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Favorite–longshot bias in pari-mutuel betting: An evolutionary explanation☆

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A B S T R A C T

Favorite–longshot bias (FLB) refers to an observed tendency whereby "longshots" are overvalued and favorites are undervalued. We offer an evolutionary explanation for FLB in pari-mutuel betting using a simple market model. A bettor is forced to quit with some probability if his total net gain in one day is negative. Because of a positive track take, the expected returns of any strategy are negative, and so every agent must eventually lose and disappear in the long run. Those who favor longshots have a better chance of getting ahead with rare but large gains, enabling them to survive for longer than those who bet on favorites. This relative advantage results in overvaluation of longshots in the long run.

Keywords:
Favorite–longshot bias
Market selection
Competitive evolutionary market
Profit maximization
Risk loving

1. Introduction

Favorite–longshot bias (FLB) refers to an observed tendency whereby "longshots" are overvalued and favorites are undervalued in various market settings. Griffith (1949) first reported that the realized average rates of return from betting on favorites tend to be robustly and significantly greater than those from betting on long shots in American horse races. A number of empirical studies have documented that FLB emerges at racetracks in different countries, and in other kinds of gambling markets as well.[1] Moreover, FLB is not restricted to gambling markets. Plott et al. (2003), Axelrod et al. (2009), and Koessler et al. (2012) investigated pari-mutuel systems experimentally to observe FLB, and Kelly et al. (2012) studied a data set from the Hurricane Futures Market to find FLB.

Intrigued and inspired by these findings, researchers have proposed several theoretical explanations.[2] In early studies, such as Griffith (1949), FLB is deemed to be a consequence of the misperception of probabilities or the irrationality of agents. Subsequent studies relate FLB to particular features of agents' preferences. For instance, Weitzman (1965) considers

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[1] See Thaler and Ziemba (1988) and Hausch and Ziemba (1995) for surveys. Snowberg and Wolfers (2010) found FLB in a very large data set of American horse races.

[2] Ottaviani and Sørensen (2008) is an excellent review of theoretical attempts.

preferences for risk taking and Ali (1977) relies on the heterogeneity of beliefs.[3] Recent studies, such as Shin (1991), Hurley and McDonough (1995), and Ottaviani and Sørensen (2010), focus on strategic behavior among both bettors and bookmakers under asymmetric information.

American racetracks use a pari-mutuel betting system. It has been pointed out that the odds determined in such a system can be regarded as competitive market prices.[4] FLB therefore challenges the classic idea of natural selection of profit maximizers in competitive markets articulated by several authors such as Alchian (1977) and Friedman (1953). If the market is dominated by arbitragers who seek expected profits, the law of demand will adjust prices to equate the expected returns from investment opportunities. In the long run, only profit maximizers will survive. Friedman (1953) states: "Let the apparent immediate determinant of business behavior be anything at all – habitual reaction, random chance, or whatnot. Whenever this determinant happens to lead to behavior consistent with rational and informed maximization of returns, the business will prosper and acquire resources with which to expand; whenever it does not, the business will tend to lose resources and can be kept in existence only by the addition of resources from outside."[5] That is, the wealth of de facto profit-maximizing investors will eventually grow to dominate the market, and hence the market will behave *as if* it consists solely of profit-maximizing traders in the long run.

The market selection process fueled by the growth of wealth was formally modeled by Blume and Easley (1992), and then further examined by several other works that followed. If the same logic is applied to pari-mutuel betting markets, those who happen to favor lower than average returns, irrespective of the underlying reasons, e.g., misperception, risk preferences, beliefs, information, or strategic behavior, will be driven out of the markets, and thus FLB will not be observed in the long run. The previous theoretical works do not explain why these characteristics are relevant to the growth dynamics.

The aim of this paper is to understand why the compelling idea of market selection does not work. That is, we want to reconcile the logic of wealth dynamics with FLB to explain why agents who choose low expected returns tend to matter in the long run. For this purpose, we establish a simple evolutionary model of pari-mutuel betting markets for horse races. There are two horses in every race, and one has a higher probability of winning than the other. There is a continuum of price-taking agents who are expected utility maximizers.[6] Agents' preferences belong to one of two types, where one is more risk loving than the other. We will analyze more general preferences, but for now, to make the exposition simpler, let us assume that all agents know the objective winning probabilities, and one type is risk neutral and the other is risk loving.

Each agent bets one unit, and after the track take is subtracted, the remaining pool of bets is paid out to the winning bettors. The resulting odds can be seen as market clearing prices, and in a competitive equilibrium, all risk-neutral agents bet on the favorite while all risk-loving agents bet on the longshot if FLB is exhibited. Thus, the risk-neutral agents tend to gain more than the risk-loving agents on average because they never bet on an overvalued horse. However, note that because of the track take, the wealth of neither type grows if the bias is only moderate. Every gambler loses in the long run. Hence, the long-run growth of wealth does not constitute a reasonable criterion for survival. This is the crucial point of departure.

We therefore postulate the following criterion: an agent is forced to exit the market with some probability if he does not win more than some specified amount. For the sake of exposition, let us assume that the specified amount is 0; that is, after betting on some races, if an agent's total net gain is positive at the end of the day, he will return to the racetrack the next day with a fresh bank. However, if not, no matter what his past experiences on the races might be, the agent will think about quitting, and there is some probability that he actually does quit. Then, there will be new agents from a pool of potential agents to replace those who quit, similar to the addition of resources from outside, and the next day of races begins.

We show that in this simple environment, FLB is exhibited in the long run, under reasonable assumptions. The basic idea is very simple. Suppose FLB does not occur, so that the expected net returns from the favorite and the long shot are the same. However, this equality implies that the common return is negative because of the track take. Note that this also implies that the *variance* in returns is larger for the longshot than for the favorite. Then, the probability of a positive return is larger for the longshot bet because of the higher variance. For instance, in an environment where a one dollar bet results in a return of 80 cents on average, if one keeps betting on a horse that wins with a very high probability, the realized average returns per race will be concentrated around 80 cents. Therefore, the chance of making more than one dollar on average is very small. Conversely, if one keeps betting on a long shot, the return will still be 80 cents on average, but the returns are more spread out because of occasional large gains (see, Fig. 1 for illustration). Therefore, the chance of achieving a return of more than one dollar per race is larger. Hence without FLB, the risk-loving agents are a better fit to the environment under our criterion. FLB must occur to reduce the market fitness of risk-loving agents so that both types are equally fit in a long-run steady state.

We emphasize that both types survive in the long run, and it is the heterogeneity of the preferences of bettors that induces FLB in a steady state. The heterogeneity in our model can arise from various combinations of heterogeneous risk attitudes and beliefs. Therefore, in principle our analysis accommodates the static analysis relying on heterogeneous risk attitudes such as Weitzman (1965) and Quandt (1986), as well as those relying on heterogeneous beliefs such as Ali (1977), Blough

[3] See also Snowberg and Wolfers (2010).

[4] See Snyder (1978).

[5] Friedman (Friedman, 1953), page 22.

[6] The model can accommodate more general non-expected utility maximizers.

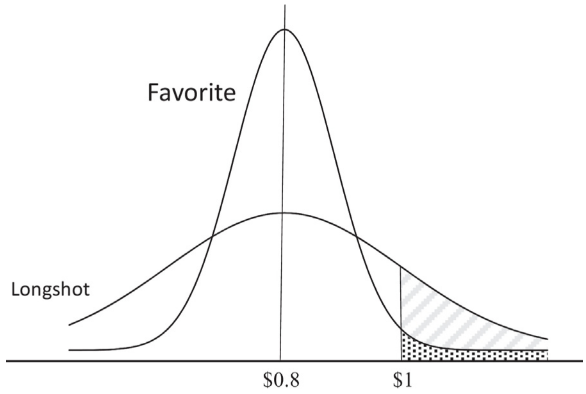

Fig. 1. Distribution of average returns without FLB.

(1994) and Watanabe (1997). In other words, our analysis *explains* why heterogeneity arises despite natural selection, while the existing literature *assumes* some degree of persistent heterogeneity a priori.

