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Theorems about quadrilaterals and conics

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We study quadrilaterals inscribed and circumscribed about conics and prove interesting theorems. Theorems are discovered by experimenting with dynamical geometry software. The Poncelet theorem for quadrilaterals is proved by elementary means together with Poncelet's grid property.

Keywords: quadrilaterals; conics; projective geometry; dynamical geometry software

2010 AMS Subject Classifications: 14N05; 51N35; 65D18; 51N05; 68R99

1. Introduction

Our research is motivated by the conjecture of Schwartz and Tabachnikov [22, Theorem 4.c]. While searching for the proof we encountered several interesting results about conics and quadrilaterals.

Conjecture 1 (Conjecture of Tabachnikov and Schwartz) Let $A_1A_2 \dots A_{12}$ be a 12-gon inscribed in a conic $\mathcal{C}$. Let π map 12-gon $X_1X_2 \dots X_{12}$ onto a new 12-gon according to the rule $\pi(X_i) = l(X_iX_{i+3}) \cap l(X_{i+1}X_{i+4})$. Then, 12-gon $A_1A_2 \dots A_{12}$ is mapped with $\pi^{(3)} = \pi \circ \pi \circ \pi$ onto a 12-gon inscribed in a conic.

It seemed that this conjecture is a perfect candidate to use the technique illustrated in the paper *Illumination of Pascal's Hexagrammum and Octagrammum Mysticum* by Baralić and Spasojević [2,3]. The problems we study are strongly influenced by the very inspirative paper [13]. Many important questions in dynamical systems and combinatorics have their equivalents in the terms of algebraic curves. Schwartz and Tabachnikov originally formulated their conjecture in [22, Theorem 4.c] in terms of a pentagram map [20].

We will explain Figure 1 carefully. We start with a 12-gon $A_1A_2 \dots A_{12}$ (the green points lying on the violet conic) inscribed in a conic and define the (yellow) points obtained by π, (blue and violet lines), $\pi^{(2)}$ the red points (green and orange lines) and $\pi^{(3)}$ the violet points (black and yellow lines). It turns out that at each step we have a 6×6 cage of curves, see [13]. But instead of dealing

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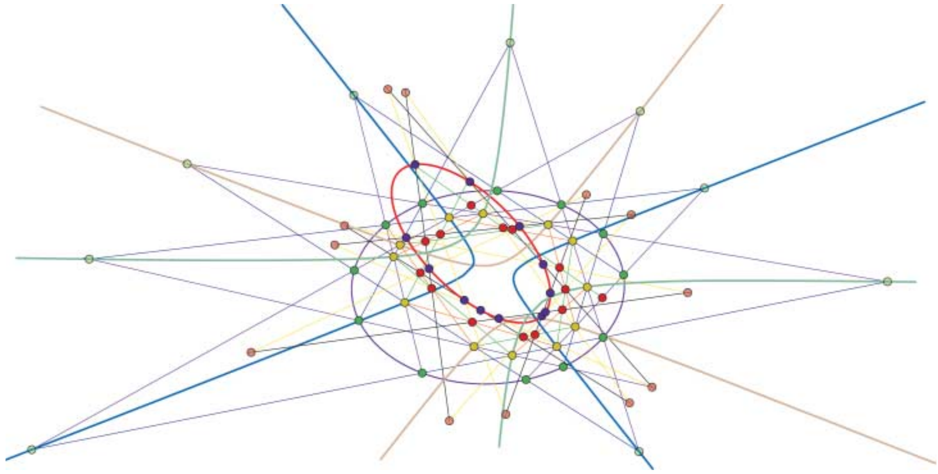

Figure 1. Conjecture of Tabachnikov and Schwartz (color online only).

with 24 points at the second step we take only 12 of them. It is not possible to catch the curves we want in the cage. By the Mystic Octagon theorem, we could catch three interesting conics and one quartic in the blue-violet cage. What to do with the curves at other steps? Definitely we should try to add some new points and then apply Bézout's theorem or a similar statement. But what are those points and how to find them? If we look more carefully, three quadrilaterals inscribed in a conic can be noticed ($A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$ and $A_3A_6A_9A_{12}$) and usually the steps are always defined as the certain intersection points of the side lines of quadrilaterals. Thus, we thought that if we want to overcome the problems we faced, we should understand the quadrilaterals inscribed in conics better.

Theorems about quadrilaterals and conics are usually known like degenerate cases of Pascal and Brianchon theorems. Baralić and Spasojević [2] proved some new results about two quadrilaterals inscribed in a conic. However, in this paper we study more complicated structures involving both tangents at the vertices and the side lines of quadrilateral. We start from the degenerate form of Pascal and Brianchon theorems for the quadrilateral and then we discover new interesting points, conics and loci. Classical projective geometry from the nineteenth century studied extensively these objects, leading to the founding of new mathematical disciplines such as algebraic geometry. Development of computer graphics, dynamical geometry, dynamical systems, etc. during the second half of the twentieth century renewed the interest of researchers for the classical projective geometry. Recently, two excellent book on this topic were published [15,19]. Theorems 3.1–3.3 we present here extend the known results about geometry of quadrilaterals inscribed in a conic.

The objects are studied by elementary means. Some of the results are in particular the corollary of the Great Poncelet Theorem for the case when n-gon is quadrilateral. Here we give a short proof for this case. Some special facts about this special case are explained as well.

Finally, we compare two theorems – the Mystic Octagon theorem for the case of two quadrilaterals and the Poncelet Theorem for the quadrilaterals. They have in common that they state that certain 8 points coming from two quadrilaterals inscribed in a conic lie on the same conic. While the first one is a pure algebro-geometric fact, the latter involves much deeper structure of the space and cannot be seen naturally as a special case of the first one. Thus, we could not find 'The theorem of all theorems for conics in projective geometry' and elementary surprises in projective geometry like those in [22,23] could come as the special cases of different general statements.

2. From Pascal to Brocard theorem

In this section we show how the Pascal theorem for hexagon [16] (1639) inscribed in a conic degenerates to the Brocard theorem for the quadrilateral inscribed in a circle. All results here are well known and are part of the standard olympiad problem solving curriculum, but our aim is to illustrate the power of degeneracy tool and prepare the background for the next sections.

LEMMA 2.1 *Let $ABCD$ be a quadrilateral inscribed in a conic $\mathcal{C}$ and let M be the intersection point of the lines AD and BC, N be the intersection point of the lines AB and CD, P be the intersection point of the tangents to $\mathcal{C}$ at A and C, and Q be the intersection point of the tangents to $\mathcal{C}$ at B and D. Then, the points M, N, P and Q are collinear* (*Figure* 2).

Proof Apply the Pascal theorem to degenerate hexagon $AABCCD$ and we get the points M, N and P are collinear. Apply the Pascal theorem to degenerate hexagon $ABBCD$ and we get that the points M, N and Q are collinear. ■

Dual statement to Lemma 2.1 is the following:

LEMMA 2.2 *Let conic $\mathcal{C}$ touch the sides AB, BC, CD and DA of a quadrilateral $ABCD$ in the points M, N, P and Q, respectively. Then the lines AC, BD, MP and NQ pass through the same point O* (*Figure* 3).

