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Optimizing (s, S) Policies for Multi-period Inventory Models with Demand Distribution Uncertainty: Robust Dynamic Programming Approaches

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Abstract

We consider a finite-horizon single-product periodic-review inventory management problem with demand distribution uncertainty. We formulate the problem as a dynamic program and prove the existence of an optimal (s, S) policy. The corresponding dynamic robust counterpart models are then developed for the box and the ellipsoid uncertainty sets. These counterpart models are transformed into tractable linear and second-order cone programs, respectively. We illustrate the effectiveness and practicality of the proposed robust optimization approaches through a numerical study.

Keywords: Inventory, periodic-review (s, S) policy, robust optimization, demand distribution uncertainty, dynamic programming

1. Introduction

Inventory management is critical to the success of all supply chains. Many researchers have made great efforts to identify effective inventory policies to determine when and how much to order a product(Zipkin, 2000). Establishing an effective inventory policy often requires an in-depth analysis of the nature of the target business. Traditional inventory models, particularly for a multi-period setting, usually assume that the demand distribution of a product and all of its parameters are completely known (Ahmed et al., 2007). These assumptions may not hold in many practical situations. The solutions based on such assumptions may lead to severe constraint violations even under very small perturbations(Beyer and Sendhoff, 2007). Demands are often volatile in practice resulting in inaccurate forecasts. This is especially true for products with short

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life cycles, large varieties, and long supply lead times such as fashion goods, electronic products, and mass-customized goods. Bertsimas and Thiele (2006) show that an optimal inventory policy heavily tuned to a particular demand distribution may perform very poorly for another demand distribution bearing the same uncertainty parameters. The desire for an effective inventory policy to deal with highly unpredictable demand with inevitable forecast errors motivates the development of robust inventory models with demand distribution uncertainty.

This study is motivated by the experience of SYE, a company selling electronics products on the Neusoft Electronics Market located in northeast China. SYE operates in a very challenging business environment caused by short product life cycles and by a volatile and unpredictable market. This environment leads to inevitable errors in demand forecasts and a considerable risk of having excessive inventory or stockout. Considering the fast-moving nature of the product and limited capital, SYE divides the planning horizon into several sales periods and then decides when and how much to order a product. The challenge faced by SYE compels the development of a robust inventory policy based on a multi-period model with demand distribution uncertainty. A periodic-review (s_t, S_t) policy has a reorder point (s_t) and an order-up-to level (S_t) for each period in a planning horizon of T periods. Under this policy, the inventory position is reviewed in every time period t. If the inventory position is equal to or below s_t, an order with a sufficient quantity of the product is placed to bring the inventory position back to the order-up-to level S_t. SYE finds that a periodic-review (s, S) policy will be useful for its inventory management.

Some of the earlier works for multi-period inventory models assume that the uncertainty parameters are random with known distributions, most studies do not present structural robust inventory policies, and others only derive the corresponding values of the parameters attached to the proposed inventory policies. In contrast, this study investigates a finite-horizon single-product periodic-review inventory management problem with uncertainty in demand probability distributions. This work has three main contributions. The first contribution is that we consider demand distribution uncertainty, which is commonly observed in practice especially for products with short life cycles or for inventory managers with limited information of demand distributions. The second contribution is that an $((s_t, S_t))$ policy for each period t is proved to be optimal even for *non-stationary distribution-free* multi-period inventory problems. Such a policy is attractive for the inventory managers. The third contribution is that two types of, i.e., box and ellipsoid, uncertainty sets are used to model demand distribution uncertainty. The resulting models are transformed into tractable linear and second-order cone programs, respectively, which can be solved efficiently to determine the reorder point (s_t) and the order-up-to level (S_t) for each period t. All the transformed versions are proved to be equivalent to the original models.

The remainder of this paper is organized as follows. Section 2 reviews some relevant literature. Section 3 describes the basic multi-period dynamic inventory model. Section 4 proves the existence of optimal (s_t, S_t) inventory policies under demand distribution uncertainty. In Section 5, robust dynamic programming approaches for finding the optimal (s_t, S_t) policies are developed for the box and ellipsoid uncertainty sets. Section 6 conducts a numerical study to show the effectiveness and practicality of the robust optimization approaches. Section 7 provides some concluding remarks and discusses future research directions.

2. Literature review

Relevant literature on multi-period inventory models is reviewed in this section. Previous works on stochastic multi-period inventory management are briefly reviewed first. Robust optimization and its application to multi-period inventory management are then surveyed.

2.1. Stochastic multi-period inventory management

To the best of our knowledge, the first study on multi-period inventory systems can go back to Wagner and Whitin (1958) for a dynamic version of the economic lot sizing model. Since then, many studies have focused on multi-period inventory models and dedicated to finding policies optimizing system performance in both deterministic(Mousavi et al., 2013; Ventura et al., 2013; Cárdenas-Barrón et al., 2015) and stochastic(Matsuyama, 2006; Wang et al., 2010; Farahvash and Altiok, 2011; Lim, 2011; Liu et al., 2012; Ning et al., 2013; Abouee-Mehrizi et al., 2015; Kim et al., 2015) market environments.

For inventory management problems with stochastic parameters, most of the previous studies were on the single-period problem known as the newsvendor model. The key difference between the single-period and the multi-period models is that the multi-period models may involve stock leftovers or shortages from previous periods, making the optimal order quantities more complicated(Zhang et al., 2009). Farahvash and Altiok (2011) used a stochastic dynamic programming model to solve a multi-period inventory problem with raw material procurements carried out via a reverse auction. Lim (2011) proposed a stochastic nonlinear mixed binary integer programming model for a multi-period inventory problem with quantity discounts based on previous orders. Chen and Wei (2012) studied the multi-period channel coordination problem in the framework of vendor-managed inventory for deteriorating goods and used a calculus-based formulation combined with dynamic programming techniques to solve this problem. Schmitt and Snyder (2012) developed an infinite-horizon inventory control model under both yield uncertainty and disruptions, and pointed out that using a single-period approximation could lead to a wrong strategy for mitigating supply risks. Janakiraman et al. (2013) analyzed the multi-period inventory model and showed that a system with an equal or longer expected lead time combined with a greater lead time variability in dilation ordering had a higher average cost. Recently, Kim et al. (2015) proposed a multi-stage stochastic programming model combining the multi-period newsvendor problems with transshipment to optimize the inventory control policy. Abouee-Mehrizi et al. (2015) considered a finite horizon multi-period inventory system where the objective was to determine the optimal joint replenishment and transshipment policies, and found that the optimal ordering policy in each period was determined based on two switching curves.

Since the extension from one period to multi-period can make the effective management of inventory systems more difficult, it is essential to provide inventory managers with a tractable policies with certain structures. The well-known periodic review (s, S) policy is accordingly proposed in which an order is placed to bring the inventory level up to S when its inventory level falls to or below s when reviewed. Using a dynamic programming approach, Scarf (1960) first showed that the (s, S) policy is optimal for finite horizon dynamic inventory systems with a linear ordering cost function and a convex holding cost function. On this basis, Song and Zipkin (1993) and Chen and Song (2001) modeled the demand level as a state of a continuous Markov chain, and showed that state-dependent (s, S) policies were optimal for a multi-period inventory problem under a fluctuating demand environment. Benkherouf and Sethi (2010) used a quasi-variational inequality approach to show the optimality of an (s, S) policy for a single-item infinite-horizon inventory model. Xu et al. (2010) further investigated the structural properties of (s, S) policies for inventory models with lost sales which could then be used to develop computational schemes for the lost sales with Erlang demands. More recently, Li and Xu (2013) studied discrete-time inventory replenishment decisions in a continuous-time dynamic-pricing setting and used a novel sample-path approach to prove the optimality of the (s, S) inventory policy in the presence of dynamic pricing. Noblesse et al. (2014a) skillfully characterized the ordering process of continuous review (s, S) and (r, nQ) inventory policies, and discussed the impact of the batching parameter on the variability in the ordering process. Using an (s, S) policy, Noblesse et al. (2014b) further examined the lot sizing decision in a production-inventory model and found that high costs would be incurred when the EOQ deviated from desirable production lot sizes. Feinberg and Lewis (2015) proved results on a Markov decision process with infinite state spaces, weakly continuous transition probabilities and one-step costs, which were applied to show the optimality of (s, S) policies for stochastic periodic review inventory control problems. Disney et al. (2016) studied the impact of stochastic lead times with order crossover on inventory costs and safety stocks in the order-up-to policy, and presented a new method for determining the distribution of the number of open orders. Song and Wang (2017) considered periodic review inventory control problems with both fixed order cost and uniform random yield, they proved that an (s, S) structure is optimal in any period.

2.2. Robust optimization and its application to multi-period inventory management

Other works related to this study are robust optimization techniques and their application to multi-period inventory control problems. Different from stochastic programming assuming full knowledge of the distribution information of the stochastic parameters, robust optimization addresses uncertainty parameters in optimization models by relaxing this assumption. Using well prespecified deterministic uncertainty sets in which all potential values of these parameters reside, the optimization models with uncertainty parameters can be transformed into tractable robust counterparts. Robust optimization employs a min-max approach that guarantees the feasibility of the obtained solution for all possible values of the uncertainty parameters in the designated uncertainty set(Bienstock and ÖZbay, 2008). Vlajic et al. (2012) believe robustness is a key property of a system or a strategy that can be used to improve performance in settings with uncertainty. More detailed discussion on robust optimization can be found in Gabrel et al. (2014).

Research on inventory control under ambiguous demand distributions can be traced back to Scarf et al. (1958), who derived the optimal order quantity using a min-max method for the classical newsvendor problem with only known mean and variance of the demand. His work was later extended by Alfares and Elmorra (2005), Yue et al. (2006), Perakis and Roels (2008), Bhattacharya et al. (2011), Jindal and Solanki (2014) and Kwon and Cheong (2014) for single period models. In multi-period settings, Gallego et al. (2001) analyzed the (s, S) policy for finite-horizon models when the demand distribution was under a linear constraint. Ben-Tal et al. (2004) introduced an adjustable robust model for linear programming problems, and applied it to a multi-stage inventory management problem. Bertsimas and Thiele (2006) developed a new approach to address demand ambiguity in a multi-period inventory control problem, which has the advantage of being computationally tractable. Bienstock and ÖZbay (2008) considered how to optimally set the basestock level for a single buffer to deal with demand uncertainty. Lin (2008) explored the EOQ model with backorder price discount by assuming known mean and variance of the demand lead time.

Ben-Tal et al. (2009) considered the problem of minimizing the overall cost of a supply chain over a possible long horizon, and proposed a globalized robust counterpart to control inventories in serial supply chains. See and Sim (2010) proposed a robust optimization approach to address a multi-period inventory control problem with only limited information of the demand distributions such as the mean, support, and some measures of deviations. Lin and Ng (2011) presented a robust model with interval demand data to determine the optimal order quantity and to select markets for products with short life cycles. Wei et al. (2011) used a robust optimization approach to solve an inventory and production planning problem with uncertainty in demand and returns over a finite planning horizon. Klabjan et al. (2013) proposed an integrated approach combining in a single step data fitting and inventory optimization for single-item multi-period stochastic lot-sizing problems. Recently, Qiu and Shang (2014) applied a robust optimization approach to derive the static order quantities for multi-period inventory models with conditional value-at-risk. Under the assumption of the (r, Q) strategy, Lin and Song (2015) developed a hybrid algorithm to find an inventory policy by minimizing the expected cost and a risk measure. Kang et al. (2015) developed a distribution-dependent robust linear optimization approach and applied it to a discrete-time stochastic inventory control problem with certain service level constraints. Using a similar interval uncertainty set proposed in Kang et al. (2015), Thorsen and Yao (2016) developed an adversarial approach based on Benders' decomposition to determine optimal robust static and basestock policies.

Lim and Wang (2016) considered a multi-product, multi-period inventory management problem with ordering capacity constraints. Demand for each product in each period is characterized by an uncertainty set. They proposed a target-oriented robust optimization approach to solve the problem. Their objective is to identify an ordering policy that maximizes the sizes of all the uncertainty sets such that all demand realizations from the sets will result in a total cost lower than a pre-specified cost target. They proved that a static decision rule was optimal for an approximate formulation of the problem, which significantly reduced the computational burden. Their numerical results suggest that, although only limited demand information is used, the proposed approach significantly outperforms traditional methods if the latter assume inaccurate demand distributions.