We do not regard the survival of risk-loving preferences as peculiar or irrational. In an environment where the objective of a decision-maker is to achieve some difficult target value within a specified period, it is often optimal to select a strategy to maximize the variance of random outcomes even for a risk-averse decision-maker, as is explained in Dubins and Savage (1965). When the target is too high to reach by playing safely, it is better to adopt a risky strategy that offers some chance of success. Therefore, risk lovers are optimizing when no FLB exists, and in this sense, it is not the irrationality of agents that creates the bias in our model.

We are also interested in how the size of the long-run bias changes as the size of the track take increases. We show that when it is sufficiently small, FLB increases as the track take increases. However, this relation is ambiguous if the track take is large. Here, the bias might decrease as the track take increases. We speculate that this might explain why FLB is not clearly observed at Japanese racetracks where the track take is larger than that in comparable countries.[7] Theoretically, this reversal occurs because a large track take forces a large proportion of agents to exit, and hence the property of the long-run steady-state population is primarily determined by the property of the potential pool of agents replacing the losers.

The rest of the paper is organized as follows. The basic pari-mutuel market model is set up in Section 2, where we discuss in detail how it can be viewed as a competitive equilibrium. Section 3 summarizes the properties of the evolutionary dynamics we consider, taking the exit rule and the replacement rule as exogenously given. The exit criterion that we propose above is formally described in Section 4, and we show that FLB occurs as a unique and stable long-run equilibrium. Section 5 is devoted to a comparative statics exercise, and we conclude with a discussion on the generality of our idea beyond racetracks in Section 6.

2. Simple pari-mutuel racetrack model

We begin with a very simple static model of racetracks with a pari-mutuel system. There are two horses, Favorite (F) and Long shot (L). Horse F wins with probability p and horse L wins with probability $1-p$, where $1/2 < p < 1$.

There is a continuum of agents of total size one who are price takers. Each agent is an expected utility maximizer with an increasing and continuous von Neumann Morgenstern utility function u with $u(0) = 0$ and $\lim_{z\to\infty} u(z) = +\infty$, and has a subjective probability $q \in (0, 1)$ on horse F to win. By assumption, each agent must bet a fixed amount β on either horse F or horse L. Because we have normalized the total size of agents to be one, the total amount of bets is β.

There is a track take of τ per betting unit, $0 \le \tau < 1$. That is, $\beta(1-\tau)$ is paid out to the winning bettors. Thus, let B be the total number of agents who bet on horse F. Then, the gross return per unit bet on F is $\beta(1-\tau)/\beta B = (1-\tau)/B$ if F wins. Similarly, the gross return per unit bet on L is $(1-\tau)/(1-B)$.

The price-taking assumption implies that each agent effectively takes the total bet B on horse F as given because B completely determines the odds. Each agent bets on the horse with a higher subjective expected utility by assumption. An agent (u, q) is indifferent between the two horses if

$$u\left(\frac{1-\tau}{B}\right) q = u\left(\frac{1-\tau}{1-B}\right)(1-q) \tag{1}$$

holds because we normalize $u(0) = 0$. Because u is increasing, continuous, and $\lim_{z\to\infty} u(z) = +\infty$, there is a unique solution for the above equation, which we denote by $B^*(u, q) \in [0, 1]$. It can be shown that $B^*(u, q)$ is increasing in q, and that it increases

[7] See Snowberg and Wolfers (2010) and references therein.

Table 1
Summary of bets of types *f* and ℓ.

Range of y	Equilibrium	Type f bet	Type ℓ bet
$B_f^* < y$	$B = B_f^*$	B_f^* on F, $y - B_f^*$ on L	on L
$B_\ell^* \le y \le B_f^*$	$B = y$	on F	on L
$y < B_\ell^*$	$B = B_\ell^*$	on F	$1 - B_\ell^*$ on L, $B_\ell^* - y$ on F

if u becomes more risk averse when $q \ge \frac{1}{2}$.[8] As u becomes more risk loving, $B^*(u, q)$ decreases to its lower bound $\frac{1}{2}$. It is convenient to regard the limit case $B^* = \frac{1}{2}$ as the variance maximizer, which is a form of extreme risk loving; if an agent is interested in maximizing the variance of the returns, he will bet on L as long as $B > \frac{1}{2}$.[9]

Example 1. CRRA utility functions

Suppose that the utility function of agents is represented by the CRRA utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$. Then,

$$B^*(u, q) = \frac{q^{1/(1-\gamma)}}{q^{1/(1-\gamma)} + (1-q)^{1/(1-\gamma)}}.$$

If the agents are risk neutral ($\gamma = 0$), $B^*(u, q) = q$. If the agents are risk loving ($\gamma < 0$), $B^*(u, q) < q$, and $\gamma \to -\infty$ implies $B^*(u, q) \to \frac{1}{2}$ (i.e., $\frac{1}{2}$ corresponds to extremely risk loving). If the agents are risk averse ($\gamma > 0$), $B^*(u, q) > q$, and $\gamma \to 1$ implies $B^*(u, q) \to 1$.

An agent (u, q) bets on F if $B < B^*(u, q)$ and bets on L if $B > B^*(u, q)$, and the specific functional forms of u or q are irrelevant to the agents' behavior.[10] Therefore, from now on, we identify each agent with the corresponding threshold value B^*.

We assume that there are two types of agents, with threshold values B_f^* and B_ℓ^*, where $B_f^* > B_\ell^*$. We call them type f and type ℓ, respectively. Thus, for instance, when all agents have the same vNM utility function, type f assigns a higher probability to horse F winning than type ℓ does. When they have the same subjective probability, a type-f agent is more risk averse than a type-ℓ agent.

Let y, $0 \le y \le 1$, be the number of type-f agents at the racetrack; thus, $1 - y$ is the number of type-ℓ agents by construction. Then, parameter y completely characterizes the property of competitive equilibria as follows.

By definition, a competitive equilibrium occurs at B where the total amount of bets on F given B coincides with B itself. The total amount of bets on L given B is exactly $1 - B$ by Walras' law. Clearly, $B \in \left[B_\ell^*, B_f^*\right]$ must hold in equilibrium otherwise all the agents will bet on the same horse. When $B_\ell^* \le y \le B_f^*$, B must be equal to y in a competitive equilibrium, where all type-f agents bet on F and all type-ℓ agents bet on L. When $y \ge B_f^*$, in particular when $y = 1$, some type-f agents must bet on L, so B must be equal to B_f^* in a competitive equilibrium to maintain indifference between F and L. Indeed, $B = B_f^*$ is an equilibrium where B_f^* type-f agents bet on F, $y - B_F^*$ type-F agents bet on L, and all the type-ℓ agents bet on L. Similarly, when $y \le B_\ell^*$, $B = B_\ell^*$ is a unique competitive equilibrium. Table 1 summarizes these cases. We emphasize that it is the threshold values B_f^* and B_ℓ^* that determine the market equilibrium, and so the market might "predict" the objective probability p even when none of the market participants has a correct belief about p.

At $B \in \left[B_\ell^*, B_f^*\right]$, the payout per unit bet on horse F is $(1 - \tau)/B$ with probability p, and that on horse L is $(1 - \tau)/(1 - B)$ with probability $1 - p$. The objective expected payouts are equalized when $B = p$ holds. This means that this static market exhibits FLB in equilibrium, i.e., the expected payout for the favorite F is higher than that for the longshot L, if and only if $B < p$.

Although the above characterization is valid for any B_ℓ^* and B_f^*, some natural requirements should be added to our model in order to understand why FLB emerges in the long run despite market selection. FLB should be exhibited in some equilibria and no bias should arise in others in our model, otherwise our analysis using this model would be vacuous. From Table 1, it can be seen that the property holds if $B_f^* \ge p > B_\ell^*$. In addition, if one type is driven out of the market by natural selection, the market should exhibit no bias, otherwise natural selection is seemingly an inappropriate explanation for FLB. The previous inequalities imply that type ℓ alone, i.e., when $y = 0$, would result in FLB, so in effect the market equilibrium should predict the objective probability when $y = 1$, which requires $B_f^* = p > B_\ell^*$. Therefore, we make the following assumption throughout the rest of this paper:

Assumption 1. $B_f^* = p > B_\ell^*$.

