

Singapore Management University

# Institutional Knowledge at Singapore Management University

---

Research Collection School Of Economics

School of Economics

---

1-2003

## Incentive compatibility in multi-unit auctions

Sushil BIKHCHANDANI

Shurojit CHATTERJI

*Singapore Management University*, shurojitc@smu.edu.sg

Arunava SEN

Follow this and additional works at: [https://ink.library.smu.edu.sg/soe\\_research](https://ink.library.smu.edu.sg/soe_research)



Part of the [Economic Theory Commons](#), and the [Strategic Management Policy Commons](#)

---

### Citation

BIKHCHANDANI, Sushil; CHATTERJI, Shurojit; and SEN, Arunava. Incentive compatibility in multi-unit auctions. (2003). 1-28.

Available at: [https://ink.library.smu.edu.sg/soe\\_research/2022](https://ink.library.smu.edu.sg/soe_research/2022)

This Working Paper is brought to you for free and open access by the School of Economics at Institutional Knowledge at Singapore Management University. It has been accepted for inclusion in Research Collection School Of Economics by an authorized administrator of Institutional Knowledge at Singapore Management University. For more information, please email [cherylds@smu.edu.sg](mailto:cherylds@smu.edu.sg).

# Incentive Compatibility in Multi-unit Auctions

by

Sushil Bikhchandani,<sup>†</sup> Shurojit Chatterji,<sup>‡</sup> and Arunava Sen<sup>\*</sup>

September 2004

(First version: June 2003)

## *Abstract*

We characterize incentive compatibility in multi-unit auctions with multi-dimensional types. An allocation mechanism is incentive compatible if and only if it is non-decreasing in marginal utilities (NDMU). The notion of incentive compatibility we adopt is dominant strategy in private value models and ex post incentive compatibility in models with interdependent values. NDMU is the following requirement: if changing one buyer's type, while keeping everyone else's types the same, changes this buyer's allocation then the new allocation must be relatively more attractive (or relatively less unattractive) to this buyer.

*JEL* Classification Numbers: D44.

*Keywords:* dominant strategy, ex post incentive compatibility, multi-unit auctions.

---

We are grateful to Joe Ostroy, Phil Reny, and Rakesh Vohra for helpful comments.

<sup>†</sup>Anderson School of Management, UCLA, <sup>‡</sup>I.T.A.M., Mexico City, and <sup>\*</sup>I.S.I., Delhi.

# 1 Introduction

We characterize incentive compatibility in multi-unit auctions with multi-dimensional types. We identify a simple new condition on allocation mechanisms, non-decreasing in marginal utilities (NDMU), which is necessary and sufficient for incentive compatibility. An allocation mechanism satisfies NDMU if changing one buyer's type (while keeping everyone else's types the same) changes this buyer's allocation under the mechanism, then the new allocation must be relatively more attractive (or relatively less unattractive) to this buyer. In effect, NDMU is a requirement that the allocation mechanism be sensitive to changes in marginal utilities.

In a private values model, a deterministic allocation mechanism is implementable in dominant strategies if and only if it is NDMU. This result extends to an interdependent values model which is informationally separable: a deterministic allocation mechanism is ex post incentive compatible if and only if it is NDMU.

The notion of incentive compatibility we adopt -- dominant strategy in private values settings and ex post incentive compatibility in the case of interdependent values -- is equivalent to requiring Bayesian incentive compatibility for all possible priors (see Ledyard 1978 and Bergemann and Morris 2001). Thus, the assumption of common knowledge of priors is not necessary for the mechanisms considered here. This weakening of common knowledge assumptions is in the spirit of the Wilson doctrine (see Wilson 1987).

Myerson (1981) showed that in a single object auction a random allocation mechanism is Bayesian incentive compatible if and only if each buyer's probability of receiving the object is non-decreasing in his reported type. In extensions of Myerson's characterization to models with multi-dimensional types, the necessary and sufficient condition for Bayesian incentive compatibility is that the random allocation rule should be the subgradient of a convex function.<sup>1</sup> This condition is not easy to work with. By focusing on deterministic dominant strategy allocation rules, we show that incentive compatibility is characterized by NDMU, which is much more intuitive than the subgradient condition. The resulting simplification of the constraint set for incentive compatible auctions will be helpful in applications such as in finding an expected revenue maximizing auction in the class of deterministic, dominant strategy auctions.<sup>2</sup>

Some of the recent literature on auctions has focused on efficiency. Maskin (1992) presented a single object model with interdependent values and multi-dimensional types and showed that any incentive compatible auction is inefficient. Jehiel and

---

<sup>1</sup>See, for example, McAfee and McMillan (1988), Williams (1999), Krishna and Perry (1997), and Jehiel and Moldovanu (2001).

<sup>2</sup>For an analysis of optimal multi-unit auctions of divisible objects see Maskin and Riley (1989).

Moldovanu (2001) established that efficiency is generically impossible in auctions with interdependent values and multi-dimensional types. An important open question is the existence and nature of a second-best auction, i.e., an auction which is efficient subject to the constraints imposed by incentive compatibility. Understanding the structure of incentive compatible auctions is a first step towards answering this question.

Even in private value settings, where the Vickrey-Clarke-Groves (VCG) mechanism is *ex post* efficient, there are reasons to be interested in other (inefficient) incentive compatible mechanisms. First, as already noted, the auctioneer may be interested in revenue (rather than efficiency). Second, it is well-known that because of its computational complexity the VCG auction is infeasible for selling more than a small number of objects. Several papers investigate computationally feasible (but inefficient) auctions in private value settings [see Nisan and Ronen 2000, Lehman et al. 1999, and Holzman and Monderer 2003]. Characterizing the set of incentive compatible auctions facilitates the selection of an auction that is preferable to the VCG auction on grounds of expected revenue or computational feasibility.

Roberts (1979) showed that in quasilinear environments with complete domain a condition called positive association of differences (PAD) is necessary and sufficient for dominant strategy incentive compatibility. A multi-unit auction maps into a much more restrictive domain of preferences than Roberts assumes. The PAD condition is vacuous in our model as all allocation rules satisfy it.

The paper is organized as follows. The characterization of incentive compatibility is developed in Section 2 for a single buyer model. In Section 3, we describe how this characterization extends easily to many buyers. In Section 4, we discuss the connections of our paper to the subgradient characterization of incentive compatibility and also to the papers of Roberts (1979) and Chung and Ely (2002). We conclude in Section 5. Some proofs are given in an Appendix.

## 2 A single buyer model

There are  $L$  indivisible, identical units of an object for sale to a buyer. The buyer's type is denoted by a  $K$  vector,  $\theta = (\theta_1, \theta_2, \dots, \theta_k, \dots, \theta_K)$ . The domain of  $\theta$  is  $D \subseteq \mathfrak{R}^K$ . We assume quasilinear preferences over (indivisible) objects and (divisible) money. The buyer's utility function when his type is  $\theta$ , he gets  $\ell$  units, and has  $m$  units of money is:

$$U(\theta, \ell, m) = V(\theta, \ell) + m.$$

The buyer has no endowment of the indivisible objects. It is convenient to assume that the buyer's initial endowment of money is normalized to zero and that the buyer

can supply any (negative) quantity required. Also,  $V(\theta, 0) = 0$  for all  $\theta$ . Hence the utility of no trade for the buyer is  $U(\theta, 0, 0) = 0$ . Throughout we assume free disposal:  $V(\theta, \ell) \geq V(\theta, \ell - 1) \geq 0, \forall \ell = 0, 1, 2, \dots, K$ .

An *auction* consists of an *allocation mechanism*  $h$  and a *payment function*  $t$  where  $h : D \rightarrow \{0, 1, 2, \dots, L\}$  is a function from the buyer's reported type to an allocation of the indivisible object to the buyer and  $t : D \rightarrow \Re$  is a function from the buyer's reported type to a money payment by the buyer.

The buyer's type is his private information. By the revelation principle, we restrict attention to direct mechanisms, where truthful revelation of his type is a best response for the buyer. Thus, we have the following definition.

An auction is *truth-telling* if truthfully reporting his type is optimal for the buyer:

$$V(\theta, h(\theta)) - t(\theta) \geq V(\theta, h(\theta')) - t(\theta'), \quad \forall \theta, \forall \theta'.$$

An allocation mechanism  $h$  is *truthful* if there exists a payment function  $t$  such that  $(h, t)$  is truth-telling;  $t$  is said to *implement*  $h$ . Consider the following restriction on the allocation mechanism.

An allocation mechanism  $h$  is *non-decreasing in marginal utilities (NDMU)* if for every  $\theta, \theta'$ , the following holds:

$$V(\theta', h(\theta')) - V(\theta', h(\theta)) \geq V(\theta, h(\theta')) - V(\theta, h(\theta)). \quad (1)$$

If  $h$  satisfies NDMU, then the difference in the buyer's utility between  $h(\theta')$  and  $h(\theta)$  at  $\theta'$  is greater than or equal to this difference at  $\theta$ .

NDMU is a simple, intuitive, and new condition on allocation mechanisms. In effect, it is a requirement that the allocation mechanism be sensitive to changes in marginal utilities. It is easy to show that NDMU is a necessary condition for truth-telling:

**Lemma 1** *If  $(h, t)$  is a truth-telling auction then  $h$  is NDMU.*

**Proof:** Let  $(h, t)$  be a truth-telling auction. Consider two types  $\theta'$  and  $\theta$  of the buyer. By the optimality of truth-telling at  $\theta$  and  $\theta'$  respectively, we have

$$\begin{aligned} V(\theta, h(\theta)) - t(\theta) &\geq V(\theta, h(\theta')) - t(\theta') \\ \text{and} \quad V(\theta', h(\theta')) - t(\theta') &\geq V(\theta', h(\theta)) - t(\theta) \end{aligned}$$

These two inequalities imply that

$$\begin{aligned} V(\theta', h(\theta')) - V(\theta', h(\theta)) &\geq t(\theta') - t(\theta) \\ &\geq V(\theta, h(\theta')) - V(\theta, h(\theta)). \end{aligned}$$

Hence  $h$  satisfies NDMU. ■

The proof of Lemma 1 does not use the fact that the objects are homogeneous --  $h(\cdot)$  may be an allocation of heterogenous objects to the buyer, or it might represent a decision that the buyer cares about. Clearly, NDMU is necessary for truth-telling in more general settings than considered in this paper.

Next, we obtain conditions on  $D$ , the domain of the buyer's type, under which NDMU is sufficient for truth-telling.

## 2.1 Sufficiency of NDMU

Without further loss of generality let  $L$ , the number of units of the object, equal  $K$ , the dimensionality of the type vector  $\theta$ . Let

$$V(\theta, k) = \begin{cases} \sum_{\ell=1}^k \theta_{\ell}, & \text{if } k = 1, 2, \dots, K, \\ 0, & \text{if } k = 0. \end{cases}$$

Thus,  $\theta_k$  is the marginal utility for  $k^{\text{th}}$  unit of the object for a buyer of type  $\theta$ .

An auction is truth-telling if

$$\sum_{\ell=1}^{h(\theta)} \theta_{\ell} - t(\theta) \geq \sum_{\ell=1}^{h(\theta')} \theta_{\ell} - t(\theta'), \quad \forall \theta, \forall \theta'. \quad (2)$$

Standard arguments imply that if  $(h, t)$  is truth-telling then  $h(\theta) = h(\theta')$  implies  $t(\theta) = t(\theta')$ . That is, the payment by a buyer depends only on his allocation.

Using (1) we see that an allocation mechanism  $h$  is NDMU if for every  $\theta$  and  $\theta'$ ,

$$\text{If } h(\theta') > h(\theta) \text{ then } \sum_{\ell=h(\theta)+1}^{h(\theta')} \theta'_{\ell} \geq \sum_{\ell=h(\theta)+1}^{h(\theta')} \theta_{\ell}. \quad (3)$$

If the buyer is allocated more units by the mechanism when his (reported) type is  $\theta'$  than when it is  $\theta$ , then it must be the case that his valuation at  $\theta'$  for these additional units is at least as large as his valuation at  $\theta$ .

