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# Production Cost Heterogeneity in a Circular-City Model

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# Production Cost Heterogeneity in a Circular-City Model

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***Abstract***

We derive the closed-form solution characterizing the equilibrium in a circular-city model with competing firms of heterogeneous production costs. Tractability issues in this setting are well known and have not been resolved in prior work. In this paper, the equilibrium solution illustrates effects of production costs on firms' strategic decisions, their aggregate profit, and consumer surplus.

Keywords: Circular-city model, cost heterogeneity, game theory.

JEL Code: D4, L1.

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## *1. Introduction*

Based on Salop's influential work (1979), applications of Salop's circular-city model have generated important insights in economics and a variety of other disciplines. However, efforts to relax assumptions of the model are challenged by analytical tractability. An important extension of Salop (1979) is to examine firm heterogeneity in the spatial context. Many recent studies have contributed to this direction of research (Syverson 2004, Vogel 2008, 2011, Alderighi and Piga 2012). However, in Syverson (2004) and Vogel (2008, 2011), rivals' heterogeneous costs do not play a role in a firm's pricing decision. The dependency of equilibrium prices on rivals' heterogeneity is meaningful because firms' strategic responses to their neighbors' heterogeneous characteristics create an effect that propagates throughout the market – an important phenomenon also observed in practice. Whereas the circular-city setup offers the power to capture such an effect, this very feature also substantially complicates the analysis.

This paper allows for dependency of firms' decisions on rivals' heterogeneous characteristics. The focus is on characterizing equilibrium pricing decisions of firms with heterogeneous production costs in a circular spatial setting. Related to our work, Alderighi and Piga (2012) also incorporate firms' heterogeneous characteristics that impact their strategic decisions. Their results identify the uniqueness of equilibrium prices but do not generate the closed-form solution of such an equilibrium. Whereas numerical results (as shown in Alderighi and Piga 2012) illustrate some properties given sets of parameter values, availability of an analytical solution in our work offers the theoretical foundation for obtaining rigorous insights through comparative statics. Moreover, our derivation of the equilibrium solution is generalizable to settings with heterogeneity other than that of production costs in a circular-city model.

Based on the closed-form equilibrium solution, we identify a *ripple effect* that connects all firms pricing strategies. Not only does a firm's cost exert a positive impact on its own and all other firms' prices, such effect diminishes as it propagates away from the originating firm, similar to the dynamics of ripples during dispersion. Furthermore, we identify effects of production cost heterogeneity on firms' equilibrium prices and profits and on consumer surplus.

## *2. The Model*

Our model inherits the standard properties of Salop (1979) circle-city model. On a circle of unit circumference, a continuum of consumers is distributed uniformly, and $n$ firms are located equidistantly. Consumers are indexed by their own locations, which represent taste. Without the loss of generality, let firm $i$ be located at $\frac{i}{n}$ and offer products of value $v$ at price $p_i$. Firms set prices simultaneously. Each consumer purchases at most one unit of product. The distance between a consumer and her chosen firm represents the misfit between the purchased product and her ideal product. Let a consumer's transportation cost be linear in the distance between locations of the firm and the consumer at rate t. Thus, a consumer located at *x* who purchases from Firm *i* derives the utility $u(x) = v - p_i - t \cdot \left|\frac{i}{n} - x\right|$.

We extend the Salop (1979) model to account for heterogeneity in firms' production costs. Firm *i* incurs a marginal production cost $c_i$ where $i \in \{0,1, \dots n-1\}$. Following the convention in the literature (Eaton and Lipsey 1978, Syverson 2004, Alderighi and Piga 2012), we examine the equilibrium in which all firms obtain a positive market share. In other words, we impose the following condition to rule out cases where an existing firm cannot actively compete with other firms.