Assumption 1 holds when type-f agents are risk neutral and have a correct belief about the winning probability of the favorite, and type-ℓ agents are risk neutral but overestimate the winning probability of the long shot, or also have a correct

[8] A proof is given in Appendix A for completeness.
[9] Thus, our model accommodates models of both heterogeneous risk attitudes, such as Weitzman (1965) and Quandt (1986), and heterogeneous beliefs, such as Ali (1977), Blough (1994), and Watanabe (1997).
[10] Thus, we can readily accommodate a more general nonlinear increasing utility function $u(z, q)$.

belief but are more risk loving than type-ℓ agents. This may hold even when both types are risk averse and overestimate the winning probability of the favorite differently: if they share the same belief, type-f agents are more risk averse than type-ℓ agents, i.e., type-ℓ agents are relative risk lovers and type-f agents are relatively risk averse. Thus Assumption 1 can accommodate various kinds of risk preferences and beliefs.

3. Simple evolutionary dynamics and FLB

3.1. The evolutionary model

We are interested in whether or not FLB arises in the long run. In the setup outlined in the previous section, it is equivalent to ask whether the fraction of type-f agents is smaller than the winning probability p of the favorite. Therefore, let y_t denote the fraction of type-f agents and $1 - y_t$ denote that of type-ℓ agents on day t where $t = 1, 2, \ldots$. Because the outcome of the competitive model is uniquely determined by the fraction of type-f agents at the time, it suffices to describe how y_t changes over time as follows.

At the end of each day, some agents are no longer able to bet and quit the racetrack for good. Agents might be forced to exit for financial reasons, or they might lose interest in betting after a bad experience. We elaborate on how these factors are related to the equilibrium odds in the betting market in the next section. Let q_f and q_ℓ denote the number of type-f and type-ℓ agents, respectively, quitting the betting market. Because, by definition, the behavior of agents is independent of past history, we might as well assume that the quit rates on day t depend only on y_t. Therefore, we can write $q_f(y)$ and $q_\ell(y)$ for the quit rates when the fraction of type-f agents is y, and we assume that these are continuous functions on [0, 1], taking values in (0, 1). By construction, the total number of agents quitting at the end of day t, z_t, is given by:

$$z_t = q_f(y_t)y_t + q_\ell(y_t)(1 - y_t). \tag{2}$$

There are some natural requirements on the quit rates. First, the performance of each bettor should depend on which horse he bets on, not his type per se. Therefore, we can use ρ_F and ρ_L to denote the chance of quitting for those who bet on horse F and horse L, respectively, and these rates should depend on the equilibrium price B. Because an increase in B makes the market less favorable for those betting on F and more favorable for those betting on L, it is reasonable to expect that ρ_F is decreasing and ρ_L is increasing in B.

As can be seen in Table 1, when $y \in \left[B_\ell^*, p\right]$, $B = y$ holds in equilibrium and all type-f agents bet on F and all type-ℓ agents bet on L. Consequently, the quit rate for type-f (type-ℓ) agents should equal that for horse F (horse L). Therefore, it is reasonable to require that $q_f = \rho_F$ and $q_\ell = \rho_\ell$ hold. In contrast, when $y > p$ or $B_\ell^* > y$, the equilibrium odds do not depend on y, and agents of the same type behave differently in equilibrium. When $y > p$, $B = p$ holds, and all type-ℓ agents bet on L. Conversely, the proportion of type-f agents betting on F is $\frac{p}{y}$, and that of agents betting on L is $1 - \frac{p}{y}$. Because the equilibrium returns on horses are the same as those in the case where $y = p$, the performance, and thus the quit rate, of an agent should only depend on which horse he bets on. Because there is a continuum of agents, it seems reasonable to assume that the quit rate for type-f agents is equal to the average quit rate, $\frac{p}{y}\rho_F(p) + (1 - \frac{p}{y})\rho_L(p)$, and that for type-$\ell$ agents is $\rho_L(p)$. A symmetric argument applies when $y < B_\ell^*$.[11]

In summary, we make the following assumption throughout the rest of the paper:

Assumption 2. There exist functions ρ_F and ρ_L, defined on $\left[B_\ell^*, p\right]$, where ρ_F is increasing and ρ_L is decreasing, such that $q_f(y)$ and $q_\ell(y)$ can be written as

$$q_f(y) = \begin{cases} \rho_F\left(B_\ell^*\right) & y < B_\ell^*, \\ \rho_F(y) & B_\ell^* \le y \le p, \\ \dfrac{p}{y}\rho_F(p) + (1 - \dfrac{p}{y})\rho_L(p) & y > p, \end{cases}$$

$$q_l(y) = \begin{cases} \dfrac{1 - B_\ell^*}{1 - y}\rho_L(B_\ell^*) + (1 - \dfrac{1 - B_\ell^*}{1 - y})\rho_F(B_\ell^*) & y < B_\ell^*, \\ \rho_L(y) & B_\ell^* \le y \le p, \\ \rho_L(p) & y > p. \end{cases}$$

At the beginning of each day, new agents arrive to keep the total population equal to one. The number of type-f agents replacing those who have just quit is denoted by r. We assume that r depends only on the returns on the horses prevailing on the day, which equivalently means that r depends only on the fraction of type-f agents when $y \in \left[B_\ell^*, p\right]$. For analytical convenience, we assume that it is a continuous function of y on [0, 1] and that $r(y) \in (0, 1)$ for any $y \in [0, 1]$. We refer to function r as the *replacement rule*.

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[11] It will become clear that the essence of our analysis remains the same as long aits is some weighted average of $\rho_F(p)$ and $\rho_L(p)$.

It seems reasonable to assume that r is non-increasing; an increase in y makes the market more crowded with type-f agents, thus making conditions less favorable for type-f agents, so type-f agents are not encouraged to enter the market. Therefore, we make the following assumption throughout the rest of the paper:

Assumption 3. (i) r is non-increasing, and (ii) $r(y) \le y$ if $y \ge p$ and $r(y) \ge y$ if $y \le B_\ell^*$.

Condition (ii) above is a mild boundary requirement in our view. For instance, if $r(y) > y$ holds when $y > p$, type-f agents, who are already dominant in the market, will welcome an even larger proportion of newcomers, and so the power of market selection will be exaggerated.

We offer two examples of the replacement rule which satisfy Assumption 3.

Example 2. simple replicator

New agents are chosen according to the relative fitness of the surviving agents on the previous day, measured by the quit rates: for $y \in \left[B_\ell^*, p\right]$,

$$r(y) = \frac{q_\ell(y)}{q_\ell(y) + q_f(y)}, \tag{3}$$

where $r(y) = r\left(B_\ell^*\right)$ if $y < B_\ell^*$ and $r(y) = r(p)$ if $y > p$. Here, the number of new type-f agents increases if type-f agents are more resilient (i.e., q_f is smaller) on the previous day. This can be regarded as an environment in which it is more likely that the newcomers will adopt the behavior of well-performing agents.[12]

Example 3. fixed rate

New agents are chosen at random from an underlying pool of potential agents where the proportion of type-f agents is δ, $\delta \in \left[B_\ell^*, p\right]$: for any $y \in [0, 1]$,

$$r(y) = \delta. \tag{4}$$

This may be interpreted as an environment in which the newcomers do not take their predecessors' performance into account.

Assumption 3 is trivially satisfied in Example 3. In Example 2, it is satisfied under Assumption 2 when $B_\ell^* \le \frac{q_\ell(B_\ell^*)}{q_\ell(B_\ell^*)+q_f(B_\ell^*)}$ and $\frac{q_\ell(p)}{q_\ell(p)+q_f(p)} \le p$ hold.

Now we are ready to describe the population dynamics. The fraction of type-f agents changes from time t to time $t+1$ as follows:

$$y_{t+1} = (1 - q_f(y_t))y_t + r(y_t)z_t. \tag{5}$$

Substituting (2) into (5), we have an evolutionary dynamics given by

$$y_{t+1} = \Psi(y_t), \; t = 1, 2, \ldots, \tag{6}$$

where the policy function Ψ is given by

$$\begin{aligned} \Psi(y) &= (1 - q_f(y))y + r(y)\{q_f(y))y + q_\ell(y)(1 - y)\} \\ &= y - (1 - r(y))q_f(y)y + r(y)q_\ell(y)(1 - y). \end{aligned} \tag{7}$$

To see the long-run outcome of our betting environment, let y^* be a steady state of dynamics (6). As discussed in the previous section, FLB is exhibited in the long run if and only if $y^* < p$ holds. Setting $y^* = y_t = y_{t+1}$ in equation (6) and collecting terms, we find that $y^* \in (0, 1)$ is obtained as a solution to the following equation:

$$y^* = \frac{r(y^*)q_\ell(y^*)}{(1 - r(y^*))q_f(y^*) + r(y^*)q_\ell(y^*)}. \tag{8}$$

Note that y^* belongs to (0, 1) because $0 < q_f, q_\ell, f < 1$. Eq. (8) can also be understood as follows: $r(y^*)q_\ell(y^*)(1 - y^*)$ can be thought as the flow of agents from the type-ℓ population to the type-f population. Similarly, $(1 - r(y^*))q_f(y^*)y^*$ is the flow of agents in the other direction. By definition, these flows must cancel each other out at a steady state, and indeed (8) says that these two flows are equated.

