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Asymptotic theory for rough fractional vasicek models

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Abstract

This paper extends the asymptotic theory for the fractional Vasicek model developed in Xiao and Yu (2018) from the case where $H \in (1/2, 1)$ to the case where $H \in (0, 1/2)$. It is found that the asymptotic theory of the persistence parameter (κ) critically depends on the sign of κ. Moreover, if $\kappa > 0$, the asymptotic distribution for the estimator of κ is different when $H \in (0, 1/2)$ from that when $H \in (1/2, 1)$.

JEL Classification: C15, C22, C32.
Keywords: Least squares; Roughness; Strong consistency; Asymptotic distribution

1 Introduction

The fractional Vasicek model (fVm), which is a Vasicek model driven by a fractional Brownian motion (fBM), has found a considerable amount of applications in economics and finance; see Comte and Renault (1998); Comte et al. (2012); Chronopoulou and Viens (2012a,b); Bayer et al. (2016) and references therein. The model is given by

$$dX_t = \kappa (\mu - X_t) \, dt + \sigma dB_t^H, \tag{1.1}$$

where σ is a positive constant, $\mu, \kappa \in \mathbb{R}$, H is the Hurst parameter, and B_t^H is an fBM. In a recent study, based on a continuous record of X_t over a time period of $[0, T]$, Xiao and Yu (2018) developed the long-span asymptotic theory for alternative estimators of κ and μ when H and σ are known and H takes a value in the range of $(1/2, 1)$.

While $H \in (1/2, 1)$ is empirically relevant for many economic time series, recent findings suggest that some time series is better modelled by an fBM with $H \in (0, 1/2)$. For example, Gatheral et al. (2017) showed that the logarithm of realized variance behaves more like an fBm with H near 0.1 than that with H bigger than 0.5, regardless of timescale sampled. Hence, it is important to study Model (1.1) with $H \in (0, 1/2)$.

The present paper extends the asymptotic results of Xiao and Yu (2018) from the case where $H \in (1/2, 1)$ to the case where $H \in (0, 1/2)$. It is found that the asymptotic theory critically depends on the sign of κ, as in Xiao and Yu (2018). However, if $\kappa > 0$ the asymptotic theory for κ is different when $H \in (0, 1/2)$ from that when $H \in (1/2, 1)$.

The rest of the paper is organized as follows. Section 2 introduces the model and discusses the least squares (LS) estimators and the ergodic-type estimators of κ and μ. Section 3 establishes strong consistency and asymptotic distributions for the LS estimators of κ and μ and those of the ergodic-type estimators of κ and μ when $\kappa > 0$. The proofs of the main results are given in Appendix.

We use the following notations throughout the paper. Let $\xrightarrow{p}$, $\xrightarrow{a.s.}$, $\xrightarrow{\mathcal{L}}$ and $\overset{a}{\sim}$ denote convergence in probability, convergence almost surely, convergence in distribution, and asymptotic equivalence, respectively, as $T \to \infty$. Let $\overset{d}{=}$ denote equivalence in distribution.

2 The Model and Estimation Methods

Before introducing the model, we first state some basic facts about the fBm (see Nualart (2006) for more details). An fBm $B^H = \{B_t^H, t \in \mathbb{R}\}$ with the Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following covariance function

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) . \tag{2.1}$$

This covariance function implies that the fBm is self-similar with the self-similarity parameter H, that is, $B_{\lambda t}^H \overset{d}{=} \lambda^H B_t^H$. A direct consequence of (2.1) is that $B_n^H - B_{n-1}^H$ is a discrete-time Gaussian process with a covariance function

$$\begin{aligned} r(n) &= \mathbb{E}\left[\left(B_{k+n}^H - B_{k+n-1}^H\right)\left(B_k^H - B_{k-1}^H\right)\right] \\ &= \frac{1}{2}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right] \overset{a}{\sim} H(2H-1)n^{2H-2} . \end{aligned}$$

By the convexity of the function $g(n) = n^{2H}$, the increments, $B_{k+n}^H - B_{k+n-1}^H$ and $B_k^H - B_{k-1}^H$, are positively correlated if $1/2 < H < 1$. However, the increments are negatively correlated if $0 < H < 1/2$, generating the feature of roughness in the sample path. Thus, B^H is persistent when $1/2 < H < 1$ and antipersistent when $0 < H < 1/2$, If $H = 1/2$, B_t^H becomes a standard Brownian motion W_t. Moreover, if $H \in (1/2, 1)$, $\sum_{n=1}^{\infty} r(n) = \infty$, suggesting that the process exhibits long-range dependence. However, if $H \in (0, 1/2)$, $\sum_{n=1}^{\infty} r(n) < \infty$. Recently Gatheral et al. (2017) showed that the logarithm of variance behaved more like an fBm with H near 0.1 than that with H bigger than 0.5 for variance coming from different financial markets and sampled at different time scales and that an fBm with H near 0.1 can produce empirically more reasonable volatility surface. However, Gatheral et al. (2017) did not provide any asymptotic theory for making statistical inference.

The model concerned in the present paper is given by (1.1). Xiao and Yu (2018) developed the long-span asymptotic theory for κ and μ when $H \in (1/2, 1)$. The goal of the present paper is to extend the results in Xiao and Yu (2018) to the case where $H \in (0, 1/2)$. This extension is important in light of the empirical results in Gatheral et al. (2017). Following Xiao and Yu, we assume that σ and H are known and the whole trajectory of X_t for $t \in [0, T]$ is available.[1] The asymptotic theory is developed by assuming $T \to \infty$.

