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Asset Pricing with Financial Bubble Risk

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Abstract

This paper characterizes systematic risk stemming from the possible occurrence of price bubbles and measures the impact of this additional risk factor on asset prices. Historical stock market behavior and recent empirical experience have led economists and policy makers to acknowledge that price bubbles in financial markets do occur and need to be accounted for in risk analysis. New econometric tools for analyzing mildly explosive behavior (Phillips and Magdalinos, 2007; Phillips, Wu and Yu, 2011) have made it possible to detect the presence of bubbles in data and to date stamp their origination and collapse, providing empirical confirmation of such episodes in recent data. The potential for price bubbles and market collapse provides another source of stock market risk and adds to the risk premium. We provide an analytic and empirical investigation of this additional risk factor. The standard present value model is extended to allow for possible price bubbles and the effects of integrating bubble behavior into a consumption-based asset pricing model are analyzed. The theory involves attention to the investor time horizon and a study of the validity of conventional log linear approximations in the presence of nonstationary and mildly explosive data. Finite decision horizons accommodate myopic investors and are a component of speculative behavior that focuses on short run market gains rather than long run effects of fundamentals. An econometric approach to estimate bubble risk effects is developed and the methods are applied to composite stock market index data, giving new model-based equity premium and market volatility estimates that more closely match the data than traditional consumption based asset pricing models.

Keywords: Asset Pricing, Bubbles, Explosive process, Equity premium puzzle, Log linear approximation, Mildly explosive time series, Present value model, Risk, Volatility puzzle.

JEL classification: G10, C22

1 Introduction

While there is still debate about the existence, source, form and even the nomenclature of price bubbles, experience such as the dot.com bubble in the 1990s and the price elevations and crashes during the subprime crisis over 2007-2008 make bubble phenomena hard to ignore, whatever their

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particular origins may be. Recent empirical evidence discussed below supports the presence of mildly explosive behavior[1] for several financial series during these episodes. Given this market phenomena, the present paper points to an additional source of systematic risk in financial markets stemming from the possibility of a (periodically collapsing) price bubble in the future. Such a possibility is not accommodated in the existing literature on asset price determination.

The models studied in the present paper extend the standard present value model and a consumption based CAPM model by incorporating the possibility that bubbles may occur, at least during some sub-period in the future. Using these extended models, we investigate the effects of price bubbles on financial asset returns and explore the relationship between price bubbles and some well known financial market anomalies.

Two anomalies that have attracted a lot of attention in the literature are the "equity premium" and "stock market volatility" puzzles. These puzzles have typically been studied in the context of standard models under the assumption that there is no "rational bubble" in asset prices. Several possible explanations for the puzzles have been proposed in the literature. But the literature still seems far from any consensus regarding a satisfactory explanation for the puzzles, although recent applied research (Barro, 2006) has pointed to the role played by rare disasters.

Recent applied econometric work (Phillips, Wu and Yu, 2011, hereafter PWY; Phillips, Shi and Yu, 2015a, hereafter PSY) has reported strong empirical evidence in support of the existence of periodically collapsing bubbles in Nasdaq and S&P 500 stock market data. That work has been extended to study the subprime crisis and many associated financial series were found to manifest episodes of mildly explosive behavior over the period 2002-2009 (Phillips and Yu, 2011, hereafter PY). This research has developed econometric methods based on mildly explosive processes for date stamping the origination and termination of bubble behavior in financial data. The present paper seeks to use some of the same methods to investigate the possible impact of bubbles on pricing and return formulae, providing some new insights into the relationship between bubble phenomena and financial market returns. The modified asset pricing theory is used to explore the empirical implications of admitting periods of possible exuberance and collapse on the equity premium and volatility puzzles.

While extending the present value model to allow for possible price exuberance, we develop a novel and flexible characterization of the investor horizon. Investor time horizons are commonly assumed to be infinite for analytic tractability (see, e.g., Campbell and Viceira, 1999), regardless of realism. The present paper introduces a new investor horizon that eliminates the need for transversality conditions and provides a realistic degree of myopia in decision making. The time horizon is formulated so that it is functionally dependent on the sample size and is at once distant

[1] Mildly explosive behavior occurs when the generating mechanism for the data has an autoregressive root (θ) that is slightly greater than unity and of the form $\theta = 1 + \frac{c}{k_n}$, where $c > 0$, $k_n \to \infty$, $\frac{k_n}{n} \to 0$, and n is the sample size. Such a root exceeds the commonly used local to unity root $\theta = 1 + \frac{c}{n}$, as $n \to \infty$, for which the corresponding process has random wandering behavior similar to that of a unit root process in contrast to explosive behavior. Mildly explosive processes are useful in modeling financial exuberance and were introduced by Phillips and Magdalinos (2007) who discuss their various properties and show how they may be used to conduct inference. Section 2.2 below develops this theory in the context of an asset pricing model when there may be periods of financial exuberance.

enough to maintain analytical convenience while at the same time accommodating some myopic decision making. Enhancing realism of the investor horizon has the additional advantage that the model can successfully incorporate the impact of price exuberance on asset prices within the framework of the present value model.

This paper contributes to the asset pricing and related econometric literatures in several ways. First, the asset pricing model developed here features aspects of stock market performance and economic agent behavior that improve realism: (i) the model accommodates the possibility of periods of price exuberance, a fact that is hard to ignore in the history of the stock market; (ii) representative agent behavior is based on decision making over a finite investor horizon and takes into account investor ignorance in advance of the precise dates of any periods of exuberance and collapse, thereby enhancing realism of investor horizons and forward information in decision making; (iii) derivation of the present value model takes into account the properties of the log linear approximation, including issues of validity of the approximation during episodes of exuberance. In these respects, we believe that our asset pricing model is not "too simple to capture the full array of governing variables that drive economic reality" (Greenspan, 2008).

Second, the new model is simple enough for analysis and econometric implementation. In particular, the new risk factor stemming from potential outbreaks of price exuberance has a simple and estimable parsimonious expression. Origination and termination dates of price exuberance are explicitly incorporated into the model framework, even though these dates are not known in advance by investors. Third, econometric dating methods enable us to determine such dates ex post and to compute the empirical contribution of bubble risk to the equity premium and market volatility, thereby contributing to the equity premium and market volatility puzzle literature.

The plan of the paper is as follows. Section 2 discusses the present value model and the effects of price bubbles on the validity of the commonly used log linear approximation. New conditions for the validity of this approximation in the presence of bubbles are developed. This section also introduces the concept of a distant investor horizon, as distinct from the usual infinite horizon framework, and considers the advantages of this framework in a model where the price process may have periods of mildly explosive behavior. Section 3 extends existing consumption-based asset pricing models to allow for bubble effects, develops a new recursive utility framework under Epstein-Zin (1991) preferences that allows for distant decision horizons, and develops new formulae for the equity premium and return volatility. Section 4 develops the econometric methodology for studying the equity premium and volatility puzzles allowing for mildly explosive behavior. Section 5 provides an empirical application of these techniques to the consumption based CAPM model and provides estimates of a new bubble premium effect that helps to explain the equity premium and volatility puzzles. Section 6 concludes. Proofs, together with some supplementary discussion on recursive utility functions and the implications of finite and distant investor horizons, are given in the Appendix.

2 Models with Exuberance

2.1 The Present Value Model and Financial Bubbles

We start from the standard accounting identity for a financial return (R_{t+1}) of an asset held over $(t, t+1)$ in terms of the dividend (D_{t+1}) and price (P_{t+1})

$$1 + R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}.$$

To facilitate analysis of the present value model, the following loglinear approximation about the sample mean to the exact relation was suggested by Campbell and Shiller (1988a) and is summarized in Campbell, Lo and Mackinlay (1997, Ch.7)

$$\begin{aligned} r_{t+1} &= \log(1 + R_{t+1}) = \log(P_{t+1} + D_{t+1}) - \log P_t \qquad (2.1) \\ &= \Delta p_{t+1} + \log\left\{1 + e^{d_{t+1} - p_{t+1}}\right\} \\ &\simeq \kappa + \rho p_{t+1} + (1-\rho) d_{t+1} - p_t, \qquad (2.2) \end{aligned}$$

where $p_t = \log P_t$, $d_t = \log D_t$, and

$$\kappa = -\rho \log \rho - (1-\rho)\log(1-\rho), \quad \rho = \frac{1}{1 + \exp(\overline{d-p})} < 1, \qquad (2.3)$$

with $\overline{d-p} = n^{-1} \sum_{t=1}^{n} (d_t - p_t)$ being the average log dividend-price ratio based on a sample of size n.

The commonly used present value relationship is obtained from (2.2) by simple manipulations under condition (2.3). First, defining $\delta_t = d_t - p_t$, transposition of (2.2) and forward recursion leads to

$$\begin{aligned} \delta_t &\simeq \rho \delta_{t+1} + r_{t+1} - \Delta d_{t+1} + \kappa \\ &= \lim_{i\to\infty} \rho^i \delta_{t+i} + \sum_{i=0}^{\infty} \rho^i (r_{t+1+i} - \Delta d_{t+1+i}) + \frac{\kappa}{1-\rho}, \qquad (2.4) \end{aligned}$$

which converges almost surely for fixed $\rho \in (0,1)$ provided $E|r_t| < \infty$ and $E|\Delta d_t| < \infty$. Taking conditional expectations at time t then leads to the present value relationship

$$p_t = \frac{\kappa}{1-\rho} + d_t + E_t \sum_{i=0}^{\infty} \rho^i (\Delta d_{t+1+i} - r_{t+1+i}) + \text{plim}_{i\to\infty} \rho^i E_t p_{t+i} - \text{plim}_{i\to\infty} \rho^i E_t d_{t+i}. \qquad (2.5)$$

Terminal conditions conventionally eliminate the last two terms of (2.5) by imposing $\text{plim}_{i\to\infty} \rho^i E_t (p_{t+i} - d_{t+i}) = 0$, which puts conditions on the permissible trajectories of the price dividend ratio. If these conditions do not hold, then the term affects the relationship in ways that we now proceed to analyze.

We start by assuming that the dividend process is not explosive, an assumption that is supported by widespread empirical evidence[2]. In this event, we can reasonably expect that $\text{plim}_{i\to\infty}\rho^i E_t(d_{t+i}) = 0$ when $\rho \in (0,1)$ is fixed. In particular, if dividends behave like an integrated process with mean dividend growth rate g, with stationary and ergodic disturbances $\varepsilon_{d,t}$ and with an initialization $d_0 = O_p(1)$, then we have $d_{t+1} = g + d_t + \varepsilon_{d,t+1} = O_p\left(\sqrt{t} + gt\right)$. It follows that

$$\begin{aligned} \text{plim}_{i\to\infty}\rho^i E_t d_{t+i} &= \text{plim}_{i\to\infty}\rho^i E_t(ig + d_t + \sum_{j=1}^{i}\varepsilon_{d,t+j}) \\ &= \text{plim}_{i\to\infty}\rho^i(ig + d_t + \sum_{j=1}^{i}E_t\varepsilon_{d,t+j}) = 0, \end{aligned} \tag{2.6}$$

since $\rho < 1$. This type of calculation is usually performed, as in (2.6), under the assumption that ρ is fixed, whereas it is in fact both sample and sample size dependent in view of (2.3). In what follows we explore the effects of this dependence and accordingly we often write $\rho = \rho_n$.

If a period of price exuberance is expected in which prices follow an explosive model of the form $p_{t+1} = \theta p_t + \varepsilon_{p,t+1}$, for some $\theta > 1$ with stationary and ergodic increments $\varepsilon_{p,t}$ then

$$\text{plim}_{i\to\infty}\rho^i E_t p_{t+i} = \text{plim}_{i\to\infty}\rho^i E_t(\theta^i p_t + \sum_{j=1}^{i}\theta^{i-j}\varepsilon_{p,t+j}) = \text{plim}_{i\to\infty}(\rho\theta)^i p_t, \tag{2.7}$$

which is divergent, convergent, or convergent to zero according as

$$\theta > \frac{1}{\rho} = 1 + \exp(\overline{d-p}), \quad \theta\rho \to_p c > 0, \quad \theta\rho \to_p 0. \tag{2.8}$$

The limit behavior in (2.7) is therefore influenced by the nature and asymptotic behavior of the sample mean $\overline{d-p}$ as well as the specific parameter values of θ and ρ and any dependencies these parameters have, including dependence on the sample size, that affect the relationship (2.8) between them. For example, ρ is clearly sample size dependent in view of its definition (2.3) and the log linear approximation; and the autoregressive coefficient θ may be sample size dependent if the price process p_t manifests local to unity or mildly explosive behavior. The time horizon contemplated in decision making also comes into play in determining the ultimate form of (2.5). The usual terminal conditions set the horizon to be infinite, but a more flexible framework may be needed to accommodate myopic investors with finite horizons or investors with distant horizons that may grow according to some measure such as the size of the available data set.

All of these potential influences on the model (2.5) will be explored in detail below. If the usual terminal condition does not hold and instead $\text{plim}_{i\to\infty}\rho^i E_t(p_{t+i} - d_{t+i}) = b_t \neq 0$, the nature of the process b_t becomes decisive in modifying the properties of the present value relationship. The properties of b_t are determined according to the specific value of $\theta\rho$ or its limiting form in cases such as those described above where θ and ρ are sample dependent and do not take on fixed values.

[2] For example, PWY (2010) show that real NASDAQ dividends do not manifest explosive behavior over the period April 1976 to June 2005.

A possible framework for the analysis of such possibilities is considered in the next section. This framework allows for potential episodes of exuberance where $\theta > 1$ leading to the presence of price bubbles over some future periods.

2.2 Financial Bubble Effects

Our intent is to allow for the possible presence of financial bubbles in a standard asset pricing model and to do so we show how the presence of bubbles affects present value calculations. Bubbles may be accommodated within the model but only under certain conditions. These conditions relate to parameter sequences that involve the generating mechanism for prices, the investor horizon and the duration of the bubble period. Intuitively, the effect of price bubbles on present value calculations will be finite provided the duration of the bubble period is not too large and the investor horizon relates to a moderately distant or foreseeable future. The latter framework replaces infinite horizon with distant horizon calculations and avoids the (commonly used) zero terminal conditions described above. An additional feature of the use of a distant horizon framework is that the calculations directly relate the investor decision making timeframe to the available data by means of sample size dependence. The precise conditions are detailed and discussed below.

The parametric framework for the price and dividend series is assumed to be of the form

$$d_{t+1} = d_t + \varepsilon_{d,t+1}, \tag{2.9}$$

$$p_{t+1} = \begin{cases} \theta p_t + \varepsilon_{p,t+1} & 0 \leq t < \tau_e, \quad \theta = 1, \quad p_0 = O_p(1) \\ \theta_n p_t + \varepsilon_{p,t+1} & \tau_e \leq t < \tau_f, \quad \theta_n = 1 + \frac{c}{k_n} \\ p_{\tau_e} + \varepsilon_p^* & t = \tau_f \\ \theta p_t + \varepsilon_{p,t+1} & \tau_f < t \leq n, \quad \theta = 1 \end{cases}. \tag{2.10}$$

We may also add a drift to (2.9) and to those components of (2.10) for which $\theta = 1$ but to keep the analysis as simple as possible we use (2.9) and (2.10). The autoregressive coefficient in the price equation shifts from unity over $[1, \tau_e] \cup (\tau_f, n]$ to

$$\theta_n = 1 + \frac{c}{k_n} > 1, \ c > 0, \tag{2.11}$$

over the bubble period $(\tau_e, \tau_f]$. Here n is the sample size and $k_n \to \infty$ as $n \to \infty$ in such a way that $\frac{k_n}{n} \to 0$ which implies that θ_n is mildly explosive, taking a value that is in an explosive neighborhood of unity but exceeds the immediate local to unity interval $\left[1, 1 + \frac{c}{n}\right]$ for some fixed $c > 0$. The bubble collapses in period $\tau_f + 1$ at which point p_t returns to within some $O_p(1)$ random quantity ε_p^* of its pre-bubble value p_{τ_e}. Hence, $p_{\tau_f+1} = p_{\tau_e} + \varepsilon_p^* = O_p\left(\sqrt{\tau_e}\right)$ provides a new initialization for the price series in the post-bubble epoch $(\tau_f, n]$.

The formulation (2.11) is used in PY (2011). It is plausible for many financial time series during periods of exuberance and it facilitates analysis, including central limit theory, as shown in Phillips and Magdalinos (2007). In what follows, we will assume that both initialization d_0 and p_0 are $O_p(1)$, although more complex initialization may be considered (e.g., Phillips and Magdalinos, 2009). At τ_f, the mildly explosive period terminates, the price series collapses and unit root behavior re-initiates at $t = \tau_f + 1$. More flexible transitions may also be accommodated but to keep the framework focused will not be considered here.

The time parameters τ_e and τ_f can be viewed as the origination and termination dates of bubble activity, allowing for structural breaks in the form of the autoregressive coefficient at these dates, switching the regime between unit root and mildly explosive episodes. The shift from $\rho = 1$ to $\rho = 1 + \frac{c}{k_n}$ is a small local change that is hardly detectable initially but it can have a major impact on the price trajectory when the shift is of moderate duration. To simplify technicalities, only a single explosive episode is included within the time horizon over which the conditional expectation E_t is taken. The assumption of a single bubble episode may also be interpreted as being consistent with what a representative investor may anticipate as a possible future trajectory for asset prices over a typical future time horizon. Generalization to the case of multiple bubble episodes is relatively straightforward but introduces additional notational complexity without leading to substantial changes in the analysis that follows at least if there is only a finite number of such episodes. For the present development, the error process $\varepsilon_{t+1} = (\varepsilon_{d,t+1}, \varepsilon_{p,t+1})$ is assumed to be generated by a linear process as

$$\varepsilon_t = C(L) e_t = \sum_{j=0}^{\infty} c_j e_{t-j}, \quad \text{with} \ \sum_{j=0}^{\infty} j |c_j| < \infty, \tag{2.12}$$

where $\{e_t\}$ is a vector of independent and identically distributed variates with variance matrix

$$\Sigma = \begin{bmatrix} \sigma_{dd} & \sigma_{dp} \\ \sigma_{pd} & \sigma_{pp} \end{bmatrix}.$$

The iid assumption is not essential and may be replaced by a martingale difference sequence framework at the cost of some additional complexity. It is assumed that $(1, -1) C(1) = 0$, so that the vector $(1, -1)$ is cointegrating in the system (2.9)-(2.10) for the non explosive period $t \notin [\tau_e, \tau_f]$, thereby producing a stationary log dividend price ratio $d_{t+1} - p_{t+1}$. If there is a drift component in the system over the non explosive period then this is also removed over those periods by the same (deterministically) cointegrating transform. It is also convenient, although not essential, to assume that ε_{pt} is iid, which further restricts the allowable form of the linear process operator $C(L)$.

