3-2020

Investing with cryptocurrencies: A liquidity constrained investment approach

Simon TRIMBORN
National University of Singapore

Mingyang LI
Xiamen University

Wolfgang Karl HARDLE
Singapore Management University

Published in Journal of Financial Econometrics, Spring 2020, 18 (2), 280-306. DOI: 10.1093/jjfinec/nbz016

Investing with cryptocurrencies - A liquidity constrained investment approach [1]

Simon Trimborn [2]
Mingyang Li [3]
Wolfgang Karl Härdle [4]

May 25, 2018

Cryptocurrencies have left the dark side of the finance universe and become an object of study for asset and portfolio management. Since they have a low liquidity compared to traditional assets, one needs to take into account liquidity issues when adding them to the same portfolio. We propose a LIquidity Bounded Risk-return Optimization (LIBRO) approach, which is a combination of risk-return portfolio optimization under liquidity constraints. In the application cryptocurrencies are included into portfolios formed with S&P 100 component stocks, US-Bonds and Commodities. We illustrate the importance of the liquidity constraints in an in-sample and out-of-sample study. LIBRO improves the weight optimization in the sense of adding cryptocurrencies only in tradable amounts depending on the intended investment amount. The return increases strongly in-sample and out-of-sample. The paper shows that including cryptocurrencies can indeed improve the risk-return trade-off of the portfolio.

JEL classification: C01, C58, G11

Keywords: crypto-currency, CRIX, portfolio investment, asset classes, blockchain

[1]Financial support from IRTG 1792 "High Dimensional Non Stationary Time Series", Humboldt-Universität zu Berlin, is gratefully acknowledged.
[2]Department of Statistics & Applied Probability, National University of Singapore, Singapore and Humboldt-Universität zu Berlin, C.A.S.E. - Center for Applied Statistics and Economics, Spandauer Str. 1, 10178 Berlin, Germany, tel: +65 6516-1245, E-Mail: simon.trimborn@nus.edu.sg
[3]Wang Yanan Institute for Studies in Economics, Xiamen University, Siming South Road, 422, Xiamen, China, E-mail: limingyang@stu.xmu.edu.cn
[4]Humboldt-Universität zu Berlin, C.A.S.E. - Center for Applied Statistics and Economics, Spandauer Str. 1, 10178 Berlin, Germany and SKBI School of Business, Singapore Management University, 50 Stamford Road, Singapore 178899, tel: +49 (0)30 2093-5630, E-Mail: haerdle@hu-berlin.de

1 Introduction

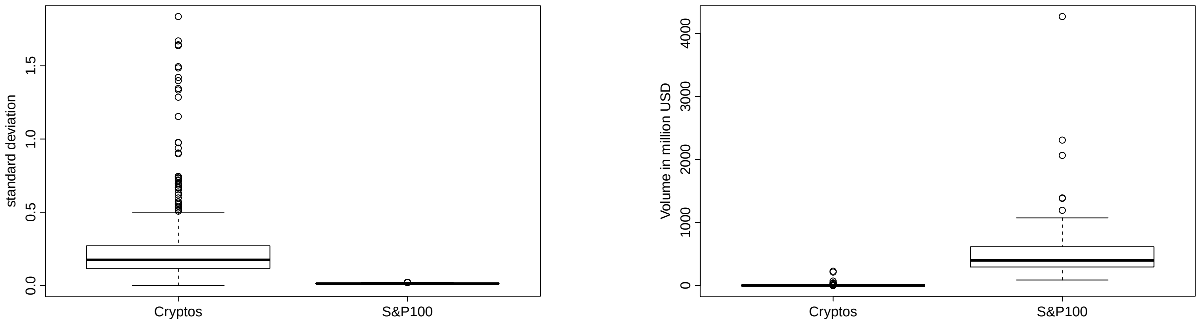
With the emergence of cryptocurrencies, not just a new kind of currencies and transaction networks arose, also a new kind of investment products. The cryptocurrency (CC) market shows a strong gain over the last years, which can be inferred from CRIX, developed by Härdle and Trimborn (2015) and visualized on crix.hu-berlin.de. The CRIX index indicates a gain of the market of 3200% over the last 1.5 years, which makes it attractive for investors. Simultaneously the market bears high risk in terms of price variations and operational risk. In the last years users and exchanges were vulnerable in many ways, e.g. the traders on the exchange Mt.Gox experienced fraud and exchanges like Bitfinex got hacked. Also single users were subject to larceny. The situation improved already a lot but still remains a trust problem since the market is not fully developed. It is often pointed out a procedure called "cold-storage" shall be used to secure ones CCs. It refers to storing the access codes for the coins in a way that they are disconnected from any device in thread of a hostile attack. While this source of risk can be managed comparably easy, the market risk is difficult to handle.

A natural question is why an investor should engage in such a risky market given the described volatility effects. Advantages beside the opportunity for strong gains need to be present to make an investment worth the risk. An important perk is the finding of CCs having a low linear dependency with each other, Elendner et al. (2017): top 10 CCs (by market capitalization) have a low linear dependency with traditional assets. Since CCs are of low correlation with each other and uncorrelated with traditional assets, they are indeed interesting for investors due to the diversification effect. Making use of this advantage, Brière et al. (2015) and Eisl et al. (2015) added Bitcoin to a portfolio of traditional assets and found an enhanced portfolio in terms of risk-return. Since alternative CCs (alt coins, other than Bitcoin), have favorable properties too, we are aiming on constructing portfolios consisting of traditional assets and several cryptocurrencies. Lee Kuo Chuen et al. (2017) worked in a related direction by investigating a portfolio mimicking CRIX, the CRyptocurrency IndeX. Treating CRIX as a financial asset, Chen et al. (2018) and Chen et al. (2017) investigated option pricing based on CRIX.

When investing with CCs, one is confronted with a higher volatility pattern than for traditional assets, see Figure 1. Markowitz (1952) developed a method - accounting for diverging variance and covariance - in terms of a minimum variance portfolio according to a target return. The approach was applied in a broad variety of applications and showed its usefulness especially in the case of Gaussian distributed data. But CCs are known to behave different from the normal distribution, Elendner et al. (2017). In particular the stronger tails come with higher risk arising from higher moments, Scaillet et al. (2018). Tail risk optimized portfolios might be worth considering in this market, like taking into account Conditional Value-at-Risk (CVaR), Rockafellar and Uryasev (2000). One further issue though can not be handled by these risk optimization methods, namely the low liquidity of the CC market. In Figure 1 (right plot), we make a comparison of liquidity measured by median daily trading amount of CCs and S&P 100 component stocks. It is obvious that the median daily trading amount of CCs are all lower than the 25% quantile of S&P 100 stocks.

If we want to include CCs and stocks into the same portfolio, we need to avoid giving CCs a too big weight since this will induce a severe liquidity problem on adjusting the position when reallocating the portfolio. For example, if we hold a long position on an asset, which equals to twice its average daily trading amount, then it is expected to take about two days to clear this position, following the same pace of the market. However, this may result in missing the trading opportunity. A proper way to deal with such a liquidity issue, is the introduction of liquidity constraints on the weights. Krokhmal et al. (2002) utilized liquidity constraints in the sense of restricting the change in a position. Darolles et al. (2012) choose a related approach by incorporating a penalty term into the optimization function, balancing the risk and change of positions in the portfolio. However we intend to be able to clear all positions at a time, which is assumed to be in the interest of an investor engaging in a risky market akin to the CC market. Instead our definition of liquidity constraints is concentrated on the entire weight given to a CC, rather than the allowed change in a position. Additionally such an approach has the advantage to tackle a drawback of Markowitz portfolios. Minimum Variance optimized portfolios often suffer from extreme positive and negative weights, Härdle et al. (2018). This may result from a single dominant factor in the covariance matrix, Green and Hollifield (1992). In an empirical study, Jagannathan and Ma (2003) find nonnegativity constraints on the weights to have an equal effect to removing the effect of a single dominant eigenvalue from the covariance matrix. Fan et al. (2012) provide theoretical insights into their findings and find constraining the weights from taking extreme positions to be more effective than nonnegativity constraints. Thus introducing weight constraints gives us the opportunity to "beat two birds with one stone".

Due to the outlined challenges and the advantage from investing with cryptocurrencies, we are aiming at a portfolio optimization method which accounts for volatility or tail risk and low liquidity. We call it LIBRO - LIquidity Bounded Risk-return Optimization, which is a combination of a risk optimization portfolio formation method and an additional restriction, which prevents big weights on low liquidity assets. The portfolios are formed with Mean-Variance (Markowitz) and Conditional Value-at-Risk as risk measures. Due to the huge dimensionality of the asset universe and limited data availability, the sample covariance matrix may not be a well-conditioned estimator of its theoretical counterpart (well-conditioned in the sense of inverting the covariance matrix does not amplify the estimation error, Ledoit and Wolf (2004)). A well-conditioned and more accurate estimator was introduced by Ledoit and Wolf (2004), which we apply for the estimation of the Markowitz portfolios. Reduced factor model approaches were e.g. investigated by Kozak et al. (2017) and sparse estimation by e.g. Friedman et al. (2008). To investigate the robustness of the results, the reallocation dates in the out-of-sample study are set to be monthly and weekly. In order to overcome estimation difficulties driven by too short time series, we work under an extending window approach. Two datasets are compared in the application, where the first one is a portfolio formed with S&P 100 components and CCs. The excess return from the portfolio with CCs to the pure stock one ranges from 13.5% to 88% (gained over 3.5 years) in the in-sample analysis, and ranges from 13.7% to 60% (gained over 2.75 years) in the out-of-sample analysis. By using stocks, bonds and commodities as the traditional assets, the results still range from 6% to 20.43% in-sample



(a) Comparison of standard deviation of CCs and S&P100 Equity Index components.

(b) Comparison of median trading volume of CCs and S&P100 Equity Index components.

Figure 1: The Figures show the boxplots of standard deviation and median trading volume (measured in US dollar) of all CCs and S&P 100 components, using the sample between 2014-04-22 and 2017-10-30. Obviously CCs have much lower daily trading volumes than S&P 100 component stocks and higher volatilities than stocks, highlighting the importance of volatility and liquidity risk management when investing on them.

(3.5 years) and 6.78% to 24.38% out-of-sample (2.75 years). Summary statistics of the return series indicate that including CCs can increase the Sharpe-Ratio, thus the paper shows that including CCs can indeed improve the risk-return trade-off of the portfolio. Furthermore the study illustrates the importance of the liquidity constraints and their effects.

The paper is organized as follows. In section 2 the data get introduced, while in section 3 the portfolio optimization methods are reviewed. Section 4 introduces the liquidity constraints and section 5 gives an in-sample and out-of-sample application with S&P 100 component stocks, Barclays Capital US Aggregate Index (US-Bonds Index), S&P GSCI (Commodities Index) and CCs. The portfolios based on stocks are abbreviated with S, while the Stock, Bonds, Commodities ones are referred to as SBC. Finally, the results are summarized in section 6. The codes used to obtain the results in this paper are available via `www.quantlet.de`, Borke and Härdle (2018) and Borke and Härdle (2017).

2 Data description

In this paper, 42 CCs are used to form portfolios with traditional financial assets, with a sample period from 2014-04-22 to 2017-10-30. The daily price (in USD) and trading volume data is downloaded from the CRIX cryptocurrencies database (crix.hu-berlin.de), kindly provided by CoinGecko. The CCs were selected such that the average market cap during the sample period is no less than $10,000$ US dollar. This criteria was applied since we target portfolios consisting of a reasonably high investment size, thus the CCs shall have enough market capitalization for being added to a portfolio.