Suppose that  $\theta = (\theta_1, \dots, \theta_k, \theta_{k+1}, \dots, \theta_K)$  is such that  $h(\theta) = k$ . If  $h$  is NDMU then we have the following conclusions:

- a. Let  $\theta' = (\theta_1, \dots, \theta_{k-1}, \theta_k + \epsilon, \theta'_{k+1}, \theta'_{k+2}, \dots, \theta'_K)$ , where  $\epsilon > 0$  and  $\theta' \in D$ .  
Then  $h(\theta') \geq k$ .
- b. Let  $\tilde{\theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_k, \theta_{k+1} - \epsilon, \theta_{k+2}, \dots, \theta_K)$ , where  $\epsilon > 0$  and  $\tilde{\theta} \in D$ . Then  $h(\tilde{\theta}) \leq k$ .
- c. Let  $\theta^* = (\theta_1, \dots, \theta_{k-1}, \theta_k + \epsilon, \theta_{k+1} - \epsilon, \theta_{k+2}, \dots, \theta_K)$ , where  $\epsilon > 0$  and  $\theta^* \in D$ .  
Combining a and b we have  $h(\theta^*) = k$ .

NDMU is stronger than the requirement that the allocation mechanism be non-decreasing, i.e., the requirement that  $h(\theta') \geq h(\theta)$  for every  $\theta'$  and  $\theta$  such that  $\theta'_\ell > \theta_\ell$ ,  $\forall \ell$ . It can be verified that (i) a mechanism that is NDMU is non-decreasing, and (ii) for  $K \geq 2$ , a non-decreasing mechanism need not be NDMU.

If the domain of the buyer's types,  $D$ , is not large enough then NDMU is not sufficient for truth-telling. This is clear from the following example with  $K = 2$  units.

**Example 1:** The buyer's marginal utilities can take one of three values:  $\theta^0 = (55, 15)$ ,  $\theta^1 = (60, 25)$ ,  $\theta^2 = (40, 35)$ . That is,  $D = \{\theta^0, \theta^1, \theta^2\}$ . Consider the allocation rule  $h(\theta^0) = 0$ ,  $h(\theta^1) = 1$ , and  $h(\theta^2) = 2$ . This allocation rule is NDMU on the set  $D$  because: (i)  $h(\theta^1) = 1 > 0 = h(\theta^0)$ ,  $\theta_1^1 = 60 > 55 = \theta_1^0$ ; (ii)  $h(\theta^2) = 2 > 1 = h(\theta^1)$ ,  $\theta_2^2 = 35 > 25 = \theta_2^1$ ; and (iii)  $h(\theta^2) = 2 > 0 = h(\theta^0)$ ,  $\theta_1^2 + \theta_2^2 = 75 > 70 = \theta_1^0 + \theta_2^0$ .

We show that there is no payment scheme that will make this allocation mechanism truthful. Suppose that the buyer payments are  $t^k$  at  $\theta^k$ ,  $k = 0, 1, 2$ . Without loss of generality, let  $t^0 = 0$ . Then we must have  $t^1 \geq 55$ , else type  $\theta^0$  would report  $\theta^1$ . Similarly,  $t^2 - t^1 \geq 25$  else type  $\theta^1$  would report  $\theta^2$ . Therefore, we must have  $t^2 \geq 80$ . But then  $\theta^2$  would report  $\theta^0$ .

Even if the domain over which the allocation rule is defined is connected, NDMU is not sufficient for truthfulness. Let  $\hat{D}$  be the sides of the triangle with corners  $\theta^0, \theta^1, \theta^2$  defined above. The allocation rule  $\hat{h}$  is as follows:  $\hat{h}(\theta) = 0$ ,  $\forall \theta \in [\theta^0, \theta^2]$ ,  $\hat{h}(\theta) = 1$ ,  $\forall \theta \in [\theta^1, \theta^0]$ , and  $\hat{h}(\theta) = 2$ ,  $\forall \theta \in [\theta^2, \theta^1]$ . It may be verified that NDMU is satisfied but there are no payments that induce truth-telling.  $\triangle$

Requiring NDMU on a larger domain strengthens this condition. In the rest of the paper, we make the following assumption on  $D$ .

**Domain Assumption:** *There exist constants  $\bar{a}_k \in (0, \infty)$ ,  $k = 1, 2, \dots, K$ , such that the domain of buyer types,  $D$ , satisfies either (A) or (B) below:*

A.  $D = \prod_{k=1}^K [0, \bar{a}_k]$

B.  $D$  is the convex hull of points  $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k-1}, \bar{a}_k, 0, \dots, 0)$ ,  $k = 0, 1, \dots, K$ .

The assumption that  $\bar{a}_k < \infty$  for all  $k$  is not essential, but does simplify the proofs. Domain assumption A does not restrict the marginal utilities to be decreasing (or increasing). We do not specifically assume that  $\bar{a}_k \geq \bar{a}_{k+1}$ , but when this inequality holds for all  $k$  and domain assumption B is satisfied, then we have diminishing marginal utilities; that is,  $\theta_k \geq \theta_{k+1}$  for all  $\theta \in D$ . Under domain assumption B,  $\theta = (\theta_1, \theta_2, \dots, \theta_K) \in D$  if and only if  $0 \leq \theta_\ell \leq \bar{a}_\ell$ ,  $\forall \ell$  and

$$\frac{\theta_\ell}{\bar{a}_\ell} \geq \frac{\theta_{\ell+1}}{\bar{a}_{\ell+1}} \quad \ell = 1, 2, \dots, K - 1. \quad (4)$$

A straightforward modification of our proofs extends all our results to the case of

increasing marginal utilities, i.e., when  $D$  is the convex hull of points  $(0, 0, \dots, 0, \bar{a}_k, \bar{a}_{k+1}, \dots, \bar{a}_K)$ ,  $k = 1, \dots, K$  and  $(0, 0, \dots, 0)$ . The assumption of increasing marginal utilities obtains when the objects are complements, such as airwave spectrum rights.

The inverse of an allocation rule  $h$  is:<sup>3</sup>

$$Y(k) \equiv \{\theta \in D \mid h(\theta) = k\}.$$

$Y(k)$  is the set of the buyer's reported types at which he is allocated  $k$  units. By definition,  $Y(k)$ ,  $k = 0, 1, 2, \dots, K$  are mutually exclusive and their union is  $D$ .

The next lemma allows us to assume, without loss of generality, that  $Y(k) \neq \emptyset$  for all  $k$ .  $D^K$  and  $D^{K-S}$  refer to subsets which satisfy the domain assumption in  $\mathfrak{R}^K$  and  $\mathfrak{R}^{K-S}$  respectively.<sup>4</sup>

**Lemma 2** *Let  $\hat{h}$  be an allocation rule that is NDMU on  $D^K$ . Suppose that  $\hat{Y}(k) = \emptyset$  for each  $k = k_i$ ,  $0 \leq k_1 < k_2 < \dots < k_S \leq K$ . There exists an allocation rule  $h$  on  $D^{K-S}$  such that  $Y(q) \neq \emptyset$ ,  $\forall q = 0, 1, \dots, K - S$  and:*

(i)  *$h$  is NDMU on  $D^{K-S}$ .*

(ii) *If  $h$  is truthfully implementable then  $\hat{h}$  is truthfully implementable.*

Essentially,  $h$  is obtained from  $\hat{h}$  by bundling a unit  $k_i$  for which  $\hat{Y}(k_i) = \emptyset$  with the next higher unit  $k$  for which  $\hat{Y}(k) \neq \emptyset$ . (From the proof of the above lemma it is clear that  $h$  achieves the same allocation as  $\hat{h}$  for almost all  $\theta$ .) Thus, if we start with an NDMU allocation rule  $\hat{h}$  which does not have full range (i.e.,  $\hat{Y}(k) = \emptyset$  for some  $k$ ), by Lemma 2(i) we can map it to another NDMU allocation rule  $h$  which has full range on a smaller commodity space. Then, it is enough to show that NDMU of the full range  $h$  is sufficient for  $h$  to be truthful, because Lemma 2(ii) implies that the original allocation rule  $\hat{h}$  (which is not full range) is truthful. Thus, without any loss of generality, we consider only full range allocation mechanisms. This simplifies the proofs.

We sketch a geometric argument for the sufficiency of NDMU for truth-telling. Restrict attention to  $K = 2$  and assume that  $D = [0, 1]^2$  satisfies domain assumption A. We first establish the structure of the sets  $Y(k)$ ,  $k = 0, 1, 2$ . Then, we obtain payments that will induce the buyer to reveal his private information,  $\theta$ .

Figure 1a shows three points  $\theta^0$ ,  $\theta^1$ , and  $\theta^2$  such that  $\theta^k \in Y(k)$ ,  $k = 0, 1, 2$ . First, consider  $\theta^0$ . Any  $\theta$  inside trapezoid A to the left of  $\theta^0$  satisfies  $\theta_1 < \theta_1^0$ ,  $\theta_1 + \theta_2 < \theta_1^0 + \theta_2^0$ . Since  $\theta^0 \in Y(0)$ , NDMU implies that all points in A belong to  $Y(0)$ . Similarly, all points in the rectangle B to the right and below  $\theta^1$  belong to  $Y(1)$ ; and all points in the trapezoid C above  $\theta^2$  belong to  $Y(2)$ . Moreover, any  $\theta$  in the unit square belongs

<sup>3</sup>The dependence of  $Y$  on the mechanism  $h$  is suppressed for notational convenience.

<sup>4</sup>The proofs of this lemma and of most subsequent results are in an appendix.



to either  $Y(0)$ ,  $Y(1)$ , or  $Y(2)$ ; if  $\theta$  is in  $Y(0)$  then to its left is a trapezoid like A which belongs to  $Y(0)$ , etc. Thus, applying the above argument to all points in the unit square we see that the sets  $Y(0)$ ,  $Y(1)$ , and  $Y(2)$  must be as shown in Figure 1b, with any point on the boundary between  $Y(k)$  and  $Y(k')$  belonging to any one of these two sets.<sup>5</sup> Consequently, the sets  $\text{cl}[Y(k)]$ ,  $k = 0, 1, 2$ , intersect at one point,  $c = (c_1, c_2)$  and<sup>6</sup>

$$\begin{aligned}\text{cl}[Y(0)] &= \{\theta \in [0, 1] \mid \theta_1 \leq c_1, \theta_1 + \theta_2 \leq c_1 + c_2\} \\ \text{cl}[Y(1)] &= \{\theta \in [0, 1] \mid \theta_1 \geq c_1, \theta_2 \leq c_2\} \\ \text{cl}[Y(2)] &= \{\theta \in [0, 1] \mid \theta_2 \geq c_2, \theta_1 + \theta_2 \geq c_1 + c_2\}.\end{aligned}$$

Let the buyer's payment when he gets 0 units [i.e., when  $\theta \in Y(0)$ ] be 0, when he gets 1 unit he pays  $c_1$ , and when he gets 2 units his payment is  $c_1 + c_2$ . It is easily verified that these payments induce truth-telling by the buyer.

In the remainder of this section, we generalize this argument to more than two objects. For  $k = 1, 2, \dots, K$  define<sup>7</sup>

$$\begin{aligned}\theta_k^m &\equiv \sup\{\theta_k \mid \theta \in Y(k-1)\}, \\ \theta_k^M &\equiv \inf\{\theta_k \mid \theta \in Y(k)\}.\end{aligned}$$

A direct consequence of NDMU is that  $\theta_k^M \geq \theta_k^m$ . Using Figures 1a and 1b, we gave a heuristic argument for the equalities  $\theta_k^m = \theta_k^M$ ,  $k = 1, 2$ , and defined  $c_k \equiv \theta_k^m = \theta_k^M$ ; the  $c_k$ 's provided the money payments that supported  $h$  as a truthful mechanism. That is the basic plan of the proof of sufficiency of NDMU for truth-telling. However, unless  $Y(k-1)$  and  $Y(k)$  have a non-empty interior,  $\theta_k^M = \theta_k^m$  need not be true without the "tie-breaking" assumption, TBB, defined below. Therefore, we first prove sufficiency of NDMU and TBB (Lemmas 3 and 4). We then show that for any NDMU allocation rule  $h$  there exists an allocation rule  $h'$  that satisfies NDMU and TBB, and  $h$  and  $h'$  are truthfully implemented by the same money payments (Lemma 5).