**Condition 1.** For all $\in \{0,1, \dots, n-2\}$, $|c_i - c_{i+1}| < \frac{t}{n}$.

We apply Condition 1 to the analysis throughout the paper. Notice that a firm's pricing strategy and profit are not only affected by its first-degree neighboring rivals—those located on its immediate left and right—but the firm's price and profits are in fact functions of pricing strategies and production costs of all remaining firms. Intuitively, the first-degree neighboring rivals strategize with consideration for their neighbors, who are second-degree neighbors to the original firm. Competition propagates around the circle and links all firms' pricing strategies together (Alderighi and Piga 2012). Each firm's product cost rides on this *ripple effect* and plays a role in all the other firms' pricing strategies. The demand for Firm *i* is then $q_i = \frac{1}{n} + \frac{1}{2t}(p_{i+1} + p_{i-1} - 2p_i)$, generating a profit of

$$\pi_i = (p_i - c_i)q_i = (p_i - c_i)\left[\frac{1}{n} + \frac{1}{2t}(p_{i+1} + p_{i-1} - 2p_i)\right] \qquad (1)$$

For notational convenience, we extend the domain of *i*, such that $i \in \mathbb{Z}$, to allow for continuous increments to firms' indices. Firms i and $i \pm n$ denote the same entity.

### *3. Analysis and Results*

**Proposition 1**: There exists a unique equilibrium among *n* firms.

For Firm *i*, $i \in \{0,1, \dots, n-1\}$, $p_i^* = \frac{t}{n} + \sum_{d=0}^{n-1} b_d c_{i-d}$

$$\text{where,} \quad b_d = \frac{(2+\sqrt{3})^d + (2+\sqrt{3})^{n-d}}{\sqrt{3}\left((2+\sqrt{3})^n - 1\right)} \qquad (2)$$

Firm *i*'s profit is $\pi_i^* = \frac{1}{t}(p_i^* - c_i)^2$.

Proof: All proofs are relegated to the Appendix.

Proposition 1 summarizes an important contribution of this work – the closed-form solution of firms' equilibrium in a circular-city model with cost heterogeneity. The functional form of equilibrium prices reaffirms aforementioned intuition that each firm's pricing strategy depends not only on its own production cost as well as those of all other firms in the market.

Eq. (2) yields a number of interesting insights into how production costs affect firms' equilibrium prices and profits. In Eq. (2), $b_d = \frac{\partial p_{i+d}}{\partial c_i}$ represents the impact of the production cost of Firm *i*'s $d$th-degree neighbor on Firm *i*'s price.

- First, $0 < b_0 < 1$ suggests that a firm's price increases with its production cost, but by a lesser magnitude than the increase in the production cost. Competition plays a role in moderating the extent of price increase as a result of higher production cost. As $\frac{\partial \pi_i^*}{\partial c_i} = \frac{2}{t}(p_i^* - c_i)(b_0 - 1) < 0$, a firm's profit decreases as its production cost increases.
- Second, $b_d > 0$, for $d = 0, 1, \ldots, n-1$, which implies that an increase in any firm's production cost leads to a price increase for every firm in the market. This follows from the first property: As an increase in a firm's production cost raises its own price, such price increase mitigates the firm's price competition with its first-degree neighbors, who then also raise price. The ripple effect passes incentives to raise price from firm to firm around the circle, resulting in price increases for all firms. Furthermore, $\frac{\partial \pi_i}{\partial c_{i'}} = \frac{2}{t}(p_i - c_i) b_{c_i' - c_i} > 0$; thus, whereas an increase in a firm's production cost reduces its own profit, the other firms' profits increase.
- Third, through ripple effect, the impact of a firm's production cost on other firms' prices weakens as it travels further away from the original firm. This is illustrated by Eq. (2): $b_d > b_{d'}$, $if \min\{d, n-d\} < \min\{d', n-d'\}$. In other words, a firm's price is affected more strongly by the production costs of the firms that offer more similar products than those offering products of greater differentiation.

We now turn to the analysis of the equilibrium price, profit and consumer surplus according to a general distribution of cost heterogeneity subject to Condition 1. Let us consider an industry where firms' production costs are independent of each other, following an identical cumulative distribution function, $F(c)$.

