12-2011

Technical Note: Cellular Bucket Brigades

Yun Fong LIM
Singapore Management University, yflim@smu.edu.sg

Published in Operations Research, 2011 December, 59, 6, 1539-1545
doi:10.1287/opre.1110.0958

Cellular Bucket Brigades

Yun Fong Lim
Lee Kong Chian School of Business, Singapore Management University,
50 Stamford Road, Singapore 178899, Singapore
yflim@smu.edu.sg

March 13, 2011

Abstract

Workers in a bucket brigade production system perform unproductive travel when they walk to get more work from their colleagues. We introduce a new design of bucket brigades to reduce unproductive travel. Under the new design, each worker works on one side of an aisle when he proceeds in one direction and works on the other side when he proceeds in the reverse direction. We propose simple rules for workers to share work under the new design and find a sufficient condition for the system to self-balance. Numerical examples suggest that the improvement in throughput by the new design can be as large as 30%. Even with a 20% reduction in labor, the new design can still increase throughput by 7%.

1 Introduction

Bartholdi and Eisenstein (1996a) introduce bucket brigades as a way to effectively coordinate workers on an assembly line. All workers in a bucket brigade follow a simple rule: Continue to assemble an item (an instance of the product) along the line until either your colleague downstream takes over it or you complete it; then you walk back to get more work, either from your colleague upstream or from a buffer at the start of the line.

Bucket brigades are effective because they require neither a detailed work-content model nor computation for work balance, which are necessary for any static work allocation policy. Due to their dynamics, bucket brigades constantly seek balance and spontaneously adapt to disruptions and seasonality. For applications and related studies on bucket brigades, see Bartholdi and Eisenstein (1996b, 2005), Bartholdi et al. (1999, 2001, 2006, 2009), Villalobos et al. (1999a,b), Eisenstein (2005), Armbruster and Gel (2006), Armbruster et al. (2007), and Lim and Yang (2009).

In this study, we present a novel design alternative that may provide significant improvement to the performance of a bucket brigade. According to the bucket brigade rule, after a worker relinquishes his item, he walks back to get another item. The throughput of bucket brigades is compromised due to this unproductive travel, which can be significant especially for long assembly lines.

We propose a new line design alternative to reduce unproductive travel. The idea is to distribute work content on both sides of an aisle. Each worker assembles an item on one side of the aisle while he proceeds in one direction and assembles, possibly, another item on the other side while he proceeds in the reverse direction. We introduce new rules for workers to share work because they are not only required to work in both directions, but also to cross the aisle. We find a sufficient condition for the system to self-balance under the new rules.

Although the new design eliminates the unproductive walk back inherent in a traditional bucket brigade, it incurs a new form of waste: It requires workers to walk across the aisle. Fortunately, this cross-aisle travel is short when the aisle is narrow. We show that a bucket brigade is more productive under the new design if the aisle width is smaller than a *critical aisle width*. Our results suggest that a higher throughput can be attained under the new design even with fewer workers. Consequently, the new design not only improves production rate, but also significantly reduces labor cost.

We first illustrate the ideas and assess the effectiveness of the new design of bucket brigades using a basic model. We then show that the new design generally remains effective under a much more general setting.

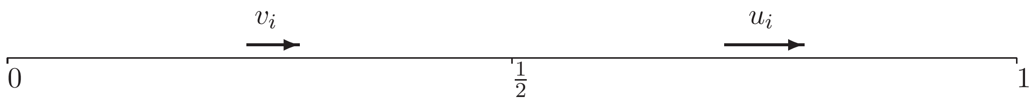


Figure 1: **An assembly line.** Each worker i works with velocities v_i and u_i in the intervals $[0, 1/2)$ and $[1/2, 1]$ respectively.

2 Cellular bucket brigades

Consider an assembly line with length 1 shown in Figure 1. The work content of the product is cumulative along the line so that points 0 and 1 correspond to the initiation and completion, respectively, of an item. We consider different work content along the line such that each worker i, for $i = 1, \ldots, n$, has finite, constant *work* velocities v_i and u_i in the intervals $[0, 1/2)$ and $[1/2, 1]$ respectively. We assume all workers walk with a finite, constant *walk* velocity w. For the case with $v_i < u_i$ (or $v_i > u_i$) for all i, the model represents an assembly line where work is more concentrated on one half of the line and is diluted on the other half. For example, it is common to have major assembly work in one section of a line while simple packaging is done in the other section.

To reduce unproductive travel, we fold the entire line at point 1/2 so that the two halves of the line form an aisle with width a in between. Figure 2 shows such a design with three workers. The *forward* line contains work in the interval $[0, 1/2)$ of the original line and the *backward* line continues the remaining work in the reverse direction. Each item is initiated at the start of the forward line and is progressively assembled until it reaches the end of the forward line where it is transferred to the backward line. The item is then assembled backward until it is completed at the end of the backward line. Each worker i works on the forward and backward lines with velocities v_i and u_i respectively.

We conceptualize the aisle as a real line such that points 0 and 1/2 represent its start and end respectively. Each worker i has a unique coordinate $x_i \in [0, 1/2]$ *along* the aisle. We require workers to remain in a fixed ordering from 1 to n so that $x_1 \leq x_2 \leq \cdots \leq x_n$. This will not

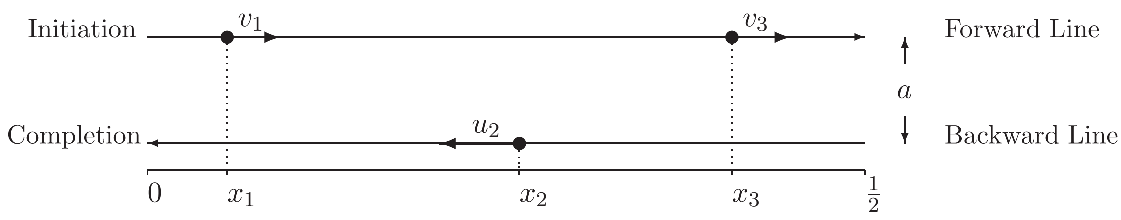


Figure 2: **A new design of the assembly line.** The two halves of the assembly line form an aisle in between. Each worker i has a coordinate x_i along the aisle and works on the forward and backward lines with velocities v_i and u_i respectively.

be a limitation of our model as the new design is effective only if the aisle is sufficiently narrow that workers usually cannot overtake or pass each other. Only worker 1 can initiate and finish an item at the start of the aisle, and only worker n can transfer an item from the forward line to the backward line at the end of the aisle. We call workers $i-1$ and $i+1$ the *predecessor* and *successor*, respectively, of worker i.

Worker i continues to work forward until a *hand-off* occurs at point $p \in [0, 1/2]$ along the aisle where he meets his successor, who is working on the backward line. Unlike in a traditional bucket brigade, a hand-off now becomes an exchange of work between the two workers and is not instantaneous: The two workers first relinquish their items at point p and then walk *across* the aisle with velocity w, which takes time a/w. After they exchange their work, worker i continues on the backward line, while his successor continues on the forward line.

When worker n reaches the end of the forward line, the system *resets* itself: Worker n walks across the aisle with his item to the backward line, which takes time a/w. He continues to work backward until he meets worker $n-1$, who is working forward. After exchanging their work, worker $n-1$ continues to work backward until he meets and exchanges work with worker $n-2$, and so on until worker 1 completes an item at point 0. Worker 1 then relinquishes the completed item, crosses the aisle, and initiates a new item. Each reset triggers a completion, which is followed by an initiation. The work-in-process is bounded by the number of workers in the system.

To keep workers in the same ordering, we need new rules to handle situations in which a worker catches up with his colleagues. Consider a worker i, who is working forward, catches up with worker $i+1$. If worker $i+1$ is working forward then worker i is *blocked* and he works forward with his successor's velocity v_{i+1}. If worker $i+1$ is crossing the aisle at point x_{i+1} then worker i *waits* for worker $i+1$ at the same point $x_i = x_{i+1}$. Similarly, consider a worker i, who is working backward, catches up with worker $i-1$. If worker $i-1$ is working backward then worker i is blocked and he works backward with velocity u_{i-1}. If worker $i-1$ is crossing the aisle at point x_{i-1} then worker i waits at $x_i = x_{i-1}$. Under the new design each worker independently follows these rules:

Work forward: Continue to assemble your item on the forward line until

1. you exchange work with your successor, then **work backward**; or
2. you reach the end of the forward line if you are the last worker, then carry the item to the backward line and **work backward**; or
3. you catch up with your successor, who is working forward, then slow down to match his velocity and **work forward**; or
4. you catch up with your successor, who is crossing the aisle, then **wait**.

Work backward: Continue to assemble your item on the backward line until

1. you exchange work with your predecessor, then **work forward**; or
2. you complete your item at the end of the backward line if you are the first worker, then initiate a new item and **work forward**; or
3. you catch up with your predecessor, who is working backward, then slow down to match his velocity and **work backward**; or
4. you catch up with your predecessor, who is crossing the aisle, then **wait**.

Wait: Continue to carry your item,

1. if you are on the forward line, remain idle until your successor has finished crossing the aisle, then **work forward**; or
2. if you are on the backward line, remain idle until your predecessor has finished crossing the aisle, then **work backward**.

The waiting rule has the potential of wasting production capacity because it requires workers to stand idle. However, as we will see from our analysis, the waiting rule will never be invoked under normal operation of a properly configured system.

Figure 3 illustrates how workers move under the new design. Let $x_i^t \in [0, 1/2]$ denote the hand-off point between worker i and his successor due to the t-th reset. The two workers cross

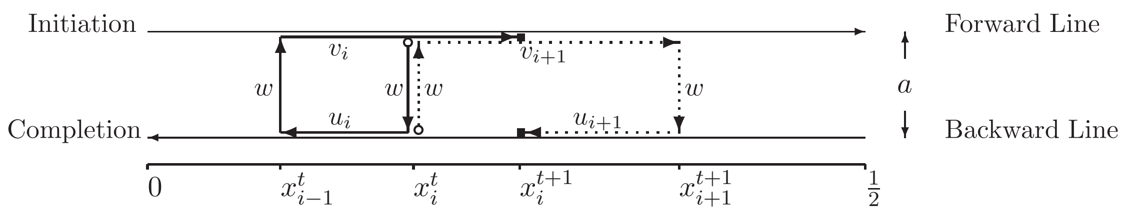


Figure 3: **A cellular bucket brigade.** This figure illustrates the paths of worker i (bold solid arrows) and worker $i+1$ (dotted arrows) between two successive hand-offs, which occur at points x_i^t and x_i^{t+1}, between the two workers in a cellular bucket brigade. The start and the end of each worker's path are represented by a circle and a square respectively.

the aisle with velocity w at point x_i^t to exchange work. Worker i then works backward with velocity u_i until he meets worker $i-1$, who is working forward, at point x_{i-1}^t. After a hand-off, worker i works forward with velocity v_i. Meanwhile, worker $i+1$ continues to work forward with velocity v_{i+1} until he meets worker $i+2$, who is working backward, at point x_{i+1}^{t+1}. After a hand-off, worker $i+1$ works backward with velocity u_{i+1}. The next hand-off between workers i and $i+1$ occurs at point x_i^{t+1}. We call this a *cellular* bucket brigade (because workers move in cells).