The LS estimators of κ and μ are,

$$\hat{\kappa}_{LS} = \frac{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}\,, \tag{2.2}$$

$$\hat{\mu}_{LS} = \frac{(X_T - X_0)\int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}\,. \tag{2.3}$$

When $H \in (0, 1/2)$, X_t is no longer a semimartingale. In this case, for $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ to consistently estimate κ and μ, we have to interpret the stochastic integral $\int_0^T X_t dX_t$ carefully. In fact, we interpret it differently when the sign of κ is different. If $\kappa > 0$, we interpret it as a divergence integral; if $\kappa < 0$, we interpret it as a Young integral; if $\kappa = 0$, we can interpret it as either a divergence integral or a Young integral. The asymptotic distribution of $\hat{\kappa}_{LS}$ is different across these three cases.

When $\kappa > 0$, one may use the ergodic-type estimator of Hu and Nualart (2010) to estimate κ and μ, which is given by

$$\hat{\kappa}_{HN} = \left(\frac{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}{T^2\sigma^2 H\Gamma(2H)}\right)^{-\frac{1}{2H}}\,, \tag{2.4}$$

$$\hat{\mu}_{HN} = \frac{1}{T}\int_0^T X_t dt\,. \tag{2.5}$$

Compared with (2.2) and (2.3) which involve the stochastic integral $\int_0^T X_t dX_t$, the ergodic-type estimators in (2.4) and (2.5) do not contain any stochastic integral.

3 Asymptotic Theory

Let us first consider the case when $\kappa > 0$.

[1] With a continuous record, both σ and H can be estimated arbitrarily accurately.

Lemma 3.1. *Let $H \in (0, 1/2)$, $X_0/\sqrt{T} = o_{a.s.}(1)$, $\kappa > 0$ in Model (1.1). As $T \to \infty$,*

$$\frac{1}{T}\int_0^T X_t dt \overset{a.s.}{\to} \mu \,, \tag{3.1}$$

$$\frac{1}{T}\int_0^T X_t^2 dt \overset{a.s.}{\to} \sigma^2\kappa^{-2H} H\Gamma\left(2H\right) + \mu^2 \,, \tag{3.2}$$

$$\frac{1}{T}\int_0^T X_t dX_t \overset{a.s.}{\to} -\sigma^2\kappa^{1-2H} H\Gamma\left(2H\right) \,. \tag{3.3}$$

Theorem 3.1. *Let $H \in (0, 1/2)$, $X_0/\sqrt{T} = o_{a.s.}(1)$, $\kappa > 0$ in Model (1.1). Then, as $T \to \infty$, $\hat{\kappa}_{LS} \overset{a.s.}{\to} \kappa$ and $\hat{\mu}_{LS} \overset{a.s.}{\to} \mu$. Moreover, let $H \in (0, 1/2)$, $X_0/\sqrt{T} = o_p(1)$, $\kappa > 0$ in Model (1.1). Then, as $T \to \infty$,*

$$\sqrt{T}\left(\hat{\kappa}_{LS} - \kappa\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \kappa\delta_{LS}^2\right) , \tag{3.4}$$

$$T^{1-H}\left(\hat{\mu}_{LS} - \mu\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right) , \tag{3.5}$$

where $\delta_{LS}^2 = (4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}$.

Theorem 3.2. *Let $H \in (0, 1/2)$, $X_0/\sqrt{T} = o_{a.s.}(1)$, $\kappa > 0$ in Model (1.1). Then, as $T \to \infty$, $\hat{\kappa}_{HN} \overset{a.s.}{\to} \kappa$ and $\hat{\mu}_{HN} \overset{a.s.}{\to} \mu$. Moreover, let $H \in (0, 1/2)$, $X_0/\sqrt{T} = o_p(1)$, $\kappa > 0$ in Model (1.1). Then, as $T \to \infty$,*

$$\sqrt{T}\left(\hat{\kappa}_{HN} - \kappa\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \kappa\delta_{HN}^2\right) , \tag{3.6}$$

$$T^{1-H}\left(\hat{\mu}_{HN} - \mu\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right) , \tag{3.7}$$

where $\delta_{HN}^2 = \frac{1}{4H^2}\left[(4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}\right]$.

Remark 3.1. *Theorem 3.1 and Theorem 3.2 extend Theorems 3.2 and 3.3 of Xiao and Yu (2018). Combining with Xiao and Yu (2018), we have provided the full coverage of asymptotic laws of the LS estimators and the ergodic-type estimators for κ and μ in fVm for all $H \in (0, 1)$.*

