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Abstract

The existence of second order moment or the finite variance is a commonly used assumption in financial time series analysis. We examine the validation of this condition for main stock index return series by applying the extreme value theory. We compare the performances of the adaptive Hill's estimator and the Smith's estimator for the tail index using Monte Carlo simulations for both *i.i.d* data and dependent data. The simulation results show that the Hill's estimator with adaptive data-based truncation number performs better in both cases. It has not only smaller bias but also smaller *MSE* when the true tail index α is not more than 2. Moreover, the Hill's estimator shows precise results for the hypothesis test of infinite variance. Applying the adaptive Hill's estimator to main stock index returns over the world, we find that for most indices, the second moment does exist for daily, weekly and monthly returns. However, an additional test for the existence of the fourth moment shows that generally the fourth moment does not exist, especially for daily returns. And these results don't change when a Gaussian-GARCH effect is removed from the original return series.

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Chapter 1 Introduction

Moment conditions are important assumptions in economic and econometric time series analysis. A lot of model buildings and the corresponding estimate methods are based on some moment assumptions of the innovations, hence the series itself. This is because various central limited theories (CLTs) (such as Lindeberg-Lévy central limit theory) are applied to show consistency of estimators and to deduce the asymptotic distribution for the estimators and test statistics. However, there are generally moment conditions for the CLTs to be valid. In most cases, the second moment of the series or the function form should exist; hence the variance should be finite. Sometimes we need higher order moment conditions, such as the finite fourth moment.

When applying these results to financial time series, such as returns series of equity and exchange rate, a natural question is whether those moment assumptions are satisfied. It is well known that many financial variables have much heavier tails than the normal distribution. This is the so called heavy-tailed stylized fact. One important consequence of heavy tails is that some moments may not exist. It has been found empirically that high order moments may not exist. For example, in several studies of exchange rate yields (Koedijk et al., 1990; Jansen and DeVries, 1991), for different currencies and at different time frequencies, there is strong evidence that the maximal moment exponent of the empirical densities for these series are less than four, which implies that the finite fourth moments of these series do not exist; and in some cases, even the second moment does not exist. Such evidence also appears in the stock return series; for example, see Loretan and Phillips (1994).

In this study our interest is in the second order moment, hence the variance of the stock index returns. This is a basic condition in most financial time series analysis and also for developing the optimal asset allocation theory. Different estimation methods have been proposed and the asymptotic distribution of estimators is developed based on this condition. Therefore, it is important to check the empirical validity of the finite variance assumption.

We empirically check the validity of this assumption for main stock index return series, including daily data, weekly data and monthly data. As a byproduct of our method, we give an estimate of the maximal moment exponent of the series. Our method applies the extreme value theory to estimate the tail index of these series, which is also the maximal moment exponent. However, in order to choose a better way to do such estimation, we compare the performances of two distribution assumptions of the tails in two Monte Carlo simulations: the Pareto distribution and the Generalized Pareto (GP) distribution. We find the Hill's estimator with data-adaptive truncation number (Hill, 1975; Hall and Welsh, 1985; Phillips et al., 1996), which is based on the Pareto distribution, performs better in both cases. While the GP distribution nests the Pareto type, the tail index estimators through the *peaks over threshold* (POT) method, which are based on GP distribution (Smith, 1987), perform less satisfactorily in our simulations.

The rest of the paper is organized as follows. In Chapter 2, we give a literature review of the methods applied in this paper. We describe details of the methods in Chapter 3. Monte Carol simulations and results are presented in Chapter 4. In Chapter 5, we present the empirical results for main stock indices return series. Finally, we conclude in Chapter 6.

Chapter 2 Literature Review

According to the characteristics of stock return series, mainly the stylized fact of heavy tails, it is useful to examine the empirical distribution directly. In fact, heavy-tailed distributions, such as the Pareto distribution, have been proved very useful in modeling such phenomena in finance and other subject areas. For further details, we refer to Embrechts et al. (1997) and papers collected in FinkenstÄadt and Rootzen eds. (2003).

Here we review several approaches that are more relevant to the present study. One is to apply the above distributions and to fit a density function to the whole series, and then use the estimated parameterized density to make judgments. There are examples of this approach applied to stock returns, see a comprehensive literature review by Mittnik and Rachev (1993). However, as mentioned by Kearns and Pagan (1992), there is a potential problem with this strategy. The fitted density could be heavily affected by the vast majority of observations that lie in the center of the density, whereas our interest is really focused upon the tails. Consequently, a literature has developed that seeks to determine the probability of large deviations by concentrating attention upon the ‘‘tails’’ and estimating that part of the density only (see Hols and de Vries (1991)). In this way, the estimation of “tail index” becomes the subject of interest. In fact, it was first suggested by Mandelbrot (1963) that many economic and financial series were best modeled as independently distributed stable processes (which are characterized by a tail index, or characteristic exponent, less than 2). Subsequent empirical papers included Fama (1965), Officer (1972), Blattberg and Gonedes (1974), Hsu et al. (1974), and Akgiray and Booth (1988), which estimated the tail index for various stock return series, investigating whether such series were consistent with stable laws. For stable laws, the tail index is restricted to be less or equal to 2, however, the tail index could take any positive number for real data. Thus it is necessary to discriminate between different probability models, e.g., Jansen and deVries (1991), Hols and deVries (1991).

Summarily, in order to estimate and make an inference about the tail index, it is necessary to apply the extreme value theory (e.g.,Leadbetter et al.(1983)), which studies the limiting distributions of large realizations (the order statistics) of a series. The extreme value theory is based on the distributions mentioned above and has been widely used to estimate tail index of financial time series. See, for example, the Hill's estimator based on Pareto distribution, the POT estimator based on GP distribution and estimators from the stable law.

Among these, a simple way is to assume that the tail shape is of the Pareto-type. Actually, based on the Pareto distribution assumption of the tails, alternative tail index estimators have been proposed. A common feature is that all these estimators are based on the order statistics. The most widely used estimator is the Hill's estimator, which was first proposed by Hill (1975) and by Weissman (1978) from a different perspective for *i.i.d* sequence. Other estimators include Pickands (1975), deHaan and Resnick (1980), Teugels (1981), et al. Kearns and Pagan (1992) compared three estimators through Monte Carlo simulations, corresponding to Pickands (1975), Hill (1975) and de Haan and Resnick (1980), and found that the Hill's estimator performs better than the others. In fact, a lot of attentions have been paid to Hill's estimator in the literature. Studies of Hill's estimator in the *i.i.d* setting include Hall (1982), Mason (1982,1988), Davis and Resnick (1984)，Hall and Welsh (1985), Haeusler and Teugels (1985), Csŏrgő, Deheuvels and Mason (1985), Beirlant and Teugels (1989), Mason and Turova (1994), Geluk et al. (1997), Resnick and Starica (1997a, 1997b), de Haan and Resnick (1998). Recent research has been focusing on applying the Hill's estimator to dependent data. For example, Hsing (1991) studied Hill's estimator under weak dependence assumption; Resnick and Starica (1995, 1998) proved the consistency of Hill's estimator for certain classes of heavy-tailed stationary processes. Other related papers which study the Hill's estimator in dependent case include Rootzen, Leabetter and de Haan (1990) , Rootzen (1995), Rootzen et al. (1998) , Drees (2000), Ling and Peng (2004).

Since the performance of the Hill's estimator is well studied for *i.i.d* sequences and in dependent cases, we apply it in our paper.

However, there is a potential problem in the use of the Hill's estimator, which is how to determine the optimal truncation number (m) of the order statistics. In fact, the choice is critical to the performance of the Hill's estimator. Both Hill (1975) and Hall (1982) pointed out that the truncation number should be determined based on tail characteristic of the data itself. Dumouchel (1983) had suggested that m should be less than 10% of the sample size. To choose the optimal truncation number, Hall and Welsh (1985) suggested an adaptive data-based procedure. However, this method has not attracted much attention in the empirical literature. Phillips et al. (1996) applied this procedure to calculate Hill's estimator. Drees and Kaufmann (1998) proposed a sequential procedure to get a consistent estimator of the truncation number. However, their method only performs well for few certain distributions and in extremely large samples. Generally, the performance of their method is poorer than that proposed by Hall and Welsh (1985). Another data-driven approach is to use re-sampling procedure where large sample size is usually necessary, see Hall (1990), Danielsson et al. (2001). In our paper we choose the optimal truncation number using the adaptive data-based procedure suggested by Phillips et al. (1996).

Except for the well-known Hill's estimator for tail index, there are also other estimators which are not based on order statistics. One estimator is proposed by Smith (1987). This estimator is based on POT method and GP distribution; see Smith (1984, 1987). Koopman and Shephard (2003) applied this method to test for infinite variance in important sampling in the context of stochastic volatility models.

As a summary, we will apply the Hill's estimator with adaptive data-based truncation number choosing procedure and Smith's (1987) method. We will use two simulation designs to compare the performance of the two methods. One is the *i.i.d* case and the other is the dependent case with a GARCH structure. We give the details of these two estimators in Chapter 3.

Chapter 3 Method Description

The methods used in this paper to estimate the tail index include the Hill's estimator and Smith's POT method as mentioned in Chapter 2. Details of the two methods are given below.