3.2. FLB in the long run

To show the existence and uniqueness of a steady state y^* smaller than p, i.e., FLB is exhibited at a unique steady state, we have the following simple criterion:

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[12] Alternatively, $r(y) = \frac{1-q_f(y)}{(1-q_\ell(y))+(1-q_f(y))}$ can be found in a similar manner.

Proposition 4. *If $q_f(p) > q_\ell(p)$, there exists a unique steady state y^* such that $y^* < p$.*

A proof is given in Appendix A. The driving force behind this result can be roughly described as follows. The condition $q_f(p) > q_\ell(p)$ says that type F is *less fit* at $y = p > \frac{1}{2}$. Therefore, type-f agents must exit the market more often than type-ℓ agents. Thus, p cannot be a steady state. It can be shown that the boundary behavior of the functions stated in Assumptions 2 and 3 does not allow the fitness of type-f agents to improve when the market is more crowded with type-f agents, and so there will be no steady state larger than p. However, if y is sufficiently small, i.e., the market is crowded with type-ℓ agents, type ℓ is less fit, and so a steady state must exist somewhere between these alternatives by continuity. The boundary behavior implies that there will be no steady state smaller than B^*_ℓ. Then, the monotonicity properties stated in Assumptions 2 and 3 imply the condition of uniqueness.

Next, we enquire whether the steady state found in Proposition 4 is stable, possibly in some restricted area. We fix a non-empty interval $\left[\underline{y}, \bar{y}\right]$ and assume that all the relevant functions are continuously differentiable on $\left(\underline{y}, \bar{y}\right)$, and report the following technical result[13]:

Lemma 5. *Assume that $q_f' > 0$ and $q_\ell' < 0$ on $\left(\underline{y}, \bar{y}\right)$. Suppose that: (a) there is a constant $\eta > 0$ such that*

$$\max\left\{|q_f'(y)|, |q_\ell'(y)|\right\} \le \eta \max\left\{q_f(y), q_\ell(y)\right\}$$

at any $y \in \left(\underline{y}, \bar{y}\right)$; (b) $|r'|$ is bounded by a constant $\kappa > 0$ on $\left(\underline{y}, \bar{y}\right)$; and (c) quitting probabilities $q_f(y)$ and $q_\ell(y)$ are sufficiently small such that

$$\max_{y \in [\underline{y}, \bar{y}]}\left(\max\left\{q_f(y), q_\ell(y)\right\}\right) < \frac{1}{1+\kappa+\eta}.$$

Then, a steady state $y^* \in (\underline{y}, \bar{y})$, if it exists, is stable in the sense that the dynamics converges monotonically to y^* from any initial value $y_0 \in \left[\underline{y}, \bar{y}\right]$.

Lemma 5 says that if quitting probabilities $q_f(y)$ and $q_\ell(y)$ are small enough uniformly in $\left(\underline{y}, \bar{y}\right)$, the steady state is stable. A proof for this result is given in Appendix A.

4. The gambler's fate under positive track take

4.1. The survival criterion

The dynamic analysis in the previous section assumes that the quitting rates, q_f and q_ℓ, as well as the replacement rule r, are exogenously given. We now relate these rates to the basic static competitive market model to argue that FLB will emerge in the betting market.

With a positive track take, all types of gamblers will lose on average, and the expected loss becomes indefinitely large if an agent continues playing and the markets are efficient. Thus, technically, a standard criterion of growth in wealth over the long run is not appropriate for survival in gambling markets. We postulate instead that the gamblers at the racetrack are there to enjoy the races, and they are satisfied if they are ahead of some target after a certain period. In other words, the gamblers who quit must be those who fall short of their target.

We contend that although our postulate of target-seeking behavior is crude, it is a plausible criterion to approximate the behavior of lovers of horse racing: the majority are rational people with self-control, and their primary purpose in betting is not to make money out of the races but rather to test their ability to evaluate horses. Most of them know, through logic or from experience, that betting is not a good way to raise money. It is the feeling of winning, not the money won, that induces them to continue betting.

Here, we illustrate the role of our postulate in the market dynamics. Assume that $B = p$, and assume that a type-f agent bets on F and a type-ℓ agent bets on L. Then, the net expected returns on F and L are the same because $B = p$, but the variance from betting on F, σ_F^2, is smaller than that from betting on L, σ_L^2, because $p > \frac{1}{2}$. For the sake of illustration, imagine that these returns are normally distributed, that is, the returns on F and L are normally distributed with a common mean μ and variances σ_F^2 and σ_L^2, respectively, where $\sigma_F^2 < \sigma_L^2$.

Suppose that the gambler's target is 0, i.e., they might quit if they experience negative returns. Recall that because of the track take, $\mu < 0$ must hold, and hence the probability of a negative return is *larger* for F than for L, because the distribution of returns from F is more concentrated around μ. Thus, the chance of quitting is higher for type-f agents, i.e., type f is less fit than type ℓ. This means that when the target is sufficiently ambitious, an agent betting on F is more likely to quit than an agent betting on L at $y = p$.

[13] Differentiability is not essential, but it makes the analysis very simple.

Table 2
Expected returns and the variance in the gross returns.

	Expected net returns	Variances
Horse F	$\mu_F = \frac{p(1-\tau)}{B} - 1$	$\sigma_F^2 = \frac{p(1-p)(1-\tau)^2}{B^2}$
Horse L	$\mu_L = \frac{(1-p)(1-\tau)}{(1-B)} - 1$	$\sigma_L^2 = \frac{p(1-p)(1-\tau)^2}{(1-B)^2}$
Indifference	$B = p$	$B = 1/2$

4.2. The gambler's ruin problem

Here, we present the simple illustration provided above more carefully and rigorously, utilizing the normal approximation of binary returns in our market model. Imagine that there are numerous races run on a given day, and let K be the number of races run each day. We regard a single day as the period after which the agents review their performance. The races are identical, and the outcomes are independent.[14] Each agent bets one unit of money on each race and has a target rate of return from gambling of $\hat{x}$. Because the races are iid, assume that the betting strategy is the same throughout the day, and the same equilibrium occurs in every race.

If an agent has won more than the target amount of $\hat{x}K$ at the end of the day, it is an enjoyable day for him and he is determined to return to the racetrack on a subsequent day. Conversely, if the agent's gain is not more than $\hat{x}K$, he is severely discouraged, and doubts whether he should ever return to the racetrack. We call such an agent a loser, and assume that some losers actually quit gambling. In reality, the size of the loss might influence the decision, but for simplicity we assume that quitting takes place with a common probability ϵ for all losers independently. Technically, the size of ϵ is related to stability, and is not crucial for eliciting FLB. If a loser happens to remain involved, he forgets all his prior troubles and returns to the racetrack with a fresh mind and a fresh bank.

Now, let us determine the number of quitting agents in this scenario. Let W_F and W_L represent the agents' terminal wealth after betting all day on horses F and L, respectively. We denote the chance of an agent exiting the racetrack by $\rho_j(B)$ when an agent bets on horse $j = F, L$ for the entire day, where $B \in \left[B_\ell^*, p\right]$. From these functions, the quit rates, $q_f(y)$ and $q_\ell(y)$, are constructed for $y \in [0, 1]$ to meet Assumption 2. According to our postulate about the quitting rule described above, $\rho_j(B)$ is given by

$$\rho_j(B) = \epsilon Pr\left[W_j \leq \hat{x}K\right]$$

for $j = F, L$.[15]

Recall that when the proportion of money bet on F is B, the mean μ_j and the variance σ_j^2 of the returns from betting on horse j ($j = F, L$) in each race are shown in Table 2.