For the traditional financial assets, we choose the S&P 100 components, Barclays Capital US Aggregate Index (US-Bonds Index), S&P GSCI (Commodities Index). The daily closing price (in USD) and trading volume, dated from 2014-04-22 to 2017-10-30, are downloaded from Datastream. To get the daily trading volume of stocks measured in US dollar, we multiply the daily trading volume by the daily close price. Three stocks were omitted, DowDuPont Inc., The Kraft Heinz Company and PayPal Holdings, Inc., since they have a shorter sample period due to company mergers or spin-offs.

We show the summary statistics for the top 10 CCs over time in Table 1, for the full list see the Table 11 in the Appendix. In both Tables, the CC statistics are arranged in decreasing order of their mean daily trading volume. For comparison, we list the summary statistics of stocks, bond index and commodity index too. The summary statistics of stocks are the average value for all individual stocks. The first five columns focus on the return series, while the remaining two list the mean trading volume and market capitalization. We will focus on Table 1 to analyze the summary statistics of CCs. Compared to the average annualized mean return of 8% for stocks and 5% for bond index, the ones of CCs can be quite shocking: except for PPC and BLK, all other eight have a positive return that exceeds 10%. What‘s more, five of them exceeds 20%, three of them exceeds 50%, and there is even one of them, DASH, that shows an annualized average return that exceeds 100%. It is witnessed that three of the alt-coins, Ripple (XRP), Dashcoin (DASH) and DigiByte (DGB), have a higher return than BTC, the dominant CC in the market, indicating that it is time to take into account these alt-coins for portfolio formation as well. However, the outstanding returns come at a price. Judging from Table 1, CCs also have much higher volatility and tail risk. All CCs have an annualized volatility that exceeds 50%, with seven of them even exceeding 100%. On the contrary, S&P100 stocks have an average annualized volatility of 20%, so does the commodity index, yet the bond index has only 5%. Thus one experiences a trade-off in terms of high return yet high standard deviation for the CCs. This finding is consistent with the reported size effect by Elendner et al. (2017). The similar picture is observed for the kurtosis: all the listed CCs, except for BTC and NXT, have a higher kurtosis than stocks (10.29), bond index (12.38) and commodity index (4.84). Among these CCs, BTC has the lowest volatility among the ten, which is not surprising since it has the largest market capitalization, largest trading volume and longest trading history, which makes it the most mature CC. Besides, BTC is the only CC that has a negative skewness, akin to stocks and bonds, while the other CCs all have positive skewness. This may infer that the other CCs are in a different development phase than BTC. For the auto-correlation, most of the top ten CCs have negative or slightly positive ρ, like stocks and bond.

The influence of liquidity is a major point of this study. The evolution of the log trading volume of the top 10 CCs by trading volume is shown in Figure 2. The trading volume shows for each of the CCs daily changes but mostly a trend behavior over time horizons. While each of them has a certain variation, they vary around a base value within a certain range. Table 2 fortifies this visual observation, showing descriptive statistics of the log returns of the turnover value. For all of the 10 CCs holds the observation, that they vary around a mean value of roughly 0. The median is slightly negative for all CCs, suggesting more frequently a decrease in the liquidity than an increase. However the 1st and 3rd quantile show more or less opposing values, being a hint for an even variation around the median value. In extreme cases - min and max values - this observation does not hold. The variance and extreme values suggest fixing liquidity weights based on the mean turnover value could result in too high boundaries, thus an approach based on a robust measure, the median, will be applied. In the next section, we are introducing the portfolio optimization methods to be used.

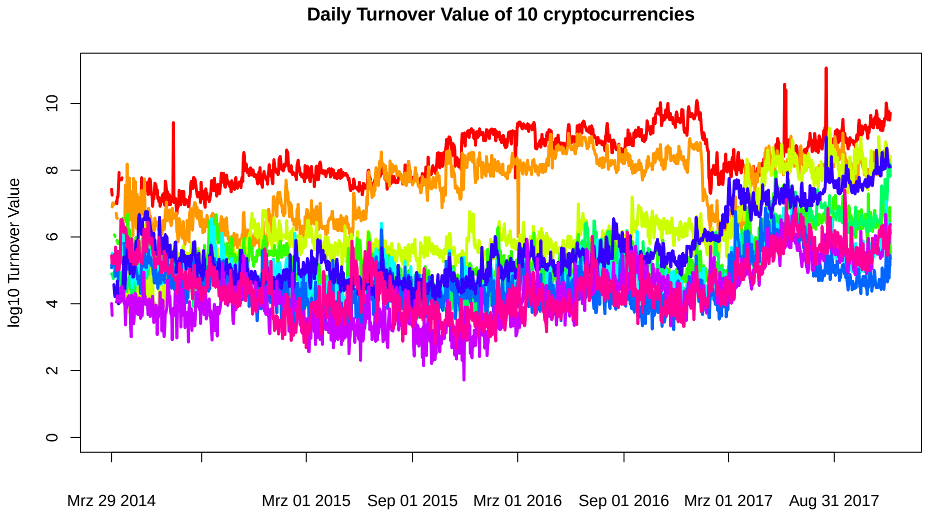


Figure 2: Log 10 turnover value of top 10 CCs by their daily trading volume, BTC, LTC, XRP, DOGE, NXT, PPC, NMC, DASH, XCP, DGB
LIBROliquidity

3 Constrained Portfolio Optimization

Markowitz (1952) introduced the theory of optimizing weights such that the variance of the portfolio is minimized according to a certain target return. When the variance serves as a risk measure, this translates into risk minimization. Consider now N assets with T returns given by an $(N \times T)$ matrix X and let $\hat{\Sigma}$ be the estimated covariance matrix of the respective assets. Then the Markowitz portfolio is defined as, Härdle and Simar (2015):

$$\begin{aligned} &\min_{w} \; w^\top \hat{\Sigma} w \\ &\text{s.t. } \mathbf{1}_N^\top w = 1, \; \mu \le x^\top w \end{aligned} \tag{1}$$

where $w = (w_1, w_2, \dots, w_N)^\top$ is the weight on assets, x the $(N \times 1)$ vector of expected returns of the assets, $\mathbf{1}_N$ is a $(N \times 1)$ matrix (vector) with all elements equals 1 and μ is the target return. The optimization problem is extended by a bound for each weight. The vector of constraints $a = (a_1, \dots, a_N)^\top$ with $a_i \in [0, \infty)$ for all $i = \{1, \dots, N\}$ is a $(N \times 1)$ vector and can be given (or estimated) upfront. Furthermore an upper bound for the sum over the absolute values of the weights gets introduced. Then, a constrained Markowitz portfolio is defined as,

	Ann.Ret	Ann.STD	skewness	kurtosis	ρ	mean volume	market cap
BTC	0.49	0.55	-0.61	10.87	-0.01	4.71e+08	1.07e+10
LTC	0.29	0.87	0.34	23.70	0.02	9.04e+07	3.22e+08
XRP	0.68	1.09	2.72	37.76	0.01	4.88e+06	8.35e+08
DASH	1.16	1.37	0.62	50.60	-0.14	1.61e+06	1.52e+08
DOGE	0.11	0.96	0.97	15.54	0.01	3.87e+05	3.11e+07
NXT	0.17	1.08	0.80	8.42	-0.03	2.28e+05	2.08e+07
DGB	0.73	1.70	2.93	29.89	-0.03	1.63e+05	7.24e+06
PPC	-0.17	1.00	0.62	12.51	-0.05	1.05e+05	1.54e+07
BLK	-0.05	1.27	1.82	17.72	-0.08	7.15e+04	4.21e+06
VTC	0.33	1.76	1.84	17.83	-0.01	5.93e+04	2.42e+06
Stocks	0.08	0.20	-0.26	10.29	0.00	5.40e+08	
Bond	0.05	0.05	-1.91	12.38	-0.04		
Commodity	-0.20	0.20	0.05	4.84	-0.07		

Table 1: Summary statistics of top 10 CCs by trading volume. Ann.Ret and Ann.STD indicates annualized mean and standard deviation of the return of each CC, which are calculated by multiplying their daily counterparts by 250 and $\sqrt{250}$ respectively. For the purpose of comparison, we list the summary statistics of traditional financial assets at the bottom part of the table as well. "Bond" and "Commodity" indicates the summary statistics of the daily return of bond index and commodity index, while "Stocks" indicates the average level of the summary statistics on the daily return of each individual stock.

$$\min\ w^\top \hat{\Sigma} w \tag{2}$$

$$\text{s.t. } \mathbf{1}_N^\top w = 1,\ \mu \leq x^\top w, \tag{3}$$
$$||w||_1 \leq c,\ |w_i| \leq a_i\ \forall i.$$

The parameter c controls the amount of shortselling, $c \in [1, \infty)$. Fan et al. (2012) showed how the risk of the estimated portfolio is influenced by the choice of c while $a_i = \infty$ for all a_i. The estimation of $\hat{\Sigma}$ is crucial for the method, yet the huge dimensionality of the asset universe and limited data availability challenge the estimation of $\hat{\Sigma}$. Thus we employ the covariance estimator of Ledoit and Wolf (2004). It is shown to be invertible, well-conditioned and is asymptotically more accurate than the sample covariance matrix. The estimator is a weighted average of the identity matrix and the sample covariance matrix. The identity matrix is a well-conditioned matrix and due to the combination with the sample covariance matrix under a quadratic loss function, the resulting estimator has

	BTC	LTC	XRP	DOGE	NXT	PPC	NMC	DASH	XCP	BLK
min	-5.50	-4.85	-3.65	-4.00	-2.06	-3.00	-2.95	-3.07	-2.64	-4.15
1st Quantile	-0.26	-0.36	-0.39	-0.37	-0.38	-0.50	-0.57	-0.35	-0.56	-0.50
mean	0.01	0.00	0.01	-0.00	0.01	0.00	-0.00	0.01	0.00	0.00
median	-0.03	-0.06	-0.02	-0.01	-0.02	-0.05	-0.04	-0.01	-0.04	-0.03
3rd Quantile	0.23	0.29	0.36	0.33	0.36	0.46	0.48	0.32	0.53	0.47
max	5.44	4.47	4.86	3.39	2.53	4.50	4.90	2.89	4.40	4.28
variance	0.26	0.41	0.48	0.41	0.38	0.63	0.84	0.33	0.83	0.67

Table 2: Summary statistics of the trading volume of top 10 CCs

the well-conditioned property and is more accurate than the sample covariance matrix, Ledoit and Wolf (2004). For more details, we refer to section 7.1 in the Appendix.

However, Markowitz portfolio optimization neglects the effect of higher moments when minimizing the risk. Due to the often occurring strong decreases in the CC market, portfolios optimized for Conditional Value-at-Risk (CVaR) will be applied to compare the performance with the Markowitz portfolio.

Defining $y(w) = w^\top X$ as the returns of the portfolio with weights w. For α being the probability level such that $0 < \alpha < 1$, the Value-at-Risk is defined

$$\mathrm{VaR}_\alpha(w) = -\inf\{y|F(y|w) \geq \alpha\} \tag{4}$$

with $F(y|w)$ being the distribution function of the portfolio returns with weights w. $\mathrm{VaR}_\alpha(w)$ is the corresponding α-quantile of the cdf, defining the loss to be expected in $(\alpha \cdot 100)\%$ of the times.