3.1 the Adaptive Hill's Estimator

Suppose we have an *i.i.d* sequence $\{X_t\}_1^n$, where n is the sample size. If the tail behavior of X_t follows the Pareto- Lévy form , for example , as generalized in Phillips et al.(1996)

$$P(X > x) = pCx^{-\alpha}(1 + a_1(x)), x > 0 \tag{1}$$

$$P(X < -x) = qCx^{-\alpha}(1 + a_2(x)), x > 0 \tag{2}$$

where $a_i(x) \to 0\,(i = 1, 2)$ as $x \to \infty$ and $p, q \geq 0$. The parameter C is a scale parameter and α , is the tail index, or the maximal moment exponent in the sense that $\alpha = \sup\left\{r > 0 : E\left|X\right|^r < \infty\right\}$, which determines the tail slope. Models (1) and (2) are corresponding to right tail and left tail.

Indeed, α is what we are concerned about. If α is not less than 2 for both right tail and left tail, then the variance should exist. However, if anyone of the two is significantly less than 2, then the second moment does not exist which means the variance is infinite.

The unknown parameters in (1) and (2) could be estimated by conditional maximum likelihood method based on order statistics of the sample under Pareto tails assumption, for example, $a_i(x) = 0$ as in (1) and (2). For simplicity, we only describe the method for parameters in (1); however, the idea is the same for (2).

Let $X_{n,1} \leq X_{n,2} \leq \cdots \leq X_{n,n}$ denote the order statistics of $\{X_t\}_1^n$ in ascending order. Then according to Hill (1975), the Hill's estimator for α and C in (1) is defined as

$$\hat{\alpha}_H = \left(m^{-1} \sum_{j=1}^{m} \ln X_{n,n-j+1} - \ln X_{n,n-m} \right)^{-1} \quad (3)$$

$$\hat{C} = (m/n) X_{n,n-m}^{\hat{\alpha}_H} \quad (4)$$

where m is the order statistic truncation number.

The asymptotic theory for the estimator was established by Hall (1982) in the case of tail distributions of forms (1) and (2). When $a_i(x) = O(x^{-\beta}), \beta > 0$, it was showed that if $m = m(n)$ so that $m \to \infty$ and $m / n^{2\beta/2\beta+\alpha} \to 0$ as $n \to \infty$, then we have the limit distribution for $\hat{\alpha}_H$

$$m^{1/2}(\hat{\alpha}_H - \alpha) \xrightarrow{d} N(0, \alpha^2) \quad (5)$$

(Theorem 2 of Hall, 1982).

The asymptotic distribution (5) holds when the truncation number m is optimally selected at order of $n^{2\beta/2\beta+\alpha}$. In fact, a central problem in estimating α is how to choose m as we have mentioned in Chapter 2. Hall and Welsh (1985) proposed a procedure to select the optimal order statistic truncation number $m = m(n)$ to minimize the asymptotic mean squared error for the estimators, which could be called adaptive data-based truncation number choosing procedure.

The details of this procedure were presented in Section 4 of Hall and Welsh (1985). A simplified version was given in Phillips et al. (1996). When the tail distributions are of forms (1) and (2) with $\alpha_i(x) = Dx^{-\alpha} + o(x^{-\alpha})$ and $p = q - 1$, then by choosing

$$m = m(n) = \left[\lambda n^{2/3} \right], with\ \lambda = (2C^2 / D)^{1/3}$$

and [ ] represents the integer part of its argument, the mean squared error of the limit distribution of $\hat{\alpha}_H$ will be minimized. The parameter λ could be estimated adaptively by

$$\hat{\lambda} = \left| \hat{\alpha}_m / 2^{1/2} (n / t_1)(\hat{\alpha}_{t_1} - \hat{\alpha}_m) \right|^{2/3}$$

where $\hat{\alpha}_m$ and $\hat{\alpha}_{t_1}$ are preliminary estimators of α using formula (3) with data truncation numbers $m(n)=\left[n^{\sigma}\right]$ and $m(n)=\left[n^{\tau}\right]$, respectively, where $0<\sigma<2/3$ and $2/3<\tau<1$. Then the optimal choice of m could be obtained by

$$\hat{m}=\left[\hat{\lambda}\, n^{2/3}\right] \tag{6}$$

Actually, this is the situation when $\rho=1$ in Hall and Welsh (1985). However, this situation is usually the case as indicated by their paper. Moreover, it is superior as suggested by Drees and Kaufmann (1998).

In sum, based on formulas (3) and (6), we could obtain the adaptive Hill's estimator of tail index for the sequence $\{X_t\}_1^n$.

3.2 the Smith's (1987) Method

Again, suppose we have an *i.i.d* sequence $\{X_t\}_1^n$, where n is the sample size. For a threshold value $u>0$, defining the excesses $Y_t = X_t - u,\ X_t \ge u, t=1,\cdots,s$, where s is the sample size of the constructed sequence $\{Y_t\}_1^s$, then as Smith (1987) had argued, as u increases, the limit distribution of Y_t is the generalized Pareto (GP) distribution. In particular, Y_t has the following asymptotic density

$$f(y)=\frac{1}{\gamma}(1+k\frac{y}{\gamma})^{-\frac{1}{k}-1},\ \gamma>0 \tag{7}$$

the range of y being $0\le y\le\infty\,(k\ge 0)$ or $0\le y\le -\gamma/k\,(k<0)$.

The case of $k<0$ means that $\{Y_t\}_1^s$, hence $\{X_t\}_1^n$ have some upper bound which guarantees the existence of all moments. However, this is quite unusual in practice as pointed out by Koopman and Shephard (2003).

For model (7), the key is that only $1/k$ moments exist. If we estimate k, then we can determine the order of moments for $\{X_t\}_1^n$ by focusing on $1/\hat{k}$.

The unknown parameter vector $\lambda = (k, \gamma)'$ could be estimated by maximum likelihood method as discussed in Smith (1987) and Koopman and Shephard (2003). In fact the log-likelihood of $\{Y_t\}_1^s$ equals

$$\log f(y;\lambda) = -s\log\gamma - (1+\frac{1}{k})\sum_{i=1}^{s}\log z_i \tag{8}$$

where $z_i = 1 + k\gamma^{-1}y_i$ .Based on the standard method of Fisher scoring where the score vector is

$$\begin{pmatrix} s_k \\ s_\gamma \end{pmatrix} = \frac{d\log f(y;\lambda)}{d\lambda} = \begin{pmatrix} k^{-2}\sum_{i=1}^{s}\log z_i - (1+k^{-1})\gamma^{-1}\sum_{i=1}^{s}\log z_i / y_i \\ -s\gamma^{-1} + (1+k)\gamma^{-2}\sum_{i=1}^{s}\log z_i / y_i \end{pmatrix}$$

and the expected information matrix nI where

$$I = \frac{1}{(1+2k)(1+k)}\begin{pmatrix} 1+k & -1 \\ -1 & 2 \end{pmatrix},$$

then we could obtain the maximum likelihood estimator $\hat{\lambda}$ and the asymptotic distribution of $\hat{\lambda}$ is given by

$$\sqrt{s}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0, I^{-1}),\ where\ I^{-1} = (1+k)\begin{pmatrix} 2 & 1 \\ 1 & 1+k \end{pmatrix} \tag{9}$$

The above method is called the *peak over threshold* (POT) approach; see Smith (1984, 1987). It had been used by Koopman and Shephard (2003) to test existence of variance of the weights in important sampling. Similar to the problem of choosing m in the Hill's estimator, there is the problem about how to choose the threshold value u . Koopman and Shephard (2003) suggested that u should be sufficient large that only a small portion of the sample is used to get better results, for example, u taking value of the 0.99 upper quantile of the series in their study.

3.3 Hypothesis Testing

Our purpose is to test existence of the second moment of $\{X_t\}_1^n$, so it is necessary to do hypothesis test and make statistic inference based on both estimators .

For the adaptive Hill's estimator, our null hypothesis should be $H_0 : \alpha = 2$ and alternative hypothesis $H_1 : \alpha < 2$. Based on formula (5), we have the test statistics

$$t_H = \sqrt{m}/2(\hat{\alpha} - 2) \tag{10}$$

The null hypothesis H_0 will be rejected when t_H takes a negative value with large $|t_H|$ compared to a standard normal distribution.

For Smith's (1987) estimator, since we are interested in whether k is significantly different from $1/2$, then the null hypothesis could be $H_0 : k = \frac{1}{2}$ and the alternative hypothesis $H_1 : k > \frac{1}{2}$, as showed in Koopman and Shephard (2003).

Therefore, a signed t-test is available for H_0 based on formula (9), which is

$$t_s = \sqrt{s/3}\left(\hat{k} - \frac{1}{2}\right) \tag{11}$$

The null hypothesis H_0 will be rejected for large positive value of t_s compared to a standard normal distribution.

So far, we have introduced the procedures for estimating the adaptive Hill's estimator and Smith's (1987) estimator, and we also establish the hypothesis test for infinite variance. In the next chapter, we examine their finite sample performance through Monte Carlo simulations.

Chapter 4 Simulation Study

We compare performances of the adaptive Hill's estimator and Smith's (1987) estimator in this chapter. Our simulation study consists of two parts[1]. In the first simulation design, we generate *i.i.d* sequences which have the Pareto distribution with different tail indices and then estimate the tail index and do the statistic test using the methods described in Chapter 3. In our second simulation design, the generated data has a GARCH (1, 1) structure with innovations being Pareto distribution and we re-do the estimation and the test for the tail index. To consider the GARCH (1, 1) situation is because this is also a typical phenomenon in financial time series, especially for equity return series, see Bollerslev et al. (1992). Since we intend to test for daily, weekly and monthly stock returns, which the sample size are typically different in practice, we construct the sample size to be large, moderate and small in both simulations, corresponding to these time frequencies.

