Specifying and estimating vector autoregressions using their Eigensystem representation

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Abstract

This article introduces the principles and mechanics of the eigensystem vector autoregression (EVAR) framework, where a VAR may be specified and estimated directly via its eigenvalue and eigenvector parameters. Using explicit constraints on the eigensystem permits control of a VAR's allowable dynamics, which is illustrated empirically with standard and time-varying VAR estimations specified to be always non-explosive.

JEL classification: C13, C22, C32

Keywords: vector autoregression (VAR); companion matrix; eigenvalues; eigenvectors

1 Introduction

In this article, I develop a framework that allows vector autoregressions (VARs) to be specified and estimated directly from their eigenvalue and eigenvector parameters. What I will hereafter refer to as the eigensystem VAR (EVAR) permits explicit control of the allowable dynamics for a VAR by using constraints on its eigensystem which, aside from unit roots imposed in the case of cointegration, is not feasible with VARs estimated via their coefficients (hereafter CVARs).

To my knowledge, the EVAR has no precedent in the literature. That is, while it is routine in applications to check the stability of an estimated CVAR by calculating the eigenvalues of its associated companion matrix (and the eigenvectors could also be calculated), I have not found any literature that reverses the process; i.e. using the eigensystem to generate the companion matrix and hence VAR coefficients during the EVAR estimation. As referenced in my overview of VARs and their eigensystem in section 2, there are two applied mathematics articles that detail the eigensystem form of a VAR underlying my EVAR framework specification, but neither develops a framework for estimation.

Following my development of an estimation method in section 3, I illustrate the empirical application of the EVAR in section 4, i.e. estimating standard and time-varying VARs subject to eigenvalue magnitude constraints that guarantee non-explosive dynamics at all times, compared to explosive dynamics that occur in the analogous CVARs. My conclusion in section 5 outlines other potential applications of the EVAR framework.

2 VARs and their eigensystem

A standard reduced-form VAR may be specified as:

$$y_t = \alpha + \beta_1 y_{t-1} + \ldots + \beta_P y_{t-P} + \varepsilon_t \tag{1}$$

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where y_t is an $N \times 1$ vector of data at time t, y_{t-p} is data at time $t-p$ with p ranging from 1 to P, α is a vector of constants, β_p are $N \times N$ matrices of coefficients associated with y_{t-p}, and ε_t is a vector of residuals with an assumed multivariate normal distribution $\varepsilon_t \sim N\left(0_{N\times 1}, \Omega_\varepsilon\right)$, with $0_{N\times 1}$ the $N \times 1$ vector of zeros and Ω_ε the $N \times N$ covariance matrix.

The companion form of a VAR is its equivalent re-expression as an NP-variable first-order VAR, i.e.:

$$Y_t = A + BY_{t-1} + E_t \tag{2}$$

where $Y_t = \left[y_t', \ldots, y_{t-P+1}'\right]'$, $A = [\alpha', 0, \ldots, 0]'$, $Y_{t-1} = \left[y_{t-1}', \ldots, y_{t-P}'\right]'$, $E_t = [\varepsilon_t', 0, \ldots, 0]'$ (all $NP \times 1$ vectors), and $E_t \sim N\left(0_{NP\times 1}, \Omega_E\right)$ with Ω_ε in the upper-left corner being the only non-zero part of the $NP \times NP$ covariance matrix Ω_E. B is the $NP \times NP$ companion matrix:

$$B = \begin{bmatrix} & \beta & \\ I_{NP-N} & & 0_{NP\times N} \end{bmatrix} \tag{3}$$

where $\beta = [\beta_1, \ldots, \beta_P]$, an $N \times NP$ matrix, and $I_{NP\quad N}$ and $0_{NP\times N}$ are respectively the identity matrix and a zero matrix with dimensions given in their subscripts. Lütkepohl (2006) chapter 2 and Hamilton (1994) chapter 11 are standard references for the aspects outlined above.