Then, the spectral risk measure Conditional Value-at-Risk is defined as, Rockafellar and Uryasev (2000),

$$\mathrm{CVaR}_\alpha(w) = -\frac{1}{1-\alpha} \int\limits_{y(w) \leq -\mathrm{VaR}_\alpha(w)} y f(y|w) dy, \tag{5}$$

with $\frac{\partial}{\partial y} F(y|w) = f(y|w)$ the probability density function for the portfolio returns y with weights vector w. Thus $\mathrm{CVaR}_\alpha(w)$ includes the expected value over the tail of the pdf left of the $\mathrm{VaR}_\alpha(w)$.

It follows the optimization problem

$$
\begin{aligned}
&\min \ \mathrm{CVaR}_\alpha(w) && (6)\\
&\text{s.t. } \mathbf{1}_N^\top w = 1,\ \mu \leq x^\top w, && (7)\\
&\qquad ||w||_1 \leq c,\ |w_i| \leq a_i \ \forall i.
\end{aligned}
$$

So far we still considered the parameter c, however short-selling is still a rare phenomenon in the CC market, thus the variable c is an issue. First exchanges started to offer it for larger CCs and the launch of Bitcoin Futures allows for it too. But still it is not possible for most CCs to short-sell. Due to the inability of short-selling in the CC market, the exposure is set to $c = 1$, which produces the no short-sell constraints combined with $\mathbf{1}_N^\top w = 1$. Surely it is only a matter of time until short-selling is common in the CC market too. In that case our optimization problem could be amended to allow for short-selling which enables one to decrease the historical risk due to hedging effects. However this approach can cause extreme weights on single assets such that the position is not tradable in a real market situation.

4 LIBRO: LIquidity Bounded Risk-return Optimization

So far, the actual measure for liquidity for the constraints was not further explained. Yet this is a central point of this study, because CCs have far lower daily trading amount than traditional financial assets, causing a liquidity problem to any portfolio construction. To react to this issue, one tries to avoid holding too many illiquid assets via weight constraints $|w_i| \leq a_i$ for all $i = 1, \ldots, N$.

Many different liquidity measures were proposed in the literature, which tackle either one aspect of liquidity or aim on several aspects at the same time, Wyss (2004). In the context of this research, we are interested in

1. being able to trade the assets on the reallocation date

2. being able to sell or buy between two reallocation dates, if necessary.

Naturally, an asset with a higher liquidity should be allowed to have a higher weight in the portfolio.

Our data set consists of pricing and daily Turnover Value data, which enables us to use the Turnover Value as a proxy for liquidity. An even better measure would be Limit Order Book based measures since they allow for a deeper look into the markets behavior. We do not possess of a sufficient history of these data to run an analysis, thus for the moment all liquidity measures using such information are not applicable. Since the time period of interest for a trading action is restricted to a daily basis, daily closing data for the standard assets and CCs are going to be used. The trading volume of asset i at time t is defined to be:

$$TV_{it} = p_{it} \cdot q_{it} \tag{8}$$

where p_{it} is the closing price of asset i at date t, and q_{it} is the volume traded at date t of asset i. The liquidity of asset i in a sample with time length T can be measured using the sample median of trading volume:

$$TV_i = \frac{1}{2}(TV_{i,u} + TV_{i,l}) \tag{9}$$

where $TV_{i,u} = TV_{i,\lceil \frac{T+1}{2} \rceil}$ and $TV_{i,l} = TV_{i,\lfloor \frac{T+1}{2} \rfloor}$

Next we construct the liquidity bound. Recall that, w_i, $i = 1, \ldots, N$ denote the weight on asset i, M is the total amount we are going to invest, so Mw_i is the market value of the position on asset i. Hence the constraint on w_i concerning the liquidity of asset i is:

$$Mw_i \leq TV_i \cdot f_i, \tag{10}$$

where f_i is a factor, controlling the maximum ratio of the position on asset i to its median trading volume, i.e. liquidity. The larger the f_i, the more bullish the investor is on asset i, and the more likely the position on asset i will suffer from a low liquidity problem when clearing or rebalancing. For example, setting $f_i = 0.1$ corresponds to a position on asset i not larger than 10% of the median trading amount of asset i. Dividing both sides of (10) by M yields the bound for w_i:

$$w_i \leq \frac{TV_i \cdot f_i}{M} = \widehat{a}_i.$$

It follows, the Markowitz portfolio optimization framework we will use in this paper is:

$$\begin{aligned} &\min\ w^\top \hat{\Sigma} w \\ &\text{s.t. } \mathbf{1}_N^\top w = 1,\ \mu \leq x^\top w,\ ||w||_1 = 1, \\ &\quad w_i \leq \frac{1}{M} \cdot \widehat{Liq}_i = \widehat{a}_i\ \forall i, \end{aligned} \tag{11}$$

where $\widehat{Liq} = (TV_1 \cdot f_1, \cdots, TV_N \cdot f_N)^\top$. The CVaR optimization problem thus reads as:

$$\begin{aligned} &\min\ \text{CVaR}_\alpha(w) \\ &\text{s.t. } \mathbf{1}_N^\top w = 1,\ \mu \leq x^\top w,\ ||w||_1 = 1, \\ &\quad w_i \leq \frac{1}{M} \cdot \widehat{Liq}_i = \widehat{a}_i\ \forall i. \end{aligned} \tag{12}$$

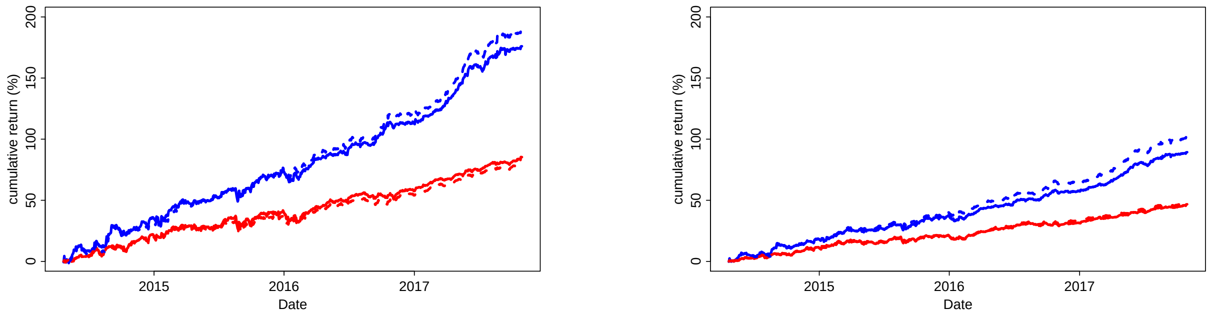
5 Application

For the application we run 3 kind of settings. In a first step we perform in-sample analysis without liquidity constraints to investigate whether including CCs to traditional financial portfolios increase the risk-return trade-off of portfolio formation. Furthermore we intend to find out whether including alt coins - CCs other than Bitcoin - are profitable and, last not but least, to confirm whether the introduction of liquidity constraints is necessary. In the second step, when liquidity constraints are included into the in-sample analysis, we set $f_i = 0.01$ for all $i = 1, \ldots, N$, i.e., we assume that our position on a certain asset can not exceed 1% of its daily trading volume. This is a quite conservative setting, and investors who want to be more aggressive can enlarge this factor. We choose to be conservative because the CC market, especially the alt coin market, exhibits swings in its daily trading volume. Thus we are securing our portfolio choice against it. Furthermore, trading on the entire daily trading volume would be a rather strict assumption. In both cases the portfolio and weights are formed over the entire time period. For the out-of-sample portfolio formation, we choose an extending window approach. The initial portfolio weights are derived over the time period 2014-04-22 until 2014-12-31. Due to the limited data availability in the CC market, the April 2014 data do not get omitted to enhance the estimation. 2 kind of portfolio rebalancing frequencies are applied. Weekly and monthly, while the underlying data period is extended. Thus for the monthly case, on the next reevaluation date, the 2015-02-01, the derivation period is extended to 2014-04-22 until 2015-01-31. For the portfolio formation under CVaR as quantile level in all cases, $\alpha = 0.05$ is chosen. For the liquidity constraint, we consider the same setting for all analysis: unbounded (without liquidity constraint), and bounded with investment amount equals to 1.0×10^5, 1.0×10^6 and 1.0×10^7 US dollar respectively, see (11) and (12). For selecting the target return μ, the Sharpe-Ratio and the Return-to-CVaR Ratio are maximized for the Markowitz and CVaR portfolio respectively. The median over the trading volume, necessary for the constraints, is chosen in-sample over the entire sample and out-of-sample over the extending window. We compare 2 different datasets consisting of traditional assets by adding CCs to them, one based solely on stocks from S&P100 (S), and one on Stocks plus US-Bonds and Commodities (SBC). We define the S and SBC plus CCs portfolios as S-CC and SBC-CC respectively.

5.1 In sample portfolio formation

5.1.1 Without liquidity bounds

The cumulative return of the portfolios formed by S&P 100 component stocks and SBC with or without CCs for both risk definitions are shown in Figure 3. It is witnessed that the improvement on return is remarkable and consistent throughout the sample. Starting from the very beginning, the portfolio with CCs (S-CC/SBC-CC) outperforms the one without, with the difference getting larger and large as time goes by. At the end of the sample, the S-CC Markowitz portfolio gives a cumulative return of 173.3%, while the S portfolio ends at 85.3%, only half of the former. However comparing the S-CC and the SBC-CC, the prior outperforms by having double the cumulative return. This is because the later



(a) A comparison of the cumulative return of the S-CC portfolios.

(b) A comparison of the cumulative return of the SBC-CC portfolios.

Figure 3: The solid and dash lines indicate the cumulative return performance of Markowitz and CVaR portfolios respectively. The red line indicates the S/SBC portfolio while the blue line stands for the S-CC/SBC-CC portfolio.

reaches the maximum Sharpe-ratio at a lower target return level. In fact, the optimal Sharpe-ratio of the later portfolio is always higher than its former counterparts, see Table 3 and 4. For the CVaR portfolio hold similar observations, while the out-performance is even larger. It is an interesting observation, that the portfolio return for CVaR portfolio is higher compared to Markowitz, since it suggests the return increase in CCs rises stronger than the risk induced from the tails. The summary statistics of the Markowitz portfolio returns are given in Table 3 and 4, where the S/SBC column is the one for the S and SBC portfolio respectively, and the "unbounded" column shows the one for the S-CC and SBC-CC respectively without implementing the liquidity constraint. Exemplary for the results of both strategies (Markowitz and CVaR) with S and SBC, we are looking in more detail at the S Markowitz portfolio results, Table 3. Coinciding with the previous finding, the annualized average return of the S-CC (48%) is twice as high compared to S portfolio (23%). Though the volatility is a bit higher, the whole risk-return trade-off is improved after adding CCs, since the Sharpe-Ratio increases from 0.12 to 0.18. Higher moments of portfolio returns are also improved: after CCs added, the skewness changes from -0.31 to 0.14, and kurtosis decreases from 5.50 to 4.55. The Maximum drawdown, which measures the downside risk of the portfolio, stays the same. Similar results can be observed when forming a CVaR portfolio, see Table 3 and 4. When using SBC portfolio, the average return is roughly halving while the standard deviation shrinks more than a half, resulting in an improvement in Sharpe-ratio.