**Tie-breaking at boundaries (TBB):** Let  $\theta_k^m$  and  $\theta_k^M$  be defined for an allocation mechanism  $h$ . Then  $h$  satisfies TBB if:

- (i)  $\theta_k > 0$  for all  $\theta \in Y(k)$ , and
- (ii)  $\theta_k < \bar{a}_k$  for all  $\theta \in Y(k-1)$ .

Consider TBB(i). If  $\theta_k^M > 0$  then the definition of  $\theta_k^M$  implies that for all  $\theta \in Y(k)$  we have  $\theta_k \geq \theta_k^M > 0$ . Thus, TBB(i) imposes no restriction. If, instead,  $\theta_k^M = 0$

<sup>5</sup>It is also possible that the boundary between  $Y(0)$  and  $Y(2)$  meets the left vertical side of the unit square, rather than the top horizontal side as shown.

<sup>6</sup>The closure of  $Y(k)$  is denoted  $\text{cl}[Y(k)]$ .

<sup>7</sup>For notational simplicity, the dependence of  $\theta_k^m$  and  $\theta_k^M$  on  $h$  is suppressed.

then the definition of  $\theta_k^M$  implies that there exists a sequence  $\theta^n \in Y(k)$  such that  $\lim_{n \rightarrow \infty} \theta_k^n = 0$ ; the existence of a point  $\theta \in Y(k)$  at which  $\theta_k = \theta_k^M = 0$  is precluded by TBB(i). Similarly, TBB(ii) imposes no restriction if  $\theta_k^m < \bar{a}_k$ , and when  $\theta_k^m = \bar{a}_k$  it requires that for any  $\theta \in Y(k-1)$ , we have  $\theta_k < \bar{a}_k$ .

In effect, TBB takes care of the difficulties that arise when  $Y(k)$  has an empty interior for some  $k$ , i.e., the point  $(c_1, c_2)$  in Figure 1b lies on the edge of the domain  $D$ . To see this, consider Figure 2, which shows the regions  $Y(0)$ ,  $Y(1)$ , and  $Y(2)$  for two NDMU allocation rules which differ only along the line segment  $ABC$ . In Figure 2a,  $Y(1)$  is the line segment  $BC$  and the line segment  $AB$  belongs to  $Y(2)$ ; there are  $\theta \in Y(2)$  along the segment  $AB$  for which  $\theta_2 = 0$  and therefore TBB(i) is not satisfied. In Figure 2b TBB(i) is satisfied. Observe that  $\theta_1^M = \theta_1^m$  in Figure 2b (but not in Figure 2a). Let  $c_1 \equiv \theta_1^M = \theta_1^m$  for the allocation rule depicted in Figure 2b. The payments  $c_1$  for 1 unit and  $c_1 + 0 = c_1$  for the second unit implement the allocation rule in Figure 2b *and* the allocation rule in Figure 2a. Starting with the NDMU allocation rule in Figure 2a, by transferring  $AB$  (or more generally transferring points in  $Y(2)$  for which  $\theta_2 = 0$ ) from  $Y(2)$  to  $Y(1)$  we get the NDMU and TBB allocation rule in Figure 2b. The prices  $c_k$  can be defined from the  $\theta_k^M$  and  $\theta_k^m$  for the allocation rule in Figure 2b which satisfies NDMU and TBB, and these prices yield money payments that truthfully implement the allocation rules in Figures 2b *and* 2a. This, essentially, is the intuition underlying the next three lemmas.

**Lemma 3** *Suppose that  $h$  is NDMU and TBB. Then  $\bar{a}_k \geq \theta_k^M = \theta_k^m \geq 0$  for all  $k$ , where  $\theta_k^M$  and  $\theta_k^m$  are defined w.r.t.  $h$ .*

For an allocation rule  $h$  satisfying NDMU and TBB, define  $c_k \equiv \theta_k^M = \theta_k^m$  for all  $k$ . Next, we show that the “prices”  $c_k$  (yield money payments that) truthfully implement the allocation rule  $h$ .

**Lemma 4** *Suppose that an allocation mechanism  $h$  is NDMU and TBB. Then for any  $k = 0, 1, 2, \dots, K$ , we have*

$$\sum_{\ell=1}^k \theta_\ell - \sum_{\ell=1}^k c_\ell \geq \sum_{\ell=1}^q \theta_\ell - \sum_{\ell=1}^q c_\ell, \quad \forall q \neq k, \forall \theta \in Y(k).$$

*Thus, the prices  $c_\ell$  truthfully implement the allocation rule  $h$ .*

The next lemma allows us to weaken the sufficient condition to NDMU.

**Lemma 5** *If an allocation mechanism  $h$  satisfies NDMU then there exists an allocation mechanism  $h'$  which satisfies NDMU and TBB such that  $h(\theta) = h'(\theta)$ , for almost all  $\theta \in D$ . Moreover, with  $c'_\ell$  defined with respect to  $h'$ , the money payments  $\sum_{\ell=1}^k c'_\ell$  truthfully implement  $h$ .*

This leads to the main result for the single buyer model.

**Theorem 1** *An allocation mechanism is truthful if and only if it is non-decreasing in marginal utilities.*

**Proof:** In view of Lemma 1 we only need to prove that NDMU is sufficient for truthfulness. Using the procedure given in the proof of Lemma 5, construct an allocation rule  $h'$  which is NDMU and TBB. By Lemma 4, we can obtain prices  $c'_\ell$  that implement  $h'$ . From Lemma 5 we know that these prices also implement  $h$ . Thus, NDMU is sufficient for truthfulness. ■

**Remark 1:** If an allocation mechanism satisfies NDMU on a utility domain  $D$  then it continues to satisfy NDMU if one adds a constant to the marginal utility of the  $k$ th unit. All the results in this section can be proved (with minor changes in the proofs) if each  $\theta_k$  takes values in  $[\underline{a}_k, \bar{a}_k]$ ,  $0 < \underline{a}_k < \bar{a}_k$ , instead of taking values in  $[0, \bar{a}_k]$ . This fact will be important in Section 3.2, where we generalize Theorem 1 to a model with many buyers and interdependent values.

**Remark 2:** NDMU implies that for any  $k, q \in \{0, 1, 2, \dots, K\}$ ,  $k < q$ , there exists a hyperplane separating the (possibly non-convex) sets  $Y(k)$  and  $Y(q)$ . This is true regardless of the domain of types  $D$ . Take any  $C_{kq}$  which satisfies

$$\sup_{\theta \in D} \left\{ \sum_{\ell=k+1}^q \theta_\ell \mid \theta \in Y(k) \right\} \leq C_{kq} \leq \inf_{\theta \in D} \left\{ \sum_{\ell=k+1}^q \theta_\ell \mid \theta \in Y(q) \right\}.$$

As NDMU implies that the sup on the left is no greater than the inf on the right, it is possible to choose such a  $C_{kq}$ . The hyperplane  $\sum_{\ell=k+1}^q \theta_\ell = C_{kq}$  (weakly) separates  $Y(k)$  and  $Y(q)$ . Any proof of sufficiency of NDMU must show that there exists a point  $c = (c_1, c_2, \dots, c_K) \in \mathfrak{R}^K$  through which passes a separating hyperplane between every pair of regions  $Y(k)$  and  $Y(q)$ . In particular, for  $k < q$ , the hyperplane  $\sum_{\ell=k+1}^q \theta_\ell = C_{kq} \equiv \sum_{\ell=k+1}^q c_\ell$  separates  $Y(k)$  and  $Y(q)$ .<sup>8</sup> The point  $c$  represents the prices with which one can truthfully implement the allocation rule. The proof of Theorem 1 shows that if an allocation rule  $h$  satisfies NDMU on domain A or B, then there exists a price vector  $(c_1, c_2, \dots, c_K)$  such that  $h$  maximizes the buyer's utility at these prices.<sup>9</sup>

After completing this paper, we became aware of Lavi et al. (2003), who independently obtain a similar characterization of dominant strategy incentive compatibility in private value models. There are some differences between Lavi et al.'s paper and ours: (i) their proofs appear to depend on an assumption that marginal utilities are

<sup>8</sup>From Example 1 we know that unless NDMU is satisfied on a large enough domain  $D$  such a point may not exist.

<sup>9</sup>The connection between parametric pricing and incentive compatibility has been noted by Barbera and Jackson (1995), who obtain a fixed-price characterization of dominant strategy allocation rules in an exchange economy.

unbounded above; and (ii) they show that NDMU together with full range implies dominant strategy implementability in multi-object auctions.

### 3 Extension to multiple buyers

There are  $b = 1, 2, \dots, B$  buyers. Buyer  $b$ 's type is denoted by  $\theta_b = (\theta_{b1}, \theta_{b2}, \dots, \theta_{bk}, \dots, \theta_{bK})$ , where each  $\theta_b \in D_b \subset \mathfrak{R}_+^K$ . Each  $D_b$  satisfies the domain assumption of Section 2.1. The characteristics of all the buyers are denoted by  $\theta = (\theta_1, \theta_2, \dots, \theta_b, \dots, \theta_B)$  or by  $(\theta_b, \theta_{-b})$ . Buyer  $b$ 's utility function when the types are  $(\theta_b, \theta_{-b})$ , he gets  $k$  units, and has  $m$  units of money is:

$$U_b(\theta_b, \theta_{-b}, k, m) = V_b(\theta_b, \theta_{-b}, \ell) + m.$$

A *feasible allocation*  $f = (f_1, f_2, \dots, f_b, \dots, f_B)$  is a  $B$  vector such that each  $f_b \in \{0, 1, \dots, K\}$  and  $\sum_b f_b \leq K$ . Let  $F$  be the set of feasible allocations. An *auction* consists of an *allocation mechanism*  $H = (h_1, h_2, \dots, h_b, \dots, h_B)$  and a *transfer function*  $T = (t_1, t_2, \dots, t_b, \dots, t_B)$  such that  $H$  is a function from the buyers' reported types to a feasible allocation and  $T$  is a function from the buyers' reported types to money payments by the buyers.

An allocation mechanism  $H$  is NDMU if for every  $b, \theta_{-b}, \theta_b, \theta'_b$ , the following holds:

$$\begin{aligned} V_b(\theta'_b, \theta_{-b}, h_b(\theta'_b, \theta_{-b})) &- V_b(\theta'_b, \theta_{-b}, h_b(\theta_b, \theta_{-b})) \\ &\geq V_b(\theta_b, \theta_{-b}, h_b(\theta'_b, \theta_{-b})) - V_b(\theta_b, \theta_{-b}, h_b(\theta_b, \theta_{-b})). \end{aligned} \quad (5)$$

The appropriate generalization of incentive compatibility in the multiple buyer case depends on whether buyers have private or interdependent values. These two cases are taken up next.