Under these conditions and if the initialization $d_0, p_0 = o_p(\sqrt{n})$, we have $n^{-1/2} d_{\lfloor n \cdot \rfloor} \Rightarrow B_d(\cdot)$ where $\lfloor \cdot \rfloor$ is the integer part of the argument and $n^{-1/2} p_{\tau_e} \Rightarrow B_p(r_e)$ when $\tau_e = \lfloor n r_e \rfloor$ for some $r_e \in (0, 1)$ and where $B = (B_d, B_p)'$ is Brownian motion with variance matrix $\Omega = C(1) \Sigma C(1)'$ (e.g., Phillips and Solo, 1992).

2.3 Loglinear Approximation in the Presence of Bubbles

We analyze the validity of the usual loglinear approximation and specify conditions under which the approximation remains valid in the presence of bubble activity. The log linear approximation (2.1) typically relies on the presumption that the sample mean of the log dividend-price ratio,

$\overline{d-p}$, converges to a constant corresponding to some population mean of $d_t - p_t$. The approximation is then a simple linearization about this value. When d_t and p_t are integrated processes, the approximation is clearly valid when the time series are cointegrated with specific cointegrating vector $[1, -1]$. In particular, if $d_t - p_t = \alpha + u_t$, then $n^{-1} \sum_{t=1}^{n} (d_t - p_t) \longrightarrow_{a.s} \alpha$, and

$$\rho = \rho_n := \frac{1}{1 + \exp(\overline{d-p})} \longrightarrow_{a.s.} \frac{1}{1 + \exp(\alpha)} < 1, \tag{2.13}$$

The non zero mean effect α may arise from the presence of an intercept in the cointegrating relation and associated deterministic components in the generating mechanism. From (2.13), the limit of ρ_n lies in the interval $(0, 1)$ and the component expressions in (2.5) will then converge, as discussed earlier.

However, if there is an episode in which p_t manifests mildly explosive behavior of the form shown in (2.10) while d_t remains integrated, then $d_t - p_t$ is no longer stationary and ergodic and the log linear approximation is valid only under certain circumstances. In particular, we note that if $\rho_n \longrightarrow_{a.s.} 1$ then the validity of (2.5) may be compromised by potentially divergent components. Intuitively, if the duration of a bubble episode is of the same order as the sample size, then the sample mean of the price process will be dominated by the explosive episode and may diverge, so that ρ_n is driven to unity. On the other hand, if the duration of the bubble episode is of smaller order than the sample size, the effect of explosive behavior on the sample mean should also be small and the approximation should be unaffected. We call the first case *long* and the second case *short* bubble duration and consider each in turn.

(i) Long bubble duration

Careful analysis of the validity of the loglinear approximation requires that we determine explicitly the orders of magnitude of components that depend on ρ_n. It is convenient (as in PWY) to parameterize the origination and termination dates of the bubble as fractions of the sample size, setting $\tau_e = \lfloor nr_e \rfloor$ as above and $\tau_f = \lfloor nr_f \rfloor$ for some fixed numbers $r_e < r_f$. Then the duration of the bubble period is sample size dependent and may be denoted by $m_n := \tau_f - \tau_e$. In this case $m_n = \lfloor nr_b \rfloor$, where $r_b = r_f - r_e > 0$, and the bubble episode duration therefore has the same order of magnitude as the overall sample.

We may now analyze the limit behavior of ρ_n and the components that depend on it. We assume that $d_t - p_t = u_t$ is stationary and ergodic over non-bubble periods with $E\left(u_t 1\left\{t \notin [\tau_e, \tau_f)\right\}\right) = \alpha$. Then, as shown in the Appendix,

$$\overline{d-p} = (r_e + 1 - r_f)\,\alpha + \frac{\tau_f - \tau_e}{n} \frac{1}{\tau_f - \tau_e} \sum_{t=\tau_e}^{\tau_f - 1} (d_t - p_t) + o_{a.s.}(1),$$

and in this case of long bubble duration

$$\frac{1}{\rho_n} = 1 + e^{\left(1-r_f+r_e\right)\alpha} \exp\left\{-\frac{k_n}{\sqrt{n}c} e^{cr_b \frac{n}{k_n}} B_p\left(r_e\right)\right\}\left\{1+o_p\left(1\right)\right\}. \tag{2.14}$$

If $k_n = n^{1-\eta}$ for some $\eta \in (0,1)$ it follows that

$$\rho_n \to_p 1_{\{B_p(r_e)>0\}} \tag{2.15}$$

as $n \to \infty$, where 1_A is the indicator of the event A. The finite sample version of $\{B_p\left(r_e\right) < 0\}$ is $\left\{n^{-1/2} p_{\tau_e} < 0\right\}$ in which case $p_{\tau_e} = \log P_{\tau_e} < 0$ and in this case there is a negative bubble over $(\tau_e, \tau_f]$. Importantly, (2.15) shows that the discount factor ρ_n tends to a Bernoulli random variable limit with outcomes $\{0,1\}$ for which unity occurs with probability $P\{B_p\left(r_e\right) > 0\} = 1/2$ because $B_p\left(r_e\right) =_d N\left(0, \sigma_{pp} r_e\right)$.

Since $\rho_n < 1$ for all fixed n by virtue of the definition (2.3), we have

$$\underset{i\to\infty}{\text{plim}} \rho_n^i E_t d_{t+i} = 0,$$

as before in (2.6) provided n is fixed. Further,

$$\rho_n^i E_t p_{t+i} = \begin{cases} \rho_n^i p_t & t < \tau_e, \quad t+i < \tau_e \\ \rho_n^i \theta_n^{t+i-\tau_e} p_t & t < \tau_e, \quad t+i \in [\tau_e, \tau_f] \\ \rho_n^i \left(p_t + E\varepsilon_p^*\right) & t < \tau_e, \quad t+i > \tau_f \\ \rho_n^i \theta_n^i p_t & t \in [\tau_e, \tau_f), \quad t+i \le \tau_f \\ \rho_n^i \left(p_{\tau_e} + E\varepsilon_p^*\right) & t \in [\tau_e, \tau_f], \quad t+i > \tau_f \\ \rho_n^i p_t & t > \tau_f \end{cases}, \tag{2.16}$$

whose behavior depends on the timing of the conditional expectation E_t and the value of $\rho_n \theta_n$. From (2.15) we have, for large (fixed) n,

$$\begin{aligned} \rho_n \theta_n &= \frac{1 + \frac{c}{k_n}}{1 + O_p\left(\exp\left\{-\frac{k_n}{\sqrt{n}c} e^{cr_b \frac{n}{k_n}} B_p\left(r_e\right)\right\}\right)} \\ &= \left\{1 + \frac{c}{k_n} + o_p\left(1\right)\right\} 1_{\{B_p(r_e)>0\}} + o_p\left(1\right), \end{aligned} \tag{2.17}$$

and so $\left(\rho_n \theta_n\right)^i$ diverges as $i \to \infty$ with probability $P\{B_p\left(r_e\right) > 0\}$. Importantly, however, as $i \to \infty$ with n fixed, it is apparent from (2.16) that eventually $t+i$ lies beyond the bubble episode and then the autoregressive parameter θ_n reverts to unity. In this event, the weighted conditional expectation is

$$\rho_n^i E_t p_{t+i} = \rho_n^i \left[\left(p_t + E_t \varepsilon_p^*\right) 1_{\{t<\tau_f\}} + \left(p_{\tau_e} + \varepsilon_p^*\right) 1_{\{t \ge \tau_f\}}\right] = O\left(\rho_n^i\right) = o\left(1\right),$$

which validates the log linear approximation and the present value relationship in the limit as $i \to \infty$. Hence, when the duration of the bubble episode has the same order as the sample size (that is when $m_n \sim nr_b$ with $r_b > 0$), the discount factor has a random limit but the log linear approximation and the present value relationship for the asset price are both still applicable.

It is apparent from the above calculations that this conclusion holds only if the bubble episode is asymptotically (as $i \to \infty$) negligible in duration and there is an infinite time horizon in present value calculations. If there are an infinite number of bubble episodes then as $i \to \infty$, we will get some effects based on (2.17) where $(\rho_n \theta_n)^i$ diverges as $i \to \infty$ with probability $P\{B_p(r_e) > 0\}$. Similarly, if investors are myopic and the time frame of investor decision making is limited, then the bubble effects play a more important role.

(ii) Short bubble duration

We consider the formulation in which the bubble duration $m_n = \tau_f - \tau_e = \lfloor md_b \rfloor$ for some $d_b > 0$ and $m \to \infty$ for which $\frac{m}{n} \to 0$. In this set up, the bubble duration m_n is of smaller order than the sample size, thereby attenuating the effect of the bubble on the sample average. The asymptotics are now quite different from the standard duration case considered above. For this parameterization, and under the rate conditions

$$\frac{m}{k_n} \to a \in [0, \infty), \ \frac{k_n}{\sqrt{n}} \to 0 \tag{2.18}$$

it is shown in the Appendix that

$$\rho_n \to_p \frac{1}{1 + e^{\alpha}} < 1, \tag{2.19}$$

which is identical to the cointegrating case where $d_t - p_t$ is stationary and ergodic. As in that case, therefore, the usual log linear approximation and present value relationship is valid. A key requirement for validity is that $\frac{m}{k_n} = O(1)$ as indicated in (2.18), so that the length of the bubble period is at most of the same order of magnitude as the localizing coefficient sequence k_n that defines the autoregressive parameter θ_n during the mildly explosive episode.

2.4 Myopic investors with a distant horizon

Let the investor horizon $I = I_n$ depend on the sample size n. The horizon is therefore finite for fixed n and may grow with the sample size, reflecting a horizon that lengthens as the sample information increases. Some flexibility in the choice of I_n can be arranged, thereby producing various degrees of myopia in the present value calculations. However, as $n \to \infty$ we assume that $I_n \to \infty$, giving an infinite horizon limit. The formulation I_n is therefore called a *distant investor horizon.* To complete the model formulation, we need to parameterize the bubble duration. In doing so, we will consider long and short durations in turn.

(i) Distant horizon and long bubble duration

In the long bubble duration case we have $\tau_e = \lfloor nr_e \rfloor$, $\tau_f = \lfloor nr_f \rfloor$, and $m_n = \lfloor nr_b \rfloor$, as above, but now I_n also depends on n. The earlier analysis leading to (2.14) continues to hold and, as shown in the Appendix, the asymptotic character of the discount factor $\rho_n^{I_n}$ depends on the expansion rate of the distant horizon I_n. In particular, we find

$$\rho_n^{I_n} \sim \begin{cases} e^{-ag} 1_{\{B_p(r_e)>0\}} & \text{for } I_n \sim a I_n^*, \ a > 0 \\ 1_{\{B_p(r_e)>0\}} & \text{for } \frac{I_n}{I_n^*} + \frac{1}{I_n} = o_p(1) \end{cases}, \tag{2.20}$$

where $g = e^{\left(1 - r_f + r_e\right)\alpha}$ and

$$I_n^* := \exp\left\{ \frac{k_n}{\sqrt{n}c} e^{cr_b \frac{n}{k_n}} B_p(r_e) \right\} \to \infty,$$

whenever $k_n = n^{1-\eta}$ for some $\eta \in (0,1)$ and $B_p(r_e) > 0$. For $I_n = o_p(I_n^*)$, we can then classify outcomes as follows:

$$\begin{array}{ll} (\rho_n \theta_n)^{I_n} \to_p e^{cb} 1_{\{B_p(r_e)>0\}} & \text{for } \frac{I_n}{k_n} \to b \in [0, \infty) \\ (\rho_n \theta_n)^{I_n} \sim e^{cbk_n^{\delta}} 1_{\{B_p(r_e)>0\}} & \text{for } I_n = bk_n^{1+\delta}, \quad \delta, b > 0 \end{array}. \tag{2.21}$$

We may now analyze the present value model under various degrees of investor myopia represented in the form $I_n = bk_n^{1+\delta}$ for $\delta \in [0, \eta)$. As shown in the Appendix we find that

$$\rho_n^{I_n} E_t p_{t+I_n} \sim \begin{cases} 1_{\{B_p(r_e)>0\}} p_t & t < \tau_e, \quad t + I_n < \tau_e \\ e^{cbk_n^{\delta}} 1_{\{B_p(r_e)>0\}} p_t & t < \tau_e, \quad t + I_n \in [\tau_e, \tau_f] \\ 1_{\{B_p(r_e)>0\}} \left(p_t + E\varepsilon_p^*\right) & t < \tau_e, \quad t + I_n > \tau_f \\ e^{cbk_n^{\delta}} 1_{\{B_p(r_e)>0\}} p_t & t \in [\tau_e, \tau_f), \quad t + I_n \leq \tau_f \\ 1_{\{B_p(r_e)>0\}} \left(p_{\tau_e} + E\varepsilon_p^*\right) & t \in [\tau_e, \tau_f], \quad t + I_n > \tau_f \\ 1_{\{B_p(r_e)>0\}} p_t & t > \tau_f \end{cases}. \tag{2.22}$$

Since $I_n = bk_n^{1+\delta} = o(n)$ for $\delta \in [0, \eta)$, the investor horizon is dominated by the sample size as $n \to \infty$ and investors with horizon I_n may therefore be considered to be myopic. When $t + I_n \in [\tau_e, \tau_f]$ we have

$$\rho_n^{I_n} E_t p_{t+I_n} \sim e^{cbk_n^{\delta}} 1_{\{B_p(r_e)>0\}} p_t \quad \begin{cases} = e^{cb} 1_{\{B_p(r_e)>0\}} p_t & \text{for } \delta = 0 \\ \to \infty & \text{for } \delta > 0 \text{ and } B_p(r_e) > 0 \end{cases} \tag{2.23}$$

The parameter δ measures the degree of myopia. When $\delta > 0$, $I_n/k_n \to \infty$ and the horizon exceeds the localizing coefficient rate $k_n = n^{1-\eta} \to \infty$ in the explosive parameter $\theta_n = 1 + c/k_n$. In this case the weighted conditional expectation $\rho_n^{I_n} E_t p_{t+I_n}$ diverges (when $B_p(r_e) > 0$) because the investor horizon gives a sufficiently long duration during an explosive episode for the factor $\theta_n^{I_n}$ to diverge.

When $t < \tau_e$ (that is when current time t predates the origination of the bubble), the investor horizon I_n may satisfy either $t + I_n < \tau_e$ or $t + I_n \geq \tau_e$, in which case the mildly explosive case $\theta = \theta_n$ may or may not influence the conditional expectation $\rho_n^{I_n} E_t p_{t+I_n}$ in (2.22). The case

$\{t < \tau_e,\ t + I_n > \tau_e\}$ is not relevant in practice and is excluded because τ_e is assumed to be unknown before its realization. On the other hand, once $t \geq \tau_e$ investors are aware that market exuberance is occurring and the mildly explosive coefficient $\theta_n = 1 + c/k_n$ influences the conditional expectation $\rho_n^{I_n} E_t p_{t+I_n}$ by way of both the parameter c and the sequence k_n. However when $\{t \geq \tau_e,\ t + I_n > \tau_e\}$ investors do not know for how long the period of exuberance will continue and their time horizon restricts them to the left of the terminal point τ_f, so that $t + I_n \leq \tau_f$. It is this ignorance of the terminal date that helps to sustain the bubble. The model therefore operates under the convention that the timing parameters τ_e and τ_f are unknown to investors in advance.

In place of (2.4), we now have under a distant investor horizon I_n the relation

$$\delta_t = \rho_n^{I_n} \delta_{t+I_n} + \sum_{i=0}^{I_n - 1} \rho_n^i (r_{t+1+i} - \Delta d_{t+1+i}) + \kappa_n \frac{1 - \rho_n^{I_n}}{1 - \rho_n}, \tag{2.24}$$

where κ_n depends on n and has the form

$$\kappa_n = -\log \rho_n - (1 - \rho_n) \log(\frac{1}{\rho_n} - 1).$$

We focus on the case $B_p(r_e) > 0$ or, in finite samples, $p_{\tau_e} > 0$ which leads to a period of mildly explosive behavior following τ_e. In this case, we show in the Appendix that as $n \to \infty$ (2.24) has the asymptotically simpler form

$$\delta_t \sim \rho_n^{I_n} \delta_{t+I_n} + \sum_{i=0}^{I_n - 1} \rho_n^i (r_{t+1+i} - \Delta d_{t+1+i}), \tag{2.25}$$

which, by taking conditional expectations, leads to the new present value relationship

$$p_t \sim d_t + E_t \sum_{i=0}^{I_n - 1} \rho_n^i (\Delta d_{t+1+i} - r_{t+1+i}) + \rho_n^{I_n} E_t p_{t+I_n} - \rho_n^{I_n} E_t d_{t+I_n}. \tag{2.26}$$

The present value relationship (2.26) depends on the investor horizon ($I_n = b k_n^{1+\delta}$) and the parameters of the mildly explosive process (c and k_n). Importantly, when the horizon is such that $t + I_n \in (\tau_e, \tau_f]$ for $t \in (\tau_e, \tau_f]$, the weighted conditional expectation $\rho_n^{I_n} E_t p_{t+I_n}$ that appears in (2.26) involves an explosive expansion path, which we can write as the *bubble array process*

$$b_{n,t} = \rho_n^{I_n} E_t p_{t+I_n} = e^{cbk_n^\delta} p_t. \tag{2.27}$$

This component materially affects present value calculations and contributes exuberance effects to these evaluations. Note that for $t > \tau_e$, we find that $b_{n,t}$ satisfies

$$b_{n,t} = \theta_n b_{n,t-1} + e^{cbk_n^\delta} \varepsilon_{pt} \tag{2.28}$$

and therefore forms a submartingale array when ε_{pt} is a martingale difference because $\theta_n > 1$. As is apparent from (2.28), when $\delta > 0$, the volatility in the bubble process component $b_{n,t}$ increases with the investor horizon I_n. Using (2.27) and (2.28) in (2.26) gives the alternate expression

$$p_t \sim d_t + E_t \sum_{i=0}^{I_n-1} \rho_n^i (\Delta d_{t+1+i} - r_{t+1+i}) + b_{n,t} - \rho_n^{I_n} E_t d_{t+I_n} \tag{2.29}$$

for the present value model that involves the bubble process $b_{n,t}$.