3. A multi-period inventory model with setup cost

Consider a finite-horizon single-product inventory system. An inventory manager reviews the inventory level periodically, and orders and sells the product over a finite planning horizon of T periods. The demand in period t is denoted by D_t, for $t = 1, 2, ..., T$, where D_t is a stochastic variable. Fig.1 shows the timeline of the events. At the beginning of each period $t = 1, 2, ..., T$, the

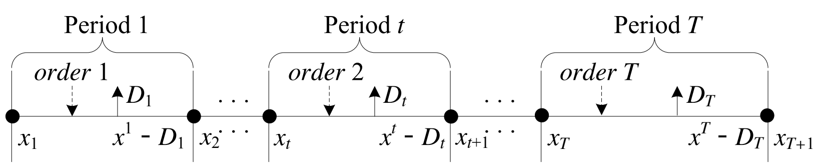

Figure 1: The dynamic inventory system over T periods.

inventory manager observes the on-hand inventory level before ordering, x_t, and then makes an ordering decision. The unit selling price and unit purchase cost in period t are denoted by r_t and c_t, respectively. The replenishment orders are assumed to be delivered instantly(Li and Xu, 2013; Abouee-Mehrizi et al., 2015; Feinberg and Lewis, 2015). The on-hand inventory level after the ordering decision is then represented by the variable x^t. The starting inventory level x_t may be positive indicating a surplus or negative indicating a shortage. The demand that cannot be satisfied is backlogged and can be met later, i.e., the unsatisfied demand does not become lost sales.

The demand D_t of each period t is assumed to be independently distributed, consistent with the assumptions of most of the studies in the literature of multi-period inventory problems(Matsuyama, 2006; See and Sim, 2010; Chen and Wei, 2012). After an ordering decision is made and the demand D_t is realized, the ending inventory level of period t, x_{t+1}, is determined. The inventory state dynamic equation can be described as (1) in the following

$$x_{t+1} = x^t - D_t, \ t = 1, 2, ..., T, \tag{1}$$

where x_1 represents the initial inventory level and is given. At the end of each period t, a holding or backorder cost is incurred. The holding cost is $h_t x_{t+1}$ if $x_{t+1} > 0$, where h_t is the unit holding cost, and the backorder cost is $-b_t x_{t+1}$ if $x_{t+1} < 0$, where b_t is the unit backorder cost. The stochastic demand D_t is discrete and belongs to a countable set of non-negative numbers, i.e., $D_t \in \{D_t^1, D_t^2, ..., D_t^{K_t}\}$, where K_t is positive and D_t^k, for $k = 1, 2, ..., K_t$, represents a possible value of D_t and is called a *demand scenario* in period t. The probability of demand scenario D_t^k is denoted by $p_t^k = \Pr\{D_t = D_t^k\}$, for $k = 1, 2, ..., K_t$. For notational convenience, let $\boldsymbol{p}_t = (p_t^1, p_t^2, ..., p_t^{K_t})'$ denote a column vector of these probabilities in period t. Given the inventory level after the ordering decision x^t and a demand scenario D_t^k, for $k = 1, 2, ..., K_t$, define the cost function for each period t in (2) in the following

$$\begin{aligned} C_t(x^t, D_t^k) &= -r_t \min\{x^t, D_t^k\} + h_t \max\{x^t - D_t^k, 0\} + b_t \max\{D_t^k - x^t, 0\} \\ &= -r_t D_t^k + \max\{h_t(x^t - D_t^k), -(r_t + b_t)(x^t - D_t^k)\}. \end{aligned} \tag{2}$$

The sales revenue is subtracted in (2) so that minimizing cost is equivalent to maximizing profit. Let $C_t(x^t) = (C_t(x^t, D_t^1), C_t(x^t, D_t^2), ..., C_t(x^t, D_t^{K_t}))'$ denote a cost vector.

Let $V^t(x_t)$ be a function representing the optimal expected cost over the periods $t, ..., T$ given the initial inventory level x_t at the start of period t. The multi-period inventory management problem is formulated as a dynamic program in (3) in the following

$$V^t(x_t) = \min_{x \geq x_t} \left\{ \mathcal{K}\delta(x - x_t) + c_t(x - x_t) + H^t(x) \right\}, \; t = 1, 2, ..., T. \tag{3}$$

where $\mathcal{K}$ represents a fixed ordering cost and $\delta(\Delta)$ is an indicator function that equals 1 if $\Delta > 0$, and 0 otherwise. The boundary condition is $V^{T+1}(x_{T+1}) \equiv 0$, for all $x_{T+1} \geq 0$. The function $H^t(x)$ in (3) is given in (4) in the following

$$H^t(x) = C_t(x)'\boldsymbol{p}_t + \gamma \tilde{V}^{t+1}(x - D_t)'\boldsymbol{p}_t, \; t = 1, 2, ..., T. \tag{4}$$

where $\gamma \in [0, 1]$ is a discount factor and $\tilde{V}^{t+1}(x - D_t) = (V^{t+1}(x - D_t^1), ..., V^{t+1}(x - D_t^{K_t}))'$ represents a vector of optimal expected costs over the periods $t + 1, ..., T$. The function $H^t(x)$ includes the expected revenue, the expected holding cost, the expected backorder cost, and the optimal expected future cost. For each period t, the decision variable x in (3) determines whether an order is placed and how much should be ordered.

To handle the expectations in (3) and (4), traditional approaches to inventory problems usually assume that the stochastic demand follows a certain, such as Poisson or normal among others, probability distribution with known parameters. This assumption is often unrealistic because of limited demand information available in practice, especially for perishable goods with short life cycles. Instead of assuming full knowledge of the underlying probability distributions, the demand probabilities, $\boldsymbol{p}_t$, are not assumed to be explicitly specified but are only assumed to belong to an uncertainty set. As a result of this assumption, robust optimization is a natural approach for solving this inventory problem. Therefore, this inventory problem becomes how to describe a tractable robust counterpart for the dynamic program in (3) and (4), and then how to find an optimal solution.

4. The optimality of the (s_t, S_t) policy with demand distribution uncertainty

In this section, the robust counterparts of the dynamic program in (3) and (4) under demand distribution uncertainty are formulated. Given the initial inventory level x_t at the start of period t, let $z^t(x_t)$ denote the optimal expected cost over the periods $t, ..., T$. This optimal expected cost can be determined by (5) in the following

$$z^t(x_t) = \min_{x \geq x_t} \left\{ \mathcal{K}\delta(x - x_t) + c_t(x - x_t) + G^t(x) \right\}, \ t = 1, 2, ..., T. \tag{5}$$

The boundary condition is $z^{T+1}(x_{T+1}) \equiv 0$ for all $x_{T+1} \geq 0$. The function $G^t(x)$ in (5) represents the worst-case expected cost over the periods $t, ..., T$. It is given by (6) in the following

$$G^t(x) = \max_{\boldsymbol{p}_t} \left\{ C_t(x)'\boldsymbol{p}_t + \gamma \tilde{z}^{t+1}(x - D_t)'\boldsymbol{p}_t \right\}, \ t = 1, 2, ..., T. \tag{6}$$

where $G^t(x) = \max_{\boldsymbol{p}_t} \left\{ C_t(x)'\boldsymbol{p}_t + \gamma \tilde{z}^{t+1}(x - D_t)'\boldsymbol{p}_t \right\}$ is a vector of optimal costs over the periods $t+1, ..., T$. The functions $z^t(x_t)$ in (5) and $G^t(x)$ in (6) are counterparts of $V^t(x_t)$ in (3) and $H^t(x)$ in (4), respectively. In (6), different worst-case demand distributions are permitted for different periods. That is, the worst-case distribution $\boldsymbol{p}_t^*$ for period t is not necessarily the same as the worst-case distribution $\boldsymbol{p}_{t+1}^*$ for period $t+1$.

The optimal value of the variable x in (5) represents the order-up-to level. From (5), the *minimal worst-case* expected cost over periods $t, ..., T$ is $\mathcal{K} + c_t(x - x_t) + G^t(x)$ if an order is placed in period t and is $G^t(x)$ if an order is not placed in period t, i.e., if $x = x_t$. Since the function $z^t(x_t)$ in (5) consists of an indicator term $\mathcal{K}\delta(x - x_t)$ with a value of $\mathcal{K}$ or 0 and a linear term $c_t(x - x_t)$ with a constant $c_t x_t$, define a function $\psi^t(x)$ as

$$\psi^t(x) = c_t x + G^t(x), \ t = 1, 2, ..., T. \tag{7}$$

To show that an (s_t, S_t) policy is optimal for the inventory problem, it is sufficient to verify that $\psi^t(x) \to \infty$ as $|x| \to \infty$, for $t = 1, 2, ..., T$, and the function $\psi^t(x)$ is $\mathcal{K}$-convex. A $\mathcal{K}$-convex function is defined below.

DEFINITION 1. *A real-valued function $f(a)$ is $\mathcal{K}$-convex for $\mathcal{K} \geq 0$, if for any $a_1 \leq a_2$ and $\lambda \in [0, 1]$,*

$$f((1 - \lambda)a_1 + \lambda a_2) \leq (1 - \lambda)f(a_1) + \lambda f(a_2) + \lambda\mathcal{K}.$$

A $\mathcal{K}$-convex function plays an important role in proving the existence of an optimal inventory policy for the multi-period inventory problem with a fixed ordering cost. The following theorem shows the optimality of an (s, S) policy. All the proofs can be found in the Online Supplement.

THEOREM 1. *(Optimality of the (s_t, S_t) policy). An (s_t, S_t) policy is optimal for the multi-period inventory problem in (5) and (6). That is, for each period $t = 1, 2, ..., T$ it is optimal to place an order to replenish the inventory level to S_t if the starting inventory level of the period is not larger than s_t, and not to place any order in the period otherwise.*

An (s_t, S_t) policy is appealing because it can be implemented easily in practice. Robust optimization approaches are developed in the next section to find the optimal s_t and S_t under demand distribution uncertainty.

5. Determining s_t and S_t using robust optimization

The key of solving the problem in (5) and (6) is to specify the uncertainty sets to which the demand distributions belong. Two types of, i.e., the box and the ellipsoid, uncertainty sets are considered in this study. In the following, $\boldsymbol{e}$ represents a vector of 1s of appropriate dimension.

DEFINITION 2. *For any period $t = 1, 2, ..., T$, the demand probability $\boldsymbol{p}_t$ belongs to a box uncertainty set*

$$\mathcal{P}_{\mathcal{B}} \triangleq \left\{ \boldsymbol{p}_t : \boldsymbol{p}_t = \bar{\boldsymbol{p}}_t + \boldsymbol{\xi}_t, \boldsymbol{e}'\boldsymbol{\xi}_t = 0, \underline{\boldsymbol{\xi}}_t \leq \boldsymbol{\xi}_t \leq \overline{\boldsymbol{\xi}}_t \right\}, \tag{8}$$

where $\bar{\boldsymbol{p}}_t$ is a vector representing the most likely or nominal distribution, and $\boldsymbol{\xi}_t$ is a vector representing disturbance terms with a known support $[\underline{\boldsymbol{\xi}}_t, \overline{\boldsymbol{\xi}}_t]$.

The restriction $\boldsymbol{e}'\boldsymbol{\xi}_t = 0$ is necessary to ensure that $\boldsymbol{p}_t$ is a probability distribution. The non-negativity requirement $\boldsymbol{p}_t \geq \boldsymbol{0}$ can be included in the restriction $\underline{\boldsymbol{\xi}}_t \leq \boldsymbol{\xi}_t \leq \overline{\boldsymbol{\xi}}_t$.