Let $\mathbf{x}^t = \left(x_1^t, x_2^t, \ldots, x_{n-1}^t\right)$. Since the system resets itself when worker n reaches point 1/2, we have $x_n^t = 1/2$ for all t. For convenience, we define $x_0^t = 0$ for all t. Let f be a function defined implicitly by the cellular bucket brigade rules such that $\mathbf{x}^{t+1} = f(\mathbf{x}^t)$. We say $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_{n-1}^*)$ is a fixed point if $\mathbf{x}^* = f(\mathbf{x}^*)$. Define $\theta_i = (1/v_i + 1/u_i)^{-1}$. Lemma 1 identifies a fixed point of hand-off locations for a cellular bucket brigade. All proofs can be found in the Online Appendix.

Lemma 1. *For a cellular bucket brigade there exists a unique fixed point* $\mathbf{x}^*$*, where*

$$x_i^* = \frac{\sum_{j=1}^{i} \theta_j}{2\sum_{j=1}^{n} \theta_j}, \quad i = 1, \ldots, n-1.$$

Figure 4 shows a three-worker cellular bucket brigade operating on its fixed point. Each worker i repeats a simple loop for each item produced: After exchanging work with his successor at

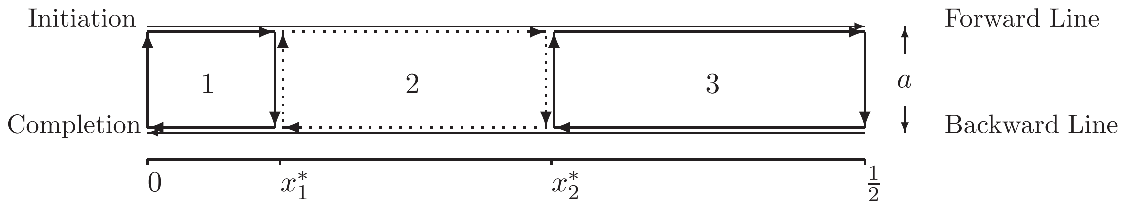


Figure 4: **Balance in a cellular bucket brigade.** Each worker repeats a simple loop when the system operates on its fixed point. Workers 1, 2, and 3 cover the loops on the left, in the middle, and on the right respectively. Hand-offs occur at points x_1^* and x_2^*.

point x_i^*, he works backward until point x_{i-1}^*, where he exchanges work with his predecessor; he then works forward until he completes the loop at point x_i^*.

On the fixed point the system assembles an item every time worker 1 completes a loop. Thus, the average throughput of a cellular bucket brigade is

$$\rho_c = \left[\frac{2a}{w} + \left(\frac{1}{v_1} + \frac{1}{u_1}\right) x_1^*\right]^{-1} = \left(\frac{2a}{w} + \frac{1}{2\sum_{j=1}^{n} \theta_j}\right)^{-1}. \tag{1}$$

Note that the throughput of a cellular bucket brigade decreases with the aisle width a. Given a team of workers, we expect a cellular bucket brigade to be more productive than its traditional counterpart when the aisle is sufficiently narrow. A narrow aisle is preferred by cellular bucket brigades also because it is easier for workers to see their successors and predecessors on the opposite side of the aisle.

If a cellular bucket brigade is configured properly then its fixed point is an attractor.

Theorem 1. *The fixed point* $\mathbf{x}^*$ *of a cellular bucket brigade is an attractor if*

$$\frac{1}{v_1} - \frac{1}{u_1} > \frac{1}{v_2} - \frac{1}{u_2} > \cdots > \frac{1}{v_n} - \frac{1}{u_n}. \tag{2}$$

The proof of Theorem 1 shows that the fixed point $\mathbf{x}^*$ is at least a local attractor: If the system is sufficiently close to the fixed point such that no blocking occurs, then it will converge to the fixed point. This implies that the system will converge to the fixed point after perturbations, as long as they are not too disruptive. An extensive set of simulations, which cover a broad range of work and walk velocities with numerous different initial states, suggest that the fixed point $\mathbf{x}^*$ is also a global attractor: If Condition (2) holds, then the system always converges to the fixed point given any initial state.

Theorem 1 shows that a worker with a small forward velocity but a large backward velocity should be assigned a low index. Consider a special case with $u_i = \lambda v_i$, where λ is a constant for all i. We say worker i is *faster* than worker j if $v_i > v_j$. Theorem 1 reduces to the following results: (1) If $\lambda > 1$ (the forward line has more work than the backward line), then workers should be sequenced from slowest to fastest in the direction of the forward line. (2) If $\lambda < 1$ (the backward line has more work content), then workers should be sequenced from slowest to fastest in the direction of the backward line. In short, for a cellular bucket brigade to self-balance in this special case, workers should be sequenced from *slowest to fastest in the direction of production flow of one side of the aisle that has more work content.*

3 Comparison with serial bucket brigades

We compare the performance of a cellular bucket brigade with its traditional counterpart. Under the traditional design, workers remain in a fixed sequence from 1 to n along the production flow shown in Figure 1. Each worker i works forward until he hands off his item to worker $i+1$. When worker n completes his item at the end of the line, the line resets itself: Worker n walks back to get work from worker $n-1$, who in turn walks back to get work from worker $n-2$, and so on until worker 1 initiates a new item at the start of the line. All assumptions are identical to that of the normative model described in Theorem 3 of Bartholdi and Eisenstein (1996a) except now each worker has two different work velocities and they walk back with a finite, constant velocity w. As a result, the reset is not instantaneous. We call this a *serial* bucket brigade.

Let $\mathbf{x}^t = \left(x_1^t, x_2^t, \ldots, x_{n-1}^t\right)$, where x_i^t is the location at which worker i hands off work to worker $i+1$ due to the t-th reset. Since worker n finishes each item at point 1, we have $x_n^t = 1$ for all t. For convenience, we define $x_0^t = 0$ for all t. Define $\psi_i = (1/v_i + 1/w)^{-1}$ and

$\phi_i = (1/u_i + 1/w)^{-1}$. Lemma 2 identifies a fixed point for a serial bucket brigade. We define worker k as the worker that repeatedly crosses point 1/2 on the line in Figure 1 when the system operates on the fixed point.

Lemma 2. *For a serial bucket brigade there exists a unique fixed point* $\mathbf{x}^*$*, which can be determined as follows: If* $1/\psi_1 \leq 1/\sum_{j=2}^{n} \phi_j$*, then* $k = 1$ *and*

$$\begin{aligned} x_1^* &= \frac{1-(1/2)(1/\psi_1 - 1/\phi_1)\sum_{j=2}^{n}\phi_j}{(1/\phi_1)\sum_{j=1}^{n}\phi_j}; \\ x_i^* &= x_{i-1}^* + \phi_i\left[(1/2)/\psi_1 + (x_1^* - 1/2)/\phi_1\right], \; i = 2, \ldots, n. \end{aligned} \tag{3}$$

Otherwise,

$$\begin{aligned} x_1^* &= \psi_1 \frac{(1/2)(1/\psi_k + 1/\phi_k)}{(1/\psi_k)\sum_{j=1}^{k}\psi_j + (1/\phi_k)\sum_{j=k+1}^{n}\phi_j}; \\ x_i^* &= x_{i-1}^* + \psi_i(x_1^*/\psi_1), \; i = 2, \ldots, k-1; \\ x_n^* &= 1; \\ x_i^* &= x_{i+1}^* - \phi_{i+1}(x_1^*/\psi_1), \; i = n-1, \ldots, k; \end{aligned} \tag{4}$$

and k *is the smallest index such that* $(x_1^*/\psi_1)\sum_{j=1}^{k}\psi_j \geq 1/2$.

Upon the fixed point, each worker $i = 1, \ldots, k-1$ repeatedly works on an interval that lies in $[0, 1/2)$ with velocity v_i, while each worker $i = k+1, \ldots, n$ executes a portion of work that lies in $[1/2, 1]$ with velocity u_i. Worker k is the only one that repeatedly works in both intervals $[0, 1/2)$ and $[1/2, 1]$ with velocities v_k and u_k respectively. Since the system assembles an item every time worker 1 completes a loop, the throughput of a serial bucket brigade is determined as follows:

$$\rho_s = \begin{cases} \left[(1/2)/\psi_1 + (x_1^* - 1/2)/\phi_1\right]^{-1}, & \text{if } k = 1; \\ (x_1^*/\psi_1)^{-1}, & \text{otherwise.} \end{cases} \tag{5}$$

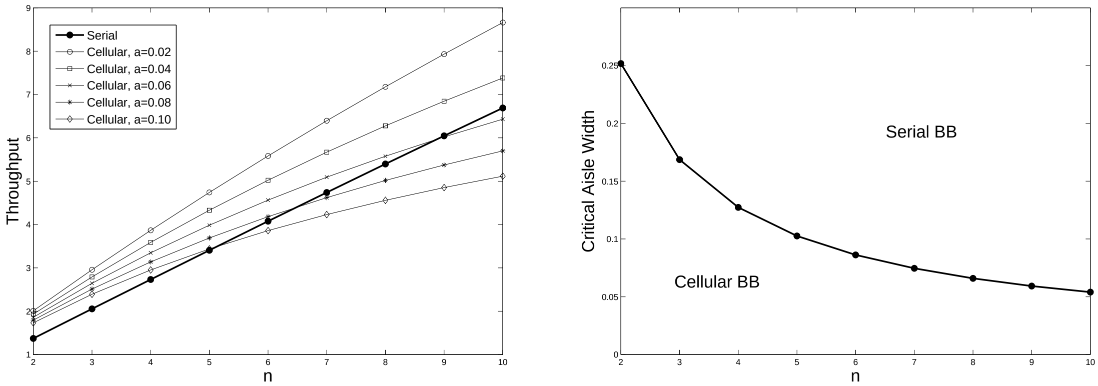
It can be shown that if $v_1 < v_2 < \cdots < v_k$ and $u_k < u_{k+1} < \cdots < u_n$, then the fixed point $\mathbf{x}^*$ is at least a local attractor (see the Online Appendix): If the system is sufficiently close to the fixed point $\mathbf{x}^*$ such that worker k is the only one that crosses point 1/2 all the time, then the serial bucket brigade will converge to the fixed point. An extensive set of simulations suggest that the fixed point is also a global attractor.

Which design is more productive? Workers in a cellular bucket brigade walk across the aisle to exchange work. This unproductive travel increases not only with the aisle width, but also with the number of workers in a team. Producing an item by a cellular bucket brigade requires $2(n-1)$ hand-offs, compared to $n-1$ hand-offs in a serial bucket brigade. As n increases, the increase in hand-offs for the cellular bucket brigade is twice as many as the increase in hand-offs for the serial bucket brigade. To find out when a cellular bucket brigade outperforms its serial counterpart, we should answer the following questions: (1) Given a team of workers and their velocities, how narrow should the aisle be? (2) Given the aisle width and workers' velocities, how small should the team size be?

Figure 5(a) shows the average throughput of both cellular and serial bucket brigades when they operate on their fixed points given the team size n. For the cellular bucket brigade we plot the throughput with different aisle width a. The velocities are $v_i = 1.0 + [i - (n+1)/2]\delta v, u_i = 1.1 + [i - (n+1)/2]\delta v,\ i = 1, \ldots, n$, where $\delta v = 0.1$, and $w = 2.0$. The throughput increases with the team size n, but due to the increase in unproductive travel, the increase in throughput is less than linear (the rate of increase is decreasing in n). Note that as $n \to \infty$, $\rho_c \to w/(2a)$ whereas ρ_s is unbounded.