		S	S-CC			
			unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.23	0.48	0.33	0.27	0.27
	Ann.STD	0.13	0.17	0.14	0.13	0.13
	Sharpe-ratio	0.12	0.18	0.15	0.14	0.13
	Skewness	-0.31	0.14	-0.24	-0.54	-0.55
	Kurtosis	5.50	4.55	5.20	5.70	5.68
	Max drawdown	0.10	0.10	0.11	0.11	0.11
	Auto correlation	-0.02	-0.04	-0.01	0.00	0.01
CVaR	Ann.Ret	0.22	0.52	0.32	0.26	0.26
	Ann.STD	0.12	0.18	0.14	0.12	0.12
	Sharpe-ratio	0.12	0.18	0.15	0.14	0.13
	Skewness	-0.27	0.28	-0.00	-0.47	-0.48
	Kurtosis	5.50	4.81	4.89	5.60	5.54
	Max drawdown	0.10	0.09	0.10	0.10	0.10
	Auto correlation	-0.03	-0.05	-0.02	-0.02	-0.02

Table 3: Summary statistics of in-sample S/S-CC Markowitz/CVaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter.

Even though the portfolio return results are delighting, however a check on the weights suggests that a liquidity constraint aiming at lowering the weight on illiquid assets is needed. The weights different from 0 given to CCs in the unbound case for the Markowitz and CVaR portfolios are shown in Tables 5, 6, 7 and 8 respectively.

The key information conveyed from the Tables is, the CCs having better liquidity are not generally given a larger weight. These biggest weight on a CC is given to NLG (Gulden), which has a medium level liquidity. However, Bitcoin (BTC), the CC that has the largest trading volume, is given a zero weight. Considering the CCs in the unbounded S-CC Markowitz portfolio, their weights account for 16.3% of the whole portfolio. The top three CCs by weights are NLG (Gulden), XRP (Ripple) and DASH (Dash coin), which accounts for 5.3%, 3.7% and 3.2% respectively. This shows alt coins are more appealing in term of variance minimization compared to BTC, at least during the sample covered by the paper. Furthermore, the inclusion of liquidity constraints appears necessary, since the high weight CCs are partly of low liquidity compared to BTC. For instance, with an investment amount $M = 10,000$ US dollar and S-CC, one needs to hold a position of 5300 US dollar, which accounts for 45.5% of NLG's average daily trading volume. Taking

		SBC	SBC-CC			
			unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.13	0.25	0.18	0.15	0.15
	Ann.STD	0.06	0.08	0.07	0.06	0.06
	Sharpe-ratio	0.14	0.19	0.17	0.16	0.15
	Skewness	-0.34	0.01	-0.27	-0.46	-0.48
	Kurtosis	4.95	4.33	4.81	5.04	5.20
	Max drawdown	0.04	0.05	0.05	0.05	0.05
	Auto correlation	0.03	-0.01	0.01	0.03	0.04
CVaR	Ann.Ret	0.13	0.28	0.19	0.16	0.15
	Ann.STD	0.06	0.09	0.07	0.06	0.06
	Sharpe-ratio	0.13	0.19	0.16	0.16	0.15
	Skewness	-0.26	0.29	0.03	-0.39	-0.38
	Kurtosis	4.77	4.59	4.69	4.90	4.86
	Max drawdown	0.05	0.05	0.05	0.05	0.05
	Auto correlation	0.01	-0.02	-0.02	0.00	0.01

Table 4: Summary statistics of in-sample SBC/SBC-CC Markowitz/CVaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter.

into the consideration of price impact, this position is neither easy to obtain nor to clear. The Table 7 showing the weights for the CVaR portfolio provide similar results while the influence of CCs and NLG in particular is even higher. However when relying on SBC portfolio, the weights given to CCs shrink strongly, Table 8. It seems the inclusion of bonds and commodities shifts the Mean-Variance and Mean-CVaR Frontier such strongly, that the resulting portfolio favors less CCs. However we observed this harms the return achieved from the portfolio.

5.1.2 Including Liquidity constraints

The cumulative return of portfolios with liquidity constraint included is shown in Figure 4, with the summary statistics of these returns in Table 3. When liquidity constraint is imposed, the cumulative return line shift downward compared to the one without liquidity constraint. The larger the investment amount, the lower the cumulative return. This is not surprising, since adding a liquidity constraint makes the global optimal Sharpe-ratio point and Return-to-CVaR point unreachable, and the larger the investment amount, the tighter the liquidity constraint. Hence, the further the constrained optimal Sharpe-ratio

	unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	5.37	8.28	9.01
XRP	3.72	4.53	0.80	0.08
DASH	3.17	2.13	0.21	0.02
DGB	0.95	0.13	0.01	0.00
VTC	0.18	0.10	0.01	0.00
NLG	5.30	0.02	0.00	0.00
FLO	1.53	0.01	0.00	0.00
RBY	0.84	0.03	0.00	0.00
NOTE	0.18	0.01	0.00	0.00
CBX	0.12	0.00	0.00	0.00
total	16.00	12.32	9.32	9.11

Table 5: Weights(in %) given to CCs in in-sample S-CC Markowitz portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The "unbounded" refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts M. Weights in red indicate that it is bounded by upper bound, i.e. its weight equals to its liquidity upper bound.

point to the unconstrained one. When investment amount is set to 1.0×10^5 or 1.0×10^6 US dollar, the cumulative return of Markowitz portfolios still outperform the one formed only by traditional assets throughout the sample, and by the end, the cumulative return is 128.5% and 117.0% respectively, still 43.2% and 31.7% higher than the S portfolio, by considering this setting as exemplary. When the investment amount is increased to 1.0×10^7 US dollar, the portfolio does not outperform the one containing only stocks until 2017, however ends at 98.8%, still 13.5% higher. For the CVaR portfolios, the constraint portfolios still outperform the S and SBC portfolio, however only after 2017 and with an excess return of 12.2% and 5.8% respectively.

Summary statistics also favor the portfolios with CCs, see the last three columns of Table 3 and 4. When investment amounts are set to 1.0×10^5, 1.0×10^6 or 1.0×10^7, compared to the S/SBC portfolios, the liquidity constrained ones have higher average return and higher Sharpe-ratio, however a slightly higher standard deviation and almost the same downside risk. Kurtosis and Skewness show a mixed picture. Mostly the absolute value of either increases, making it less akin to the Gaussian distribution. Yet this observation is not surprising, taking into account the strong deviations of CC return series from the Gaussian distribution. Overall the summary statistics provide support for adding CCs to

	unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	2.11	3.14	3.68
XRP	1.47	1.77	0.80	0.08
DASH	1.34	1.41	0.21	0.02
DGB	0.43	0.13	0.01	0.00
VTC	0.13	0.10	0.01	0.00
NLG	2.26	0.02	0.00	0.00
FLO	0.64	0.01	0.00	0.00
RBY	0.37	0.03	0.00	0.00
total	6.65	5.57	4.18	3.79

Table 6: Weights(in %) given to CCs in in-sample SBC-CC Markowitz portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The "unbounded" refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts M. Weights in red indicate that it is bounded by upper bound, i.e. its weight equals to its liquidity upper bound.

portfolios consisting of the chosen traditional assets.

To see how the liquidity constraint affect the optimization procedure, we turn to analysis on the weights. For the comparison we focus on weights given to CCs since the impact of CCs to the portfolio is of higher interest and no liquidity bound on the traditional assets is binding in any situation. The weights are shown in Table 5, 6, 7 and 8, where we show the ones for the CCs included in the portfolio in any considered situation. The first column shows the weights when no liquidity constraint is implemented, while the remaining three columns show those when liquidity constraints are included with three different investment amounts. The weights colored red indicates that its liquidity upper bound is binding, i.e. the weight given to this CC just equals to its liquidity upper bound. Before implementing the liquidity upper bound, the CCs account for 16% of the total position for S-CC Markowitz portfolio, with the largest weight 5.3% given to NLG, and zero weight to Bitcoin. After including a liquidity constraint, the total weight on CCs decreases to 12.3%, 9.3% and 9.1% as the investment amount increasing from 1.0×10^5 to 1.0×10^7. This is not surprising, since the liquidity upper bounds limits the weight given to CCs. When investment amount equals to 100,000 US dollar, the lower 8 CCs are binding, including NLG, which only has a weight of 0.02%. When investment increases to 1,000,000 and 10,000,000 US dollar, only the bound on Bitcoin is not binding.

	unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	3.82	8.71	9.61
XRP	3.30	5.57	0.80	0.08
DASH	3.72	2.15	0.22	0.02
DGB	0.00	0.12	0.01	0.00
NLG	6.22	0.02	0.00	0.00
FLO	2.79	0.01	0.00	0.00
RBY	1.11	0.03	0.00	0.00
MAX	0.50	0.01	0.00	0.00
CBX	0.28	0.00	0.00	0.00
ZEIT	0.12	0.00	0.00	0.00
total	18.03	11.73	9.74	9.71

Table 7: Weights(in %) given to CCs in in-sample S-CC CVaR portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The "unbounded" refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts M. Weights in red indicate that it is bounded by upper bound, i.e. its weight equals to its liquidity upper bound.

As the liquidity constraints tightening, the weight on Bitcoin increases from 5.4% to 9.0%, which shows the great investment potential of the Bitcoin market, since it can account for about 9% of the portfolio when formed together with S&P 100 stocks, while not being constrained. When considering the CVaR portfolio, the cumulative weight on CCs in the unbound case is 18%, thus even larger, however in the constraint cases it is shrinking up to 9.71%. For the SBC portfolios with either risk definition - Markowitz and CVaR - the weight given to CCs is considerably lower, thus the portfolios favor bonds and commodities stronger. Still no weight is given to BTC in the unbound case however when optimizing with liquidity constraints, BTC receives weight and the constraints become active on a variety of CCs. This shows that the constraints are necessary for achieving the goal set in this study, namely constructing portfolios with CCs in which the positions on the CCs can be easily cleared. After illustrating the potential of CCs in portfolio performance and the effect of liquidity constraints in the in-sample analysis, next we turn to an out-of-sample study to investigate the performance of the portfolios under pseudo real conditions.

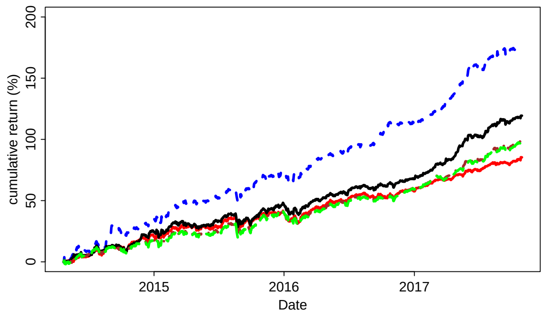
	unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
BTC	0.00	1.27	4.38	4.73
XRP	1.32	2.33	0.80	0.08
DASH	1.80	2.15	0.22	0.02
DGB	0.00	0.12	0.01	0.00
NLG	3.18	0.02	0.00	0.00
FLO	1.35	0.01	0.00	0.00
RBY	0.53	0.03	0.00	0.00
MAX	0.26	0.01	0.00	0.00
total	8.44	5.94	5.40	4.84

Table 8: Weights(in %) given to CCs in in-sample SBC-CC CVaR portfolios. Only CCs that have a positive weight in at least one portfolio are shown in the Table. The "unbounded" refers to the portfolio formed without liquidity constraint included; the remaining three are all formed under liquidity constraint with different investment amounts M. Weights in red indicate that it is bounded by upper bound, i.e. its weight equals to its liquidity upper bound.