DEFINITION 3. *For any period $t = 1, 2, ..., T$, the demand probability $\boldsymbol{p}_t$ belongs to a box uncertainty set*

$$\mathcal{P}_{\mathcal{E}} \triangleq \{ \boldsymbol{p}_t : \boldsymbol{p}_t = \bar{\boldsymbol{p}}_t + \boldsymbol{A}_t\boldsymbol{\xi}_t, \boldsymbol{e}'\boldsymbol{A}_t\boldsymbol{\xi}_t = 0, \bar{\boldsymbol{p}}_t + \boldsymbol{A}_t\boldsymbol{\xi}_t \geq \boldsymbol{0}, \|\boldsymbol{\xi}_t\| \leq 1 \}, \tag{9}$$

where $\bar{\boldsymbol{p}}_t$ is a vector representing the most likely or nominal distribution corresponding to the center of the ellipsoid, $\|\cdot\|$ is the standard Euclidean norm with dual norm $\|\cdot\|_*$, $\boldsymbol{\xi}_t$ is a vector representing disturbance terms with $\|\boldsymbol{\xi}_t\| = \sqrt{\boldsymbol{\xi}_t^T\boldsymbol{\xi}_t}$, and $\boldsymbol{A}_t \in R^{n\times n}$ is a known scaling matrix of the ellipsoid.

The conditions $\boldsymbol{e}'\boldsymbol{A}_t\boldsymbol{\xi}_t = 0$ and $\bar{\boldsymbol{p}}_t + \boldsymbol{A}_t\boldsymbol{\xi}_t \geq \boldsymbol{0}$ are necessary to ensure that $\boldsymbol{p}_t$ is a probability distribution. These conditions have a similar purpose to that of $\boldsymbol{e}'\boldsymbol{\xi}_t = 0$ in (8).

The two uncertainty sets $\mathcal{P}_{\mathcal{B}}$ and $\mathcal{P}_{\mathcal{E}}$ defined above are widely used to describe uncertain parameters(Ben-Tal et al., 2005; Zhu and Fukushima, 2009; Qiu and Shang, 2014). Due to incomplete data or lack of forecast expertise, it is reasonable to consider these two types of uncertainty sets for the multi-period inventory management problem as the actual demand distribution $\boldsymbol{p}_t$ can be approximated by introducing the disturbance vector $\boldsymbol{\xi}_t$.

5.1. The optimal (s_t, S_t) policy under the box uncertainty set

Assume the discrete demand probability distribution belongs to a box uncertainty set $\mathcal{P}_{\mathcal{B}}$ as defined in (8). The last period $t = T$ will be considered first in obtaining an optimal (s_t, S_t) policy for each period $t = 1, 2, ..., T$. Since $z^{T+1}(x_{T+1}) \equiv 0$, then (6) becomes

$$G^T(x_T) = \max_{\boldsymbol{p}_T \in \mathcal{P}_{\mathcal{B}}} C_T(x_T)'\boldsymbol{p}_T,$$

and (5) becomes

$$z^T(x_T) = \min_{x \geq x_T} \left\{ \mathcal{K}\delta(x - x_T) + c_T(x - x_T) + G^T(x) \right\}$$

To obtain an optimal solution to the above problem, the function $\psi^T(x)$ defined in (7) needs to be minimized. According to the proof of Theorem 1, for $t = T$, the (s_T, S_T) policy is optimal with $S_T = \arg\min_x \ \psi^T(x)$, where

$$\psi^T(x) = c_T x + G^T(x) = c_T x + \max_{\boldsymbol{p}_T \in \mathcal{P}_{\mathcal{B}}} \left\{ C_T(x)'\boldsymbol{p}_T \right\}. \tag{10}$$

Therefore, the order-up-to level $S_T = x^*$ can be obtained by solving the following problem

$$\begin{aligned} &\min_x \max_{\boldsymbol{p}_T \in \mathcal{P}_{\mathcal{B}}} c_T x + C_T(x)'\boldsymbol{p}_T \\ &= \min_x \left\{ c_T x + C_T(x)'\bar{\boldsymbol{p}}_T + \Upsilon^*(x) \right\} \end{aligned} \tag{11}$$

where $\Upsilon^*(x)$ is the optimal objective value of the following linear program

$$\max_{\boldsymbol{\xi}_T} \left\{ \Upsilon(x) = C_T(x)'\boldsymbol{\xi}_T \,\middle|\, \boldsymbol{e}'\boldsymbol{\xi}_T = 0, \underline{\boldsymbol{\xi}}_T \leq \boldsymbol{\xi}_T \leq \overline{\boldsymbol{\xi}}_T \right\}. \tag{12}$$

The dual of the linear program (12) is given by

$$\min_{\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T} \left\{ \underline{\boldsymbol{\xi}}_T'\boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T'\boldsymbol{\nu}_T \,\middle|\, \boldsymbol{e}'\delta_T + \boldsymbol{\tau}_T + \boldsymbol{\nu}_T = C_T(x), \boldsymbol{\tau}_T \leq \boldsymbol{0}, \boldsymbol{\nu}_T \geq \boldsymbol{0} \right\}, \tag{13}$$

where δ_T, $\boldsymbol{\tau}_T$, and $\boldsymbol{\nu}_T$ are the dual variables corresponding to the constraints in (12). Consider the following optimization problem with variables $(x, \delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T) \in R \times R \times R^{K_T} \times R^{K_T}$

$$\begin{aligned} \min_{x, \delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T} \quad & c_T x + C_T(x)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T'\boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T'\boldsymbol{\nu}_T \\ s.t. \quad & \boldsymbol{e}'\delta_T + \boldsymbol{\tau}_T + \boldsymbol{\nu}_T = C_T(x) \\ & \boldsymbol{\tau}_T \leq \boldsymbol{0}, \boldsymbol{\nu}_T \geq \boldsymbol{0}. \end{aligned} \tag{14}$$

Theorem 2 shows that solving Problem (14) is equivalent to solving Problem (11).

THEOREM 2. *(Finding the order-up-to level S_T). If $(x^*, \delta_T^*, \boldsymbol{\tau}_T^*, \boldsymbol{\nu}_T^*)$ is an optimal solution to Problem (14), then x^* solves Problem (11). Conversely, if $\hat{x}^*$ solves Problem (11), then $(\hat{x}^*, \hat{\delta}_T^*, \hat{\boldsymbol{\tau}}_T^*, \hat{\boldsymbol{\nu}}_T^*)$ solves Problem (14), where $(\hat{\delta}_T^*, \hat{\boldsymbol{\tau}}_T^*, \hat{\boldsymbol{\nu}}_T^*)$ is an optimal solution to Problem (13).*

According to Theorem 2, the order-up-to level S_T for the last period can be obtained by finding a solution x^* of a minimization problem (14). The inventory manager can adjust $\bar{\boldsymbol{p}}_T$ t$T+1$o ensure $\underline{\boldsymbol{\xi}}_T \leq \mathbf{0}$ and $\overline{\boldsymbol{\xi}}_T \geq \mathbf{0}$, implying that the term $\underline{\boldsymbol{\xi}}_T' \boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T' \boldsymbol{\nu}_T$ in the objective function of Problem (14) is positive because $\boldsymbol{\tau}_T \leq \mathbf{0}$ and $\boldsymbol{\nu}_T \geq \mathbf{0}$. The optimal value of $\underline{\boldsymbol{\xi}}_T' \boldsymbol{\tau}_T^* + \overline{\boldsymbol{\xi}}_T' \boldsymbol{\nu}_T^*$ measures the cost induced by the demand distribution uncertainty.

Obviously, Problem (14) is a convex programming model. The piecewise linearity of the components of $C_T(x)$ ensures that Problem (14) is a piecewise linear program. Furthermore, each component of $C_T(x)$ can be linearized by introducing an auxiliary variable κ_T with the restrictions $\kappa_T \geq h_T(x - D_T)$ and $\kappa_T \geq -(r_T + b_T)(x - D_T)$. As before, $x - D_T$ is the starting inventory level of period $T+1$. Therefore, Problem (14) becomes a linear programming model that can be solved efficiently. The order-up-to level is equal to the optimal value of x, i.e., $S_T = x^*$. Thus, the optimal objective value of Problem (11) is

$$\psi^T(S_T) = c_T S_T + C_T(S_T)' \bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T' \boldsymbol{\tau}_T^* + \overline{\boldsymbol{\xi}}_T' \boldsymbol{\nu}_T^*,$$

implying that $G^T(S_T) = C_T(S_T)' \bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T' \boldsymbol{\tau}_T^* + \overline{\boldsymbol{\xi}}_T' \boldsymbol{\nu}_T^*$.

The reorder point s_T is derived in the following. Recall that $G^T(x_T)$ is the worst-case expected cost with the starting inventory x_T but without an order placed in period T. Furthermore, s_T is the reorder point at or below which an order will be triggered to raise the inventory level to S_T. Given S_T, s_T is the largest value of y with $y \leq S_T$ such that

$$G^T(y) = \mathcal{K} + c_T(S_T - y) + G^T(S_T). \tag{15}$$

The right hand side of (15) is the worst-case expected cost when the starting inventory level is y and an order is placed to replenish the inventory level to S_T. This implies that s_T is the threshold at which the cost associated with ordering $S_T - y$ equals the cost associated with not placing an order. The reorder point s_T can be found by solving

$$\min \left\{ y \,\middle|\, G^T(y) \leq \mathcal{K} + c_T(S_T - y) + G^T(S_T),\ y \leq S_T \right\}.$$

Since an accurate expression of the function $G^T(y)$ cannot be derived, the above problem cannot be solved directly. Fortunately, Theorem 3 below provides an effective approach of finding the reorder point, which equals the optimal value of y, i.e., $s_T = y^*$.

THEOREM 3. *(Finding the reorder point s_T). The optimal solution y^* of the following problem, with variables $(y, \delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T) \in R \times R \times R^{K_T} \times R^{K_T}$, is the reorder point s_T*

$$\begin{aligned}
\max_{y \le S_T,\, \delta_T,\, \boldsymbol{\tau}_T,\, \boldsymbol{\nu}_T} \quad & C_T(y)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T'\boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T'\boldsymbol{\nu}_T \\
s.t. \quad & \boldsymbol{e}'\delta_T + \boldsymbol{\tau}_T + \boldsymbol{\nu}_T = C_T(y) \\
& C_T(y)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T'\boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T'\boldsymbol{\nu}_T \le \mathcal{K} + c_T(S_T - y) + G^T(S_T) \\
& \boldsymbol{\tau}_T \le \mathbf{0}, \boldsymbol{\nu}_T \ge \mathbf{0}.
\end{aligned} \tag{16}$$

Similar to Problem (14), Problem (16) can be converted to a linear program after linearizing each component of $C_T(y)$ and can be solved efficiently. Likewise, the term $\underline{\boldsymbol{\xi}}_T'\boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T'\boldsymbol{\nu}_T$ in the objective function measures the cost caused by the demand distribution uncertainty.

The two fundamental problems (14) and (16) are solved to find S_T and s_T. The cost incurred in period T with the initial inventory x_T is given by

$$z^T(x_T) = \begin{cases} \mathcal{K} + c_T(S_T - x_T) + G^T(S_T), & \text{if } x_T \le s_T \\ G^T(x_T), & \text{otherwise.} \end{cases} \tag{17}$$

Since x_T can be observed at the beginning of period T, the cost $z^T(x_T)$ with $x_T > s_T$ can be obtained by solving Problem (39) in Appendix A.3.