Remark 3.2. *The rate of convergence for $\hat{\kappa}_{HN}$ and $\hat{\kappa}_{LS}$ is the same (i.e. $\sqrt{T}$) and independent of H, but their asymptotic variances depend on H. Since $\lim_{z\to 0} z\Gamma(z) = 1$, $\lim_{H\to 1/2}\delta_{LS}^2 = \lim_{H\to 1/2}\delta_{HN}^2 = 2$, suggesting that, when $H \to 1/2$, $\hat{\kappa}_{LS}$ and $\hat{\kappa}_{HN}$ have the same asymptotic variance of 2κ. In this case, the asymptotic distribution is identical to that in Feigin (1976). When $0 < H < 1/2$, $4H^2 < 1$ and the asymptotic variance of $\hat{\kappa}_{LS}$ is smaller than that of $\hat{\kappa}_{HN}$, suggesting $\hat{\kappa}_{LS}$ is asymptotically more efficient than $\hat{\kappa}_{HN}$. Figure 1 plots δ_{LS}^2 and δ_{HN}^2 as a function of H. It can be seen that as H increases, δ_{LS}^2 monotonically increases while δ_{HN}^2 monotonically decreases. They both converge to 2 when H approaches 1/2. The relative asymptotic efficiency increases as H decreases. When $H = 0.1$ which is an empirically realistic value for H according to Gatheral et al. (2017), the relative asymptotic efficiency is 25, favoring $\hat{\kappa}_{LS}$. This difference is very significant. The direction of relative asymptotic efficiency is different from that in Xiao and Yu (2018) where $\hat{\kappa}_{LS}$ is found to be asymptotically less efficient than $\hat{\kappa}_{HN}$ when $H > 1/2$.*

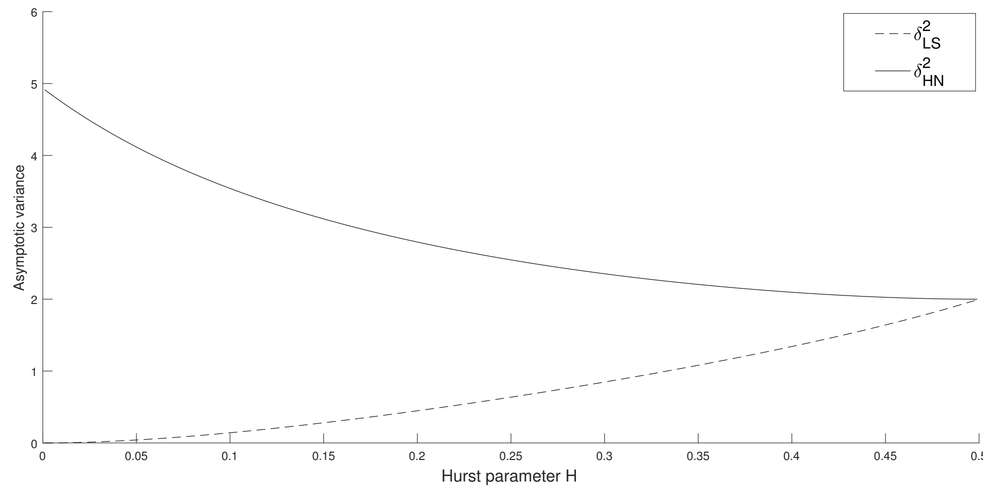

Figure 1. Plots of δ^2_{LS} and δ^2_{HN} as functions of H

Remark 3.3. *Unlike κ, the asymptotic distribution for $\hat{\mu}_{LS}$ is identical to that of $\hat{\mu}_{HN}$, which is also the same as those obtained in Xiao and Yu (2018) when $H > 1/2$.*

Remark 3.4. *When $0 < H < 1/2$, paths generated from an fBm are irregular. In this case, the stochastic integration with respect to fBm should be interpreted as a divergence integral introduced by Cheridito and Nualart (2005). If we interpret the integral $\int_0^T X_t dX_t$ in (2.2) as a Young integral, then, as $T \to \infty$,*

$$\hat{\kappa}_{LS} = \frac{\frac{X_T - X_0}{T} \frac{\int_0^T X_t dt}{T} - \frac{1}{2} \frac{X_T^2 - X_0^2}{T}}{\frac{1}{T} \int_0^T X_t^2 dt - \left(\frac{1}{T} \int_0^T X_t dt \right)^2} \xrightarrow{a.s.} 0 \, , \tag{3.8}$$

by (3.1), (3.2) and (A.23), implying that $\hat{\kappa}_{LS}$ would be inconsistent.

Now, we consider the case where $\kappa < 0$. Applying the Young integral to (2.2) and (2.3), we can rewrite $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ as

$$\begin{aligned} \hat{\kappa}_{LS} &= \frac{(X_T - X_0) \int_0^T X_t dt - \frac{T}{2} (X_T^2 - X_0^2)}{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt \right)^2} \\ &= \frac{\frac{X_T}{T} e^{\kappa T} e^{\kappa T} \int_0^T X_t dt - \frac{X_0}{T} e^{\kappa T} e^{\kappa T} \int_0^T X_t dt - \frac{1}{2} X_T^2 e^{2\kappa T} + \frac{1}{2} X_0^2 e^{2\kappa T}}{e^{2\kappa T} \int_0^T X_t^2 dt - e^{2\kappa T} \frac{1}{T} \left(\int_0^T X_t dt \right)^2} \, , \end{aligned}$$

$$\hat{\mu}_{LS} = \frac{(X_T - X_0) \int_0^T X_t^2 dt - \frac{X_T^2 - X_0^2}{2} \int_0^T X_t dt}{(X_T - X_0) \int_0^T X_t dt - T \frac{X_T^2 - X_0^2}{2}} = \frac{\frac{e^{\kappa T}}{T} \int_0^T X_t^2 dt - \frac{X_T + X_0}{2T} e^{\kappa T} \int_0^T X_t dt}{\frac{e^{\kappa T}}{T} \int_0^T X_t dt - \frac{X_T + X_0}{2} e^{\kappa T}} .$$