Lemmas 2.1 and 2.2 will be used to prove other interesting relations among the lines and points that naturally occur in a quadrilateral inscribed in conics configurations. Many points are going to be introduced so we are going to organize labels of our points.

Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a conic $\mathcal{C}$ and let M_1 be the intersection point of the lines A_1A_2 and A_3A_4, M_2 of A_2A_3 and A_4A_1 and M_3 of A_3A_1 and A_2A_4. Let N_3 be the intersection point of the tangent lines to the conic at A_1 and A_3, P_3 of the tangents at A_2 and A_4, N_2 of the tangents at A_1 and A_4, P_2 of the tangents at A_2 and A_3, N_1 of the tangents at A_1 and A_2 and P_1 of

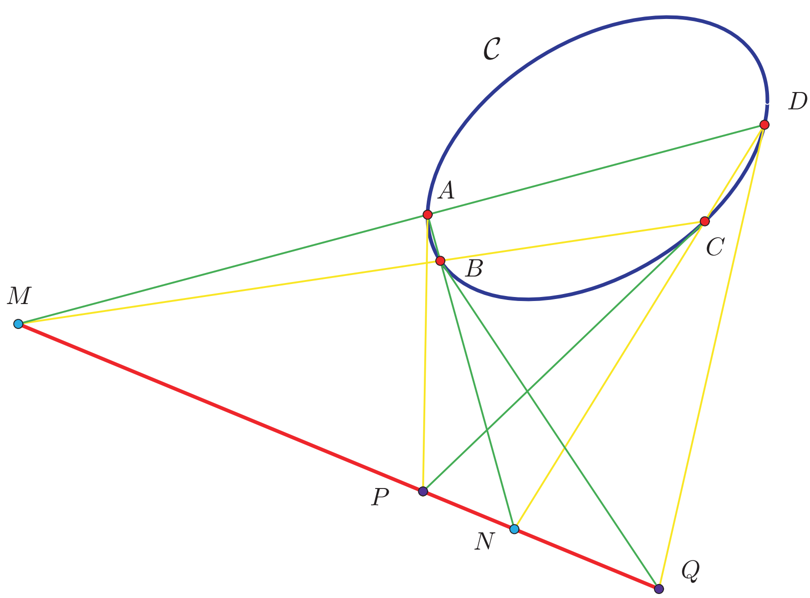

Figure 2. Lemma 2.1.

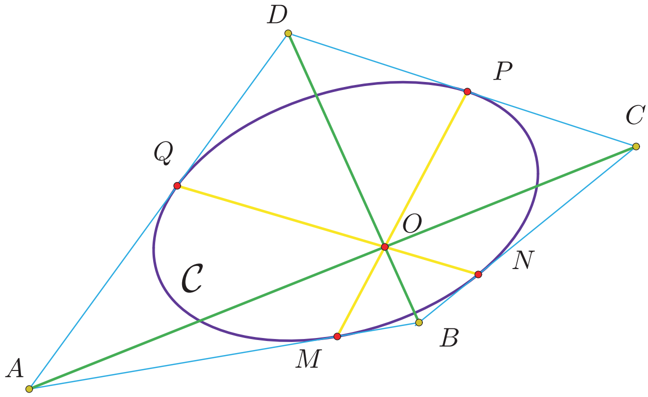

Figure 3. Lemma 2.2.

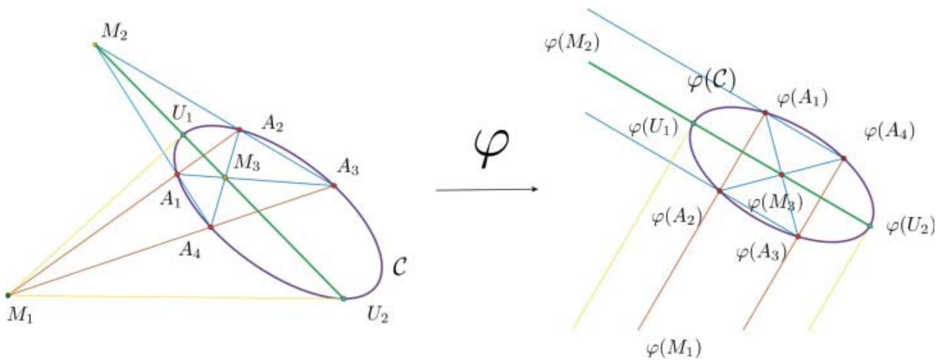

Figure 4. Lemma 2.3.

the tangents at A_3 and A_4. Let U_1 and U_2 be the points where the tangents from M_1 touch $\mathcal{C}$, and analogously V_1, V_2 and W_1, W_2 for the points M_2 and M_3, respectively.

Lemma 2.1 states that the points M_1, M_2, N_3, P_3 are collinear, as well as the points M_2, M_3, N_1, P_1 and M_3, M_1, N_2 and P_2. Denote these three lines by m_3, m_1 and m_2, respectively. We are going to prove that U_1 and U_2 lie on the line m_3, V_1 and V_2 on the line m_2 and W_1 and W_2 on m_1 – so that m_1, m_2 and m_3 are the polar lines of the points M_1, M_2 and M_3 with respect to $\mathcal{C}$.

The following lemma is a well-known result about poles and polars. A classical proof using harmonic division could be found in [10]. However, for the reader's convenience we give a proof based on a different, well-known idea. Indeed, moving the configuration into a special position will be a central idea in the proof of Theorem 3.3.

Lemma 2.3 *The points U_1, M_2, U_2 and M_3 are collinear* (*Figure* 4).

Proof There is a projective transformation φ that maps the points A_1, A_2, A_3 and A_4 onto the vertices of a square. Thus, $\varphi(M_3)$ is the centre of a square with vertices $\varphi(A_1)$, $\varphi(A_2)$, $\varphi(A_3)$ and $\varphi(A_4)$. The points $\varphi(M_1)$ and $\varphi(M_2)$ are at infinity. There is a unique way to inscribe a square into a conic, and the lines $\varphi(A_1)\varphi(A_2)$ and $\varphi(A_1)\varphi(A_4)$ are parallel to the axes of the conic $\varphi(\mathcal{C})$, see [1]. The points $\varphi(U_1)$ and $\varphi(U_2)$ must be mapped onto the axis parallel to the line $\varphi(A_1)\varphi(A_4)$.