The process $b_{n,t} = \rho_n^{I_n} E_t p_{t+I_n}$ is defined not only over the explosive price period $t \in [\tau_e, \tau_f)$ but also for other periods such as $t < \tau_e$ and $t \geq \tau_f$. We use the term *bubble process* for $b_{n,t}$ because it includes the bubble period and is consistent with the existing rational bubble literature in that sense (e.g., Blanchard and Watson, 1982) even though the process has a meaning and non zero value in other periods. In particular, according to the asymptotic theory developed in this paper, the process $b_{n,t}$ corresponds to the price process p_t for $t \notin [\tau_e, \tau_f)$ upon some reasonable restrictions on the investor horizon I_n. Later in the paper we show that the contribution of the process $b_{n,t} = \rho_n^{I_n} E_t p_{t+I_n}$ to asset price determination is equivalent to the effects of traditional systematic market risk during normal market periods such as $t \notin [\tau_e, \tau_f)$, but transforms into a new form of bubble risk during the period of price exuberance $t \in [\tau_e, \tau_f)$. Therefore, although the process $b_{n,t} = \rho_n^{I_n} E_t p_{t+I_n}$ might well be referred to by a more general term, we continue to use the terminology bubble process because of its special significance during periods of exuberance.

As shown in the Appendix, expression (2.26) simplifies to

$$p_t \sim E_t \sum_{i=0}^{I_n-1} \rho_n^i (\Delta d_{t+1+i} - r_{t+1+i}) + b_{n,t} - \left(I_n g + \sum_{i=1}^{I_n} E_t \varepsilon_{dt+i} \right), \tag{2.30}$$

or further to

$$p_t \sim E_t \sum_{i=0}^{I_n-1} \rho_n^i (\Delta d_{t+1+i} - r_{t+1+i}) + b_{n,t}, \tag{2.31}$$

when $g = 0$ and $E_t \varepsilon_{dt+i} = 0$ for $i \geq 1$. In both expressions the bubble array process $b_{n,t}$ figures prominently in the new present value formulae.

(ii) Distant horizon and short bubble duration

As earlier, let the bubble duration be $m_n = \tau_f - \tau_e = \lfloor m d_b \rfloor$ for some $d_b > 0$ and $m \to \infty$ for which $\frac{m}{n} \to 0$. Then, from (2.19) we have $\rho_n \to_p \frac{1}{1+e^{\alpha}} < 1$. It follows that

$$\rho_n^{I_n} E_t d_{t+I_n} = o_p(1),$$

and then (2.26) reduces to

$$p_t \sim d_t + E_t \sum_{i=0}^{I_n-1} \rho_n^i (\Delta d_{t+1+i} - r_{t+1+i}) + \rho_n^{I_n} E_t p_{t+I_n}.$$

It remains to analyze $\rho_n^{I_n} E_t p_{t+I_n}$. We consider the case where $\frac{k_n}{m_n} \to 0$, so the bubble duration is long in relation to the explosive coefficient rate k_n but short in relation to the sample size. Once again we set the horizon to be $I_n = bk_n^{1+\delta}$ for $\delta \in [0, \eta)$. In view of (2.19) we have

$$\rho_n \theta_n \sim \frac{1 + \frac{c}{k_n}}{1 + e^{\alpha}} < 1$$

so the factor

$$(\rho_n \theta_n)^{I_n} \sim \frac{e^{cbk_n^{\delta}}}{(1 + e^{\alpha})^{bk_n^{1+\delta}}} = o\,(1)$$

will be negligible as $n \to \infty$. In this case the usual present value model holds asymptotically.

These analyses show that the timing profile of investor decision making and bubble duration can play a big role in the form of the present value relationship. As case (i) above demonstrates, a moderate investor horizon I_n can lead to price bubble effects manifesting in the present value relationship. In particular, for these effects to occur I_n should not pass to infinity "too fast" (which would make a single bubble episode negligible in expectation formation) and the bubble duration should be "long enough" to influence expectations following the origination of the bubble.

Intuitively, if investors had perfect foresight over an infinite horizon then temporal mis-pricing events would never happen. However, all investors have finite horizons. Empirical evidence of price bubbles as well as the realities of the market implies that some investors are "myopic". Myopia is incorporated into the above framework by means of the finite horizon I_n and accompanying conditions on I_n that control the manner of its expansion as the sample data increases. This framework enables the model to interlock several realistic features of the market, such as finite horizon investment and possible periods of market exuberance, which take on greater significance in a finite investor horizon context.

The econometric methods used in PWY (2011) enable us to test whether there is evidence of bubble activity in financial data. These methods confirm the existence of mildly explosive periods in NASDAQ prices, revealing an empirical bubble in 1990s prices that collapsed in 2001. The presence of price bubbles may reasonably be considered as a magnifying source of stock market return risk. Correspondingly, we expect that some portion of asset prices may be explained by the presence of a bubble component in the present value framework.

3 The Asset Pricing Model with Bubble Effects

This section investigates the impact of bubble activity on asset price returns. To this end, we use a return-based version of (2.29), following Campbell (2003), or Campbell, Lo and MacKinlay (1997, Ch.7). As shown in the Appendix, the resulting model for excess returns has the form

$$\begin{aligned} r_{t+1} - E_t(r_{t+1}) &= (E_{t+1} - E_t) \sum_{i=0}^{I_n} \rho_n^i \Delta d_{t+1+i} - (E_{t+1} - E_t) \sum_{i=1}^{I_n} \rho_n^i r_{t+1+i} \\ &\quad + \rho_n (b_{n,t+1} - E_t b_{n,t+1}) - \rho_n^{I_n+1} (E_{t+1} - E_t) d_{t+I_n+1}, \end{aligned} \tag{3.1}$$

When $I_n = \infty$ and terminal conditions are imposed that ensure the final two terms of (3.1) are zero, the equation reduces to that given in Campbell (2003), viz.,

$$r_{t+1} - E_t(r_{t+1}) = (E_{t+1} - E_t) \sum_{i=0}^{\infty} \rho_n^i \Delta d_{t+1+i} - (E_{t+1} - E_t) \sum_{i=1}^{\infty} \rho_n^i r_{t+1+i}.$$

Equation (3.1) follows by approximating the nonlinear return identity, solving forward to the horizon I_n, allowing for the presence of bubbles, and taking conditional expectations. The resulting equation shows that with the possibility of financial exuberance in the future, unexpected stock returns are associated with changes in expectations of future dividend cash flows, real returns and some fraction of the unanticipated bubble effects less the discounted change in expectations of dividends in the terminal period.

3.1 Equilibrium Analysis with Recursive Utility

The previous expression can be related to optimal consumption and portfolio investor choice by means of a consumption-based asset pricing model. Here we follow derivations in Campbell (2003) but allow for the presence of price bubbles. Each household is assumed to have Epstein-Zin (EZ) preferences[3]

$$V_t = \left\{ (1-\delta) C_t^{\frac{1-\gamma}{\varphi}} + \delta (E_t V_{t+1}^{1-\gamma})^{\frac{1}{\varphi}} \right\}^{\frac{\varphi}{1-\gamma}} = \left\{ (1-\delta) C_t^{1-1/\psi} + \delta (E_t V_{t+1}^{1-\gamma})^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}} \tag{3.2}$$

where $\varphi = (1-\gamma)/(1-1/\psi)$, and $\psi \neq 1$. This preference function breaks the link between the elasticity of intertemporal substitution (EIS) parameter (ψ) and the relative risk aversion (RRA) parameter (γ), in contrast to the power utility case. It is widely accepted that this flexibility regarding attitudes towards risk and propensity for intertemporal substitution is necessary to capture key characteristics in financial markets.

The intertemporal budget constraint for a representative agent can be written as

$$W_{t+1} = (1 + R_{m,t+1})(W_t - C_t)$$

where W_{t+1} is the representative agent's wealth, and $(1 + R_{m,t+1})$ is the gross return on the market portfolio. (Here and subsequently, the subscript m signifies market quantities.) As in Epstein and

[3] Some discussion of recursive utility and finite (or distant) investor horizon is provided in Appendix 7.8.

Zin (1989, 1991) and Weil (1989), dynamic programming arguments lead to an Euler equation of the form[4]

$$1 = E_t\left[\left\{\delta(\frac{C_{t+1}}{C_t})^{-\frac{1}{\psi}}\right\}^{\varphi}\left\{\frac{1}{1+R_{m,t+1}}\right\}^{1-\varphi}(1+R_{i,t+1})\right] \tag{3.3}$$

for any asset i. Appendix 7.9 shows the derivation of (3.3) as an approximating Euler equation allowing for a finite and distant (rather than infinite) horizon for utility maximization by a representative agent.

We assume that the joint conditional distribution of asset returns $(1+R_{m,t+1}, 1+R_{i,t+1})$ and consumption growth $(\frac{C_{t+1}}{C_t})$ is lognormal and homoskedastic[5], following standard practice (e.g., Campbell, 2003; and Mehra, 2003). These assumptions enable us to derive a closed form solution for the main sources of stock market volatility and excess mean returns.

The Euler equation (3.3) for each asset i and conditional lognormality lead to the key relation between expected market returns and aggregate consumption growth, as given in Campbell (2003),

$$E_t(r_{m,t+1}) = \mu + \frac{1}{\psi}E_t(\Delta c_{t+1}). \tag{3.4}$$

Under the additional assumption of a Lucas tree-type endowment economy (as in Lucas, 1978, and Mehra and Prescott, 1985) we can link the return accounting identity and the consumer's optimal choice problem, so that aggregate consumption is the dividend on the portfolio of all invested wealth (i.e., the market return):

$$d_{m,t} = c_t \tag{3.5}$$

Using (3.4) and (3.5) in (3.1) leads to the equation

$$\begin{aligned} r_{m,t+1} - E_t(r_{m,t+1}) &= (\Delta c_{t+1} - E_t\Delta c_{t+1}) + (1-\frac{1}{\psi})(E_{t+1}-E_t)\sum_{i=1}^{I_n}\rho_n^i \Delta c_{t+1+i} \\ &\quad +\rho_n(b_{m,n,t+1} - E_t(b_{m,n,t+1})) - \rho_n^{I_n+1}(E_{t+1}-E_t)\, d_{t+I_n+1}. \end{aligned} \tag{3.6}$$

In view of (3.5) $\Delta d_{m,t+1} = \Delta c_{t+1} = g + \varepsilon_{md,t+1}$, where g is the consumption growth rate. Thus, $E_{t+1}(\Delta c_{t+1+i}) - E_t(\Delta c_{t+1+i}) = 0$ for all $i \geq 1$, so the second term on the right side of (3.6) vanishes. If $\varepsilon_{md,t+1}$ is a martingale difference sequence, then

$$\begin{aligned} (E_{t+1}-E_t)\, d_{t+I_n+1} &= c_{t+1} - c_t - g = \varepsilon_{md,t+1}, \\ \Delta c_{t+1} - E_t\Delta c_{t+1} &= \varepsilon_{md,t+1} \end{aligned}$$

[4] In Epstein and Zin (1991) there are several typographical errors in the derivation that do not affect the final Euler equation. For completeness, the Euler equation derivation with correct modifications and allowance for distant (not infinite) decision horizons is provided in Appendix 7.9.

[5] The assumptions of conditional lognormality and homoskedasticity yield convenient closed form expressions. Without lognormality, the relation still holds approximately, as discussed in Epstein and Zin (1991). The conditional homoskedasticity assumption can be relaxed, but that relaxation is not needed here under *iid* dividend and price innovations (see section 4.2).

Therefore,

$$
\begin{aligned}
r_{m,t+1} - E_t(r_{m,t+1}) &= (\Delta c_{t+1} - E_t \Delta c_{t+1}) + \rho_n (b_{m,n,t+1} - E_t(b_{m,n,t+1})) - \rho_n^{I_n+1} \varepsilon_{md,t+1} \\
&= \rho_n (b_{m,n,t+1} - E_t(b_{m,n,t+1})) + \left(1 - \rho_n^{I_n+1}\right) \varepsilon_{md,t+1} \qquad (3.7) \\
&= \rho_n (b_{m,n,t+1} - E_t(b_{m,n,t+1})) + c_n (\Delta c_{t+1} - E_t \Delta c_{t+1}) \qquad (3.8)
\end{aligned}
$$

where $c_n = 1 - \rho_n^{I_n+1}$ depends on the horizon length I_n for the given sample size n. Accordingly, (3.8) shows that for a finite investment horizon I_n, the determining factors for market excess returns involve a linear combination of market exuberance and consumption growth. The consumption based element that appears in the traditional consumption-based asset pricing model, viz.,

$$
r_{m,t+1} - E_t(r_{m,t+1}) = (\Delta c_{t+1} - E_t \Delta c_{t+1}) \qquad (3.9)
$$

is now augmented in (3.8) by an additional term involving a price bubble effect in the data.

Note that equation (3.8) transforms into (3.9) under traditional assumptions. In particular, if we let $I_n \to \infty$ with fixed $|\rho_n| = |\rho| < 1$, then $b_{m,n,t+1} = \rho_n^{I_n} E_t p_{t+I_n} \to 0$, $c_n = 1 - \rho_n^{I_n+1} \to 1$ which leads to (3.9). A dramatic consequence of the finite investor horizon I_n used here is that $\rho_n^{I_n+1}$ may remain close to unity. In that case, c_n is close to zero, implying a limited role for the consumption growth component in explaining asset return movements. One might regard (3.8) as a generalized form of (3.9). But in our framework we do not impose either an infinite investor horizon condition or a no-bubble condition. Even though expression (3.8) does not directly subsume (3.9), at least without side conditions involving $I_n \to \infty$, later in the paper we show that the asset pricing implication of (3.8) generalizes the traditional contribution from (3.9) during normal time periods. In particular, the (conditional) second moments of $\rho_n(b_{m,n,t+1} - E_t(b_{m,n,t+1}))$ correspond to the traditional asset pricing contributions of the market returns.

Our empirical work, discussed below, reveals that the additional term $\rho_n(b_{m,n,t+1} - E_t(b_{m,n,t+1}))$ in (3.8) has a much larger impact on asset returns than the usual consumption based element, indicating that price bubble effects or exuberance can play a dominant role in determining unexpected asset market returns. This result is in accordance with empirical and historical evidence that speculative bubble behavior can dominate the effects of economic fundamentals.

One criticism of rational asset pricing theory is that we sometimes do observe dramatic movements in asset prices without any apparent fundamental change in real economic variables. Behavioral economists have postulated psychological explanations for this phenomena introducing notions such as “excessively optimistic investors” and the “social contagion of boom thinking” (Shiller, 2008). This line of behavioral research has received much attention, is growing fast, and resonates with recent historical analyses of financial crises (Ferguson, 2008; Ahamed; 2008). On the other hand, speculative bubbles can also be explained in terms of rational behavior in response to variable discount factors and the formation of “rational bubbles” under rational expectations assumptions. Our reasoning is consistent with the rational bubble literature but adds an identifiable and estimable expression representing the effects of bubble risk on asset prices using reasonable assumptions on the price process and the investor horizon. This econometric tractability of bubble risk, which is detailed and discussed below, is one of the novel contributions of this paper.

Standard consumption-based asset pricing models of the form (3.9) are well known to fail empirically. This failure comes from the fact that stock market risk is explained only by way of consumption in (3.9) and that model implies the covariance between consumption and asset returns determines the degree of risk in asset return, so that undiversifiable market risk is determined solely by consumption risk.

By contrast, when the investor horizon is limited and there is market exuberance, the asset pricing model takes on a very different form as seen in (3.8) where an additional (and potentially dominant) factor in explaining stock market returns is the market bubble effect $b_{m,n,t+1} - E_t(b_{m,n,t+1})$.

3.1.1 The Volatility Puzzle

Stock market volatility, which is measured by the variance of market returns, is explained in the standard consumption-based asset pricing framework by the variation of aggregate consumption growth, as indicated in (3.9). The empirical failure of this theory, due to consumption growth being too smooth, is called the "stock market volatility puzzle".

In the presence of market exuberance, quite a different model applies in view of (3.8), which leads to the following expression for conditional volatility

$$Var_t(r_{m,t+1}) = \rho_n^2 Var_t(b_{m,n,t+1}) + c_n^2 Var_t(\Delta c_{t+1}) + \rho_n c_n Cov_t(b_{m,n,t+1}, \Delta c_{t+1}). \tag{3.10}$$

In this model, consumption growth is still present as a determinant of return volatility but the contribution of the consumption series depends on the coefficient c_n. This coefficient makes the contribution of the real economy fundamental variable (the consumption growth component) even smaller than the traditional model of (3.9). Instead, market volatility is mainly determined by the bubble risk component in prices, so exuberance becomes the dominant factor in the return distribution.

3.1.2 The Equity Premium Puzzle

We can also explore the contribution of bubble risk to the risk premium. From the Euler equation (3.3), we have

$$\begin{aligned} 1 &= E_t\left[\exp\left\{s_{t+1} + r_{i,t+1}\right\}\right], \\ s_{t+1} &= \varphi \log \delta - \frac{\varphi}{\psi}\Delta c_{t+1} + (\varphi - 1) r_{m,t+1}, \end{aligned}$$

where $r_{i,t+1} = \log(1 + R_{i,t+1})$, $s_{t+1} = \log S_{t+1}$, and S_{t+1} is the stochastic discount factor such that $E_t\left[S_{t+1}(1 + R_{i,t+1})\right] = 1$.