The eigensystem representation of B is:

$$B = VDV^{-1} \; ; \; D = \text{diag}\left([D_1, \ldots, D_{NP}]\right) \; ; \; V = [V_1, \ldots, V_{NP}] \tag{4}$$

where D and the V are respectively the eigenvector and eigenvector matrices, both $NP \times NP$. I adopt the standard assumption of unique eigenvalues, which is also consistent with my subsequent empirical application, so the NP eigenvalues $[D_1, \ldots, D_{NP}]$ on the leading diagonal are the only non-zero entries of D. From Wilkinson (1965) p. 33-34, as referenced by Neumaier and Schneider (2001) in the context of decomposing a CVAR into its eigensystem components, V has the following columns:

$$V_k = \begin{bmatrix} S_k D_k^{P-1} \\ \vdots \\ S_k D_k \\ S_k \end{bmatrix} \text{ with } S_k = \begin{bmatrix} S_{1,k} \\ \vdots \\ S_{N-1,k} \\ 1 \end{bmatrix} \tag{5}$$

Each S_k is an $N \times 1$ vector and I have set the last elements of all S_k to 1 as the most convenient eigenvector normalization for this article.

The exposition above shows that NP eigenvalues plus $(N-1)\,NP$ unique values of $S = [S_1, \ldots, S_{NP}]$, i.e. S_u which excludes the normalizing row of ones, map exactly to the N^2P coefficients in β, and vice versa. An unconstrained VAR can therefore be estimated equivalently via its coefficients or its eigensystem parameters. Boshnakov (2002) mentions the potential for the latter but does not develop a framework for doing so. Additionally, as noted in the following section, an unconstrained EVAR estimation would be computationally inefficient relative to a CVAR estimation.

3 VAR and EVAR estimation

The log-likelihood function $\mathcal{L}\left(\theta, \Omega_\varepsilon\right)$ for a VAR conditioned on the initial P observations of the $N \times (P+T)$ dataset $\{y\}_{1-P}^{T}$ is (e.g. see Hamilton (1994) p. 293):

$$\mathcal{L}\left(\theta, \Omega_\varepsilon\right) = -\frac{NT}{2}\log\left(2\pi\right) - \frac{T}{2}\log\left(\det\left[\Omega_\varepsilon\right]\right) - \frac{1}{2}\sum_{t=1}^{T} \varepsilon_t' \Omega_\varepsilon^{-1} \varepsilon_t \tag{6}$$

where the parameter set θ determines the residuals $\varepsilon_t = y_t - \widehat{y}_t(\theta)$ and $\Omega_\varepsilon = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t'\varepsilon_t$. With $\theta = \beta$ and therefore $\varepsilon_t = y_t - \beta Y_{t\ 1}$, the linear form of $\mathcal{L}(\theta, \Omega_\varepsilon)$ with respect to the parameters allows analytic maximization, and the closed-form solution for β may be estimated using a series of OLS regressions (e.g. see Hamilton (1994) pp. 293-96).

Conversely, the EVAR is inherently non-linear with respect to the eigensystem parameters in D and S, so maximizing $\mathcal{L}(\theta, \Omega_\varepsilon)$ requires computationally expensive numerical methods. An EVAR should therefore only be applied when eigensystem constraints are considered to provide net benefits relative to a CVAR, such as the non-explosiveness constraint examples in this article. For that application, the constrained maximization, $\text{argmax}_{[\theta,\Omega_\varepsilon]}\mathcal{L}(\theta, \Omega_\varepsilon)$ subject to all $|D_k| < \gamma$, may be converted to unconstrained optimization using a parameter set $\theta = [x, W_u]$ which, as detailed below, produces the eigensystem parameters $[D(x|\gamma), S_u(W_u)]$, hence $\beta[D(x), S_u(W_u)]$ as outlined in section 2, and therefore $\varepsilon_t = y_t - \beta[D(x|\gamma), S_u(W_u)]Y_{t-1}$.