For the cellular bucket brigade the rate of increase in throughput improves as the aisle becomes narrower. The cellular bucket brigade outperforms the serial bucket brigade for all team sizes when $a \leq 0.04$ (4% of the length of the original assembly line). The throughput improvement given by the cellular bucket brigade can be as large as 30% (for $n = 10$). Note that even with fewer workers the cellular bucket brigade may outperform the serial bucket brigade. For example, in Figure 5(a) the cellular bucket brigade with 8 workers and aisle width $a = 0.02$ is 7% more productive than the serial bucket brigade with 10 workers. For this example, the cellular bucket brigade not only attains a 7% higher production rate but also saves 20% of labor cost.

For wider aisles ($a > 0.04$), the cellular bucket brigade remains more productive when the team size is sufficiently small. As the team size gets larger, the cellular bucket brigade



(a) (b)

Figure 5: **Throughput and critical aisle width.** (a) The cellular bucket brigade is more productive than its serial counterpart when the aisle width and/or team size are sufficiently small. (b) The critical aisle width decreases with the team size. The cellular bucket brigade outperforms the serial bucket brigade when the aisle width falls below this curve and vice versa.

requires more hand-offs to produce an item. This results in more unproductive travel and undermines its performance. Figure 5(a) shows that when the team size is larger than some critical value the cellular bucket brigade becomes less productive than its serial counterpart. Fortunately, this critical team size can be increased by reducing the aisle width. Figure 5(a) can also be interpreted in another way. For each team size n the cellular bucket brigade is more productive than the serial bucket brigade if the aisle width is smaller than the critical aisle width $a^* = \frac{w}{2}\left(\frac{1}{\rho_s} - \frac{1}{2\sum_{j=1}^{n}\theta_j}\right)$, which can be determined by comparing Equations (1) and (5).

Figure 5(b) shows the critical aisle width given the team size n. The cellular bucket brigade dominates below the curve and the serial bucket brigade is preferred above the curve. The critical aisle width decreases with the team size. The larger the team the narrower the aisle should be to maintain the effectiveness of the cellular bucket brigade. Figure 5(b) is especially useful when we consider using the cellular bucket brigade. For example, if we have a team of 5 workers the figure indicates that the aisle width must be smaller than 10% of the total length of the original assembly line to justify the new design. On the other hand, if the aisle width is 10% of the line length then the figure suggests that the cellular bucket brigade is preferred if the team size is smaller than 5.

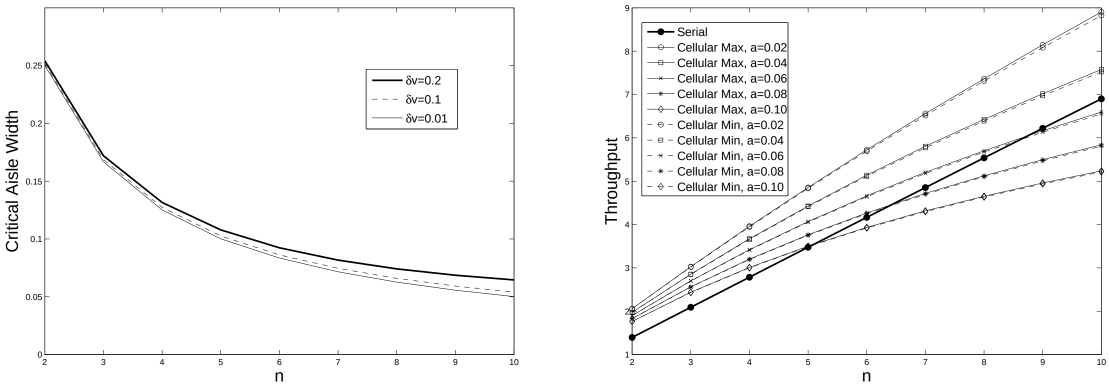
Figure 6(a) illustrates the sensitivity of the critical aisle width to the difference in work velocities. We set v_i, u_i and w as in Figure 5(a) but vary δv from 0.01 to 0.2. The critical aisle width increases with δv, implying that large difference in skill level favors the cellular bucket brigade. This is because in the serial bucket brigade workers who work faster cover larger work intervals (see Equations (3) and (4)) and have more unproductive walk than their slower colleagues. This wastes production capacity as we want workers who work fast to focus on assembling rather than walking. This problem is overcome by the cellular bucket brigade in which *every* worker has the same amount of unproductive, cross-aisle travel to produce an item. As a result, workers who work fast are more utilized for assembling. This advantage is especially significant when different workers possess very different skill levels.

4 Generalization of the basic model

We also compare the performance of cellular and serial bucket brigades based on a generalized assembly line where each worker i changes his work velocity from v_i to u_i at point $\beta \in [0, 1]$ and walks with velocity w_i, for $i = 1, \ldots, n$. To construct a cellular bucket brigade, we fold the assembly line at point 1/2. We obtain generalized expressions of fixed points, throughput, and convergence conditions for both cellular and serial bucket brigades (see the Online Appendix). All these expressions reduce to the results of the basic model discussed in the previous sections if we set $\beta = 1/2$ and $w_i = w$ for all i.

Since worker i walks with velocity w_i to cross the aisle in a cellular bucket brigade, he may cause his colleagues to wait if he walks too slow. In this case, the system may have multiple fixed points given the same ordering of workers and its capacity is not fully used due to waiting. Fortunately, we identify a sufficient condition to ensure the system has a unique fixed point and no waiting occurs on the fixed point. This condition can be satisfied as long as the walk velocities of workers are not too different.

The generalized convergence condition for the cellular bucket brigade may be satisfied as long as β is neither too small nor too large (that is, in the cellular bucket brigade the point



(a) (b)

Figure 6: **Sensitivity and generalization.** (a) The critical aisle width increases with the difference in work velocities. (b) The throughput of bucket brigades based on a generalized assembly line with $\beta = 0.25$.

where workers change their work velocities cannot be too close to the start of the aisle). There may exist different orderings of workers that satisfy this generalized condition and result in different fixed points and throughput. In such cases we record both the maximum and the minimum throughput of the cellular bucket brigade. Figure 6(b) compares the throughput of both designs with $\beta = 0.25$. We set v_i and u_i as in Figure 5(a) and $w_i = 2.0 + [i - (n+1)/2] \times 0.1$, for $i = 1, \ldots, n$. Similar results are observed if different values are used for these parameters. The gap between the maximum and the minimum throughput of the cellular bucket brigade is marginal. The cellular bucket brigade is more productive when the aisle is narrow and/or the team size is small. Details of these generalized results can be found in the Online Appendix.

Although the generalized model leads to more complex theoretical results and an extra condition to ensure the uniqueness of the fixed point, the cellular bucket brigade generally remains effective for boosting productivity given a team of workers.

5 Conclusions

The main contribution of this paper is to introduce cellular bucket brigades so that workers can share work in a more productive manner. Under this new design, work content is distributed on both sides of an aisle. Workers work on one side of the aisle when they proceed in one direction, and they work on the other side when they proceed in the reverse direction. This eliminates the unproductive walk back inherent in traditional bucket brigades. However, a cellular bucket brigade requires workers to walk across the aisle to exchange work. Although this travel is unproductive, we show that the new design is still preferable when the aisle width and/or team size are sufficiently small. Numerical examples suggest that the throughput improvement by the new design can be as large as 30%. Even with a 20% reduction in labor, the new design can still increase throughput by 7%.

We introduce simple rules for workers to share work under the new design. Based on the rules, we identify a sufficient condition for the system to self-balance. All attractive characteristics of traditional bucket brigades such as repetition of work, workers are constantly busy, and regular output are preserved under this new design.

Given a team of workers and their velocities, a cellular bucket brigade is more productive than its traditional counterpart if the aisle width is below a critical value, which increases with the difference in work velocities. Thus, a team of workers with very different skill levels favors the cellular bucket brigade in which workers who work fast are more utilized to assemble items.

In practice, workers may find it difficult to see each other if they work on opposite sides of an aisle. This can be overcome by installing mirrors on each side of the aisle. Also, this will not be an issue if the aisle is sufficiently narrow.

Acknowledgments

The author thanks the associate editor and two anonymous referees for their comments that substantially improved the paper. Special thanks must go to John Bartholdi, III and Donald Eisenstein for their feedback on an earlier draft. The author is supported by the Lee Kong Chian School of Business, Singapore Management University.

References

Alligood, K.T., T.D. Sauer, J.A. Yorke. 1996. *Chaos: An Introduction to Dynamical Systems.* Springer, New York. ISBN 0-387-94677-2.

Armbruster, D., E.S. Gel. 2006. Bucket brigades revisited: Are they always effective? *Eur. J. Oper. Res.* **172**(1) 213–229.

Armbruster, D., E.S. Gel, J. Murakami. 2007. Bucket brigades with worker learning. *Eur. J. Oper. Res.* **176**(1) 264–274.

Bartholdi, J.J. III, L.A. Bunimovich, D.D. Eisenstein. 1999. Dynamics of two- and three-worker "bucket brigade" production lines. *Oper. Res.* **47**(3) 488–491.

Bartholdi, J.J. III, D.D. Eisenstein. 1996a. A production line that balances itself. *Oper. Res.* **44**(1) 21–34.

Bartholdi, J.J. III, D.D. Eisenstein. 1996b. The bucket brigade web page. URL `http://www.BucketBrigades.com`.

Bartholdi, J.J. III, D.D. Eisenstein. 2005. Using bucket brigades to migrate from craft manufacturing to assembly lines. *Manufacturing Service Oper. Management* **7**(2) 121–129.

Bartholdi, J.J. III, D.D. Eisenstein, R.D. Foley. 2001. Performance of bucket brigades when work is stochastic. *Oper. Res.* **49**(5) 710–719.

Bartholdi, J.J. III, D.D. Eisenstein, Y.F. Lim. 2006. Bucket brigades on in-tree assembly networks. *Eur. J. Oper. Res.* **168**(3) 870–879.

Bartholdi, J.J. III, D.D. Eisenstein, Y.F. Lim. 2009. Deterministic chaos in a model of discrete manufacturing. *Naval Res. Logist.* **56**(4) 293–299.

Eisenstein, D.D. 2005. Recovering cyclic schedules using dynamic produce-up-to policies. *Oper. Res.* **53**(4) 675–688.

Lim, Y.F., K.K. Yang. 2009. Maximizing throughput of bucket brigades on discrete work stations. *Production Oper. Management* **18**(1) 48–59.

Villalobos, J.R., L.F. Muñoz, L. Mar. 1999a. Assembly line designs that reduce the impact of personnel turnover. *Proc. of IIE Solutions Conference*, Phoenix, AZ.

Villalobos, J.R., F. Estrada, L.F. Muñoz, L. Mar. 1999b. Bucket brigade: A new way to boost production. *Twin Plant News* **14**(12) 57–61.