Under the assumption of joint conditional lognormality, we have

$$E_t(r_{i,t+1}) - r_{f,t+1} + \frac{1}{2}Var(r_{i,t+1}) = -Cov(s_{t+1}, r_{i,t+1}),$$

which leads to the following result

$$\begin{aligned} & E_t(r_{i,t+1}) - r_{f,t+1} + \frac{1}{2}Var_t(r_{i,t+1}) \\ = \quad & \frac{\varphi}{\psi}Cov(r_{i,t+1},_t \Delta c_{t+1}) + (1-\varphi)Cov_t(r_{i,t+1}, r_{m,t+1}). \end{aligned} \tag{3.11}$$

The explicit influence of "bubble risk" to the equity premium follows from (3.8). For any risky asset $r_{i,t+1}$,

$$Cov_t(r_{i,t+1}, r_{m,t+1}) = \rho_n Cov_t(r_{i,t+1}, b_{m,n,t+1}) + c_n Cov_t(r_{i,t+1}, \Delta c_{t+1}),$$

and then

$$\begin{aligned} & E_t(r_{i,t+1}) - r_{f,t+1} + \frac{1}{2}Var_t(r_{i,t+1}) \\ = \quad & \frac{\varphi}{\psi}Cov_t(r_{i,t+1}, \Delta c_{t+1}) + (1-\varphi)\rho_n Cov_t(r_{i,t+1}, b_{m,n,t+1}) + (1-\varphi)c_n Cov_t(r_{i,t+1}, \Delta c_{t+1}) \\ = \quad & (\frac{\varphi}{\psi} + (1-\varphi)c_n)Cov_t(r_{i,t+1}, \Delta c_{t+1}) + (1-\varphi)\rho_n Cov_t(r_{i,t+1}, b_{m,n,t+1}) \end{aligned} \tag{3.12}$$

Again, without the final bubble term in (3.12), the equity premium is explained by the covariance between consumption and asset returns multiplied by the combination of utility parameters of the representative investor. Because of the low correlation of consumption growth with asset returns, this logic is generally unsupported by empirical evidence, leading to the "equity premium puzzle".

It is interesting to compare the new equation (3.12) to the traditional expression (3.11). In equation (3.11), there is no uncertainty about the data generating process (DGP) of real variables. Under rational expectations, the investor knows all relevant information about the return and the consumption growth process. However, equation (3.12) features a new bubble variable $b_{m,n,t+1}$, which features according to the unknown origination and termination parameters τ_e and τ_f. Therefore, investors bear an additional source of risk arising from the "fear of ignorance" regarding these parameters. The corresponding structural uncertainty increases the magnitude of the risk, producing a new "bubble risk" element to the equity premium. This argument is explored more fully in the Section 3.2.

In view of (2.22), we can characterize the bubble risk more explicitly. When $B_p\left(r_e\right) > 0$, simple substitution using (2.22) leads to

$$\begin{aligned} Cov_t(r_{i,t+1}, b_{m,n,t+1}) &= Cov_t(r_{i,t+1}, p_{m,n,t+1}) \qquad t \notin [\tau_e, \tau_f) \\ Cov_t(r_{i,t+1}, b_{m,n,t+1}) &= Cov_t(r_{i,t+1}, e^{cbk_n^\delta} p_{m,n,t+1}) \quad t \in [\tau_e, \tau_f). \end{aligned} \tag{3.13}$$

This characterization implies that the risk stemming from the presence of bubbles corresponds to traditional systematic market risk $Cov_t(r_{i,t+1}, p_{m,n,t+1})$ for normal market periods when $t \notin [\tau_e, \tau_f)$ but transforms into the bubble risk

$$Cov_t(r_{i,t+1}, e^{cbk_n^\delta} p_{m,n,t+1}) = Cov_t\left(r_{i,t+1}, e^{(\theta-1)I_n} p_{m,t+1}\right)$$

in the period of price exuberance $t \in [\tau_e, \tau_f)$. This new risk measure therefore subsumes systematic market risk and bubble risk, so the measure might be considered a general financial market risk. Similar to our earlier nomenclature involving the term "bubble" in Section 2.4, it is convenient to use the term "bubble risk" here for the term $Cov_t(r_{i,t+1}, b_{m,n,t+1})$, while allowing for its broader definition (3.13) in terms of a general financial market risk as it covers both normal and exuberant price periods.

The new equation (3.12) for the equity premium provides for an additional source of risk arising from the presence of price bubbles. We can therefore expect some portion of the equity premium to be explained by such additional risk factors. The new contribution relies on the magnitude and sign of the composite parameter

$$(1-\varphi)\rho_n \sim \frac{(\gamma - \frac{1}{\psi})}{(1-\frac{1}{\psi})}.$$

The meaning of this coefficient is illuminating. First, as the relative risk aversion (RRA) parameter γ increases, the new contribution to bubble risk rises. This is intuitively clear because more risk averse agents will respond more actively to additional sources of risk. Second, for conventional values of the RRA parameter γ, the intertemporal elasticity of substitution (EIS) parameter ψ is expected to exceed unity and then $1-\varphi > 0$, leading to a positive contribution to the equity premium. There has been much discussion about reasonable empirical values of ψ, although no general consensus. For the power utility case, some of empirical macroeconomics literature have provided support for an EIS value close to zero (e.g., Hall, 1988). Other researchers (e.g., Jones et al., 2000), use a value close to unity to match aggregate data dynamics. With EZ preferences, estimates of the EIS parameter that exceed unity have been reported recently (e.g., Bansal, Gallant and Tauchen, 2007; Chen, Favilukis and Ludvigson, 2013). Epstein and Zin (1989) argue that early resolution of uncertainty is preferable than late resolution if $\gamma > \frac{1}{\psi}$. In this case, bubbles magnify the risk of agents who prefer early resolution, and vice versa, which accords with our theory of the role of the investor horizon in the fundamental asset price equation.

It is also noteworthy that the bubble risk manifesting in (3.12) is systematic and may reasonably be expected to arise intermittently over time in relation to market exuberance. It is not diversifiable by any particular hedging strategy. This theory therefore introduces another systematic and periodic source of risk into the standard asset pricing mechanism.

3.2 Investor Information and Uncertainty

All asset pricing models involve assumptions concerning the structure of investor information and the manner in which this information is used. The standard mechanism is based on a rational expectation equilibrium (REE) formulation in which the parameters of the data generating process (DGP) and the past filtration of information are known to agents. Some recent research has used the REE framework with more general assumptions about consumption growth. In particular, Bansal and Yaron (2004) introduce a persistent stochastic component to reflect long run risk in consumption growth and raise investor risk, leading to some improvement in the empirical performance of the consumption based CAPM model. Another approach is to allow for rare disaster states in consumption growth, which again leads to higher investor risk (see Rietz, 1988, and Barro, 2006). Recent work by Weitzman (2007) used a Bayesian-learning approach to assist in explaining stock market puzzles. In that framework and in contrast to REE, agents do not know the DGP and the evolution in parameters produces a "structural uncertainty" in which investors have subjective beliefs and a "fear of ignorance", which leads to a left tail-thickened posterior distribution of the consumption growth rate.

The present paper uses the REE framework in which the DGP is known to all agents up to uncertainty concerning the origination and termination dates (τ_e and τ_f) of potential explosive behavior, which are not known to investors prior to their realizations. In this sense, the model framework involves an important structural uncertainty, which affects the risk taking behavior of investors. Accordingly, the present framework has a feature in common with the Bayesian-learning approach. In our model, investors have information about the generating mechanism, including the value of the parameter θ_n (upon the emergence of exuberance) and their own investor horizon I_n, but do not know in advance the timing of the structural breaks τ_e and τ_f. This restriction rules out the possibility of abnormal arbitrage opportunities. Thus, although the model involves a departure from strict REE because of the uncertainty in the dating parameters τ_e and τ_f, the framework is justified by no arbitrage pricing theory and conforms in other respects with the REE framework. In particular, similar to REE, all processes known up to time t are included in the conditioning information set at t, including the price process $(p_s)_{s \leq t}$ and innovation process $(\varepsilon_{ps})_{s \leq t}$. The investor horizon I_n is finite and assumed to be a small order of infinity in the asymptotic framework. This condition is justified by the fact that τ_e and τ_f are not known to the investor before their realizations, so if τ_e and τ_f are of order $O(n)$ in a sample of size n, then I_n is a smaller order of n, or $o(n)$, as $n \to \infty$. Accordingly, we assume that $t + I_n < \tau_e$ for $t < \tau_e$, and $t + I_n \leq \tau_f$ for $t \in [\tau_e, \tau_f)$ in the implementation of the asset pricing formulae, which leads to a reasonable characterization of investor information and behavior prior to and following the emergence of a bubble episode.

4 Econometric Issues

4.1 Identifying Price Bubbles

Empirical implementation of this theory requires the use of the bubble series in (3.12). From (2.27) we have

$$b_{n,t} = \rho_n^{I_n} E_t p_{t+I_n} = e^{cbk_n^{\delta}} p_t = e^{(\theta_n - 1) I_n} p_t, \tag{4.1}$$

which can be estimated by $e^{(\hat{\theta}-1)I_n}p_t$ using an empirical estimate $\hat{\theta}$ of the autoregressive coefficient during a period of exuberance. Since $b_{n,t}$ also depends on the investor horizon I_n, we can parameterize the impact of the bubble effect on risk in terms of this parameter.

Using (2.22) and (4.1), we have the following characterization of the bubble process for the case $B_p(r_e) > 0$ and with a (power) parameterization of the investor horizon I_n in terms of the sample size

$$b_{m,n,t} = \begin{cases} p_{m,t} & t < \tau_e, \quad t + I_n \notin [\tau_e, \tau_f] \\ e^{(\theta-1)I_n} p_{m,t} & t < \tau_e, \quad t + I_n \in [\tau_e, \tau_f] \\ e^{(\theta-1)I_n} p_{m,t} & t \in [\tau_e, \tau_f), \quad t + I_n \leq \tau_f \\ p_{m,t} & t \in [\tau_e, \tau_f), \quad t + I_n > \tau_f \\ p_{m,t} & t \geq \tau_f. \end{cases} . \tag{4.2}$$

In Section 2.4, the investor horizon had the form $I_n = bk_n^{1+\delta}$ with $k_n = n^{\alpha_k}$ for some $\alpha_k \in (0,1)$. To simplify the characterization we now use the formulation

$$I_n = n^{\alpha_I}, \tag{4.3}$$

where $\alpha_I = \alpha_k(1+\delta) < 1$ with $\alpha_I \in (\alpha_k, 1)$. This simplification does not cause any material change in the previous asymptotic theory but is easier to implement in what follows. According to the earlier discussion on investor information, since τ_e and τ_f are assumed unknown to investors prior to their realizations, we rule out the cases $t+I_n \in [\tau_e, \tau_f]$ for $t < \tau_e$, and $t+I_n > \tau_f$ for $t \in [\tau_e, \tau_f)$, from (4.2). These reductions lead to the following simplified characterization of the bubble process

$$b_{m,n,t} = \begin{cases} p_{m,t} & t < \tau_e \\ e^{(\theta-1)I_n} p_{m,t} & t \in [\tau_e, \tau_f) \\ p_{m,t} & t \geq \tau_f \end{cases} . \tag{4.4}$$

Equation (4.4) holds when I_n is a suitably small infinity and we apply the convention concerning investor ignorance of τ_e and τ_f. In particular, we require $t + I_n < \tau_e$ for $t < \tau_e$, and $t + I_n \leq \tau_f$ for $t \in [\tau_e, \tau_f)$. For example, suppose $t = \tau_e - o(n) = \tau_e - n^{\beta}$, for some β satisfying $\alpha_I < \beta < 1$. Then $t + I_n = \tau_e - n^{\beta} + n^{\alpha_I} < \tau_e$ holds. Similarly, for $t = \tau_f - n^{\beta}$, with $\alpha_I < \beta < 1$ again, we have $t + n^{\alpha_I} \leq \tau_f$ for $t \in [\tau_e, \tau_f)$. The power law investor horizon $I_n = n^{\alpha_I}$ meets these requirements with $\alpha_I \in (0,1)$ and with the investor information convention. When $\theta = 1$, (4.4) reduces to $b_{m,n,t} = e^{(\theta-1)I_n} p_{m,t} = p_{m,t}$. It is also consistent with the parametric characterization of the autoregressive coefficient θ given in (2.10).

In spite of a long history of discussion concerning rational bubbles and exuberance, there appears to be no formally identified characterization of the bubble process in terms of observed data and model parameters that can be used in econometric practice. Many existing studies treat the bubble process as a latent process and rely on some estimation procedure to recover it (see, among others, Wu, 1997). The expression given in (4.4) implies that the bubble process is identified using observable data on price, the investor horizon $I_n = n^{\alpha_I}$, and an estimate of the AR coefficient θ. This formulation is conveniently tractable for econometric implementation.

4.2 Estimation of the Conditional Second Moments

Practical implementation of the formulae (3.10) and (3.12) requires estimation of the conditional variance $Var_t(b_{m,n,t+1})$ and the conditional covariances $Cov_t(r_{i,t+1}, b_{m,n,t+1})$ and $Cov_t(b_{m,n,t+1}, \Delta c_{t+1})$. While the bubble process is nonstationary, its conditional second moments are finite and can be estimated using estimated parameters, the price series and the stationary price innovation process $\varepsilon_{p,t}$.

4.2.1 Bubble Conditional Variance

The conditional variation $Var_{t-1}(b_{m,n,t})$ can be characterized in terms of the the price series and the variance of the innovations as follows. From (4.4),

$$Var_{t-1}(b_{m,n,t}) = \begin{cases} Var_{t-1}(p_{m,t}) & t < \tau_e \\ e^{2(\theta-1)I_n} Var_{t-1}(p_{m,t}) & t \in [\tau_e, \tau_f) \\ Var_{t-1}(p_{m,t}) & t \geq \tau_f \end{cases}.$$

Observe that

$$\begin{aligned} E_{t-1}(p_{m,t}) &= \begin{cases} E_{t-1}(p_{m,t-1} + \varepsilon_{p,t}) = p_{m,t-1} & t \notin [\tau_e, \tau_f) \\ E_{t-1}(\theta_n p_{m,t-1} + \varepsilon_{p,t}) = \theta_n p_{m,t-1} & t \in [\tau_e, \tau_f) \end{cases}, \\ E_{t-1}(p_{m,t}^2) &= \begin{cases} E_{t-1}(p_{m,t-1}^2 + 2p_{t-1}\varepsilon_{p,t} + \varepsilon_{p,t}^2) = p_{m,t-1}^2 + E_{t-1}(\varepsilon_{p,t}^2) & t \notin [\tau_e, \tau_f) \\ E_{t-1}(\theta_n^2 p_{m,t-1}^2 + 2\theta_n p_{m,t-1}\varepsilon_{p,t} + \varepsilon_{p,t}^2) = \theta_n^2 p_{m,t-1}^2 + E_{t-1}(\varepsilon_{p,t}^2) & t \in [\tau_e, \tau_f) \end{cases}, \end{aligned}$$

which leads to

$$\begin{aligned} Var_{t-1}(p_{m,t}) &= \begin{cases} \left(p_{m,t-1}^2 + E_{t-1}(\varepsilon_{p,t}^2)\right) - p_{m,t-1}^2 = E_{t-1}(\varepsilon_{p,t}^2) & t \notin [\tau_e, \tau_f) \\ \left(\theta_n^2 p_{m,t-1}^2 + E_{t-1}(\varepsilon_{p,t}^2)\right) - (\theta_n p_{m,t-1})^2 = E_{t-1}(\varepsilon_{p,t}^2) & t \in [\tau_e, \tau_f) \end{cases} \\ &= Var_{t-1}(\varepsilon_{p,t}) = Var(\varepsilon_{p,t}). \end{aligned}$$

The final equality follows when the innovation process is a martingale difference and this assumption is used here to simplify implementation[6]. Since the nonstationary component is present only in the conditional mean, the effects of nonstationarity are removed by centering in the conditional variance. Hence, the conditional bubble variation can be characterized in terms of the variance of

[6] The martingale difference assumption for the price and dividend innovations $\varepsilon_{p,t}$ and $\varepsilon_{d,t}$ with constant contemporary covariance matrix accords with commonly used assumptions (including conditional homoskedasticity) for returns and consumption growth in traditional forms of this model. Weaker conditions are obviously important and worth exploring but are outside the scope of the present work.

the innovations as

$$Var_{t-1}(b_{m,n,t}) = \begin{cases} Var_{t-1}(\varepsilon_{p,t}) & t < \tau_e \\ e^{2(\theta-1)I_n} Var_{t-1}(\varepsilon_{p,t}) & t \in [\tau_e, \tau_f) \\ Var_{t-1}(\varepsilon_{p,t}) & t \geq \tau_f \end{cases}.$$

Since we observe the average over the entire sample period, we estimate $E\left(Var_{t-1}(\varepsilon_{p,t})\right) = Var(\varepsilon_{p,t})$. The sample analogue is $\hat{E}\left(Var_{t-1}(\varepsilon_{p,t})\right) = n^{-1}\sum_{t=1}^{n}(\hat{\varepsilon}_{p,t} - \bar{\varepsilon}_p)^2$. Using this estimate, the conditional variance of the bubble process can be estimated as

$$\hat{E}\left(Var_{t-1}(b_{m,n,t})\right) = \begin{cases} \hat{E}\left(Var_{t-1}(\varepsilon_{p,t})\right)\mu\left(t < \tau_e\right) \\ +e^{2(\hat{\theta}-1)I_n}\hat{E}\left(Var_{t-1}(\varepsilon_{p,t})\right)\mu\left(t \in [\tau_e, \tau_f)\right) \\ +\hat{E}\left(Var_{t-1}(\varepsilon_{p,t})\right)\mu\left(t \geq \tau_f\right). \end{cases}$$

with some measure μ. A natural candidate for $\mu(A)$ is the simple counting measure recording the relative number of observations in period A.