Using similar arguments as those in Xiao and Yu (2018), we can obtain asymptotic properties of $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$. In particular, let $H \in (0, 1/2)$, $X_0 = O_p(1)$ and $\kappa < 0$ in Model (1.1). Then, as $T \to \infty$, $\hat{\kappa}_{LS} \stackrel{a.s.}{\to} \kappa$, $\hat{\mu}_{LS} \stackrel{a.s.}{\to} \mu$ and

$$T^{1-H}\left(\hat{\mu}_{LS}-\mu\right) \stackrel{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right), \quad \frac{e^{-\kappa T}}{2\kappa}\left(\hat{\kappa}_{LS}-\kappa\right) \stackrel{\mathcal{L}}{\to} \frac{\sigma \frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\nu}{X_0-\mu+\sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\omega},$$

where ν and ω are two independent standard normal variables. The asymptotic law for $\hat{\kappa}_{LS}$ is a Cauchy-type and is similar to that developed in the explosive discrete-time and continuous-time models in Phillips and Magdalinos (2007); Wang and Yu (2015, 2016). It is the same as that in Xiao and Yu (2018) for the fVm when $H \in (1/2, 1)$.

Finally, we consider the case where $\kappa = 0$. In this case, μ vanishes and the fVm reduces to an fBm without drift. In this case $X_t = X_0 + \sigma B_t^H$. Using the relationship between the divergence integral and the Stratonovich integral and applying the divergence integral to (2.2), we can rewrite the LS estimator of κ as

$$\hat{\kappa}_{1,LS} = \frac{B_T^H \int_0^T B_t^H dt - \frac{T}{2}\left(\left(B_T^H\right)^2 - T^{2H}\right)}{T\int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2},$$

where the equality $\int_0^T B_t^H dB_t^H = \int_0^T B_t^H \circ dB_t^H - \mathbb{E}\left[\int_0^T B_t^H \circ dB_t^H\right] = (B_T^H)^2/2 - \mathbb{E}\left[(B_T^H)^2/2\right] = (B_T^H)^2/2 - T^{2H}/2$ is used. If we interpret $\int_0^T B_t^H dB_t^H$ in (2.2) as a Young integral, then the LS estimator of κ can be rewritten as

$$\hat{\kappa}_{2,LS} = \frac{B_T^H \int_0^T B_t^H dt - \frac{T}{2}\left(B_T^H\right)^2}{T\int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2}.$$

Let $H \in (0, 1/2)$, $X_0 = O_p(1)$, $\kappa = 0$ in (1.1). Then, as $T \to \infty$, using similar arguments as in Theorem 3.6 of Xiao and Yu (2018), we have $\hat{\kappa}_{1,LS} \stackrel{a.s.}{\to} 0$ and $\hat{\kappa}_{2,LS} \stackrel{a.s.}{\to} 0$. Moreover, for any T,

$$T\hat{\kappa}_{1,LS} \stackrel{d}{=} T\hat{\kappa}_{2,LS} \stackrel{d}{=} -\frac{\int_0^1 \overline{B}_u^H dB_u^H}{\int_0^1 \left(\overline{B}_u^H\right)^2 du},$$

where $\overline{B}_u^H = B_u^H - \int_0^1 B_t^H dt$. This is the Dickey-Fuller-Phillips type of distribution of Phillips (1987) and the same as that in Xiao and Yu (2018) for fVm when $H \in (1/2, 1)$.

4 Concluding Remarks

Based on a continuous record of observations with an increasing time span from an fVm with $H < 1/2$, this paper develops asymptotic theory for the two parameters in the drift function, κ and μ. When $\kappa > 0$, two type estimators are considered, the LS estimators and the ergodic-type estimators. When $\kappa = 0$ or < 0, the LS estimators are considered. It is shown that the when $\kappa > 0$, the two estimators of κ and μ are asymptotically normally distributed. However, the LS estimator of κ is asymptotically more efficient than that of the ergodic-type estimator of κ. The relative efficiency is especially large when H takes a value close to zero. When $\kappa < 0$, the LS estimator follows a Cauchy-type distribution asymptotically. When $\kappa = 0$, the LS estimator follows the Dickey-Fuller-Phillips type of distribution.

It is assumed that a continuous record of an increasing time span is available for the development of asymptotic theory. In practice, data is typically discretely sampled at, say $(0, h, 2h, ..., Nh(:= T))$ where h is the sampling interval and T is the time span. When high frequency data over a long span of time period is available, one may consider using a double asymptotic scheme by assuming $h \to 0$ and $T \to \infty$. The discretized model corresponding to (1.1) is given by

$$y_{th} = \mu + \exp(-\kappa h)(y_{(t-1)h} - \mu) + u_t,\ (1 - L)^d u_t = \varepsilon_t,\ t = 1, ..., N,$$

where L is the lag operator, $d = H - 1/2$. As shown in Wang and Yu (2016), under the double asymptotic scheme, $\exp(-\kappa h) = \exp\{-\kappa/k_N\} = 1 - \kappa/k_N + O(k_N^{-2}) \to 1$ where $k_N := 1/h \to \infty$ as $h \to 0$ and $k_N/N = 1/T \to 0$ as $T \to \infty$. This implies an autoregressive (AR) model with an AR root being moderately deviated from unity and with a fractionally integrated error term with $d \in (-1/2, 0)$. This model is closely related to a model considered in Magdalinos (2012) where it is assumed that $d \in (0, 1/2)$. Developing double asymptotic theory based on discretely sampled data will allow one to extend the results of Magdalinos (2012) to the case where $d \in (-1/2, 0)$. This analysis will be reported in later work.