We set $\sigma = 0.4, \tau = 0.8$ when using formula (6) to choose the optimal truncation number s in estimating the Hill's estimator for both simulations. The values are determined arbitrarily from the ranges $0<\sigma<2/3$ and $2/3 < \tau < 1$. In fact, as showed in Phillips et al. (1996), the results are fairly stable to different choices of these two parameters. For Smith's estimator, we follow the suggestion by Koopman and Shephard (2003) by setting the threshold value u taking value of 0.95 upper quantile and 0.99 upper quantile for each series respectively, which means 5% and 1% largest values of the original sample (unordered) are used.

4.1 Simulation Ⅰ

In this simulation, we generate *i.i.d* sequence which has the following Pareto density

$$f(x) = \alpha x^{-\alpha-1}$$

where α is the tail shape parameter or the tail index[2].

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[1] All simulations in this Chapterand Empirical study in Chapter 5 are implemented in Matlab; matlab codes are provided in Appendix B.

[2] The true data is generated from the uniform distribution in [0, 1] after density transformation.

We consider the situations when α takes value from 0.5 to 4.5 by 0.5, the critical case is that when α equals 2. Three different sample sizes are taken into account, for $n = 500$, 3000, 10000, corresponding to small, moderate and large sample size, respectively. For each case, 1000 experiments are repeated. The simulation results are presented in Table 1—3.

From Table 1—3, we could see that as the sample size increases, the performances for both estimators become better and better. The pattern is particularly obvious for the Smith's estimator. In fact, when the sample size is sufficiently large (i.e. $n = 10000$), the results are almost perfect for $\hat{\alpha}_H$ and $\hat{k}_{s,1}$; the two estimators ($\hat{\alpha}_H$ and $1/\hat{k}_{s,1}$) are very close to the true value of α, and the bias and the standard deviation of the two estimators are rather small relative to the size of each value of α.

For the adaptive Hill's estimator $\hat{\alpha}_H$, as showed in Table 1—3, the results are pretty good for each value of α and for each sample size, although for large sample size the performance is better. Under every situation, both the real standard deviation ($R-std$) and theoretical standard deviation ($T-std$) increases as the value of true tail index α increases, and the real standard deviation ($R-std$) of $\hat{\alpha}_H$ is larger than the theoretical standard deviation ($T-std$) which is computed from the asymptotic distribution of formula (5). However, the distance between these two standard deviations becomes smaller as the sample size n increases. In fact, the two are almost the same for larger sample as we could see in Table 3. This result is consistent with Kearns and Pagan (1992) that the theoretical standard deviation for tail index estimator is generally underestimated. The adaptive data-based truncation number m increases as the sample size n becomes larger. When the sample size n is fixed, $\overline{m}$, the average value of m ,is the same for every value of α. However, the ratio of the truncation number to sample size ($\overline{m}/n$) becomes smaller and smaller as the sample size increases, from roughly 0.17 to about 0.08, see Table 1—3, which is consistent with condition in Hall and Welsh (1985) .

For Smith's estimator $\hat{k}_1$, the inverse $1/\hat{k}_1$ is really the estimator for α.As we could see from Table 1—3, when the threshold u takes value of 0.95 upper quantile, the performance of $\hat{k}_1$ is much better than $\hat{k}_2$ that when u takes value of 0.99 upper quantile. This is true for each value of α and every case of sample size. Both the bias and the real standard deviation for $\hat{k}_1$ are smaller than that for $\hat{k}_2$. In fact, the performances of $\hat{k}_2$ are rather bad for small and moderate sample sizes as we could see from Table 1 and Table 2. Unlike the Hill's estimator, the real standard deviations $(R_1 - std)$ and $(R_2 - std)$ decreases as the sample size n increases.

Comparing the adaptive Hill's estimator with the Smith's estimator, we find that under the *i.i.d* setting with the Pareto distribution, generally the adaptive Hill's estimator is better than the Smith's estimator for each sample size. The adaptive Hill's estimator has a surprisingly small bias for each value of α in each case whereas the Smith's estimator only performs well when the sample size is large (see Table 3) or the value of α is not greater than 2 (see Table 1 and Table 2). The bias of the Smith's estimator is substantially large when α is large, especially when the sample size is small. Generally, although the standard deviation of the Hill's estimator is slight larger than that of the Smith's estimator when α is larger than 2.5, it is smaller when α is less than 2. For the cases around $\alpha = 2$, the two are very close. In terms of mean square error (MSE), generally, when α is less than 2, $RMSE$ of $\hat{\alpha}_H$ is smaller and for $\alpha > 2.5$, $RMSE$ of $\hat{k}_1$ is smaller. The two are close around $\alpha = 2$.

Importantly, when we do the hypothesis testing as described in Chapter 3, the test based on the Hill's estimator gives the correct answer for each value of α and each sample size at the 1% significant level as showed in the tables, whereas the test based on the Smith's estimator only performs well when the sample size is large. For small and moderate sample sizes, the test gives correct answers for only a very narrow range of α, however , the significant level is much lower than that for the Hill's estimator. The difference is extremely obvious around the case when α takes a true value of 2, see Table 1 and Table 2. In sum, the test based on the Hill's estimator gives much better results than that based on the Smith's estimator.

In conclusion, for the *i.i.d* data with the Pareto distribution, the adaptive Hill's estimator has much smaller bias and smaller MSE (for α less 2.5) than the Smith's estimator, especially when the sample size is not large. And test for the adaptive Hill's estimator shows much higher significant level in the hypothesis test universally. The above results imply that the adaptive Hill's estimator outperforms the Smith's estimator.

Table 1 : Simulation Results with i.i.d Pareto Distribution ($n=500$)

α	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$\hat{\alpha}_H$	0.494	0.987	1.481	1.974	2.468	2.961	3.455	3.949	4.442
$R-std$	0.109	0.217	0.326	0.435	0.543	0.652	0.761	0.869	0.978
$T-std$	0.060	0.120	0.180	0.240	0.300	0.360	0.420	0.480	0.540
$RMSE(\hat{\alpha}_H)$	0.109	0.218	0.327	0.435	0.544	0.653	0.762	0.871	0.980
$\overline{m}$	87	87	87	87	87	87	87	87	87
t_H	**-6.478*****	**-4.273*****	**-2.068****	**0.137**	2.342***	4.547***	6.752***	8.958***	11.163***
$\hat{k}_1$	1.920	0.937	0.603	0.435	0.332	0.263	0.214	0.177	0.149
$1/\hat{k}_1$	0.521	1.067	1.657	2.301	3.016	3.798	4.669	5.647	6.730
R_1-std	0.685	0.450	0.378	0.346	0.331	0.319	0.312	0.307	0.302
$RMSE(\hat{k}_1)$	0.690	0.454	0.384	0.352	0.338	0.327	0.320	0.316	0.311
$t_{s.1}$	**4.137*****	**1.300***	**0.338**	**-0.150**	-0.447	-0.644	-0.786	-0.893	-0.975
$\hat{k}_2$	1.533	0.488	0.136	-0.056	-0.177	-0.250	-0.307	-0.343	-0.374

$1/\hat{k}_2$	0.652	2.047	7.345	-17.903	-5.644	-3.996	-3.256	-2.919	-2.674
R_2-std	1.789	1.375	1.232	1.152	1.094	1.062	1.036	1.014	1.001
$RMSE(\hat{k}_2)$	1.849	1.467	1.341	1.279	1.237	1.212	1.194	1.175	1.165
$t_{s.2}$	**1.351***	**0.003**	**-0.452**	**-0.700**	-0.857	-0.951	-1.025	-1.070	-1.111

Notes: 1) for each value of α, 1000 replications are implemented;

2) α_H, m, t_H, k, t_s are defined as that in Chapter3, $R-std$ is the real standard deviation from simulation and $T-std$ is theoretical standard deviation computed from asymptotic distribution (5) corresponding to the true value of each α;RMSE is the root of mean square error;

3) the subscription "1" corresponding to the threshold value u being 0.95 upper quantile and "2" corresponding to the threshold value u being 0.99 upper quantile;

*4) the one-side 90%, 95% and 99% critical values for standard normal distribution are $\pm$1.28, $\pm$1.65 and $\pm$ 2.33, respectively. ***: 1% significant level; **: 5% significant level; *:10% significant level.*

Table 2 : Simulation Results with *i.i.d* Pareto Distribution ($n=3000$)

α	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$\hat{\alpha}_H$	0.499	0.998	1.497	1.995	2.494	2.993	3.492	3.991	4.490
$R-std$	0.051	0.102	0.153	0.204	0.255	0.306	0.357	0.408	0.459
$T-std$	0.034	0.067	0.101	0.134	0.168	0.201	0.235	0.268	0.302
$RMSE(\hat{\alpha}_H)$	0.051	0.102	0.153	0.204	0.255	0.306	0.357	0.408	0.459
$\overline{m}$	322	322	322	322	322	322	322	322	322
t_H	**-12.397***	**-8.246***	**-4.095***	**0.057**	4.208***	8.359***	12.510***	16.661***	20.812***
$\hat{k}_1$	1.984	0.987	0.654	0.487	0.387	0.320	0.272	0.236	0.208

$1/\hat{k}_1$	0.504	1.013	1.529	2.052	2.584	3.124	3.674	4.233	4.802
$R_1 - std$	0.280	0.177	0.144	0.129	0.120	0.114	0.110	0.107	0.105
$RMSE(\hat{k}_1)$	0.280	0.178	0.145	0.129	0.120	0.115	0.111	0.108	0.105
$t_{s.1}$	**10.513***	**3.463***	**1.108**	**-0.072**	-0.781	-1.255*	-1.593*	-1.848**	-2.046**
$\hat{k}_2$	1.970	0.959	0.617	0.445	0.340	0.270	0.220	0.182	0.153
$1/\hat{k}_2$	0.508	1.043	1.620	2.249	2.939	3.699	4.542	5.480	6.532
$R_2 - std$	0.574	0.391	0.335	0.309	0.294	0.285	0.278	0.274	0.270
$RMSE(\hat{k}_2)$	0.574	0.393	0.339	0.314	0.300	0.292	0.286	0.282	0.279
$t_{s.2}$	**4.656***	**1.460***	**0.379**	**-0.167**	-0.497	-0.718	-0.877	-0.996	-1.089

Notes: Notations are the same as in Table 1.