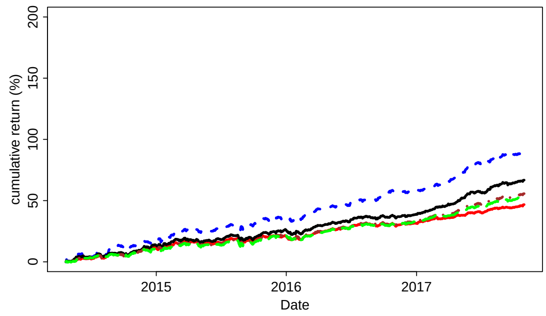
5.2 Out-of-sample portfolio formation

After having analyzed the potential of CCs regarding their performance enhancement in combined portfolios, out-of-sample analysis remains to justify the applicability of them in real-world investment. The S portfolio and the SBC portfolio will be constructed with monthly rebalanced weights, which are calculated using all the sample data before the rebalancing day. At first, the portfolio is formed on 2015-01-01 and hold to 2015-01-31, with weights calculated using sample from 2014-04-22 to 2014-12-31. Then, data from 2014-04-22 to 2015-01-31 is used to calculate the new weights, accordingly the portfolio is rebalanced at 2015-02-01. For subsequent periods, the portfolio will be rebalanced at the first day of each month, with the weights calculated using all the sample data before that day. We choose this extending window approach to calculate the rebalancing weights due to the limited amount of data in the sample.

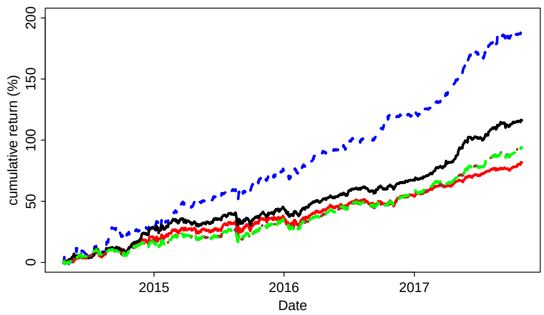
The performances of the cumulative return of Markowitz portfolios are visualized in Figure 5, where panel (a) shows those on S/S-CC portfolios, and panel (c) shows those on S/SBC-CC portfolios. For each panel, it is obvious that, the S-CC/SBC-CC portfolio, no matter liquidity bounded or not, outperform their counterpart without CCs at the end of the sample. For S-CC portfolio, when no liquidity constraint applied, the cumulative return exceeds the S portfolio at the very beginning of 2015, with the difference continuing



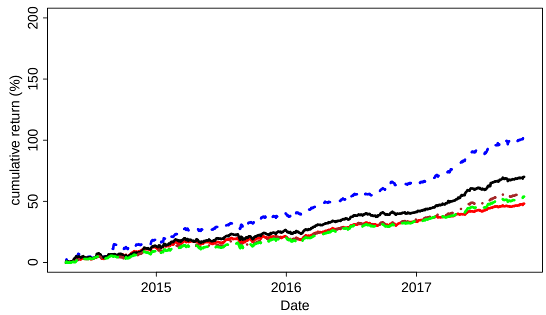
(a) In sample cumulative return of S-CC Markowitz portfolios



(b) In sample cumulative return of SBC-CC Markowitz portfolios



(c) In sample cumulative return of S-CC CVaR portfolios



(d) In sample cumulative return of SBC-CC CVaR portfolios

Figure 4: The solid and dash lines indicate the cumulative return performance of Markowitz and CVaR portfolios. The red line and blue line stand for S/SBC and S-CC/SBC-CC without liquidity constraints respectively. The remaining 3 portfolios are S-CC/SBC-CC ones containing the bounds $M = 1 \times 10^5$ USD, $M = 1 \times 10^6$ USD, $M = 1 \times 10^7$ USD.

LIBROinsample

to increase until the end of the sample. In the whole out-of-sample period, the cumulative return of S-CC portfolio without liquidity constraints is over 80%, while that of the S portfolio is 15.5%, a quite substantial improvement. For the SBC-CC portfolio without liquidity constraints, it outperforms the SBC portfolio from March 2016 onwards, and the difference keeps enlarging in the remaining periods. At the end of the sample, the cumulative return of SBC-CC portfolio reaches 37.4%, which is substantially higher than the 12.1% of SBC portfolio. The big improvement of S-CC/SBC-CC portfolio on S/SBC one indicates the huge potential of investment gain that can be obtained by including CCs into the portfolio. However, as stated in the in-sample cases, only when these improvements persist to exist when liquidity constraints are included, one can infer that the profits are feasible in practice.

		S	S-CC unbounded	S-CC $M = 1.0 \times 10^5$	S-CC $M = 1.0 \times 10^6$	S-CC $M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.05	0.28	0.13	0.10	0.09
	Ann.STD	0.13	0.19	0.15	0.14	0.14
	Sharpe-ratio	0.03	0.09	0.06	0.04	0.04
	Skewness	-0.38	0.09	-0.29	-0.44	-0.44
	Kurtosis	5.98	5.23	5.95	6.20	6.15
	Max drawdown	0.17	0.17	0.16	0.16	0.16
	Auto correlation	0.03	-0.02	0.03	0.04	0.04
CVaR	Ann.Ret	0.07	0.28	0.12	0.08	0.16
	Ann.STD	0.14	0.21	0.16	0.17	0.18
	Sharpe-ratio	0.03	0.08	0.05	0.03	0.06
	Skewness	-0.21	0.12	-0.48	-0.31	-0.11
	Kurtosis	5.83	5.13	6.39	9.36	9.44
	Max drawdown	0.16	0.19	0.18	0.25	0.23
	Auto correlation	-0.00	-0.05	0.04	0.10	0.12

Table 9: Summary statistics of out-of-sample monthly rebalanced S/S-CC Markowitz/C-VaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter.

Now comes the situation when liquidity constraints are included. In Figure 5, we label the cumulative return calculated with liquidity constraints by black, brown and green, indicating an investment amount equals to 1×10^5\$, 1×10^6\$, 1×10^7\$ respectively. For all these three investment amounts, S-CC portfolios exceed the S one starting from March of 2016, and the difference does not go large until 2017. At the end of the sample, the liquidity bounded portfolio with investment amount 1×10^5\$, 1×10^6\$, 1×10^7\$ ends at a cumulative return equal to 40.0%, 28.6% and 29.2%, which is 24.5%, 13.1% and 13.7% higher than the pure stock portfolio. When bond and commodity indexes are included, the liquidity bounded cumulative return under investment amount 1×10^5\$, 1×10^6\$, 1×10^7\$ reaches 21.2%, 17.7% and 16.4%, all of which outperform the SBC portfolio. All in all, adding CCs into portfolios is profitable even after controlling for low liquidity by constraints.

Summary statistics also favor the portfolios with CCs added, see Table 9 and 10. In either S-CC or SBC-CC case, the portfolios with CCs always dominate the one without, regarding return and Sharpe-ratio. When no liquidity bound is applied or bounded with an investment amount of 100,000 US dollar, the portfolio show less negative skewness and

		SBC	SBC-CC unbounded	$M = 1.0 \times 10^5$	$M = 1.0 \times 10^6$	$M = 1.0 \times 10^7$
Markowitz	Ann.Ret	0.04	0.13	0.07	0.06	0.06
	Ann.STD	0.07	0.10	0.08	0.07	0.07
	Sharpe-ratio	0.04	0.08	0.06	0.06	0.05
	Skewness	-0.33	-0.10	-0.34	-0.37	-0.38
	Kurtosis	5.98	5.63	6.33	6.31	6.40
	Max drawdown	0.08	0.10	0.08	0.07	0.07
	Auto correlation	0.08	0.03	0.07	0.08	0.08
CVaR	Ann.Ret	0.03	0.17	0.07	0.05	0.11
	Ann.STD	0.07	0.11	0.11	0.14	0.14
	Sharpe-ratio	0.03	0.10	0.04	0.02	0.05
	Skewness	-0.36	0.30	-0.94	-0.92	-0.51
	Kurtosis	5.47	5.87	14.16	14.93	15.48
	Max drawdown	0.11	0.06	0.14	0.22	0.23
	Auto correlation	0.05	-0.02	0.09	0.19	0.22

Table 10: Summary statistics of out-of-sample monthly rebalanced SBC/SBC-CC Markowitz/CVaR portfolio return. All indices are calculated using daily returns. Ann.Ret and Ann.STD indicate annualized mean return and standard deviation, ρ refers to the autocorrelation parameter.

less heavy tails. Though Skewness and Kurtosis may get worse under a tighter liquidity constraint when investment amount gets larger, the maximum drawdown improves after CCs added, which is somehow surprising, since the CC‘ market is considered highly risky. Interestingly the mean return on SBC/SBC-CC is lower than S/S-CC, yet in combination with a lower volatility as well. The SBC-CC portfolio outperforms S-CC one on two aspects: first, it only has about half the max drawdown, which is a substantial decrease on the downside risk; second, at larger investment amounts, it has higher Sharpe-ratio.

Having a look at the weights of the CCs, gives an answer on how CCs influence the performance of the portfolios. For the monthly rebalanced weights, see Figure 7 and 8. Obviously the CCs taking strong positions in the portfolios in the unbound case, e.g. NLG, which even reaches 8% in the fourth quarter of 2016, get ruled out when liquidity constraints are added. Furthermore, by the red rectangles we visualize the weights reaching its respective upper bound on the reallocation date. The constraints are mostly in place, giving support for their introduction into the methodology. For the S-CC Markowitz portfolios, when investment amount equals to 1.0×10^5\$, 6 CCs are included over time. When investment amount increases to 1.0×10^6\$ and 1.0×10^7\$, only 3 and 1 CCs are included over time respectively and BTC becomes the only one that is not affected by the liquidity bound. A further observation is the absence of Bitcoin, the largest and most liquid CC, in the unbounded portfolio. Yet with liquidity constraints it is always part of the portfolio while not reaching its respective upper bound. Additionally it becomes apparent Bitcoin and also Ripple receive higher weights in 2017, due to their better Sharpe-ratio, thus adding value to the portfolios due to the strong gains in the CC market in this period.

Shifting our analysis to the CVaR portfolios provides partly different observations. Still the unbound portfolios with CCs clearly outperform the one without CCs, and in the constraint portfolios, the cumulative returns perform better at the end of the sample, Figure 6. However, for the constrained case with investment amount $M = 1.0 \times 10^5$ and $M = 1.0 \times 10^6$, the improvements are not consistent throughout the sample: the cumulative return of pure S portfolio still outperforms until March of 2017 and August of 2017. It is interesting to observe, that in this case the portfolio having the highest investment amount, and therefore the strongest constraints, performs the best among the constrained ones, with a stable improvement compared to the pure S portfolio, and the highest cumulative return at the sample end. A similar observation can be made for the SBC-CC portfolio.Having a look at the weights for the monthly reallocation, Figures 9 and 10, one observes only Bitcoin being included in the strongest restricted portfolio. Since CVaR gives less weight to assets having high tail risk, it can be inferred, that for larger investment amounts only the tail risk of Bitcoin is sufficiently low to be adequate for the portfolio. Interestingly this causes the portfolio to outperform other constraint ones, however the unbounded one still outperforms.

Having a look at the summary statistics, Table 9 and 10, the SBC-CC unbounded portfolio performs better, annualized return 0.17, than the corresponding Markowitz portfolio, annualized return 0.13. Also the Sharpe-Ratio enhances. It is quite remarkable that the annualized return is high for $M = 1.0 \times 10^7$, yet the Sharpe-Ratio (0.05) is the same than when compared to the corresponding Markowitz portfolio. For the S-CC portfolios, again they perform better than the SBC-CC CVaR counterparts, as such in terms of annualized return and Sharpe-Ratio. Again the $M = 1.0 \times 10^7$ portfolio shows a remarkably high annualized return. Comparing the weights, Figure 7-10, one observes that in the portfolios only BTC was included, yet for the CVaR portfolios with higher values only from mid of 2016 onwards. For the Markowitz portfolios, BTC was also included in 2015.