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APPENDIX

A.1. Proof of Lemma 3.1

Proof. For $t \geq 0$, we define

$$Y_t = \sigma \int_{-\infty}^{t} e^{-\kappa(t-s)} dB_s^H \,. \tag{A.1}$$

Cheridito et al. (2003) showed that Y_t is Gaussian, stationary, and ergodic. The integral with respect to fBm exists as a path-wise Riemann-Stieltjes integral, and can be calculated using integration by parts (see Prop. A.1 in Cheridito et al. (2003)). To avoid integration with respect to fBm for $0 < H < 1/2$, using integration by parts and (A.1), we write the solution of (1.1) as

$$X_t = Y_t + \left(1 - e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma\kappa e^{-\kappa t}\int_{-\infty}^{0} e^{\kappa s} B_s^H ds\,. \tag{A.2}$$

Using (A.2), we have

$$\begin{aligned}\frac{1}{T}\int_0^T X_t dt &= \frac{1}{T}\int_0^T Y_t dt + \frac{\mu}{T}\int_0^T \left(1 - e^{-\kappa t}\right) dt + \frac{X_0}{T}\int_0^T e^{-\kappa t} dt \\ &\quad + \frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^{0} e^{\kappa s} B_s^H ds dt\,. \end{aligned}\tag{A.3}$$

For the first term in (A.3), using the ergodic theorem and the fact $\mathbb{E}\left[Y_0\right] = 0$, we obtain

$$\frac{1}{T}\int_0^T Y_t dt \overset{a.s.}{\to} \mathbb{E}\left(Y_0\right) = 0\,. \tag{A.4}$$

For the second term in (A.3), it is obvious that

$$\frac{\mu}{T}\int_0^T \left(1 - e^{-\kappa t}\right) dt \to \mu\,. \tag{A.5}$$

Using the fact that $X_0 = o_{a.s.}(\sqrt{T})$, we can easily obtain

$$\frac{X_0}{T}\int_0^T e^{-\kappa t} dt \overset{a.s.}{\to} 0\,. \tag{A.6}$$

Moreover, a straightforward calculation shows that

$$\begin{aligned} &\mathbb{E}\left[\frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^{0} e^{\kappa s} B_s^H ds dt\right]^2 \\ =\ &\mathbb{E}\left[\frac{\sigma^2\kappa^2}{T^2}\int_0^T e^{-\kappa t} dt\int_0^T e^{-\kappa v} dv\int_{-\infty}^{0} e^{\kappa s} B_s^H ds\int_{-\infty}^{0} e^{\kappa u} B_u^H du\right] \\ =\ &\frac{\sigma^2}{T^2}\left(1 - e^{-\kappa T}\right)^2 \kappa^{-2H-2} H\Gamma(2H). \end{aligned}\tag{A.7}$$

From (A.7), we can easily deduce

$$\left\|\frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^{0} e^{\kappa s} B_s^H ds dt\right\|_{L^2(\Omega)} \leq C T^{-1}, \tag{A.8}$$

where C denotes a suitable positive constant. Consequently, from (A.8) and Lemma 2.1 of Kloeden and Neuenkirch (2007), we obtain

$$\frac{\sigma\kappa}{T}\int_0^T e^{-\kappa t}\int_{-\infty}^{0} e^{\kappa s} B_s^H ds dt \overset{a.s.}{\to} 0\,. \tag{A.9}$$

Substituting (A.4), (A.5), (A.6) and (A.9) into (A.3), we obtain (3.1).

Next, using (A.2), we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T X_t^2 dt &= \frac{1}{T}\int_0^T \left[Y_t + \left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma\kappa e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s} B_s^H ds\right]^2 dt \\ &= \frac{1}{T}\int_0^T Y_t^2 dt + \frac{1}{T}\int_0^T \left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right]^2 dt \\ &\quad + \frac{\sigma^2\kappa^2}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s} B_s^H ds\right)^2 dt + \frac{2}{T}\int_0^T Y_t \left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right] dt \\ &\quad + \frac{2\sigma\kappa}{T}\int_0^T Y_t \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s} B_s^H ds\right) dt \\ &\quad + \frac{2\sigma\kappa}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s} B_s^H ds\right)\left[\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t}\right] dt\,. \qquad (A.10)\end{aligned}$$

Using the ergodic theorem, we can obtain

$$\frac{1}{T}\int_0^T Y_t^2 dt \overset{a.s.}{\to} \mathbb{E}\left(Y_0^2\right)\,. \qquad (A.11)$$

