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A simple test for multivariate conditional symmetry

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Abstract

This paper proposes a simple consistent nonparametric test of multivariate conditional symmetry based on the principle of characteristic functions. The test statistic is shown to be asymptotically normal under the null and consistent against any conditional asymmetric distributions.

Keywords: Characteristic function; Conditional symmetry; Test

JEL classification: C12; C14

1. Introduction

Conditional symmetry is important for the purpose of identification (see Newey, 1990). It is also of interest in modelling time series data in business and finance (e.g., Brännäs and De Gooijer, 1992) and in constructing predictive regions for nonlinear time series (e.g., De Gooijer and Grannoun, 2000; Polonik and Yao, 2000).

Despite the wide use of the property of conditional symmetry, tests for conditional symmetry are far and few in between. A few exceptions are Zheng (1998), Bai and Ng (2001), and Hyndman and Yao (2002). In this paper we propose a simple test for conditional symmetry based on the principle of characteristic functions. Unlike the above tests, our test is applicable when both the dependent and conditioning variables are multivariate and it is easy to implement.

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The paper is organized as follows. We describe the test statistic in Section 2 and derive its asymptotic distributions in Section 3. All technical details are relegated to the Appendix.

2. The hypothesis and test statistic

Let (X, Y) , (X_1, Y_1) , …, (X_n, Y_n) be independent observations with common joint probability f_{XY} and distribution F_{XY}. Let $f_{Y|X}(\cdot|x)$ and $F_{Y|X}(\cdot|x)$ denote the conditional density and distribution of Y given $X = qx \in R^{d_1}$, respectively. Based on the data $\{X_i, Y_i\}_{i=1}^n$, we are interested in testing whether Y is symmetric around zero conditional on X : $Pr[f_{Y|X}(y|X)=f_{Y|X}(-y|X)]=1$ for all $y\in\mathbb{R}^{d_2}$. In terms of the joint probability, the null hypothesis is

$$H_0 : Pr[f_{XY}(X,y) = f_{XY}(X,-y)] = 1 \text{ for all } y\in\mathbb{R}^{d_2}, \tag{2.1}$$

and the alternative hypothesis is

$$H_1 : Pr[f_{XY}(X,y) = f_{XY}(X,-y)] < 1 \text{ for some } y\in\mathbb{R}^{d_2}. \tag{2.2}$$

The proposed test is based on the principle of characteristic functions (ch.f's). It is well known that two distribution functions are equal if and only if their respective ch.f's are equal. Let $\phi_{XY}(\cdot,\cdot)$ be the ch.f of (X, Y) : $\phi_{XY}(u, v)=E[\exp(iu'X+iv'Y)]$, where $i = \sqrt{-1}$, $u\in\mathbb{R}^{d_1}$ and $v\in\mathbb{R}^{d_2}$. Let $\psi(u, v)=\phi_{XY}(u, v)-\phi_{XY}(u, -v)$. Y is symmetric about zero conditioning on X if and only if $\psi(u, v)=0$. This motivates us to consider the following smooth functional

$$\Gamma\equiv\frac{1}{2}\int\int|\phi_{XY}(u,v)-\phi_{XY}(u,-v)|^2\mathrm{d}G_1(u)\mathrm{d}G_2(v), \tag{2.3}$$

where $\mathrm{d}G_i(u)=g_i(u)\mathrm{d}u$ and we choose g_i to be a density function with full support on $\mathbb{R}^{d_i}$, $i=1, 2$.

Let $h_i(z) = \int \exp(iu'z)\mathrm{d}G_i(u)$, the ch.f of $\mathrm{d}G_i(u)$. Assume that h_i is symmetric. By the Fubini Theorem and the formula for change of variables, we have

$$\begin{aligned}\Gamma &= \frac{1}{2}\int\int\int\int \exp(iu'x + iv'y)\{f_{XY}(x,y)-f_{XY}(x,-y)\}\exp(-iu'\tilde{x}-iv'\tilde{y})\\ &\quad\times\{f_{XY}(\tilde{x},\tilde{y})-f_{XY}(\tilde{x},-\tilde{y})\}\mathrm{d}G_1(u)\mathrm{d}G_2(v)\mathrm{d}(x,y)\mathrm{d}(\tilde{x},\tilde{y})\\ &= \int\int h_1(x-\tilde{x})h_2(y-\tilde{y})\{f_{XY}(x,y)f_{XY}(\tilde{x},\tilde{y})-f_{XY}(x,-y)f_{XY}(\tilde{x},-\tilde{y})\}\mathrm{d}(x,y)\mathrm{d}(\tilde{x},\tilde{y})\\ &= E[h_1(X_1-X_2)\{h_2(Y_1-Y_2)-h_2(Y_1+Y_2)\}].\end{aligned}$$

To introduce the test statistic, let K be a kernel function on $\mathbb{R}^{d_1}$ and $B\equiv B(n)$ be the $d_1\times d_1$ bandwidth matrix. Define $K_B(u)\equiv|B|^{-1}K(B^{-1}u)$, where $|B|$ is the determinant of B. The test statistic is

$$\Gamma_n = \frac{2}{n(n-1)|B|^{1/2}}\sum_{1\le i<j\le n} H_n(Z_i,Z_j), \tag{2.4}$$

where $Z_i=(X_i, Y_i)$, and $H_n(Z_i, Z_j)=|B|^{1/2}[h_2(Y_i-Y_j)-h_2(Y_i+Y_j)]\,h_1(X_i-X_j)\,K_B(X_i-X_j)$.