To allow for the general case of eigenvalues being either complex conjugate pairs (CCPs) or real values, they should fall within the magnitude constraint defined by a circle with radius γ in the complex plane, as illustrated in panel 2 of figure 1. This constraint may be achieved by mapping unconstrained real-valued pairs (x_k, x_{k+1}) to coefficients (ϕ_k, ϕ_{k+1}) within a generalized version of the standard AR(2) stability triangle, as illustrated in panel 2 of figure 1, which are then converted to eigenvalue pairs $(D_k, D_{k+1}) = \frac{1}{2}\phi_k \pm \frac{1}{2}\sqrt{\phi_k^2 + 4\phi_{k+1}}$ (i.e. the quadratic solution for AR(2) eigenvalues from AR(2) coefficients; e.g. see Hamilton (1994) p. 10).

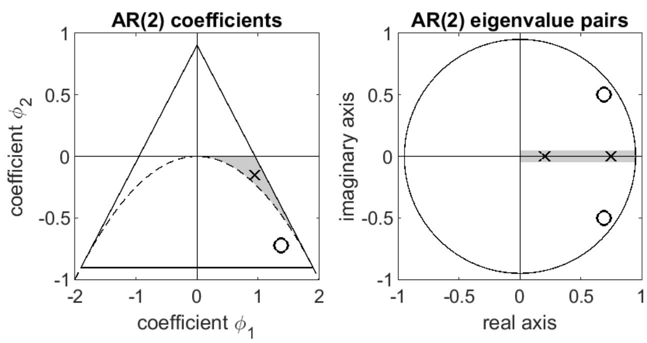

Figure 1: The generalized AR(2) triangle is used to obtain real or complex conjugate pairs of eigenvalues with an arbitrary magnitude constraint of γ. In this example $\gamma = 0.95$.

The generalized AR(2) triangle that maps (ϕ_k, ϕ_{k+1}) to $(|D_k|, |D_{k+1}|) < \gamma$ is defined by $\left|\frac{1}{2}\phi_k \pm \frac{1}{2}\sqrt{\phi_k^2 + 4\phi_{k+1}}\right| < \gamma$, which results in the lines defined by $|\phi_k| < 2\gamma$ (the triangle base) and $-\gamma^2 < \phi_{k+1} < \gamma^2 - \gamma|\phi_k|$ (the allowable height given ϕ_k). (ϕ_k, ϕ_{k+1}) pairs within the generalized AR(2) triangle are obtained from scaled shifted logistic functions of (x_k, x_{k+1}), i.e.:

$$\phi_k = 2\gamma\left(\frac{2}{1+\exp(-x_k)} - 1\right) \tag{7a}$$

$$\phi_{k+1} = \frac{\gamma|\phi_k|}{1+\exp(-x_{k+1})} - \gamma^2 \tag{7b}$$

When NP is odd, the last eigenvalue must be real, obtained as $D_{NP} = \gamma/[1+\exp(-x_{NP})]$.

There is no constraint on the parameters S_u for the application of the EVAR framework in this article, so an unconstrained real-valued $(N-1)\times NP$ matrix W_u is used to generate CCPs or real values of S_u according to their corresponding eigenvalues. That is, $S_u = W_u U$ where, if NP is even, U is the block-diagonal matrix:

$$U = \text{diag}\left(\begin{bmatrix} D_1 & D_2 \\ 1 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} D_{NP-1} & D_{NP} \\ 1 & 1 \end{bmatrix}\right) \tag{8}$$

Hence, each pair of real vectors $[W_{u,k}, W_{u,k+1}]$ multiplied into U gives $[W_{u,k}, W_{u,k+1}]U = [W_{u,k}D_k + W_{u,k+1}, W_{u,k+1}D_{k+1} + W_{u,k+1}]$, which are CCPs when (D_k, D_{k+1}) is a CCP, and real pairs when (D_k, D_{k+1}) is a real pair. If NP is odd, the last diagonal element of U is the real eigenvalue D_{NP}.

4 Empirical application

Figure 2 and table 1 illustrate the first example of applying the EVAR framework to end-quarter United States data for unemployment u, annual CPI inflation π, and the 3-month Treasury bill rate r (all from https://fred.stlouisfed.org). The sample is from March 1948 (the first complete period) to March 1980, which I have deliberately selected to make the comparisons most visually apparent. To ensure that all differences in the results are attributable only to the eigenvalue constraints, for all estimations I use mean-adjusted data (e.g. see Lütkepohl (2006) pp. 83-85, so $\overline{y}_t = y_t - \mu = [u_t, \pi_t, r_t]' - [\mu_u, \mu_\pi, \mu_r]'$ and α is set to 0), and the imposed lag length is $P = 2$ (as suggested by the Schwatz criterion for the CVAR).