Integrating by parts together with similar arguments as those of (A.7) yields

$$\begin{aligned}\mathbb{E}\left(Y_0^2\right) &= \mathbb{E}\left[\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)^2\right] \\ &= \lim_{T\to-\infty}\mathbb{E}\left[\sigma^2\left(-e^{\kappa T}B_T^H - \kappa\int_T^0 e^{\kappa s} B_s^H ds\right)^2\right] \\ &= \sigma^2\lim_{T\to\infty}\Bigg[e^{-2\kappa T}(-T)^{2H} + \kappa^2\int_0^T\int_0^T e^{-\kappa s}e^{-\kappa t}\mathbb{E}\left[B_{-s}^H B_{-t}^H\right] ds dt \\ &\quad + 2\kappa e^{-\kappa T}\int_0^T e^{-\kappa s}\mathbb{E}\left[B_{-T}^H B_{-s}^H\right] ds\Bigg] \\ &= \sigma^2\kappa^{-2H} H\Gamma(2H). \qquad (A.12)\end{aligned}$$

Combining (A.11) and (A.12), we obtain

$$\frac{1}{T}\int_0^T Y_t^2 dt \overset{a.s.}{\to} \sigma^2\kappa^{-2H} H\Gamma(2H)\,. \qquad (A.13)$$

A straightforward calculation shows that

$$\frac{1}{T}\int_0^T \left[\mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]^2 dt \overset{a.s.}{\to} \mu^2\,. \qquad (A.14)$$

Using similar arguments as those in (A.7), we can obtain

$$\frac{\sigma^2\kappa^2}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s} B_s^H ds\right)^2 dt \overset{a.s.}{\to} 0\,. \qquad (A.15)$$

Moreover, by the Cauchy-Schwarz inequality and the same arguments as those in (A.12),

$$\frac{2}{T}\int_0^T Y_t\left[\left(1-e^{-\kappa t}\right)\mu+X_0e^{-\kappa t}\right]dt \stackrel{a.s.}{\to} 0\,, \tag{A.16}$$

$$\frac{2\sigma\kappa}{T}\int_0^T Y_t\left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right)dt \stackrel{a.s.}{\to} 0\,, \tag{A.17}$$

$$\frac{2\sigma\kappa}{T}\int_0^T \left(e^{-\kappa t}\int_{-\infty}^0 e^{\kappa s}B_s^H ds\right)\left[\left(1-e^{-\kappa t}\right)\mu+X_0e^{-\kappa t}\right]dt \stackrel{a.s.}{\to} 0\,. \tag{A.18}$$

Substituting (A.13)-(A.18) into (A.10), we obtain (3.2).

Now, using (1.1), we can write

$$\frac{1}{T}\int_0^T X_t dX_t = \frac{\kappa\mu}{T}\int_0^T X_t dt - \frac{\kappa}{T}\int_0^T X_t^2 dt + \frac{\sigma}{T}\int_0^T X_t dB_t^H\,. \tag{A.19}$$

Using the relationship between the divergence integral and the Stratonovich integral (see Proposition 5.2.4 in Nualart (2006)), we have

$$\begin{aligned}\frac{\sigma}{T}\int_0^T X_t dB_t^H &= \frac{\sigma}{T}\int_0^T X_t\circ dB_t^H - \mathbb{E}\left[\frac{\sigma}{T}\int_0^T X_t\circ dB_t^H\right] \\ &= \frac{1}{T}\int_0^T X_t\circ\left[dX_t-\kappa\left(\mu-X_t\right)dt\right] - \mathbb{E}\left[\frac{\sigma}{T}\int_0^T X_t\circ dB_t^H\right] \\ &= \frac{X_T^2}{2T} - \frac{\kappa\mu}{T}\int_0^T X_t dt + \frac{\kappa}{T}\int_0^T X_t^2 dt - \mathbb{E}\frac{\sigma}{T}\int_0^T \left(1-e^{-\kappa t}\right)\mu\circ dB_t^H \\ &\quad -\mathbb{E}\left[\frac{\sigma}{T}\int_0^T\left[X_0e^{-\kappa t}+\sigma\left(B_t^H-\kappa\int_0^t B_s^He^{-\kappa(t-s)}ds\right)\right]\circ dB_t^H\right],\end{aligned} \tag{A.20}$$

where $\int_0^T X_t\circ dB_t^H$ denotes the Stratonovich integral.

From Eq. (3.7) of Hu et al. (2018), we can see that

$$\lim_{T\to\infty}\mathbb{E}\left[\frac{\sigma}{T}\int_0^T\left[\sigma\left(B_t^H-\kappa\int_0^t B_s^He^{-\kappa(t-s)}ds\right)\right]\circ dB_t^H\right] = \sigma^2H\kappa^{1-2H}\Gamma(2H). \tag{A.21}$$

A straightforward calculation shows that

$$\lim_{T\to\infty}\mathbb{E}\left[\frac{\sigma}{T}\int_0^T\left[\left(1-e^{-\kappa t}\right)\mu+X_0e^{-\kappa t}\right]\circ dB_t^H\right] = 0\,. \tag{A.22}$$

Using Lemma 18 of Hu et al. (2018), we can see that, for any $\epsilon>0$,

$$\frac{X_T}{T^\epsilon} \stackrel{a.s.}{\to} 0\,. \tag{A.23}$$

Substituting (3.1), (3.2), (A.21)-(A.23) into (A.20), we have

$$\frac{\sigma}{T}\int_0^T X_t dB_t^H \stackrel{a.s.}{\to} 0\,. \tag{A.24}$$

Finally, combining (3.1), (3.2), (A.19) and (A.24), we obtain (3.3). □

A.2. Proof of Theorem 3.1

Proof. From (2.2), we can write $\hat{\kappa}_{LS}$ as

$$\hat{\kappa}_{LS} = \frac{\frac{X_T - X_0}{T}\frac{1}{T}\int_0^T X_t dt - \frac{1}{T}\int_0^T X_t dX_t}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \,. \tag{A.25}$$

