \mu \quad (1) \\ U_H e^{-\theta t} &= -\mu v \quad (2) \\ \dot{\mu} &= -r\mu \quad (3) \end{cases}$$

Intratemporal condition from (1) and (2):

$$v_t = -\frac{U_H(C_t, H_t)}{U_C(C_t, H_t)}$$
$$= -\frac{\frac{-B}{\bar{H}-H_t}}{\frac{1}{C_t}} = \frac{BC_t}{\bar{H}-H_t}$$

The Euler equation:

From (1) we have:

$$\dot{\mu} = \frac{dU_C}{dt}e^{-\theta t} - \theta e^{-\theta t}U_C$$
$$= (U_{CC}\dot{C} + U_{CH}\dot{H})e^{-\theta t} - \theta e^{-\theta t}U_C$$
$$\Leftrightarrow \frac{\dot{\mu}}{\mu} = \frac{(U_{CC}\dot{C} + U_{CH}\dot{H})e^{-\theta t} - \theta e^{-\theta t}U_C}{U_c e^{-\theta t}} = -r$$
$$= \left(\frac{U_{CC}\dot{C} + U_{CH}\dot{H}}{U_C} - \theta\right)$$
$$\Leftrightarrow r = \theta - \frac{U_{CC}}{U_C}\dot{C} - \frac{U_{CH}}{U_C}\dot{H}$$

For a separable C and H utility function, $U_{CH} = 0$ hence we have the Euler equation exactly the same with fixed H model.

$$r = \theta - \frac{U_{CC}}{U_C}\dot{C} = \theta - \frac{\dot{C}}{C}$$

$$\Leftrightarrow \frac{\dot{C}}{C} = r - \theta$$

A.2.2 Firm's Profit Maximization

Let $k_t = \frac{K_t}{H_t}$ then output per unit labor is: $f(k_t) = k_t^{\alpha}$. Hence, wage and return of conventional machine is still same with the baseline model.

$$\begin{cases} r^K & = f'(k) \\ w & = f(k) - kf'(k) \end{cases}$$

However, now the with the flow of effective capital depends on the change in labor as well:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{H}}{H}$$

Hence,

$$\frac{\dot{c}}{c} = f'(k) - \delta - \theta = \alpha k^{\alpha-1} - \delta - \theta - \frac{\dot{H}}{H}$$

and since $K_t = w_t^n$ we have:

$$\frac{\dot{k}}{k} = \frac{\dot{w}^n}{w^n} - \frac{\dot{H}}{H} = r + \frac{vH}{w^n} - \frac{C}{w^n} - \frac{\dot{H}}{H} = r + v\frac{H}{K} - \frac{C}{K} - \frac{\dot{H}}{H} = (r^K - \delta) + \frac{v}{k} - \frac{c}{k} - \frac{\dot{H}}{H}$$

Substitute equation for wage and return on conventional machine, we have

$$\frac{\dot{k}}{k} = (f'(k) - \delta) + \frac{1}{k}(f(k) - kf'(k)) - \frac{c}{k} - \frac{\dot{H}}{H} = \frac{f(k)}{k} - \delta - \frac{c}{k} - \frac{\dot{H}}{H}$$

$$\Leftrightarrow \dot{k} = f(k) - \delta k - c - \frac{\dot{H}}{H}k$$

A.2.3 Initial Steady State

At steady state $\frac{\dot{c}}{c} = \frac{\dot{H}}{H} = \frac{\dot{k}}{k} = 0$ then:

$$\alpha k_{SS}^{\alpha-1} - \delta - \theta = 0 \Leftrightarrow k_{SS} = \left(\frac{\alpha}{\delta+\theta}\right)^{\frac{1}{1-\alpha}}$$

$$f(k_{SS}) - \delta k_{SS} = c_{SS} = \frac{C_{SS}}{H_{SS}} = k_{SS}^{\alpha} - \delta k_{SS}$$

$$f(k_{SS}) - k_{SS}f'(k_{SS}) = (1-\alpha)k_{SS}^{\alpha} = v_{SS}$$

From the intratemporal condition between consumption and work:

$$v_{SS} = \frac{BC_{SS}}{\bar{H} - H_{SS}} \Leftrightarrow v_{SS}\left(\frac{\bar{H}}{H_{SS}} - 1\right) = B\frac{C_{SS}}{H_{SS}}$$

$$\Leftrightarrow H_{SS} = \bar{H}\left(B\frac{C_{SS}}{H_{SS}}\frac{1}{v_{SS}} + 1\right)^{-1}$$

Then the labor and consumption level at steady state will be

$$H_{SS} = \bar{H}\left(B\frac{(k_{SS}^{\alpha} - \delta k_{SS})}{(1-\alpha)k_{SS}^{\alpha}} + 1\right)^{-1}$$

$$= \bar{H}\left(B\frac{1-\delta k_{SS}^{1-\alpha}}{1-\alpha} + 1\right)^{-1} = \bar{H}\left(\frac{(1-\alpha)}{(B+1-\alpha) - B\delta k_{SS}^{1-\alpha}}\right)$$

$$C_{SS} = H_{SS}\left(f(k_{SS}) - \delta k_{SS}\right)$$

$$= \bar{H}\left(\frac{(1-\alpha)}{(B+1-\alpha) - B\delta k_{SS}^{1-\alpha}}\right)\left(k_{SS}^{\alpha} - \delta k_{SS}\right)$$

Plug in the formula for k_{SS}, labor and consumption in terms of model's parameters are:

$$H_{SS} = \bar{H}\left(B\frac{1-\delta\frac{\alpha}{\delta+\theta}}{1-\alpha}+1\right)^{-1}$$

$$C_{SS} = \bar{H}\left(\frac{(1-\alpha)}{(B+1-\alpha)-B\delta\frac{\alpha}{\delta+\theta}}\right)\left(\left(\frac{\alpha}{\delta+\theta}\right)^{\frac{\alpha}{1-\alpha}}-\delta\left(\frac{\alpha}{\delta+\theta}\right)^{\frac{1}{1-\alpha}}\right)$$

A.2.4 Algorithm loop for consumption and labor path after Additive Robots

Now since H does not stay constant anymore, we cannot have a close form solution for human wealth w_t^h at the initial period. Instead we need to guess the initial consumption c_0 and induce the series of $\{H_t\}_{t=1}^T$ to have w_0^h and recalculate the initial consumption. If it does not match with the guess, we need to update the guess.

1. Guess C_0 and generate $\{C_t\}_{t=1}^T$ using the growth rate of consumption.
2. Calculate H_t using the intratemporal equation
   - If $\left(\bar{H}-\frac{B}{v}C\right)>0$ then $H_{t+\kappa}=\bar{H}-\frac{B}{v}C$
   - If $\left(\bar{H}-\frac{B}{v}C\right)<0$ then $H_{t+\kappa}=0$
3. Calculate human wealth using labor path and fixed wage rate W_t^n
4. Recalculate $C_t' = \theta(W_t^h + W_t^n)$
5. Compare C_t' with initial guess. If not match then update the guess.

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