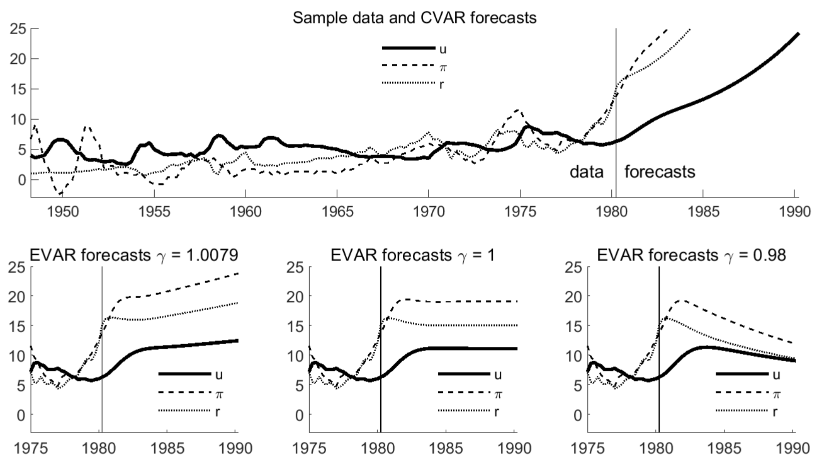

Figure 2: Forecasts from CVAR and EVAR estimations using the March 1948 to March 1981 sample.

The CVAR estimated with OLS results in an explosive model, as apparent from the forecasts in panel 1 of figure 2. The EVARs estimated with quasi-Newton optimization, with forecasts in the remaining panels of figure 2, use: (1) a local-to-unity constraint $\gamma = 1 + 1/T$, which ensures only mildly explosive dynamics (e.g. see Phillips and Magdalinos (2007) in the univariate context); (2) a unit constraint, which ensures non-explosive dynamics; and (3) a constraint less than unity, which ensures mean-reverting dynamics.

The right side of table 1 compares the eigensystem calculated from the CVAR estimation to the eigensystem from the EVAR estimation with $\gamma = 0.98$. The most notable difference is the largest CVAR real eigenvalue of 1.04 versus 0.98 for the EVAR (i.e. essentially equal to γ), which accounts for the distinctly different dynamics of explosiveness versus mean-reversion in figure 2. However, the left side of table 1 shows that the differences between the CVAR coefficients and the coefficients calculated from the EVAR estimation are small and non-systematic. Hence, it is not apparent how the eigensystem of a VAR could be explicitly controlled via coefficient constraints in a CVAR.

Table 1: Coefficients and eigensystems for CVAR and EVAR $\gamma = 0.98$ estimations

CVAR coefficient estimates β: β_1			β_2			eigensystem from CVAR: D_1 & $S_{u,1}$	$D_{2,3}$ & $S_{u,2,3}$	$D_{4,5}$ & $S_{u,4,5}$	D_6 & $S_{u,6}$
1.35	-0.03	-0.04	-0.46	0.05	0.06	1.04	0.72±0.22i	0.59±0.20i	0.19
-0.32	1.19	0.16	0.34	-0.40	0.12	0.34	-1.81±0.71i	-1.10±0.66i	0.48
-0.19	0.03	1.29	0.22	-0.01	-0.29	1.23	1.50±0.44i	-1.36±1.65i	1.38

coefficients from EVAR β: β_1			β_2			EVAR eigensystem estimates: D_1 & $S_{u,1}$	D_2 & $S_{u,2}$	$D_{3,4}$ & $S_{u,3,4}$	D_5 & $S_{u,5}$	D_6 & $S_{u,6}$
1.35	-0.03	-0.04	-0.45	0.06	0.07	0.98	0.73	0.66±0.23i	0.58	0.24
-0.31	1.20	0.17	0.32	-0.40	0.12	0.72	-3.36	-1.86±1.34i	-2.01	0.82
-0.18	0.03	1.30	0.16	-0.02	-0.31	1.51	-1.25	0.86±2.92i	-1.51	2.03