#### 3.1 Private values

In this case  $V_b(\theta_b, \theta_{-b}, \ell) = V_b(\theta_b, \ell)$  and we assume that

$$V_b(\theta_b, k) = \begin{cases} \sum_{\ell=1}^k \theta_{b\ell}, & \text{if } k = 1, 2, \dots, K, \\ 0, & \text{if } k = 0. \end{cases}$$

A *dominant strategy auction* is one in which truth-telling is a dominant strategy for all buyers. That is,

$$\sum_{\ell=1}^{h_b(\theta_b, \theta_{-b})} \theta_{b\ell} - t_b(\theta_b, \theta_{-b}) \geq \sum_{\ell=1}^{h_b(\theta'_b, \theta_{-b})} \theta_{b\ell} - t_b(\theta'_b, \theta_{-b}), \quad \forall \theta_b, \forall \theta'_b, \forall \theta_{-b}, \forall b. \quad (6)$$

An allocation mechanism  $H$  is NDMU if for every  $b$ ,  $\theta_{-b}$ ,  $\theta_b$ ,  $\theta'_b$ , the following holds:

$$\text{If } h_b(\theta'_b, \theta_{-b}) > h_b(\theta_b, \theta_{-b}) \text{ then } \sum_{\ell=h_b(\theta_b, \theta_{-b})+1}^{h_b(\theta'_b, \theta_{-b})} \theta'_{b\ell} \geq \sum_{\ell=h_b(\theta_b, \theta_{-b})+1}^{h_b(\theta'_b, \theta_{-b})} \theta_{b\ell}. \quad (7)$$

Observe that the requirement of dominant strategy in (6) is the same as requiring truth-telling ( i.e. (2) ) for each buyer  $b$  and for each value of  $\theta_{-b}$ . Moreover (7), the multi-buyer version of NDMU, is equivalent to requiring (3) for each buyer  $b$  and for each value of  $\theta_{-b}$ . Fix  $\theta_{-b}$ , the types of other buyers. Because each  $D_b$  satisfies the domain assumption of Section 2.1, all the results of the single buyer model extend to the multi-buyer case under private values. Thus, the counterpart of Theorem 1 is

**Theorem 2** *An allocation mechanism is dominant strategy implementable if and only if it is non-decreasing in marginal utilities.*

### 3.2 Interdependent values

Each buyer's utility may depend on other buyers' types. We restrict attention to the *informationally separable* case.<sup>10</sup> That is, each  $\theta_{bk}$  conveys information (only) about the marginal utility of consuming a  $k^{\text{th}}$  unit of the object. This allows us to simplify  $V_b(\theta_b, \theta_{-b}, k)$ , the reservation value for consuming  $k$  units, as follows:

$$V_b(\theta_b, \theta_{-b}, k) = \begin{cases} \sum_{\ell=1}^k V_{b\ell}(\theta_{b\ell}, \theta_{-b\ell}), & \text{if } k \geq 1, \\ 0, & \text{if } k = 0. \end{cases}$$

where  $\theta_{-b\ell} = (\theta_{1\ell}, \theta_{2\ell}, \dots, \theta_{b-1\ell}, \theta_{b+1\ell}, \dots, \theta_{B\ell})$ . Buyer  $b$ 's marginal valuation for the  $k$ th unit is  $V_{bk}(\theta_{bk}, \theta_{-bk})$ . Free disposal implies that  $V_{bk}(\theta_{bk}, \theta_{-bk}) \geq 0$ . We assume that  $V_{bk}(\theta_{bk}, \theta_{-bk})$  is increasing and continuous in  $\theta_{bk}$  for each  $b, k$ .<sup>11</sup>

The appropriate generalization of truth-telling in this setting is *ex post incentive compatibility*. An allocation mechanism is ex post incentive compatible if:

$$\sum_{\ell=1}^{h_b(\theta_b, \theta_{-b})} V_{b\ell}(\theta_{b\ell}, \theta_{-b\ell}) - t_b(\theta_b, \theta_{-b}) \geq \sum_{\ell=1}^{h_b(\theta'_b, \theta_{-b})} V_{b\ell}(\theta_{b\ell}, \theta_{-b\ell}) - t_b(\theta'_b, \theta_{-b}), \quad \forall \theta_b, \forall \theta'_b, \forall \theta_{-b}, \forall b. \quad (8)$$

<sup>10</sup>See Krishna (2000, p. 248) for a heterogeneous object auction model with informationally separable interdependent values.

<sup>11</sup>We do not make any assumption about the dependence of  $V_{bk}(\theta_{bk}, \theta_{-bk})$  on  $\theta_{-bk}$ . Thus, neither the single crossing condition nor the assumption that  $V_{bk}(\theta_{bk}, \theta_{-bk})$  be non-decreasing in  $\theta_{-bk}$  is required.

Ex post incentive compatibility is the same as uniform equilibrium of D'Aspremont and Gerard-Varet (1979) and uniform incentive compatibility of Holmstrom and Myerson (1983).

An allocation rule is NDMU if for every  $b, \theta_{-b}, \theta_b, \theta'_b$

$$\text{If } h_b(\theta'_b, \theta_{-b}) > h_b(\theta_b, \theta_{-b}) \text{ then } \sum_{\ell=h_b(\theta_b, \theta_{-b})+1}^{h_b(\theta'_b, \theta_{-b})} V_{b\ell}(\theta'_{b\ell}, \theta_{-b\ell}) \geq \sum_{\ell=h_b(\theta_b, \theta_{-b})+1}^{h_b(\theta'_b, \theta_{-b})} V_{b\ell}(\theta_{b\ell}, \theta_{-b\ell}). \quad (9)$$

As in the private values case, (8) and (9) generalize single buyer versions of truth-telling and NDMU for each buyer, for each realization of other buyers' types. It needs to be checked that the domain assumption, as modified in the Remark 1 following Theorem 1, is satisfied.

Fix a buyer  $b$  and the types of the other buyers,  $\theta_{-b}$ . Recall that the smallest and largest values for  $\theta_{bk}$  are 0 and  $\bar{a}_{bk}$  respectively. Let  $\underline{V}_{bk}(\theta_{-bk}) \equiv V_{bk}(0, \theta_{-bk})$  and  $\bar{V}_{bk}(\theta_{-bk}) \equiv V_{bk}(\bar{a}_{bk}, \theta_{-bk})$ . As  $V_{bk}(\theta_{bk}, \theta_{-bk})$  is increasing in  $\theta_{bk}$ ,  $\bar{V}_{bk}(\theta_{-bk}) > \underline{V}_{bk}(\theta_{-bk}) \geq 0$ . The continuity of  $V_{bk}(\theta_{bk}, \theta_{-bk})$  in  $\theta_{bk}$  implies that as  $\theta_{bk}$  increases from 0 to  $\bar{a}_{bk}$ , buyer  $b$ 's marginal utility for the  $k$ th unit increases continuously from  $\underline{V}_{bk}(\theta_{-bk})$  to  $\bar{V}_{bk}(\theta_{-bk})$ . Thus, as the domain of  $\theta_b$  satisfies the domain assumption, so does the domain of the marginal utilities of buyer  $b$  for a fixed value of  $\theta_{-b}$ , with the qualification that the smallest marginal utility for the  $k$ th unit may be strictly positive rather than 0. But under this modified domain assumption NDMU is sufficient to induce truth-telling (see Remark 1). Thus, we have

**Theorem 3** *In informationally separable, interdependent values models, an allocation mechanism is ex post implementable if and only if it is non-decreasing in marginal utilities.*

Whether or not the model is informationally separable, there always exist non-trivial mechanisms satisfying NDMU. It is the domain assumption on marginal utilities that is essential to our proof of sufficiency of NDMU. One set of primitive assumptions under which this domain requirement of Section 2.1 is met is informational separability. Jehiel et al. (2004) show that in a non-informationally separable, interdependent values model with multi-dimensional types *and* consumption externalities, the set of ex post incentive compatible mechanisms consist of trivial (constant) allocation mechanisms, except in non-generic cases. As there are no consumption externalities in our setting, the Jehiel et al. (2004) result does not apply.<sup>12</sup> Therefore, the possibility that the domain assumption on marginal utilities holds (and therefore NDMU is sufficient for ex post incentive compatibility) in a non-informationally separable model with interdependent values remains open.

---

<sup>12</sup>See Bikhchandani (2004) for more on this issue.

## 4 Relationship to earlier work

We discuss how our paper relates to the literature on auctions with private values. Myerson (1981) showed that a necessary and sufficient condition for incentive compatibility of a single object auction is that each buyer's probability of receiving the object is non-decreasing in his reported valuation.<sup>13</sup> Several authors, including McAfee and McMillan (1988), Williams (1999), Krishna and Perry (1997), and Jehiel and Moldovanu (2001), have extended Myerson's analysis to obtain necessary and sufficient conditions for Bayesian incentive compatible mechanisms in the presence of multi-dimensional types. These results are easily adapted to dominant strategy mechanisms.

To place our results in the context of this earlier work, consider a probabilistic allocation rule  $G = (g_1, g_2, \dots, g_b, \dots, g_B)$  where  $g_b = (g_{b1}, g_{b2}, \dots, g_{bk}, \dots, g_{bK})$  and  $g_{bk}(\theta_b, \theta_{-b})$  is the probability that buyer  $b$  gets at least  $k$  units when buyer types are  $(\theta_b, \theta_{-b})$ .<sup>14</sup> An auction  $(G, T)$  induces the following payoff function for buyer  $b$ :

$$\Pi_b(\theta_b, \theta_{-b}) \equiv g_b(\theta_b, \theta_{-b}) \cdot \theta_b - t_b(\theta_b, \theta_{-b}),$$

where  $x \cdot y$  denotes the dot product of two vectors  $x$  and  $y$ . Dominant strategy incentive compatibility implies that for all  $b, \theta_b, \theta'_b, \theta_{-b}$ ,

$$\begin{aligned} \Pi_b(\theta_b, \theta_{-b}) &\geq g_b(\theta'_b, \theta_{-b}) \cdot \theta_b - t_b(\theta'_b, \theta_{-b}) \\ &= \Pi_b(\theta'_b, \theta_{-b}) + g_b(\theta'_b, \theta_{-b}) \cdot (\theta_b - \theta'_b) \\ \implies \Pi_b(\theta_b, \theta_{-b}) &= \max_{\theta'_b} \{g_b(\theta'_b, \theta_{-b}) \cdot \theta_b - t_b(\theta'_b, \theta_{-b})\} \end{aligned} \quad (10)$$

As  $\Pi_b(\cdot, \theta_{-b})$  is the maximum of a family of linear functions it is a convex function of  $\theta_b$ . Further,  $g_b(\cdot, \theta_{-b})$  is a subgradient of  $\Pi_b(\cdot, \theta_{-b})$ . This leads to the following characterization for allocation rules. A probabilistic allocation rule is incentive compatible if and only if it is the subgradient of a convex function.

Our contribution is to show that when one restricts attention to deterministic allocations rules, the subgradient condition simplifies considerably to NDMU. Thus, our results lead to a transparent necessary and sufficient condition for dominant strategy incentive compatibility in multi-unit auctions. To verify directly that the subgradient condition implies NDMU for a deterministic mechanism, define for a (deterministic) allocation rule  $H$ :

$$g_{bk}(\theta_b, \theta_{-b}) \equiv \begin{cases} 1, & \text{if } h_b(\theta_b, \theta_{-b}) \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

<sup>13</sup>Myerson characterized Bayesian incentive compatibility; simple modifications to his proofs yield a similar characterization for dominant strategy incentive compatibility. For deterministic allocation mechanisms, this characterization coincides with NDMU specialized to single object settings.

<sup>14</sup>Feasibility of the allocation rule  $G$  implies that  $1 \geq g_{b1}(\theta) \geq g_{b2}(\theta) \geq \dots \geq g_{bK}(\theta) \geq 0, \forall b, \forall \theta$  and  $\sum_b \sum_k g_{bk}(\theta) \leq K, \forall \theta$ .

Next, observe that (10) implies

$$g_b(\theta'_b, \theta_{-b}) \cdot (\theta'_b - \theta_b) \geq \Pi_b(\theta'_b, \theta_{-b}) - \Pi_b(\theta_b, \theta_{-b}) \geq g_b(\theta_b, \theta_{-b}) \cdot (\theta'_b - \theta_b)$$

which implies

$$[g_b(\theta'_b, \theta_{-b}) - g_b(\theta_b, \theta_{-b})] \cdot (\theta'_b - \theta_b) \geq 0. \quad (11)$$

Using (7), it is easily verified that a deterministic allocation mechanism  $H$  satisfies NDMU if and only if its associated  $G$  function satisfies (11). It is much harder to establish directly that NDMU implies the subgradient condition.

Although our characterization of incentive compatibility is significantly simpler, the restriction to deterministic mechanisms may be a limitation. Manelli and Vincent (2003) and Thanassoulis (2004) show that a multi-product monopolist can strictly increase profits by using a random, rather than deterministic, mechanism. Their examples involve heterogeneous goods, but it remains possible that a seller of identical goods may prefer a random selling scheme. Therefore, it is worthwhile investigating whether our approach generalizes to random mechanisms.

Define NDMU to be (11) for a random allocation rule  $G = (g_1, g_2, \dots, g_B)$ . The following example establishes that for random allocation rules NDMU is not sufficient for truth-telling.<sup>15</sup> There are two units and one buyer whose valuation  $\theta$  is in the unit square. Define  $g(\theta) = \frac{1}{3}A\theta$ , where  $A$  is the matrix  $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ . We have

$$[g(\theta') - g(\theta)] \cdot (\theta' - \theta) = 1/3[A(\theta' - \theta)] \cdot (\theta' - \theta) = 1/3(\theta' - \theta)^T A^T (\theta' - \theta) \geq 0,$$

where the inequality follows as  $A$  is positive semi-definite. Thus,  $g$  is NDMU. But  $g$  cannot be the subgradient of a convex function as the matrix of second partials of this convex function would then be  $A$ , which is not possible as  $A$  is not symmetric. Whether there is an intuitive condition, which in conjunction with NDMU, is sufficient for incentive compatibility of random allocation rules is an open question.