By (3.1), (3.2), (3.3), (A.23), (A.25) and the arithmetic rule of convergence, we obtain the almost sure convergence of $\hat{\kappa}_{LS}$ defined in (2.2), i.e., $\hat{\kappa}_{LS} \overset{a.s.}{\to} \kappa$. Now, using (2.3), we can rewrite $\hat{\mu}_{LS}$ as

$$\hat{\mu}_{LS} = \frac{\frac{X_T - X_0}{T}\frac{1}{T}\int_0^T X_t^2 dt - \frac{1}{T}\int_0^T X_t dX_t \frac{1}{T}\int_0^T X_t dt}{\frac{X_T - X_0}{T}\frac{1}{T}\int_0^T X_t dt - \frac{1}{T}\int_0^T X_t dX_t} \,. \tag{A.26}$$

Similarly, using (3.1), (3.2), (3.3), (A.23) and (A.26), we obtain the strong consistency of $\hat{\mu}_{LS}$ defined in (2.3), i.e., $\hat{\mu}_{LS} \overset{a.s.}{\to} \mu$. This proves the first part of the theorem.

To prove the second part, let us first consider (3.4). Based on (1.1), (2.2), (A.2) and integration by parts, we can write

$$\sqrt{T}\left(\hat{\kappa}_{LS} - \kappa\right) = I_1 + I_2 + I_3 \,, \tag{A.27}$$

where

$$\begin{aligned}
I_1 &= \frac{\sigma\left(\frac{\mu - X_0}{\sqrt{T}}\left(e^{-\kappa T}B_T^H + \kappa\int_0^T B_t^H e^{-\kappa t}dt\right) - \frac{\sigma}{\sqrt{T}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right)}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \,, \\
I_2 &= \frac{\left(\frac{X_T - X_0}{\sqrt{T}} + \frac{\kappa(X_0 - \mu)}{\sqrt{T}}\int_0^T e^{-\kappa t}dt - \frac{\sigma}{\sqrt{T}}B_T^H + \frac{\sigma\kappa}{\sqrt{T}}e^{-\kappa T}\int_0^T B_s^H e^{\kappa s}ds\right)\frac{1}{T}\int_0^T X_t dt}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \,, \\
I_3 &= \frac{\left(-\mu\sigma + \frac{\sigma}{T}\int_0^T X_t dt\right)\frac{B_T^H}{\sqrt{T}}}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \,.
\end{aligned}$$

First, by the law of the iterated logarithm for fBm (see e.g. Corollary A1 in Taqqu (1977)), we have

$$\frac{\sigma\left(\mu - X_0\right)}{\sqrt{T}}e^{-\kappa T}B_T^H \overset{a.s.}{\to} 0 \,.$$

Using similar arguments as those in (A.7), we have

$$\frac{\sigma\mu\kappa}{\sqrt{T}}\int_0^T B_t^H e^{-\kappa t}dt \overset{p}{\to} 0 \,.$$

Similarly, using the assumption $X_0/\sqrt{T} = o_p(1)$, we obtain

$$\frac{\sigma X_0\kappa}{\sqrt{T}}\int_0^T B_t^H e^{-\kappa t}dt \overset{p}{\to} 0 \,.$$

Moreover, from Theorem 5 of Hu et al. (2018), we can easily obtain

$$\frac{\sigma^2}{\sqrt{T}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\sigma^4 H^2\kappa^{1-4H}\Gamma^2(2H)\delta_{LS}^2\right).$$

Combining all these convergency results, (3.1), (3.2) and applying Slutsky's theorem

$$I_1 \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\kappa\delta_{LS}^2\right) . \tag{A.28}$$

Using (A.23) and by the law of the iterated logarithm for fBm, we obtain

$$\frac{X_T}{\sqrt{T}} \xrightarrow{a.s.} 0\,,\quad \frac{X_0}{\sqrt{T}} \xrightarrow{p} 0\,,\quad \frac{\kappa\left(X_0-\mu\right)}{\sqrt{T}}\int_0^T e^{-\kappa t}dt \xrightarrow{p} 0\,,\quad \frac{\sigma}{\sqrt{T}}B_T^H \xrightarrow{a.s.} 0\,. \tag{A.29}$$

Furthermore, since $\int_0^\infty e^{-\kappa z}z^{2H-1}dz=\kappa^{-2H}\Gamma(2H)$, we have

$$\begin{aligned}
&\frac{\sigma^2\kappa^2}{T}e^{-2\kappa T}\int_0^T\int_0^T \mathbb{E}\left[B_s^H B_u^H\right]e^{\kappa s}e^{\kappa u}dsdu\\
=\;&\frac{\sigma^2\kappa^2}{T}\frac{e^{-2\kappa T}}{2}\int_0^T\int_0^T e^{\kappa(s+u)}\left(s^{2H}+u^{2H}-|s-u|^{2H}\right)dsdu\\
=\;&\frac{\sigma^2}{T}\kappa e^{-2\kappa T}\left(e^{\kappa T}-1\right)\int_0^T e^{\kappa s}s^{2H}ds-\frac{\sigma^2}{T}\frac{\kappa^2 e^{-2\kappa T}}{2}\int_0^T\int_0^T e^{\kappa(s+u)}|s-u|^{2H}dsdu\\
=\;&\frac{\sigma^2\kappa\left(e^{-\kappa T}-e^{-2\kappa T}\right)}{T}\int_0^T e^{\kappa s}s^{2H}ds-\frac{\sigma^2\kappa}{2T}\int_0^T e^{-\kappa v}v^{2H}dv+\frac{\sigma^2\kappa}{2T}e^{-2\kappa T}\int_0^T e^{\kappa v}v^{2H}dv\\
\to\;& 0\,.
\end{aligned}$$