4.2.2 Conditional Covariance Relation

We also need an estimable expression of the conditional covariance between returns and the bubble process. From (4.4) we have

$$Cov_{t-1}(r_{i,t}, b_{m,n,t}) = \begin{cases} Cov_{t-1}(r_{i,t}, p_{m,t}) & t < \tau_e \\ e^{(\theta-1)I_n} Cov_{t-1}(r_{i,t}, p_{m,t}) & t \in [\tau_e, \tau_f) \\ Cov_{t-1}(r_{i,t}, p_{m,t}) & t \geq \tau_f \end{cases},$$

and direct computation leads to

$$\begin{aligned} & Cov_{t-1}(r_{i,t}, p_{m,t}) = E_{t-1}(r_{i,t}p_{m,t}) - E_{t-1}(r_{i,t})E_{t-1}(p_{m,t}) \\ = & \begin{cases} E_{t-1}(r_{i,t}(p_{m,t-1} + \varepsilon_{p,t})) - E_{t-1}(r_{i,t})E_{t-1}(p_{m,t-1} + \varepsilon_{p,t}) & t \in [\tau_e, \tau_f) \\ E_{t-1}(r_{i,t}(\theta_n p_{m,t-1} + \varepsilon_{p,t})) - E_{t-1}(r_{i,t})E_{t-1}(\theta_n p_{m,t-1} + \varepsilon_{p,t}) & t \notin [\tau_e, \tau_f) \end{cases} \\ = & E_{t-1}(r_{i,t}\varepsilon_{p,t}) = Cov_{t-1}(r_{i,t}, \varepsilon_{p,t}). \end{aligned} \quad (4.5)$$

Since we are dealing with deviations from conditional means and we observe the entire sample period for estimation instead of looking at a particular period, the parameter of interest is $E\left(Cov_{t-1}(r_{i,t}, p_{m,t})\right) = E\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right) = Cov(r_{i,t}, \varepsilon_{p,t})$. The covariance can then be estimated by the sample quantity

$$\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right) = \frac{1}{n}\sum(r_{i,t} - \bar{r}_i)(\hat{\varepsilon}_{p,t} - \bar{\varepsilon}_p).$$

Using these estimates, the contribution to the equity premium from the bubble risk is estimated as

$$\hat{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right) = \begin{cases} \hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu(t < \tau_e) \\ +e^{(\hat{\theta}-1)I_n}\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu(t \in [\tau_e, \tau_f)) \\ +\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu(t \geq \tau_f) \end{cases}. \tag{4.6}$$

Similar arguments lead to the estimate

$$\hat{E}\left(Cov_{t-1}(\Delta c_t, b_{m,n,t})\right) = \begin{cases} \hat{E}\left(Cov_{t-1}(\Delta c_t, \varepsilon_{p,t})\right)\mu(t < \tau_e) \\ +e^{(\hat{\theta}-1)I_n}\hat{E}\left(Cov_{t-1}(\Delta c_t, \varepsilon_{p,t})\right)\mu(t \in [\tau_e, \tau_f)) \\ +\hat{E}\left(Cov_{t-1}(\Delta c_t, \varepsilon_{p,t})\right)\mu(t \geq \tau_f) \end{cases}. \tag{4.7}$$

4.2.3 Characterization of the Price Innovation Process

Since price innovations are used to estimate moments involving the bubble process, we need to specify the innovation process. By definition

$$\varepsilon_{p,t} = \begin{cases} \Delta p_{m,t} & t \notin [\tau_e, \tau_f) \\ p_{m,t} - \theta_n p_{m,t-1} & t \in [\tau_e, \tau_f) \end{cases},$$

which identifies the price innovation process. The price innovation process is determined by returns (without dividend) for the non-explosive price period $t \notin [\tau_e, \tau_f)$, and equals the error term in the price autoregression during the episode of exuberance $t \in [\tau_e, \tau_f)$.

5 Empirical Implementation

This section explores the empirical performance of this asset pricing framework against historical data. To evaluate the risk of exuberance in equations (3.10) and (3.12), we first estimate the dates of any episodes of exuberance in the data together with the autoregressive coefficient of the price process during these episodes. Next, we estimate the utility parameters in the Epstein-Zin (EZ) preference function. Upon estimating the moment expressions in (3.10) and (3.12), we then calculate the final contribution of the bubble risk components to the asset return equation.

5.1 Data

We use quarterly observations as aggregate quarterly consumption data is measured more accurately than its monthly proxies. For the market return and price level, we use value-weighted returns and level data from CRSP (Quarterly Return and Level based on the NYSE/AMEX/NASDAQ composite) over the period 1947:01 to 2009:04, giving n=252 observations in total. The data are available from Wharton Research Data Services (WRDS)[7]. For the real price series, we use the (seasonally adjusted) CPI index obtained from the St.Louis Fed[8]. Aggregate per capita consumption data is

[7] http://wrds.wharton.upenn.edu/
[8] http://research.stlouisfed.org

obtained by expenditure on nondurables and services divided by total population[9]. Aggregate real per capita consumption follows by dividing aggregate per capita consumption by the implicit price deflator obtained from the same source. The series for $\frac{C_{t+1}}{C_t}$ covers the period 1947:02 to 2009:04. The three-month Treasury bill rate is used as the risk free asset, and six size/book-market sorted returns (hence, in total 7 asset returns) are used as individual riskless and risky assets for the utility parameter estimation. These are directly taken from Kenneth French's homepage[10].

5.2 Bubble Dating

Since the bubble process shows regime changes in this paper, we need to find the period of exuberance $[\tau_e, \tau_f]$ and the length of the explosive episode ($m_n = \tau_f - \tau_e$). Using the forward recursive regression method of PWY, we detect two significant exuberance episodes (see Fig. 1). The first period is $t = 31, ..., 40$ ($m_{n1} = 10$) and the second one is $t = 202, ..., 216$ ($m_{n2} = 15$). The corresponding periods are 1954:03 to 1956:04 and 1997:02 to 2000:04, respectively. This result does not conflict with the periodically collapsing bubble framework of this paper.

Our results are similar to those in PWY. The first bubble episode is not in the range of the PWY data. The second bubble period is included in the bubble period detected in PWY, which spans from July 1995 to September 2000, based upon monthly NASDAQ price data. This paper defines a shorter bubble period than PWY, possibly because of the lower data frequency or the use of the composite market price index (NYSE/AMEX/NASDAQ). Nevertheless, these results and those of PWY are very similar, showing that the methodology is robust to a variety of monthly or quarterly data sets in detecting bubbles.

The second episode captures the U.S. market surge and collapse associated with the 1997 Asian financial crisis, which eventually influenced the dot.com bubble. The first bubble episode is associated with the economic boom following the end of the Korean War. Other than these two episodes the recursive series show some short lived explosive coefficients, producing minor episodes that are not recorded as bubbles here because they do not satisfy the $\log(n)$ duration condition discussed in PWY[11], as is evident in Figure 1.

We also run a more recently developed date-stamping technique developed in PSY (2015a&b). As shown in Figure 2, the PSY test detects essentially the same periods corresponding to two distinct bubble episodes. PWY test reports slightly shorter duration periods for these bubbles than the PSY test, which is consistent with the findings of PSY (2015a; Figures 7 and 8). In what follows, we use the more conservative results of PWY for our purpose of analyzing the empirical impact of these bubbles on asset pricing and financial market risk[12]. **[FOOTNOTE ADDED]**

[9] Total population is computed by dividing real total disposable income by real per capita disposable income. Both series as well as expenditure data are obtained from the homepage of the Bureau of Economic Analysis (www.bea.gov).

[10] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[11] Here, $n = 252$ and $\log(252) = 5.53$, requiring a sustained mildly explosive coefficient over 6 periods for inclusion as a bubble episode.

[12] **In Appendix 7.10, we also report the results based on the PSY tests, which confirms the robustness of the main empirical results.**

In sum, we detect two significant bubble episodes in our data period and these findings corroborate historical evidence about these events. The durations of these bubbles are long enough to warrant incorporating their effects into our modeling framework for risk assessment and volatility analysis.

Figure 1: Bubble dating using PWY (2011) test
NYSE/AMEX/NASDAQ index from 1947:Q1 to 2009:Q4

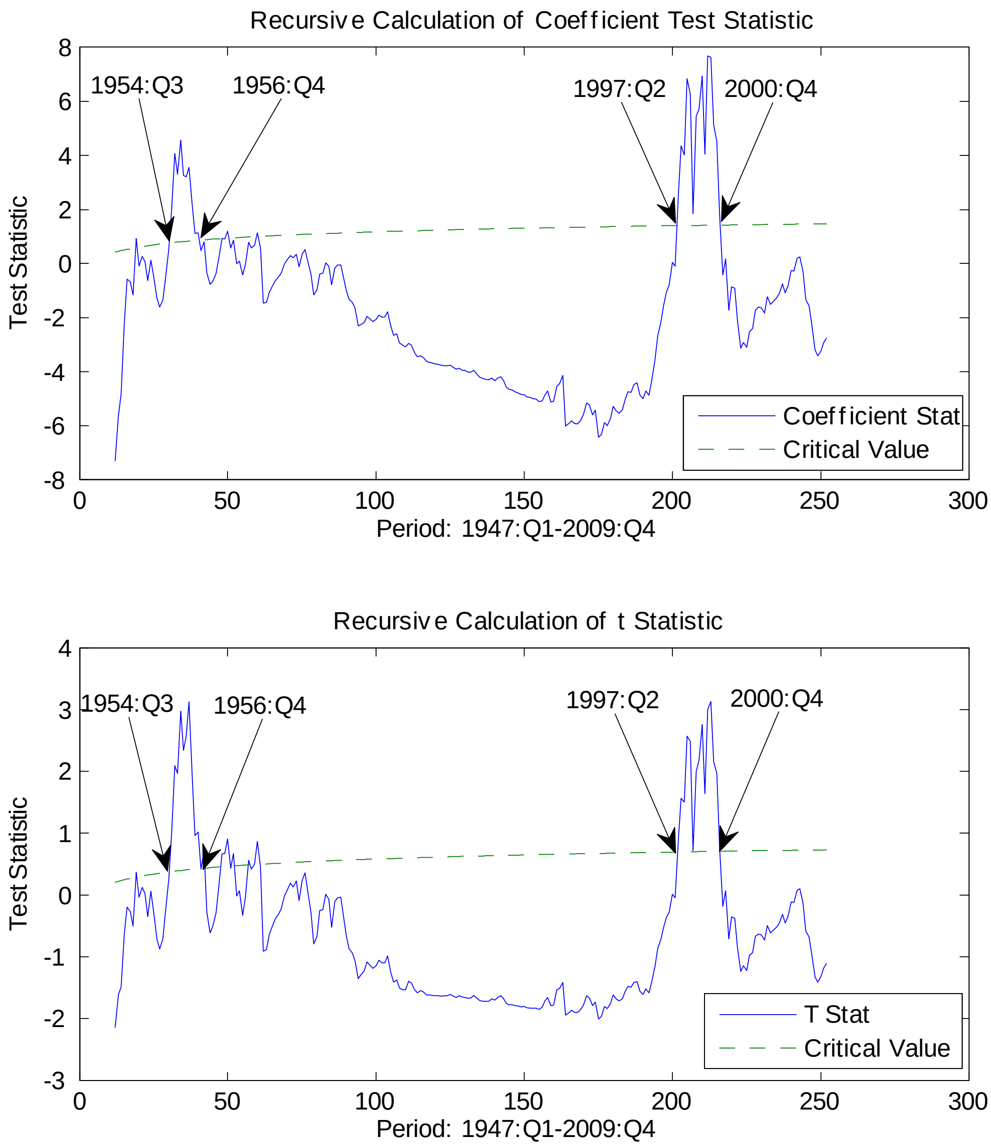
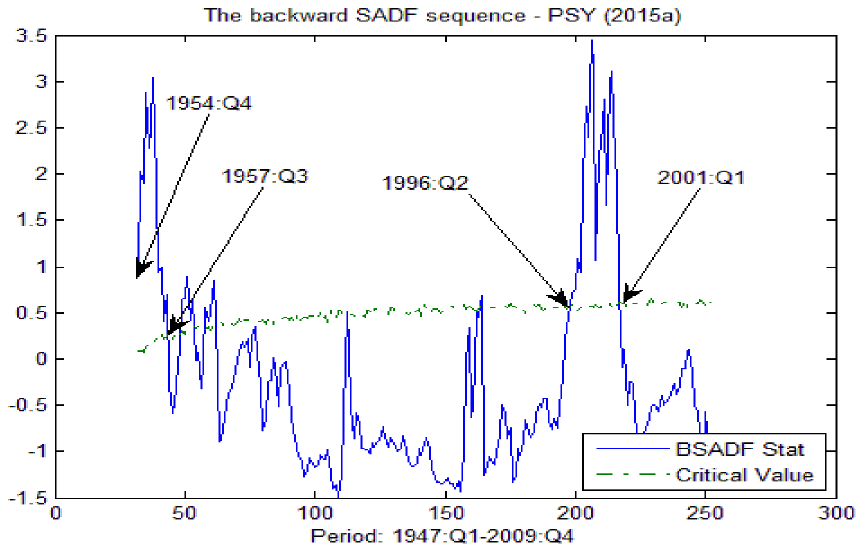


Figure 2: Bubble dating using PSY (2015a) test

NYSE/AMEX/NASDAQ index from 1947:Q1 to 2009:Q4



5.3 Estimation of the Utility Parameters

After detecting the bubble episodes, estimating the utility parameters (δ , φ and ψ) is the next step. There is an extensive literature on the estimation of the EZ utility parameters and some conflicting evidence on realistic estimates. In our application, the value of the EIS parameter (ψ) is important because we expect bubble risk to explain some fraction of stock market risk. From the definition of φ in (3.2) we find that

$$1 - \varphi = \frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}},$$

which is positive when $\psi, \gamma > 1$. Bansal and Yaron (2004) and Bansal, Kiku and Yaron (2007) have argued that $\psi < 1$ implies that asset valuations rise with higher economic uncertainty.

We follow the standard estimating procedure (Epstein and Zin, 1991) who use GMM, as in Hansen and Singleton (1982), to estimate the Euler equation (3.3). In particular, given information Ω_t we have

$$\begin{aligned} & E\left[\left\{\delta(\frac{C_{t+1}}{C_t})^{-\frac{1}{\psi}}\right\}^{\varphi}\left\{\frac{1}{1+R_{m,t+1}}\right\}^{1-\varphi}(1+R_{i,t+1}) - 1|\Omega_t\right] \\ = \; & E\left[\exp(s_{t+1})(1+R_{i,t+1}) - 1|\Omega_t\right] = 0, \end{aligned}$$

where

$$
\begin{aligned}
s_{t+1} &= \varphi \log \delta - \frac{\varphi}{\psi} \Delta c_{t+1} + (\varphi - 1) r_{m,t+1} \\
\varphi &= \frac{(1-\gamma)}{(1-\frac{1}{\psi})},
\end{aligned}
$$

as described above. With instruments x_t, the moment conditions are

$$
E[h_i(W_{t+1}, \alpha) \otimes x_t] = 0, \tag{5.1}
$$

where

$$
W_{t+1} = (\frac{C_{t+1}}{C_t}, 1 + R_{m,t+1}, 1 + R_{i,t+1}), \quad \alpha = (\delta, \varphi, \psi),
$$

and

$$
h_i(W_{t+1}, \alpha) = \exp(s_{t+1})(1 + R_{i,t+1}) - 1.
$$

The GMM estimate is then the extremum estimator

$$
\begin{aligned}
\hat{\alpha} &= \arg\min Q_T(\alpha), \\
Q_T(\alpha) &= [\frac{1}{n} \sum h(W_{t+1}, \alpha) \otimes x_t]' W [\frac{1}{n} \sum h(W_{t+1}, \alpha) \otimes x_t], \\
h &= (h_1, h_2, ..., h_N)'.
\end{aligned}
$$

Here we estimate (δ, φ, ψ) directly rather (δ, γ, ψ), following the suggestion in Epstein and Zin (1991) for more reliable numerical results and because $(1-\varphi)$ is the crucial parameter determining the direction and degree of the bubble contribution to the equity premium (3.12). This approach provides direct estimate of φ and its statistical significance.

There is some flexibility in the choice of instruments. Among several different combinations, we use a constant, lagged and twice lagged consumption growth, and the market return. Linear and squared terms and all pair-wise cross products of each instrument $(\frac{C_t}{C_{t-1}}, \frac{C_{t-1}}{C_{t-2}}, 1+R_{m,t}, 1+R_{m,t-1})$, are used giving a total of 15 instruments. Combined with 7 asset returns, we have overall 105 orthogonality conditions.

Since we use twice lagged consumption growth, our sample size shrinks to $n = 249$. Moreover, we also exclude the period of explosive prices in GMM estimation based on the results from Section 5.2. Since (5.1) utilizes the unconditional moment restrictions, nonstationarity of the market price which is embedded in the market return $r_{m,t+1}$ can cause some nonstandard asymptotic behavior in GMM estimation. Analyzing the behavior of GMM estimation under nonstationarity is an interesting topic of research but is not pursued here. Instead, we follow the simple alternative of removing the data corresponding to the explosive period (which is consistently dated) in the GMM implementation. Thus, the total number of effective observations in the sample is $n = 224$, after subtracting the period of exuberance with 25 observations, as discussed in the next section.

Table 1: GMM Estimation Results for the Utility Parameters

	GMM Parameter Estimates			
Parameters	Estimates	Standard errors	t-stat	p-val
δ	0.9619	0.0148	65.2041	0
φ	-0.7376	0.1380	-5.3441	0
$1-\varphi$	1.7376		6.3441	0
ψ	2.4217	16.3635	0.1480	0.8825
γ (indirect)	1.4330			

Table 1 shows that (δ, φ) are estimated with good accuracy. For the EIS parameter ψ, the point estimate 2.42 is reasonable and accords with earlier findings but has a large standard error. The problem of imprecise estimation of the EIS parameter is common in the empirical literature, as discussed recently by Kim et al. (2010), and the outcome here is similar to other recent empirical work. Our emphasis focuses not on the EIS parameter ψ but on φ and in particular $(1-\varphi)$, since ψ only arises in the consumption risk component, which turns out to be empirically negligible in the equity premium equation (3.12). The estimate of $(1-\varphi)$ is positive and strongly significant. Indirect calculation of the RRA parameter γ gives an estimate of 1.4330. This value of the RRA parameter lies in the usual historical range. It is noteworthy that the estimate of ψ exceeds unity, which is considered to be a critical value for the EIS parameter in explaining the dynamics of the aggregate stock market within EZ preferences.