Table 3 : Simulation Results with *i.i.d* Pareto Distribution ($n = 10000$)

α	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$\hat{\alpha}_H$	0.501	1.002	1.503	2.005	2.506	3.007	3.508	4.009	4.510
$R - std$	0.033	0.066	0.100	0.133	0.166	0.199	0.232	0.266	0.299
$T - std$	0.022	0.044	0.067	0.089	0.111	0.133	0.156	0.178	0.200
$RMSE(\hat{\alpha}_H)$	0.033	0.066	0.100	0.133	0.166	0.199	0.232	0.266	0.299
$\bar{m}$	770	770	770	770	770	770	770	770	770
t_H	**-18.984***	**-12.624***	**-6.264***	**0.096**	6.456***	12.816***	19.176***	25.536***	31.896***
$\hat{k}_1$	1.992	0.994	0.661	0.495	0.395	0.328	0.281	0.245	0.217
$1/\hat{k}_1$	0.502	1.006	1.512	2.021	2.533	3.047	3.564	4.084	4.607

$R_1 - std$	0.159	0.098	0.079	0.069	0.064	0.060	0.058	0.056	0.055
$RMSE(\hat{k}_1)$	0.159	0.099	0.079	0.070	0.064	0.060	0.058	0.056	0.055
$t_{s.1}$	**19.270*** **	**6.389*** **	**2.094** **	**-0.055**	-1.345*	-2.205**	-2.820***	-3.281***	-3.640***
$\hat{k}_2$	1.983	0.985	0.652	0.484	0.384	0.317	0.268	0.232	0.204
$1/\hat{k}_2$	0.504	1.015	1.534	2.064	2.605	3.159	3.725	4.304	4.897
$R_2 - std$	0.311	0.205	0.172	0.156	0.147	0.141	0.136	0.133	0.131
$RMSE(\hat{k}_2)$	0.312	0.206	0.173	0.157	0.148	0.142	0.138	0.135	0.132
$t_{s.2}$	**8.565*** **	**2.808*** **	**0.882**	**-0.084**	-0.665	-1.053	-1.331*	-1.540*	-1.702**

Notes: notations are the same as in Table 1.

4.2 Simulation Ⅱ

In this simulation, the generated data come from a GARCH (1, 1) structure with Pareto distributed innovations, that is

$$x_t = \sqrt{h_t}\varepsilon_t$$
$$h_t = 0.001 + 0.1x_{t-1}^2 + 0.85h_{t-1}$$

where ε_t is *i.i.d* Pareto distribution with different tail shape values as in Simulation Ⅰ. The three different situations of sample size have been considered, too. For the GARCH (1, 1) structure, we set the parameters unchanged for every case in the simulation. The values of these three parameters are selected artificially, however, we do consider that GARCH effect is usually larger than ARCH effect and the sum of coefficients is close to one in practice. The simulation results are presented in Table 4—6.

From Table 4—6, we could see that the results are similar as that in Simulation

Ⅰ. When the sample size u increases, both estimators improve. A slight difference is that when α takes value of 0.5, both estimators are poor. This is particularly obvious in Table 6 where we cannot get a reasonable result. In fact, this is the case when none of the moments exist. And the real standard deviation ($R-std$) and the theoretical standard deviation ($T-std$) for the Hill's estimator are very close in each case of sample size, which is different from results in Kearns and Pagan (1992) who found the distance between those two standard deviation becomes larger when dependent relationship exists. However, in their study, the truncation number m was not selected by the adaptive data-based method used in this paper. The ratio of $\overline{m}/n$ decreases as the sample size n increases. Again, the Smith's estimator performs better when the threshold value u takes value of 0.95 upper quantile of the series than that when u is 0.99 upper quantile for every situation.

Similar as results in Simulation Ⅰ, in general the adaptive Hill's estimator $\hat{\alpha}_H$ has much smaller bias for each value of α and each case of sample size than the Smith's estimator, especially when sample size is not large, see Table 4 and Table 5. Actually, in those cases, the bias for Smith's estimator is unacceptably large when α is large than 2. If we look at the mean square error (MSE), the adaptive Hill's estimator has smaller MSE when α is not greater than 2. Although the Smith's estimator performs very well for in large samples as showed in Table 6, the adaptive Hill's estimator always outperforms it in the hypothesis testing for each case considered. In fact, the test performance the adaptive Hill's estimator is pretty good for each value of α, even when the sample size is small.

Table 4 : Simulation Results for GARCH (1,1) series with Pareto Distribution ($n=500$)

α	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$\hat{\alpha}_H$	0.575	0.998	1.516	2.017	2.523	3.021	3.525	4.030	4.528
$R-std$	0.163	0.169	0.231	0.307	0.384	0.464	0.529	0.613	0.674

$T-std$	0.089	0.119	0.183	0.243	0.303	0.364	0.425	0.483	0.545
$RMSE(\hat{\alpha}_H)$	0.180	0.169	0.231	0.308	0.385	0.464	0.530	0.613	0.674
$\overline{m}$	35	94	92	95	97	96	97	97	97
t_H	**-4.153*** **	**-4.608*** **	**-2.211** **	**0.024**	2.340***	4.566***	6.874***	9.197***	11.447***
$\hat{k}_1$	1.541	0.968	0.634	0.457	0.351	0.280	0.228	0.192	0.161
$1/\hat{k}_1$	0.649	1.033	1.577	2.188	2.847	3.572	4.390	5.222	6.216
R_1-std	1.001	0.483	0.379	0.346	0.334	0.322	0.313	0.307	0.303
$RMSE(\hat{k}_1)$	1.101	0.484	0.381	0.349	0.338	0.327	0.318	0.313	0.309
$t_{s.1}$	**3.006*** **	**1.351***	**0.387**	**-0.124**	-0.429	-0.635	-0.786	-0.891	-0.979
$\hat{k}_2$	0.320	0.398	0.139	-0.042	-0.179	-0.270	-0.315	-0.355	-0.396
$1/\hat{k}_2$	3.125	2.511	7.201	-23.811	-5.584	-3.700	-3.180	-2.817	-2.522
R_2-std	1.291	1.274	1.225	1.156	1.112	1.079	1.054	1.031	1.006
$RMSE(\hat{k}_2)$	2.119	1.409	1.334	1.277	1.253	1.236	1.213	1.195	1.181
$t_{s.2}$	**-0.232**	**-0.131**	**-0.466**	**-0.700**	-0.877	-0.994	-1.052	-1.104	-1.157

Notes: 1) for each value of α, 1000 replications are implemented;

2) α_H, m, t_H, k, t_s are defined as that in Chapter3, $R-std$ is the real standard deviation from simulation and $T-std$ is theoretical standard deviation computed from asymptotic distribution (5) corresponding to the true value of each α; RMSE is the root of mean square error.

3) the subscription "1" corresponding to the threshold value u being 0.95 upper quantile and "2" corresponding to the threshold value u being 0.99 upper quantile;

*4) the one-side 90%, 95% and 99% critical values for standard normal distribution are $\pm$1.28, $\pm$1.65 and $\pm$ 2.33, respectively. ***: 1% significant level; **: 5% significant level; *:10% significant level.*

Table 5 : Simulation Results for GARCH (1,1) series with Pareto Distribution ($n = 3000$)

α	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$\hat{\alpha}_H$	0.472	0.974	1.498	2.003	2.504	3.001	3.503	4.005	4.510
$R-std$	0.165	0.117	0.130	0.158	0.203	0.246	0.291	0.321	0.345
$T-std$	0.063	0.064	0.101	0.134	0.167	0.201	0.234	0.267	0.301
$RMSE(\hat{\alpha}_H)$	0.168	0.120	0.130	0.158	0.203	0.246	0.291	0.321	0.345
$\overline{m}$	65	334	312	328	327	328	333	331	332
t_H	**-6.160*** **	**-8.727*** **	**-4.135*** **	**0.016**	4.174***	8.268***	12.488***	16.654***	20.965***
$\hat{k}_1$	3.664	1.042	0.664	0.493	0.392	0.325	0.276	0.240	0.211
$1/\hat{k}_1$	0.273	0.960	1.505	2.030	2.550	3.080	3.618	4.169	4.733
R_1-std	3.052	0.658	0.155	0.129	0.123	0.115	0.112	0.107	0.104
$RMSE(\hat{k}_1)$	3.476	0.659	0.155	0.129	0.123	0.115	0.112	0.108	0.105
$t_{s.1}$	**22.370*** **	**3.834*** **	**1.163**	**-0.052**	-0.762	-1.240*	-1.581*	-1.840**	-2.041**
$\hat{k}_2$	1.563	0.950	0.629	0.452	0.347	0.276	0.226	0.187	0.156
$1/\hat{k}_2$	0.640	1.053	1.590	2.214	2.886	3.624	4.435	5.354	6.430
R_2-std	0.973	0.382	0.344	0.310	0.296	0.286	0.280	0.276	0.272
$RMSE(\hat{k}_2)$	1.067	0.385	0.346	0.314	0.301	0.291	0.286	0.283	0.281
$t_{s.2}$	**3.360*** **	**1.423***	**0.408**	**-0.153**	-0.485	-0.709	-0.868	-0.991	-1.089

Notes: notations are the same as in Table 4.