5.3 Robustness: weekly vs. monthly rebalancing

Since the CCs‘ market is of variation, it is an interesting question to ask how will the portfolio perform if we readjust the portfolio more frequently, for example weekly rather than monthly. In this section, a weekly rebalancing portfolio is constructed, with weights updated every Wednesday. Again, weights are calculated using an extending window approach: all the data before the readjustment date are utilized for calculation, and the first portfolio is formed on 2015-01-01.

The cumulative return plots of Markowitz and CVaR portfolios are shown in panel (b) and (d) of Figure 5 and 6. An overview of the results gives the conclusion: the cumulative return of weekly rebalanced portfolios show almost the same pattern as those of the monthly ones, however, in most cases, they perform worse. For Markowitz method, the cumulative return at the end of the sample of S and S-CC portfolio are 9.4% and 72.6% respectively, which are 6.7% and 8.5% smaller than that of their monthly readjusted counterpart. When liquidity constrained, the cumulative return is 4.9%, 4.7% and 5.3% lower than for the monthly readjusted case at investment amounts 1×10^5\$, 1×10^6\$, 1×10^7\$. The same deterioration happens when bond and commodity indices are included. For the case with CVaR portfolios, the situation is similar: in both the S-CC and SBC-CC case, all portfolios have a smaller cumulative return than their monthly counterpart.

Furthermore rebalancing the portfolio weekly instead of monthly, harms the performance of the portfolios. Apart from this issue, the return performance curves appear almost similar, suggesting robust results regarding the reallocation frequency. However the better performance with monthly rebalancing gives support for the interpretation that at times swings in the return series of CCs have to be endured to ensure a better performance in the end of the day.

6 Conclusion

In this paper, we explore the potential gain of including CCs into risk optimized portfolios considering low liquidity of the CC market. On one hand, the rapid increasing cryptocurrencies (CCs) are promising investment assets, while on the other hand, these CCs are more volatile, have heavy tails and relatively low liquidity, so investing on them is somewhat challenging. To control the risk as well as liquidity problem, we propose LIBRO - LIquidity Bounded Risk-return Optimization method, which extend the framework employed in Fan et al. (2012) to contain an additional liquidity constraint, depending on the intended investment amount. Applying the methodology to monthly and weekly reallocated Markowitz and CVaR portfolios consisting of S&P 100 component stocks, Barclays Capital US Aggregate Index (US-Bonds Index), S&P GSCI (Commodities Index) and adding CCs to it, the results show a strong improvement in terms of volatility/quantile risk to return. However, it is worth noting that Bitcoin (BTC), the earliest and most dominant CC is given a zero weight when no liquidity constraint is include under both risk definitions, volatility and quantile risk. Two key information can be inferred from this result: firstly, though mostly discussed in literature, BTC is not the most appealing CC in terms of risk-return optimization, at least during the sample covered by the paper, which highlights our contribution to include CC other than BTC for portfolio formation. Secondly, the inclusion of liquidity constraints appear necessary, since some high-weight-CCs are partly of low liquidity compared to BTC. In this situation, one can no longer assume that the positions on these CCs would not distort the market or being tradable in the necessary amount. To improve the applicability of the portfolio formation strategy, including an upper bound becomes necessary, which is correlated with the daily trading volume of these assets.

The results of the in-sample analysis are already remarkable, while the out-of-sample analysis provides impressive results too. In the Markowitz portfolio the cumulative return gain reaches up to 80%. When including the liquidity upper bounds, the S-CC and SBC-CC portfolios still outperform the ones without constraints. For the Markowitz portfolios with monthly and weekly reallocation, the cumulative excess return ranges from

10% to 22% with investment amount equals to 1×10^5, 1×10^6, 1×10^7 US dollars. Over an investment period of roughly three years, this is a substantial gain. For the largest investment amount which is by construction the strongest restricted portfolio, one observes the only CC being included is BTC. Since CVaR gives less weight to assets having high tail risk, it can be inferred, that for larger investment amounts only the tail risk of BTC is sufficiently low to be adequate for the portfolio. Furthermore the monthly reallocated portfolios clearly outperformed the weekly adjusted ones. The better performance with monthly rebalancing gives support for the interpretation that at times swings in the return series of CCs have to be endured to ensure a better performance in the end of the day.

The main implications of the paper are, including CCs into the portfolio can bring huge gain to the investor, even under the situation with the largest investment amount which infers the tightest liquidity constraint. What's more, investing in alt coins can provide much more return gain than just including BTC, but it is more likely to encounter a liquidity problem, thus we propose LIBRO to tackle the low liquidity issue of certain CCs.

References

Borke, L. and W. K. Härdle (2017). "GitHub API based QuantNet Mining infrastructure in R". *SFB 649 Discussion Paper.*

Borke, L. and W. K. Härdle (2018). "Q3-D3-LSA". *Handbook of Big data Analytics.* Springer Verlag.

Brière, M., K. Oosterlinck, and A. Szafarz (2015). "Virtual currency, tangible return: Portfolio diversification with bitcoin". *Journal of Asset Management* 16.6, pp. 365–373.

Chen, C. Y., W. K. Härdle, A. J. Hou, and W. Wang (2018). "Pricing Cryptocurrency options: the case of CRIX and Bitcoin". *IRTG 1792 Discussion Paper 2018-004, Journal of Financial Econometrics submitted.*

Chen, S., C. Y.-H. Chen, W. K. Härdle, T. M. Lee, and B. Ong (2017). "CRIX Index: Analysis of a Cryptocurrency Index for Portfolio Investment." *Handbook of Digital Finance and Financial Inclusion: Cryptocurrency, FinTech, InsurTech, and Regulation.* Ed. by D. Lee Kuo Chuen and R. Deng. Vol. 1. Elsevier.

Darolles, S., C. Gourieroux, and E. Jay (2012). *Robust Portfolio Allocation with Systematic Risk Contribution Restrictions.* SSRN Scholarly Paper ID 2192399. Rochester, NY: Social Science Research Network.

Eisl, A., S. M. Gasser, and K. Weinmayer (2015). *Caveat Emptor: Does Bitcoin Improve Portfolio Diversification?* SSRN Scholarly Paper ID 2408997. Rochester, NY: Social Science Research Network.

Elendner, H., S. Trimborn, B. Ong, and T. M. Lee (2017). "The Cross-Section of Crypto-Currencies as Financial Assets". *Handbook of Digital Finance and Financial Inclusion: Cryptocurrency, FinTech, InsurTech, and Regulation.* Ed. by D. Lee Kuo Chuen and R. Deng. Vol. 1. Elsevier.

Fan, J., J. Zhang, and K. Yu (2012). "Vast Portfolio Selection With Gross-Exposure Constraints". *Journal of the American Statistical Association* 107.498, pp. 592–606.

Friedman, J., T. Hastie, and R. Tibshirani (2008). "Sparse inverse covariance estimation with the graphical lasso". *Biostatistics* 9.3, pp. 432–441.

Green, R. C. and B. Hollifield (1992). "When Will Mean-Variance Efficient Portfolios be Well Diversified?" *The Journal of Finance* 47.5, pp. 1785–1809.

Härdle, W. K., D. K. C. Lee, S. Nasekin, and A. Petukhina (2018). "Tail Event Driven ASset allocation: evidence from equity and mutual funds' markets". *Journal of Asset Management* 19.1, pp. 49–63.

Härdle, W. K. and L. Simar (2015). "Applications in Finance". *Applied Multivariate Statistical Analysis*. 4th ed. Springer, pp. 487–499.

Härdle, W. K. and S. Trimborn (2015). "CRIX or evaluating Blockchain based currencies". *Oberwolfach Report No. 42/2015 "The Mathematics and Statistics of Quantitative Risk"*.

Jagannathan, R. and T. Ma (2003). "Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps". en. *The Journal of Finance* 58.4, pp. 1651–1683.

Kozak, S., S. Nagel, and S. Santosh (2017). "Interpreting factor models". *Journal of Finance (forthcoming).*

Krokhmal, P., J. Palmquist, and S. Uryasev (2002). "Portfolio optimization with conditional value-at-risk objective and constraints". *Journal of risk* 4, pp. 43–68.

Ledoit, O. and M. Wolf (2004). "A well-conditioned estimator for large-dimensional covariance matrices". *Journal of multivariate analysis* 88.2, pp. 365–411.

Lee Kuo Chuen, D., L. Guo, and Y. Wang (2017). "Cryptocurrency: A New Investment Opportunity?" *The Journal of Alternative Investments* 20.3, pp. 16–40.

Markowitz, H. (1952). "Portfolio Selection". *The Journal of Finance* 7.1, pp. 77–91.

Rockafellar, R. T. and S. Uryasev (2000). “Optimization of conditional value-at-risk”. *Journal of risk* 2, pp. 21–42.

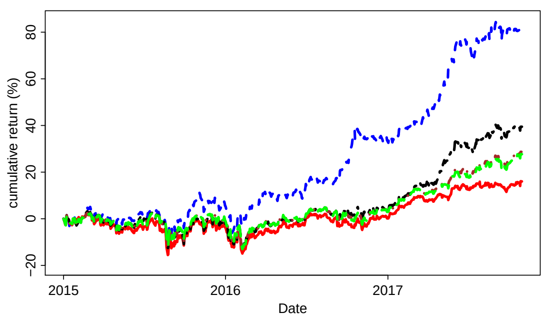
Scaillet, O., A. Treccani, and C. Trevisan (2018). “High-frequency jump analysis of the bitcoin market”. *Journal of Financial Econometrics (forthcoming).*

Wyss, R. v. (2004). “Measuring and Predicting Liquidity in the Stock Market”. *Doctoral thesis.*