After assuming that there are numerous races, let us further assume that W_F and W_L are independent normal random variables; that is, W_j can be regarded as the normal distribution of the mean $\mu_j K$ and the variance $\sigma_j^2 K$.[16] Standardizing the wealth per bet by

$$z_j(B) = \frac{KB - \mu_j K}{\sigma_j \sqrt{K}}, \tag{9}$$

one can express $\rho_j(B)$ as

$$\rho_j(B) = \epsilon \Phi(z_j(B)) = \epsilon \int_{-\infty}^{z_j(B)} \phi(u) du$$

for $j = F, L$, where Φ and ϕ are the cumulative probability distribution function and the density function for the standard normal distribution, respectively. Substituting (2) into (9), z_j can be written as

$$z_F(B) = \sqrt{\frac{K}{p(1-p)}} \left\{ \frac{1+\hat{x}}{1-\tau} B - p \right\},$$
$$z_L(B) = \sqrt{\frac{K}{p(1-p)}} \left\{ \frac{1+\hat{x}}{1-\tau} (1-B) - (1-p) \right\}. \tag{10}$$

[14] Obviously, in a particular race, the outcomes of horses F and L are perfectly negatively correlated, so implicitly we are assuming that this model is an abstraction of numerous races at several different racetracks.

[15] This may be seen as an extremely simplified version of the so-called *Gambler's Ruin problem*. In principle, one should be able to replace ρ_j with the probability of reaching $\hat{x}k$ at some $k \leq K$, which will make the analysis more complicated. However, we conjecture that the basic message remains the same.

[16] Of course, if these agents bet on exactly the same races, their wealth will be correlated. In principle, the analysis can be carried out while considering correlation, but we believe that such an analysis only blurs our message.

It can then be readily verified that ρ_F is increasing and ρ_L is decreasing on $\left[B^*_\ell, p\right]$. Therefore, we have constructed quit rates, q_f and q_ℓ, that satisfy Assumption 2.

4.3. Emergence of FLB

In the above setup, we verify that FLB emerges as a steady state of evolutionary dynamics when the replacement rule satisfies Assumption 3. Because the quit rates constructed in the previous subsection satisfy Assumption 2, Proposition 4 reduces the problem to finding a condition that implies $q_f(p) > q_\ell(p)$. In fact, there is a simple and intuitive condition. As a reference point, we find $\hat{y}$ using the rule

$$\hat{y} = \frac{1}{2} + \frac{(2p-1)(1-\tau)}{2(1+\hat{x})},$$

and then $z_F(\hat{y}) = z_L(\hat{y})$ by (10). That is, $\hat{y}$ is a unique number where $\rho_F\left(\hat{y}\right) = \rho_L\left(\hat{y}\right)$, and $y > \hat{y}$ implies that $\rho_F(y) > \rho_L(y)$. Moreover, $\hat{x} > -\tau$ implies that $p > \hat{y}$, hence $\rho_F(p) > \rho_L(p)$ holds. Thus, we have the following result.

Proposition 6. *Suppose that $\hat{x} > -\tau$. Then, there exists a unique steady state y^* such that $y^* < p$.*

Proof. By Proposition 4, it suffices to show that $q_f(p) > q_\ell(p)$. By construction, we have $q_f(p) = \rho_F(p)$ and $q_\ell(p) = \rho_L(p)$. If $\hat{x} > -\tau$, $\rho_F(p) > \rho_L(p)$ holds, as seen above. ■

Recall that when $p = B$, i.e., there is no bias, the expected net rate of return is $-\tau$. Hence, if $\hat{x} \le -\tau$, the target can be thought of as modest. Therefore, Proposition 6 states that a unique steady state exhibiting FLB exists if the target $\hat{x}$ is sufficiently ambitious. This is intuitive, and the logic is identical to that in our earlier illustration because both distributions of returns are symmetric around $-\tau$. Therefore, Proposition 6 formally provides the reason why FLB emerges despite the natural market selection of the target-seeking behavior and the survival rule we postulated.

To complete our analysis, we investigate the case where the unique steady state is stable, applying Lemma 5 by setting $\left[\underline{y}, \bar{y}\right] = \left[B^*_\ell, p\right]$. Then, we need to assume that $q'_f > 0$ and $q'_\ell < 0$ on $\left(B^*_\ell, p\right)$, thus $\rho'_F > 0$ and $\rho'_L < 0$ on $\left(B^*_\ell, p\right)$ by construction. Because they are normally distributed in the setup of this section, it is sufficient to require that the reference point $\hat{y}$ belongs to $\left(B^*_\ell, p\right)$. Recall that $\hat{y} < p$ is warranted by $\hat{x} > -\tau$.

We are now ready to state a stability result, which says that if the chance of quitting, ϵ, for losers is sufficiently small, the unique steady state is stable.

Proposition 7. *Suppose that $\hat{x} > -\tau$ and $\hat{y} \ge B^*_\ell$ holds, and $|r'|$ is bounded on $\left[B^*_\ell, p\right]$. Then, there is a unique steady state $y^* \in (B^*_\ell, p)$, and there exists $\bar{\epsilon} > 0$ such that y^* is globally stable in (B^*_ℓ, p) for any chance of quitting $\epsilon < \bar{\epsilon}$.*

A proof, which is an application of Lemma 5, can be found in Appendix A.

The requirement $\hat{y} \ge B^*_\ell$ in Proposition 7 is not necessarily restrictive. Recall that $\hat{y} \ge B^*_\ell$ holds if the degree of risk loving is sufficiently large or overestimates (underestimates) the winning probability of the longshot (favorite). As a benchmark, consider the case where type-ℓ agents have a correct belief and are extremely risk loving such that $B^*_\ell = \frac{1}{2}$. Thus, $\hat{y} \ge B^*_\ell$ automatically holds because $\hat{y} < 1/2$ for $\hat{x} > -\tau$.

The boundedness condition for r' in Proposition 7 is also not very restrictive. For instance, it trivially holds in the case of constant replacement. For the simple replicator, note that r is independent of ϵ, as is r', because both $q_\ell(y)$ and $q_f(y)$ are given by the standard normal distribution multiplied by the same ϵ. In addition, as $\frac{1}{\epsilon}\left(q_\ell(y) + q_f(y)\right)$ is bounded away from zero, $r'(y)$ can be continuously extended on $[B^*_\ell, p)$, and thus it is bounded on $\left[B^*_\ell, p\right]$.

5. Comparative statics: the role of the track take

A positive track take together with our behavioral postulate is the main driving force behind our FLB results. Therefore, it is interesting to examine how the size of the track take affects the magnitude of the bias. Because the track take is the source of the bias, one might expect that the bias would increase as the track take increases. However, this is not necessarily the case. In the following section, we concentrate on the extreme case of $B^*_\ell = 1/2$ to explain why the comparative statics might be delicate.

To facilitate a comparative statics analysis of the steady state, let functions q_f, q_ℓ, and r depend on an exogenous variable α in some prespecified interval $I \subseteq \mathbb{R}$, and denote them by $q_f(y, \alpha)$, $q_\ell(y, \alpha)$, and $r(y, \alpha)$. The corresponding policy function (7) is denoted by $\Psi(y, \alpha)$. We use $y^*(\alpha)$ to represent the unique steady state when the exogenous parameter is set at α. Letting $\xi(y, \alpha) = \Psi(y, \alpha) - y$, equation (8) implies that the steady state $y^*(\alpha)$ satisfies

$$\xi(y^*(\alpha), \alpha) = 0. \tag{11}$$

Applying the Implicit Function Theorem to (11), we have

$$\xi_y(y^*(\alpha), \alpha)\frac{d}{d\alpha}y^*(\alpha) + \xi_\alpha(y^*(\alpha), \alpha) = 0,$$

where ξ_y and ξ_α are partial derivatives of ξ by y and α, respectively, as long as ξ_y does not vanish at $(y^*(\alpha), \alpha)$. Assume $\xi_y(y^*(\alpha), \alpha) < 0$, which is the case if $q_f\left(\frac{1}{2}\right) < q_\ell\left(\frac{1}{2}\right)$.[17] Hence, $\frac{d}{d\alpha}y^*(\alpha)$ and $\xi_\alpha(y^*(\alpha), \alpha)$ have the same sign, and thus we have the following result.

