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How to describe objects?

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How to Describe Objects? *

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Abstract

This paper addresses the problem of randomly allocating n indivisible objects to n agents where each object can be evaluated according to a set of characteristics. The planner chooses a subset of characteristics and a ranking of them. Then she describes each object as a list of values according to the ranking of those chosen characteristics. Being informed of such a description, each agent figures out her preference that is lexicographically separable according to the characteristics chosen and ranked by the planner. Hence a description of the objects induces a collection of admissible preferences. We call a description good if it induces a preference domain that admits an sd-strategy-proof, sd-efficient, and equal-treatment-of-equals rule.

When problem size n satisfies two technical assumptions, a description is good if and only if it is a binary tree, i.e., for each feasible combination of values of the top-t ranked characteristics, the following-up characteristic takes at most two feasible values.[1] In addition, whenever the description is a binary tree, the probabilistic serial rule (Bogomolnaia and Moulin (2001)) satisfies all three axioms.

Keywords: Random assignment; sd-strategy-proofness; sd-efficiency; equal treatment of equals; lexicographically separable preferences;

JEL Classification: C78, D71.

1 Introduction

A random assignment problem (Bogomolnaia and Moulin (2001)) deals with the situation where n indivisible objects are to be allocated to n agents, each agent receiving exactly one object. Each agent reports a strict preference on objects to a planner and then the planner assigns a lottery to each agent according to some prescribed (random assignment) rule. Examples of assignment problems include allocating houses to residents (Shapley and Scarf (1974)), tasks

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[1] I provide the proof subject to two technical assumptions in Appendix A. Although I can not verify these two assumptions analytically, I conjecture them to be true. In addition, I provide Matlab codes to verify them given specific n and I have verified the assumptions with these codes for all the cases where $n \leqslant 1000$.

to workers (Hylland and Zeckhauser (1979)), and college seats to applicants (Gale and Shapley (1962)).

We are interested in designing a satisfactory random assignment rule to allocate the objects. Particularly we want the rule to attain some nice properties. Since agents are reporting ordinal preferences on objects but assigned lotteries on the objects, we need to extend the ordinal preferences on objects to preferences on lotteries before evaluating the lotteries. A standard approach is to adopt the *stochastic dominance extension* (or sd-extension), which declares that a lottery is at least as good as another if the former (first-order) stochastically dominates the latter according to the ordinal preference on objects. Under the assumption that agents are expected utility maximizers, the sd-extension is equivalent to saying that an agent prefers a lottery to another if the former delivers an expected utility as high as the amount delivered by the latter with whichever the Bernoulli utility that represents the agent's preference on objects. In this sense, sd-extension can be seen as a cautious approach.

With sd-extension, properties on rules can be defined. The first important property is sd-strategy-proofness[2], which requires that reporting the true preference always leads to a lottery at least as good as any lottery delivered by reporting a false preference. A classical rule satisfying sd-strategy-proofness is the random serial dictatorship rule (Abdulkadiroğlu and Sönmez (1998)). However, the random serial dictatorship rule is not sd-efficient (Bogomolnaia and Moulin (2001)), which is the second property we require on the rule. A rule is sd-efficient if it always specifies a random assignment which can not be Pareto improved. Bogomolnaia and Moulin (2001) characterizes an sd-efficient rule to be generated by a simultaneous eating algorithm. Within these sd-efficient rules, one is of special interest since it satisfies a basic fairness property: equal treatment of equals which requires that whenever two agents report the same preference they receive the same lottery. This rule is generated by the simultaneous eating algorithm with uniform speed and called the probabilistic serial (PS) rule. We call a rule satisfying these three properties an acceptable rule.

Unfortunately according to Bogomolnaia and Moulin (2001), when all strict preferences are admissible and $n > 3$, there does not exist an acceptable rule.[3] Further more, this impossibility is strengthened progressively by a series of works. Kasajima (2013) proves the impossibility on the single-peaked domains. Chang and Chun (2016) prove the impossibility on the single-peaked domains when people have a common peak. Liu and Zeng (2016) prove the impossibility on any domain which involves a structure what they call elevating property. In addition, Liu and Zeng (2016) propose a class of domains, restricted tier domains, on which the PS rule is sd-strategy-proof and hence acceptable by avoiding the structure leading to impossibility. Liu (2016) then defines another class of preference domains, sequentially dichotomous domains, which are much larger and more flexible than restricted tier domains. It is shown that on a sequentially dichotomous domain the PS is sd-strategy-proof and hence acceptable.

From these results, it is the preference domain that constrains the scope for designing an acceptable rule. In theory, we simply assume what preferences are admissible and a larger

[2]Henceforth, we add prefix "sd-" to emphasize that the corresponding property is in ex-ante sense and it's established with respect to the notion of stochastic dominance extension.