In sum, the estimated values of the utility parameters are within the range of existing studies and the positive estimate for $(1-\varphi)$ provides support for the presence of additional bubble risk in asset price determination, as suggested in (3.12).

5.4 Contribution of Bubble Risk

This section calculates the empirical contribution of the presence of bubble factors on asset prices and stock market risk. The sample analogues of the conditional second moments are estimated, and these are combined with the results of the estimated utility parameters and the estimated ρ[13].

We start with the equity premium formula from (3.12)

$$\begin{aligned} & E_t(r_{i,t+1}) - r_{f,t+1} + \frac{1}{2}Var_t(r_{i,t+1}) \\ = \quad & (\frac{\varphi}{\psi} + (1-\varphi)c_n)Cov_t(r_{i,t+1}, \Delta c_{t+1}) + (1-\varphi)\rho_n Cov_t(r_{i,t+1}, b_{m,n,t+1}). \end{aligned}$$

Using (4.6) we have

$$\hat{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right) = \begin{cases} \hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu\left(t \notin [\tau_e, \tau_f)\right) \\ +e^{(\hat{\theta}-1)I_n}\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu\left(t \in [\tau_e, \tau_f)\right). \end{cases}$$

[13] Following Kim, Lee, Park and Yeo (2010), real dividends are obtained using the formula (returns with dividend−returns without dividend)×market price index/CPI.

For empirical evaluation, we need to set a value to the investor horizon I_n in this expression. We use $I_n = n^{0.3}$, which corresponds to a horizon of 5-6 periods or around 16 months with quarterly data. The sensitivity of our empirical results to this setting of I_n is investigated later (see Table 4).

Since we have two episodes of price exuberance in our application we consider the following natural extension of (4.6),

$$\hat{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right) = \begin{cases} \hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu\left(t \notin [\tau_{1e}, \tau_{1f}) \text{ and } t \notin [\tau_{2e}, \tau_{2f})\right) \\ \quad +e^{\left(\hat{\theta}_1 - 1\right)I_n}\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu\left(t \in [\tau_{1e}, \tau_{1f})\right) \\ \quad +e^{\left(\hat{\theta}_2 - 1\right)I_n}\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)\mu\left(t \in [\tau_{2e}, \tau_{2f})\right). \end{cases} \tag{5.2}$$

If we consider $\varepsilon_{p,t} \simeq \Delta p_{m,t}$ as a proxy for the market return without dividend[14], the general financial market risk corresponds to the estimated systematic market risk $\hat{E}\left(Cov(r_{i,t}, \varepsilon_{p,t})\right)$ for the normal periods $t \notin [\tau_{1e}, \tau_{1f})$ and $t \notin [\tau_{2e}, \tau_{2f})$, compounded by $e^{\left(\hat{\theta}_1 - 1\right)I_n}$ during the bubble periods. So, the new risk factor applies beyond explosive price periods in the overall calculation of financial market risk. The counting measure μ in (5.2) has the following empirical form here: $\mu\left(t \in [\tau_{1e}, \tau_{1f})\right) = \frac{\tau_{1f} - \tau_{1e}}{n} = \frac{m_1}{n}$, $\mu\left(t \in [\tau_{2e}, \tau_{2f})\right) = \frac{\tau_{2f} - \tau_{2e}}{n} = \frac{m_2}{n}$ and $\mu\left(t \notin [\tau_{1e}, \tau_{1f}) \text{ or } t \notin [\tau_{2e}, \tau_{2f})\right) = 1 - \frac{m_1}{n} - \frac{m_2}{n}$, where m_1 and m_2 represent the length of the first and second bubble episodes, respectively.

Finally, the conditional covariance between the asset return and the bubble series is estimated as,

$$\tilde{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right) = \left(\frac{n - m_1 - m_2}{n} + \frac{m_1}{n}e^{\left(\hat{\theta}_1 - 1\right)I_n} + \frac{m_2}{n}e^{\left(\hat{\theta}_2 - 1\right)I_n}\right)\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right) \tag{5.3}$$

where $\tilde{E}\left(\cdot\right)$ denotes an estimated empirical expectation using the present data.

We confirm the effect of structural uncertainty discussed in section 3.2. The extra risk stemming from investor ignorance about τ_e and τ_f is embedded in the coefficients $m_1 = \tau_{1f} - \tau_{1e}$ and $m_2 = \tau_{2f} - \tau_{2e}$. Equation (5.3) implies that the estimated financial market risk from price bubbles is equal to the estimated systematic market risk $\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)$ amplified by the estimated coefficient

$$N_p = \left(\frac{n - m_1 - m_2}{n} + \frac{m_1}{n}e^{\left(\hat{\theta}_1 - 1\right)I_n} + \frac{m_2}{n}e^{\left(\hat{\theta}_2 - 1\right)I_n}\right) > 1.$$

Strict inequality holds in this evaluation because implementation of the bubble-dating technology rejects the hypothesis of $\theta_i = 1$, and indicates two periods with $\theta_i > 1$, $i = 1, 2$. The estimated coefficient N_p increases as the intensity of the bubbles increases (via larger θ), as the investor has a longer horizon (longer I_n), and the estimated bubble duration gets longer (larger $\tau_f - \tau_e$). Financial market risk is the same as the usual systematic market risk if there is no price exuberance

[14] It is not exactly same, however, during the explosive episodes. See section 4.2.3.

($m_1 = m_2 = 0$, or equivalently, $N_p = 1$). Symbolically

$$\underbrace{\tilde{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right)}_{\text{Financial Market Risk}} = \underbrace{\left(\frac{n-m_1-m_2}{n} + \frac{m_1}{n}e^{(\hat{\theta}_1-1)I_n} + \frac{m_2}{n}e^{(\hat{\theta}_2-1)I_n}\right)}_{\text{Amplified by Exuberance}} \times \underbrace{\hat{E}\left(Cov_{t-1}(r_{i,t}, \varepsilon_{p,t})\right)}_{\text{Systematic Risk}}$$

We may represent (3.12) by estimation of unconditional moments and with an empirical version of the expectation using the entire sample period.

$$\begin{aligned} & \hat{E}(E_{t-1}\left(r_{i,t} - r_{f,t}\right)) + \frac{1}{2}\hat{E}\left(Var_{t-1}(r_{i,t})\right) \\ = \quad & (\frac{\varphi}{\psi} + (1-\varphi)c_n)\hat{E}\left(Cov_{t-1}(r_{i,t}, \Delta c_t)\right) + (1-\varphi)\rho_n\tilde{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right) \end{aligned} \tag{5.4}$$

Table2: Estimation Result for Equity Premium

LHS $= E(r_{i,t} - r_{f,t}) + \frac{1}{2}Var(r_{i,t})$	RHS $= (\frac{\varphi}{\psi} + (1-\varphi)c_n)Cov(r_{i,t}, \Delta c_t)$ $+(1-\varphi)\rho_n\tilde{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right)$	
1.7445 (%)	$\frac{\varphi}{\psi} + (1-\varphi)c_n$	-0.2184
	$Cov(r_{i,t}, \Delta c_t)$	0.0039 (%)
	$(1-\varphi)$	1.7376
	ρ	0.9919
	N_p[15]	1.0163
	$Cov(r_{i,t}, \varepsilon_{p,t})$	0.6785 (%)
	$(\frac{\varphi}{\psi} + (1-\varphi)c_n)Cov(r_{i,t}, \Delta c_t)$	-0.0009 (%)
	$(1-\varphi)\rho_n\tilde{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right)$	1.1884 (%)
	$I_n = n^{0.3}$	5.2532
	θ_1	1.0553
	θ_2	1.0091
	c_n	0.0496
Equity Premium 1.7445 (%)	Estimation (RHS total)	1.1875 (%)

Table 2 shows the estimation results for the components of the equity premium. The equity premium is defined by $E(r_{m,t} - r_{f,t}) + \frac{1}{2}Var(r_{m,t})$. The sample analogue of this corresponding quantity is computed as approximately 1.7%. This is smaller than the historical equity premium of around 6%, which is measured by annual data from the late 1800's to the late 1900's (see, among others, Mehra (2003) and Mehra and Prescott (1985) who used data over 1889-1978). However, their results are consistent with the most recent findings which report a declining equity premium of around 0.7% after the 1970's (See, Jagannathan et al (2000)). Importantly, the classical con-

[15] $N_p = \frac{n-m_1-m_2}{n} + \frac{m_1}{n}e^{(\theta_1-1)I_n} + \frac{m_2}{n}e^{(\theta_2-1)I_n}$.

tribution from the component $(\frac{\varphi}{\psi} + (1-\varphi)c_n)Cov(r_{i,t}, \Delta c_t)$ on the right side of (5.4) is negligible (-0.0007%). So the equity premium cannot be explained by this classical components, leaving the equity premium puzzle unexplained. However, the new contribution from price exuberance $(1-\varphi)\rho_n \tilde{E}\left(Cov_{t-1}(r_{i,t}, b_{m,n,t})\right)$ is much larger (1.1875 %) than the traditional consumption based component. In fact, as Table 2 shows, the new contribution explains approximately 70% of the equity premium.

Table 3: Estimation Results for Market Volatility

LHS $= Var(r_{m,t})$		RHS $= \rho_n^2 \tilde{E}\left(Var_{t-1}(b_{m,n,t})\right) + c_n^2 Var(\Delta c_t) + \rho_n c_n \tilde{E}\left(Cov_{t-1}(b_{m,n,t}, \Delta c_t)\right)$	
	0.6728 (%)	ρ	0.9919
		N_v[16]	1.0372
		N_p[17]	1.0163
		$I_n = n^{0.3}$	5.2532
		c_n	0.0496
		$Var(\varepsilon_{p,t})$	0.6926 (%)
		$Var(\Delta c_t)$	0.0063 (%)
		$Cov(\varepsilon_{p,t}, \Delta c_t)$	-0.0009 (%)
		$\tilde{E}\left(Var_{t-1}(b_{m,n,t})\right)$	0.7184 (%)
		$\rho_n^2 \tilde{E}\left(Var_{t-1}(b_{m,n,t})\right)$	0.7068 (%)
		$c_n^2 Var(\Delta c_t)$	0.0000(%)
		$\rho_n c_n \tilde{E}\left(Cov_{t-1}(b_{m,n,t}, \Delta c_t)\right)$	0.0000(%)
Realized Market Volatility	0.6728 (%)	Estimation (RHS total)	0.7068 (%)

In a similar way, we can estimate the components in the market volatility equation (3.10)

$$\begin{aligned}
\tilde{E}\left(Var_{t-1}(b_{m,n,t})\right) &= \left(\frac{n-m_1-m_2}{n} + \frac{m_1}{n}e^{2(\hat{\theta}_1-1)I_n} + \frac{m_2}{n}e^{2(\hat{\theta}_2-1)I_n}\right)\hat{E}\left(Var_{t-1}(\varepsilon_{p,t})\right), \\
\tilde{E}\left(Cov_{t-1}(b_{m,n,t}, \Delta c_t)\right) &= \left(\frac{n-m_1-m_2}{n} + \frac{m_1}{n}e^{(\hat{\theta}_1-1)I_n} + \frac{m_2}{n}e^{(\hat{\theta}_2-1)I_n}\right)\hat{E}\left(Cov_{t-1}(\varepsilon_{p,t}, \Delta c_t)\right),
\end{aligned}$$

giving an estimable version of (3.10) using unconditional moments and empirical expectation.

$$Var(r_{m,t}) = \rho_n^2 \tilde{E}\left(Var_{t-1}(b_{m,n,t})\right) + c_n^2 Var(\Delta c_t) + \rho_n c_n \tilde{E}\left(Cov_{t-1}(b_{m,n,t}, \Delta c_t)\right). \qquad (5.5)$$

[16] $N_v = \left(\frac{n-m_1-m_2}{n} + \frac{m_1}{n}e^{2(\theta_1-1)I_n} + \frac{m_2}{n}e^{2(\theta_2-1)I_n}\right)$

[17] $N_p = \left(\frac{n-m_1-m_2}{n} + \frac{m_1}{n}e^{(\theta_1-1)I_n} + \frac{m_2}{n}e^{(\theta_2-1)I_n}\right)$

Table 3 shows the empirical results for the stock market volatility puzzle. The traditional components involving the consumption growth rate have only a negligible role in explaining stock market volatility, whereas stock market volatility is fully explained in our model affirming the dominant role of the speculative bubble impact on stock market volatility.

5.5 Effects of the investor horizon

In the above calculations we used a distant formulation of the investor horizon setting $I_n = n^{0.3}$ and $\alpha_I = 0.3$ in (4.3). It is instructive to observe the effects of different investor horizons on these results. Since the key parameter of the investor horizon is the expansion rate parameter α_I, we calculate the impact of different choices of α_I in the $(0, 1)$ interval.

Table 4 shows the empirical impact in (3.10) and (3.12) arising from different investor horizons. The range of $[0.1, 0.4]$ for α_I gives empirically reasonable performance in our model and corresponds to an investor horizon in the range of 5 to 28 months. A reasonable assumption for a given finite sample size n is that $I_n \leq \min\{m_{n1}, m_{n2}\}$ where $\min\{m_{n1}, m_{n2}\}$ represents the minimum duration of the explosive episode. This assumption implies that the representative investor cannot have a longer horizon than the length of any period of financial exuberance. This condition seems appropriate in a no-arbitrage pricing framework and must be satisfied because investors never know the exact timing of τ_e and τ_f before their realization.

Table 4: Contributions with Changing Investor Horizons

	Equity Premium (%)	Market Volatility (%)	Changing Investor Horizons	
LHS	1.7445	0.6728	α_I	$I_n = n^{\alpha_I}$
RHS	1.1741	0.6885	0.1	1.7384
	1.1787	0.6945	0.2	3.0219
	1.1875	0.7068	0.3	5.2532
	1.2052	0.7360	0.4	9.1319

6 Conclusion

This paper extends the standard present value and consumption-based asset pricing models by incorporating the possibility of periodically collapsing price exuberance and investor uncertainty about the origination and termination dates of exuberance. It is shown that temporary explosive behavior can arise and be incorporated in the extended model under some reasonable assumptions on the price process and investor horizon. The model developed here allows for an explicit characterization that quantifies the effect of potential speculative bubbles on risk. The expression is easy to estimate, adds a new component to conventional asset pricing, and allows for an empirical assessment of the impact of speculative behavior.

One aspect of our asset pricing theory is that it incorporates a finite, instead of an infinite, investor horizon. Finite decision horizons accommodate myopic investors and are a component of speculative behavior, emphasizing short run market gains rather than long run effects of economic fundamentals. These conditions, in place of the usual terminal transversality conditions, produce further structural uncertainty in pricing that leads to extra risk bearing by agents and associated market volatility. Our empirical evaluation of the new model using US composite stock market data confirms that exuberance is an important contributor to asset prices and stock market risk, corroborating less formal historical analysis such as that of Ferguson (2008) and Ahamed (2009).

Asset pricing models do not usually allow for mispricing in relation to fundamentals or the existence of exuberance in equilibrium analysis. Even in the context of the simplest iid mechanism for consumption growth, our theoretical model and empirical results show that the presence of bubbles can have a considerable impact on asset prices. Future research may usefully extend the simple framework considered here to models of exuberance of greater complexity, more realistic consumption growth processes, volatility in economic fundamentals, and variable discounting regimes.