**Proposition 2**: The average equilibrium price in the market is made up of the average production cost plus a constant markup: $E(p_i^*) = \frac{t}{n} + E(c_i)$.

Intuitively, the average price decreases when more firms are in the market (a higher *n*) due to the competition effect; meanwhile, the average price increases with a higher degree of differentiation, which is implicit in any increase in consumers' transportation cost *t*. More interestingly, this result implies that the magnitude of any increase/decrease in the production cost (of one or multiple firms) is fully accounted in equilibrium prices. Firms are able to completely transfer this shift in cost to consumers, and not only to those buying from the firms that incur such cost shift. Whereas a firm's price changes to a lesser extend relative to the changes in its production cost (Proposition 1), the remaining difference in the production cost is accounted for in the total adjustments of the other firms' prices, as a result of competition under ripple effect.

**Proposition 3**: The price variance is increasing in the variance of production costs; furthermore, $Var(p) < Var(c)$. The expected aggregate profit of firms is increasing in the variance of production costs and independent of the expected production cost.

Not surprisingly, production cost heterogeneity causes dispersion in equilibrium prices; however, the variance of production costs always dominates that of prices. An increase in the variance in production cost allows either high-cost firms (which are also firms with higher prices) to charge more or low-cost firms to charge less, because a firm's equilibrium price is increasing in its own cost (Proposition 1). The variance of prices in turn increases. On the other hand, the ripple effect distributes the impact of the cost change on *one* firm to price changes among *all* firms (Proposition 1). This propagated price change tightens variation in price relative to that in cost.

Furthermore, shifting the average production cost while holding the variance fixed has no impact on firms' competition intensity and expected aggregate profit, which, instead, increases in the variance of costs. An increase in the variance of production cost leads to a higher degree of asymmetry in the competition. Given that $0 < b_0 < 1$ (Proposition 1), firms with increased costs suffer a loss in its margin as well as market share, whereas the opposite applies for those with reduced costs. Therefore, as the variance of costs increases, among the expected aggregate profit the proportion of transactions with increased margin overtakes those with reduced margins under competition with the ripple effect; as a result, the expected aggregate profit goes up.

**Proposition 4**: The expected consumer surplus is decreasing in the average production cost and increasing in the variance of production costs.

A higher variance of production costs not only raises the aggregate profit (Proposition 3), it also increases the expected consumer surplus. Clearly, the latter requires that the average production cost does not shift up. Because increases in costs are fully transferred to price (proposition 2), which offsets the positive effect of variance on consumer surplus. Suppose the average cost is fixed, a higher variance can be achieved by increasing a higher cost while decreasing a lower cost. Whereas the locations of the firms whose costs change generally matter, the *expected* consumer surplus accounts for uncertainties in firms' locations. This mean-preserving spread in productions costs will lead to a wider expected price gap between any two neighboring firms. In expectation, the consumer segment that benefits from the widened price gap is larger than the segment that becomes worse off, because reductions in the lower cost lead to price cut which expands the firm's market share.

From both Propositions 3 and 4, the scenario of increasing cost variance while fixing the average cost results in a higher social welfare. It suggests a potential mechanism to increase profitability of an industry without taxing consumers. Whereas overall improvements in production efficiency may actually intensify firms' competition, balanced shocks such as policies to regulate supplier contracts to induce dispersion of production efficiencies or capabilities may improve social welfare.

## *4. Conclusion*

This paper makes an important theoretical contribution to the literature on spatial competition with heterogeneous firms by characterizing the closed-form equilibrium. The results illustrate the ripple effect among competing firms and explain the dependence of equilibrium prices on the distribution of production costs. Furthermore, the analysis on the aggregate profit and consumer surplus shed light on the impact of policies and mechanisms that may alter industry-wide production costs.

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**Appendix**

Proof of Proposition 1.