A Technical details

A.1 Proof of Lemma 1

Proof. On the fixed point each worker i repeats a simple loop for each item produced: He exchanges work with his successor at point x_i^*. He then works backward with velocity u_i until he reaches point x_{i-1}^*, where he exchanges work with his predecessor. After the hand-off he works forward with velocity v_i until he completes the loop at point x_i^*. The fixed point $\mathbf{x}^*$ can be found by solving the following equations:

$$\frac{x_i^* - x_{i-1}^*}{u_i} + \frac{x_i^* - x_{i-1}^*}{v_i} = \frac{x_{i+1}^* - x_i^*}{v_{i+1}} + \frac{x_{i+1}^* - x_i^*}{u_{i+1}}, \quad i = 1, \ldots, n-1.$$

Lemma 1 follows by simple algebra. □

A.2 Proof of Theorem 1

Proof. Iteration t follows the sequence of hand-off points from the end to the start of the aisle caused by the t-th reset. A hand-off occurs at point x_i^t where worker i, who is working forward, meets worker $i+1$, who is working backward on the opposite side of the aisle. Since the time spent in the movement from one iteration to the next (depicted in Figure 3) is the same for both workers, we have

$$\frac{a}{w} + \frac{x_i^t - x_{i-1}^t}{u_i} + \frac{a}{w} + \frac{x_i^{t+1} - x_{i-1}^t}{v_i} = \frac{a}{w} + \frac{x_{i+1}^{t+1} - x_i^t}{v_{i+1}} + \frac{a}{w} + \frac{x_{i+1}^{t+1} - x_i^{t+1}}{u_{i+1}}, \tag{6}$$

for $i = 1, \ldots, n-1$.

Rewriting Equation (6) yields:

$$x_i^{t+1} = \frac{1/v_i + 1/u_i}{1/v_i + 1/u_{i+1}} x_{i-1}^t - \frac{1/v_{i+1} + 1/u_i}{1/v_i + 1/u_{i+1}} x_i^t + \frac{1/v_{i+1} + 1/u_{i+1}}{1/v_i + 1/u_{i+1}} x_{i+1}^{t+1},$$

for $i = 1, \ldots, n-1$. Or we can write

$$x_i^{t+1} = (1+\alpha_i)\gamma_i x_{i-1}^t - \alpha_i x_i^t + (1+\alpha_i)(1-\gamma_i) x_{i+1}^{t+1}, \tag{7}$$

where

$$\begin{aligned}\alpha_i &= \frac{1/v_{i+1} + 1/u_i}{1/v_i + 1/u_{i+1}},\\ \gamma_i &= \frac{1/v_i + 1/u_i}{1/v_i + 1/u_i + 1/v_{i+1} + 1/u_{i+1}},\end{aligned}$$

for $i = 1, \ldots, n-1$. Note that Condition (2) implies $0 < \alpha_i < 1$, $i = 1, \ldots, n-1$.

Equation (7) can be expressed as an affine system (Martelli 1999):

$$\mathbf{y}^{t+1} = A\mathbf{y}^t + \mathbf{b},$$

where $\mathbf{y}^t = \left(x_1^t, x_2^t, \ldots, x_{n-2}^t, x_{n-1}^{t+1}\right)^T$. The first $n-2$ components of the vector $\mathbf{y}^t$ correspond to the last $n-2$ hand-offs of iteration t and the last component corresponds to the first hand-off of iteration $t+1$. (This is to accommodate the vector $\mathbf{b}$, which affects only the updating of x_{n-1}^{t+1}.) The matrix A can be factored as $A = A_{n-1}A_1A_2 \ldots A_{n-2}$, where each matrix A_i updates x_i^t according to Equation (7), and

$$\mathbf{b} = \left(0, 0, \ldots, 0, \frac{(1+\alpha_{n-1})(1-\gamma_{n-1})}{2}\right)^T.$$

In this way we first update x_{n-2}^t, then x_{n-3}^t, and so on until x_1^t, and then finally x_{n-1}^{t+1}, which uses the last component of $\mathbf{b}$.

Each matrix A_i is an identity matrix except for row i. Each $A_2, A_3, \ldots, A_{n-2}$ has three non-zero terms in row i that sum to 1, with values $(1+\alpha_i)\gamma_i$, $-\alpha_i$, and $(1+\alpha_i)(1-\gamma_i)$ in columns $i-1$, i, and $i+1$ respectively. For A_1 the first term $(1+\alpha_1)\gamma_1 > 0$ is omitted from row 1, and thus the sum of the first row has absolute value less than 1. For A_{n-1} the last term $(1+\alpha_{n-1})(1-\gamma_{n-1}) > 0$ is omitted from row $n-1$, thus the sum of the last row has absolute value less than 1.

For the full transition matrix A, all eigenvalues have modulus less than one. In short, this follows because each $A_2, A_3, \ldots, A_{n-2}$ can be replaced by a stochastic matrix, while both A_1 and A_{n-1} can be replaced by a strictly sub-stochastic matrix. Since all states communicate, the system tends to the zero matrix. Thus, the orbit $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2, \ldots$ converges to the unique fixed point $\mathbf{y}^*$ of hand-off locations. (See, for example, Martelli 1999 for dynamics of affine systems.) □

A.3 Proof of Lemma 2

Proof. On the fixed point each worker i repeats a simple loop for each item produced: He receives work from worker $i-1$ at point x^*_{i-1} and relinquishes work for worker $i+1$ at point x^*_i. Recall that the loop of worker k overlaps with both intervals $[0, 1/2)$ and $[1/2, 1]$. We first assume $k > 1$. The fixed point $\mathbf{x}^*$ in Equations (4) can be found by solving the following equations:

$$\begin{aligned}
\left(\frac{1}{v_1}+\frac{1}{w}\right)x_1^* &= \left(\frac{1}{v_i}+\frac{1}{w}\right)(x_i^*-x_{i-1}^*), \quad i=2,\ldots,k-1;\\
\left(\frac{1}{v_1}+\frac{1}{w}\right)x_1^* &= \frac{1/2-x_{k-1}^*}{v_k}+\frac{x_k^*-1/2}{u_k}+\frac{x_k^*-x_{k-1}^*}{w};\\
\left(\frac{1}{v_1}+\frac{1}{w}\right)x_1^* &= \left(\frac{1}{u_i}+\frac{1}{w}\right)(x_i^*-x_{i-1}^*), \quad i=k+1,\ldots,n.
\end{aligned}$$

The solution follows by simple algebra. Note that if $k = 2$ and the corresponding $x^*_1 \geq 1/2$, which imply $1/\psi_1 \leq 1/\sum_{j=2}^n \phi_j$, then worker 1 is the one that repeatedly crosses point 1/2. In this case, the fixed point $\mathbf{x}^*$ in Equations (3) can be found by solving the following equations:

$$\frac{1/2}{v_1}+\frac{x_1^*-1/2}{u_1}+\frac{x_1^*}{w} = \left(\frac{1}{u_i}+\frac{1}{w}\right)(x_i^*-x_{i-1}^*), \quad i=2,\ldots,n.$$

The solution follows by simple algebra.

□

A.4 Convergence of serial bucket brigades

Theorem 2. *The fixed point $\mathbf{x}^*$ of a serial bucket brigade is an attractor if $v_1 < v_2 < \cdots < v_k$ and $u_k < u_{k+1} < \cdots < u_n$.*

Proof. Iteration t follows the sequence of hand-off points from the end to the start of the assembly line caused by the t-th reset. When the system operates sufficiently close to the fixed point $\mathbf{x}^*$ there is only one worker (we assume worker k) that crosses point 1/2 in each iteration.

A hand-off occurs at point x_i^t where worker i, who is working forward, meets worker $i+1$, who is walking backward. Since the time spent in the movement from one iteration to the next is the same for both workers, we have

$$\begin{aligned}
\frac{x_i^t - x_{i-1}^t}{w} + \frac{x_i^{t+1} - x_{i-1}^t}{v_i} &= \frac{x_{i+1}^{t+1} - x_i^t}{v_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^{t+1}}{w}, \text{ for } i = 1, \ldots, k-2;\\
\frac{x_{k-1}^t - x_{k-2}^t}{w} + \frac{x_{k-1}^{t+1} - x_{k-2}^t}{v_{k-1}} &= \frac{1/2 - x_{k-1}^t}{v_k} + \frac{x_k^{t+1} - 1/2}{u_k} + \frac{x_k^{t+1} - x_{k-1}^{t+1}}{w};\\
\frac{x_k^t - x_{k-1}^t}{w} + \frac{1/2 - x_{k-1}^t}{v_k} + \frac{x_k^{t+1} - 1/2}{u_k} &= \frac{x_{k+1}^{t+1} - x_k^t}{u_{k+1}} + \frac{x_{k+1}^{t+1} - x_k^{t+1}}{w};\\
\frac{x_i^t - x_{i-1}^t}{w} + \frac{x_i^{t+1} - x_{i-1}^t}{u_i} &= \frac{x_{i+1}^{t+1} - x_i^t}{u_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^{t+1}}{w}, \text{ for } i = k+1, \ldots, n-1.
\end{aligned}$$

Rewriting the above equations yields:

$$\begin{aligned}
x_i^{t+1} &= x_{i-1}^t - \frac{\psi_i}{\psi_{i+1}} x_i^t + \frac{\psi_i}{\psi_{i+1}} x_{i+1}^{t+1}, \text{ for } i = 1, \ldots, k-2;\\
x_{k-1}^{t+1} &= x_{k-2}^t - \frac{\psi_{k-1}}{\psi_k} x_{k-1}^t + \frac{\psi_{k-1}}{\phi_k} x_k^{t+1} + \frac{\psi_{k-1}}{2}\left(\frac{1}{\psi_k} - \frac{1}{\phi_k}\right);\\
x_k^{t+1} &= \frac{\phi_k}{\psi_k} x_{k-1}^t - \frac{\phi_k}{\phi_{k+1}} x_k^t + \frac{\phi_k}{\phi_{k+1}} x_{k+1}^{t+1} - \frac{\phi_k}{2}\left(\frac{1}{\psi_k} - \frac{1}{\phi_k}\right);\\
x_i^{t+1} &= x_{i-1}^t - \frac{\phi_i}{\phi_{i+1}} x_i^t + \frac{\phi_i}{\phi_{i+1}} x_{i+1}^{t+1}, \text{ for } i = k+1, \ldots, n-1.
\end{aligned}$$

Since $x_n^t = 1$ for all t,

$$x_{n-1}^{t+1} = x_{n-2}^t + \frac{\phi_{n-1}}{\phi_n}\left(1 - x_{n-1}^t\right).$$

Substituting x_{i+1}^{t+1} into the equation for x_i^{t+1}, for $i = n-2$ to $i = 1$ yields:

$$\begin{aligned}
x_i^{t+1} &= x_{i-1}^t + \frac{\psi_i}{\phi_n}\left(1 - x_{n-1}^t\right), \text{ for } i = 1, \ldots, k-1;\\
x_k^{t+1} &= \frac{\phi_k}{\psi_k} x_{k-1}^t + \frac{\phi_k}{\phi_n}\left(1 - x_{n-1}^t\right) - \frac{\phi_k}{2}\left(\frac{1}{\psi_k} - \frac{1}{\phi_k}\right);\\
x_i^{t+1} &= x_{i-1}^t + \frac{\phi_i}{\phi_n}\left(1 - x_{n-1}^t\right), \text{ for } i = k+1, \ldots, n-1.
\end{aligned}$$