Figures 3 and 4 respectively illustrate the results from a time-varying (TV) CVAR and a TV-EVAR estimated over the sample from March 1948 to September 2008 (which avoids changes to the data-generating process for interest rates when they were constrained by the lower bound). For clarity and comparability, I use classical estimations with minimal parametrization, and I fix all aspects other than the state equation dynamics. Hence, based on the specification noted in Lubik and Matthes (2015), the state and measurement equations for the TV-CVAR are:

$$\begin{array}{rcll} \text{vec}\left(\beta_t'\right) & = & \text{vec}\left(\beta_{t-1}'\right) + \eta_t & ; \quad \eta_t \sim N\left(0, \kappa I_P\right) \\ \overline{y}_t & = & \left(I_N \otimes \overline{Y}_{t-1}'\right) \text{vec}\left(\beta_t'\right) + \nu_t & ; \quad \nu_t \sim N\left(0, \Omega_\nu\right) \end{array} \tag{9}$$

where $\text{vec}\left(\beta_t'\right)$ is the $N^2P \times 1$ vector of β_t coefficients, $\left(I_N \otimes \overline{Y}_{t-1}'\right)\text{vec}\left(\beta_t'\right)$ is the $N \times N^2P$ VAR data in seemingly unrelated regression form, $\kappa = 0.001$ is a single calibrated rate for the independent random-walk diffusions underlying the evolution of β_t, and the initial state variance matrix is the N^2P identity matrix. The initial values for β_t, the lag length $P = 2$, and the fixed value for Ω_v are those for a CVAR estimation over the full sample. I estimate the TV-CVAR with the Kalman filter. Panel 1 of figure 3 plots, for clarity, just the three diagonal coefficients of $\beta_{1,t}$ (i.e. the own-variable first-lag coefficients), and panel 2 plots the three largest eigenvalues magnitudes associated with β_t. The largest magnitude often evolves above 1, indicating that the VAR estimate in these periods is explosive.

The analogous TV-EVAR is specified as:

$$\begin{array}{rcll} \theta_t & = & \theta_{t-1} + \eta_t & ; \quad \eta_t \sim N\left(0, \kappa I_P\right) \\ \overline{y}_t & = & \left(I_N \otimes Y_{t-1}'\right) \text{vec}\left\{\beta\left[D\left(x_t|\gamma = 1\right), S_u\left(W_{u,t}\right)\right]\right\} + \nu_t & ; \quad \nu_t \sim N\left(0, \Omega_\nu\right) \end{array} \tag{10}$$

where $\theta_t = \left[x_t, \left\{\text{vec}\left(W_{u,t}'\right)'\right\}\right]'$, an $N^2P \times 1$ vector, and the eigenvalue magnitude constraint $\gamma = 1$ imposes non-explosive dynamics for the EVAR at all points in time. All other aspects are as for the TV-CVAR, including the initial values for $[x_t, W_{u,t}]$ that are calculated to replicate the initial β_t for the TV-CVAR . The measurement equation is now non-linear, so I estimate this model with the extended Kalman filter, i.e. by linearizing $\beta\left[D\left(x_t\right), S_u\left(W_{u,t}\right)\right]$ at each step t using a numerically calculated Jacobian with respect to the state variables.