Table 6 : Simulation Results for GARCH (1,1) series with Pareto Distribution ($n = 10000$)

α	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$\hat{\alpha}_H$	–	0.964	1.500	2.005	2.506	3.009	3.513	4.013	4.515
$R-std$	–	0.112	0.089	0.110	0.145	0.154	0.176	0.205	0.230
$T-std$	–	0.044	0.067	0.090	0.112	0.134	0.156	0.179	0.201
$RMSE(\hat{\alpha}_H)$	–	0.118	0.089	0.110	0.145	0.154	0.176	0.205	0.231
$\overline{m}$	–	801	741	745	768	792	791	775	772
t_H	–	**-13.331*** **	**-6.313*** **	**-0.003**	6.312***	12.773***	19.225***	25.445***	31.804***
$\hat{k}_1$	–	1.023	0.666	0.497	0.397	0.330	0.282	0.246	0.218
$1/\hat{k}_1$	–	0.978	1.502	2.011	2.517	3.035	3.552	4.068	4.593
R_1-std	–	0.175	0.079	0.068	0.065	0.059	0.057	0.055	0.054
$RMSE(\hat{k}_1)$	–	0.176	0.079	0.068	0.065	0.059	0.057	0.055	0.054
$t_{s.1}$	–	**6.785*** **	**2.138** **	**-0.034**	-1.326*	-2.201**	-2.821***	-3.282***	-3.644***
$\hat{k}_2$	–	0.983	0.657	0.487	0.386	0.318	0.269	0.233	0.205
$1/\hat{k}_2$	–	1.017	1.523	2.053	2.590	3.148	3.718	4.295	4.887
R_2-std	–	0.229	0.174	0.156	0.148	0.140	0.136	0.133	0.131
$RMSE(\hat{k}_2)$	–	0.230	0.174	0.157	0.149	0.141	0.137	0.134	0.132
$t_{s.2}$	–	**2.807*** **	**0.905**	**-0.075**	-0.657	-1.053	-1.334*	-1.543*	-1.705**

Notes: notations are the same as in Table 4.

Summary for our simulation study, we find that the adaptive Hill's estimator outperforms the Smith's estimator for both *i.i.d* series and dependent series where a

GARCH (1, 1) structure exists. In both simulations, the adaptive Hill's estimator has smaller bias for each value of α and every case of sample size, it also has smaller MSE than that of the Smith's estimator when α is less than 2.5. Though the Smith's estimator does well when the sample size is large (10000 in our simulations), it is poor when we do the hypothesis test for infinite variance. The performance of the adaptive Hill's estimator is pretty good in the hypothesis test and keeps at 1% significant level even for small sample.

The results presented above imply that it is better to use the adaptive Hill's estimator in our empirical study.

Chapter 5 Empirical Study

We apply the above adaptive Hill's estimator to main stock index returns in order to test whether or not the second moment of these series exists. In addition, we also test for the existence of the fourth moment as a byproduct.

Our empirical data consists of eleven stock index returns series: the Dow Jones Industrial Average index (Dow), Standard& Poor 500 index (S&P 500) and Nasdaq index for the American market; the CAC 40 index, FTSE 100 index and DAX index for the European stock markets; the Nikkei 225 (N 225), Hang Seng index (HSI), Straits Times index (STI), Taiwan Weighted index (TWII), and Seoul Composite index (KS11) for the Asian markets[3]. These include the main stock indices over the world. For each stock index, there are three different time frequencies of returns: daily, weekly and monthly. The returns are defined as $100*(\log P_t - \log P_{t-1})$, where P_t is the closing price at time t. So in all, we have thirty-three time series. The length for these series ranges from about 11 to 59 years according to data availability. The daily returns series has the largest sample size and the monthly returns series has the smallest sample size. Number of observations for each series could be seen in Table 7.

Summary statistics for each returns series are presented in Table 7. We could see clearly the existence of heavy tails in these series. The Kurtosis of these returns, at all three frequencies, is far larger than three as showed in the table. An extreme case is the HSI: all the three series have Kurtosis larger than ten. And the daily returns has the largest Kurtosis value for almost each index, but the weekly returns and monthly returns have close Kurtosis except several cases (for example, Nasdaq, HSI). Additionally, there is commonly negative Skewness for every index at all frequencies (except monthly returns for FTST 100). However, the absolute value of Skewness is typically very small and close to zero in fact, except the case of HSI.

[3] All data are from yahoo.finance.com.

Table 7: Descriptive Statistics

	Dow							S&P500						
	N	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*	*N*	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*
D	14716	0.03	0.91	9.67	-25.63	-1.61	50.32	14716	0.03	0.90	8.71	-22.90	-1.27	36.38
W	3051	0.13	2.01	11.86	-15.39	-0.43	6.63	3051	0.14	1.97	13.21	-13.01	-0.39	6.31
M	701	0.57	4.12	13.47	-26.42	-0.67	6.16	701	0.62	4.09	15.10	-24.54	-0.58	5.44

	Nasdaq							CAC 40						
	N	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*	*N*	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*
D	9435	0.03	1.19	13.26	-12.04	-0.30	13.27	4624	0.02	1.32	7.00	-7.68	-0.13	5.92
W	1951	0.16	2.73	17.38	-29.18	-1.16	14.30	957	0.09	2.70	11.03	-12.13	-0.14	3.95
M	448	0.70	6.35	19.87	-31.79	-0.85	5.96	219	0.37	5.62	12.59	-19.23	-0.57	3.61

	FTSE100							DAX						
	N	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*	*N*	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*
D	6123	0.03	1.03	7.60	-13.03	-0.51	10.47	4436	0.03	1.39	7.55	-9.87	-0.23	6.87
W	1265	0.13	2.18	10.07	-19.29	-0.71	8.83	918	0.16	2.91	12.89	-14.08	-0.30	5.14
M	290	-0.55	4.62	30.17	-13.49	1.27	9.27	211	0.71	6.21	19.37	-29.33	-0.91	6.08

	N225							HSI						
	N	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*	*N*	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*
D	6025	0.01	1.38	12.43	-16.14	-0.15	10.05	5328	0.04	1.73	17.25	-40.54	-3.08	73.48
W	1268	0.02	2.73	11.05	-12.79	-0.28	4.79	1121	0.19	3.76	13.92	-54.01	-3.01	42.75
M	293	0.10	6.00	18.29	-21.35	-0.37	3.60	258	0.83	8.32	26.45	-58.16	-1.52	13.28

	STI							TWII						
	N	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*	*N*	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*
D	5115	0.02	1.26	12.87	-10.55	-0.13	12.38	2697	-0.01	1.62	8.52	-9.94	-0.13	5.42
W	1067	0.12	2.91	19.87	-25.51	-0.51	12.28	561	-0.04	3.65	18.32	-14.29	-0.06	5.18

M	246	0.52	6.70	24.86	-21.72	-0.34	5.73	131	-0.22	7.73	22.42	-21.50	-0.02	3.35

	KS11						
	N	*Me.*	*Std.*	*Max.*	*Min.*	*Skew.*	*Kurto.*
D	2700	0.03	2.14	10.02	-12.81	-0.13	6.19
W	570	0.13	4.66	17.44	-21.35	-0.38	5.67
M	131	0.64	10.06	39.76	-31.81	0.24	4.41

Notes: 1) D : daily returns ;W : weekly returns ; M :monthly returns;

2) N : number of observations; Me.: mean; Std.: standard deviation; Max..: maximal value; Min..: minimal value; Skew.: skewness; Kurto.:Kurtosis;

3)Time period: Dow:1950.01.01-2008.06.30 ; S&P500 :1950.01.01-2008.06.30 ; Nasdaq :1971.02.01-2008.06.30 ; CAC 40 :1990.03.01-2008.06.30 ;FTSE 100 :1984.04.01-2008.06.30 ;DAX :1990.11.26-2008.06.30 ; N 225 :1984.01.04-2008.06.30 ; HSI :1986.12.31-2008.06.30 ;STI :1987.12.28-2008.06.30 ; TWII: 1997.07.01-2008.06.30; KS11:1997.07.01-2008.06.30.

The tail index estimator and test results are showed in Table 8. We consider three kinds of tails for each series: right-tail which is corresponding to positive extreme values in the returns; left-tail which is corresponding to negative extreme values and two-tail which combines all the extreme values. If any of these three tail indices is less than 2, then the second moment does not exist for that series. Except for testing the second moment or the infinite variance where $\alpha = 2$, we also test the existence for the fourth moment. The values of test statistics are expressed in t_2 and t_4 respectively, see Table 8.