Lemma 8. *The steady state $y^*(\alpha)$ is increasing (decreasing) as the exogenous parameter $\alpha \in I$ increases if $\xi_\alpha(y^*(\alpha), \alpha)$ is positive (negative).*

Now we are ready to examine how the track take affects FLB in the setup of normally distributed wealth. Let

$$\alpha = (1 + \hat{x})/(1 - \tau) \tag{12}$$

and substitute α into (10). Then, the critical values z_F and z_L can be rewritten as

$$\begin{aligned} z_F(B, \alpha) &= \sqrt{\frac{K}{p(1-p)}}(\alpha B - p), \\ z_L(B, \alpha) &= \sqrt{\frac{K}{p(1-p)}}(\alpha(1 - B) - (1 - p)). \end{aligned} \tag{13}$$

We shall regard ρ_F and ρ_L as functions of y and α from now on.

When the replacement rule is a constant (4), we have the following result.

Lemma 9. *Suppose that $r(y, \alpha) = \delta$ for any (y, α), where $\delta \in \left[\frac{1}{2}, p\right]$, and let $I \subseteq [1, +\infty)$ be an open interval. Then, $y^*(\cdot)$ is decreasing (increasing) at α in I if $-(1-\delta)\phi(z_F(y^*(\alpha), \alpha))(y^*(\alpha))^2 + \delta\phi(z_L(y^*(\alpha), \alpha))(1 - y^*(\alpha))^2 < 0$ holds (> holds).*

Now we focus on the special case of $\delta = p$, i.e., the arrival rate of type-f agents is equal to the probability that horse F wins. Although the ratio of arriving agents can sustain the efficiency of the betting market, Proposition 6 shows that FLB emerges when $\hat{x} > -\tau$.[18] In this special case, we have a clear comparative statics result as follows:

Proposition 10. *Suppose that $r(y, \alpha) = p$ for any (y, α). Then, for sufficiently small $\eta > 0$, $y^*(\alpha)$ is decreasing in $\alpha \in [1, 1 + \eta]$.*

Proposition 10 is a local result, and does not assert that the bias is increasing globally in τ. One might expect that such an assertion would be true because the track take represents market friction. However, in general, this is not the case. In fact, the intuition is very simple. Consider an extreme case where τ is so high that it is impossible to win at all. Then, the market is always filled with newly arriving agents, even in the long run, and hence the steady state will inherit the property of the pool of potential agents, and the results of past races will not matter much; that is, the property of the steady state will be governed by the property of r. In particular, FLB caused by market friction will diminish in the range where τ is extremely large.

For instance, under a constant replacement rule with $\delta = p$, the steady state will approach p, which means that at some point the bias starts to decrease as the track take τ increases. Indeed, let $\tau \to 1$, i.e., let $\alpha \to +\infty$ in (12), and observe that $z_F(y^*(\alpha), \alpha)$ and $z_L(y^*(\alpha), \alpha)$ approach ∞ by (13), and $q_f(y^*(\alpha), \alpha)$ and $q_\ell(y^*(\alpha), \alpha)$ approach 1. Then, by (8), $y^*(\alpha)$ approaches $r(y^*(\alpha))$, and so $y^*(\alpha) \to p$ holds when $r(y) = p$ everywhere.

To show this phenomenon numerically, Fig. 2 depicts the relationship between τ and y^* when $p = r(y, \alpha) = 0.9$, $\epsilon = 0.1$, $K = 30$, and $\hat{x} = 0$. Therefore, FLB occurs if y^* is less than 0.9. As can be seen from the figure, FLB does not exist when $\tau = 0$. Then, y^* decreases and the degree of FLB increases in the neighborhood of $\tau = 0$ when τ increases, as Proposition 10 showed. In contrast, y^* increases when $\tau \geq 1.3$ and the size of *FLB* decreases in τ.

Of course, that FLB vanishes in this case does not imply that the market learns more and becomes more efficient as the track take increases. The limit probability approaches δ, rather than p. Thus, it is more accurate to say that as τ increases, the market loses its learning power and the property of the entering population matters more.

The idea is similar in the case of a replicator type replacement. Both q_ℓ and q_f go to one, as nobody can survive in the limit. This means that both types are extremely unsuccessful at the racetrack, but are still equally fit. Hence, the replacement rule supplies roughly the same amounts of each type from the population. Consequently, when the target for survival is severe, the steady state y^* will approach 0.5. Fig. 3 depicts the relationship between τ (see (12)) and y^* when $p = 0.75$, $\epsilon = 0.2$, and $K = 30$ when the replacement rule is given by (3). Thus, FLB occurs if y^* is less than 0.75. In contrast to the case of $p = r(y, \alpha)$, FLB exists even when $\tau = 0$. As τ rises, i.e., as the target becomes increasingly severe, FLB increases monotonically and y^* approaches 0.5.

6. Concluding remarks

Using a simple evolutionary market model, we demonstrate that FLB arises in the long run. The driving force behind this result is the fact that under the exit rule we postulated, relative risk lovers are a better fit than relatively risk-averse bettors when the markets are not biased.

[17] See the proof of Proposition 4 where $\Psi(y, \alpha) - y$ is shown to be decreasing in the relevant domain.

[18] When $\tau = 0$ and $\hat{x} = 0$, $q_\ell(p) = q_f(p)$, and so FLB does not occur.

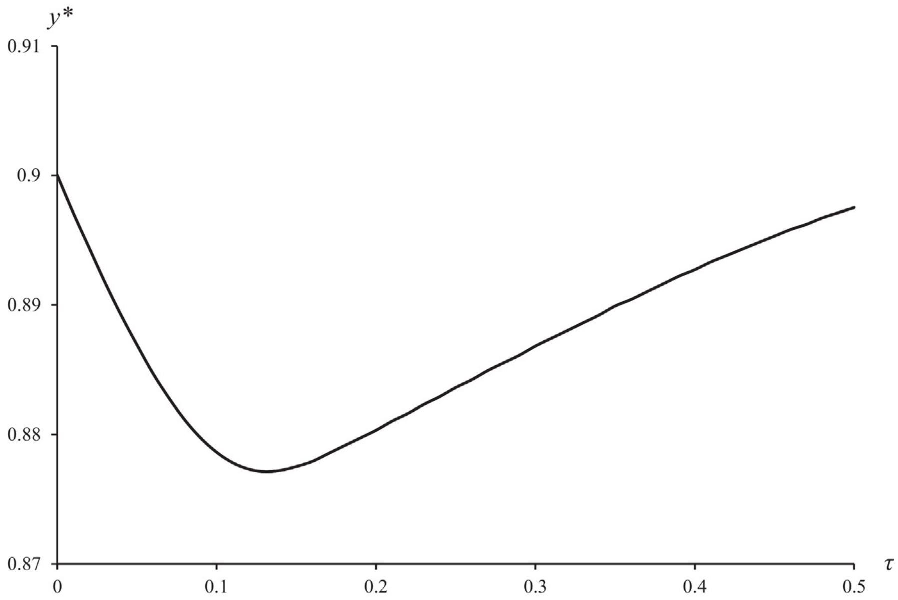

Fig. 2. Equilibrium odds and track take for a fixed replacement rate.

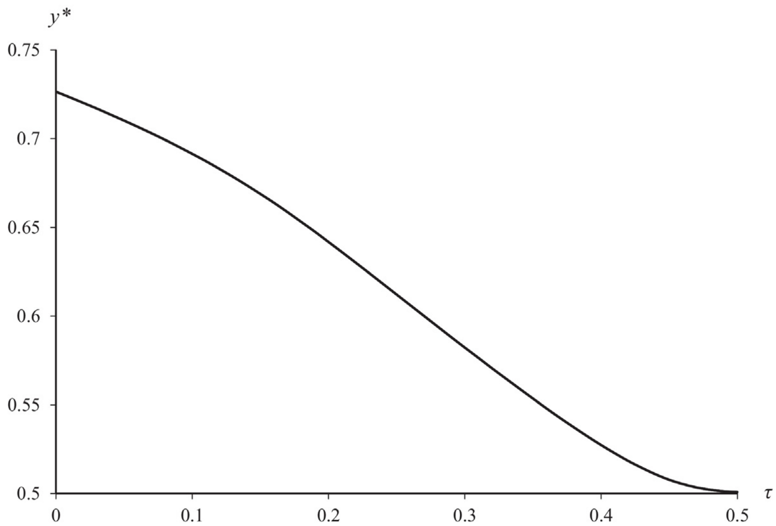

Fig. 3. Equilibrium odds and track take for the replicator rule.