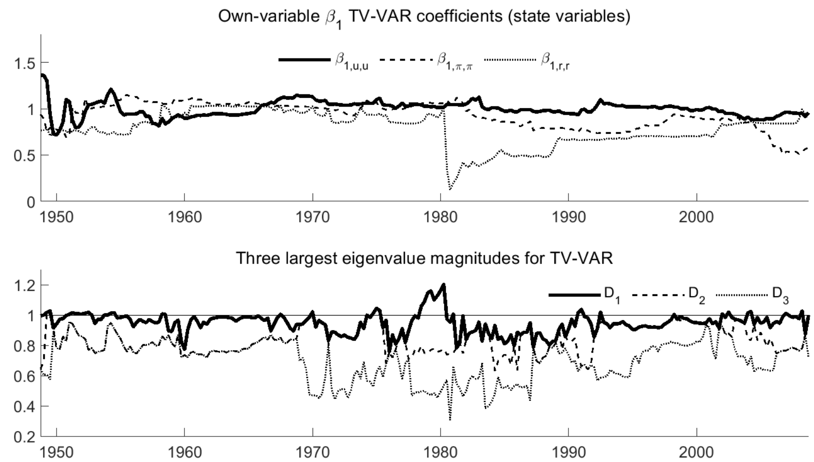

Figure 3: Estimated time series of coefficients and eigenvalue magnitudes for the TV-CVAR. Note that CCP eigenvalue magnitudes overlap.

For the EVAR, panels 1 to 3 of figure 4 respectively plot the first three state variables, the three largest eigenvalues, and the three diagonal coefficients of $\beta_{1,t}$ from $\beta\left[D\left(x\right), S_u\left(W_u\right)\right]_{1,t}$. The largest eigenvalue magnitude always remains below 1, consistent with the constraint $\gamma = 1$, indicating that the VAR estimate in all periods remains non-explosive.

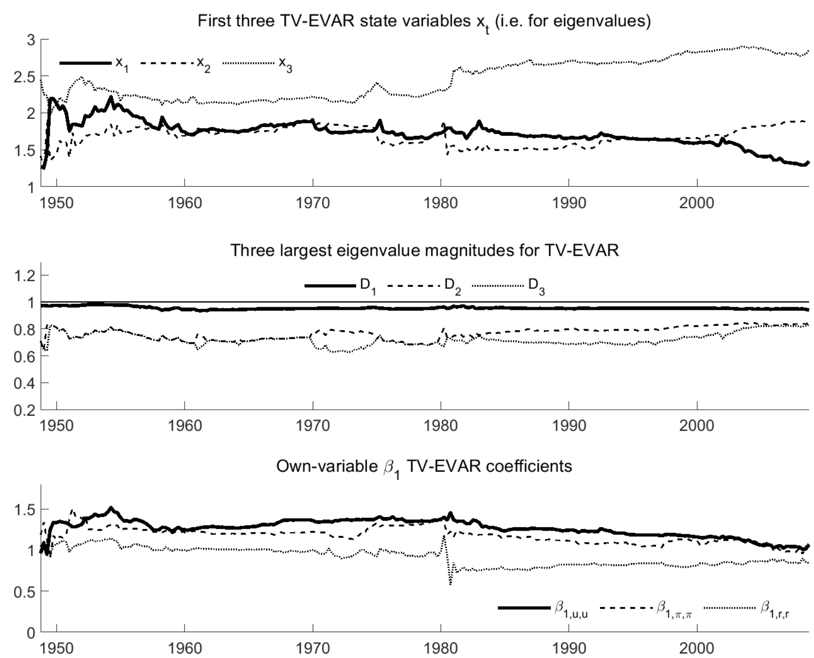

Figure 4: Estimated time series of state variables, eigenvalue magnitudes, and coefficients for the TV-EVAR. Note that CCP eigenvalue magnitudes overlap.

5 Conclusion

Specifying and estimating VARs directly from their eigenvalue and eigenvector parameters, as developed in this article, provides a unique framework for controlling the allowable dynamics of estimated VARs in ways that should prove useful for economic applications. For example, I illustrate how otherwise explosive forecasts from a CVAR may be avoided using eigenvalue magnitude constraints within an EVAR. Straightforward extensions to the EVAR application in this article are imposing that all eigenvalues be real, to avoid pronounced oscillatory dynamics, and testing zero and/or repeated eigenvalue restrictions to obtain parsimonious models. A related extension is imposing appropriate restrictions on the eigenvector parameters within the EVAR. With further development, the EVAR framework also offers an avenue for structural VAR identification, i.e. by using eigensystem specifications to control the allowable variable inter-relationships in ways consistent with economic principles.

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