The test statistic Γ_n has the advantage that it has zero mean under H_0 and hence it does not have a finite sample bias term. We will show that after being appropriately scaled, Γ_n is asymptotically normally distributed under H_0.

3. The asymptotic distributions

We first make the following assumptions.

A1. $f_{XY}(x, y)$ is continuous and has uniformly bounded second order derivatives with respect to x.

A2. The densities g_i, $i=1, 2$, are uniformly bounded on $\mathbb{R}^{\mathrm{d}_i}$ with symmetric ch.f's h_i.

A3. The kernel function $K(\cdot)$ is a symmetric, bounded and continuous density on $\mathbb{R}^{\mathrm{d}_1}$ satisfying $\int ||u||^2 K(u)\mathrm{d}u < \infty$.

A4. As $n\to\infty, ||B||\to 0$, and $n|B|\to\infty$, where $||B||=\{tr\ (B'B)\}^{1/2}$.

Assumption A1 imposes the smoothness condition on f_{XY} and it can be weakened to Lipschitz continuity with little modification on the proofs. The symmetry of h_i in A2 can be easily satisfied, say, by choosing g_i either from the normal family or the double exponential family. Both A3 and A4 are standard in the nonparametric literature. In practice, one frequently chooses B to be a diagonal matrix: $B=\mathrm{diag}(b_1, \ldots, b_{\mathrm{d}_1})$.

We now state our first result, the proof of which is outlined in the Appendix.

Theorem 3.1. *Under* Assumptions A1, A2, A3, A4 *and under* H_0, $T_n\equiv n|B|^{1/2}\Gamma_n/\hat{\sigma} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1)$, *where* $\hat{\sigma}^2 = 2(n(n-1))^{-1}\sum_{i\neq j} H_n^2(Z_i, Zj)$ *is a consistent estimator for*

$$\sigma^2 = 2\int h_1^2(u)K^2(u)\mathrm{d}u\left\{\int\int\int[h_2(y_1-y_2)-h_2(y_1+y_2)]^2 f_{XY}(x,y_1)f_{XY}(x,y_2)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\right\}.$$

Note $\int h_1^2(u)K^2(u)\mathrm{d}u < \infty$ by the uniform boundedness of ch.f's and Assumption A3. To implement the test, we compare T_n with z_α, the αth upper percentile of the standard normal distribution, and reject H_0 if $T_n>z_\alpha$.

The following result shows that our test is consistent.

Theorem 3.2. *Under* Assumptions A1, A2, A3, A4 *and under* H_1, $T_n/(n|B|^{1/2}) = \Gamma_n/\hat{\sigma} \xrightarrow{\mathrm{p}} \tau$, *where* $\tau\equiv\frac{1}{2\sigma}\int h_1(u)K(u)\mathrm{d}u\int\int|\phi_{Y|X}(u|x)-\phi_{Y|X}(-u|x)|^2\mathrm{d}G_2(u)f_X^2(x)\mathrm{d}x > 0$, *and* $\phi_{Y|X}(\cdot|x)$ *is the conditional ch.f of Y given X=x.*

To study the local power of the test, we specify the local alternative in terms of conditional ch.f's:

$$H_1(\alpha_n) : \phi_{Y|X}(u|x) = \phi_{Y|X}(-u|x) + \alpha_n\Delta(u,x), \tag{3.1}$$

where $\Delta(u, x)$ satisfies $\gamma\equiv\frac{1}{2}\int\int|\Delta(u,x)|^2\mathrm{d}G_2(u)f_X^2(x)\mathrm{d}x < \infty$, and $\alpha_n\to 0$ as $n\to\infty$.

The following theorem shows that our test can distinguish local alternatives $H_1(\alpha_n)$ at rate $\alpha_n=n^{-1/2}\ |B|^{-1/4}$.

Theorem 3.3. *Under* Assumptions A1, A2, A3, A4 *and* $H_1(n^{-1/2}\ |B|^{-1/4})$, $Pr(T_n \geq z_\alpha | H_1(\alpha_n)) \rightarrow 1-\Phi(z_\alpha - \gamma \int h_1(u)K(u)du/\sigma)$, *where* Φ *is the cdf of the standard normal distribution.*

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Appendix A

Let $A \approx C$ denote $A = C\ \{1 + o\ (1)\}$ componentwise for any matrices A, C of the same dimension. Let $Z = (X,\ Y)$, and $z_i = (x_i,\ y_i)$, $i = 1,\ 2$. Denote the marginal distribution of X by F_X.