7 Appendix

7.1 Derivation of Equation (2.15)

Since $\rho_n^{-1} = 1 + \exp(\overline{d-p})$, we start by analyzing the sample mean

$$\overline{d-p} = \frac{1}{n}\sum_{t=1}^{n}(d_t - p_t) = \frac{1}{n}\left\{\sum_{t=1}^{\tau_e-1}(d_t-p_t) + \sum_{t=\tau_e}^{\tau_f-1}(d_t-p_t) + \sum_{t=\tau_f}^{n}(d_t-p_t)\right\}.$$

Assuming that $d_t - p_t = u_t$ is stationary and ergodic over non-bubble periods with $E\left(u_t 1\left\{t \notin [\tau_e, \tau_f)\right\}\right) = \alpha$, we have

$$\begin{aligned}\overline{d-p} &= \frac{\tau_e - 1}{n}\frac{1}{\tau_e - 1}\sum_{t=1}^{\tau_e-1}u_t + \frac{\tau_f-\tau_e}{n}\frac{1}{\tau_f-\tau_e}\sum_{t=\tau_e}^{\tau_f-1}(d_t-p_t) + \frac{n-\tau_f+1}{n}\frac{1}{n-\tau_f+1}\sum_{t=\tau_f}^{n}u_t \\ &= (r_e + 1 - r_f)\,\alpha + \frac{\tau_f-\tau_e}{n}\frac{1}{\tau_f-\tau_e}\sum_{t=\tau_e}^{\tau_f-1}(d_t-p_t) + o_{a.s.}(1)\,.\end{aligned}$$

Then

$$\frac{1}{\rho_n} = 1 + e^{(1-r_f+r_e)\alpha}\frac{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n}d_{s-1+\tau_e})}{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n}p_{s-1+\tau_e})}\left\{1 + o_{a.s.}(1)\right\}. \tag{7.1}$$

With the parameter settings $\tau_e = \lfloor nr_e \rfloor$ and $\tau_f = \lfloor nr_f \rfloor$ for some fixed numbers $r_e < r_f$, standard functional limit theory gives $n^{-3/2}\sum_{s=1}^{m_n}d_{s-1+\tau_e} \Rightarrow \int_{r_e}^{r_f}B_d$. It follows that

$$\exp\left(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n}d_{s-1+\tau_e}\right) = O_p\left(\exp\left\{n^{1/2}\int_{r_e}^{r_f}B_d\right\}\right). \tag{7.2}$$

Further,

$$\frac{1}{\sqrt{k_n}}\frac{p_{t+\tau_e}}{\theta_n^t}=\frac{1}{\sqrt{k_n}}\sum_{s=1}^{t}\frac{\varepsilon_{p,\tau_e+s}}{\theta_n^{t-s}}+\frac{p_{\tau_e}}{\sqrt{k_n}}=O_p\left(\sqrt{\frac{n}{k_n}}B_p\left(r_e\right)\right)$$

since $n^{-1/2}p_{\tau_e}\Rightarrow B_p\left(r_e\right)$ and

$$k_n^{-1/2}\sum_{s=1}^{t}\frac{\varepsilon_{p,\tau_e+s}}{\theta_n^{t-s}}=O_p\left(1\right),$$

by virtue of Lemma 4.2 of Phillips and Magdalinos (2007, hereafter PM). Hence

$$\begin{aligned}\sum_{t=1}^{m_n}p_{t-1+\tau_e} &= k_n^{1/2}\sum_{t=1}^{m_n}\left\{\frac{1}{\sqrt{k_n}}\frac{p_{t-1+\tau_e}}{\theta_n^{t-1}}\right\}\theta_n^{t-1}\sim p_{\tau_e}\sum_{t=1}^{m_n}\theta_n^{t-1}\\ &\sim O_p(n^{1/2}\frac{k_n}{c}\theta_n^{m_n}B_p\left(r_e\right))\sim O_p\left(\frac{n^{1/2}k_n}{c}\exp\left(c\frac{m_n}{k_n}\right)B_p\left(r_e\right)\right),\end{aligned} \tag{7.3}$$

and then

$$\exp\left(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n}p_{s-1+\tau_e}\right)=O_p\left(\exp\left(\frac{k_n}{\sqrt{n}c}\exp\left(c\frac{m_n}{k_n}\right)B_p\left(r_e\right)\right)\right). \tag{7.4}$$

Combining (7.2) - (7.4) we have

$$\begin{aligned}\frac{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n}d_{s-1+\tau_e})}{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n}p_{s-1+\tau_e})} &= O_p\left(\frac{\exp\left\{n^{1/2}\int_{r_e}^{r_f}B_d\right\}}{\exp\left(\frac{k_n}{\sqrt{n}c}\exp\left(c\frac{m_n}{k_n}\right)B_p\left(r_e\right)\right)}\right)\\ &= O_p\left(\exp\left\{-\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)+n^{1/2}\int_{r_e}^{r_f}B_d\right\}\right)\\ &= O_p\left(\exp\left\{-\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)\right\}\right),\end{aligned} \tag{7.5}$$

when $k_n=n^{1-\eta}$ for some $\eta\in(0,1)$. It follows that

$$\frac{1}{\rho_n}=1+O_p\left(\exp\left\{-\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)\right\}\right), \tag{7.6}$$

and so

$$\rho_n\rightarrow_p 1_{\{B_p(r_e)>0\}}, \tag{7.7}$$

as given in (2.15). .

7.2 Derivation of Equation (2.19)

For this parameterization of m_n, we have

$$n^{-1/2}m_n^{-1}\sum_{s=1}^{m_n}d_{s-1+\tau_e}=\frac{m_n^{1/2}}{n^{1/2}m_n}\sum_{s=1}^{m_n}\frac{d_{s-1+\tau_e}-d_{\tau_e}}{m_n^{1/2}}+n^{-1/2}d_{\tau_e}\Rightarrow B_d\left(r_e\right),$$

and in place of (7.2) we get

$$\exp\left(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n} d_{s-1+\tau_e}\right) = O_p\left(\exp\left\{\frac{m_n}{\sqrt{n}}B_d\left(r_e\right)\right\}\right).$$

Additionally, as in (7.3) we have $\sum_{t=1}^{m_n} p_{t-1+\tau_e} \sim O_p\left(\frac{n^{1/2}k_n}{c}\exp\left(c\frac{m_n}{k_n}\right)B_p\left(r_e\right)\right)$, and (7.4) continues to hold, so that

$$\frac{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n} d_{s-1+\tau_e})}{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n} p_{s-1+\tau_e})} = O_p\left(\frac{\exp\left\{\frac{m_n}{\sqrt{n}}B_d\left(r_e\right)\right\}}{\exp\left(\frac{k_n}{\sqrt{n}c}\exp\left(c\frac{m_n}{k_n}\right)B_p\left(r_e\right)\right)}\right) = 1 + o_p\left(1\right),$$

provided $\frac{m_n}{\sqrt{n}} \to 0$ and $\frac{k_n}{\sqrt{n}}\exp\left(c\frac{m_n}{k_n}\right) = o\left(1\right)$ which will be so if

$$\frac{m}{k_n} \to a \in [0,\infty), \quad \frac{k_n}{\sqrt{n}} \to 0. \tag{7.8}$$

Under these rate conditions, we have

$$\frac{1}{\rho_n} = 1 + \exp\left(\frac{n-\tau_f+\tau_e}{n}\alpha\right)\left\{1 + o_p\left(1\right)\right\}.$$

Hence,

$$\rho_n \to_p \frac{1}{1+e^{\alpha}} < 1, \tag{7.9}$$

as in the cointegrating case where $d_t - p_t$ is stationary and ergodic and validating the usual log linear approximation and present value relationship.

7.3 Derivation of Equation (2.20)

The earlier analysis holds, including (7.1) and (7.5), which lead to the representation

$$\begin{aligned}\frac{1}{\rho_n} &= 1 + e^{\left(1-r_f+r_e\right)\alpha}\frac{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n} d_{s-1+\tau_e})}{\exp(\frac{m_n}{n}\frac{1}{m_n}\sum_{s=1}^{m_n} p_{s-1+\tau_e})}\left\{1 + o_{a.s.}\left(1\right)\right\} \\ &= 1 + e^{\left(1-r_f+r_e\right)\alpha}\exp\left\{-\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)\right\}\left\{1 + o_p\left(1\right)\right\}. \end{aligned} \tag{7.10}$$

Then, for $i = I_n$, we get

$$\begin{aligned}\rho_n^{I_n} &\sim 1/\left[1+e^{\left(1-r_f+r_e\right)\alpha}\exp\left\{-\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)\right\}\right]^{I_n} \\ &= 1/\left[1+\frac{g}{\exp\left\{\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)\right\}}\right]^{I_n} \\ &\sim e^{-ag}1_{\{B_p(r_e)>0\}},\end{aligned}$$

with $g = e^{\left(1-r_f+r_e\right)\alpha}$ and $I_n \sim aI_n^*$ for some $a > 0$, where

$$I_n^* := \exp\left\{\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)\right\} \to \infty, \tag{7.11}$$

where the divergence holds when $k_n = n^{1-\eta}$ for some $\eta \in (0,1)$ and $B_p\left(r_e\right) > 0$. Further, if $I_n = o_p\left(I_n^*\right)$ then $\rho_n^{I_n} \to_p 1_{\{B_p(r_e)>0\}}$. If $B_p\left(r_e\right) < 0$, then $\rho_n^i \to_p 0$. In short, we have

$$\rho_n^{I_n} \sim \begin{cases} e^{-ag}1_{\{B_p(r_e)>0\}} & \text{for } I_n \sim aI_n^*,\ a>0 \\ 1_{\{B_p(r_e)>0\}} & \text{for } I_n = o_p\left(I_n^*\right)\end{cases}, \tag{7.12}$$

giving (2.20).

7.4 Derivation of Equation (2.22)

When $B_p\left(r_e\right) > 0$ we have from (7.10)

$$\rho_n\theta_n = \frac{1+\frac{c}{k_n}}{1+\frac{g}{\exp\left\{\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p(r_e)\right\}}} + o_p\left(1\right). \tag{7.13}$$

Then

$$\left(\rho_n\theta_n\right)^i = \left(1+\frac{c}{k_n}\right)^i\left(1+\frac{g}{\exp\left\{\frac{k_n}{\sqrt{n}c}e^{cr_b\frac{n}{k_n}}B_p\left(r_e\right)\right\}}\right)^{-i} + o_p\left(1\right)$$

Since $k_n^p = o_p\left(I_n^*\right)$ for all finite $p > 0$ we can classify outcomes as follows:

$$\begin{aligned}\left(\rho_n\theta_n\right)^{I_n} &\to_p e^{cb}1_{\{B_p(r_e)>0\}} && \text{for } \tfrac{I_n}{k_n} \to b \in [0,\infty) \\ \left(\rho_n\theta_n\right)^{I_n} &\sim e^{cbk_n^\delta}1_{\{B_p(r_e)>0\}} && \text{for } I_n = bk_n^{1+\delta},\quad \delta, b > 0\end{aligned} \tag{7.14}$$

Combine (7.14) with (2.16) and, if the time horizon satisfies

$$t < \tau_e,\ t + I_n < \tau_e, \tag{7.15}$$

then

$$\rho_n^{I_n}E_tp_{t+I_n} = \rho_n^{I_n}p_t \sim 1_{\{B_p(r_e)>0\}}p_t \quad t < \tau_e,\quad t + I_n < \tau_e\ . \tag{7.16}$$

The mildly explosive coefficient $\theta = \theta_n$ is irrelevant in this case because from the present time t the investor horizon does not extend beyond the origination point τ_e and investors are unaware of the impending bubble. However, when

$$t < \tau_e, \ t + i \in [\tau_e, \tau_f], \tag{7.17}$$

the mildly explosive case $\theta = \theta_n$ becomes relevant because the horizon extends into the bubble period. We analyze this case here for completeness, although as discussed in the text the origination date τ_e is unknown to investors so by convention the conditional expectation in this case will be the same as for (7.16). With this caveat, we evaluate the conditional expectation factors over the relevant interval, which in this case is

$$\rho_n^i \theta_n^{t+i-\tau_e} p_t \quad t < \tau_e, \quad t + i \in [\tau_e, \tau_f] \ . \tag{7.18}$$

If $I_n = bk_n^{1+\delta}$ then the horizon $I_n = o(n)$ for $\delta \in [0, \frac{\eta}{1-\eta})$ when $k_n = n^{1-\eta}$ for some $\eta > 0$. With this horizon I_n we must have $\tau_e - t < bk_n^{1+\delta}$ if the dating conditions $t < \tau_e$ and $t + i \in [\tau_e, \tau_f]$ are to hold for $i \leq I_n$. Hence, when $t = \tau_e - o\left(k_n^{1+\delta}\right)$, we get

$$\theta_n^{t+I_n-\tau_e} = \left(1 + \frac{c}{k_n}\right)^{t+I_n-\tau_e} \sim \left(1 + \frac{c}{k_n}\right)^{bk_n^{1+\delta}\{1+o(1)\}} \sim e^{cbk_n^\delta},$$

which leads to the following asymptotic form

$$\rho_n^{I_n} \theta_n^{t+I_n-\tau_e} \sim e^{cbk_n^\delta} 1_{\{B_p(r_e)>0\}} \ \text{ over } t \geq \tau_e - o\left(k_n^{1+\delta}\right), \text{ and } t + I_n \in [\tau_e, \tau_f],$$

for (7.18).

Depending on whether the investor horizon I_n satisfies (7.15) or (7.17), the mildly explosive case $\theta = \theta_n$ may or may not appear for the bubble characterization when $t < \tau_e$. However, as indicated, (7.17) is generally not relevant in practice because τ_e is unknown before its realization. This is why the case (7.15) is more important since τ_e is realized and known only when $t \geq \tau_e$.

For the case

$$t \in [\tau_e, \tau_f), \ t + I_n \leq \tau_f \tag{7.19}$$

the mildly explosive case $\theta = \theta_n$ is relevant in the conditional expectation formulae given in (2.16). So we need to examine the factor involving $\rho_n^i \theta_n^i$, viz.,

$$\rho_n^i \theta_n^i p_t \quad t \in [\tau_e, \tau_f), \quad t + i \leq \tau_f \ . \tag{7.20}$$

We use a similar asymptotic argument as before. If $I_n = bk_n^{1+\delta}$, then the horizon $I_n = o(n)$ for $\delta \in [0, \frac{\eta}{1-\eta})$ when $k_n = n^{1-\eta}$ for some $\eta > 0$. With this horizon I_n , $\tau_f - t > bk_n^{1+\delta}$ should hold to ensure $t \in [\tau_e, \tau_f)$ and $t + i \leq \tau_f$ for $i \leq I_n$. When $t = \tau_f - n^{\beta}$, we can have

$$\tau_f - t = n^{\beta} > bk_n^{1+\delta} \text{ for some } b,\ \delta \text{ and } k_n. \tag{7.21}$$

In this case the factor in (7.20) for the full horizon I_n is

$$(\rho_n \theta_n)^{I_n} \sim e^{cbk_n^{\delta}} 1_{\{B_p(r_e)>0\}}, \text{ over } t \in [\tau_e, \tau_f),\ t + I_n \leq \tau_f. \tag{7.22}$$

For other combinations of time t and investor horizon I_n, the bubble characterizations are straightforwardly calculated using (7.12).

Using (7.12), (7.16) and (7.22) in (2.16) we find that

$$\rho_n^{I_n} E_t p_{t+I_n} \sim \begin{cases} 1_{\{B_p(r_e)>0\}} p_t & t < \tau_e,\quad t + I_n < \tau_e \\ e^{cbk_n^{\delta}} 1_{\{B_p(r_e)>0\}} p_t & t < \tau_e,\quad t + I_n \in [\tau_e, \tau_f] \\ 1_{\{B_p(r_e)>0\}} \left(p_t + E\varepsilon_p^*\right) & t < \tau_e,\quad t + I_n > \tau_f \\ e^{cbk_n^{\delta}} 1_{\{B_p(r_e)>0\}} p_t & t \in [\tau_e, \tau_f),\quad t + I_n \leq \tau_f \\ 1_{\{B_p(r_e)>0\}} \left(p_{\tau_e} + E\varepsilon_p^*\right) & t \in [\tau_e, \tau_f],\quad t + I_n > \tau_f \\ 1_{\{B_p(r_e)>0\}} p_t & t > \tau_f \end{cases},$$

giving (2.22) as stated. These formulae enable us to analyze the present value model under various degrees of investor myopia represented in the form $I_n = bk_n^{1+\delta}$ for $\delta \in [0, \eta)$.

7.5 Derivation of Equation (2.25)

From (2.4) and when the investor horizon is I_n we have

$$\delta_t = \rho_n^{I_n} \delta_{t+I_n} + \sum_{i=0}^{I_n - 1} \rho_n^i (r_{t+1+i} - \Delta d_{t+1+i}) + \kappa_n \frac{1 - \rho_n^{I_n}}{1 - \rho_n}, \tag{7.23}$$

where

$$\kappa_n = -\log \rho_n - (1 - \rho_n) \log(\frac{1}{\rho_n} - 1).$$

We focus on the case $B_p(r_e) > 0$. We show that as $n \to \infty$

$$\kappa_n \frac{1 - \rho_n^{I_n}}{1 - \rho_n} \sim \kappa_n I_n \rho_n^{I_n - 1} = -I_n \rho_n^{I_n - 1} \log \rho_n - I_n \rho_n^{I_n - 1} (1 - \rho_n) \log(\frac{1}{\rho_n} - 1) \to 0, \tag{7.24}$$

which simplifies (7.23). First

$$\log \rho_n = -\log \left(1 + \frac{g}{\exp\left\{\frac{k_n}{\sqrt{n}c} e^{cr_b \frac{n}{k_n}} B_p(r_e)\right\}}\right) \sim -\frac{g}{\exp\left\{\frac{k_n}{\sqrt{n}c} e^{cr_b \frac{n}{k_n}} B_p(r_e)\right\}},$$

so that as $n \to \infty$

$$-I_n \rho_n^{I_n - 1} \log \rho_n \sim \frac{g b k_n^{1+\delta} \rho_n^{I_n - 1}}{\exp\left\{ \frac{k_n}{\sqrt{nc}} e^{c r_b \frac{n}{k_n}} B_p\left(r_e\right) \right\}} \to 0,$$

and next

$$I_n \rho_n^{I_n - 1} (1 - \rho_n) \log(\frac{1}{\rho_n} - 1) \sim \frac{g b k_n^{1+\delta} \rho_n^{I_n - 1}}{\exp\left\{ \frac{k_n}{\sqrt{nc}} e^{c r_b \frac{n}{k_n}} B_p\left(r_e\right) \right\}} \log\left(\frac{g}{\exp\left\{ \frac{k_n}{\sqrt{nc}} e^{c r_b \frac{n}{k_n}} B_p\left(r_e\right) \right\}} \right) \to 0,$$

giving (7.24). Hence, (7.23) becomes

$$\delta_t \sim \rho_n^{I_n} \delta_{t+I_n} + \sum_{i=0}^{I_n - 1} \rho_n^i (r_{t+1+i} - \Delta d_{t+1+i}),$$

giving (2.25) as stated.

7.6 Derivation of Equation (2.30)

In view of (7.12), $\rho_n^{I_n} \sim 1$ when $B_p\left(r_e\right) > 0$ and so the final term of (2.29) has the form $\rho_n^{I_n} E_t d_{t+I_n} \sim I_n g + d_t + \sum_{i=1}^{I_n} E_t \varepsilon_{dt+i}$, leading to the stated result

$$\begin{aligned} p_t &\sim E_t \sum_{i=0}^{I_n - 1} \rho_n^i (\Delta d_{t+1+i} - r_{t+1+i}) + b_{n,t} - \left(I_n g + \sum_{i=1}^{I_n} E_t \varepsilon_{dt+i} \right) && (7.25) \\ &= E_t \sum_{i=0}^{I_n - 1} \rho_n^i (\Delta d_{t+1+i} - r_{t+1+i}) + b_{n,t}, && (7.26) \end{aligned}$$

when $g = 0$ and $E_t \varepsilon_{dt+i} = 0$ for $i \geq 1$.