It follows by simple algebra that for $i = 1, \ldots, k-1$,

$$\begin{aligned}
x_i^{t+1} - x_{i-1}^{t+1} &= x_{i-1}^t - x_{i-2}^t + (\psi_i - \psi_{i-1})\frac{1}{\phi_n}\left(1 - x_{n-1}^t\right),\\
\frac{x_i^{t+1} - x_{i-1}^{t+1}}{\psi_i} &= \left(\frac{\psi_{i-1}}{\psi_i}\right)\frac{x_{i-1}^t - x_{i-2}^t}{\psi_{i-1}} + \left(1 - \frac{\psi_{i-1}}{\psi_i}\right)\frac{1 - x_{n-1}^t}{\phi_n}. \qquad (8)
\end{aligned}$$

For $i = k$,

$$\frac{x_k^{t+1}}{\phi_k} - \frac{x_{k-1}^{t+1}}{\psi_k} = \left(\frac{\psi_{k-1}}{\psi_k}\right)\frac{x_{k-1}^t - x_{k-2}^t}{\psi_{k-1}} + \left(1 - \frac{\psi_{k-1}}{\psi_k}\right)\frac{1 - x_{n-1}^t}{\phi_n} - \frac{1}{2}\left(\frac{1}{\psi_k} - \frac{1}{\phi_k}\right),$$

$$\frac{1/2 - x_{k-1}^{t+1}}{\psi_k} + \frac{x_k^{t+1} - 1/2}{\phi_k} = \left(\frac{\psi_{k-1}}{\psi_k}\right)\frac{x_{k-1}^t - x_{k-2}^t}{\psi_{k-1}} + \left(1 - \frac{\psi_{k-1}}{\psi_k}\right)\frac{1 - x_{n-1}^t}{\phi_n}. \tag{9}$$

For $i = k + 1$,

$$\begin{aligned} x_{k+1}^{t+1} - x_k^{t+1} &= x_k^t - \frac{\phi_k}{\psi_k}x_{k-1}^t + (\phi_{k+1} - \phi_k)\frac{1 - x_{n-1}^t}{\phi_n} + \frac{\phi_k}{2}\left(\frac{1}{\psi_k} - \frac{1}{\phi_k}\right), \\ \frac{x_{k+1}^{t+1} - x_k^{t+1}}{\phi_{k+1}} &= \left(\frac{\phi_k}{\phi_{k+1}}\right)\left(\frac{1/2 - x_{k-1}^t}{\psi_k} + \frac{x_k^t - 1/2}{\phi_k}\right) + \left(1 - \frac{\phi_k}{\phi_{k+1}}\right)\frac{1 - x_{n-1}^t}{\phi_n}. \end{aligned} \tag{10}$$

Similarly, for $i = k + 2, \ldots, n$,

$$\begin{aligned} x_i^{t+1} - x_{i-1}^{t+1} &= x_{i-1}^t - x_{i-2}^t + (\phi_i - \phi_{i-1})\frac{1 - x_{n-1}^t}{\phi_n}, \\ \frac{x_i^{t+1} - x_{i-1}^{t+1}}{\phi_i} &= \left(\frac{\phi_{i-1}}{\phi_i}\right)\frac{x_{i-1}^t - x_{i-2}^t}{\phi_{i-1}} + \left(1 - \frac{\phi_{i-1}}{\phi_i}\right)\frac{1 - x_{n-1}^t}{\phi_n}. \end{aligned} \tag{11}$$

Let

$$\begin{aligned} y_i^t &= \frac{x_i^t - x_{i-1}^t}{\psi_i}, \text{ for } i = 1, \ldots, k-1; \\ y_k^t &= \frac{1/2 - x_{k-1}^t}{\psi_k} + \frac{x_k^t - 1/2}{\phi_k}; \\ y_i^t &= \frac{x_i^t - x_{i-1}^t}{\phi_i}, \text{ for } i = k+1, \ldots, n. \end{aligned}$$

Equations (8–11) become

$$\begin{aligned} y_i^{t+1} &= \left(\frac{\psi_{i-1}}{\psi_i}\right)y_{i-1}^t + \left(1 - \frac{\psi_{i-1}}{\psi_i}\right)y_n^t, \text{ for } i = 1, \ldots, k; \\ y_i^{t+1} &= \left(\frac{\phi_{i-1}}{\phi_i}\right)y_{i-1}^t + \left(1 - \frac{\phi_{i-1}}{\phi_i}\right)y_n^t, \text{ for } i = k+1, \ldots, n. \end{aligned}$$

These equations can be expressed as a linear system

$$\mathbf{y}^{t+1} = A\mathbf{y}^t,$$

where $\mathbf{y}^t = \left(y_1^t, y_2^t, \ldots, y_n^t\right)^T$ and A is a transition matrix of a finite state Markov chain. Since the Markov chain is irreducible and aperiodic (see Ross 1996), $A^t \to A^*$ as $t \to \infty$. Thus, the orbit $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2, \ldots$ converges to the unique fixed point $\mathbf{y}^*$. It can be shown that the hand-off points converge to the fixed point $\mathbf{x}^*$ by simple algebra. □

A.5 Analysis of generalized cellular bucket brigades

We fold the assembly line at point 1/2 so that the two halves of the line form an aisle with width a in between. If $\beta \leq 1/2$ each worker i works forward with velocities v_i and u_i in the intervals $[0, \beta)$ and $[\beta, 1/2]$, respectively, along the aisle, and works backward with velocity u_i. If $\beta > 1/2$ each worker i works forward with velocity v_i and works backward with velocities u_i and v_i in the intervals $[0, 1-\beta]$ and $(1-\beta, 1/2]$, respectively, along the aisle. Define

$$\mu_i = \begin{cases} (2/u_i)^{-1}, & \text{if } \beta \leq 1/2; \\ (2/v_i)^{-1}, & \text{otherwise;} \end{cases}$$

for $i = 1, \ldots, n$.

We only consider the case with $\beta \leq 1/2$. (The analysis of the case with $\beta > 1/2$ is similar and all the results can be obtained by replacing β with $1-\beta$.) Let worker k be the one that repeatedly crosses point $\beta \in [0, 1/2]$ on a fixed point. Thus, upon the fixed point, worker $i = 1, \ldots, k-1$ repeatedly works on an interval that lies in $[0, \beta)$ with forward velocity v_i and backward velocity u_i, and worker $i = k+1, \ldots, n$ repeatedly works forward and backward with velocity u_i on an interval that lies in $[\beta, 1/2]$. Worker k is the only one that repeatedly works in both intervals $[0, \beta)$ and $[\beta, 1/2]$ with forward velocities v_k and u_k respectively and backward velocity u_k.

It is straightforward to see that there is no blocking when the system operates on a fixed point. However, since each worker i walks with velocity w_i in the generalized model, different workers may spend different amounts of time to cross the aisle. As a result, worker i may cause his predecessor or successor to wait if worker i walks too slow to cross the aisle. The following condition ensures that there exists a unique fixed point $\mathbf{x}^*$ and there is no waiting on $\mathbf{x}^*$: For $i = 1, \ldots, n$,

$$\begin{aligned} &\tfrac{1}{\theta_k} \sum_{j=1}^{k} \theta_j \left(\tfrac{2a}{w_i} - \tfrac{2a}{w_j} \right) + \tfrac{1}{\mu_k} \sum_{j=k+1}^{n} \mu_j \left(\tfrac{2a}{w_i} - \tfrac{2a}{w_j} \right) \leq \beta \tfrac{1}{\theta_k} + (\tfrac{1}{2} - \beta) \tfrac{1}{\mu_k}, \quad i \neq k; \\ &\sum_{j=1}^{k} \theta_j \left(\tfrac{2a}{w_i} - \tfrac{2a}{w_j} \right) + \sum_{j=k+1}^{n} \mu_j \left(\tfrac{2a}{w_i} - \tfrac{2a}{w_j} \right) \leq \tfrac{1}{2}, \quad i = k. \end{aligned} \tag{12}$$

Lemma 3. *If Condition (12) holds, then no waiting occurs in a generalized cellular bucket brigade on a fixed point* $\mathbf{x}^*$*, which is unique and can be determined as follows: If* $\sum_{j=2}^{n} 2\mu_j(2a/w_1 + \beta/\theta_1 - 2a/w_j) \leq 1 - 2\beta$*, then* $k = 1$ *and*

$$
\begin{aligned}
x_1^* &= \frac{\frac{1}{2} - \sum\limits_{j=2}^{n} \mu_j \left[\beta\left(\frac{1}{\theta_1} - \frac{1}{\mu_1}\right) + \left(\frac{2a}{w_1} - \frac{2a}{w_j}\right)\right]}{\frac{1}{\mu_1}\sum\limits_{j=1}^{n} \mu_j}; \\
x_i^* &= x_{i-1}^* + \mu_i \left(\frac{\beta}{\theta_1} + \frac{x_1^* - \beta}{\mu_1} + \frac{2a}{w_1} - \frac{2a}{w_i}\right), \ i = 2, \ldots, n.
\end{aligned} \tag{13}
$$

Otherwise,

$$
\begin{aligned}
x_1^* &= \theta_1 \frac{\frac{1}{2}\frac{1}{\mu_k} + \beta\left(\frac{1}{\theta_k} - \frac{1}{\mu_k}\right) - \left[\frac{1}{\theta_k}\sum\limits_{j=2}^{k} \theta_j \left(\frac{2a}{w_1} - \frac{2a}{w_j}\right) + \frac{1}{\mu_k}\sum\limits_{j=k+1}^{n} \mu_j \left(\frac{2a}{w_1} - \frac{2a}{w_j}\right)\right]}{\frac{1}{\theta_k}\sum\limits_{j=1}^{k} \theta_j + \frac{1}{\mu_k}\sum\limits_{j=k+1}^{n} \mu_j}; \\
x_i^* &= x_{i-1}^* + \theta_i \left(\frac{x_1^*}{\theta_1} + \frac{2a}{w_1} - \frac{2a}{w_i}\right), \ i = 2, \ldots, k-1; \\
x_n^* &= \frac{1}{2}; \\
x_i^* &= x_{i+1}^* - \mu_{i+1} \left(\frac{x_1^*}{\theta_1} + \frac{2a}{w_1} - \frac{2a}{w_{i+1}}\right), \ i = n-1, \ldots, k;
\end{aligned} \tag{14}
$$

where k *is the smallest index such that* $(x_1^*/\theta_1)\sum_{j=1}^{k} \theta_j + \sum_{j=2}^{k} \theta_j(2a/w_1 - 2a/w_j) \geq \beta$*.*

Proof. We first show that if Condition (12) holds then no waiting occurs on a fixed point $\mathbf{x}^*$. We prove by contradiction. Assume Condition (12) holds and suppose there is a worker i, $1 \leq i \leq k-1$, who causes his predecessor or successor to wait (that is, $2a/w_i$ is very large). As a result, $x_{i-1}^* = x_i^*$. Without loss of generality, assume worker $i-1$ waits for worker i and this implies that

$$
\begin{aligned}
\frac{2a}{w_i} &> \frac{a}{w_{i-1}} + \frac{x_{i-1}^* - x_{i-2}^*}{v_{i-1}} + \frac{a}{w_{i-1}} + \frac{x_{i-1}^* - x_{i-2}^*}{u_{i-1}}; \\
x_{i-1}^* &< x_{i-2}^* + \theta_{i-1}\left(\frac{2a}{w_i} - \frac{2a}{w_{i-1}}\right).
\end{aligned} \tag{15}
$$

Since other workers might also wait for their predecessors or successors on the fixed point