The problem in the last period T has been solved so far. Without loss of generality, assume the problem in period has been solved. That is, an optimal (s_t, S_t) policy together with the cost $z^t(x_t)$ has been obtained in period t, for $1 < t < T$. For period $t-1$, the following problem, from the definition of $\psi^t(x)$ in (7), is solved to find an order-up-to level S_{t-1}

$$\begin{aligned}
\min_x \psi^{t-1}(x) &= c_{t-1}x + G^{t-1}(x) \\
&= c_{t-1}x + \max_{\boldsymbol{p}_{t-1} \in \mathcal{P}_{\mathcal{B}}} \left\{ C_{t-1}(x)'\boldsymbol{p}_{t-1} + \gamma \tilde{z}^t(x - D_{t-1})'\boldsymbol{p}_{t-1} \right\}
\end{aligned} \tag{18}$$

where $\tilde{z}^t(x - D_{t-1}) = (z^t(x - D_{t-1}^1), \cdots, z^t(x - D_{t-1}^{K_t}))'$ is the vector of optimal costs for period . Since the decision variable x is the inventory level after an ordering decision is made in period $t-1$, the term $x - D_{t-1}$ is the initial inventory level of period t, i.e., $x_t = x - D_{t-1}$. Similar to the process for period t, S_{t-1} can be obtained by solving Problem (18), which is equivalent to the following problem

$$\begin{aligned}
\min_{\substack{x,\, \delta_{t-1} \\ \boldsymbol{\tau}_{t-1},\, \boldsymbol{\nu}_{t-1}}} \quad & c_{t-1}x + [C_{t-1}(x) + \gamma \tilde{z}^t(x - D_{t-1})]'\bar{\boldsymbol{p}}_{t-1} + \underline{\boldsymbol{\xi}}_{t-1}'\boldsymbol{\tau}_{t-1} + \overline{\boldsymbol{\xi}}_{t-1}'\boldsymbol{\nu}_{t-1} \\
s.t. \quad & \boldsymbol{e}'\delta_{t-1} + \boldsymbol{\tau}_{t-1} + \boldsymbol{\nu}_{t-1} = C_{t-1}(x) + \gamma \tilde{z}^t(x - D_{t-1}) \\
& \boldsymbol{\tau}_{t-1} \le \mathbf{0}, \boldsymbol{\nu}_{t-1} \ge \mathbf{0}.
\end{aligned} \tag{19}$$

Thus, the order-up-to level $S_{t-1} = x^*$ is found for period $t-1$. The reorder point for period $t-1$ is found next. Recall that $\psi^{t-1}(S_{t-1})$ denote the optimal objective value of Problem (19) and the expression of $G^{t-1}(S_{t-1})$ can be derived accordingly. Similarly, the reorder point s_{t-1} is the largest value of y with $y \le S_{t-1}$ such that

$$G^{t-1}(y) = \mathcal{K} + c_{t-1}(S_{t-1} - y) + G^{t-1}(S_{t-1}), \tag{20}$$

where $G^{t-1}(y)$ is the optimal objective value of the following problem

$$\begin{aligned} \min_{\delta_{t-1}, \boldsymbol{\tau}_{t-1}, \boldsymbol{\nu}_{t-1}} \quad & [C_{t-1}(y) + \gamma \tilde{z}^t(y - D_{t-1})]' \bar{\boldsymbol{p}}_{t-1} + \underline{\boldsymbol{\xi}}'_{t-1} \boldsymbol{\tau}_{t-1} + \overline{\boldsymbol{\xi}}'_{t-1} \boldsymbol{\nu}_{t-1} \\ s.t. \quad & \boldsymbol{e}' \delta_{t-1} + \boldsymbol{\tau}_{t-1} + \boldsymbol{\nu}_{t-1} = C_{t-1}(y) + \gamma \tilde{z}^t(y - D_{t-1}) \\ & \boldsymbol{\tau}_{t-1} \le \mathbf{0}, \boldsymbol{\nu}_{t-1} \ge \mathbf{0}. \end{aligned} \tag{21}$$

Similar to that in period t, the reorder point s_{t-1} can be found by solving the following problem with variables $(y, \delta_{t-1}, \boldsymbol{\tau}_{t-1}, \boldsymbol{\nu}_{t-1}) \in R \times R \times R^{K_{t-1}} \times R^{K_{t-1}}$

$$\begin{aligned} \max_{\substack{y \le S_{t-1}, \delta_{t-1} \\ \boldsymbol{\tau}_{t-1}, \boldsymbol{\nu}_{t-1}}} \quad & [C_{t-1}(y) + \gamma \tilde{z}^t(y - D_{t-1})]' \bar{\boldsymbol{p}}_{t-1} + \underline{\boldsymbol{\xi}}'_{t-1} \boldsymbol{\tau}_{t-1} + \overline{\boldsymbol{\xi}}'_{t-1} \boldsymbol{\nu}_{t-1} \\ s.t. \quad & \boldsymbol{e}' \delta_{t-1} + \boldsymbol{\tau}_{t-1} + \boldsymbol{\nu}_{t-1} = C_{t-1}(y) + \gamma \tilde{z}^t(y - D_{t-1}) \\ & [C_{t-1}(y) + \gamma \tilde{z}^t(y - D_{t-1})]' \bar{\boldsymbol{p}}_{t-1} + \underline{\boldsymbol{\xi}}'_{t-1} \boldsymbol{\tau}_{t-1} + \overline{\boldsymbol{\xi}}'_{t-1} \boldsymbol{\nu}_{t-1} \\ & \qquad \le \mathcal{K} + c_{t-1}(S_{t-1} - y) + G^{t-1}(S_{t-1}) \\ & \boldsymbol{\tau}_{t-1} \le \mathbf{0}, \boldsymbol{\nu}_{t-1} \ge \mathbf{0}, \end{aligned} \tag{22}$$

with $s_{t-1} = y^*$, the optimal solution. The optimal cost incurred in period $t-1$ with the initial inventory x_{t-1} has the form

$$z^{t-1}(x_{t-1}) = \begin{cases} \mathcal{K} + c_{t-1}(S_{t-1} - x_{t-1}) + G^{t-1}(S_{t-1}), & \text{if } x_{t-1} \le s_{t-1} \\ G^{t-1}(x_{t-1}), & \text{otherwise.} \end{cases} \tag{23}$$

The optimal (s_t, S_t) policy for period $t = T-1, T-2, ..., 1$ can be determined recursively. The optimal cost incurred in period $t = 1$ is the total cost over the T periods, which equals

$$z^1(x_1) = \begin{cases} \mathcal{K} + c_1(S_1 - x_1) + G^1(S_1), & \text{if } x_1 \le s_1 \\ G^1(x_1), & \text{otherwise.} \end{cases} \tag{24}$$

The above discussion provides a solution procedure for finding the optimal (s_t, S_t) inventory policy. In this procedure, two linear programming problems need to be solved for each period t to find s_t and S_t, respectively. Due to the linearity of these problems, the (s_t, S_t) policy can be determined efficiently.

5.2. The optimal (s_t, S_t) policy under the ellipsoid uncertainty set

Assume the discrete demand probability distribution belongs to an ellipsoid uncertainty set $\mathcal{P}_{\mathcal{E}}$ as defined in (9). The last period T is considered first in finding an optimal (s_t, S_t) policy for each period $t = 1, 2, ..., T$. Since $z^{T+1}(x_{T+1}) \equiv 0$, the worst-case expected cost is given by

$$G^T(x_T) = \max_{\boldsymbol{p}_T \in \mathcal{P}_{\mathcal{E}}} C_T(x_T)'\boldsymbol{p}_T$$

According to Theorem 1, for $t = T$, an (s_t, S_t) policy is optimal. The order-up-to level S_T can be obtained by solving

$$\begin{aligned} \min_x \psi^T(x) &= c_T x + G^T(x) \\ &= \min_x \max_{\boldsymbol{p}_T \in \mathcal{P}_{\mathcal{E}}} c_T x + C_T(x)'\boldsymbol{p}_T \\ &= \min_x \{c_T x + C_T(x)'\bar{\boldsymbol{p}}_T - \Gamma^*(x)\} \end{aligned} \tag{25}$$

where $\Gamma^*(x)$ is the optimal objective value of the following problem

$$\min_{\boldsymbol{\xi}_T} \{\, \Gamma(x) = -C_T(x)'\boldsymbol{A}_T\boldsymbol{\xi}_T \,|\, \boldsymbol{e}'\boldsymbol{A}_T\boldsymbol{\xi}_T = 0, \bar{\boldsymbol{p}}_T + \boldsymbol{A}_T\boldsymbol{\xi}_T \geq \boldsymbol{0}, \|\boldsymbol{\xi}_T\| \leq 1\}\,. \tag{26}$$

The order-up-to level S_T is equal to $x^* = \arg\min_x \psi^T(x)$.

The Lagrangian dual function associated with (26) is

$$\begin{aligned} g(\boldsymbol{\lambda}_T, \rho_T, \gamma_T) &= \min_{\boldsymbol{\xi}_T} \mathcal{L}(\boldsymbol{\xi}_T; \boldsymbol{\lambda}_T, \rho_T, \gamma_T) \\ &= \min_{\boldsymbol{\xi}_T} \left\{-C_T(x)'\boldsymbol{A}_T\boldsymbol{\xi}_T + \boldsymbol{\lambda}_T'(-\bar{\boldsymbol{p}}_T - \boldsymbol{A}_T\boldsymbol{\xi}_T) + \rho_T(\|\boldsymbol{\xi}_T\| - 1) + \gamma_T\boldsymbol{e}'\boldsymbol{A}_T\boldsymbol{\xi}_T\right\} \\ &= -(\boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T) - \max_{\boldsymbol{\xi}_T} \left\{[\boldsymbol{A}_T'C_T(x) + \boldsymbol{A}_T'\boldsymbol{\lambda}_T - \gamma_T\boldsymbol{A}_T'\boldsymbol{e}]'\boldsymbol{\xi}_T - \rho_T\|\boldsymbol{\xi}_T\|\right\} \\ &= -(\boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T) - f_T^*(\boldsymbol{A}_T'C_T(x) + \boldsymbol{A}_T'\boldsymbol{\lambda}_T - \gamma_T\boldsymbol{A}_T'\boldsymbol{e}), \end{aligned} \tag{27}$$

where $f_T^*(y)$ is the conjugate function of $f_T(\boldsymbol{\xi}) = \rho_T\|\boldsymbol{\xi}\|$ with $f_T^*(y) = 0$ if $\|y\|_* \leq \rho_T$ and $f_T^*(y) = \infty$ otherwise, and $\|\cdot\|_*$ is a dual norm of $\|\cdot\|$ with $\|\cdot\|_* = \|\cdot\|$. Since the Lagrangian dual function yields lower bounds for any $\boldsymbol{\lambda}_T \geq 0$ and $\rho_T \geq 0$, an equivalent formulation of (26) is

$$\begin{aligned} &\max_{\boldsymbol{\lambda}_T, \rho_T, \gamma_T} g(\boldsymbol{\lambda}_T, \rho_T, \gamma_T) \\ &= \max_{\boldsymbol{\lambda}_T, \rho_T, \gamma_T} \left\{ -\boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T - \rho_T \;\middle|\; \begin{array}{l} \|\boldsymbol{A}_T'C_T(x) + \boldsymbol{A}_T'\boldsymbol{\lambda}_T - \gamma_T\boldsymbol{A}_T'\boldsymbol{e}\| \leq \rho_T, \\ \boldsymbol{\lambda}_T \geq \boldsymbol{0}, \rho_T \geq 0 \end{array} \right\} \end{aligned} \tag{28}$$

Consider the following problem with variables $(x, \boldsymbol{\lambda}_T, \rho_T, \gamma_T) \in R \times R^{K_T} \times R \times R$

$$\begin{aligned} \min_{x, \boldsymbol{\lambda}_T, \rho_T, \gamma_T} \quad & c_T x + C_T(x)'\bar{\boldsymbol{p}}_T + \boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T \\ s.t. \quad & \|\boldsymbol{A}_T'C_T(x) + \boldsymbol{A}_T'\boldsymbol{\lambda}_T - \gamma_T\boldsymbol{A}_T'\boldsymbol{e}\| \leq \rho_T \\ & \boldsymbol{\lambda}_T \geq \boldsymbol{0}, \rho_T \geq 0. \end{aligned} \tag{29}$$

Theorem 4 shows that solving Problem (29) is equivalent to solving Problem (25).

THEOREM 4. *(Finding the order-up-to level S_T): If $(x^*, \boldsymbol{\lambda}_T^*, \rho_T^*, \gamma_T^*)$ is an optimal solution to Problem (29), then x^* solves Problem (25). Conversely, if $\hat{x}^*$ solves Problem (25), then $(\hat{x}^*, \hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is an optimal solution to Problem (29), where $(\hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is an optimal solution to Problem (2828) with $x = \hat{x}^*$.*

Similar to the box uncertainty set, the demand distribution uncertainty also leads to a positive cost $\boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T$ when the demand probability distribution belongs to an ellipsoid uncertainty set. The term $\boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T$ in (29) has a similar meaning to that of $\underline{\boldsymbol{\xi}}_T'\boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T'\boldsymbol{\nu}_T$ in (14).