Now the points $\varphi(U_1)$, $\varphi(U_2)$, $\varphi(M_2)$ and $\varphi(M_3)$ lie on the axis of the conic $\varphi(\mathcal{C})$. Consequently, the points U_1, M_2, U_2 and M_3 then lie on the same line. ∎

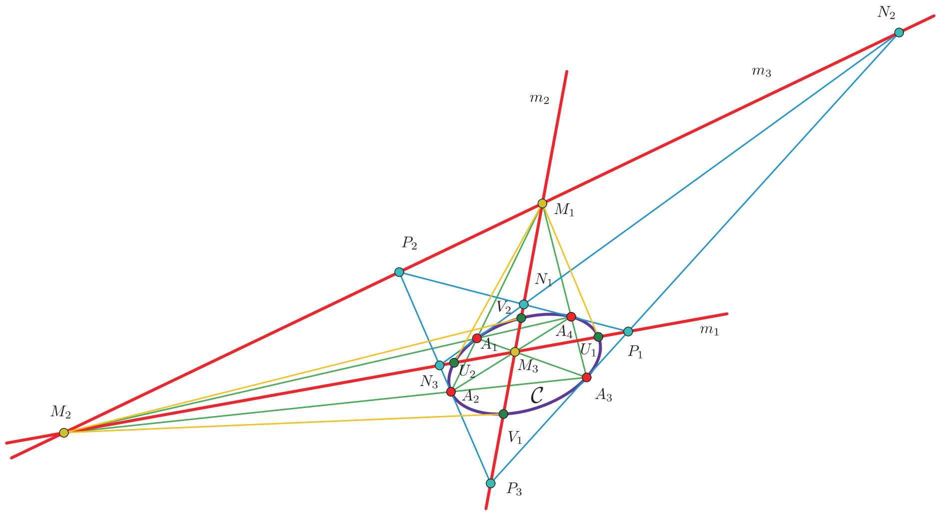

Figure 5. Quadrilateral inscribed in a conic.

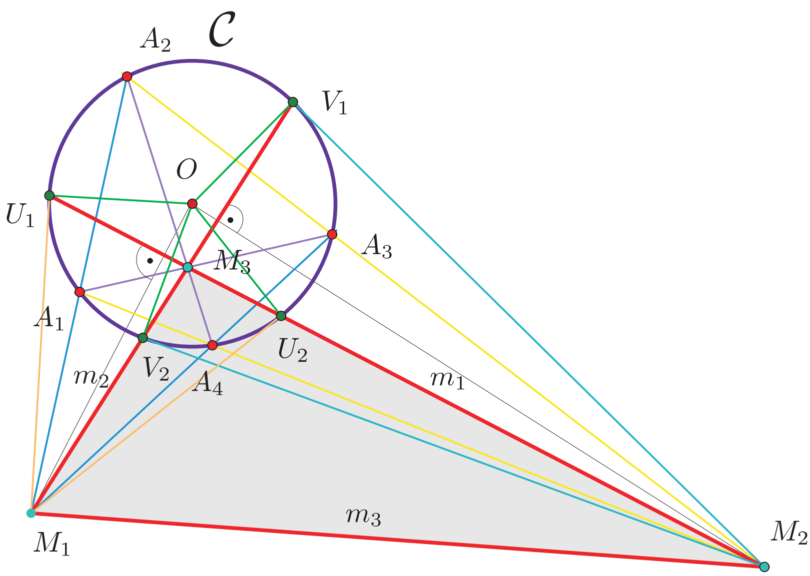

Figure 6. Brocard theorem.

Lemma 2.3 clearly implies the analogous statement for the lines m_2 and m_3. This is the classical theorem of the projective geometry and a very useful tool (Figure 5).

We treat one very special case – when the conic $\mathcal{C}$ is a circle. Projective geometry gives us a plenty of techniques. For example, in the proof of Lemma 2.3 we used the projective transformation. We have already described degeneracy tool when we take some limit cases of polygons inscribed (or circumscribed) in a conic. It is good to keep in mind that a conic could degenerate itself for example to the two lines. This is a way to get interesting configurations of points and lines.

The configuration 5 in the case of a circle has a nice property which is known as the Brocard theorem. Let O be the centre of a circle $\mathcal{C}$. Then the quadrilateral $M_1U_1OU_2$ is deltoid and we get $M_1O \perp m_1$. Similarly, $M_2O \perp m_2$. Thus:

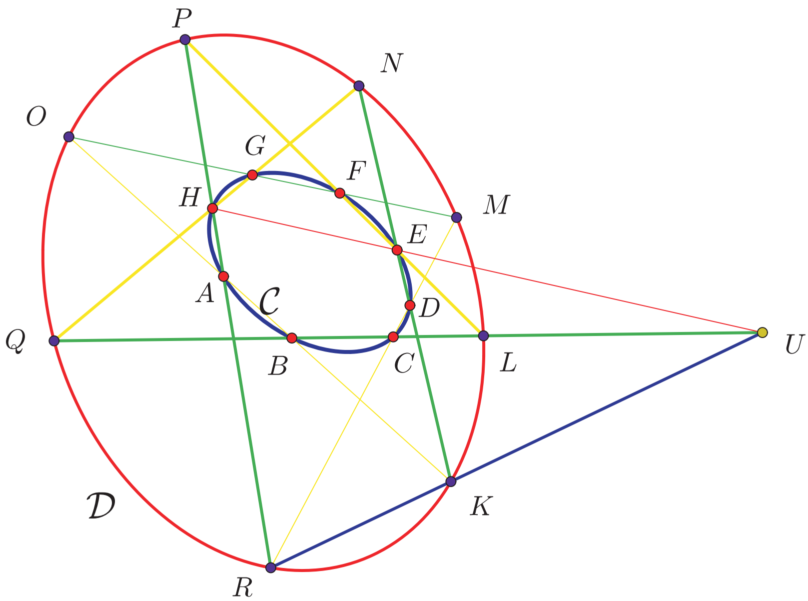

Figure 7. Mystic Octagon Theorem 2.2.

THEOREM 2.1 (Brocard theorem) *Let O be the centre of circumscribed circle of a cyclic quadrilateral $A_1A_2A_3A_4$. Then O is the orthocentre of triangle $\triangle M_1M_2M_3$ (Figure 6).*

THEOREM 2.2 (Mystic Octagon Theorem) *Let $ABCDEFGH$ be an octagon inscribed in a conic $\mathcal{C}$ and let the lines AB, CD, EF and GH intersect the lines BC, DE, FG and HA in the points K, L, M, N, O, P, Q and R. Then the eight points K, L, M, N, O, P, Q and R lie on the same conic (Figure 7).*

Proof This theorem was formulated by Wilkinson [25]. The proof we present uses only the Pascal theorem and is given by Evans and Rigby [11]. It could also be found in the monograph [4] by Bix.

Let U be the intersection point of the lines BC and EH. By Pascal's theorem the points K, R and U are collinear, see Figure 7. Then by the converse of Pascal's theorem, the points K, L, N, O, Q and R lie on the same conic. Analogously, we prove that the points L, M, N, O, Q and R lie on the same conic. There is a unique conic through some five points, so the points K, L, M, N, O, Q and R lie on the same conic. In the same manner we can prove that the point P also belongs to this conic. ■

3. More lines, pencils of lines and surprising conics

We continue in the same manner. The lines and the pencils of lines we study came from various degenerations of the vertices of hexagon inscribed in a conic. Let us note that configuration associated with 60 Pascal lines has been described in [2,14,24]. All results from this section could be obtained as certain degenerate cases. But we are going to treat them by elementary means.