As we could see from Table 8, the value of $\hat{\alpha}$ is in the range of (2.630, 5.050) for daily data; and (2.078, 4.324) for weekly data, (1.853, 5.312) for monthly data respectively. These results are similar as that in Loretan and Phillips (1994). The smallest value of $\hat{\alpha}$, which is also the only case that $\hat{\alpha} < 2$, appears in the left-tail for monthly returns of STI which is about 1.853 and the biggest value of $\hat{\alpha}$ is about

5.312 which is the left-tail index for monthly data of TWII. Most values of $\hat{\alpha}$ is larger than 3. And typically the left-tail value is smaller than that of the other two, however, there are cases where the right-tail values is larger, for example, daily returns for Nasdaq and CAC 40, weekly returns for STI and TWII , and monthly returns for TWII. The values of $\hat{\alpha}$ imply that for most series, the first three moment should exist but the fourth moment may not exist.

When we look at the results for the hypothesis test of $\alpha = 2$, the evidence is obvious. The null hypothesis could not be rejected for any case. The value of test statistics t_2 for the second moment is positive for almost every series, and in fact takes relatively large values in most cases. The only negative value of t_2 happens for the left tail of monthly returns for STI which is negative of -0.294, however, we can't reject the null hypothesis, too. The above results show that we should accept the null hypothesis that $H_0 : \alpha = 2$, thus universally the second moment exists for all the indices and at all frequencies.

However, when we perform an additional test for the existence of the fourth moment, there is strong evidence that the null hypothesis should be rejected, especially for daily returns. If we look at the values of t_4 which is the test statistics for the null hypothesis $H_0 : \alpha = 4$, we could find that it is negative in most cases as showed in Table 8. And the null hypothesis is rejected at three time frequencies for different indices. For example, it is been rejected for daily returns of Dow index, S&P 500, Nasdaq, CAC 40, FTSE100, DAX, HSI and STI, all at 1% or 5% significant level. For weekly returns, it's been rejected for Nasdaq, HSI, STI, TWII, KS11 etc; and for monthly returns, DAX, STI, KS11. There are situations where the null hypothesis can't be rejected for several series, for example, weekly returns of Dow, CAC 40 and FTSE 100, and monthly returns of Dow and FTSE 100. These series are mainly concentrated on weekly and monthly frequencies. It seems that the null hypothesis has been rejected generally for daily returns and less frequently for weekly returns and monthly returns. A special case is N 225 index where the null hypothesis could not be rejected for any of these three time frequencies. The test results imply that the fourth moment generally does not exist, especially for daily returns.

Further, since the GARCH effect exists commonly in these return series, to check the robustness of our results, we remove the GARCH effect from the original return series. A Gaussian-GARCH model has been estimated to each series, and then we re-estimate the Hill's estimator to the residuals[4]. The results are presented in Table 9. From the table, we could see clearly that we obtain very close tail index estimators as results in Table 8 for almost every series. The tail index is universally larger than 2 and less than 4, mainly in the range of (3, 4). The evidence for general existence of second moment is overwhelming. For the existence of fourth moment, the pattern with respective to time frequency is similar as that in Table 8. The null hypothesis is more likely to be rejected for daily returns and less likely for weekly returns and monthly returns. Seemly the GARCH effect doesn't have much impact on the results.

In sum, the results in Table 8 and Table 9 imply that for stock index returns, the second moment does exist at time frequencies of daily, weekly and monthly; hence the variance is finite for these series. However, the fourth moment seldom exists, especially for daily returns. The maximal moment exponent seems less than 4 generally for stock index returns over the world. The results are robust to the existence of GARCH effect. The evidence is obvious and small difference among these indices may be induced by gap in sample size partly.

[4] For these return series, the existence of GARCH effect is commonly. We estimate the Gaussian GARCH model for every series using matlab. The specification of the GARCH structure is chosen based on BIC. However, for most series, a GARCH (1, 1) specification is sufficient enough to explore the GARCH effect.

Table 8: Empirical Results for Tail Index

		Dow				S&P 500			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.574	209	11.375	-1.541*	3.668	185	11.343	-1.129
	left-tail	3.525	286	12.897	-2.007**	3.388	260	11.194	-2.465***
	two-tail	3.672	308	14.672	-1.439 *	3.730	294	14.833	-1.157
W	right-tail	3.962	78	8.662	-0.085	3.919	83	8.740	-0.185
	left-tail	3.531	85	7.056	-1.082	3.394	96	6.831	-1.484*
	two-tail	3.703	102	8.602	-0.749	3.827	85	8.422	-0.399
M	right-tail	4.024	33	5.813	0.034	3.551	43	5.085	-0.736
	left-tail	3.201	35	3.554	-1.181	2.923	32	2.609	-1.524*
	two-tail	3.804	50	6.378	-0.346	4.241	43	7.346	0.395

		Nasdaq				CAC 40			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	2.760	168	4.926	-4.018***	3.307	110	6.855	-1.817**
	left-tail	3.385	160	8.760	-1.944**	3.527	92	7.325	-1.133
	two-tail	3.212	171	7.924	-2.576***	4.002	100	10.011	0.006
W	right-tail	3.631	71	6.872	-0.777	4.293	46	7.776	0.497
	left-tail	2.859	60	3.326	-2.210**	4.264	39	7.070	0.413
	two-tail	3.190	84	5.455	-1.855*	4.338	67	9.569	0.692
M	right-tail	3.490	27	3.872	-0.662	4.462	17	5.076	0.476
	left-tail	2.359	28	**0.951**	-2.170**	2.482	15	**0.934**	-1.470*
	two-tail	3.643	34	4.791	-0.520	3.905	19	4.151	-0.104

		FTSE100				DAX			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.734	127	9.769	-0.750	3.264	118	6.864	-1.999**
	left-tail	3.099	132	6.312	-2.589***	3.394	89	6.574	-1.430
	two-tail	3.751	139	10.320	-0.735	3.792	94	8.686	-0.505
W	right-tail	4.090	63	8.294	0.178	3.364	43	4.471	-1.043
	left-tail	3.522	48	5.272	-0.828	4.197	36	6.590	0.295
	two-tail	3.865	65	7.518	-0.272	3.968	58	7.493	-0.061
M	right-tail	3.260	18	2.672	-0.785	3.193	23	2.861	-0.968
	left-tail	4.113	20	4.726	0.127	2.069	17	**0.143**	-1.990**
	two-tail	3.951	26	4.975	-0.062	2.663	20	1.484	-1.494*

		N225				HSI			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.732	125	9.681	-0.750	3.155	121	6.353	-2.324**
	left-tail	4.053	116	11.054	0.142	2.630	159	3.969	-4.320***
	two-tail	3.851	146	11.182	-0.450	2.939	238	7.244	-4.092***
W	right-tail	3.541	59	5.918	-0.881	3.289	51	4.601	-1.270
	left-tail	3.610	51	5.748	-0.697	2.644	58	2.450	-2.583***
	two-tail	3.458	60	5.648	-1.049	2.911	102	4.600	-2.750***
M	right-tail	3.071	25	2.676	-1.162	3.062	25	2.656	-1.172
	left-tail	4.019	19	4.401	0.021	2.650	18	1.379	-1.432*
	two-tail	3.574	23	3.774	-0.511	2.217	82	**0.983**	-4.036***

		STI				TWII			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	2.809	149	4.937	-3.635***	3.923	62	7.571	-0.152
	left-tail	2.771	154	4.782	-3.814***	3.831	73	7.823	-0.360
	two-tail	3.134	143	6.778	-2.590***	4.258	75	9.777	0.558
W	right-tail	2.633	48	2.193	-2.368***	2.590	33	1.695	-2.025**
	left-tail	3.066	48	3.693	-1.618**	3.922	28	5.085	-0.103
	two-tail	3.011	83	4.607	-2.252**	4.324	41	7.440	0.518
M	right-tail	2.095	31	**0.266**	-2.651***	2.731	13	1.318	-1.144
	left-tail	1.853	16	**-0.294**	-2.147**	5.312	12	5.736	1.136
	two-tail	3.617	17	3.333	-0.395	4.598	17	5.357	0.617

		KS11			
		$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.800	66	7.310	-0.407
	left-tail	3.425	72	6.043	-1.221
	two-tail	5.050	71	12.850	2.212
W	right-tail	2.941	35	2.784	-1.566**
	left-tail	2.078	30	**0.213**	-2.632***
	two-tail	3.111	35	3.285	-1.316*
M	right-tail	2.021	13	**0.038**	-1.784**
	left-tail	4.436	12	4.219	0.377
	two-tail	2.760	19	1.657	-1.351*

Notes: 1) D : daily returns ;W : weekly returns ; M :monthly returns;

2) right tail is defined as tail of $\{X_t\}_1^n$;left tail is defined as $\{-X_t\}_1^n$;two tail is defined as $\{|X_t|\}_1^n$;

3) $\hat{\alpha}$: tail index estimated; m :adaptive truncation number ; t_2 : the test statistics for $\alpha = 2$; t_4 : the test statistics for $\alpha = 4$;

4) Time period : Dow 1950.01.01-2008.06.30 ; S&P 500 :1950.01.01-2008.06.30 ;
Nasdaq :1971.02.01-2008.06.30 ; CAC 40 :1990.03.01-2008.06.30;
FTSE 100 :1984.04.01-2008.06.30 ; DAX :1990.11.26-2008.06.30 ;
N 225 :1984.01.04-2008.06.30 ; HSI :1986.12.31-2008.06.30 ;
STI :1987.12.28-2008.06.30 ; TWII: 1997.07.01-2008.06.30;
KS11:1997.07.01-2008.06.30.