By linearizing the components of $C_T(x)$, Problem (29) becomes a second-order cone programming model and can be solved efficiently. The optimal order-up-to level in the last period T equals the optimal value of x in Problem (29), i.e., $S_T = x^*$. The optimal objective value of Problem (29) is

$$\psi^T(S_T) = c_T S_T + C_T(S_T)'\bar{\boldsymbol{p}}_T + (\boldsymbol{\lambda}_T^*)'\bar{\boldsymbol{p}}_T + \rho_T^*, \tag{30}$$

where $G^T(y)$ is the optimal objective value of the following problem with variables $(\boldsymbol{\lambda}_T, \rho_T, \gamma_T) \in R^{K_T} \times R \times R$

$$\begin{aligned} \min_{\boldsymbol{\lambda}_T, \rho_T, \gamma_T} \quad & C_T(y)'\bar{\boldsymbol{p}}_T + \boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T \\ s.t. \quad & \|\boldsymbol{A}_T' C_T(y) + \boldsymbol{A}_T'\boldsymbol{\lambda}_T - \gamma_T \boldsymbol{A}_T'\boldsymbol{e}\| \le \rho_T \\ & \boldsymbol{\lambda}_T \ge \boldsymbol{0}, \rho_T \ge 0. \end{aligned} \tag{31}$$

Similar to that in Theorem 3 for the box uncertainty set, the following tractable second-order cone program can be used to find s_T

$$\begin{aligned} \max_{y \le S_T, \boldsymbol{\lambda}_T, \rho_T, \gamma_T} \quad & C_T(y)'\bar{\boldsymbol{p}}_T + \boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T \\ s.t. \quad & \|\boldsymbol{A}_T' C_T(y) + \boldsymbol{A}_T'\boldsymbol{\lambda}_T - \gamma_T \boldsymbol{A}_T'\boldsymbol{e}\| \le \rho_T \\ & C_T(y)'\bar{\boldsymbol{p}}_T + \boldsymbol{\lambda}_T'\bar{\boldsymbol{p}}_T + \rho_T \le K + c_T(S_T - y) + G^T(S_T) \\ & \boldsymbol{\lambda}_T \ge \boldsymbol{0}, \rho_T \ge 0. \end{aligned} \tag{32}$$

The optimal reorder point is equal to the optimal value of y in Problem (32), i.e., $s_T = y^*$. Thus, an optimal (s_T, S_T) policy for period T can be constructed by solving Problems (29) and (32). The cost incurred in period T with an initial inventory level x_T is

$$z^T(x_T) = \begin{cases} \mathcal{K} + c_T(S_T - x_T) + G^T(S_T), & \text{if } x_T \le s_T \\ G^T(x_T), & \text{otherwise.} \end{cases}$$

Up to now, the problem in the last period has been solved. Assume the problem has been solved for period t, and an optimal (s_T, S_T) policy together with the associated cost $z^t(x_t)$ has been found for $1 < t < T$. The problem for period $t-1$ is then considered. The following problem is solved to find S_{t-1}

$$\begin{aligned}\min_x \psi^{t-1}(x) &= c_{t-1}x + G^{t-1}(x)\\ &= c_{t-1}x + \max_{\boldsymbol{p}_{t-1}\in\mathcal{P}_{\mathcal{E}}} \left\{C_{t-1}(x)'\boldsymbol{p}_{t-1} + \gamma\tilde{z}^t(x-D_{t-1})'\boldsymbol{p}_{t-1}\right\}\end{aligned}$$

where $\tilde{z}^t(x–D_{t-1})=(z^t(x–D^1_{t-1}),\cdots,z^t(x–D^{K_t}_{t-1}))'$ is the vector of optimal costs for period t. As above, $x–D_{t-1}$ is the starting inventory level of period t. Similar to the analysis for the box uncertainty set, S_{t-1} can be determined by solving the following problem with variables $(x,\boldsymbol{\lambda}_{t-1},\rho_{t-1},\gamma_{t-1})\in R\times R^{K_{t-1}}\times R\times R$

$$\begin{aligned}\min_{x,\boldsymbol{\lambda}_{t-1},\rho_{t-1},\gamma_{t-1}} \quad & c_{t-1}x + [C_{t-1}(x)+\gamma\tilde{z}^t(x-D_{t-1})]'\bar{\boldsymbol{p}}_{t-1}+\boldsymbol{\lambda}'_{t-1}\bar{\boldsymbol{p}}_{t-1}+\rho_{t-1}\\ s.t. \quad & \|\boldsymbol{A}'_{t-1}[C_{t-1}(x)+\gamma\tilde{z}^t(x-D_{t-1})+\boldsymbol{\lambda}_{t-1}-\gamma_{t-1}\boldsymbol{e}]\| \le \rho_{t-1}\\ & \boldsymbol{\lambda}_{t-1}\ge\boldsymbol{0}, \rho_{t-1}\ge 0.\end{aligned} \tag{33}$$

The optimal order-up-to level for period $t-1$ is equal to the optimal value of x in Problem (33), i.e., $S_{t-1}=x^*$. Recall that $\psi^{t-1}(S_{t-1})$ denotes the optimal objective value of Problem (33) implying $G^{t-1}(S_{t-1})=\psi^{t-1}(S_{t-1})-c_{t-1}S_{t-1}$. Similarly, s_{t-1} can be found by solving the following problem with variables $(y,\boldsymbol{\lambda}_{t-1},\rho_{t-1},\gamma_{t-1})\in R\times R^{K_{t-1}}\times R\times R$

$$\begin{aligned}\max_{y\le S_{t-1},\boldsymbol{\lambda}_{t-1},\rho_{t-1},\gamma_{t-1}} \quad & [C_{t-1}(y)+\gamma\tilde{z}^t(y-D_{t-1})]'\bar{\boldsymbol{p}}_{t-1}+\boldsymbol{\lambda}'_{t-1}\bar{\boldsymbol{p}}_{t-1}+\rho_{t-1}\\ s.t. \quad & \|\boldsymbol{A}'_{t-1}[C_{t-1}(y)+\gamma\tilde{z}^t(y-D_{t-1})+\boldsymbol{\lambda}_{t-1}-\gamma_{t-1}\boldsymbol{e}]\| \le \rho_{t-1}\\ & [C_{t-1}(y)+\gamma\tilde{z}^t(y-D_{t-1})]'\bar{\boldsymbol{p}}_{t-1}+\boldsymbol{\lambda}'_{t-1}\bar{\boldsymbol{p}}_{t-1}+\rho_{t-1}\\ & \qquad \le \mathcal{K}+c_{t-1}(S_{t-1}-y)+G^{t-1}(S_{t-1})\\ & \boldsymbol{\lambda}_{t-1}\ge\boldsymbol{0}, \rho_{t-1}\ge 0.\end{aligned} \tag{34}$$

The optimal reorder point is equal to the optimal value of y in Problem (34), i.e., $s_{t-1}=y^*$.

The optimal cost incurred with the initial inventory x_{t-1} in period $t-1$ is

$$z^{t-1}(x_{t-1}) = \begin{cases}\mathcal{K}+c_{t-1}(S_{t-1}-x_{t-1})+G^{t-1}(S_{t-1}), & \text{if } x_{t-1}\le s_{t-1}\\ G^{t-1}(x_{t-1}), & \text{otherwise,}\end{cases} \tag{35}$$

where $G^{t-1}(x_{t-1})$ is the optimal objective value of the following problem with $x_{t-1} > s_{t-1}$

$$\begin{aligned}\min_{\boldsymbol{\lambda}_{t-1},\rho_{t-1},\gamma_{t-1}} \quad & [C_{t-1}(x_{t-1})+\gamma\tilde{z}^t(x_{t-1}-D_{t-1})]'\bar{\boldsymbol{p}}_{t-1}+\boldsymbol{\lambda}'_{t-1}\bar{\boldsymbol{p}}_{t-1}+\rho_{t-1}\\ s.t. & \|\boldsymbol{A}'_{t-1}[C_{t-1}(x_{t-1})+\gamma\tilde{z}^t(x_{t-1}-D_{t-1})]+\boldsymbol{A}'_{t-1}\boldsymbol{\lambda}_{t-1}-\gamma_{t-1}\boldsymbol{A}'_{t-1}\boldsymbol{e}\| \le \rho_{t-1}\\ & \boldsymbol{\lambda}_{t-1}\ge\boldsymbol{0}, \rho_{t-1}\ge 0.\end{aligned}$$

The optimal (s_t, S_t) policy can be determined recursively for each period $t = T-1, T-2, ..., 1$. The cost in period 1 is given by

$$z^1(x_1) = \begin{cases} \mathcal{K} + c_1(S_1 - x_1) + G^1(S_1), & \text{if } x_1 \leq s_1 \\ G^1(x_1), & \text{otherwise,} \end{cases} \tag{36}$$

representing the total cost over all the T periods.

6. Numerical study

In this section, the effectiveness and practicality of the proposed robust optimization approaches are demonstrated through a numerical study. The solution approaches are first applied to a multi-period inventory problem. A single-period inventory problem is then used to analyze the impact of uncertainty levels on the performance of the solution approaches. For the problem instances in the numerical study, the initial inventory level and the discount factor are set to $x_1 = 0$ and $\gamma = 1$, respectively. Furthermore, $r_t = 20$, $c_t = 10$, $h_t = 2$, $b_t = 15$, $K_t = 10$ and $\mathcal{K} = 100$ are used. The demand scenarios are sampled uniformly from the interval $[100, 200]$ and are then sorted to provide K_t demand values. The demand scenarios $D_t \in \{110, 113, 128, 144, 155, 163, 181, 185, 191, 196\}$ are randomly generated. According to Andersson et al. (2013), the resulting distribution will be too specialized if K_t is too small, or will essentially resemble a uniform distribution if K_t is too large. Hence, $K_t = 10$ is chosen. Unless specifically mentioned, the nominal distribution randomly generated and used in the numerical study is $\bar{\boldsymbol{p}}_t = (0.04, 0.24, 0.18, 0.10, 0.15, 0.11, 0.02, 0.07, 0.04, 0.05)'$, for $t = 1, 2, ..., T$.

For the box uncertainty set, the uncertainty disturbance vector $\boldsymbol{\xi}_t$ takes values from the interval $[\underline{\boldsymbol{\xi}}_t, \overline{\boldsymbol{\xi}}_t]$ with $\overline{\boldsymbol{\xi}}_t = \alpha \boldsymbol{e}$ and $\underline{\boldsymbol{\xi}}_t = -\overline{\boldsymbol{\xi}}_t$, where α is a scalar that controls the uncertainty levels of the demand distributions. In this numerical study, $\alpha = 0.04$ is used. For the ellipsoid uncertainty set, the scaling matrix is $\boldsymbol{A}_t = \beta \boldsymbol{I}$, where $\boldsymbol{I}$ is an identity matrix of appropriate dimension and β is a scalar. In this numerical study, $\beta = 0.15$ is used and the levels of uncertainty can be adjusted by using different values of β. In order to evaluate the effectiveness and practicality of the proposed robust optimization approaches in dealing with demand distribution uncertainty, the actual demand in each period t is assumed to follow the nominal distribution $\bar{\boldsymbol{p}}_t$.

6.1. The multi-period inventory problem

The length of the planning horizon is one year and each period corresponds to a one month sales cycle, i.e., $T = 12$. The parameters r_t, c_t, $\mathcal{K}$, b_t, h_t, D_t, and $\bar{\boldsymbol{p}}_t$ are first assumed to be the same over the periods. This assumption will be relaxed later.

Table 1 shows the (s_t, S_t) policies and their cost performance. The second column shows the results when the actual demand distribution is assumed to be known, i.e., the nominal distribution. Thus, there is no uncertainty in the demand distribution in this case. The third and fourth columns show the results, with demand distribution uncertainty, under the box and the ellipsoid uncertainty sets, respectively. The terms per^s and per^S represent the objective values of the problems to find the reorder point s_t and the order-up-to level S_t, respectively. They represent the objective values of Problems (16) and (14) for the box uncertainty set and the objective values of Problems (32) and (29) for the ellipsoid uncertainty set. The terms $\text{per}^s|\bar{\boldsymbol{p}}_t$ and $\text{per}^S|\bar{\boldsymbol{p}}_t$ represent the corresponding objective values when a given (s_t, S_t) policy is applied to the multi-period model with the actual demand distribution.