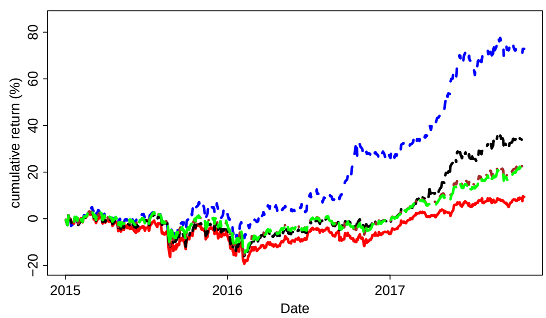
7 Appendix

	Ann.Ret	Ann.STD	skewness	kurtosis	ρ	mean volume	market cap
BTC	0.49	0.55	-0.61	10.87	-0.01	4.71e+08	1.07e+10
LTC	0.29	0.87	0.34	23.70	0.02	9.04e+07	3.22e+08
XRP	0.68	1.09	2.72	37.76	0.01	4.88e+06	8.35e+08
DASH	1.16	1.37	0.62	50.60	-0.14	1.61e+06	1.52e+08
DOGE	0.11	0.96	0.97	15.54	0.01	3.87e+05	3.11e+07
NXT	0.17	1.08	0.80	8.42	-0.03	2.28e+05	2.08e+07
DGB	0.73	1.70	2.93	29.89	-0.03	1.63e+05	7.24e+06
PPC	-0.17	1.00	0.62	12.51	-0.05	1.05e+05	1.54e+07
BLK	-0.05	1.27	1.82	17.72	-0.08	7.15e+04	4.21e+06
VTC	0.33	1.76	1.84	17.83	-0.01	5.93e+04	2.42e+06
NMC	-0.17	1.07	1.30	25.81	-0.09	5.26e+04	8.60e+06
POT	0.44	1.57	0.79	19.40	-0.05	3.95e+04	2.30e+06
XCP	0.36	1.46	1.16	9.52	-0.10	3.69e+04	7.51e+06
RDD	0.90	2.38	0.36	14.59	-0.25	2.55e+04	2.83e+06
XPM	-0.39	1.29	0.85	16.66	-0.07	2.05e+04	1.54e+06
NVC	-0.07	1.09	2.39	24.07	-0.01	1.73e+04	1.58e+06
EMC2	0.56	1.93	1.63	14.47	-0.02	1.57e+04	7.16e+05
EAC	-0.04	2.70	1.08	25.71	-0.27	1.50e+04	7.92e+05
IFC	-0.09	2.15	1.73	19.11	-0.13	1.26e+04	8.37e+05
NLG	1.00	1.46	0.90	12.35	-0.08	1.12e+04	4.28e+06
FTC	-0.00	1.76	0.59	13.66	-0.04	1.02e+04	1.84e+06
FLO	0.77	1.83	1.35	10.87	-0.07	8.35e+03	7.74e+05
ZET	-0.35	2.03	0.96	19.83	-0.16	5.80e+03	7.54e+05
WDC	-0.33	1.86	1.45	29.43	-0.12	5.65e+03	7.94e+05
RBY	1.16	2.18	0.98	28.41	-0.26	5.16e+03	2.83e+06
NOTE	0.79	1.60	1.59	15.01	-0.10	4.76e+03	1.11e+06
QRK	-0.27	2.91	-0.05	36.12	-0.35	4.38e+03	1.29e+06
MAX	-0.48	4.37	0.54	96.94	-0.38	3.69e+03	4.13e+05
HUC	-0.04	2.31	0.42	8.39	-0.19	3.25e+03	2.31e+05
SLR	0.72	1.92	0.44	9.59	-0.21	3.01e+03	2.39e+06
AUR	-0.03	1.51	1.08	19.12	-0.08	2.55e+03	1.17e+06
UNO	0.57	1.53	-0.24	15.64	-0.21	2.10e+03	9.14e+05
DMD	0.66	1.47	0.85	12.79	-0.20	1.57e+03	6.89e+05
GRS	0.99	2.86	1.52	15.80	-0.22	1.45e+03	4.72e+05
MINT	-0.01	3.63	0.27	7.76	-0.32	1.04e+03	9.71e+05
DGC	-0.43	1.92	0.71	27.86	-0.15	8.42e+02	2.95e+05
MOON	0.66	2.76	0.24	18.84	-0.16	6.83e+02	1.09e+06
EFL	0.58	1.87	-0.56	18.73	-0.16	6.19e+02	2.24e+05
NET	1.76	5.84	7.84	201.18	-0.22	5.45e+02	2.84e+05
CBX	0.17	3.33	0.81	28.53	-0.33	2.45e+02	1.80e+05
ZEIT	0.15	4.47	0.24	63.85	-0.25	2.10e+02	3.78e+05
AC	-0.30	3.05	1.17	26.52	-0.20	7.66e+01	4.13e+05
S&P stocks average	0.08	0.20	-0.26	10.29	0.00	5.40e+08	
Bond index	0.05	0.05	-1.91	12.38	-0.04		
Commodity index	-0.20	0.20	0.05	4.84	-0.07		

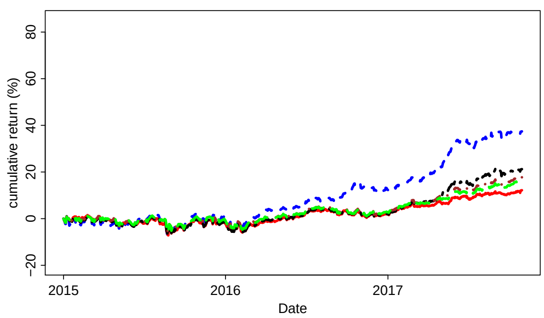
Table 11: Summary statistics of CCs. Ann.Ret and Ann.STD indicates annualized mean return and standard deviation, which are calculated by multiplying 250 and $\sqrt{250}$ to their daily counterparts. mean volume and market cap are measured in US dollar.



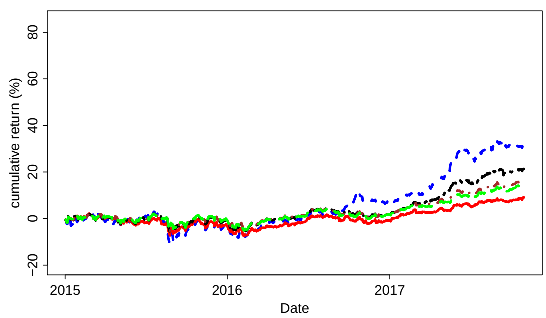
(a) Out-of-sample cumulative returns for monthly adjusted S-CC Markowitz portfolios



(b) Out-of-sample cumulative returns for weekly adjusted S-CC Markowitz portfolios

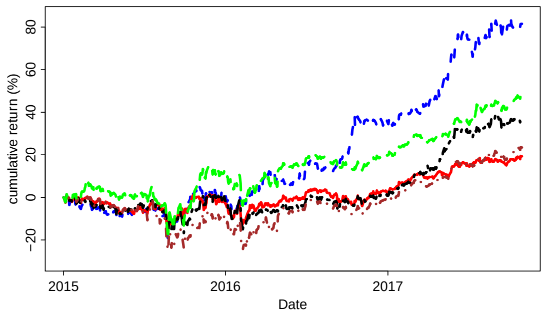


(c) Out-of-sample cumulative returns for monthly adjusted SBC-CC Markowitz portfolios

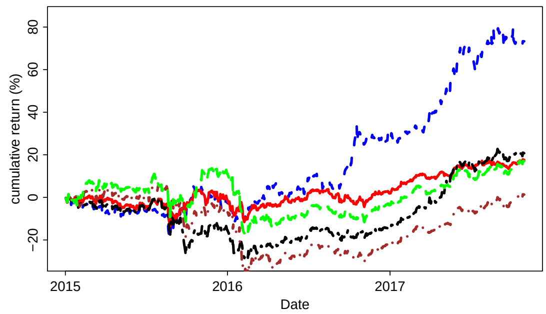


(d) Out-of-sample cumulative returns for weekly adjusted SBC-CC Markowitz portfolios

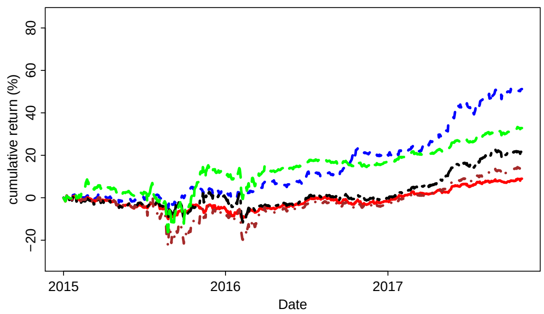
Figure 5: Out-of-sample cumulative returns with monthly (upper) and weekly (lower) adjusted Markowitz portfolios. The red line and blue line stand for S/SBC and S-CC/SBC-CC without liquidity constraints respectively. The remaining 3 portfolios are S-CC/SBC-CC ones containing the bounds $M = 1 \times 10^5$ USD, $M = 1 \times 10^6$ USD, $M = 1 \times 10^7$ USD



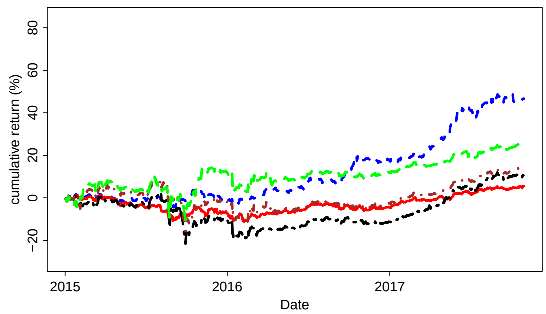
(a) Out-of-sample cumulative returns for monthly adjusted S-CC CVaR portfolios



(b) Out-of-sample cumulative returns for weekly adjusted S-CC CVaR portfolios



(c) Out-of-sample cumulative returns for monthly adjusted SBC-CC CVaR portfolios



(d) Out-of-sample cumulative returns for weekly adjusted SBC-CC CVaR portfolios

Figure 6: Out-of-sample cumulative returns with monthly (upper) and weekly (lower) adjusted CVaR portfolios. The red line and blue line stand for S/SBC and S-CC/SBC-CC without liquidity constraints respectively. The remaining 3 portfolios are S-CC/SBC-CC ones containing the bounds $M = 1 \times 10^5$ USD, $M = 1 \times 10^6$ USD, $M = 1 \times 10^7$ USD

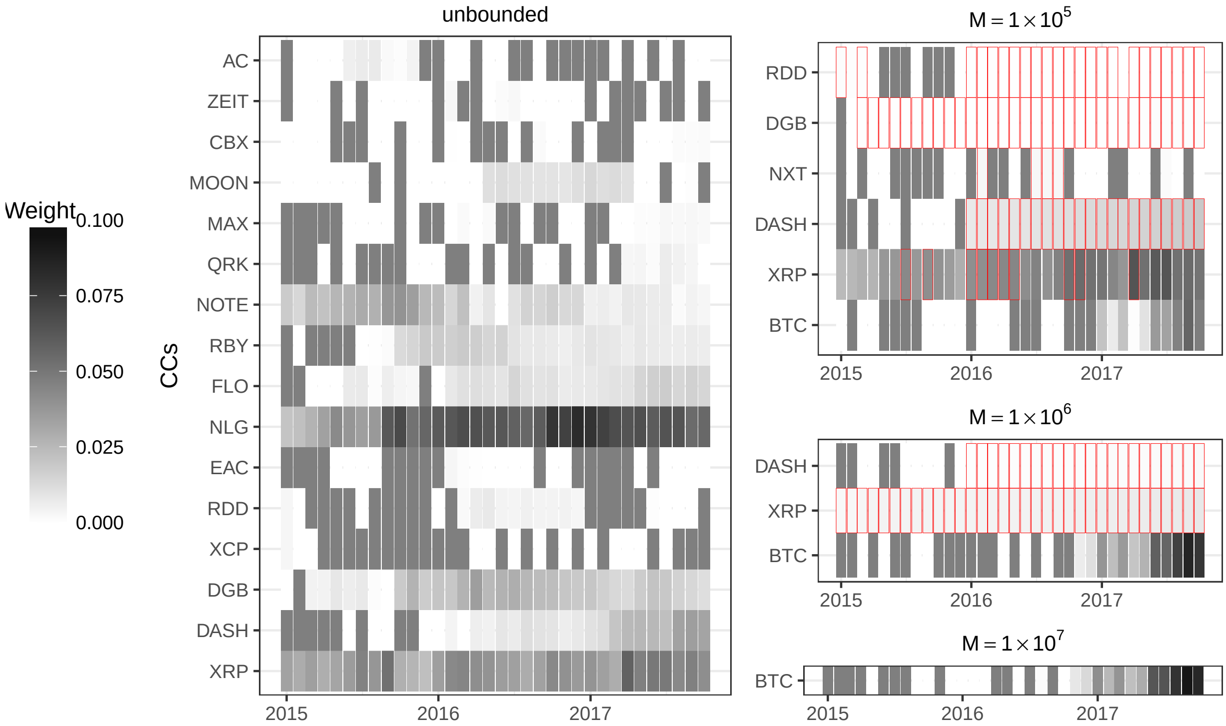


Figure 7: Weights given to CCs for S-CC Markowitz portfolios at each monthly rebalancing date under the 3 different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing data are given. The darker the color, the larger the weight.