$\mathbf{x}^*$, we have

$$\begin{aligned}
\frac{2a}{w_i} &\geq \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{v_j} + \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j}, \quad j = 1, \ldots, i-2; && (16)\\
\frac{2a}{w_i} &\geq \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{v_j} + \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j}, \quad j = i+1, \ldots, k-1; && (17)\\
\frac{2a}{w_i} &\geq \frac{a}{w_j} + \frac{\beta - x_{j-1}^*}{v_j} + \frac{x_j^* - \beta}{u_j} + \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j}, \quad j = k; && (18)\\
\frac{2a}{w_i} &\geq \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j} + \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j}, \quad j = k+1, \ldots, n. && (19)
\end{aligned}$$

Inequalities (16) lead to the following result:

$$x_{i-2}^* \leq \sum_{j=1}^{i-2} \theta_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right).$$

Since $x_n^* = 1/2$, Inequalities (17-19) lead to the following result:

$$x_i^* \geq \beta - \sum_{j=i+1}^{k} \theta_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right) + \frac{\theta_k}{\mu_k} \left[\frac{1}{2} - \beta - \sum_{j=k+1}^{n} \mu_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right) \right].$$

Since $x_{i-1}^* = x_i^*$, combining the last two inequalities with Inequality (15), we have

$$\beta - \sum_{j=i+1}^{k} \theta_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right) + \frac{\theta_k}{\mu_k} \left[\frac{1}{2} - \beta - \sum_{j=k+1}^{n} \mu_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right) \right] < \sum_{j=1}^{i-1} \theta_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right);$$

$$\frac{1}{\theta_k} \sum_{j=1}^{k} \theta_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right) + \frac{1}{\mu_k} \sum_{j=k+1}^{n} \mu_j \left(\frac{2a}{w_i} - \frac{2a}{w_j} \right) > \beta \frac{1}{\theta_k} + \left(\frac{1}{2} - \beta \right) \frac{1}{\mu_k}.$$

This contradicts Condition (12). It can be proved in a similar way that if Condition (12) holds then worker i, $k+1 \leq i \leq n$, will not cause his predecessor or successor to wait on the fixed point $\mathbf{x}^*$.

Now we show that if Condition (12) holds then worker k will not cause his predecessor or successor to wait on the fixed point $\mathbf{x}^*$. We prove by contradiction. Assume Condition (12) holds and suppose worker k causes his predecessor or successor to wait (that is, $2a/w_k$ is very large). As a result, $x_{k-1}^* = x_k^* = \beta$. Without loss of generality, assume worker $k-1$ waits for worker k and this implies that

$$\begin{aligned}
\frac{2a}{w_k} &> \frac{a}{w_{k-1}} + \frac{x_{k-1}^* - x_{k-2}^*}{v_{k-1}} + \frac{a}{w_{k-1}} + \frac{x_{k-1}^* - x_{k-2}^*}{u_{k-1}};\\
x_{k-1}^* &< x_{k-2}^* + \theta_{k-1} \left(\frac{2a}{w_k} - \frac{2a}{w_{k-1}} \right). && (20)
\end{aligned}$$

Since other workers might also wait for their predecessors or successors on the fixed point $\mathbf{x}^*$, we have

$$\frac{2a}{w_k} \geq \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{v_j} + \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j}, \quad j = 1, \ldots, k-2; \tag{21}$$

$$\frac{2a}{w_k} \geq \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j} + \frac{a}{w_j} + \frac{x_j^* - x_{j-1}^*}{u_j}, \quad j = k+1, \ldots, n. \tag{22}$$

Inequalities (21) lead to the following result:

$$x_{k-2}^* \leq \sum_{j=1}^{k-2} \theta_j \left(\frac{2a}{w_k} - \frac{2a}{w_j} \right).$$

Since $x_n^* = 1/2$, Inequalities (22) lead to the following result:

$$x_k^* \geq \frac{1}{2} - \sum_{j=k+1}^{n} \mu_j \left(\frac{2a}{w_k} - \frac{2a}{w_j} \right).$$

Since $x_{k-1}^* = x_k^*$, combining the last two inequalities with Inequality (20), we have

$$\sum_{j=1}^{k-1} \theta_j \left(\frac{2a}{w_k} - \frac{2a}{w_j} \right) > \frac{1}{2} - \sum_{j=k+1}^{n} \mu_j \left(\frac{2a}{w_k} - \frac{2a}{w_j} \right);$$

$$\sum_{j=1}^{k} \theta_j \left(\frac{2a}{w_k} - \frac{2a}{w_j} \right) + \sum_{j=k+1}^{n} \mu_j \left(\frac{2a}{w_k} - \frac{2a}{w_j} \right) > \frac{1}{2}.$$

This contradicts Condition (12).

Thus, if Condition (12) holds there is no waiting on the fixed point $\mathbf{x}^*$ on which each worker i repeats a simple loop for each item assembled: He exchanges work with worker $i-1$ at point x_{i-1}^*, which takes time a/w_i, and exchanges work with worker $i+1$ at point x_i^*, which takes time a/w_i. Recall that the loop of worker k overlaps with both intervals $[0, \beta)$ and $[\beta, 1/2]$. We first assume $k > 1$. The fixed point $\mathbf{x}^*$ in Equations (14) can be found by solving the following equations:

$$\frac{a}{w_1} + \frac{x_1^*}{v_1} + \frac{a}{w_1} + \frac{x_1^*}{u_1} = \frac{a}{w_i} + \frac{x_i^* - x_{i-1}^*}{v_i} + \frac{a}{w_i} + \frac{x_i^* - x_{i-1}^*}{u_i}, \quad i = 2, \ldots, k-1;$$

$$\frac{a}{w_1} + \frac{x_1^*}{v_1} + \frac{a}{w_1} + \frac{x_1^*}{u_1} = \frac{a}{w_k} + \frac{\beta - x_{k-1}^*}{v_k} + \frac{x_k^* - \beta}{u_k} + \frac{a}{w_k} + \frac{x_k^* - x_{k-1}^*}{u_k};$$

$$\frac{a}{w_1} + \frac{x_1^*}{v_1} + \frac{a}{w_1} + \frac{x_1^*}{u_1} = \frac{a}{w_i} + \frac{x_i^* - x_{i-1}^*}{u_i} + \frac{a}{w_i} + \frac{x_i^* - x_{i-1}^*}{u_i}, \quad i = k+1, \ldots, n.$$

The solution follows by simple algebra. Note that if $k = 2$ and the corresponding $x_1^* \geq \beta$, which imply $\sum_{j=2}^{n} 2\mu_j(2a/w_1 + \beta/\theta_1 - 2a/w_j) \leq 1 - 2\beta$, then worker 1 is the one that repeatedly crosses point β. In this case, the fixed point $\mathbf{x}^*$ in Equations (13) can be found by solving the following equations:

$$\frac{a}{w_1} + \frac{\beta}{v_1} + \frac{x_1^* - \beta}{u_1} + \frac{a}{w_1} + \frac{x_1^*}{u_1} \quad = \quad \frac{a}{w_i} + \frac{x_i^* - x_{i-1}^*}{u_i} + \frac{a}{w_i} + \frac{x_i^* - x_{i-1}^*}{u_i}, \quad i = 2, \ldots, n.$$

The solution follows by simple algebra. □

Note that if Condition (12) does not hold then a cellular bucket brigade may have multiple fixed points. These fixed points depend on the initial state of the system (initial locations and directions of workers). The production capacity of the system is not fully used because waiting occurs on these fixed points.

The following lemma determines the throughput of a cellular bucket brigade when it operates on its fixed point.

Lemma 4. *The average throughput of a generalized cellular bucket brigade on the fixed point* $\mathbf{x}^*$ *is*

$$\rho_c \quad = \quad \begin{cases} \left(\frac{\beta}{\theta_1} + \frac{x_1^* - \beta}{\mu_1} + \frac{2a}{w_1}\right)^{-1}, & \textit{if } k = 1; \\ \left(\frac{x_1^*}{\theta_1} + \frac{2a}{w_1}\right)^{-1}, & \textit{otherwise.} \end{cases}$$

Proof. On the fixed point the system assembles an item every time worker 1 completes a simple loop. If $k = 1$ the average throughput is

$$\rho_c = \left(\frac{a}{w_1} + \frac{\beta}{v_1} + \frac{x_1^* - \beta}{u_1} + \frac{a}{w_1} + \frac{x_1^*}{u_1}\right)^{-1};$$

otherwise,

$$\rho_c = \left(\frac{a}{w_1} + \frac{x_1^*}{v_1} + \frac{a}{w_1} + \frac{x_1^*}{u_1}\right)^{-1}.$$

□

The following theorem shows that if Condition (12) holds, then a cellular bucket brigade may be configured such that its unique fixed point $\mathbf{x}^*$ is an attractor.

Theorem 3. *The fixed point* $\mathbf{x}^*$ *of a generalized cellular bucket brigade is an attractor if Condition (12) holds and*

$$k > 1 \text{ with } \frac{1}{v_1} - \frac{1}{u_1} > \frac{1}{v_2} - \frac{1}{u_2} > \cdots > \frac{1}{v_{k-1}} - \frac{1}{u_{k-1}} > \frac{1}{v_k} - \frac{1}{u_k}. \tag{23}$$

Proof. Condition (12) implies the fixed point $\mathbf{x}^*$ is unique. We will show that if Condition (23) holds, then the fixed point $\mathbf{x}^*$ is at least a local attractor: If the system is sufficiently close to the fixed point such that there is only one worker constantly crosses point β and neither blocking nor waiting occurs in the system, then the system will converge to the fixed point.

Iteration t follows the sequence of hand-off points from the end to the start of the aisle caused by the t-th reset. Assume the system operates sufficiently close to the fixed point $\mathbf{x}^*$ such that only worker k crosses point β and neither blocking nor waiting occurs in each iteration. A hand-off occurs at point x_i^t where worker i, who is working forward, meets worker $i+1$, who is working backward. Since the time spent by each worker from one iteration to the next is the same, we have

$$\begin{aligned}
\frac{2a}{w_i} + \frac{x_i^t - x_{i-1}^t}{u_i} + \frac{x_i^{t+1} - x_{i-1}^t}{v_i} &= \frac{2a}{w_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^t}{v_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^{t+1}}{u_{i+1}}, \\
&\qquad\qquad i = 1, \ldots, k-2; \\
\frac{2a}{w_i} + \frac{x_i^t - x_{i-1}^t}{u_i} + \frac{x_i^{t+1} - x_{i-1}^t}{v_i} &= \frac{2a}{w_{i+1}} + \frac{\beta - x_i^t}{v_{i+1}} + \frac{x_{i+1}^{t+1} - \beta}{u_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^{t+1}}{u_{i+1}}, \\
&\qquad\qquad i = k-1; \\
\frac{2a}{w_i} + \frac{x_i^t - x_{i-1}^t}{u_i} + \frac{\beta - x_{i-1}^t}{v_i} + \frac{x_i^{t+1} - \beta}{u_i} &= \frac{2a}{w_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^t}{u_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^{t+1}}{u_{i+1}}, \quad i = k; \\
\frac{2a}{w_i} + \frac{x_i^t - x_{i-1}^t}{u_i} + \frac{x_i^{t+1} - x_{i-1}^t}{u_i} &= \frac{2a}{w_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^t}{u_{i+1}} + \frac{x_{i+1}^{t+1} - x_i^{t+1}}{u_{i+1}}, \\
&\qquad\qquad i = k+1, \ldots, n-1.
\end{aligned}$$