Proof:

From (1) F.O.C. w.r.t. $p_i$: $p_{i+1} + p_{i-1} - 4p_i + \frac{2t}{n} + 2c_i = 0$ (3)

From (3) $p_{i+1} - (2-\sqrt{3})p_i = (2+\sqrt{3})(p_i - (2-\sqrt{3})p_{i-1}) - \frac{2t}{n} - 2c_i$, for all *i*; Thus,

$(2+\sqrt{3})(p_i - (2-\sqrt{3})p_{i-1}) = (2+\sqrt{3})\left((2+\sqrt{3})(p_{i-1} - (2-\sqrt{3})p_{i-2}) - \frac{2t}{n} - 2c_{i-1}\right)$.

Similarly, $(2+\sqrt{3})^{n-1}(p_{i+2} - (2-\sqrt{3})p_{i+1}) = (2+\sqrt{3})^{n-1}\left((2+\sqrt{3})(p_{i+1} - (2-\sqrt{3})p_i) - \frac{2t}{n} - 2c_{i+1}\right)$. Summing up these equations for all *i* gives:

$p_{i+1} - (2-\sqrt{3})p_i = (2+\sqrt{3})^n(p_{i+1} - (2-\sqrt{3})p_i) - \frac{2t}{n}\sum_{j=0}^{n-1}(2+\sqrt{3})^j - 2\sum_{j=0}^{n-1}(2+\sqrt{3})^j c_{i-j}$ .

Thus, $p_{i+1} = (2-\sqrt{3})p_i + \frac{2t}{n(1+\sqrt{3})} - \frac{2}{1-(2+\sqrt{3})^n}\sum_{j=0}^{n-1}(2+\sqrt{3})^j c_{i-j}$ . Applying this equation form for all i:

$$(2-\sqrt{3})p_i = (2-\sqrt{3})\left((2-\sqrt{3})p_{i-1} + \frac{2t}{n(1+\sqrt{3})} - \frac{2}{1-(2+\sqrt{3})^n}\sum_{j=0}^{n-1}(2+\sqrt{3})^j c_{i-j-1}\right)$$

…

$(2-\sqrt{3})^{n-1}p_{i+2} = (2-\sqrt{3})^{n-1}\left((2-\sqrt{3})p_{i+1} + \frac{2t}{n(1+\sqrt{3})} - \frac{2}{1-(2+\sqrt{3})^n}\sum_{j=0}^{n-1}(2+\sqrt{3})^j c_{i-j}\right)$,

and then summing them up yields:

$p_{i+1} = (2-\sqrt{3})^n p_{i+1} + \frac{2t}{n(1+\sqrt{3})}\sum_{k=0}^{n-1}(2-\sqrt{3})^k - \frac{2}{1-(2+\sqrt{3})^n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}(2-\sqrt{3})^k(2+\sqrt{3})^j c_{i-j-k}$

Thus, $p_i^* = \frac{t}{n} - \frac{2}{(1-(2+\sqrt{3})^n)(1-(2-\sqrt{3})^n)}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}(2-\sqrt{3})^k(2+\sqrt{3})^j c_{i-j-k-1} = \frac{t}{n} + \sum_{d=0}^{n-1} b_d c_{i-d}$, (4)

where $b_d = \frac{(2+\sqrt{3})^d + (2+\sqrt{3})^{n-d}}{\sqrt{3}((2+\sqrt{3})^n - 1)} > 0$. Notice that $b_0 = \frac{(2+\sqrt{3})^n + 1}{\sqrt{3}((2+\sqrt{3})^n - 1)} < 1$ because:

for $n > 1$ , $\left(2+\sqrt{3}\right)^n > 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$ , then, $(\sqrt{3}-1)\left(2+\sqrt{3}\right)^n > 1+\sqrt{3}$ ,

Thus, $\sqrt{3}\left(\left(2+\sqrt{3}\right)^n - 1\right) > \left(2+\sqrt{3}\right)^n + 1$ , or, $\frac{\left(2+\sqrt{3}\right)^n+1}{\sqrt{3}\left(\left(2+\sqrt{3}\right)^n-1\right)} < 1$.