Let T_1 be the point of intersection of the line A_3A_4 and the tangent at A_1 to $\mathcal{C}$, T_2 of A_4A_1 and the tangent at A_2, T_3 of A_1A_2 and the tangent at A_3 and T_4 of A_2A_3 and the tangent at A_4. Let X_1 be the point of intersection of the line A_2A_3 and the tangent at A_1 to $\mathcal{C}$, X_2 of A_3A_4 and the tangent at

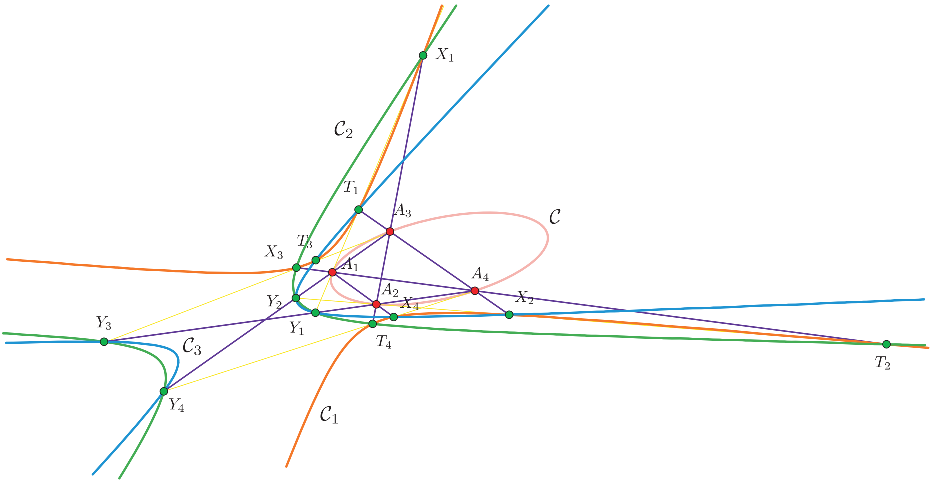

Figure 8. Lemma 3.1.

A_2, X_3 of A_4A_1 and the tangent at A_3 and X_4 of A_1A_2 and the tangent at A_4. Let Y_1 be the point of intersection of the line A_2A_3 and the tangent at A_1, Y_2 of A_1A_4 and the tangent at A_2, Y_3 of A_1A_4 and the tangent at A_3 and Y_4 of A_2A_3 and the tangent at A_4.

LEMMA 3.1 *The points X_1, X_2, X_3, X_4, T_1, T_2, T_3 and T_4 lie on the same conic $\mathcal{C}_1$; Y_1, Y_2, Y_3, Y_4, X_1, X_3, T_2, and T_4 lie on the same conic $\mathcal{C}_2$; T_1, T_3, X_2, X_4, Y_1, Y_2, Y_3 and Y_4 lie on the same conic $\mathcal{C}_3$ (Figure 8).*

Proof This statement is a special case of the Mystic Octagon theorem. The first conic appears when we consider degenerate octagon $A_1A_2A_2A_3A_3A_4A_4A_1$, the second for $A_1A_3A_3A_2A_2A_4A_4A_1$, and the third for $A_1A_3A_3A_4A_4A_2A_2A_1$. ■

PROPOSITION 3.1 *The following 16 triples of points are collinear*: (M_1, Y_1, Y_2), (M_1, Y_3, Y_4), (M_1, X_3, T_4), (M_1, X_1, T_2), (M_2, Y_1, Y_4), (M_2, Y_2, Y_3), (M_2, X_4, T_1), (M_2, X_2, T_3), (M_3, T_1, T_3), (M_3, X_2, X_4), (M_3, X_1, X_3), (M_3, T_2, T_4), (X_2, Y_3, T_4), (X_1, Y_2, T_3), (X_3, Y_4, T_1), (X_4, Y_1, T_2) (*Figure* 9).

Proof The collinearity of the points M_1, X_3 and T_4 follows from Pascal's theorem for degenerate hexagon $A_1A_4A_4A_3A_3A_2$, the collinearity of the points M_1, Y_3 and Y_4 from degenerate hexagon $A_1A_3A_3A_4A_4A_2$ and the collinearity of the points X_2, Y_3 and T_4 from degenerate hexagon $A_2A_3A_3A_4A_4A_2$. The proof for the rest is analogous. ■

PROPOSITION 3.2 *The following six triples of lines are concurrent*: (M_2M_3, X_2Y_3, X_3Y_4), (M_1M_3, X_1Y_2, X_2Y_3), (M_1M_2, X_1Y_2, X_3Y_4), (M_2M_3, X_1Y_2, X_4Y_1), (M_1M_3, X_4Y_1, X_3Y_4), (M_1M_2, X_4Y_1, X_2Y_3) *(Figure 9).*

Proof By Lemma 3.1 the points X_1, X_2, X_3, X_4, T_1, T_2, T_3 and T_4 lie on the same conic. From the Pascal theorem for the hexagon $T_1X_3X_1T_2X_4X_2$ we get that lines M_1M_3, X_4Y_1 and X_3Y_4 are concurrent. Analogously for other triples. ■

Define the points as the intersections of the lines: $B_1 = l(A_2V_1) \cap l(A_1V_2)$, $C_1 = l(A_1V_1) \cap l(A_2V_2)$, $D_1 = l(A_3V_1) \cap l(A_4V_2)$, $E_1 = l(A_4V_1) \cap l(A_3V_2)$, $B_3 = l(A_4V_1) \cap l(A_2V_2)$,

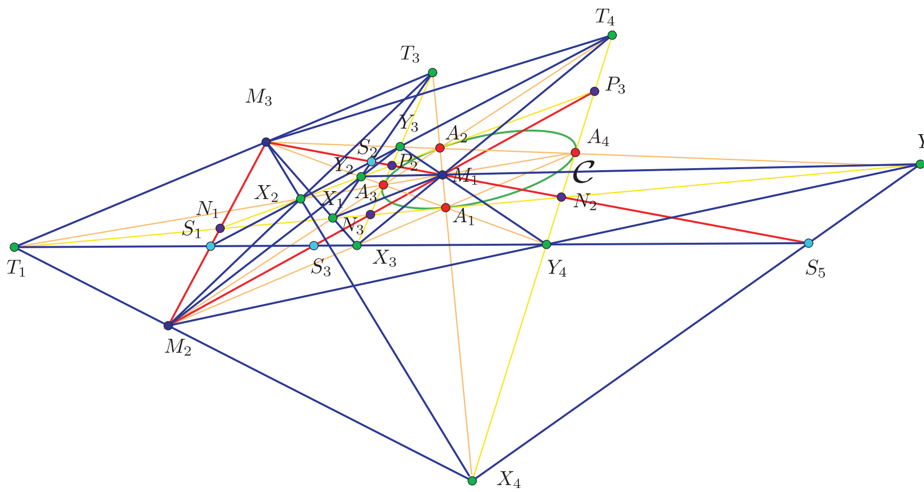

Figure 9. Propositions 3.1 and 3.2.