Rewriting the above equations yields:

$$
\begin{aligned}
x_i^{t+1} &= \frac{1/v_i+1/u_i}{1/v_i+1/u_{i+1}}x_{i-1}^t - \frac{1/v_{i+1}+1/u_i}{1/v_i+1/u_{i+1}}x_i^t + \frac{1/v_{i+1}+1/u_{i+1}}{1/v_i+1/u_{i+1}}x_{i+1}^{t+1} + \\
&\quad 2a\frac{1/w_{i+1}-1/w_i}{1/v_i+1/u_{i+1}}, \qquad i=1,\ldots,k-2; \\
x_i^{t+1} &= \frac{1/v_i+1/u_i}{1/v_i+1/u_{i+1}}x_{i-1}^t - \frac{1/v_{i+1}+1/u_i}{1/v_i+1/u_{i+1}}x_i^t + \frac{2/u_{i+1}}{1/v_i+1/u_{i+1}}x_{i+1}^{t+1} + \\
&\quad 2a\frac{1/w_{i+1}-1/w_i}{1/v_i+1/u_{i+1}} + \beta\frac{1/v_{i+1}-1/u_{i+1}}{1/v_i+1/u_{i+1}}, \qquad i=k-1; \\
x_i^{t+1} &= \frac{1/v_i+1/u_i}{1/u_i+1/u_{i+1}}x_{i-1}^t - x_i^t + \frac{2/u_{i+1}}{1/u_i+1/u_{i+1}}x_{i+1}^{t+1} + \\
&\quad 2a\frac{1/w_{i+1}-1/w_i}{1/u_i+1/u_{i+1}} - \beta\frac{1/v_i-1/u_i}{1/u_i+1/u_{i+1}}, \qquad i=k; \\
x_i^{t+1} &= \frac{2/u_i}{1/u_i+1/u_{i+1}}x_{i-1}^t - x_i^t + \frac{2/u_{i+1}}{1/u_i+1/u_{i+1}}x_{i+1}^{t+1} + \\
&\quad 2a\frac{1/w_{i+1}-1/w_i}{1/u_i+1/u_{i+1}}, \qquad i=k+1,\ldots,n-1.
\end{aligned}
$$

Or we can write

$$
\bar{x}_i^{t+1} = (1+\alpha_i)\gamma_i\bar{x}_{i-1}^t - \alpha_i\bar{x}_i^t + (1+\alpha_i)(1-\gamma_i)\bar{x}_{i+1}^{t+1} + \epsilon_i, \tag{24}
$$

for $i=1,\ldots,n-1$, where

$$
\begin{aligned}
\bar{x}_i^t &= \begin{cases} x_i^t, & \text{if } i=1,\ldots,k-1; \\ \frac{\theta_k}{\mu_k}x_i^t, & \text{otherwise;} \end{cases} \\
\alpha_i &= \begin{cases} \frac{1/v_{i+1}+1/u_i}{1/v_i+1/u_{i+1}}, & \text{if } i=1,\ldots,k-1; \\ 1, & \text{otherwise;} \end{cases} \\
\gamma_i &= \begin{cases} \frac{1/\theta_i}{1/\theta_i+1/\theta_{i+1}}, & \text{if } i=1,\ldots,k-1; \\ \frac{1/\mu_i}{1/\mu_i+1/\mu_{i+1}}, & \text{otherwise;} \end{cases} \\
\epsilon_i &= \begin{cases} \frac{2a/w_{i+1}-2a/w_i}{1/v_i+1/u_{i+1}}, & \text{if } i=1,\ldots,k-2; \\ \frac{2a/w_{i+1}-2a/w_i}{1/v_i+1/u_{i+1}} + \beta\frac{1/v_{i+1}-1/u_{i+1}}{1/v_i+1/u_{i+1}}, & \text{if } i=k-1; \\ \frac{\theta_k}{\mu_k}\left(\frac{2a/w_{i+1}-2a/w_i}{1/u_i+1/u_{i+1}} - \beta\frac{1/v_i-1/u_i}{1/u_i+1/u_{i+1}}\right), & \text{if } i=k; \\ \frac{\theta_k}{\mu_k}\left(\frac{2a/w_{i+1}-2a/w_i}{1/u_i+1/u_{i+1}}\right), & \text{otherwise.} \end{cases}
\end{aligned}
$$

Note that Condition (23) implies $0<\alpha_i<1$, $i=1,\ldots,k-1$.

Equations (24) can be expressed as an affine system:

$$\mathbf{y}^{t+1} \quad = \quad A\mathbf{y}^t + \mathbf{b},$$

where $\mathbf{y}^t = \left(\bar{x}_1^t, \bar{x}_2^t, \ldots, \bar{x}_{n-2}^t, \bar{x}_{n-1}^{t+1}\right)^T$. The first $n-2$ components of the vector $\mathbf{y}^t$ correspond to the last $n-2$ hand-offs of iteration t and the last component corresponds to the first hand-off of iteration $t+1$. The matrix A can be factored as $A = A_{n-1}A_1A_2 \ldots A_{n-2}$, where each matrix A_i updates $\bar{x}_i^t$ according to Equations (24), and

$$\mathbf{b} \quad = \quad A_{n-1}\left[\sum_{i=1}^{n-2}\left(\prod_{j=1}^{i-1} A_j\right)\mathbf{b}_i\right] + \mathbf{b}_{n-1},$$

where $\mathbf{b}_i$ is a zero vector except for the i-th component, which equals ϵ_i, for $i = 1, \ldots, n-2$, and $\mathbf{b}_{n-1}$ is also a zero vector except for the $(n-1)$-st component, which equals $\epsilon_{n-1} + (1 + \alpha_{n-1})(1 - \gamma_{n-1})(\theta_k/\mu_k)/2$. In this way we first update $\bar{x}_{n-2}^t$, then $\bar{x}_{n-3}^t$, and so on until $\bar{x}_1^t$, and then finally $\bar{x}_{n-1}^{t+1}$.

Each matrix A_i is an identity matrix except for row i. Each $A_2, A_3, \ldots, A_{n-2}$ has three non-zero terms in row i that sum to 1, with values $(1 + \alpha_i)\gamma_i$, $-\alpha_i$, and $(1 + \alpha_i)(1 - \gamma_i)$ in columns $i-1$, i, and $i+1$ respectively. For A_1 the first term $(1 + \alpha_1)\gamma_1 > 0$ is omitted from row 1, and thus the sum of the first row has absolute value less than 1. For A_{n-1} the last term $(1 + \alpha_{n-1})(1 - \gamma_{n-1}) > 0$ is omitted from row $n-1$, and thus the sum of the last row has absolute value less than or equal to 1.

For the full transition matrix A, all eigenvalues have modulus less than one. In short, this follows because each $A_2, A_3, \ldots, A_{n-2}$ can be replaced by a stochastic matrix, A_1 can be replaced by a strictly sub-stochastic matrix, and A_{n-1} can be replaced by a strictly sub-stochastic matrix or a stochastic matrix. Since all states communicate, the system tends to the zero matrix. Thus, the orbit $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2, \ldots$ converges to the unique fixed point $\mathbf{y}^*$ of hand-off locations. $\square$

The proof of Theorem 3 shows that the fixed point $\mathbf{x}^*$ is at least a local attractor: If the system is sufficiently close to the fixed point, then it will converge to the fixed point. An extensive set of simulations suggest that the fixed point $\mathbf{x}^*$ is also a global attractor: If Conditions (12) and (23) hold, then the system always converges to the fixed point given any initial state. Note that $k = 1$ and thus Condition (23) does not hold if β is too small (or if β is too large for the case with $\beta > 1/2$).

A.6 Analysis of generalized serial bucket brigades

Define $\psi_i = (1/v_i + 1/w_i)^{-1}$ and $\phi_i = (1/u_i + 1/w_i)^{-1}$, for $i = 1, \ldots, n$. Let worker k be the one that repeatedly crosses point $\beta \in [0, 1]$ on a fixed point. Thus, upon the fixed point, worker $i = 1, \ldots, k-1$ repeatedly works on an interval that lies in $[0, \beta)$ with forward velocity v_i and backward velocity w_i, and worker $i = k+1, \ldots, n$ repeatedly works on an interval that lies in $[\beta, 1]$ with forward velocity u_i and backward velocity w_i . Worker k is the only one that repeatedly works in both intervals $[0, \beta)$ and $[\beta, 1]$ with forward velocities v_k and u_k respectively and backward velocity w_k.

Lemma 5 identifies a fixed point of hand-off locations for a serial bucket brigade.

Lemma 5. *For a generalized serial bucket brigade there exists a unique fixed point* $\mathbf{x}^*$*, which can be determined as follows: If* $\beta/\psi_1 \leq (1-\beta)/\sum_{j=2}^{n} \phi_j$*, then* $k = 1$ *and*

$$\begin{aligned} x_1^* &= \frac{1-\beta(1/\psi_1 - 1/\phi_1)\sum_{j=2}^{n}\phi_j}{(1/\phi_1)\sum_{j=1}^{n}\phi_j}; \\ x_i^* &= x_{i-1}^* + \phi_i\left[\beta/\psi_1 + (x_1^* - \beta)/\phi_1\right], \ i = 2, \ldots, n. \end{aligned} \tag{25}$$

Otherwise,

$$\begin{aligned} x_1^* &= \psi_1 \frac{1/\phi_k + \beta(1/\psi_k - 1/\phi_k)}{(1/\psi_k)\sum_{j=1}^{k}\psi_j + (1/\phi_k)\sum_{j=k+1}^{n}\phi_j}; \\ x_i^* &= x_{i-1}^* + \psi_i(x_1^*/\psi_1), \ i = 2, \ldots, k-1; \\ x_n^* &= 1; \\ x_i^* &= x_{i+1}^* - \phi_{i+1}(x_1^*/\psi_1), \ i = n-1, \ldots, k; \end{aligned} \tag{26}$$

where k *is the smallest index such that* $(x_1^*/\psi_1)\sum_{j=1}^{k}\psi_j \geq \beta$.

Proof. When the system operates on the fixed point $\mathbf{x}^*$, each worker i repeats a simple loop for each item assembled: He takes over work from worker $i-1$ at point x_{i-1}^* and passes work to worker $i+1$ at point x_i^*. Recall that the loop of worker k overlaps with both intervals $[0, \beta)$ and

$[\beta, 1]$. We first assume $k > 1$. The fixed point $\mathbf{x}^*$ in Equations (26) can be found by solving the following equations:

$$
\begin{aligned}
\frac{x_1^*}{v_1} + \frac{x_1^*}{w_1} &= \frac{x_i^* - x_{i-1}^*}{v_i} + \frac{x_i^* - x_{i-1}^*}{w_i}, \quad i = 2, \ldots, k-1; \\
\frac{x_1^*}{v_1} + \frac{x_1^*}{w_1} &= \frac{\beta - x_{k-1}^*}{v_k} + \frac{x_k^* - \beta}{u_k} + \frac{x_k^* - x_{k-1}^*}{w_k}; \\
\frac{x_1^*}{v_1} + \frac{x_1^*}{w_1} &= \frac{x_i^* - x_{i-1}^*}{u_i} + \frac{x_i^* - x_{i-1}^*}{w_i}, \quad i = k+1, \ldots, n.
\end{aligned}
$$

The solution follows by simple algebra. Note that if $k = 2$ and the corresponding $x_1^* \geq \beta$, which imply $\beta/\psi_1 \leq (1-\beta)/\sum_{j=2}^n \phi_j$, then worker 1 is the one that repeatedly crosses point β. In this case, the fixed point $\mathbf{x}^*$ in Equations (25) can be found by solving the following equations:

$$
\frac{\beta}{v_1} + \frac{x_1^* - \beta}{u_1} + \frac{x_1^*}{w_1} = \frac{x_i^* - x_{i-1}^*}{u_i} + \frac{x_i^* - x_{i-1}^*}{w_i}, \quad i = 2, \ldots, n.
$$

The solution follows by simple algebra. □

The following lemma determines the throughput of a serial bucket brigade when it operates on its fixed point.