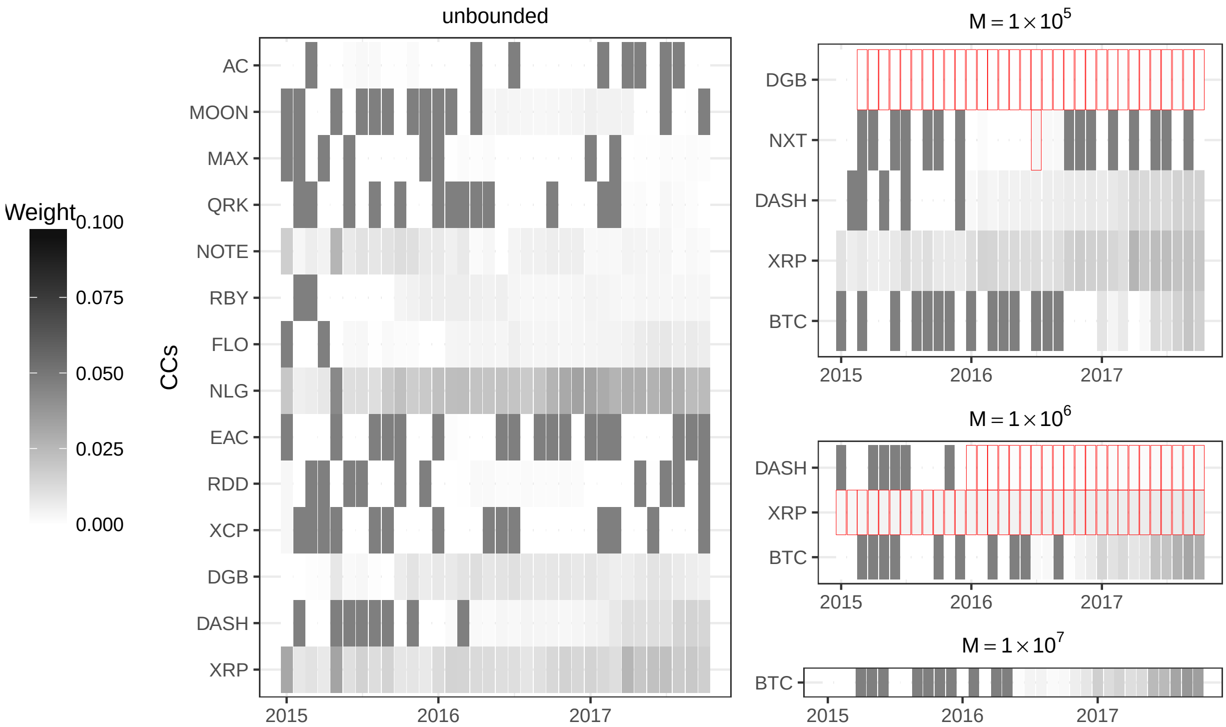


Figure 8: Weights given to CCs for SBC-CC Markowitz portfolio at each monthly rebalancing date under the 3 different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing data are given. The darker the color, the larger the weight.

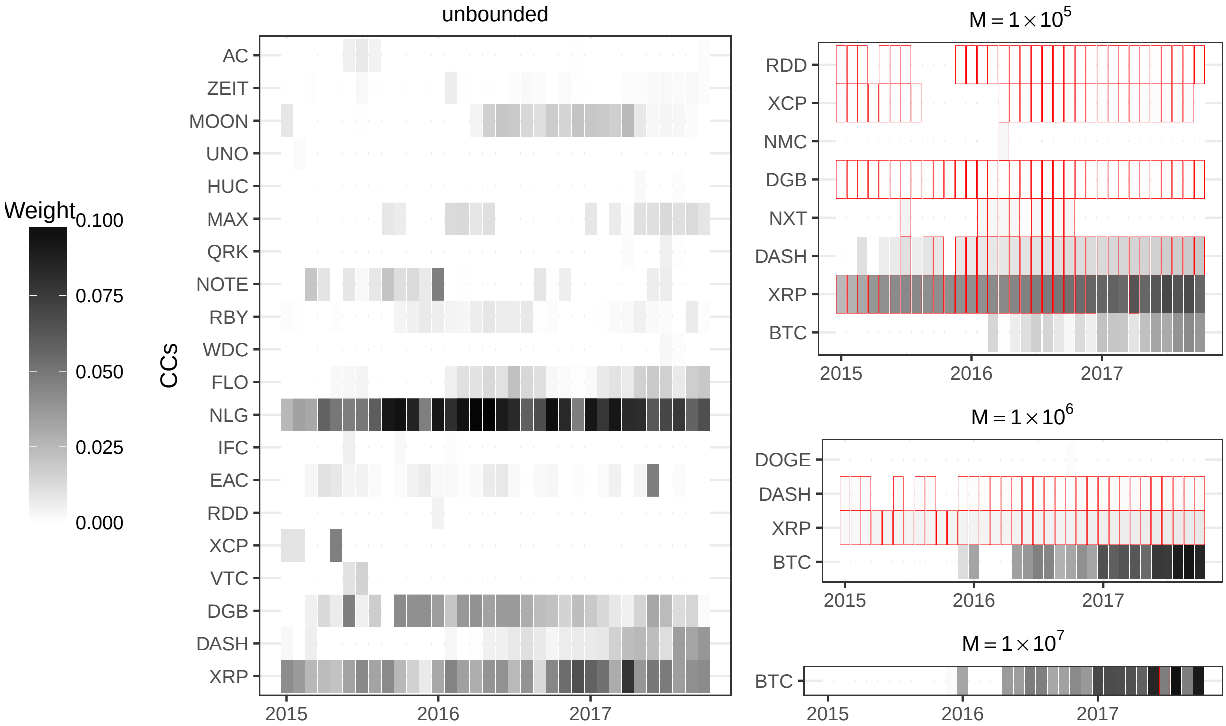


Figure 9: Weights given to CCs for S-CC CVaR portfolio at each monthly rebalancing date under the 3 different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing data are given. The darker the color, the larger the weight.

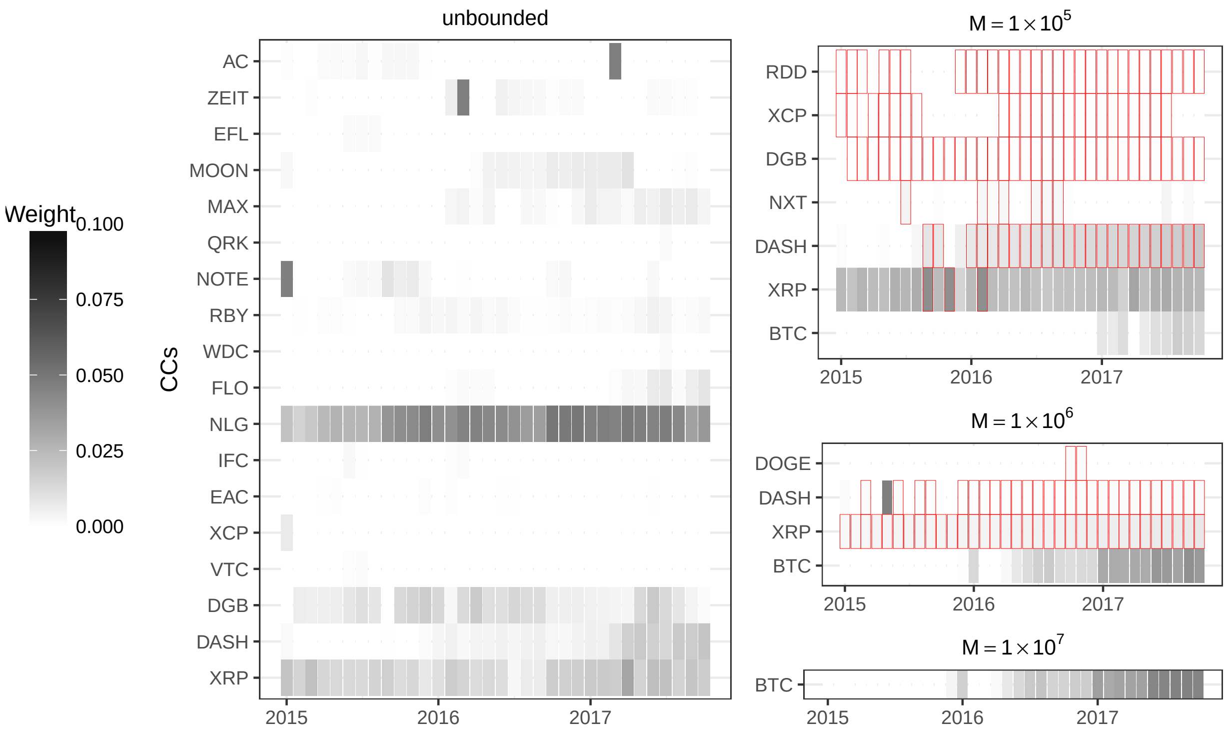


Figure 10: Weights given to CCs for SBC-CC CVaR portfolio at each monthly rebalancing date under the 3 different investment amounts. Only CCs that have a non-zero weight on at least one rebalancing data are given. The darker the color, the larger the weight.

7.1 A well-conditioned estimator for large-dimensional covariance matrices

For the $(N \times T)$ demeaned matrix X with T iid observations, define S as the sample covariance matrix $S = XX^\top/T$, I_N as the N-dimensional identity matrix and $\mathrm{tr}(\cdot)$ as the trace of a matrix. Define $||\cdot||$ as the Frobenius norm normalized by the dimensionality N: $||X|| = \sqrt{\mathrm{tr}(XX^\top)/N}$. Further define $\zeta = \mathrm{tr}(\Sigma I_N)/N$, $\gamma^2 = ||\Sigma - \zeta I_N||^2$, $\beta^2 = \mathrm{E}(||S - \Sigma||^2)$ and $\delta^2 = \mathrm{E}(S - \zeta I_N)$. Under the assumption of $\mathrm{E}(X^4) < \infty$, the optimization problem considered is

$$\min_{\rho_1 \rho_2} = \mathrm{E}(||\widehat{\Sigma} - \Sigma||^2)$$
$$\text{s.t. } \widehat{\Sigma} = \rho_1 I_N + \rho_2 S$$

which solves to $\rho_1 = \frac{\beta^2}{\delta^2}\zeta$ and $\rho_2 = \frac{\gamma^2}{\delta^2}$, thus the estimator is

$$\widehat{\Sigma} = \frac{\beta^2}{\delta^2}\zeta I_N + \frac{\gamma^2}{\delta^2} S.$$

Since the estimator depends on the true covariance matrix Σ, a consistent estimator $\widehat{\Sigma}^*$ got introduced. Define $x_{\cdot k}$ as the $(N \times 1)$ column k of X and rewrite the sample covariance matrix S as $S = \frac{1}{T}\sum_{k=1}^{T} x_{\cdot k} x_{\cdot k}^\top$. Since the matrices $x_{\cdot k} x_{\cdot k}^\top$ are iid across k, β^2 can be estimated by $\bar{b}^2 = 1/N^2 \sum_{i=1}^{N} ||x_{\cdot i} x_{\cdot i}^\top - S||^2$, Ledoit and Wolf (2004). Further define $m = \mathrm{tr}(S I_N)/N$, $d^2 = ||S - m I_N||^2$, $\bar{b}^2 = \min(\bar{b}^2, d^2)$ and $k^2 = d^2 - b^2$. A consistent estimator is then

$$\widehat{\Sigma}^* = \frac{b^2}{d^2} m I + \frac{k^2}{d^2} S.$$