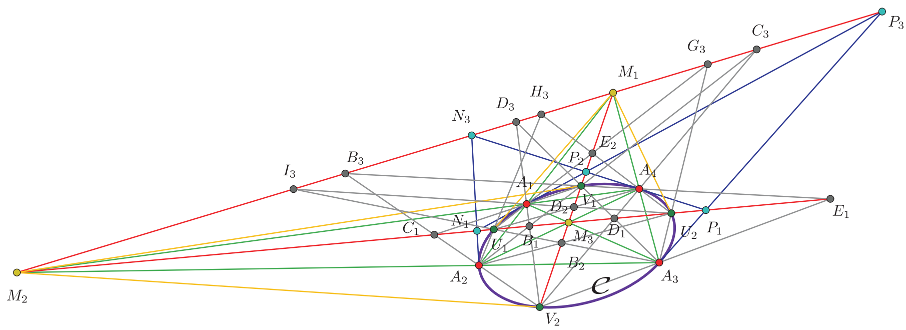

Figure 10. Propositions 3.3.

$C_3 = l(A_4V_2) \cap l(A_2V_1)$, $D_3 = l(A_1V_1) \cap l(A_3V_2)$, $E_3 = l(A_1V_2) \cap l(A_3V_1)$, $D_2 = l(A_4U_1) \cap l(A_1U_2)$, $E_2 = l(A_1U_1) \cap l(A_4U_2)$, $B_2 = l(A_3U_1) \cap l(A_2U_2)$, $C_2 = l(A_2U_1) \cap l(A_3U_2)$, $F_3 = l(A_4U_1) \cap l(A_2U_2)$, $H_3 = l(A_4U_2) \cap l(A_2U_1)$, $G_3 = l(A_1U_1) \cap l(A_3U_2)$, $I_3 = l(A_1U_2) \cap l(A_3U_1)$, $E_1 = l(A_2W_1) \cap l(A_1W_2)$, $F_1 = l(A_1W_1) \cap l(A_2W_2)$, $G_1 = l(A_3W_1) \cap l(A_4W_2)$, $H_1 = l(A_4W_1) \cap l(A_3W_2)$, $H_2 = l(A_4W_1) \cap l(A_1W_2)$, $I_2 = l(A_4W_2) \cap l(A_1W_1)$, $F_2 = l(A_2W_1) \cap l(A_3W_2)$ and $G_2 = l(A_2W_2) \cap l(A_3W_1)$.

PROPOSITION 3.3 *The points B_1, C_1, D_1, E_1, F_1, G_1, H_1, I_1 lie on the line M_2M_3. Similarly, the points B_2, C_2, D_2, E_2, F_2, G_2, H_2, I_2 lie on the line M_3M_1 and the points B_3, C_3, D_3, E_3, F_3, G_3, H_3, I_3 lie on the line M_1M_2.*

Proof Consider the quadrilateral formed by the tangent lines to the conic $\mathcal{C}$ at the points A_4, A_2, V_1 and V_2. Applying Lemma 2.2, we get that the point B_3 lies on the line M_1M_2. Analogously for other points. ■

We introduced many points and showed that some of them are collinear while some are the intersections of certain lines. But some of them lie on the conics that we are going to introduce (Figure 10).

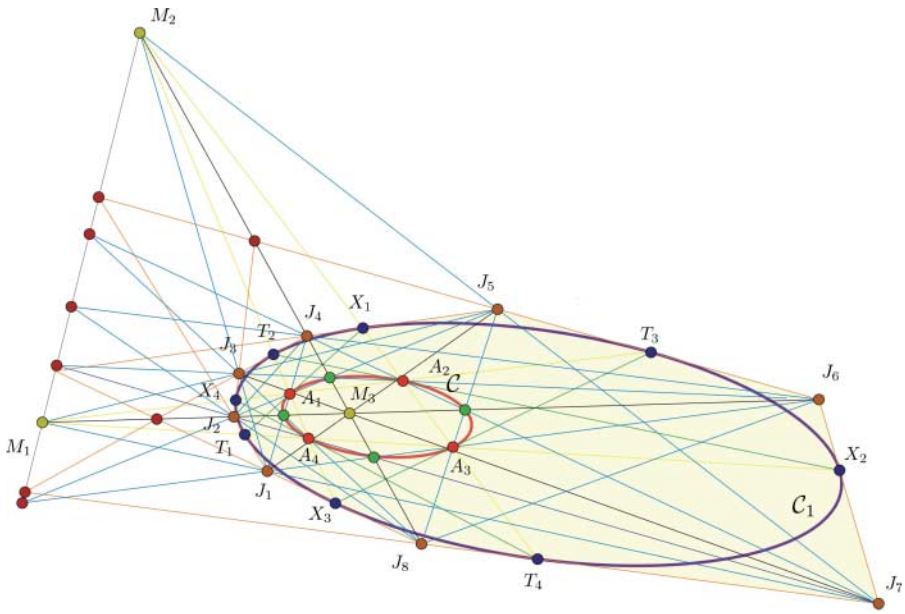

Figure 11. Theorem 3.1.

Let J_{2i-1} be the intersection point of the tangents at X_{i-2} and T_i on the conic $\mathcal{C}_1$, and J_{2i} be the intersection point of the tangents at X_{i-1} and T_i (modulo 4), for $i = 1, 2, 3, 4$. Then the following claim is true:

THEOREM 3.1

- *The lines* J_iJ_{i+4}, *for* $i = 1, 2, 3, 4$ *intersect at the point* M_3.
- *The lines* J_1J_7, J_2J_6 *and* J_3J_5 *intersect at* M_1 *and the lines* J_1J_3, J_4J_8 *and* J_5J_7 *intersect at* M_2.
- *The lines* J_1J_4 *and* J_2J_5 *intersect at* A_1, *the lines* J_4J_7 *and* J_3J_6 *at* A_2, *the lines* J_6J_1 *and* J_5J_8 *at* A_3 *and the lines* J_3J_8 *and* J_2J_7 *at* A_4.
- *The intersection points* $l(J_2J_4) \cap l(J_6J_8)$, $l(J_2J_8) \cap l(J_4J_6)$, $l(J_3J_6) \cap l(J_2J_7)$, $l(J_5J_8) \cap l(J_1J_4)$, $l(J_3J_8) \cap l(J_4J_7)$, $l(J_2J_5) \cap l(J_1J_6)$ *and* $l(J_iJ_{i+1}) \cap l(J_{i+4}J_{i+5})$ *for* $i = 1, 2, 3, 4$ *lie on the same line* M_1M_2.
- *The intersection points* $l(J_4J_5) \cap l(J_7J_8)$ *and* $l(J_3J_4) \cap l(J_1J_8)$ *lie on the same line* M_1M_3, *and the intersection points* $l(J_2J_3) \cap l(J_5J_6)$ *and* $l(J_1J_2) \cap l(J_6J_7)$ *lie on the same line* M_2M_3.
- *The point* P_3 *lies on the line* J_3J_7 *and the point* N_3 *on the line* J_1J_5.
- *Three lines* $J_{2i}J_{2i+4}$, $J_{2i+1}J_{2i-2}$ *and* $J_{2i-1}J_{2i+2}$ (*modulo 8*) *are concurrent for* $i = 1, 2, 3, 4$ (*Figure* 11).