From (3), $\frac{1}{n} + \frac{1}{2t}(p_{i+1} + p_{i-1} - 2p_i) = p_i - c_i$ (5)

Substitute (5) into (1), $\pi_i^* = (p_i^* - c_i)\left[\frac{1}{n} + \frac{1}{2t}(p_{i+1}^* + p_{i-1}^* - 2p_i)\right] = \frac{1}{t}(p_i^* - c_i)^2$

Q.E.D.

Proof of Proposition 2.

Proof: Since $p_i^* = \frac{t}{n} + \sum_{d=0}^{n-1} b_d c_{i-d}$, then $E(p_i^*) = \frac{t}{n} + \sum_{d=0}^{n-1} b_d\, E(c_i)$, where $\sum_{d=0}^{n-1} b_d = 1$.
Q.E.D.

Proof of Proposition 3.

Proof:

As $p_i^* = \frac{t}{n} + \sum_{d=0}^{n-1} b_d c_{i-d}$, $Var(p) = Var(c) \cdot \sum_{d=0}^{n-1} b_d^2$. ($c_i$ are independent of each other).

$\sum_{d=0}^{n-1} b_d^2 = \frac{2}{3}\frac{\frac{1}{\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{2n}-1\right)+n\left(2+\sqrt{3}\right)^n}{\left(\left(2+\sqrt{3}\right)^n-1\right)^2} < 1$. Thus, $Var(p) < Var(c)$.

$E(\sum_{i=0}^{n-1} \pi_i) = E\left(\sum_{i=0}^{n-1} \frac{1}{t}(p_i^* - c_i)^2\right) = E\left(\sum_{i=0}^{n-1} \frac{1}{t}\left(\frac{t}{n} + \sum_{d=0}^{n-1} b_d c_{i-d} - c_i\right)^2\right)$

$= \frac{1}{t} E\left(\sum_{i=0}^{n-1}\left(\frac{t^2}{n^2} + \frac{2t}{n}(\sum_{d=0}^{n-1} b_d c_{i-d} - c_i) + (\sum_{d=0}^{n-1} b_d c_{i-d} - c_i)^2\right)\right)$

$= \frac{1}{t}\sum_{i=0}^{n-1}\left(\frac{t^2}{n^2} + \frac{2t}{n}E(\sum_{d=0}^{n-1} b_d c_{i-d} - c_i) + E(\sum_{d=0}^{n-1} b_d c_{i-d} - c_i)^2\right)$

$= \frac{n}{t}\left[\frac{t^2}{n^2} + \frac{2t}{n}E(c)(\sum_{d=0}^{n-1} b_d - 1) + E[c^2]((1-b_0)^2 + \sum_{d=1}^{n-1} b_d^2) + E^2[c](\sum_{d\neq k} b_d b_k - 2\sum_{d=1}^{n-1} b_d)\right]$

$= \frac{n}{t}\left[\frac{t^2}{n^2} + \frac{2t}{n}E(c)(\sum_{d=0}^{n-1} b_d - 1) + (Var(c) + E^2[c])((1-b_0)^2 + \sum_{d=1}^{n-1} b_d^2) + E^2[c](\sum_{d\neq k} b_d b_k - 2\sum_{d=1}^{n-1} b_d)\right]$

$= \frac{n}{t}\left[\frac{t^2}{n^2} + \frac{2t}{n}E(c)(\sum_{d=0}^{n-1} b_d - 1) + Var(c)((1-b_0)^2 + \sum_{d=1}^{n-1} b_d^2) + E^2[c](\sum_{d=0}^{n-1} b_d - 1)^2\right]$

As $\sum_{d=0}^{n-1} b_d = 1$, $E(\sum_{i=0}^{n-1} \pi_i) = \frac{t}{n} + \frac{n}{t} Var(c)((1-b_0)^2 + \sum_{d=1}^{n-1} b_d^2)$

Thus, $\frac{\partial E(\sum_{i=0}^{n-1} \pi_i)}{\partial Var(c)} > 0$

Q.E.D.