7.7 Derivation of Equation (3.1)

Equation (3.1) without the bubble term is discussed in Campbell (2003). We pursue this derivation allowing for the existence of price bubbles. From (2.2)

$$r_{t+1} - E_t(r_{t+1}) \sim \rho(p_{t+1} - E_t(p_{t+1})) + (1 - \rho)(d_{t+1} - E_t(d_{t+1})) \tag{7.27}$$

and moving one period forward from (2.29), we have

$$p_{t+1} \sim d_{t+1} + E_{t+1} \sum_{i=0}^{I_n - 1} \rho_n^i (\Delta d_{t+2+i} - r_{t+2+i}) + b_{n,t+1} - \rho_n^{I_n} E_{t+1} d_{t+I_n+1}. \tag{7.28}$$

Taking expectations at time t, we have

$$E_t(p_{t+1}) \sim E_t(d_{t+1}) + E_t \sum_{i=0}^{I_n - 1} \rho_n^i (\Delta d_{t+2+i} - r_{t+2+i}) + E_t b_{n,t+1} - \rho_n^{I_n} E_t d_{t+I_n+1}. \tag{7.29}$$

Subtracting (7.29) from (7.28)

$$\begin{aligned}(p_{t+1}-E_t(p_{t+1})) &= (d_{t+1}-E_t(d_{t+1}))+(E_{t+1}-E_t)\sum_{i=0}^{I_n-1}\rho_n^i(\Delta d_{t+2+i}-r_{t+2+i}) \\ &\quad +(b_{n,t+1}-E_t b_{n,t+1})-\rho_n^{I_n}(E_{t+1}-E_t)\,d_{t+I_n+1}\end{aligned}$$

and substituting into (7.27) we obtain

$$\begin{aligned}r_{t+1}-E_t(r_{t+1}) &= (E_{t+1}-E_t)\sum_{i=1}^{I_n}\rho_n^i(\Delta d_{t+1+i}-r_{t+1+i})+(d_{t+1}-E_t(d_{t+1})) \\ &\quad +\rho_n\,(b_{n,t+1}-E_t b_{n,t+1})-\rho_n^{I_n+1}(E_{t+1}-E_t)\,d_{t+I_n+1} \\ &= (E_{t+1}-E_t)\sum_{i=1}^{I_n}\rho_n^i\Delta d_{t+1+i}-(E_{t+1}-E_t)\sum_{i=1}^{I_n}\rho_n^i r_{t+1+i}+(d_{t+1}-E_t(d_{t+1})) \\ &\quad +\rho_n\,(b_{n,t+1}-E_t b_{n,t+1})-\rho_n^{I_n+1}(E_{t+1}-E_t)\,d_{t+I_n+1},\end{aligned}$$

giving (3.1).

7.8 The investor horizon and life time utility maximization

7.8.1 Two different horizons

A common assumption in asset pricing models is that the representative investor is infinitely lived. This assumption is convenient analytically and allows us to ignore the effects of a "finite horizon" on portfolio choice so that we can solve the optimization problem using dynamic programming via the Bellman equation. This condition can be relaxed (e.g. Brandt,1999) and we may also retain the assumption in our framework. To avoid notational confusion, we emphasize that the investor horizon I_n defined in the present value model is not necessarily the same concept as the "horizon" employed in the lifetime utility maximization problem. The latter refers to the time horizon used in discounted utility maximization from expected future consumption and usually represents the life span of the investor. For example, if we use time separable CRRA utility ($\gamma = 1/\psi$) and a discount factor of δ, the finite horizon problem at date 0 is

$$\max E_0\left[\sum_{j=0}^{T}\delta^j u(C_j)\right],$$

and the infinite horizon problem is

$$\max E_0\left[\sum_{j=0}^{\infty}\delta^j u(C_j)\right].$$

The investor horizon I_n is an expectation horizon used in making investment decisions at any given t regarding the future price process with a discount factor ρ_n, and need not be directly related to the lifespan of the investor. (Mathematically, I_n does not depend on t.) If these two horizons may differ, then we may obtain the Euler equation (3.3) by allowing the infinitely lived agent ($T = \infty$) to make investment decisions using a finite (or distant) investor horizon (I_n) relating to the future price process.

7.8.2 Same horizons

Different discount rates for the price process (ρ_n) and for the utilities from consumption (δ) have been used in many past studies (e.g. Campbell, 2003) and cause no analytic difficulty. However, making allowance for possibly different expectation horizons (I_n and T) for the same representative agent is not so straightfoward but is still analytically tractable. In fact, if we use the same horizon I_n for investment decisions and lifetime utility maximization, we can still justify the use of the traditional Bellman equation. We demonstrate this for the special case of time separable utility (where $\gamma = 1/\psi$) but the same logic holds for the general recursive EZ utility. Intuitively, the restriction that arises from the use of a finite horizon T for lifetime utility maximization is not binding with a distant horizon I_n since $I_n \to \infty$ as $n \to \infty$. Consider the case where $\gamma = 1/\psi$ in the utility function

$$\begin{aligned}
V_t &= \left\{(1-\delta)C_t^{\frac{1-\gamma}{\varphi}} + \delta(E_t V_{t+1}^{1-\gamma})^{\frac{1}{\varphi}}\right\}^{\frac{\varphi}{1-\gamma}} = \left\{(1-\delta)C_t^{1-1/\psi} + \delta(E_t V_{t+1}^{1-\gamma})^{\frac{1-1/\psi}{1-\gamma}}\right\}^{\frac{1}{1-1/\psi}} \\
&= \left\{(1-\delta)C_t^{1-\gamma} + \delta(E_t V_{t+1}^{1-\gamma})\right\}^{\frac{1}{1-\gamma}} \\
&= \left\{(1-\delta)C_t^{1-\gamma} + \delta(E_t\left[(1-\delta)C_{t+1}^{1-\gamma} + \delta(E_{t+1}V_{t+2}^{1-\gamma})\right])\right\}^{\frac{1}{1-\gamma}}.
\end{aligned}$$

Since utility is invariant with respect to any monotone transformation, we can ignore the exponent $\frac{1}{1-\gamma}$. We consider the following form after forward recursion

$$V_t = \left[\sum_{j=0}^{\infty} \delta^j E_t(C_{t+j}^{1-\gamma}))\right],$$

and so the maximization problem is

$$\begin{aligned}
V_t(W_t) &= \max_{\{C_{t+j}\}_{j=0}^{\infty}} \left[\sum_{j=0}^{\infty} \delta^j E_t(C_{t+j}^{1-\gamma}))\right] = \max_{\{C_{t+j}\}_{j=0}^{\infty}} \left[\sum_{j=0}^{\infty} \delta^j E_t u(C_{t+j})\right], \\
W_{t+1} &= (W_t - C_t)(1 + R_{m,t+1}).
\end{aligned}$$

This lifetime maximization problem reduces to the two-period decision problem,

$$\begin{aligned} V_t(W_t) &= \max_{C_t}[E_t u(C_t)] + \max_{\{C_{t+j}\}_{j=1}^{\infty}} \left[\sum_{j=1}^{\infty} \delta^j E_t u(C_{t+j})\right], \text{ let } j = i+1, \\ &= \max_{C_t}[E_t u(C_t)] + \delta \max_{\{C_{t+1+i}\}_{i=0}^{\infty}} \left[\sum_{i=0}^{\infty} \delta^i E_t u(C_{t+1+i})\right] \\ &= \max_{C_t}[E_t u(C_t)] + \delta V_{t+1}(W_{t+1}). \end{aligned}$$

Therefore, we can derive the Euler equation by assuming the stationary value function $V_t = V_{t+1} = V$. Brandt (1999) analyzed this approach for the finitely lived agent who maximizes discounted utility as

$$V_t(W_t) = \max_{\{C_{t+j}\}_{j=0}^{T-t}} \left[\sum_{j=0}^{T} \delta^j E_t(C_{t+j}^{1-\gamma}))\right].$$

In this case, since the investor consumes everything at the end of his life (T is fixed), he has to modify the stationary value function using the properties of CRRA utility. Our framework is less restrictive since the investor horizon I_n is a small infinity rather than a fixed lifetime T. Moreover it does not depend on t, so at any given time t, the investor still has to consider future portfolio and consumption choices up to $t + I_n$. In particular,

$$\begin{aligned} V_t(W_t, I_n) &= \max_{\{C_{t+j}\}_{j=0}^{I_n}} \left[\sum_{j=0}^{I_n} \delta^j E_t u(C_{t+j})\right] \\ &= \max_{C_t}[E_t u(C_t)] + \delta \max_{\{C_{t+j}\}_{j=1}^{I_n}} \left[\sum_{j=1}^{I_n} \delta^j E_t u(C_{t+j})\right], \\ &= \max_{C_t}[E_t u(C_t)] + \delta \max_{\{C_{t+1+i}\}_{i=0}^{I_n-1}} \left[\sum_{i=0}^{I_n-1} \delta^i E_t u(C_{t+1+i})\right], \quad \text{setting } j = i+1, \\ &= \max_{C_t}[E_t u(C_t)] + \delta \max_{\{C_{t+1+i}\}_{i=0}^{I_n}} \left[\sum_{i=0}^{I_n} \delta^i E_t u(C_{t+1+i})\right] - \delta^{I_n} \max_{C_{t+I_n+1}} E_t u(C_{t+I_n+1}) \\ &= \max_{C_t}[E_t u(C_t)] + \delta \max_{\{C_{t+1+i}\}_{i=0}^{I_n}} \left[\sum_{i=0}^{I_n} \delta^i E_t u(C_{t+1+i})\right] + O_p(\delta^{I_n}) \\ &= \max_{C_t}[E_t u(C_t)] + \delta \max_{\{C_{t+1+i}\}_{i=0}^{I_n}} \left[\sum_{i=0}^{I_n} \delta^i E_t u(C_{t+1+i})\right] + o(1), \end{aligned}$$

since the discount rate δ is fixed (unlike ρ_n) and less than unity and $I_n \to \infty$. We can restrict the support of C by using the budget constraint so that $\max_{C_{t+I_n+1}} E_t u(C_{t++I_n+1})$ is bounded. Therefore, the conventional Bellman equation holds approximately up to an error order $O_p(\delta^{I_n})$, which is exponentially negligible for large n. That is

$$V_t(W_t, I_n) \simeq \max_{C_t} [E_t u(C_t)] + \delta V_{t+1}(W_{t+1}, I_n).$$

This argument allows us to use conventional dynamic programming arguments to derive the Euler equation (3.3). Using the same arguments we can also justify the use of general recursive EZ utility with a distant investor horizon I_n.

7.9 Derivation of the Epstein-Zin Euler equations

Equations (10), (11) and (12) in Epstein and Zin (1991, page 268, hereafter EZ) do not seem to be correct, although the final results (13), (15) and (16) in that paper are correct. In what follows we derive the Euler equation (3.3) with appropriate modifications of (10), (11) and (12) in EZ. We have the following value function:

$$V_t = \left\{(1-\delta)C_t^{\frac{1-\gamma}{\varphi}} + \delta(E_t V_{t+1}^{1-\gamma})^{\frac{1}{\varphi}}\right\}^{\frac{\varphi}{1-\gamma}} = \left\{(1-\delta)C_t^{1-1/\psi} + \delta(E_t V_{t+1}^{1-\gamma})^{\frac{1-1/\psi}{1-\gamma}}\right\}^{\frac{1}{1-1/\psi}}. \quad (7.30)$$

For the time being, we redefine parameters to simplify the exposition, as in EZ. Let $1 - 1/\psi = \rho$, $1 - \gamma = \alpha$, and $\alpha \neq 0, \rho \neq 0$. We write the optimization problem again as

$$\begin{aligned} V(W_t, \Omega_t) &= \max_{C_t, \omega_t} \left\{(1-\delta)C_t^{\rho} + \delta(E_t V(W_{t+1}, \Omega_{t+1})^{\alpha})^{\frac{\rho}{\alpha}}\right\}^{\frac{1}{\rho}} \quad (7.31) \\ W_{t+1} &= (W_t - C_t)\, \omega_t^T \tilde{R}_{t+1},\ \omega_t^T \mathbf{1} = 1 \end{aligned}$$

where $\omega_t = (\omega_{1t}, ..., \omega_{Nt})$ represents the N-vector of portfolio weights and Ω_t is the information set. In our earlier notation, $\omega_t^T \tilde{R}_{t+1} = (1 + R_{m,t+1}) = \tilde{R}_{m,t+1}$ is a market portfolio return with the optimal choice of ω_t.

By virtue of the homogeneous value function, both the (optimized) value function and consumption will be linear in wealth. Specifically,

$$V(W_t, \Omega_t) = \phi_t W_t,\ C_t = \Psi_t W_t. \quad (7.32)$$

First, consider the optimal choice of C_t given $\tilde{R}_{m,t+1}$. Then, by the first order condition of (7.31) with respect to C_t, and using (7.32), we have

$$(1-\delta)C_t^{\rho-1} = \delta\,(W_t - C_t)^{\rho-1}\,\mu^{*\rho}$$

where $\mu^{*\rho} = E_t(\phi_{t+1}\tilde{R}_{m,t+1})^{\alpha})^{\frac{\rho}{\alpha}}$. Using $C_t = \Psi_t W_t$,

$$(1-\delta)\Psi_t^{\rho-1} = \delta\,(1-\Psi_t)^{\rho-1}\,\mu^{*\rho}, \quad (7.33)$$

and therefore

$$\mu^{*\rho} = \left(\frac{1-\delta}{\delta}\right)\left(\frac{\Psi_t}{1-\Psi_t}\right)^{\rho-1}.$$

Using this expression and $C_t = \Psi_t W_t$ in (7.31) gives

$$\begin{aligned}
\phi_t W_t &= \left[(1-\delta)\left(\Psi_t W_t\right)^{\rho} + (1-\delta)\left(\frac{\Psi_t}{1-\Psi_t}\right)^{\rho-1}\left(1-\Psi_t\right)^{\rho} W_t^{\rho}\right]^{\frac{1}{\rho}}, \\
\phi_t &= \left[(1-\delta)\left(\Psi_t\right)^{\rho} + (1-\delta)\left(\Psi_t\right)^{\rho-1}\left(1-\Psi_t\right)\right]^{\frac{1}{\rho}},
\end{aligned}$$

so that

$$\phi_t = \left[(1-\delta)\left(\frac{C_t}{W_t}\right)^{\rho-1}\right]^{\frac{1}{\rho}}. \tag{7.34}$$

Note that (7.33) is

$$\begin{aligned}
(1-\delta)\Psi_t^{\rho-1} &= \delta\left(1-\Psi_t\right)^{\rho-1} E_t(\phi_{t+1}\tilde{R}_{m,t+1})^{\alpha})^{\frac{\rho}{\alpha}} \\
&= \delta\left(1-\Psi_t\right)^{\rho-1} E_t\left[(1-\delta)^{\frac{\alpha}{\rho}}\left(\frac{C_{t+1}}{W_{t+1}}\right)^{(\rho-1)\frac{\alpha}{\rho}} \tilde{R}_{m,t+1}{}^{\alpha}\right]^{\frac{\rho}{\alpha}}.
\end{aligned}$$

Using the budget constraint $W_{t+1} = (W_t - C_t)\,\tilde{R}_{m,t+1}$, this expression becomes

$$E_t\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{\rho-1}\tilde{R}_{m,t+1}\right]^{\frac{\alpha}{\rho}} = 1. \tag{7.35}$$

Next, maximizing (7.31) with respect ω_t is equivalent to

$$\max_{\omega_t} E_t\left[\left(\frac{C_{t+1}}{(A_t - C_t)\tilde{R}_{m,t+1}}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(\omega_t^T\tilde{R}_{t+1}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \quad \text{such that } \omega_t^T\mathbf{1} = 1,$$

or, equivalently, since A_t and C_t are known at time t, in Lagrangian form

$$\max_{\omega_t,\lambda_t} E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(\omega_t^T\tilde{R}_{t+1}\right)^{\frac{\alpha}{\rho}}\right] - \lambda_t(\omega_t^T\mathbf{1} - 1).$$

Therefore, the first order condition for any asset $i \neq j \in \{1, ..., N\}$, (multiplied by a constant $\delta^{\frac{\alpha}{\rho}}$, to use (7.35)) is

$$\delta^{\frac{\alpha}{\rho}} E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(\tilde{R}_{m,t+1}\right)^{\frac{\alpha}{\rho}-1}\tilde{R}_{i,t+1}\right] = \delta^{\frac{\alpha}{\rho}} E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(\tilde{R}_{m,t+1}\right)^{\frac{\alpha}{\rho}-1}\tilde{R}_{j,t+1}\right]. \tag{7.36}$$

Multiplying by ω_{it} and summing over i gives, with (7.35),

$$E_t\left[\delta^{\frac{\alpha}{\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(\tilde{R}_{m,t+1}\right)^{\frac{\alpha}{\rho}}\right]=1=E_t\left[\delta^{\frac{\alpha}{\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(\tilde{R}_{m,t+1}\right)^{\frac{\alpha}{\rho}-1}\tilde{R}_{j,t+1}\right](\omega_t^T\mathbf{1}).$$

Therefore,

$$\begin{aligned} & E_t\left[\delta^{\frac{\alpha}{\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\alpha}{\rho}(\rho-1)}\left(\tilde{R}_{m,t+1}\right)^{\frac{\alpha}{\rho}-1}\tilde{R}_{j,t+1}\right] \\ = \quad & E_t\left[\left\{\delta(\frac{C_{t+1}}{C_t})^{-\frac{1}{\psi}}\right\}^{\varphi}\left\{\frac{1}{1+R_{m,t+1}}\right\}^{1-\varphi}(1+R_{j,t+1})\right]=1. \end{aligned}$$

which is (3.3).

7.10 Empirical results with PSY bubble dating

We report the results for the utility parameter estimation based on PSY bubble dating methods, which confirms the robustness of the empirical results. The empirical magnitude of bubble risk contribution (Section 5.4) based on the new estimation is essentially same to the one with PWY (e.g., the corresponding new contribution to equity premium is 1.1523 % hence 0.03% lower than the one reported in Table 2).

Table 1A: GMM Estimation Results for the Utility Parameters
Based on PSY Bubble Dating

	GMM Parameter Estimates			
Parameters	Estimates	Standard errors	t-stat	p-val
δ	0.9628	0.0155	62.0191	0
φ	-0.6859	0.1298	-5.855	0
$1-\varphi$	1.6859			0
ψ	2.3980	16.3875	0.1463	0.8838
γ (indirect)	1.3999			

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