Proof Consider the quadrilateral formed by tangents to $\mathcal{C}_1$ at J_2 and J_6. By Lemma 2.2 and Proposition 3.1 the points M_3 and M_2 lie on the line J_2J_6 (we could take different orders of points). Analogously, the lines J_1J_5, J_3J_7 and J_4J_8 pass through the point M_3. In a similar manner we prove other statements for the points M_1 and M_2, as well as the points N_3 and P_3.

Lemma 2.2 applied to the quadrilateral formed by the tangents to $\mathcal{C}_1$ at J_2 and J_5 implies that the line J_2J_5 passes through A_1. Similarly, A_1 belongs to the line J_1J_4. Analogously, we prove the corresponding statements for the points A_2, A_3 and A_4.

From Lemma 2.1 applied to the quadrilateral $T_2X_1T_4X_3$ and Proposition 3.1, it follows that the intersection point of the lines J_3J_4 and J_7J_8 and the intersection point of the lines J_4J_5 and J_8J_1

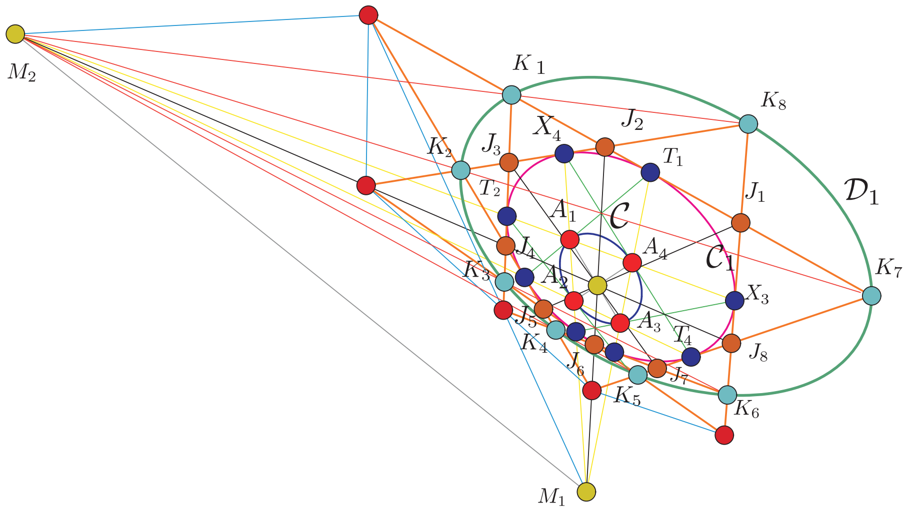

Figure 12. Theorem 3.2.

lie on the line M_1M_2. Then by Brianchon's theorem for the hexagon formed by the tangents to $\mathcal{C}_1$ at T_2, X_1, T_3, T_1, X_3 and T_4 the intersection point of the lines J_1J_4 and J_5J_8 lies on the line M_1M_2. Analogously, we prove the same statement for other points.

The Brianchon theorem for the hexagon formed by the tangents to $\mathcal{C}_1$ at T_2, X_1, X_4, T_1, X_3 and T_4 gives the concurrency of the lines J_2J_6, J_1J_4 and J_5J_8. We use the similar argument for the rest of the proof. ■

Let K_i be the intersection point of the lines J_iJ_{i+1} and $J_{i+2}J_{i+3}$ (modulo 8) for $i = 1, \ldots, 8$.

THEOREM 3.2 *The points K_i lie on the same conic $\mathcal{D}_1$ (Figure 12).*

Proof It is not hard to prove that the lines K_1K_5, K_2K_6, K_3K_7 and K_4K_8 pass through the point M_3, the lines K_2K_3, K_1K_4, K_5K_8 and K_6K_7 pass through the point M_1 and the lines K_2K_7, K_1K_8, K_3K_6 and K_4K_5 pass through the point M_2. From the collinearity of the points M_1, J_2 and $l(J_4J_5) \cap l(J_7J_8)$ the points K_1, K_2, K_4, K_5, K_7 and K_8 lie on the same conic. Using the similar argument we show that K_2, K_4, K_5, K_6, K_7 and K_8 lie on the same conic as well. Because there is a unique conic that passes through some 5 points, the points K_1, K_2, K_4, K_5, K_6, K_7 and K_8 are on the same conic. Then it is easy to prove that K_3 also lies on the conic. ■

Let $Z_1 = l(M_1U_1) \cap l(M_2V_1)$, $Z_2 = l(M_1U_1) \cap l(M_2V_2)$, $Z_3 = l(M_1U_2) \cap l(M_2V_2)$ and $Z_1 = l(M_1U_2) \cap l(M_2V_1)$.

THEOREM 3.3 *The points N_1, N_2, P_1, P_2, Z_1, Z_2, Z_3 and Z_4 lie on the same conic.*

Proof There exists a projective transformation φ that maps the vertices A_1, A_2, A_3 and A_4 onto the vertices of a square. Then the point $\varphi(M_3)$ is mapped onto the centre of a conic $\varphi(\mathcal{C})$ and the lines $\varphi(N_1)\varphi(P_1)$ and $\varphi(N_2)\varphi(P_2)$ are the axes of this conic. The points $\varphi(U_1)$, $\varphi(U_2)$, $\varphi(V_1)$ and $\varphi(V_2)$ also lie on the axes. As we could see from Figure 13, everything is symmetric with respect to the axes and it is easy to conclude that there is a conic through $\varphi(Z_1)$, $\varphi(Z_2)$, $\varphi(Z_3)$, $\varphi(Z_4)$, $\varphi(N_1)$, $\varphi(N_2)$, $\varphi(P_1)$ and $\varphi(P_2)$. ■

Theorems 3.1–3.3 associate new conics to the quadrilateral inscribed in a conic. They have interesting properties which will be explained in the following section.

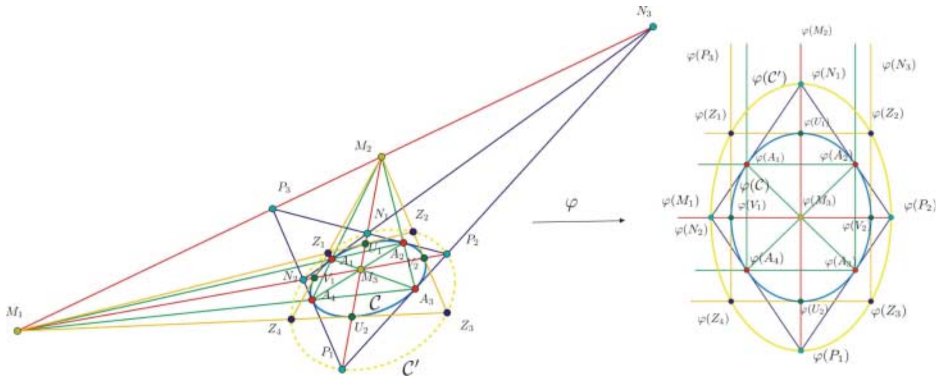

Figure 13. Theorem 3.3.