Proof of Proposition 4.

Without loss of generality, the expected consumer surplus on the circle is $E[u] = n \cdot E\left[u|_{[0,\frac{1}{n}]}\right]$.

Consider the marginal consumer between firm 0 and firm 1, $x^* = \frac{1}{2t}(p_1 - p_0) + \frac{1}{2n}$. $0 \le x < x^*$buys from Firm 0; $x^* \le x \le \frac{1}{n}$buys from Firm 1.

Thus, $E\left[u|_{[0,\frac{1}{n}]}\right] = E\left[\int_0^{x^*}(v - p_0 - tx)dx + \int_{x^*}^{\frac{1}{n}^*}\left(v - p_1 - t\left(\frac{1}{n} - x\right)\right)dx\right]$

$= E\left[-t\left(\frac{1}{2t}(p_1 - p_0) + \frac{1}{2n}\right)^2 + \left(\frac{1}{2t}(p_1 - p_0) + \frac{1}{2n}\right)\left(p_1 - p_0 + \frac{t}{n}\right) + \frac{1}{n}(v - p_1) - \frac{t}{2n^2}\right]$

Define $a \equiv p_1 - p_0$. As $E(p_1) = E(p_2) = \frac{t}{n} + E(c)$, $E[a] = 0$.

then, $E\left[u|_{[0,\frac{1}{n}]}\right] = E\left[-t\left(\frac{1}{2t}a + \frac{1}{2n}\right)^2 + \left(\frac{1}{2t}a + \frac{1}{2n}\right)\left(a + \frac{t}{n}\right) + \frac{1}{n}(v - p_1) - \frac{t}{2n^2}\right]$

$= \frac{v}{n} - \frac{5}{4}\frac{t}{n^2} + \frac{1}{4t}E(a^2) - \frac{1}{n}E(c)$, where

$E(a^2) = E[(p_1 - p_0)^2] = E[(\sum_{d=0}^{n-1} b_d c_{1-d} - \sum_{d=0}^{n-1} b_d c_{0-d})^2]$

$= E\left[\left(b_0c_1 + b_1c_0 + \cdots + b_{n-1}c_{2-n} - (b_0c_0 + b_1c_{-1} + \cdots + b_{n-1}c_{1-n})\right)^2\right]$

$= E[\left(\left(c_0(b_1 - b_0) + c_{-1}(b_2 - b_1) + \cdots + c_{2-n}(b_{n-1} - b_{n-2}) + c_{1-n}(b_n - b_{n-1})\right)^2\right]$

$= E[(\sum_{i=1-n}^{0} c_i(b_{1-i} - b_{-i}))^2]$

$= E\left[\sum_{i=1-n}^{0} c_i^2(b_{1-i} - b_{-i})^2 + \sum_{i\neq j} c_i c_j(b_{1-i} - b_{-i})(b_{1-j} - b_{-j})\right]$

$= E(c^2)\sum_{i=1-n}^{0}(b_{1-i} - b_{-i})^2 + E^2(c)\sum_{i\neq j}(b_{1-i} - b_{-i})(b_{1-j} - b_{-j})$

$= Var(c)\sum_{i=1-n}^{0}(b_{1-i} - b_{-i})^2 + E^2(c)\sum_i\sum_j(b_{1-i} - b_{-i})(b_{1-j} - b_{-j})$

$= Var(c)\sum_{i=1-n}^{0}(b_{1-i} - b_{-i})^2$

$= Var(c) \sum_{i=0}^{n-1} (b_i - b_{i-1})^2$

Thus, $E(u) = n \cdot E\left[ u|_{\left[0,\frac{1}{n}\right]} \right] = v - \frac{5}{4}\frac{t}{n} + \frac{n}{4t} Var(c) \sum_{i=0}^{n-1} (b_i - b_{i-1})^2 - E(c)$ Q.E.D.