Roberts (1979) characterizes dominant strategy mechanisms in quasilinear environments with a complete domain. He considers a model where the set of alternatives  $F$  is finite and for all  $f \in F$ , any real number  $\alpha$ , and any agent  $b \in B$ , there exists a type  $\theta_b$  of agent  $b$  such that  $V_b(\theta_b, f) = \alpha$ . Roberts identifies a condition called *positive association of differences* (PAD) which is satisfied by an allocation rule  $H$  if for all  $\theta$  and  $\theta'$

$$\begin{aligned} &\text{if } V_b(\theta'_b, H(\theta)) - V_b(\theta'_b, f) > V_b(\theta_b, H(\theta)) - V_b(\theta_b, f), \quad \forall f \neq H(\theta), \forall b, \\ &\text{then } H(\theta) = H(\theta'). \end{aligned} \quad (12)$$

---

<sup>15</sup>We are grateful to an anonymous referee for this example.



An allocation rule  $H$  is an *affine maximizer* if there exist constants  $\gamma_b \geq 0$ , with at least one  $\gamma_b > 0$ , and a function  $V_0 : F \rightarrow \Re$  such that

$$H(\theta) \in \arg \max_{f \in F} \left( V_0(f) + \sum_{b=1}^B \gamma_b V_b(\theta_b, f) \right).$$

Roberts (1979) shows that  $H$  is a (deterministic) dominant strategy mechanism if and only if  $H$  satisfies PAD if and only if  $H$  is an affine maximizer.

What is the relationship between Roberts' work and ours? The fundamental difference is that Roberts assumes a complete domain of preferences while we operate in a *restricted* domain. In our setting the set of alternatives  $F$  is the set of feasible allocations. In a feasible allocation  $f = (f_1, f_2, \dots, f_B)$ , buyer  $b$ 's allocation is  $f_b$ . Buyer  $b$  is indifferent between two feasible allocations (alternatives)  $f$  and  $f'$  whenever  $f_b = f'_b$ , i.e., there are no externalities in consumption. A further restriction is implied by free disposal in our model: if  $f_b \leq f'_b$  then  $b$  weakly prefers  $f'$  to  $f$ .

An immediate consequence of our restricted domain is that PAD becomes vacuous. It is easy to verify that if there are at least two buyers, PAD is satisfied by all allocation rules.<sup>16</sup> Even in a one buyer model, our Theorem 1 is not a corollary of Roberts' characterization. To use Roberts' theorem, we would have to assume that (i) the utility of getting  $k$  units can either be more or less than  $k - 1$  units and (ii) the utility of being allocated a bundle  $k$  can be any arbitrary number, negative or positive.<sup>17</sup> Even with these assumptions, one would not be able to generalize Roberts' theorem from a single buyer to a multi-buyer model because we assume there are no consumption externalities.

Chung and Ely (2002) obtain a characterization of incentive compatibility which they call *pseudo-efficiency*. They show that  $H$  is dominant strategy implementable if and only if there exist real-valued functions  $w_b(\theta_{-b}, f)$  such that for each  $b$  and  $\theta$ ,

$$h_b(\theta) \in \arg \max_f \left( V_b(\theta_b, f_b) + w_b(\theta_{-b}, f) \right),$$

and  $H(\theta) = (h_1(\theta), h_2(\theta), \dots, h_B(\theta))$  is a feasible allocation. Our NDMU condition must therefore be equivalent to pseudo-efficiency. However, we believe that NDMU is, in some ways, a more insightful condition than pseudo-efficiency. For instance, the definition of the latter involves an existential quantifier which makes it hard to verify.

Pseudo-efficiency is related to the affine maximizer condition of Roberts. In particular, an affine maximizer is pseudo-efficient but the converse is not true. The domain restrictions inherent in the auctions model imply that a wider class of allocation rules

---

<sup>16</sup>Let  $f$  differ from  $H(\theta)$  in the allocation to exactly one buyer. Then the hypothesis in (12) is false as the inequality holds for at most one and not for all buyers.

<sup>17</sup>Complete domain is essential to Roberts' proofs.

is incentive compatible. In fact, an affine maximizer  $h$  together with the associated transfer functions

$$t_b(\theta) = \begin{cases} \frac{1}{\gamma_b} [V_0(h(\theta)) + \sum_{b' \neq b} \gamma_{b'} V_{b'}(\theta_{b'}, h(\theta))], & \text{if } \gamma_b > 0, \\ 0, & \text{if } \gamma_b = 0. \end{cases}$$

is incentive compatible over *all* quasilinear domains.

Dictatorial social choice functions are incentive compatible on *all* domains, including non-quasilinear environments. Affine maximizers, together with their associated transfer functions, are incentive compatible over *all* quasilinear domains. Thus, affine maximizers can be thought of as the counterpart of dictatorial social choice functions in quasilinear environments.<sup>18</sup>

## 5 Concluding remarks

In the literature on incentive compatibility with multi-dimensional types, characterizations of Bayesian incentive compatibility are far from simple. By restricting attention to deterministic, dominant strategy mechanisms, we obtain a characterization of incentive compatibility that is a considerable simplification. The NDMU condition clarifies the structure of incentive compatible auctions. This in turn will be of use in identifying revenue-maximizing auctions within the class of deterministic dominant strategy auctions and shed light on the role of randomization in increasing revenue for the seller. Further, it may help in establishing the nature and existence of a second best auction in environments where efficiency is generically impossible, i.e., when buyers have interdependent valuations and multi-dimensional types.

Observe that NDMU is a monotonicity condition on the difference in a buyer's reservation values between truth-telling and lying (see 1). This appears to be different from monotonicity conditions previously shown to be necessary for incentive compatibility; these conditions are on buyer reservation values itself (see, for instance, Krishna 2002, pp. 143, 246) rather than on differences in reservation values. The discrepancy is reconciled by the fact that the necessary conditions that appear in the literature are for single object auctions. In a single object auction, when  $h(\theta') \neq h(\theta)$ , either  $h(\theta') = 0$  or  $h(\theta) = 0$ ; thus, as the reservation value of not obtaining an object is zero, (1) reduces to a condition on buyer reservation values. We show that when there are multiple units the appropriate monotonicity condition is on differences of reservation values; moreover, this condition is both necessary and sufficient.

Our strategy has been to start with NDMU, a monotonicity condition implied by dominant strategy incentive compatibility, and show that when applied to a large

---

<sup>18</sup>Note that a dictatorial social choice function is also an affine maximizer in the special case where only the dictator's  $\gamma_b$  is non-zero.

enough domain of buyer types NDMU is also sufficient for incentive compatibility. In principle, this approach can also be applied to Bayesian incentive compatibility. However, first the problem of extending this approach to random mechanisms must be solved (see Section 4). The distinction between random and deterministic allocation rules is less useful for Bayesian mechanisms since, as a function of his type alone, a buyer's allocation is a probability (whether or not the allocation rule as a function of all buyers' type is deterministic).

The literature on strategy proof mechanism design suggests strongly that unless very specific assumptions on the domain of preferences are made, impossibility results of the Gibbard-Satterthwaite type hold. One assumption which, besides being reasonable, has been particularly fruitful in this regard is that of transferable utilities. Our paper is a first step in a larger project of understanding how further restrictions on the domain of preferences, restrictions that are suited to the application at hand, expands the set of incentive compatible allocation functions. A very general theory is probably unattainable but we plan to extend our analysis to double auctions, possibly with heterogenous objects.

## 6 Appendix: Section 2.1 proofs

**Proof of Lemma 2:** First, we give a procedure that reduces the dimension of the commodity space by one, eliminating a  $k^{\text{th}}$  unit for which  $Y(k) = \emptyset$ . A new rule  $h^1$  is defined on the new commodity space such that if  $\hat{h}$  is NDMU on  $D^K$  then  $h^1$  is NDMU on  $D^{K-1}$ , and if  $h^1$  is truth-telling then so is  $\hat{h}$ . There are two cases to consider:

**Case I:**  $\hat{Y}(k) = \emptyset$ ,  $k < K$ , and  $Y(k+1) \neq \emptyset$ . Map each point  $\theta$  in  $D^K$  to a point  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{K-1})$  in  $D^{K-1}$  as follows:

$$\gamma_\ell(\theta) \equiv \begin{cases} \theta_\ell, & \text{if } \ell < k, \\ \theta_k + \theta_{k+1}, & \text{if } \ell = k, \\ \theta_{\ell+1}, & \text{if } \ell > k. \end{cases}$$

Basically, we bundle the  $k^{\text{th}}$  and  $(k+1)^{\text{st}}$  unit together as the new  $k^{\text{th}}$  unit and renumber the units accordingly. It may be verified that  $D^{K-1}$  satisfies the domain assumption A [B] if  $D^K$  satisfies domain assumption A [resp. B]. Define  $\Phi : \{0, 1, \dots, k-1, k+1, \dots, K\} \rightarrow \{0, 1, \dots, K-1\}$  as

$$\Phi(q) \equiv \begin{cases} q, & \text{if } q < k, \\ q-1, & \text{if } q > k. \end{cases}$$

The function  $\Phi$  is strictly increasing and  $\Phi^{-1}$  is well-defined. Define an allocation rule  $h^1$  on  $D^{K-1}$  as follows.<sup>19</sup>

$$h^1(\gamma) \equiv \Phi\left(\min_{\theta \in D^K} \{\hat{h}(\theta) \mid \gamma(\theta) = \gamma\}\right).$$

The rule  $h^1$  yields the same allocation as  $\hat{h}$  except when there exist  $\theta, \theta'$  such that  $\gamma(\theta) = \gamma(\theta')$  and  $\hat{h}(\theta) \neq \hat{h}(\theta')$ .<sup>20</sup> This occurs on a set of (Lebesgue) measure zero in  $D^K$ .

First, we show that NDMU of  $\hat{h}$  implies NDMU of  $h^1$ . Take any  $\gamma, \hat{\gamma} \in D^{K-1}$ . Let  $\theta, \hat{\theta} \in D^K$  be such that  $\gamma = \gamma(\theta)$ ,  $\hat{\gamma} = \gamma(\hat{\theta})$  and  $h^1(\gamma) = \Phi(\hat{h}(\theta))$ ,  $h^1(\hat{\gamma}) = \Phi(\hat{h}(\hat{\theta}))$ .

$$\begin{aligned} h^1(\gamma) > h^1(\hat{\gamma}) &\iff \hat{h}(\theta) > \hat{h}(\hat{\theta}) \\ \implies \sum_{\ell=\hat{h}(\hat{\theta})+1}^{\hat{h}(\theta)} \theta_\ell &\geq \sum_{\ell=\hat{h}(\hat{\theta})+1}^{\hat{h}(\theta)} \hat{\theta}_\ell \implies \sum_{\ell=h^1(\hat{\gamma})+1}^{h^1(\gamma)} \gamma_\ell &\geq \sum_{\ell=h^1(\hat{\gamma})+1}^{h^1(\gamma)} \hat{\gamma}_\ell \end{aligned}$$

<sup>19</sup>The set  $\{\hat{h}(\theta) \mid \theta \in D^K, \gamma(\theta) = \gamma\}$  is a singleton for almost all  $\gamma$ ; when  $\gamma = \gamma(\theta)$  for some  $\theta$  in the boundary between two regions  $\hat{Y}(q)$  and  $\hat{Y}(q')$ , then it is possible that both  $q$  and  $q'$  belong to this set. One can define  $h^1(\gamma)$  using any selection from  $\{\hat{h}(\theta) \mid \theta \in D^K, \gamma(\theta) = \gamma\}$ ; for concreteness, we define it as the smallest element of this set.

<sup>20</sup>Thus,  $\theta_\ell = \theta'_\ell$ ,  $\forall \ell \neq k, k+1$ , and  $\theta_k + \theta_{k+1} = \theta'_k + \theta'_{k+1}$ .

where we use the following facts:  $\Phi$  is strictly increasing,  $\hat{h}$  is NDMU, and  $\hat{h}(\theta), \hat{h}(\theta') \neq k$ , ( $Y(k)$  being an empty set). Thus  $h^1$  is NDMU on  $D^{K-1}$ .