The competitive equilibrium prices do not fully elicit the underlying information about the winning probability of the horses in our model. On the other hand, the so called efficient market hypothesis (EMH) asserts that in competitive markets, the market price of a financial asset would contain all relevant information about its returns. Thus we have an instance of robust failure of the EMH. However, a horse is a very peculiar financial asset and the agents are completely myopic in our model, and hence the implication of this phenomenon is limited. We speculate nonetheless that a dynamic analysis with some exit rule such as ours is worth pursuing to ask the informativeness of the prices in more sophisticated financial markets.

Under our exit rule, one must earn more than a target value to survive. We justified this rule in the context of gambling markets, but it might be questionable when the market returns are positive. Nevertheless, let us tentatively suppose that the track take τ is negative, and so the market returns are positive in the absence of FLB, but the same exit rule applies. Note that the steady state y^* is effectively determined by α in (12), and so even if τ is negative, FLB can still arise with a high target $\hat{x}$. The intuition is simple, and the same as before: without FLB, both types will earn the same average returns, and the distribution of returns for the relative risk lovers has a fatter tail. Therefore, if the target is higher than average, the relative risk lovers have a better chance of meeting the target, and so they are a better fit. A bias must arise to offset this survival advantage. That is, we have an evolutionary model whereby wealth increases but some bias persists. Returning to the discussion on natural selection in Section 1, the growth-based argument against FLB fails once the target-based survival criterion is accepted.

We are therefore tempted to speculate more generally that in a market environment where the survival of some large proportion of agents is conditional on achieving a higher than average target value, the markets tend to exhibit some degree of bias in favor of low-risk alternatives. For instance, imagine an environment in which the performance of fund managers is evaluated first in terms of whether the fund outperformed some benchmark number that represents some form of average returns. Those managers who do not achieve the target face the possibility of having their career terminated. To outperform the average, the fund managers are naturally interested in riskier alternatives, and the logic of our analysis indicates that such preferences will exaggerate the market returns of less risky assets.

In conclusion, we contend that the implications of target-driven behavior are worth investigating beyond the racetrack.

Appendix A. Property of function B^*

Property of function B^*

We show that $B^*(u, q)$, defined by (1), increases in q and also increases as utility function u becomes more risk averse when $q \geq \frac{1}{2}$. To simplify the notation, we write $B^* = B^*(u, q)$.

If $q' > q$, then $u\left(\frac{1-\tau}{B^*}\right) q' > u\left(\frac{1-\tau}{1-B^*}\right)(1-q')$, and so the threshold value must increase to obtain the equality.

Let h be an increasing and concave function with $h(0) = 0$, and set $\hat{u} = h \circ u$. It suffices to show that $\hat{u}\left(\frac{1-\tau}{B^*}\right) \geq \hat{u}\left(\frac{1-\tau}{1-B^*}\right)\frac{(1-q)}{q}$. The concavity of h and $h(0) = 0$ imply that $h(tx) \geq th(x)$ for any $t \in [0, 1]$ and $x > 0$. By construction $u\left(\frac{1-\tau}{B^*}\right) = u\left(\frac{1-\tau}{1-B^*}\right)\frac{(1-q)}{q}$ and $\frac{(1-q)}{q} \in [0, 1]$. Thus, $\hat{u}\left(\frac{1-\tau}{B^*}\right) = h\left(u\left(\frac{1-\tau}{B^*}\right)\right) = h\left(u\left(\frac{1-\tau}{1-B^*}\right)\frac{(1-q)}{q}\right) \geq h\left(u\left(\frac{1-\tau}{1-B^*}\right)\right)\frac{(1-q)}{q} = \hat{u}\left(\frac{1-\tau}{1-B^*}\right)\frac{(1-q)}{q}$.

Proofs for the results in Section 3

Proof of Proposition 4

If $q_f(p) > q_\ell(p)$, from (7),

$$\begin{aligned}\Psi(p) - p &= -(1 - r(p))q_f(p)p + r(p)q_\ell(p)(1-p)\\ &< -(1 - r(p))q_f(p)p + r(p)q_f(p)(1-p)\\ &= q_f(p)(r(p) - p).\end{aligned}$$

Since $r(p) \leq p$ by Assumption 3, we conclude that $\Psi(p) - p < 0$. This shows that p is not a steady state. Moreover, because $\Psi(0) = r(0)q_\ell(0) \geq 0$, a continuous function $y \mapsto \Psi(y) - y$ must attain 0 at some y^* in $[0, p)$ by the intermediate value theorem.

Because a steady state is a zero of the function $y \mapsto \Psi(y) - y$, it suffices to show that the function crosses zero only once from above somewhere in $(0, p)$. Recall that $\Psi(y) - y$ is expressed as

$$\Psi(y) - y = -(1 - r(y))q_f(y)y + r(y)q_\ell(y)(1-y).$$

Observe that $\Psi(0) - 0 = r(0)q_\ell(0) > 0$.

First, we show that $\Psi(y) - y < 0$ for any $y \in [p, 1]$, so that there is no steady state in $[p, 1]$. Because $q_f(p) > q_\ell(p)$, by Assumption 2, $q_\ell(y) = q_\ell(p)$ and $q_f(y)$ is decreasing on $[p, 1]$. Therefore,

$$\begin{aligned}\Psi(y) - y &< -(1 - r(y))q_f(p)y + r(y)q_f(p)(1-y)\\ &= q_f(p)(r(y) - y).\end{aligned}$$

Because $r(y) - y \leq 0$ on $[p, 1]$ by Assumption 3, we have shown the desired inequality.

Next, consider the case where $q_f\left(B_\ell^*\right) \geq q_\ell\left(B_\ell^*\right)$. By Assumption 2, q_f is increasing and q_ℓ is decreasing on $\left[B_\ell^*, p\right]$. On $[0, B_\ell^*]$, $q_f(y) = q_f(B_\ell^*)$ and $q_\ell(y)$ is non-increasing because $q_f\left(B_\ell^*\right) \geq q_\ell\left(B_\ell^*\right)$. Therefore, q_f is non-decreasing and q_ℓ is non-increasing on $[0, p]$. Because the product of positive decreasing functions is decreasing, we conclude that $\Psi(y) - y$ is decreasing on $[0, p]$. Because $\Psi(0) - 0 > 0$, we have shown the desired property.

It remains to consider the case where $q_f\left(B_\ell^*\right) < q_\ell\left(B_\ell^*\right)$. Arguing as above, $\Psi(y) - y$ is decreasing on $\left[B_\ell^*, p\right]$. Because $q_f\left(B_\ell^*\right) < q_\ell\left(B_\ell^*\right)$, we have

$$\begin{aligned}\Psi(B_\ell^*) - B_\ell^* &> -(1 - r(B_\ell^*))q_f(B_\ell^*)y + r(y)q_f(B_\ell^*)(1-y)\\ &= q_f(B_\ell^*)(r(B_\ell^*) - B_\ell^*).\end{aligned}$$