Lemma 6. *The average throughput of a generalized serial bucket brigade on the fixed point* $\mathbf{x}^*$ *is*

$$
\rho_s = \begin{cases} [\beta/\psi_1 + (x_1^* - \beta)/\phi_1]^{-1}, & \text{if } k = 1; \\ (x_1^*/\psi_1)^{-1}, & \text{otherwise.} \end{cases}
$$

Proof. On the fixed point the system assembles an item every time worker 1 completes a simple loop. If $k = 1$ the average throughput is $\rho_s = [\beta/v_1 + (x_1^* - \beta)/u_1 + x_1^*/w_1]^{-1}$; otherwise, $\rho_s = (x_1^*/v_1 + x_1^*/w_1)^{-1}$. □

The following theorem shows that a serial bucket brigade may be configured such that its unique fixed point $\mathbf{x}^*$ is an attractor.

Theorem 4. *The fixed point* $\mathbf{x}^*$ *of a generalized serial bucket brigade is an attractor if*

$$\begin{aligned}&\tfrac{1}{v_1}-\tfrac{1}{w_1}>\tfrac{1}{v_2}-\tfrac{1}{w_2}>\cdots>\tfrac{1}{v_{k-1}}-\tfrac{1}{w_{k-1}}>\tfrac{1}{v_k}-\tfrac{1}{w_k},\ \textit{and}\\ &\tfrac{1}{u_k}-\tfrac{1}{w_k}>\tfrac{1}{u_{k+1}}-\tfrac{1}{w_{k+1}}>\cdots>\tfrac{1}{u_{n-1}}-\tfrac{1}{w_{n-1}}>\tfrac{1}{u_n}-\tfrac{1}{w_n}.\end{aligned} \tag{27}$$

Proof. We will show that if Condition (27) holds, then the fixed point $\mathbf{x}^*$ is at least a local attractor: If the system is sufficiently close to the fixed point such that there is only one worker constantly crosses point β and there is no blocking in the system, then the system will converge to the fixed point.

Iteration t follows the sequence of hand-off points from the end to the start of the line caused by the t-th reset. Assume the system operates sufficiently close to the fixed point $\mathbf{x}^*$ such that only worker k crosses point β and there is no blocking in each iteration. A hand-off occurs at point x_i^t where worker i, who is working forward, meets worker $i+1$, who is working backward. Since the time spent by each worker from one iteration to the next is the same, we have

$$\begin{aligned}
\frac{x_i^t-x_{i-1}^t}{w_i}+\frac{x_i^{t+1}-x_{i-1}^t}{v_i} &= \frac{x_{i+1}^{t+1}-x_i^t}{v_{i+1}}+\frac{x_{i+1}^{t+1}-x_i^{t+1}}{w_{i+1}}, && i=1,\ldots,k-2;\\
\frac{x_i^t-x_{i-1}^t}{w_i}+\frac{x_i^{t+1}-x_{i-1}^t}{v_i} &= \frac{\beta-x_i^t}{v_{i+1}}+\frac{x_{i+1}^{t+1}-\beta}{u_{i+1}}+\frac{x_{i+1}^{t+1}-x_i^{t+1}}{w_{i+1}}, && i=k-1;\\
\frac{x_i^t-x_{i-1}^t}{w_i}+\frac{\beta-x_{i-1}^t}{v_i}+\frac{x_i^{t+1}-\beta}{u_i} &= \frac{x_{i+1}^{t+1}-x_i^t}{u_{i+1}}+\frac{x_{i+1}^{t+1}-x_i^{t+1}}{w_{i+1}}, && i=k;\\
\frac{x_i^t-x_{i-1}^t}{w_i}+\frac{x_i^{t+1}-x_{i-1}^t}{u_i} &= \frac{x_{i+1}^{t+1}-x_i^t}{u_{i+1}}+\frac{x_{i+1}^{t+1}-x_i^{t+1}}{w_{i+1}}, && i=k+1,\ldots,n-1.
\end{aligned}$$

Rewriting the above equations yields:

$$\begin{aligned}
x_i^{t+1} &= \frac{1/v_i+1/w_i}{1/v_i+1/w_{i+1}}x_{i-1}^t - \frac{1/v_{i+1}+1/w_i}{1/v_i+1/w_{i+1}}x_i^t + \frac{1/v_{i+1}+1/w_{i+1}}{1/v_i+1/w_{i+1}}x_{i+1}^{t+1}, \\
&\hfill i=1,\ldots,k-2; \\
x_i^{t+1} &= \frac{1/v_i+1/w_i}{1/v_i+1/w_{i+1}}x_{i-1}^t - \frac{1/v_{i+1}+1/w_i}{1/v_i+1/w_{i+1}}x_i^t + \frac{1/u_{i+1}+1/w_{i+1}}{1/v_i+1/w_{i+1}}x_{i+1}^{t+1} + \\
& \beta\frac{1/v_{i+1}-1/u_{i+1}}{1/v_i+1/w_{i+1}}, \qquad i=k-1; \\
x_i^{t+1} &= \frac{1/v_i+1/w_i}{1/u_i+1/w_{i+1}}x_{i-1}^t - \frac{1/u_{i+1}+1/w_i}{1/u_i+1/w_{i+1}}x_i^t + \frac{1/u_{i+1}+1/w_{i+1}}{1/u_i+1/w_{i+1}}x_{i+1}^{t+1} - \\
& \beta\frac{1/v_i-1/u_i}{1/u_i+1/w_{i+1}}, \qquad i=k; \\
x_i^{t+1} &= \frac{1/u_i+1/w_i}{1/u_i+1/w_{i+1}}x_{i-1}^t - \frac{1/u_{i+1}+1/w_i}{1/u_i+1/w_{i+1}}x_i^t + \frac{1/u_{i+1}+1/w_{i+1}}{1/u_i+1/w_{i+1}}x_{i+1}^{t+1}, \\
&\hfill i=k+1,\ldots,n-1.
\end{aligned}$$

Or we can write

$$\bar{x}_i^{t+1} = (1+\alpha_i)\gamma_i\bar{x}_{i-1}^t - \alpha_i\bar{x}_i^t + (1+\alpha_i)(1-\gamma_i)\bar{x}_{i+1}^{t+1} + \epsilon_i, \tag{28}$$

for $i=1,\ldots,n-1$, where

$$\begin{aligned}
\bar{x}_i^t &= \begin{cases} x_i^t, & \text{if } i=1,\ldots,k-1; \\ \frac{\psi_k}{\phi_k}x_i^t, & \text{otherwise}; \end{cases} \\
\alpha_i &= \begin{cases} \frac{1/v_{i+1}+1/w_i}{1/v_i+1/w_{i+1}}, & \text{if } i=1,\ldots,k-1; \\ \frac{1/u_{i+1}+1/w_i}{1/u_i+1/w_{i+1}}, & \text{otherwise}; \end{cases} \\
\gamma_i &= \begin{cases} \frac{1/\psi_i}{1/\psi_i+1/\psi_{i+1}}, & \text{if } i=1,\ldots,k-1; \\ \frac{1/\phi_i}{1/\phi_i+1/\phi_{i+1}}, & \text{otherwise}; \end{cases} \\
\epsilon_i &= \begin{cases} \beta\frac{1/v_{i+1}-1/u_{i+1}}{1/v_i+1/w_{i+1}}, & \text{if } i=k-1; \\ -\beta\frac{\psi_i}{\phi_i}\frac{1/v_i-1/u_i}{1/u_i+1/w_{i+1}}, & \text{if } i=k; \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}$$

Note that Condition (27) implies $0<\alpha_i<1$, $i=1,\ldots,n-1$.

Equations (28) can be expressed as an affine system:

$$\mathbf{y}^{t+1} = A\mathbf{y}^t + \mathbf{b},$$

where $\mathbf{y}^t = \left(\bar{x}_1^t, \bar{x}_2^t, \ldots, \bar{x}_{n-2}^t, \bar{x}_{n-1}^{t+1}\right)^T$. The first $n-2$ components of the vector $\mathbf{y}^t$ correspond to the last $n-2$ hand-offs of iteration t and the last component corresponds to the first hand-off of iteration $t+1$. The matrix A can be factored as $A = A_{n-1}A_1A_2 \ldots A_{n-2}$, where each matrix A_i updates $\bar{x}_i^t$ according to Equations (28), and

$$\mathbf{b} \quad = \quad A_{n-1}\left(A_1A_2 \cdots A_{k-1}\mathbf{b}_k + A_1A_2 \cdots A_{k-2}\mathbf{b}_{k-1}\right) + \mathbf{b}_{n-1},$$

where $\mathbf{b}_i$ is a zero vector except for the i-th component, which equals ϵ_i, for $i = k-1, k$, and $\mathbf{b}_{n-1}$ is also a zero vector except for the $(n-1)$-st component, which equals $(1+\alpha_{n-1})(1-\gamma_{n-1})(\psi_k/\phi_k)$. In this way we first update $\bar{x}_{n-2}^t$, then $\bar{x}_{n-3}^t$, and so on until $\bar{x}_1^t$, and then finally $\bar{x}_{n-1}^{t+1}$.

Each matrix A_i is an identity matrix except for row i. Each $A_2, A_3, \ldots, A_{n-2}$ has three non-zero terms in row i that sum to 1, with values $(1+\alpha_i)\gamma_i$, $-\alpha_i$, and $(1+\alpha_i)(1-\gamma_i)$ in columns $i-1$, i, and $i+1$ respectively. For A_1 the first term $(1+\alpha_1)\gamma_1 > 0$ is omitted from row 1, and thus the sum of the first row has absolute value less than 1. For A_{n-1} the last term $(1+\alpha_{n-1})(1-\gamma_{n-1}) > 0$ is omitted from row $n-1$, and thus the sum of the last row has absolute value less than 1.

For the full transition matrix A, all eigenvalues have modulus less than one. In short, this follows because each $A_2, A_3, \ldots, A_{n-2}$ can be replaced by a stochastic matrix, while both A_1 and A_{n-1} can be replaced by a strictly sub-stochastic matrix. Since all states communicate, the system tends to the zero matrix. Thus, the orbit $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2, \ldots$ converges to the unique fixed point $\mathbf{y}^*$ of hand-off locations. $\square$

The proof of Theorem 4 shows that the fixed point $\mathbf{x}^*$ is at least a local attractor. An extensive set of simulations suggest that the fixed point $\mathbf{x}^*$ is also a global attractor.