Next, suppose that payments  $t_0^1, t_1^1, t_2^1, \dots, t_{K-1}^1$  truthfully implement  $h^1$ . We may assume that  $t_0^1 = 0$ , i.e., if the buyer gets nothing he pays nothing. Thus,  $t_q^1$  is the infimum of  $\sum_{\ell=1}^q \gamma_\ell$  among  $\gamma \in Y^1(q)$ . For  $\ell \neq k$ , define  $\hat{t}_\ell \equiv t_{\Phi^{-1}(\ell)}^1$ .<sup>21</sup> Using the mapping from  $\theta$  to  $\gamma(\theta)$ , it may be verified that  $t_q$  is the infimum of  $\sum_{\ell=1}^q \theta_\ell$  among  $\theta \in \hat{Y}(q)$ . Thus,  $\hat{t}_1, \dots, \hat{t}_{k-1}, \hat{t}_{k+1}, \dots, \hat{t}_K$  truthfully implement  $\hat{h}$ .

**Case II:**  $\hat{Y}(K) = \emptyset$ . Remove the  $K^{\text{th}}$  unit from the commodity space and define  $\gamma_\ell(\theta) = \theta_\ell$ ,  $\ell = 0, 1, \dots, K-1$ . Let

$$h^1(\gamma_1, \gamma_2, \dots, \gamma_{K-1}) \equiv \Phi \left( \min_{\theta \in D^K} \{ \hat{h}(\theta) \mid \gamma(\theta) = \gamma \} \right).$$

The rest of the proof is identical to that of case I.

This procedure is applied successively  $S$  times to  $\hat{h}$  and new allocation rules are created at each stage as follows. Suppose that  $Y(k) = \emptyset$  for  $k = k_{i-1}+1, \dots, k_i-2, k_i-1$  and  $Y(k_i) \neq \emptyset$ . Then, using Case I of the above procedure, new allocation rules with the requisite properties are successively created by first lumping units  $k_i-1$  and  $k_i$  together, then  $k_i-2$  with the bundled unit  $k_i-1$  and  $k_i$ , etc. Suppose, instead, that  $Y(k) = \emptyset$  for  $k = k_S+1, K_S+2, \dots, K$ . Then, applying Case II, we can successively remove commodities  $K, K-1, \dots$ , and  $k_S+1$ . This proves the lemma.  $\blacksquare$

**Proof of Lemma 3:** From the definitions of  $\theta_k^M$  and  $\theta_k^m$ , and the fact that  $\theta \in D$ , it is clear that  $\bar{a}_k \geq \theta_k^m \geq 0$  and  $\bar{a}_k \geq \theta_k^M \geq 0$ . Let  $\theta^{k-1} \in Y(k-1)$  and  $\theta^k \in Y(k)$ . Then (3) implies that  $\theta_k^k \geq \theta_k^{k-1}$ . As this is true for any  $\theta^{k-1} \in Y(k-1)$  and  $\theta^k \in Y(k)$ , we have  $\theta_k^M \geq \theta_k^m$ .

To complete the proof, we establish that  $\theta_k^M = \theta_k^m$ . Let  $\theta^k \equiv (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_k, 0, \dots, 0)$  for any  $k = 0, 1, 2, \dots, K$  (with  $\theta^0 \equiv (0, 0, \dots, 0)$ ). Observe that under either domain assumption,  $\theta^k \in D$ . Thus,  $\theta^k \in Y(q)$  for some  $q$ . TBB(i) implies that  $\theta^k \notin Y(q)$  for any  $q > k$ , and TBB(ii) implies that  $\theta^k \notin Y(q)$  for any  $q < k$ . Therefore,  $\theta^k \in Y(k)$ . Next, let  $\theta(t) = (1-t)\theta^{k-1} + t\theta^k$ , where  $t \in [0, 1]$ , be a point on the straight line joining  $\theta^k$  and  $\theta^{k-1}$ ,  $k \geq 1$ . Observe that  $\theta(t) = (\bar{a}_1, \dots, \bar{a}_{k-1}, t\bar{a}_k, 0, \dots, 0) \in D$ ,  $\forall t \in [0, 1]$ . Thus,  $\theta(t) \in Y(q)$  for some  $q$ . TBB implies that  $\theta(t) \in Y(k-1) \cup Y(k)$ . Because  $\theta_k(t) = t\bar{a}_k$  increases in  $t$ , there exists a  $t^*$  such that  $\theta(t) \in Y(k-1)$  for all  $t \in [0, t^*]$  and  $\theta(t) \in Y(k)$  for all  $t \in (t^*, 1]$ . Thus,

$$t^* \bar{a}_k = \lim_{t \uparrow t^*} \theta_k(t) \leq \theta_k^m \leq \theta_k^M \leq \lim_{t \downarrow t^*} \theta_k(t) = t^* \bar{a}_k$$

Hence,  $\theta_k^M = \theta_k^m$  and we have proved  $\bar{a}_k \geq \theta_k^M = \theta_k^m \geq 0$ .  $\blacksquare$

<sup>21</sup>It is not necessary to define  $t_k$  as  $Y(k) = \emptyset$ .

The following implication of NDMU is used in the next proof:

$$\text{If } \sum_{\ell=h(\theta)+1}^q \theta'_\ell < \sum_{\ell=h(\theta)+1}^q \theta_\ell, \quad \forall q > h(\theta) \quad \text{then } h(\theta') \leq h(\theta). \quad (13)$$

$$\text{If } \sum_{\ell=q+1}^{h(\theta)} \theta'_\ell > \sum_{\ell=q+1}^{h(\theta)} \theta_\ell, \quad \forall q < h(\theta) \quad \text{then } h(\theta') \geq h(\theta). \quad (14)$$

Observe that if  $\theta'$ ,  $\theta$ , satisfy the hypotheses in (13) and (14) then  $h(\theta') = h(\theta)$ .

**Proof of Lemma 4:** First, we prove that for any  $k = 0, 1, 2, \dots, K$ ,

$$\left\{ \theta \in D \mid \sum_{\ell=q}^k \theta_\ell \geq \sum_{\ell=q}^k c_\ell, \forall q \leq k, \sum_{\ell=k+1}^q \theta_\ell \leq \sum_{\ell=k+1}^q c_\ell, \forall q > k \right\} \subseteq \text{cl}[Y(k)]. \quad (15)$$

There are two cases to consider.

**Case A:**  $(c_1, c_2, \dots, c_K) \in D$ .<sup>22</sup>

Consider the point  $\hat{\theta}^k(\epsilon) = (c_1 + \epsilon_1, \dots, c_k + \epsilon_k, c_{k+1} - \epsilon_{k+1}, \dots, c_K - \epsilon_K)$  where  $\epsilon_1, \epsilon_2, \dots, \epsilon_K$  satisfy the following conditions:

- (i) If  $[q \leq k \text{ and } c_q = \bar{a}_q]$  or  $[q > k \text{ and } c_q = 0]$  then  $\epsilon_q = 0$ .
- (ii) If  $[q \leq k \text{ and } c_q < \bar{a}_q]$  or  $[q > k \text{ and } c_q > 0]$  then  $\epsilon_q > 0$ .

As  $(c_1, c_2, \dots, c_K) \in D$ , there exist  $\epsilon_1, \epsilon_2, \dots, \epsilon_K$  satisfying (i) and (ii) above such that  $\hat{\theta}^k(\epsilon) \in D$ .<sup>23</sup> We claim that  $\hat{\theta}^k(\epsilon) \in Y(k)$ . Consider any  $q < k$ . If  $c_{q+1} < \bar{a}_{q+1}$  then as  $\hat{\theta}_{q+1}^k(\epsilon) > c_{q+1}$ , we know that  $\hat{\theta}^k(\epsilon) \notin Y(q)$ . If, instead,  $c_{q+1} = \bar{a}_{q+1}$  then (as  $\epsilon_{q+1} = 0$ ) we have  $\hat{\theta}_{q+1}^k(\epsilon) = \bar{a}_{q+1}$ . Thus, TBB(ii) implies that  $\hat{\theta}^k(\epsilon) \notin Y(q)$ . Next consider any  $q > k$ . If  $c_q > 0$  then as  $\hat{\theta}_q^k(\epsilon) < c_q$ , we know that  $\hat{\theta}^k(\epsilon) \notin Y(q)$ . If, instead,  $c_q = 0$  then (as  $\epsilon_q = 0$ ) we have  $\hat{\theta}_q^k(\epsilon) = 0$ . Thus, TBB(i) implies that  $\hat{\theta}^k(\epsilon) \notin Y(q)$ . But  $\hat{\theta}^k(\epsilon) \in D$  implies  $\hat{\theta}^k(\epsilon) \in Y(q)$  for some  $q$ . Hence  $\hat{\theta}^k(\epsilon) \in Y(k)$ . Next, (13) and (14) imply that<sup>24</sup>

$$\left\{ \theta \in D \mid \sum_{\ell=q}^k \theta_\ell > \sum_{\ell=q}^k (c_\ell + \epsilon_\ell), \forall q \leq k, \sum_{\ell=k+1}^q \theta_\ell < \sum_{\ell=k+1}^q (c_\ell - \epsilon_\ell), \forall q > k \right\} \subset Y(k).$$

<sup>22</sup>Lemma 3 implies that if domain assumption A is satisfied then we are in Case A.

<sup>23</sup>If domain assumption A is satisfied, this is easy to verify. If, instead, domain assumption B is satisfied then  $\epsilon_1, \epsilon_2, \dots, \epsilon_K$  must be chosen to ensure that  $\hat{\theta}^k(\epsilon)$  satisfies (4).

<sup>24</sup>If for some  $q \leq k$ ,  $c_\ell = \bar{a}_\ell$  for all  $\ell = q, q+1, \dots, k$  then the strict inequality in the set on the left hand side is replaced by a weak inequality. A similar change is made if for some  $q > k$ ,  $c_\ell = 0$  for all  $\ell = q, q+1, \dots, k$ . In either case, (i) implies that  $\epsilon_\ell = 0$  in the relevant range. This ensures that the set on the left hand side is non-empty; the inclusion of this set in  $Y(k)$  is implied by TBB together with (13) and (14).

Further, one can construct a sequence  $(\epsilon_1^n, \epsilon_2^n, \dots, \epsilon_K^n) \rightarrow 0$  such that  $\hat{\theta}^k(\epsilon^n) \in D$ . Taking limits as  $\epsilon^n \rightarrow 0$ , we get

$$\left\{ \theta \in D \mid \sum_{\ell=q}^k \theta_\ell > \sum_{\ell=q}^k c_\ell, \forall q \leq k, \sum_{\ell=k+1}^q \theta_\ell < \sum_{\ell=k+1}^q c_\ell, \forall q > k \right\} \subset Y(k),$$

which in turn implies (15).