4. Poncelet's quadrilateral porism

Jean-Victor Poncelet's famous *Closure theorem* states that if there exists one n-gon inscribed in a conic $\mathcal{C}$ and circumscribed about a conic $\mathcal{D}$ then any point on $\mathcal{C}$ is the vertex of some n-gon inscribed in a conic $\mathcal{C}$ and circumscribed about a conic $\mathcal{D}$. Poncelet published his theorem in [18]. However, this result influenced mathematics until nowadays. In a recent book [9] by Dragovic and Radnovic there are several proofs of the Closure theorem, its generalizations as well as its relations with elliptic functions theory. The proof is not elementary for an arbitrary n, although in the case $n = 3$ an elegant proof can be found in almost every monograph in projective geometry, see [5,17].

Theorems 3.2 and 3.3 are the special cases of the Poncelet theorem for $n = 4$. Actually, quadrilaterals and conics in them have a poristic property. An elementary proof using harmonic locus of two conics can be found in [12]. We kept the spirit of elementarity through our paper and our agenda was: At first, we experiment in *Cinderella*, after that the proof is recovered by elementary tools (again directly guided by *Cinderella*'s tools). In the same style we continue and offer a direct analytic proof of the Poncelet theorem for quadrilaterals without using differentials and elliptic functions. A synthetic version of this proof is presented in [15].

LEMMA 4.1 *Let λ, μ be such that the conics $\mathcal{C} : \lambda x^2 + (1-\lambda)y^2 - 1 = 0$ and $\mathcal{D} : x^2 + \mu xy + y^2 + (\mu^2 - 1)/4 = 0$ are non-degenerate. Let A be a point on $\mathcal{C}$ and B and B' be the intersections of the tangent lines from A to $\mathcal{D}$ with the conic $\mathcal{C}$. Then the points B and B' are symmetric with respect to the origin.*

Proof Let $t : y = kx + n$ be a tangent line to the conic $\mathcal{D}$ (Figure 14). The condition of tangency between t and $\mathcal{D}$ is

$$n^2 = k^2 + mk + 1. \tag{1}$$

The coordinates of the intersection points of t and $\mathcal{C}$ are

$$(x_1, y_1) = \left(\frac{-2(1-\lambda)kn - \sqrt{D}}{2(\lambda + (1-\lambda)k^2)}, \quad k \cdot \left(\frac{-2(1-\lambda)kn - \sqrt{D}}{2(\lambda + (1-\lambda)k^2)} \right) + n \right)$$

and

$$(x_2, y_2) = \left(\frac{-2(1-\lambda)kn + \sqrt{D}}{2(\lambda + (1-\lambda)k^2)}, \quad k \cdot \left(\frac{-2(1-\lambda)kn + \sqrt{D}}{2(\lambda + (1-\lambda)k^2)} \right) + n \right),$$

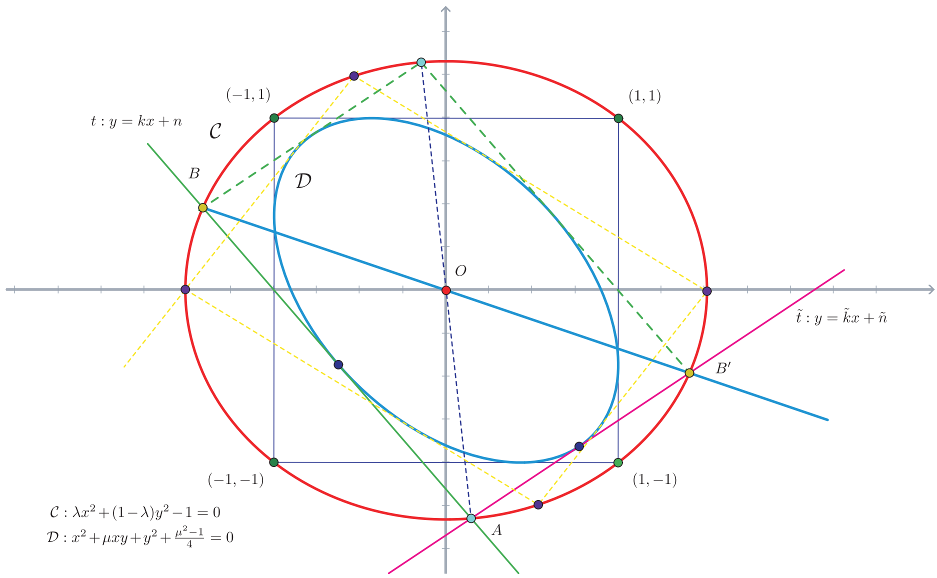

Figure 14. Lemma 4.1 and Theorem 4.1.

where $D = 4(\lambda - \lambda(1-\lambda)n^2 + (1-\lambda)k^2)$. It is necessary and enough to prove that a line through the points $(-x_1, -y_1)$ and (x_2, y_2) is tangent to $\mathcal{D}$. This line has the equation $y = \tilde{k}x + \tilde{n}$ where $\tilde{k}$ and $\tilde{n}$ can be calculated as

$$\tilde{k} = \frac{-\lambda}{(1-\lambda)k} \quad \text{and} \quad \tilde{n} = \frac{\sqrt{D}}{2k(1-\lambda)}. \tag{2}$$

We need to check if

$$\tilde{n}^2 = \tilde{k}^2 + m\tilde{k} + 1.$$

It is directly verified that condition (1) multiplied by $\lambda(1-\lambda)/k^2(1-\lambda)^2$ finishes our proof. ∎

THEOREM 4.1 *Let $\mathcal{C}$ and $\mathcal{D}$ be conics such that there exists one quadrilateral inscribed in a conic $\mathcal{C}$ and circumscribed about a conic $\mathcal{D}$. Then any point on $\mathcal{C}$ is the vertex of some quadrilateral inscribed in a conic $\mathcal{C}$ and circumscribed about a conic $\mathcal{D}$.*

Proof There exists a projective transformation that maps the vertices of the quadrilateral inscribed in a conic $\mathcal{C}$ and circumscribed about a conic $\mathcal{D}$ onto the points $(1, 1)$, $(1, -1)$, $(-1, -1)$ and $(-1, 1)$ (in the standard chart). Thus, the conics $\mathcal{C}$ and $\mathcal{D}$ are transformed in those with the equations as in Lemma 4.1. Now the claim follows. ∎

In fact, we proved more. All quadrilaterals with poristic property with respect to $\mathcal{C}$ and $\mathcal{D}$ have a common point of the intersection of diagonals (lines joining opposite vertices) and a common line passing through the intersections of opposite side lines. Our work in previous section now could be reviewed in a new light.