Proof of Theorem 3.1. The proof of the first part follows directly by applying Theorem 1 of Hall (1984), and we only sketch the proof. By construction and Assumptions A2–A3, $H_n(z_1, z_2) = H_n(z_2, z_1)$. $E[H_n(z_1, Z_2)] = |B|^{1/2} \int \{\int [h_2(y_1 - y) - h_2(y_1 + y)] \mathrm{d}F_{Y|X}(y|x)\} h_1(x_1 - x) K_B(x_1 - x) \mathrm{d}F_X(x) = 0$ under H_0. $E[H_n^2(Z_1, Z_2)] = \int h_1^2(u) K^2(u) \mathrm{d}u \{\int\int\int [h_2(y_1 - y_2) - h_2(y_1 + y_2)]^2 f_{XY}(x, y_1) f_{XY}(x, y_2) \mathrm{d}x \mathrm{d}y_1 \mathrm{d}y_2\} + O(||B||^2) = \sigma^2/2 + O(||\mathrm{B}||^2)$. Let $G_n\ (z_1, z_2) = E\ [H_n\ (Z, z_1)\ H_n\ (Z, z_2)]$. Then it is easy to verify that $E\ [G_n^2\ (Z_1, Z_2)] = O\ (|B|)$, and $E\ [H_n^4\ (Z_1, Z_2)] = O\ (|B|^{-1})$. So $\{E\ [G_n^2\ (Z_1, Z_2)] + n^{-1} E\ [H_n^4\ (Z_1, Z_2)]\} / \{E\ [H_n^2\ (Z_1, Z_2)]\}^2 \rightarrow 0$ as $n \rightarrow \infty$. The result follows.

That $\hat{\sigma}^2$ is a consistent estimator for σ^2 follows from the fact that $E\ (\hat{\sigma}^2) = 2E\ [H_n^2\ (Z_1,\ Z_2)] = \sigma^2 + O\ (||\mathrm{B}||^2)$ and $E\ (\hat{\sigma}^2)^2 = \sigma^4 + O\ (n^{-1}) + \mathrm{O}\ (n^{-2}\ |B|^{-1}) + O\ (||B||^2)$ so that $\mathrm{var}(\hat{\sigma}^2) = o(1)$. □

Proof of Theorem 3.2. Under Assumptions A1, A2, A3, A4 and H_1,

$$
\begin{aligned}
E(\Gamma_n) &= E[|B|^{-1/2} H_n(Z_1, Z_2)] \approx \mu \int\int\int [h_2(y_1 - y_2) - h_2(y_1 + y_2)] \mathrm{d}F_{Y|X}(y_1|x) \mathrm{d}F_{Y|X}(y_2|x) f_X^2(x) \mathrm{d}x \\
&= \frac{\mu}{2} \int\int\int [h_2(y_1 - y_2) + h_2(-y_1 + y_2) - h_2(y_1 + y_2) - h_2(-y_1 - y_2)] \mathrm{d}F_{Y|X}(y_1|x) \\
&\quad \times \mathrm{d}F_{Y|X}(y_2|x) f_X^2(x) \mathrm{d}x \\
&= \frac{\mu}{2} \int\int\int\int [\exp(iu'y_1) - \exp(-iu'y_1)][\exp(-iu'y_2) - \exp(iu'y_2)] \mathrm{d}G_2(u) \mathrm{d}F_{Y|X}(y_1|x) \\
&\quad \times \mathrm{d}F_{Y|X}(y_2|x) f_X^2(x) \mathrm{d}x \\
&= \frac{\mu}{2} \int\int \left| \int [\exp(iu'y) - \exp(-iu'y)] f_{Y|X}(y|x) \mathrm{d}y \right|^2 \mathrm{d}G_2(u) f_X^2(x) \mathrm{d}x \\
&= \frac{\mu}{2} \int\int \left| \phi_{Y|X}(u|x) - \phi_{Y|X}(-u|x) \right|^2 \mathrm{d}G_2(u) f_X^2(x) \mathrm{d}x
\end{aligned}
$$

where $\mu = \int h_1(u) K(u) \mathrm{d}u$. Simple but tedious calculations show $\mathrm{var}(\Gamma_n) = o(1)$. $\Gamma_n/\hat{\sigma} \xrightarrow{\mathrm{p}} \tau$ by the Chebyshev's inequality and the fact that $\hat{\sigma}^2 = \sigma^2 + o_p\ (1)$ also holds under H_1. □

Proof of Theorem 3.3. The proof is similar to that of Theorem 3.1. The only difference is that now under $H_1\ (\alpha_n)$, $E[n|B|^{1/2}\Gamma_n] = n|B|^{1/2} \frac{\mu}{2} \int\int |\phi_{Y|X}(u|x) - \phi_{Y|X}(-u|x)|^2 \mathrm{d}G(u) f_X^2(x) \mathrm{d}x = n|B|^{1/2} \alpha_n^2 \mu\gamma = \mu\gamma$. □

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