**Case B:**  $(c_1, c_2, \dots, c_K) \notin D$ .<sup>25</sup>

For each  $k = 0, 1, 2, \dots, K$  define

$$\Theta^k(\epsilon) = \{ \theta \mid c_k + \epsilon_k \leq \theta_k \leq \bar{a}_k, \theta_q = \max[\theta_{q+1} \frac{\bar{a}_q}{\bar{a}_{q+1}}, c_q + \epsilon_q], \forall q < k, \theta_q = \min[\theta_{q-1} \frac{\bar{a}_q}{\bar{a}_{q-1}}, c_q - \epsilon_q], \forall q > k \}.$$

Any  $\theta \in \Theta^k(\epsilon)$  satisfies (4). Thus, provided  $\epsilon_1, \epsilon_2, \dots, \epsilon_K$  satisfy (i) and (ii) defined in Case A, and are small enough,  $\Theta^k(\epsilon) \subset D [= \cup_{q=0}^K Y(q)]$ . For any  $\theta \in \Theta^k(\epsilon)$ , we have  $\theta_q \geq c_q + \epsilon_q$  for any  $q \leq k$ ; thus  $\Theta^k(\epsilon) \cap Y(q-1) = \emptyset$ . Similarly, for any  $q > k$ ,  $\Theta^k(\epsilon) \cap Y(q) = \emptyset$ . Thus,  $\Theta^k(\epsilon) \subset Y(k)$  for small enough  $\epsilon_\ell$ 's. From (13) and (14) applied to each  $\theta \in \Theta^k(\epsilon)$ , we know that

$$\left\{ \theta \in D \mid \theta_k > c_k + \epsilon_k, \sum_{\ell=q}^k \theta_\ell > c_k + \epsilon_k + \sum_{\ell=q}^{k-1} \max[\theta_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, c_\ell + \epsilon_\ell], \forall q < k, \sum_{\ell=k+1}^q \theta_\ell < \sum_{\ell=k+1}^q \min[\theta_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, c_\ell - \epsilon_\ell], \forall q > k \right\} \subset Y(k).$$

Taking limits as  $(\epsilon_1, \epsilon_2, \dots, \epsilon_K) \rightarrow 0$ , we see that

$$\left\{ \theta \in D \mid \theta_k > c_k, \sum_{\ell=q}^k \theta_\ell > c_k + \sum_{\ell=q}^{k-1} \max[\theta_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, c_\ell], \forall q < k, \sum_{\ell=k+1}^q \theta_\ell < \sum_{\ell=k+1}^q \min[\theta_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, c_\ell], \forall q > k \right\} \subseteq Y(k)$$

and therefore

$$\left\{ \theta \in D \mid \theta_k \geq c_k, \sum_{\ell=q}^k \theta_\ell \geq c_k + \sum_{\ell=q}^{k-1} \max[\theta_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, c_\ell], \forall q < k, \sum_{\ell=k+1}^q \theta_\ell \leq \sum_{\ell=k+1}^q \min[\theta_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, c_\ell], \forall q > k \right\} \subseteq \text{cl}[Y(k)].$$

That this last set inclusion is equivalent to (15) follows from the observation that (4) implies that if  $c_k + \sum_{\ell=q}^{k-1} \max[\theta_{\ell+1} \frac{\bar{a}_\ell}{\bar{a}_{\ell+1}}, c_\ell] > \sum_{\ell=q}^k \theta_\ell \geq \sum_{\ell=q}^k c_\ell$  for some  $q < k$  or if  $\sum_{\ell=q+1}^k \min[\theta_{\ell-1} \frac{\bar{a}_\ell}{\bar{a}_{\ell-1}}, c_\ell] < \sum_{\ell=q+1}^k \theta_\ell \leq \sum_{\ell=q+1}^k c_\ell$  for some  $q > k$ , then  $\theta \notin D$ . This establishes (15) for the case when  $\hat{\theta}^k(\epsilon) \notin D$  for any  $\epsilon_1, \epsilon_2, \dots, \epsilon_K$ .

<sup>25</sup>Domain assumption B must hold and (4) is violated by  $(c_1, c_2, \dots, c_K)$ .

Next, suppose that the set inclusion in (15) is strict. In particular, there exists  $k, \theta \in \text{cl}[Y(k)]$  such that  $\sum_{\ell=q'}^k \theta_\ell < \sum_{\ell=q'}^k c_\ell$ , for some  $q' < k$ .<sup>26</sup> We assume that  $\theta \in Y(k)$ .<sup>27</sup> Let  $q' < k$  be such that  $\sum_{\ell=q}^k \theta_\ell \geq \sum_{\ell=q}^k c_\ell, \forall q = q' + 1, \dots, k$ . Therefore,  $\theta_{q'} < c_{q'} \leq \bar{a}_{q'}$  and  $\sum_{\ell=q'}^q \theta_\ell < \sum_{\ell=q'}^q c_\ell, \forall q = q', q' + 1, \dots, k$ . Consider the point  $\hat{\theta} \equiv (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{q'-1}, \theta_{q'} + \hat{\epsilon}, \theta_{q'+1}, \dots, \theta_k, 0, \dots, 0)$  where  $\hat{\epsilon} > 0$  is small enough that  $\hat{\theta} \in D$  and  $\sum_{\ell=q'}^q \hat{\theta}_\ell < \sum_{\ell=q'}^q c_\ell, \forall q = q', q' + 1, \dots, k$ . Thus, (15) implies that  $\hat{\theta} \in \text{cl}[Y(q' - 1)]$ . Suppose that  $\hat{\theta} \in Y(q' - 1)$ . But this violates (3) as  $\sum_{\ell=q'}^k \hat{\theta}_\ell > \sum_{\ell=q'}^k \theta_\ell$  and  $\theta \in Y(k)$ . If, instead,  $\hat{\theta} \in \text{cl}[Y(q' - 1)] \setminus Y(q' - 1)$  then there exists  $\theta^* \in Y(q' - 1)$  which is arbitrarily close to  $\hat{\theta}$  and again we get a violation of (3).

Thus, for any  $\theta \in \text{cl}[Y(k)]$  we have  $\sum_{\ell=q}^k \theta_\ell \geq \sum_{\ell=q}^k c_\ell, \forall q \leq k$ . A similar proof establishes that if  $\theta \in \text{cl}[Y(k)]$  then  $\forall q > k, \sum_{\ell=k+1}^q \theta_\ell \leq \sum_{\ell=k+1}^q c_\ell$ . Therefore, the set inclusion in (15) can be replaced by an equality, i.e.,

$$\text{cl}[Y(k)] = \left\{ \theta \in D \mid \sum_{\ell=q}^k \theta_\ell \geq \sum_{\ell=q}^k c_\ell, \forall q \leq k, \& \sum_{\ell=k+1}^q \theta_\ell \leq \sum_{\ell=k+1}^q c_\ell, \forall q > k \right\}. \quad (16)$$

For any  $\theta \in Y(k)$  and any  $q < k$ ,

$$\begin{aligned} \sum_{\ell=1}^k \theta_\ell - \sum_{\ell=1}^k c_\ell &\geq \sum_{\ell=1}^q \theta_\ell - \sum_{\ell=1}^q c_\ell \\ \iff \sum_{\ell=q+1}^k \theta_\ell &\geq \sum_{\ell=q+1}^k c_\ell. \end{aligned} \quad (17)$$

The last inequality follows from (16). Thus, (17) is true; when  $\theta \in Y(k)$  the buyer cannot increase his payoffs by reporting a type  $\theta' \in Y(q), q < k$ . A similar argument establishes that (17) is true for  $q > k$ . Thus, the payments

$$t(\theta) = \begin{cases} \sum_{\ell=1}^k c_\ell, & \text{if } \theta \in Y(k), k = 1, 2, \dots, K \\ 0, & \text{if } \theta \in Y(0). \end{cases}$$

implement  $h$  truthfully. ■

**Proof of Lemma 5:** Before describing a procedure which converts  $h$  to an  $h'$  with the stated properties, we need the following result.

**CLAIM:** *Let  $h$  be an allocation rule that is NDMU but not TBB. That is there exists  $\theta^k \in Y(k)$  and  $\theta^{k-1} \in Y(k-1)$ <sup>28</sup> such that either  $\theta_k^k = \theta_k^{k-1} = 0$  or  $\theta_k^k = \theta_k^{k-1} = \bar{a}_k$ . Define a new allocation rule which is identical to  $h$  except that:*

<sup>26</sup>From the definition of  $c_k$  we know that  $q' \neq k$ .

<sup>27</sup>If  $\theta \in \text{cl}[Y(k)] \setminus Y(k)$ , then there exists  $\theta' \in Y(k)$ ,  $\theta'$  close to  $\theta$ , such that  $\sum_{\ell=q'}^k \theta'_\ell < \sum_{\ell=q'}^k c_\ell$ .

<sup>28</sup>Throughout this Claim,  $Y(\cdot)$  is defined with respect to the allocation rule  $h$ .



- (a) If  $\theta_k^k = \theta_k^{k-1} = 0$  then allocate  $k - 1$  (instead of  $k$ ) units at  $\theta^k$ .  
(b) If  $\theta_k^k = \theta_k^{k-1} = \bar{a}_k$  then allocate  $k$  (instead of  $k - 1$ ) units at  $\theta^{k-1}$ .

Then, the new allocation rule is NDMU.

PROOF: (a) Suppose that  $\theta_k^k = \theta_k^{k-1} = 0$ . At  $\theta^k$  the buyer is allocated  $k - 1$  units in the new allocation rule. Since  $h$  is NDMU, all we need to check is that  $\theta^k$  satisfies NDMU inequalities in the new allocation rule. Observe that  $0 = \theta_k^M = \theta_k^k \leq \theta_k$ ,  $\forall \theta \in Y(k)$ . Thus  $\theta^k$  satisfies the NDMU inequalities with respect to all  $\theta \in Y(k)$ . Therefore, we need to show that for any  $\theta \in Y(q)$ ,  $q \neq k, k - 1$ ,

$$\sum_{\ell=q+1}^{k-1} \theta_\ell^k \geq \sum_{\ell=q+1}^{k-1} \theta_\ell, \quad \text{if } q < k - 1 \quad \text{and} \quad \sum_{\ell=k}^q \theta_\ell^k \leq \sum_{\ell=k}^q \theta_\ell, \quad \text{if } q > k. \quad (18)$$

From NDMU of  $h$  we know that for any  $\theta \in Y(q)$ ,  $q \neq k, k - 1$ ,

$$\sum_{\ell=q+1}^k \theta_\ell^k \geq \sum_{\ell=q+1}^k \theta_\ell, \quad \text{if } q < k - 1 \quad \text{and} \quad \sum_{\ell=k+1}^q \theta_\ell^k \leq \sum_{\ell=k+1}^q \theta_\ell, \quad \text{if } q > k.$$

This, together with  $\theta_k^k = 0$ , implies (18).

(b) The proof is similar. ■

Consider any  $h$  that satisfies NDMU. From  $h$  we obtain an allocation rule  $h'$  using the following procedure. First, let  $h'(\theta) \equiv h(\theta)$ ,  $\forall \theta$ , and then make the following changes to  $h'$ :<sup>29</sup>

1. Let  $k = K$ .
2. If  $\theta_k^M = 0$  then for all  $\theta \in Y(k)$  such that  $\theta_k = 0$ , let  $h'(\theta) = k - 1$ .
3. Decrease  $k$  by 1. If  $k \geq 1$  then go to step 2; otherwise, go to Step 4.
4. Let  $k = 1$ .
5. If  $\theta_k^m = \bar{a}_k$  then for all  $\theta \in Y(k - 1)$  such that  $\theta_k = \bar{a}_k$ , let  $h'(\theta) = k$ .
6. Increase  $k$  by 1. If  $k \leq K$  then go to step 5; otherwise, stop.

NDMU of  $h$  implies that  $\theta_k^M \geq \theta_k^m$ . Thus, if at Step 2 of the procedure, we transfer some  $\theta$  from  $Y(k)$  to  $Y'(k - 1)$ , then in Step 5 we will not transfer any  $\theta$ 's from  $Y(k - 1)$  to  $Y'(k)$ , and vice versa. The Claim assures us that each time we make changes to  $h'$  in Steps 2 or 5,  $h'$  continues to satisfy NDMU; thus the  $h'$  obtained at the end of this procedure is NDMU. By construction, the final  $h'$  satisfies TBB. Further,  $h(\theta) = h'(\theta)$  for almost all  $\theta \in D$ .

<sup>29</sup>In Steps 2 and 5,  $\theta_k^M$ ,  $\theta_k^m$ ,  $Y(k)$ , and  $Y(k - 1)$  are defined with respect to  $h$ .

By Lemma 4, we know that there exist prices  $c'_\ell$  that truthfully implement  $h'$ . We show that for any  $\theta \in D$ , assuming truthful reporting under either mechanism, the buyer's payoffs are the same under  $h$  or  $h'$  implemented with prices  $c'_\ell$ . Therefore, it must also be optimal to tell the truth when  $h$  is implemented with prices  $c'_\ell$ . Let  $h(\theta) = k$  and  $h'(\theta) = k'$ . We establish that

$$\sum_{\ell=1}^k (\theta_\ell - c'_\ell) = \sum_{\ell=1}^{k'} (\theta_\ell - c'_\ell). \quad (19)$$

If  $k = k'$ , then clearly (19) is true. If, instead,  $k' < k$  then, from the above construction,  $\theta_\ell = c'_\ell = 0$ ,  $\ell = k' + 1, k' + 2, \dots, k$ . Similarly, if  $k' > k$  then  $\theta_\ell = c'_\ell = \bar{a}_\ell$ ,  $\ell = k + 1, k + 2, \dots, k'$ . Thus, (19) holds. Therefore, since the prices  $c'_\ell$  truthfully implement  $h'$ , they also truthfully implement  $h$ . ■

## 7 References

BARBERA, S. AND M. JACKSON (1995), “Strategy-proof Exchange,” *Econometrica*, 63, 51-87.