The result above implies

$$\frac{\sigma\kappa}{\sqrt{T}}e^{-\kappa T}\int_0^T B_s^H e^{\kappa s}ds \xrightarrow{p} 0\,. \tag{A.30}$$

Combining (3.1), (3.2), (A.29) and (A.30), we have

$$I_2 \xrightarrow{p} 0\,. \tag{A.31}$$

Finally, using (3.1), (3.2) and by the law of the iterated logarithm for fBm (see e.g. Corollary A1 in Taqqu (1977)), we have

$$I_3 \xrightarrow{p} 0\,. \tag{A.32}$$

By (A.27), (A.28), (A.31), (A.32) and Slutsky's theorem, we obtain (3.4).

In what follows, we consider (3.5). Using (A.2), we have

$$\begin{aligned}
\frac{1}{T^H}\int_0^T X_t dt-T^{1-H}\mu &= \frac{1}{T^H}\int_0^T\left[\left(1-e^{-\kappa t}\right)\mu+X_0e^{-\kappa t}+\sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right]dt-T^{1-H}\mu\\
&= \frac{X_0-\mu}{T^H}\int_0^T e^{-\kappa t}dt+\frac{\sigma}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dt\\
&= \frac{X_0-\mu}{T^H}\int_0^T e^{-\kappa t}dt+\frac{\sigma B_T^H}{\kappa T^H}-\frac{\sigma}{\kappa T^H}\int_0^T e^{-\kappa(T-s)}dB_s^H\,.
\end{aligned} \tag{A.33}$$

A straightforward calculation shows that

$$\frac{X_0-\mu}{T^H}\int_0^T e^{-\kappa t}dt \overset{a.s.}{\to} 0\,, \tag{A.34}$$

$$\frac{\sigma B_T^H}{\kappa T^H} \overset{\mathcal{L}}{\to} \mathcal{N}\left(0,\frac{\sigma^2}{\kappa^2}\right)\,. \tag{A.35}$$

Moreover, from Lemma 18 of Hu et al. (2018), we can see that

$$\frac{\sigma}{\kappa T^H}\int_0^T e^{-\kappa(T-s)}dB_s^H \overset{a.s.}{\to} 0\,. \tag{A.36}$$

On the other hand, a straightforward calculation shows that

$$T^{1-H}\left(\hat{\mu}_{LS}-\mu\right)=\frac{\frac{X_T-X_0}{T^H}\frac{1}{T}\int_0^T X_t^2dt-\frac{1}{T}\int_0^T X_tdX_t\frac{1}{T^H}\int_0^T X_tdt}{\frac{X_T-X_0}{T}\frac{1}{T}\int_0^T X_tdt-\frac{1}{T}\int_0^T X_tdX_t}-T^{1-H}\mu\,. \tag{A.37}$$

Combining (A.23), (3.1), (3.2), (3.3), (A.33)-(A.37) and Slutsky's theorem, we obtain (3.5). □

A.4. Proof of Theorem 3.2

Proof. Consistency of $\hat{\kappa}_{HN}$ and $\hat{\mu}_{HN}$ can be easily obtained by Lemma 3.1. Moreover, the asymptotic law of $\hat{\mu}_{HN}$ can be obtained by using Slutsky's theorem and (A.33)-(A.36). Hence, we only consider (3.6) here.

Using (2.2) and (2.4), we have

$$\begin{aligned}\sqrt{T}\left(\hat{\kappa}_{HN}-\kappa\right) &= \sqrt{T}\left(\hat{\kappa}_{HN}-\kappa^{1-\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}}+\kappa^{1-\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}}-\kappa\right)\\ &= \sqrt{T}\hat{\kappa}_{LS}^{\frac{1}{2H}}\left[\left(\frac{\sigma^2H\Gamma\left(2H\right)}{\frac{X_T-X_0}{T}\frac{1}{T}\int_0^T X_tdt-\frac{1}{T}\int_0^T X_tdX_t}\right)^{\frac{1}{2H}}-\kappa^{1-\frac{1}{2H}}\right]\\ &\quad+\sqrt{T}\kappa^{1-\frac{1}{2H}}\left(\hat{\kappa}_{LS}^{\frac{1}{2H}}-\kappa^{\frac{1}{2H}}\right)\,.\end{aligned} \tag{A.38}$$

By Theorem 3.1 and the delta method, we get

$$\sqrt{T}\left(\hat{\kappa}_{LS}^{\frac{1}{2H}}-\kappa^{\frac{1}{2H}}\right)\overset{\mathcal{L}}{\to}\mathcal{N}\left(0,\kappa^{\frac{1}{H}-1}\delta_{HN}^2\right)\,. \tag{A.39}$$

Using (3.1), (3.3), (A.23), (A.38), (A.39) and Slutsky's theorem, we obtain (3.6). □