5) the one-side 90%, 95% and 99% critical values for standard normal distribution are $\pm$1.28, $\pm$1.65 and $\pm$2.33, respectively. ***: 1% significant level; **: 5% significant level; *:10% significant level.

Table 9: Empirical Results for Tail Index after Removing the GARCH Effect

		Dow				S&P 500			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.580	198	11.116	-1.478*	3.455	194	10.135	-1.897 **
	left-tail	3.496	326	13.504	-2.276**	3.535	275	12.728	-1.927**
	two-tail	3.612	336	14.773	-1.779**	3.662	312	14.681	-1.491*
W	right-tail	3.742	76	7.593	-0.562	3.876	80	8.388	-0.278
	left-tail	3.466	88	6.875	-1.253	3.386	101	6.964	-1.543*
	two-tail	3.966	122	10.859	-0.093	3.927	90	9.138	-0.174
M	right-tail	3.704	32	4.820	-0.418	3.448	41	4.634	-0.884

	left-tail	3.162	38	3.582	-1.291*	3.271	34	3.704	-1.063
	two-tail	3.767	59	6.786	-0.448	3.569	44	5.204	-0.715

		Nasdaq				CAC 40			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	2.730	164	4.672	-4.067***	3.250	109	6.522	-1.959**
	left-tail	3.388	145	8.357	-1.842**	3.569	92	7.522	-1.035
	two-tail	3.238	165	7.953	-2.446***	4.015	102	10.176	0.038
W	right-tail	3.441	68	5.940	-1.153	4.307	45	7.737	**0.514**
	left-tail	2.884	63	3.507	-2.215**	4.377	39	7.421	**0.588**
	two-tail	3.110	89	5.238	-2.098**	4.307	58	8.786	**0.585**
M	right-tail	3.488	26	3.795	-0.652	4.746	16	5.493	0.746
	left-tail	2.555	30	1.520	-1.979**	2.724	16	1.447	-1.276*
	two-tail	3.112	33	3.193	-1.276	4.256	20	5.045	0.286

		FTSE100				DAX			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.735	125	9.700	-0.740	3.211	115	6.492	-2.116**
	left-tail	3.115	134	6.454	-2.561***	3.446	89	6.821	-1.307*
	two-tail	3.669	144	10.013	-0.994	3.664	95	8.109	-0.819
W	right-tail	3.883	59	7.230	-0.226	3.501	41	4.807	-0.798
	left-tail	3.782	50	6.300	-0.385	4.147	37	6.530	0.224
	two-tail	3.666	65	6.714	-0.674	3.949	57	7.358	-0.096
M	right-tail	3.588	19	3.461	-0.449	2.810	21	1.856	-1.363
	left-tail	3.604	19	3.495	-0.432	2.306	18	0.648	-1.797**

	two-tail	4.077	27	5.397	0.101	2.477	25	1.192	-1.904**

		N225				HSI			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.631	123	9.045	-1.023	3.137	117	6.148	-2.334***
	left-tail	4.026	118	11.003	0.070	2.663	171	4.336	-4.371***
	two-tail	3.944	147	11.782	-0.171	2.851	280	7.119	-4.807***
W	right-tail	3.359	57	5.129	-1.210	3.277	48	4.424	-1.252
	left-tail	3.733	52	6.250	-0.481	2.706	66	2.869	-2.628***
	two-tail	3.566	59	6.015	-0.833	3.094	98	5.414	-2.243**
M	right-tail	2.942	24	2.308	-1.296*	2.916	24	2.243	-1.328*
	left-tail	3.957	20	4.377	-0.048	2.865	19	1.884	-1.237
	two-tail	3.754	22	4.114	-0.288	2.801	50	2.830	-2.121**

		STI				TWII			
		$\hat{\alpha}$	m	t_2	t_4	$\hat{\alpha}$	m	t_2	t_4
D	right-tail	2.738	138	4.334	-3.707***	3.877	62	7.390	-0.242
	left-tail	3.071	146	6.472	-2.805***	3.925	74	8.278	-0.162
	two-tail	3.019	134	5.897	-2.839***	4.297	77	10.077	0.651
W	right-tail	2.489	47	1.675	-2.591***	2.568	33	1.631	-2.057**
	left-tail	3.016	50	3.591	-1.740**	3.950	28	5.158	-0.067
	two-tail	2.606	110	3.177	-3.656***	4.149	42	6.965	0.242
M	right-tail	2.202	27	0.524	-2.336***	2.662	13	1.193	-1.206
	left-tail	1.929	17	**-0.147**	-2.135**	5.421	12	5.926	1.231
	two-tail	2.905	17	1.867	-1.128	5.018	17	6.222	1.050

		KS11			
		$\hat{\alpha}$	m	t_2	t_4
D	right-tail	3.732	66	7.035	-0.545
	left-tail	3.502	73	6.415	-1.065
	two-tail	4.860	73	12.217	**1.837**
W	right-tail	2.894	34	2.607	-1.612*
	left-tail	2.140	30	0.384	-2.547***
	two-tail	2.932	35	2.756	-1.580*
M	right-tail	1.863	12	-0.238	-1.851**
	left-tail	4.742	13	4.943	0.669
	two-tail	3.628	20	3.641	-0.416

Notes: notations are the same as in Table 8.

Chapter 6 Summary of Conclusion

The problem that the variance of stock returns is finite or infinite is really related to extreme events in the stock returns. Based on extreme value theory, we examine this problem in this paper.

We compared two tail index estimators through Monte Carlo simulations: Hill's estimator with adaptive data-based truncation number and Smith's estimator through POT method. We find that the adaptive Hill's estimator performs better in both *i.i.d* setting and dependent environment with GARCH (1, 1) structure. It has not only much smaller bias for all cases but also smaller MSE when the true tail index α is not more than 2. And importantly, the Hill's estimator shows undoubted results for the test of infinite variance. The Smith's estimator does perform well when the sample size is large; however, the performance is poor when sample size is small.

When we apply the adaptive Hill's estimator to main stock index returns over the world, the results show that for most indices, the second moment does exist for daily, weekly and monthly returns. Thus the variance of stock index returns is finite commonly. However, an additional test for the existence of the fourth moment shows that generally the fourth moment does not exist, especially for daily returns. These conclusions don't change when the GARCH effect is removed from the original series. The results imply that for most stock returns, the maximal moment existed is around three, and difference among different time frequencies (i.e. daily, weekly, monthly) is small.