Theorems 3.1–3.3 are obtained after we defined certain points. If we apply the same procedure for defining new points on the points and conics in theorems, again we come to similar conclusions. Thus, by repeating this procedure, we obtain an infinite sequence of conics (Figure 15). Every two consecutive conics in this sequence are Poncelet 4-connected.

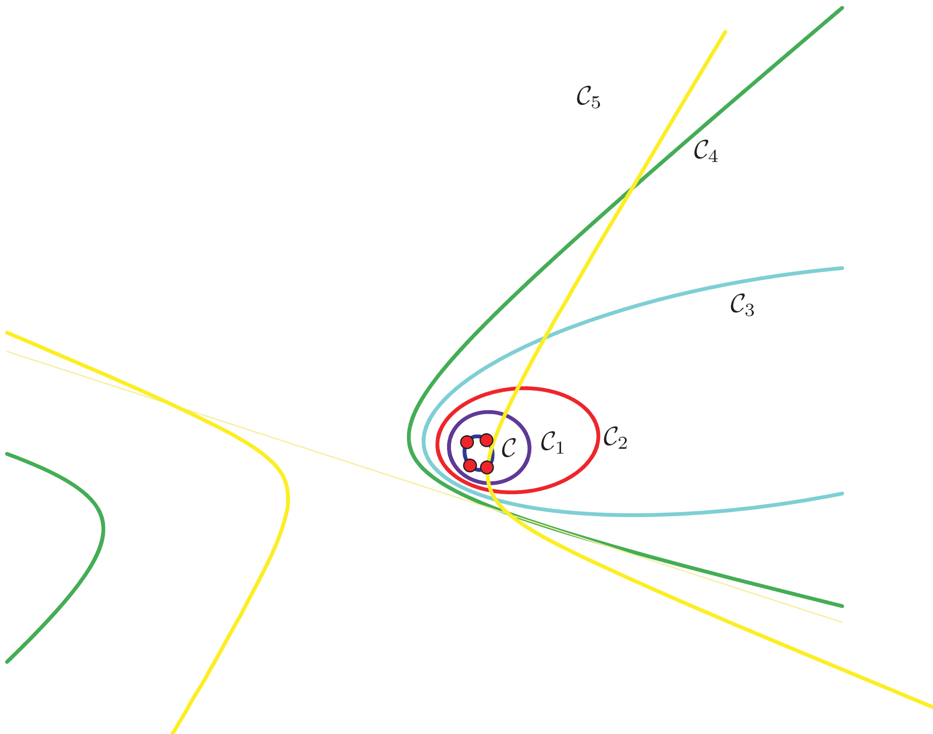

Figure 15. The first five conics in the sequence.

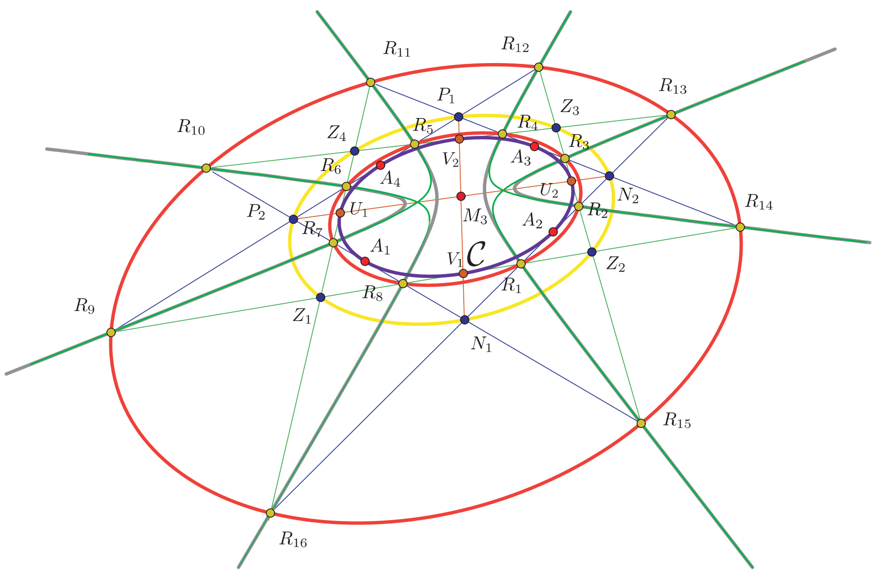

Figure 16. Theorem 4.2.

Our theorems resemble Darboux's theorem, see [6–8]. They could be seen as a very special case of Dragović–Radnović theorem 8.38 [9]. Such constructions are also studied in the paper of Schwartz, see [21]. The following result further explains their connection, but first we define 16 points of the intersections $R_1 = l(Z_1Z_2) \cap l(N_1N_2)$, $R_2 = l(Z_1Z_2) \cap l(N_1P_2)$,

$R_3 = l(Z_2Z_3) \cap l(N_1P_2), R_4 = l(Z_2Z_3) \cap l(P_1P_2), R_5 = l(Z_3Z_4) \cap l(P_1P_2), R_6 = l(Z_3Z_4) \cap l(P_1N_2)$, $R_7 = l(Z_1Z_4) \cap l(P_1N_2)$, $R_8 = l(Z_1Z_4) \cap l(N_1N_2)$, $R_9 = l(Z_1Z_2) \cap l(P_1P_2)$, $R_{10} = l(Z_3Z_4) \cap l(N_1P_2)$, $R_{11} = l(Z_2Z_3) \cap l(P_1N_2)$, $R_{12} = l(Z_1Z_4) \cap l(P_1P_2)$, $R_{13} = l(Z_3Z_4) \cap l(N_1N_2)$, $R_{14} = l(Z_1Z_2) \cap l(P_1N_2)$, $R_{15} = l(Z_1Z_4) \cap l(N_1P_2)$ and $R_{16} = l(Z_2Z_3) \cap l(N_1N_2)$ (Figure 16).

THEOREM 4.2 *The next groups of* 8 *points lie on the same conic:* $\{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$, $\{R_9, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}\}$, $\{R_1, R_2, R_5, R_6, R_{11}, R_{12}, R_{15}, R_{16}\}$, $\{R_3, R_4, R_7, R_8, R_9, R_{10}, R_{13}, R_{14}\}$, $\{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\}$ *and* $\{R_2, R_3, R_4, R_6, R_8, R_{10}, R_{12}, R_{14}, R_{16}\}$.

The proof of Theorem 4.2 uses the same arguments we used in the previous proofs so we omit it.

If we look at the conic $\mathcal{C}$ and a conic $\mathcal{F}$ through the points $\{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$ we see they are Poncelet 8-connected and appropriate conics from Theorem 4.2, the conic from Theorem 3.3 with the line M_1M_2 form Poncelet-Darboux grid. Two conics $\{R_2, R_3, R_4, R_6, R_8, R_{10}, R_{12}, R_{14}, R_{16}\}$ and $\{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\}$ are not coming from Poncelet-Darboux grid, but they could be directly obtained from Dragović–Radnović theorem 8.38, [9]. This result improves the result of Schwartz [21] in a particular case.

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