BERGEMANN, D. AND S. MORRIS (2001), “Robust Mechanism Design,” working paper, Dept. of Economics, Yale University.

BIKHCHANDANI, S. (2004), “The Limits of Ex-post Implementation Revisited,” working paper.

BIKHCHANDANI, S. AND J. M. OSTROY (2002), “The Package Assignment Model,” *Journal of Economic Theory*, 107, pp. 377-406, 2002.

CHUNG, K-S. AND J. ELY (2002), “Ex-post Incentive Compatible Mechanisms,” working paper, Dept. of Economics, Northwestern University.

D’ASPREMONT, C. AND L-A. GERARD-VARET (1979), “Incentives and Incomplete Information,” *Journal of Public Economics*, 11, 25-45.

HOLMSTROM, B. AND R. MYERSON (1983), “Efficient and Durable Decision Rules with Incomplete Information,” *Econometrica*, 1799-1819.

HOLZMAN, R. AND D. MONDERER (2004), “Characterization of Ex Post Equilibrium in VCG Combinatorial Auctions,” *Games and Economic Behavior*, 47, 87-103.

JEHIEL, P. AND B. MOLDOVANU (2001), “Efficient Design with Interdependent Valuations,” *Econometrica*, 69, 1237-1259.

JEHIEL, P., M. MEYER-TER-VEHN, AND B. MOLDOVANU (2004), “The Limits of Ex-post Implementation,” working paper.

KRISHNA, V. (2002), *Auction Theory*, Academic Press.

KRISHNA, V. AND M. PERRY (1997), “Efficient Mechanism Design”, working paper, Dept. of Economics, Pennsylvania State University.

LAVI, R., A. MU’ALEM, AND N. NISAN (2003), “Towards a Characterization of Truthful Combinatorial Auctions,” working paper, Hebrew University.

LEDYARD, J. (1978), “Incentive Compatibility and Incomplete Information,” *Journal of Economic Theory*, 18, 171-189.

LEHMANN, D., L.I. O’CALLAGHAN, AND Y. SHOHAM (1999), “Truth Revelation in Rapid, Approximately Efficient Combinatorial Auctions,” *ACM Conference on*

*Electronic Commerce.*

MANELLI, A.M. AND D.R. VINCENT (2003), "Bundling as an Optimal Selling Mechanism for a Multi-good Monopolist," working paper.

MASKIN, E. (1992), "Auctions and Privatizations," in *Privatization*, ed. H. Siebert. Kiel: Institut fuer Weltwirtschaft der Universitaet Kiel, pp. 115-136.

MASKIN, E. AND J.G. RILEY (1989), "Optimal Multi-Unit Auctions." In Frank Hahn (ed.), *The Economics of Missing Markets, Information and Games*, 312-335, Oxford University Press, Oxford, U.K.

MCAFEE, R. P. AND J. MCMILLAN (1988), "Multidimensional Incentive Compatibility and Mechanism Design," *Journal of Economic Theory*, 46, 335-54.

MYERSON, R. (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.

NISAN, N. AND A. RONEN (2000), "Computationally Feasible VCG-based Mechanisms," *ACM Conference on Electronic Commerce*.

ROBERTS, K. (1979), "The Characterization of Implementable Choice Rules," in *Aggregation and Revelation of Preferences*, J-J. Laffont (ed.), North Holland Publishing Company.

THANASSOULIS, J. (2004), "Haggling over Substitutes," *Journal of Economic Theory*, 117, 217-245.

WILLIAMS, S. (1999), "A Characterization of Efficient, Bayesian, Incentive Compatible Mechanisms," *Economic Theory*, 14, 155-180.

WILSON, R. (1987), "Game-Theoretic Analyses of Trading Processes," in *Advances in Economic Theory*, ed. Truman Bewley. Cambridge: Cambridge University Press.

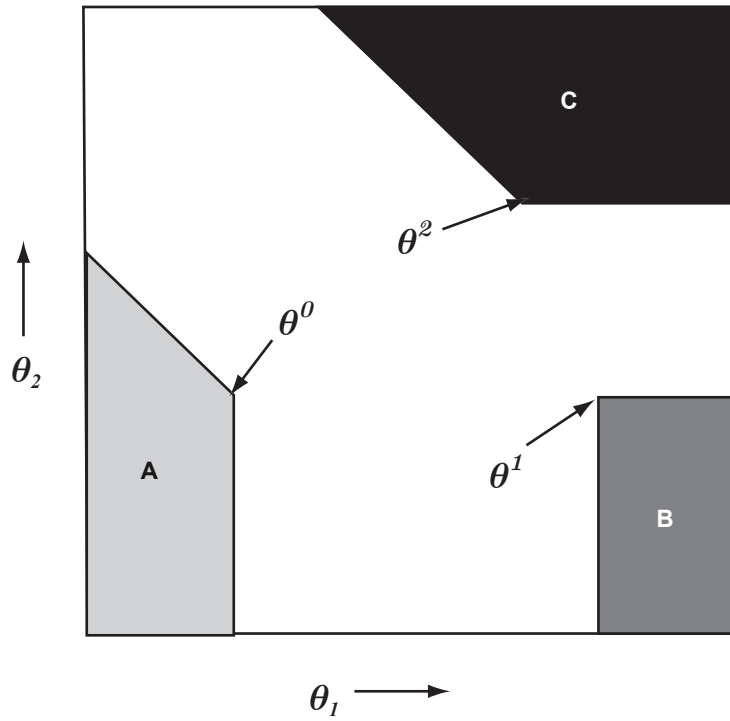


Figure 1a: Sufficiency of NDMU

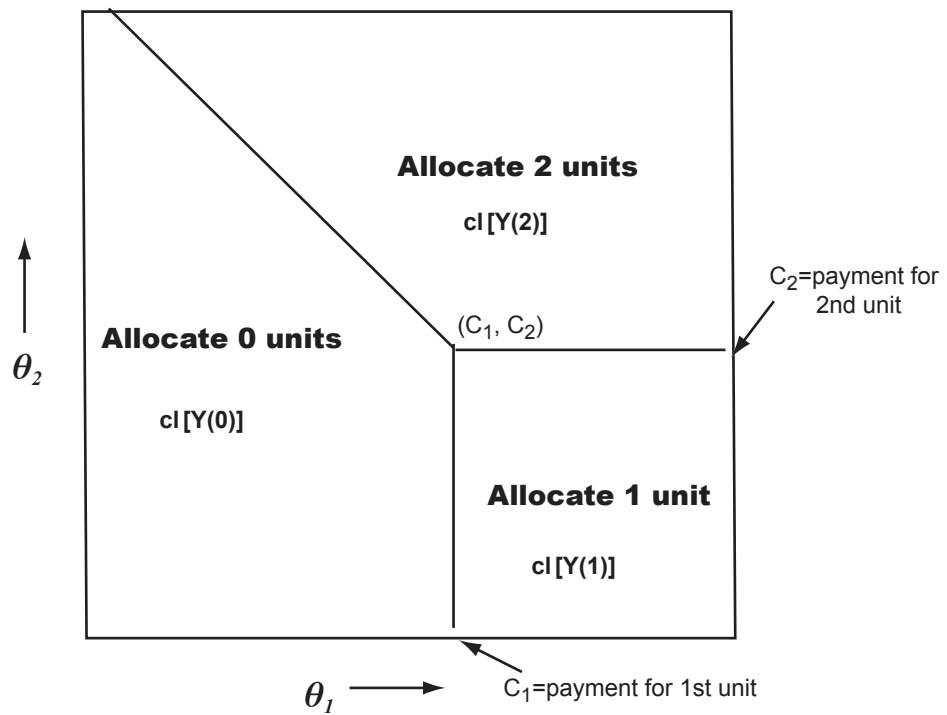


Figure 1b: Sufficiency of NDMU

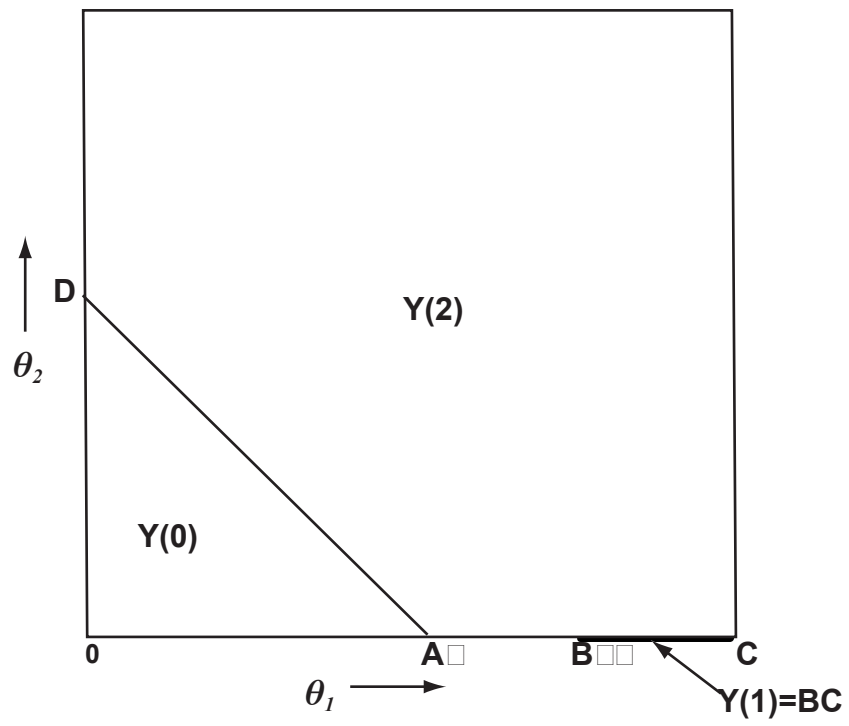


Figure 2a: TBB(i) is not satisfied

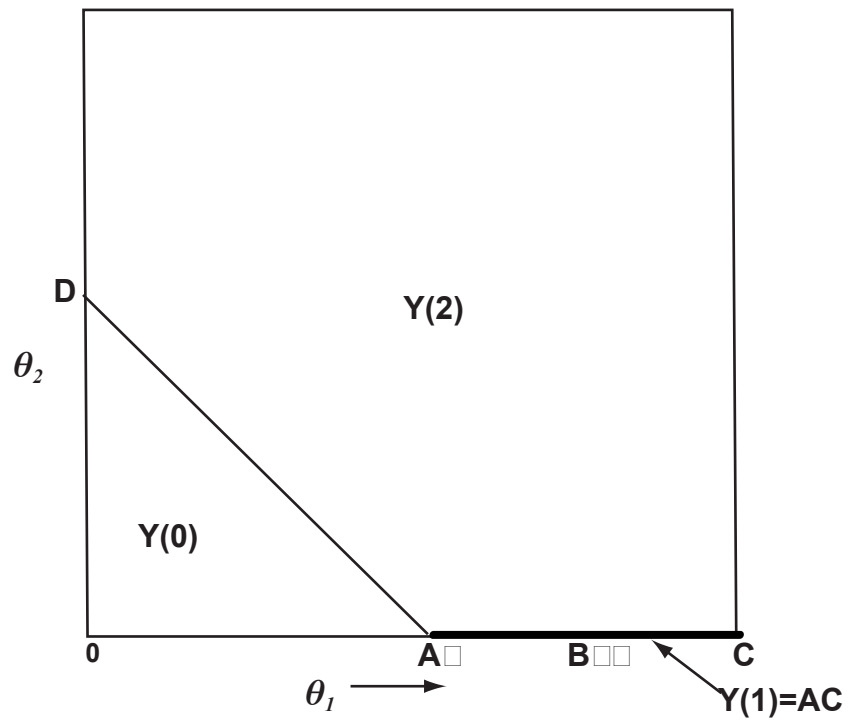


Figure 2b: TBB(i) is satisfied