Table 1: The (s_t, S_t) policies and their cost performance

Inventory policy and costs	**Nominal**	**Box**	**Ellipsoid**
(s_t, S_t)	$(165, 191)$	$(162, 183)$	$(162, 180)$
$(\text{per}^s, \text{per}^S)$	$(-16508.78, -14962.60)$	$(-15243.23, -13725.82)$	$(-14740.48, -13225.38)$
$(\text{per}^s\|\bar{\boldsymbol{p}}_t, \text{per}^S\|\bar{\boldsymbol{p}}_t)$	$(-16508.78, -14962.60)$	$(-15325.78, -13802.91)$	$(-14864.71, -13339.20)$

The (s_t, S_t) policy in each column of Table 1 is the same for each period because all the parameters are the same over the whole planning horizon. However, a different uncertainty set yields a different (s_t, S_t) policy, and these policies are different from the optimal policy when the actual distribution is known. Furthermore, the objective values of the optimal (s_t, S_t) policy when the actual distribution is known (column 2) are lower than those under the box and the ellipsoid uncertainty sets.

To obtain an (s_t, S_t) policy for a *non-stationary* distribution-free model, the components of $\bar{\boldsymbol{p}}_t$ are varied in a systematic way to create a nominal distribution for each $t = 1, 2, ..., T$. Three groups of nominal distributions, labeled as ND-I, ND-II and ND-III, are depicted in Figs. 2, 4, and 6, respectively. The expected demands of these three groups of nominal distributions correspond to three different, i.e., an increasing (Fig. 3), a decreasing (Fig. 5) and a random (Fig. 7), patterns, respectively, covering a wide range of situations in practice. The (s_t, S_t) policies and the corresponding cost performance for the three groups of nominal distributions are presented in Tables 2-7. The results in Table 2 show that increasing expected demands over time lead to non-decreasing s_t and S_t under all the three cases i.e., known actual distribution, box uncertainty set and ellipsoid uncertainty set. This implies that a larger expected demand yields a higher reorder point and a higher order-up-to level. Similarly, the results in Table 4 show that with decreasing expected demands, both s_t and S_t are non-increasing under all the three cases. Furthermore, due to the demand distribution uncertainty, the reorder point s_t and the order-up-to level S_t for each period under both the box and the ellipsoid uncertainty sets are not higher than that when the actual demand distribution is known. Results in Tables 3 and 5 show that the costs under the box and the ellipsoid uncertainty sets are higher than that when the actual demand distribution is known. These results are consistent with the results in Table 1.

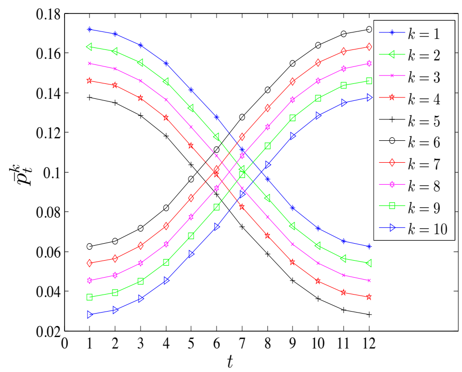

Figure 2: ND-I: Nominal distributions with increasing expected demands over time

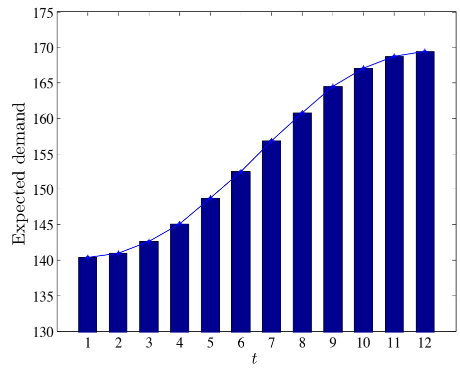

Figure 3: Expected demands under the ND-I nominal distributions for different time periods

Table 2: (s_t, S_t) inventory policies for each period t under the ND-I nominal distributions

s_t, S_t / **Period** t	**Nominal**	**Box**	**Ellipsoid**
1	(161, 191)	(159, 183)	(160, 178)
2	(162 191)	(160, 183)	(160, 178)
3	(166, 191)	(162, 183)	(162, 180)
4	(170, 191)	(165, 183)	(164, 184)
5	(174, 196)	(169, 185)	(167, 187)
6	(177, 196)	(172, 191)	(171, 191)
7	(179, 196)	(175, 191)	(174, 191)
8	(180, 196)	(177, 196)	(176, 193)
9	(181, 196)	(179, 196)	(178, 196)
10	(182, 196)	(179, 196)	(179, 196)
11	(182, 196)	(180, 196)	(179, 196)
12	(183, 196)	(180, 196)	(180, 196)

Table 3: Cost performance under the ND-I nominal distributions

Costs	**Nominal**	**Box**	**Ellipsoid**
$(\text{per}^s, \text{per}^S)$	$(-18009.65, -16053.63)$	$(-16570.14, -15078.03)$	$(-16056.38, -14560.93)$
$(\text{per}^s \vert \bar{\boldsymbol{p}}_t, \text{per}^S \vert \bar{\boldsymbol{p}}_t)$	$(-18009.65, -16053.63)$	$(-16660.43, -15165.35)$	$(-16185.94, -14671.51)$

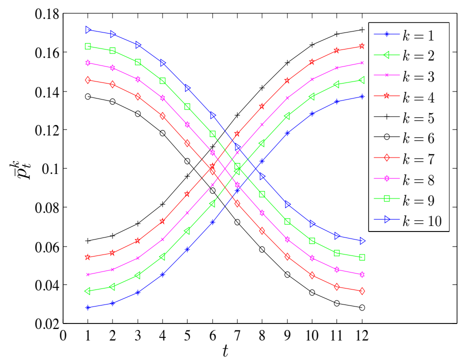

Figure 4: ND-II: Nominal distributions with decreasing expected demands over time

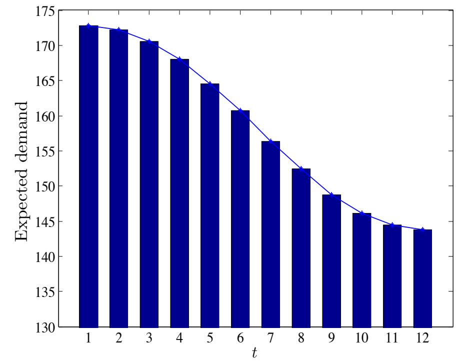

Figure 5: Expected demands under the ND-II nominal distributions for different time periods

Table 4: (s_t, S_t) inventory policies for each period t under the ND-II nominal distributions

s_t, S_t / **Period** t	**Nominal**	**Box**	**Ellipsoid**
1	(184, 196)	(182, 196)	(181, 196)
2	(184, 196)	(182, 196)	(181, 196)
3	(184, 196)	(181, 196)	(181, 196)
4	(183, 196)	(180, 196)	(180, 196)
5	(182, 196)	(179, 196)	(179, 196)
6	(181, 196)	(178, 196)	(177, 196)
7	(180, 196)	(175, 196)	(174, 195)
8	(178, 196)	(172, 196)	(170, 191)
9	(175, 196)	(169, 191)	(167, 191)
10	(172, 196)	(165, 191)	(164, 187)
11	(169, 196)	(163, 183)	(162, 183)
12	(168, 196)	(162, 183)	(161, 181)

Table 5: Cost performance under the ND-II nominal distributions

Costs	**Nominal**	**Box**	**Ellipsoid**
$(\text{per}^s, \text{per}^S)$	$(-18743.30, -17003.18)$	$(-17217.43, -15500.17)$	$(-16687.83, -14973.62)$
$(\text{per}^s \vert \bar{\boldsymbol{p}}_t, \text{per}^S \vert \bar{\boldsymbol{p}}_t)$	$(-18743.30, -17003.18)$	$(-17308.31, -15627.85)$	$(-16819.02, -15146.56)$

For the ND-III nominal distributions, more varying (s_t, S_t) policies are obtained for different periods due to the variations in the components of $\bar{\boldsymbol{p}}_t$. Results in Fig. 7 and Table 6 show that higher expected demands in some periods, e.g., $t = 5, 9, 11$, lead to higher s_t and S_t, which is consistent with the results for the ND-I and ND-II nominal distributions. Similarly, due to the demand distribution uncertainty, the values of s_t and S_t for each period under the box and the ellipsoid uncertainty sets are not higher than that when the actual demand distribution is known. Results in Table 7 show that the costs under the box and the ellipsoid uncertainty sets are higher than that when the actual demand distribution is known.

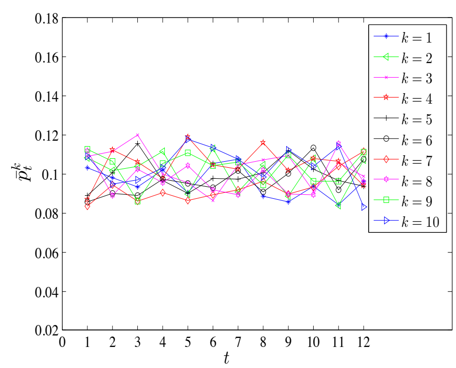

Figure 6: ND-III: Nominal distributions with varying expected demands over time

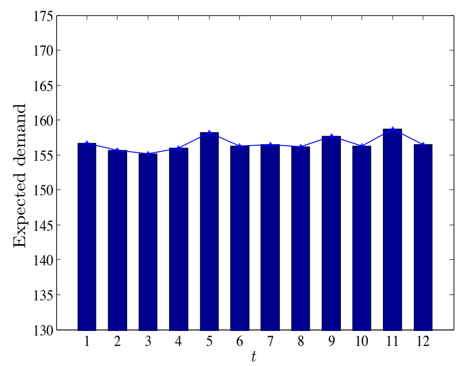

Figure 7: Expected demands under the ND-III nominal distributions for different time periods

Table 6: (s_t, S_t) inventory policies for each period t under the ND-III nominal distributions

s_t, S_t / **Period** t	**Nominal**	**Box**	**Ellipsoid**
1	(180, 196)	(176, 196)	(175, 194)
2	(179, 196)	(175, 196)	(173, 191)
3	(179, 196)	(174, 196)	(172, 191)
4	(179, 196)	(175, 196)	(174, 193)
5	(180, 196)	(177, 196)	(175, 196)
6	(180, 196)	(176, 196)	(174, 195)
7	(180, 196)	(175, 196)	(174, 194)
8	(179, 196)	(175, 196)	(173, 192)
9	(180, 196)	(176, 196)	(174, 195)
10	(179, 196)	(175, 196)	(173, 193)
11	(180, 196)	(176, 196)	(175, 196)
12	(179, 196)	(175, 191)	(173, 191)

Using the same parameters for all the periods only affects the values of and as well as the cost performance, but does not affect the optimality of the (s_t, S_t) policy or the effectiveness and practicality of the solution approaches. This is

Table 7: Cost performance under the ND-III nominal distributions

Costs	Nominal	Box	Ellipsoid
$(\text{per}^s, \text{per}^S)$	$(-18450.36, -16748.88)$	$(-16882.05, -15218.83)$	$(-16336.04, -14685.46)$
$(\text{per}^s\vert\bar{\boldsymbol{p}}_t, \text{per}^S\vert\bar{\boldsymbol{p}}_t)$	$(-18450.36, -16748.88)$	$(-16958.00, -15346.51)$	$(-16440.05, -14854.61)$

because the multi-period model with demand distribution uncertainty is non-stationary distribution free and the parameters are allowed to vary from period to period.

6.2. Sensitivity to the uncertainty level

More experiments are conducted based on a single-period inventory model to further test the effectiveness and practicality of the robust optimization approaches in coping with demand distribution uncertainty. When only a single period is involved, the subscript t is dropped from the notations. These experiments also serve as a sensitivity analysis of the impact of the uncertainty levels on the (s, S) policies and on the cost performance. Conducting these experiments is quite convenient by setting $T = 1$ and designing a special mechanism to settle accounts at the end of the period, i.e., by replacing h with $h' = h - c$ and b by $b' = b + c$(Zipkin, 2000). Thus, Problems (14) and (16) with $T = 1$ are solved for the box uncertainty set, and Problems (29) and (32) with $T = 1$ are solved for the ellipsoid uncertainty set to find the (s, S) policies.