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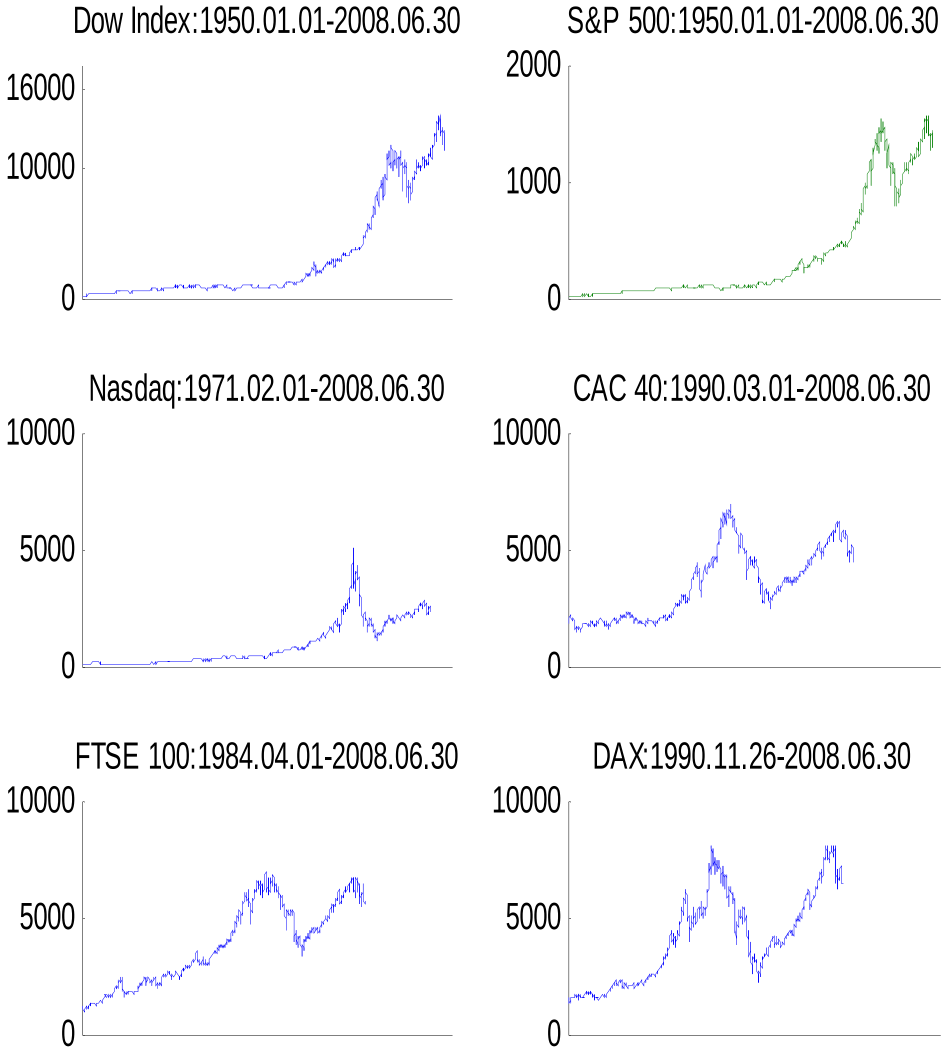
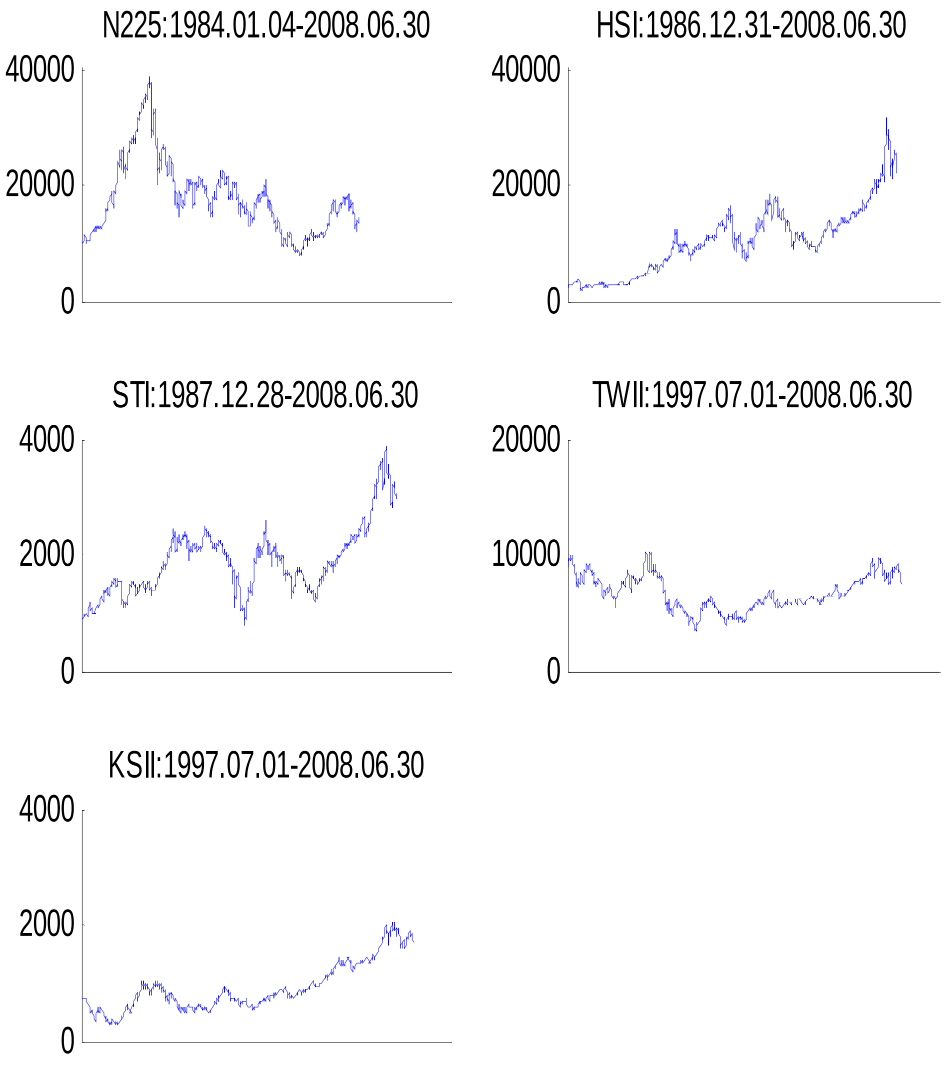
Appendices

Appendix A.

Time Plots of Close Price for All the Series

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40000
20000
0
HSI:1986.12.31-2008.06.30
40000
20000
0
STI:1987.12.28-2008.06.30
4000
2000
0
TWII:1997.07.01-2008.06.30
20000
10000
0
KSII:1997.07.01-2008.06.30
4000
2000
0

Appendix B.

The following are matlab codes used in this paper.

1. Codes for simulation Ⅰ and simulation Ⅱ.

```
function    result=result(nob,rep);
% This function returns the retults of the adaptive Hill's estimator and GP estimator.
alpha=[0.5 2;1 2;1.5 2;2 2;2.5 2;3 2;3.5 2;4 2;4.5 2];
for j=1:9;
    a=alpha(j,1);
    data=simulate_par(nob,a,rep);    % data=simulate_garchpar(nob,a,rep);
    for i=1:rep;
        x=data(:,i);
        y=sort(x,'descend');
        n=length(y);
        m1=floor(n^0.4);
        m2=floor(n^0.8)
        y1=log(y(1:m1));;
        y2=log(y(1:m2));
        a1=1/(sum(y1(1:m1-1))/(m1-1)-y1(m1));
        a2=1/(sum(y2(1:m2-1))/(m2-1)-y2(m2));
        c=abs(a1/(sqrt(2)*(n/m2)*(a2-a1)));
        lad=c^(2/3);
        m=floor(n^(2/3)*lad);
        if( m<n);
        h=log(y(1:m));
        est=1/(sum(h(1:m-1))/(m-1)-h(m));
        ad=2/sqrt(m);
        t1=(est-2)/ad;
        pa(:,i)=[est;m;ad;t1];
        gp(:,i)=gpest(x);
        continue;
        end;
    end;
    result(:,j)=[mean(pa(1,:));    std(pa(1,:)); mean(pa(3,:)); mean(pa(4,:));mean(pa(2,:));
std(pa(2,:));mean(gp(1,:));std(gp(1,:));mean(gp(2,:));mean(gp(3,:));std(gp(3,:));mean(gp(4,:))];
end;

result;
return

function pa=Hill(x)
% This function returns the adaptive Hill's estimator.
        y=sort(x,'descend');
```

```
        n=length(y);
        m1=floor(n^0.4);
        m2=floor(n^0.8);
        y1=log(y(1:m1));
        y2=log(y(1:m2));
        a1=1/(sum(y1(1:m1-1))/(m1-1)-y1(m1));
        a2=1/(sum(y2(1:m2-1))/(m2-1)-y2(m2));
        c=abs(a1/(sqrt(2)*(n/m2)*(a2-a1)));
        lad=c^(2/3);
        m=floor(n^(2/3)*lad);
        if( m<n);
        h=log(y(1:m));
        est=1/(sum(h(1:m-1))/(m-1)-h(m));
        ad1=2/sqrt(m);
        ad2=4/sqrt(m);
        t1=(est-2)/ad1;
        t2=(est-4)/ad2
        pa=[est;m;t1;t2];
        end;
        pa;
       return

function y=gpest(x);
% This function returns the GP estimators of tail index and the t-test statistics.
data=x;
d=data;
nd=length(d);
u1=quantile(d,0.95);
u2=quantile(d,0.99);
j=1;
k=1;
for i=1:nd;
    if (d(i)>=u1);
        d1(j)=d(i);


     j=j+1;
    end;
end;

for i=1:nd;
     if (d(i)>=u2);
          d2(k)=d(i);
          k=k+1;
     end;
```

```
end;

z1=d1-u1;
z2=d2-u2;
n1=length(z1);
n2=length(z2);

beta1=gpfit(z1);
beta2=gpfit(z2);

delt1=beta1(1);
t1=sqrt(n1/3)*(delt1-1/2);

delt2=beta2(1);
t2=sqrt(n2/3)*(delt2-1/2);
y=[delt1;t1;delt2;t2];
return

function y=simulate_par(nob,alpha,rep)
% This function generates Pareto distribution data.
rand('seed',1);
for i=1:rep;
    x=rand(nob,1);
    y(:,i)=(1-x).^(-1/alpha);
end;
return

% function y=simulate_garchpar(nob,alpha,rep)
% This funcion generates data with GARCH structure and Pareto innovations.
beta=[0.001 0.10 0.85];
w=beta(1);
a1=beta(2);
b1=beta(3);
rand('seed',1);
ep=simulate_par(nob,alpha,rep);
for i=1:rep;
    yp(1)=sqrt(w/(1-a1-b1));
    h(1)=w/(1-a1-b1);
    e=ep(:,i)./100;
        for j=2:nob;
        h(j)=w+a1*yp(j-1)^2+b1*h(j-1);
        yp(j)=e(j)*sqrt(h(j));
        end;
        y1=yp(2:nob)';
```

```
y(:,i)=y1;
end;
y;
return
```

2. Codes for empirical study.

```
function pa=Hill(x)
% This function returns the adaptive Hill's estimator.
        y=sort(x,'descend');
        n=length(y);
        m1=floor(n^0.4);
        m2=floor(n^0.8);
        y1=log(y(1:m1));
        y2=log(y(1:m2));
        a1=1/(sum(y1(1:m1-1))/(m1-1)-y1(m1));
        a2=1/(sum(y2(1:m2-1))/(m2-1)-y2(m2));
        c=abs(a1/(sqrt(2)*(n/m2)*(a2-a1)));
        lad=c^(2/3);
        m=floor(n^(2/3)*lad);
        if( m<n);
        h=log(y(1:m));
        est=1/(sum(h(1:m-1))/(m-1)-h(m));
        ad1=2/sqrt(m);
        ad2=4/sqrt(m);
        t1=(est-2)/ad1;
        t2=(est-4)/ad2
        pa=[est;m;t1;t2];
        end;
        pa;
       return
```