Because $r(B_\ell^*) - B_\ell^* \geq 0$ by Assumption 3, we conclude that $\Psi(B_\ell^*) - B_\ell^* > 0$, and consequently $\Psi(y) - y$ crosses zero only once from above in $\left(B_\ell^*, p\right)$. Thus, the proof is complete if we can show that $\Psi(y) - y > 0$ for $y \in \left[0, B_\ell^*\right]$. Assumption 2 implies that $q_f(y) = q_f\left(B^*_\ell\right)$ and $q_\ell(y) = \frac{1-B^*_\ell}{1-y}\rho_L(B^*_\ell) + \left(1 - \frac{1-B^*_\ell}{1-y}\right)\rho_F(B^*_\ell)$ is increasing on $\left[0, B^*_\ell\right]$ and $q_\ell(0) = \left(1 - B^*_\ell\right) q_\ell(B^*_\ell) + B^*_\ell q_f\left(B^*_\ell\right) > q_f\left(B^*_\ell\right)$ because $q_f\left(B^*_\ell\right) < q_\ell\left(B^*_\ell\right)$. Therefore, for $y \in \left[0, B^*_\ell\right]$, we have

$$
\begin{aligned}
\Psi(y) - y &= -(1 - r(y))q_f(y)y + r(y)q_\ell(y)(1 - y) \\
&> -(1 - r(y))q_f(B^*_\ell)y + r(y)q_f(B^*_\ell)(1 - y) \\
&= q_f(B^*_\ell)(r(y) - y),
\end{aligned}
$$

and $r(y) - y \geq 0$ for $y \in \left[0, B^*_\ell\right]$ by Assumption 3. ■

Proof of Lemma 5

As is well known, if $0 < \Psi'(y) < 1$ on $\left(\underline{y}, \bar{y}\right)$ and $y^* = \Psi(y^*) \in \left(\underline{y}, \bar{y}\right)$, then starting from any initial point y_0 in $\left(\underline{y}, \bar{y}\right)$, the dynamics $y(t+1) = \Psi(y(t))$ $t = 0, 1, 2, \ldots$, converges monotonically to a unique steady state y^*. It therefore suffices to show that the following inequalities hold for any $y \in \left(\underline{y}, \bar{y}\right)$:

$$
1 > \Psi'(y) > 1 - (1 + \kappa + \eta)\max_{y \in [\underline{y}, \bar{y}]}\left(\max\left\{q_f(y), q_\ell(y)\right\}\right).
$$

Because r is non-increasing, $0 \leq -r'(y) \leq \kappa$ must hold for all $y \in \left(\underline{y}, \bar{y}\right)$. Define $\theta = \max_{y \in [\underline{y}, \bar{y}]}\left(\max\left\{q_f(y), q_\ell(y)\right\}\right)$. So at any $y \in [\underline{y}, \bar{y}]$, $0 \leq q_f(y), q_\ell(y) \leq \theta$ and $0 \leq q'_f(y) \leq \eta\theta$, $0 \geq q'_\ell(y) \geq -\eta\theta$.

From (7), $\Psi'(y)$ can be computed directly as follows:

$$
\begin{aligned}
\Psi'(y) &= 1 - \left\{(1 - r(y))q_f(y) + r(y)q_\ell(y)\right\} \\
&+ \left\{r'(y)q_f(y)y + r'(y)q_\ell(y)(1 - y)\right\} \\
&- \left\{(1 - r(y))yq'_f(y) - r(y)q'_\ell(y)(1 - y)\right\}.
\end{aligned}
$$

Observe that because $q_f(y), q_\ell(y) \leq \theta$, we have

$$
\begin{aligned}
-\left\{(1 - r(y))q_f(y) + r(y)q_\ell(y)\right\} &\geq -\theta, \\
\left\{yq_f(y) + (1 - y)q_\ell(y)\right\} &\leq \theta,
\end{aligned}
$$

and because $-r'$ is bounded, as pointed out above, the second inequality gives us:

$$
\left\{r'(y)q_f(y)y + r'(y)q_\ell(y)(1 - y)\right\} \geq -\kappa\theta.
$$

Finally, note that using $0 \leq q'_f(y) \leq \eta\theta$, $0 \geq q'_\ell(y) \geq -\eta\theta$, we obtain

$$
\begin{aligned}
&-\left\{(1 - r(y))yq'_f(y) - r(y)q'_\ell(y)(1 - y)\right\} \\
&\geq -(1 - r(y))y\eta\theta - r(y)(1 - y)\eta\theta \\
&= -\eta\theta\left\{(1 - r(y))y + r(y)(1 - y)\right\} \\
&\geq -\eta\theta\left\{(1 - r(y))\max\left\{y, 1 - y\right\} + r(y)\max\left\{y, 1 - y\right\}\right\} \\
&\geq -\eta\theta.
\end{aligned}
$$

The desired result is established by combining the above inequalities.

Proofs for the results in Section 4

Proof of Proposition 7

We apply Lemma 5 for $\left[\underline{y}, \bar{y}\right] = \left[B^*_\ell, p\right]$. Because the constructed q_f and q_ℓ satisfy Assumption 2, $q'_f > 0$ and $q'_\ell < 0$ on $\left(\underline{y}, \bar{y}\right)$ holds in particular. Moreover, they are given by the quitting probability ϵ times the cumulative distribution function of the standard normal distribution, so we can find a constant $\eta > 0$ independently of ϵ such that $\max\left\{|q'_f(y)|, |q'_\ell(y)|\right\} \leq \eta\max\left\{q_f(y), q_\ell(y)\right\}$ on (B^*_ℓ, p).

If we choose a positive number $\bar{\epsilon}$ with $\bar{\epsilon} < \frac{1}{1+\kappa+\eta}$, all the conditions in Lemma 5 are satisfied because $\max_{y \in [\underline{y}, \bar{y}]}\left(\max\left\{q_f(y), q_\ell(y)\right\}\right) \leq \epsilon \leq \bar{\epsilon}$.

(14) Proofs for the results in Section 5

(14) Proof of Lemma 9

By Lemma 8, we only have to examine the sign of $\xi_\alpha(y^*(\alpha), \alpha)$. Because y^* belongs to $[\frac{1}{2}, p]$, we set $q_f(y^*(\alpha), \alpha) = \rho_F(y^*(\alpha), \alpha)$ and $q_\ell(y^*(\alpha), \alpha) = \rho_L(y^*(\alpha), \alpha)$. Hence, $\xi(y, \alpha)$ is expressed as

$$\xi(y, \alpha) = -(1-\delta)\rho_F(y, \alpha)y + \delta\rho_L(y, \alpha)(1-y).$$

Differentiating $\rho_F(y, \alpha)$ and $\rho_L(y, \alpha)$ by α, we have

$$\frac{\partial \rho_F}{\partial \alpha} = \phi(z_F(y, \alpha))\frac{\epsilon\sqrt{K}}{\sqrt{p(1-p)}}y, \quad \frac{\partial \rho_L}{\partial \alpha} = \phi(z_L(y, \alpha))\frac{\epsilon\sqrt{K}}{\sqrt{p(1-p)}}(1-y). \tag{14}$$

Therefore, differentiating $\xi(y, \alpha)$ with respect to α and evaluating it at $y = y^*(\alpha)$,

$$\begin{aligned}
&\xi_\alpha(y^*(\alpha), \alpha)\\
&= -(1-\delta)\frac{\partial}{\partial\alpha}\rho_F(y^*(\alpha), \alpha)y^*(\alpha) + \delta\frac{\partial}{\partial\alpha}\rho_L(y^*(\alpha), \alpha)(1-y^*(\alpha))\\
&= \frac{\epsilon\sqrt{K}}{\sqrt{p(1-p)}}\left\{-(1-\delta)\phi(z_F(y^*(\alpha), \alpha))(y^*(\alpha))^2\right.\\
&\left.+\delta\phi(z_L(y^*(\alpha), \alpha))(1-y^*(\alpha))^2\right\}.
\end{aligned}$$

Hence, the sign of $\xi_\alpha(y^*(\alpha), \alpha)$ is determined as stated.

Proof of Proposition 10

Note that $\alpha = 1$ in (12) implies that $\hat{x} = -\tau$. It is readily confirmed that $y^* = p$ is a unique solution to $\xi(y, 1) = 0$. Therefore, $y^*(1) = p$, and $z_F(y^*(1), 1) = z_L(y^*(1), 1)$.

To apply Lemma 9, it suffices to show that $-(1-p)\phi(z_F(y^*(\alpha), \alpha))(y^*(\alpha))^2 + p\phi(z_L(y^*(\alpha), \alpha))(1-y^*(\alpha))^2$ is negative for α close to one. By direct computation, we have:

$$\begin{aligned}
&-(1-p)\phi(z_F(y^*(1), 1))(y^*(1))^2 + p\phi(z_L(y^*(1), 1))(1-y^*(1))^2\\
&= -p^2(1-p)\phi(z_F(y^*(1), 1)) + p(1-p)^2\phi(z_F(y^*(1), 1))\\
&= p(1-p)(1-2p)\phi(z_F(y^*(1), 1)) < 0.
\end{aligned}$$

Hence, the desired conclusion follows by continuity.

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