The optimal cost performance can be achieved when the actual, i.e., the nominal, distribution is known. A *cost performance loss* is defined as the difference between the optimal cost performance under the nominal distribution and the cost performance produced by a robust (s, S) policy under demand distribution uncertainty. The cost performance loss can be interpreted as a fee that an inventory manager is willing to pay to gain access to the perfect demand information. The cost performance ratio in percentage is defined as the cost performance loss over the optimal cost under the nominal distribution.

6.2.1. Effects on the inventory policies and their cost performance

In this section, all the parameter values are the same as those in the multi-period problem, except for that $T = 1$ and the values of α and β are varied for the box and the ellipsoid uncertainty sets, respectively.

For the box uncertainty set, the scalar α measuring the uncertainty levels in demand distributions varies from 0 to 0.1 in an increment of 0.01. As α increases, the distribution uncertainty levels also increase. By solving Problems (14) and (16 with $T = 1$, the results obtained are shown in Figs. 8 and 9. Fig. 8 displays the relationships between α and the reorder point s, and between α and the order-up-to level S. The optimal (s, S) policy when the actual distribution is known corresponds to that at $\alpha = 0$. As α increases, the reorder point stays approximately the same but the order-up-to level decreases. This suggests that a higher uncertainty level in demand distribution leads to a smaller order quantity, and can thus affect the inventory cost performance.

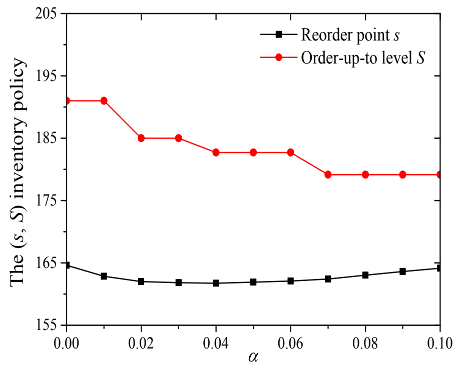

Figure 8: (s, S) policies under different values of α for the box uncertainty set

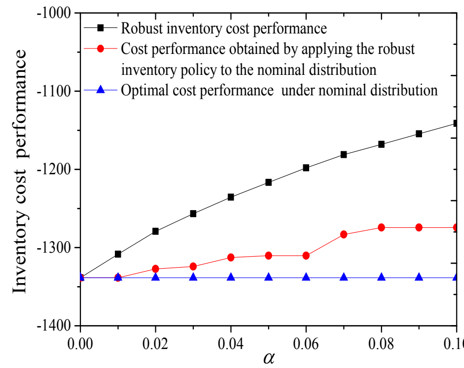

Figure 9: Cost performance under different values of α for the box uncertainty set

Fig. 9 shows the relationships between α and the robust inventory cost performance and between and the cost performance obtained when applying the robust inventory policy to the single-period model with the nominal distribution. A larger value of α, corresponding to a higher distribution uncertainty level, yields a larger cost. As α increases, compared with the more substantial changes in the robust cost performance, the changes in the cost performance of applying a robust (s, S) policy to the model with the nominal distribution are much smaller. The largest and the smallest loss ratios are $\frac{-1338.55-(-1274.45)}{-1338.55} \times 100\% = 4.79\%$ at $\alpha = 0.10$ and $\frac{-1338.55-(-1338.55)}{-1338.55} \times 100\% = 0.00\%$ at $\alpha = 0.01$, respectively. These results suggest that the robust optimization approach with the box uncertainty set can effectively curtail the impact of demand distribution uncertainty on the inventory cost performance.

For the ellipsoid uncertainty set, the scalar β varies from 0 to 0.5 in an increment of 0.05. As β increases, the distribution uncertainty levels increase. By solving Problems (29) and (32) with $T = 1$, the results obtained are shown in Figs. 10 and 11. The results in Fig. 10 are similar to those in Fig. 8. They suggest that the order quantities decrease as the distribution uncertainty levels increase. Specifically, due to demand distribution uncertainty, the values of s and S under different values of β are all smaller than their respective optimal values at $\beta = 0$.

Fig. 11 shows that the cost performance obtained by applying the robust (s, S) policy to the model with the nominal distribution is superior to the robust inventory cost performance. The largest and the smallest cost performance loss ratios are $\frac{-1338.55-(-1204.44)}{-1338.55} \times 100\% = 10.2\%$ at $\beta = 0.50$ and $\frac{-1338.55-(-1338.55)}{-1338.55} \times 100\% = 0.00\%$ at $\beta = 0.05$, respectively. These results suggest that the proposed robust optimization approach with the ellipsoid uncertainty set is also effective in dealing with demand distribution uncertainty.

The results in Figs. 9 and 11 should motivate inventory managers to improve their forecast accuracy in demand distributions. Smaller values of α and β represent more accurate forecasts on the demand distributions and yield better

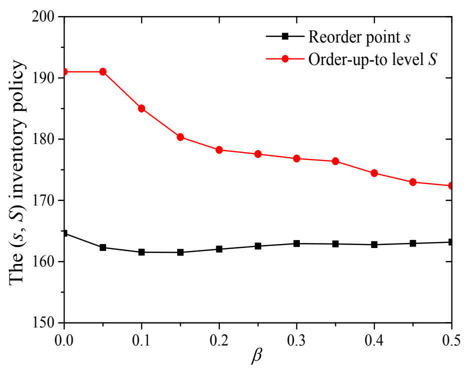

Figure 10: (s, S) policies under different values of β for the ellipsoid uncertainty set

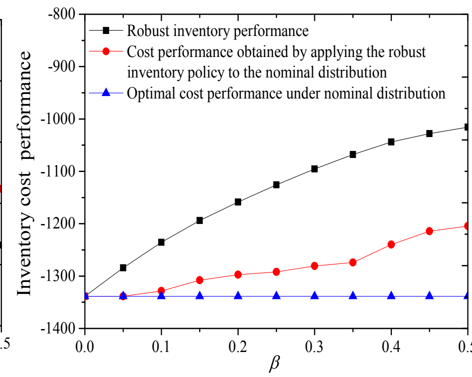

Figure 11: Cost performance under different values of β for the ellipsoid uncertainty set

inventory cost performance.

6.2.2. Further examination

To further test the effectiveness and practicality of the proposed robust optimization approaches, another nominal demand distribution $\bar{\boldsymbol{p}} = (0.03, 0.23, 0.19, 0.11, 0.16, 0.10, 0.01, 0.08, 0.05, 0.04)'$ is used. The actual distribution for the single-period inventory problem is assumed to follow this nominal distribution. All other parameters remain the same as above. The optimal (s, S) policy for this nominal distribution is $(s, S) = (165, 191)$, leading to an optimal cost of -1345.20.

Fig. 12 shows the reorder point s and the order-up-to level S, while Fig. 13 shows the cost performance, obtained with the robust optimization approach with the box uncertainty set under different distribution uncertainty levels. The results in Figs. 12 and 13 are similar to those in Figs. 8 and 9. The cost performance losses are also quite small. The largest and the smallest loss ratios are $\frac{-1345.20-(-1268.22)}{-1345.20} \times 100\% = 5.72\%$ at $\alpha = 0.10$ and $\frac{-1345.20-(-1343.77)}{-1345.20} \times 100\% = 0.11\%$ at $\alpha = 0.01$, respectively.

Figs. 14 and 15 show the results for the ellipsoid uncertainty set. The results are very similar to those in Figs. 10 and 11. The largest and smallest loss ratios are $\frac{-1345.20-(-1211.60)}{-1345.20} \times 100\% = 9.93\%$ at $\beta = 0.50$ and $\frac{-1345.20-(-1344.62)}{-1345.20} \times 100\% = 0.04\%$ at $\beta = 0.05$, respectively. The results in Figs. 13 and 15 suggest that the proposed robust optimization approaches are effective in dealing with demand distribution uncertainty for different nominal distributions.

7. Conclusion

New models are proposed for the multi-period inventory management problem with fixed ordering costs under uncertainty in discrete demand distributions. The optimality is proved for (s_t, S_t) policies for each period. Both the box and the ellipsoid uncertainty sets are used to describe the demand distribution uncertainty, and tractable robust counterpart models are developed. The proposed

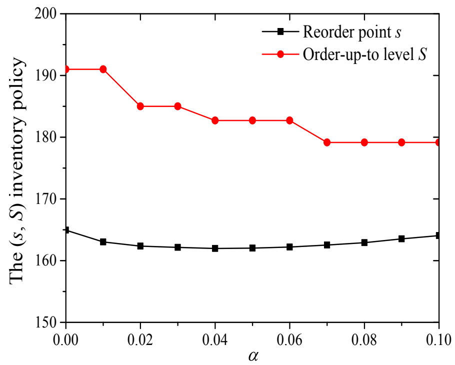

Figure 12: (s, S) policies under different values of α for the box uncertainty set

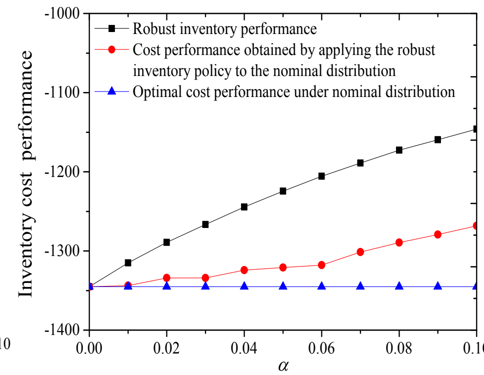

Figure 13: Cost performance under different values of α for the box uncertainty set

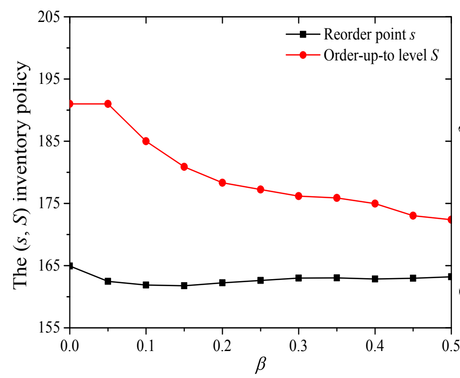

Figure 14: (s, S) policies under different values of β for the ellipsoid uncertainty set

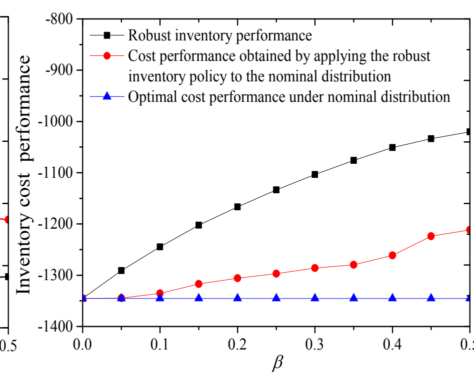

Figure 15: Cost performance under different values of β for the ellipsoid uncertainty set

models under both the box and the ellipsoid uncertainty sets can be reformulated as tractable mathematical programming models, and can thus be solved efficiently using robust optimization techniques. The robustness of the multi-period inventory models and the effectiveness and practicality of the proposed robust optimization approaches in coping with demand distribution uncertainty are validated through a numerical study. The obtained results suggest that the total cost changes stably as the uncertainty levels change and is close to the optimal cost. This implies that the proposed models are robust and the corresponding solution approaches are powerful for solving inventory management problems without known demand distributions. These robust optimization approaches can also incorporate various demand distribution uncertainty levels.

The proposed models and the solution approaches are only validated for the box and the ellipsoid uncertainty sets. Other techniques may be used to construct other types of uncertainty sets. For example, $\phi-$divergence based on statistical inference or likelihood estimate can be used to build a confidence interval of the uncertain demand probabilities with a certain confidence level(Ben-

Tal et al., 2013; Wang et al., 2013; Bayraksan and Love, 2015). Furthermore, multi-item problems with budget constraints can be incorporated into the multi-period inventory models. Value-based indicators bridging the physical and the financial operations, such as the economic value added, can be used to measure the performance of a multi-period inventory policy in medium-term operations.

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Supplementary Materials for "Optimizing (s, S) Policies for Multi-period Inventory Models with Demand Distribution Uncertainty: Robust Dynamic Programming Approaches"

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A.1 Proof of Theorem 1

The following two lemmas using the definition of a $\mathcal{K}$-convex function are used to prove Theorem 1.

Lemma 1. *For any* $t = 1, 2, ..., T$*, the function* $\psi^t(x)$ *(7) is continuous with respect to* x *and* $\lim_{|x|\to\infty} \psi^t(x) = \infty$*. Specifically,* $\psi^t(x)$ *is a* $\mathcal{K}$*-convex function of* x*.*

Lemma 2. *Under Lemma 1, let* S_t *be a minimum point of* $\psi^t(x)$ *and* s_t *be any element of the set*

$$\left\{x \,\middle|\, x \leq S_t, \psi^t(x) = \psi^t(S_t) + \mathcal{K}\right\} \tag{37}$$

The following results hold

(i) $\psi^t(S_t) \leq \psi^t(x)$*, for all* $x \in R$*.*

(ii) $\psi^t(S_t) + \mathcal{K} = \psi^t(s_t) \leq \psi^t(x)$*, for all* $x \leq s_t$*.*

(iii) $\psi^t(x) \leq \psi^t(y) + \mathcal{K}$*, for all* x *and* y *with* $s_t \leq x \leq y$*.*

The proof of Lemma 1 follows a similar logic in Scarf (1960). A different $\mathcal{K}$-convex function from that in Scarf (1960) and mathematical induction are used to complete the proof. First, consider the last period, i.e., $t = T$, $\psi^T(x) = c_T x + G^T(x) = c_T x + \max_{\boldsymbol{p}_T} \{C_T(x)'\boldsymbol{p}_T\}$. Obviously, $\psi^T(x)$ is continuous and $\lim_{|x|\to\infty} \psi^T(x) = \infty$. Specifically, $\psi^T(x)$ is convex and, hence, a $\mathcal{K}$-convex function.

Next, assume $\psi^t(x)$ is continuous, $\mathcal{K}$-convex and $\lim_{|x|\to\infty} \psi^t(x) = \infty$, then there exist two parameters s_t and S_t with $s_t < S_t$ such that S_t minimizes $\psi^t(x)$ and $\psi^t(s_t) = \psi^t(S_t) + \mathcal{K}$. From the definitions of $G^t(x)$ and $z^t(x)$, $z^t(x) = \psi^t(S_t) + \mathcal{K} - c_t x$ if $x \le s_t$ and $z^t(x) = \psi^t(x) - c_t x$ otherwise. Since $\psi^t(s_t) = \psi^t(S_t) + \mathcal{K}$, $z^t(x)$ is continuous and $\mathcal{K}$-convex.

Finally, consider period $t-1$

$$\begin{aligned}\psi^{t-1}(x) &= c_{t-1}x + G^{t-1}(x) \\ &= c_{t-1}x + \max_{\boldsymbol{p}_{t-1}} \left\{C_{t-1}(x)'\boldsymbol{p}_{t-1} + \gamma z^t(x - D_{t-1})\right\}\end{aligned}$$

Therefore, $\psi^{t-1}(x)$ is continuous. Since $z^t(x)$ is continuous and -convex, according to the properties of the $\mathcal{K}$-convex functions (Zipkin, 2000, P398), $z^t(x - D_{t-1})$ is -convex and thus $\psi^{t-1}(x)$ is also -convex. Furthermore, $\lim_{|x|\to\infty} \psi^{t-1}(x) = \infty$.

Based on Lemma 1 and the properties of a -convex function, Lemma 2 is straightforward. The proof is omitted and readers are referred to Simchi et al. (2014). Lemmas 1 and 2 then lead to Theorem 1. □

A.2 Proof of Theorem 2

Let $(x^*, \delta_T^*, \boldsymbol{\tau}_T^*, \boldsymbol{\nu}_T^*)$ be an optimal solution to Problem (14) with an optimal objective value θ^*. It can be found by comparing the constraints of Problems (13) and (14) that $(\delta_T^*, \boldsymbol{\tau}_T^*, \boldsymbol{\nu}_T^*)$ is also feasible to Problem (13). Given x^*, $\Upsilon^*(x^*) = \underline{\boldsymbol{\xi}}_T \boldsymbol{\tau}_T^* + \overline{\boldsymbol{\xi}}_T \boldsymbol{\nu}_T^*$ by the strong duality of linear programming. If x^* is not optimal to Problem (11), there exists another solution $\tilde{x}^*$ to Problem (11) such that

$$\begin{aligned}c_T\tilde{x}^* + C_T(\tilde{x}^*)'\bar{\boldsymbol{p}}_T + \Upsilon^*(\tilde{x}^*) &\le c_T x^* + C_T(x^*)'\bar{\boldsymbol{p}}_T + \Upsilon^*(x^*) \\ &= c_T x^* + C_T(x^*)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T \boldsymbol{\tau}_T^* + \overline{\boldsymbol{\xi}}_T \boldsymbol{\nu}_T^* \\ &= \theta^*.\end{aligned}$$

Given $\tilde{x}^*$, let $(\tilde{\delta}_T^*, \tilde{\boldsymbol{\tau}}_T^*, \tilde{\boldsymbol{\nu}}_T^*)$ be an optimal solution to Problem (13). By comparing the constraints of Problems (13) and (14), it can be found that $(\tilde{x}^*, \tilde{\delta}_T^*, \tilde{\boldsymbol{\tau}}_T^*, \tilde{\boldsymbol{\nu}}_T^*)$ is also feasible to Problem (14). Similarly, $\Upsilon^*(\tilde{x}^*) = \underline{\boldsymbol{\xi}}_T \tilde{\boldsymbol{\tau}}_T^* + \overline{\boldsymbol{\xi}}_T \tilde{\boldsymbol{\nu}}_T^*$ by the strong duality of linear programming. Therefore, the objective value of Problem (14) at $(\tilde{x}^*, \tilde{\delta}_T^*, \tilde{\boldsymbol{\tau}}_T^*, \tilde{\boldsymbol{\nu}}_T^*)$, denoted by $\tilde{\theta}^*$, is not larger than θ^*, i.e., $\tilde{\theta}^* \le \theta^*$. This contradicts the assumption that $(x^*, \delta_T^*, \boldsymbol{\tau}_T^*, \boldsymbol{\nu}_T^*)$ is an optimal solution to Problem (14). Hence, x^* is an optimal solution to Problem (11).

Conversely, if $\hat{x}^*$ solves Problem (11), $(\hat{\delta}_T^*, \hat{\boldsymbol{\tau}}_T^*, \hat{\boldsymbol{\nu}}_T^*)$ is an optimal solution to Problem (13) with $x = \hat{x}^*$. If $(\hat{x}^*, \hat{\delta}_T^*, \hat{\boldsymbol{\tau}}_T^*, \hat{\boldsymbol{\nu}}_T^*)$ is not an optimal solution to Problem (14), there exists another solution $(\tilde{\hat{x}}^*, \tilde{\hat{\delta}}_T^*, \tilde{\hat{\boldsymbol{\tau}}}_T^*, \tilde{\hat{\boldsymbol{\nu}}}_T^*)$ that solves Problem (14). According to the discussion above, $\tilde{\hat{x}}^*$ is an optimal solution to Problem (11), contradicting the assumption that $\hat{x}^*$ solves Problem (11). Therefore, solving Problem (14) is equivalent to solving Problem (11). □

A.3 Proof of Theorem 3

The reorder point s_T can be found by solving

$$\begin{aligned} \max_{y \le S_T} \ & G^T(y) \\ s.t. \ & G^T(y) \le \mathcal{K} + c_T(S_T - y) + G^T(S_T), \end{aligned} \tag{38}$$

where the optimal value of the variable y is the reorder point s_T, i.e., $s_T = y^*$, and $G^T(y)$ is the optimal objective value of the following problem with variables $(\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T) \in R \times R^{K_T} \times R^{K_T}$

$$\begin{aligned} \min_{\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T} \ & C_T(y)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T \boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T \boldsymbol{\nu}_T \\ s.t. \ & \boldsymbol{e}'\delta_T + \boldsymbol{\tau}_T + \boldsymbol{\nu}_T = C_T(y) \\ & \boldsymbol{\tau}_T \le \mathbf{0}, \boldsymbol{\nu}_T \ge \mathbf{0}. \end{aligned} \tag{39}$$

The constraint in Problem (38) is satisfied at equality at s_T. Therefore, Problem (38) can be rewritten as

$$\begin{aligned} \max_{y \le S_T} \min_{\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T} \ & \mathcal{K} + c_T(S_T - y) + G^T(S_T) \\ s.t. \ & \boldsymbol{e}'\delta_T + \boldsymbol{\tau}_T + \boldsymbol{\nu}_T = C_T(y) \\ & \boldsymbol{\tau}_T \le \mathbf{0}, \boldsymbol{\nu}_T \ge \mathbf{0} \\ & C_T(y)'\bar{\boldsymbol{p}}_T + \underline{\boldsymbol{\xi}}_T \boldsymbol{\tau}_T + \overline{\boldsymbol{\xi}}_T \boldsymbol{\nu}_T = \mathcal{K} + c_T(S_T - y) + G^T(S_T). \end{aligned}$$

In this optimization problem, the objective function is composed of a constant $\mathcal{K} + c_T S_T + G^T(S_T)$ and a linear term $-c_T y$ that are both independent of the variables $(\delta_T, \boldsymbol{\tau}_T, \boldsymbol{\nu}_T)$. Therefore, s_T can be found by solving Problem (16). □

A.4 Proof of Theorem 4

Let $(x^*, \boldsymbol{\lambda}_T^*, \rho_T^*, \gamma_T^*)$ and θ^* be an optimal solution and the optimal objective value to Problem (29), respectively. Then x^* is also feasible to Problem (25). Denote by θ_0^* the objective value of Problem (25) at $x = x^*$. If x^* is not optimal to Problem (25), there exists another solution $\tilde{x}^*$ that solves Problem (25) such that $\tilde{\theta}_0^* < \theta_0^*$. Given $x = \tilde{x}^*$, $(\tilde{\boldsymbol{\lambda}}_T^*, \tilde{\rho}_T^*, \tilde{\gamma}_T^*)$ is obtained by solving Problem (28). By the strong duality of the Lagrangian, $\Gamma^*(x^*) = -(\boldsymbol{\lambda}_T^*)'\bar{\boldsymbol{p}}_T - \rho_T^*$ and $\Gamma^*(\tilde{x}^*) = -(\tilde{\boldsymbol{\lambda}}_T^*)'\bar{\boldsymbol{p}}_T - \tilde{\rho}_T^*$. Therefore, $\tilde{\theta} = \tilde{\theta}_0 < \theta_0^* = \theta^*$. Because Problems (28) and (29) have the same set of constraints, $(\tilde{x}^*, \tilde{\boldsymbol{\lambda}}_T^*, \tilde{\rho}_T^*, \tilde{\gamma}_T^*)$ is also feasible to Problem (29). This contradicts the assumption that $(x^*, \boldsymbol{\lambda}_T^*, \rho_T^*, \gamma_T^*)$ is an optimal solution to Problem (29) because $\tilde{\theta}^* < \theta^*$. Thus, x^* is an optimal solution to Problem (25).

Conversely, if $\hat{x}^*$ is an optimal solution to Problem (25), $(\hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ can be obtained by solving Problem (28). Since Problems (28) and (29) have the same set of constraints, $(\hat{x}^*, \hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is also feasible to Problem (29). If $(\hat{x}^*, \hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is not an optimal solution to Problem (29), there exists another solution $(\tilde{\hat{x}}^*, \tilde{\hat{\boldsymbol{\lambda}}}_T^*, \tilde{\hat{\rho}}_T^*, \tilde{\hat{\gamma}}_T^*)$ that solves Problem (29). From the discussion above, $\tilde{\hat{x}}^*$ is an optimal solution to Problem (25), contradicting the assumption that $\hat{x}^*$ is an optimal solution to Problem (25). Therefore, $(\hat{x}^*, \hat{\boldsymbol{\lambda}}_T^*, \hat{\rho}_T^*, \hat{\gamma}_T^*)$ is an optimal solution to Problem (28). □

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