[3]When $n \leqslant 3$ the random serial dictatorship rule is good. Throughout the paper we deal with the situation where $n > 3$.

domain is preferred. It is interesting to ask, in reality, how might preference restrictions arise and what is implies for the scope of designing acceptable rules. A first observation is that, in reality, the way people figure out their preferences on the objects is fundamentally affected by the way these objects are described. Since in reality, when people are required to submit their preferences on the objects, the information they have is usually a description of the objects. Take house allocation as an example, people are usually required to express their preferences on houses before they can really live in the houses and consume the housing service. Rather, the information they have for them to figure out their preferences is usually a description of the houses, which is usually provided by the authority who organize the allocation. Take the task allocation as another example, it is impossible that workers express their preferences on tasks after they have experienced every task. Rather, the information they have for them to figure out their preferences is usually a description of the tasks, provided by the manager or their team leader.

Hence let's take a closer look on the descriptions of objects in reality and the way descriptions affect agents' formulation of their preferences on the objects. A frequently seen description in reality is in the following format:

Table 1: A typical description of objects in reality

Characteristic \ House	#1	#2	#3	#4	#5	#6
1. Type (no. of rooms)	2-Room	2-Room	3-Room	3-Room	4-Room	5-Room
2. floor area (approximately in m^2)	30	35	60	65	90	110
3. no. of bedrooms	1	2	3	3	4	3
4. no. of bathrooms	1	2	2	2	3	2

- Source: Website of Housing Development Board of Singapore. See Flat Type.

That is, each object, a house in this setup, is described as a combination of various characteristic values. Notice that the characteristics presented are deliberately chosen from a much larger set of characteristics. Essentially a house can be evaluated on many dimensions, for example how long does it take to the nearest subway station, which floor the house is on, how many public primary schools those are in the community, etc. In addition, even the characteristics chosen here can be expressed in different ways. For example, the floor area may be expressed as a series of binary choices: is the area larger than 30, 40, 50, and so on.

The reason we need to take such a close look on the descriptions of objects is as we mentioned, different descriptions induce people to formulate their preferences in different ways. Observing that objects are described as combinations of various characteristic values, we make the central behavioral assumption in this paper that *people figure out their preferences on objects that are lexicographically separable with respect to the given description.* As illustrated by the above table, a typical resident tends to compare a pair of houses first by their type and then floor area if they are of the same type and then number of bedrooms if they have the same values of both type and area, and so on.

Here are two arguments that support this behavioral assumption. First, in reality sometimes people are required to submit their marginal preferences immediately when they are informed of the characteristic values in a sequential way. For example, a frequently observed practice is as follows. First they choose a specific type on a webpage and then they are redirected to another webpage where they choose a floor area from a collection of admissible choices which depend on which type they choose. After they choose a floor area, they are led to a third page to choose a number of bedrooms and so on until the end of the characteristic list.

Second, the set of feasible combinations of characteristic values is usually very sparse relative to the whole Cartesian product. For example, in Table 1, the whole Cartesian product has $4 \times 6 \times 4 \times 3 = 288$ elements. However, the feasible set has only 6 combinations. In other words, the characteristics are heavily interdependent. Such interdependence makes our behavioral assumption not as restrictive as it appears. For example, according to Table 1, knowing that a house with more rooms always has a larger area, there is not much difference in whether a person compares two houses according to first the number of rooms and then area or first the area and then the number of rooms.

According to the behavioral assumption, by choosing a specific description of the objects, i.e., a subset of characteristics and a ranking of them, the planner actually imposes a preference restriction. In this paper, we model the situation as follows. The set of available objects is a subset of the Cartesian product of a finite set of characteristics. The planner chooses a specific description, i.e., a subset of the characteristics and a ranking of them. Facing the description provided by the planner, the agents report lexicographically separable preferences. Then the planner assigns a lottery to each agent according to a prescribed rule.

Within this setting, we investigate how choices of descriptions affect the scope for designing an acceptable rule. Specifically we ask the following three questions one by one.

1. Given an arbitrary object set, does each possible description induces a preference domain on which an acceptable rule exists?

2. If the answer to the above question is negative, what characterizes a good description so that an acceptable rule exists on the induced domain?

3. Given a good description, what allocation rule should we use?

Let us examine the questions with a specific object set illustrated in Table 2. That is, each object can be evaluated according to each of three characteristics, $c = \{1, 2\}$ $c' = \{a, b, c\}$ and $c'' = \{x, y\}$.

Table 2: An object set

Characteristic \ Object	o_1	o_2	o_3	o_4	o_5
c	1	1	1	2	2
c'	a	b	c	a	b
c''	x	y	y	x	y

The answer to the first question is obviously negative. Consider the description such that c and c' are chosen and c is ranked above c', illustrated by Table 3.

Table 3: A bad description

Characteristic \ Object	o_1	o_2	o_3	o_4	o_5
1: c	1	1	1	2	2
2: c'	a	b	c	a	b

Then according to the behavioral assumption, the following three preferences are admissible.

$$\begin{array}{ccccccccc} o_1 & \succ & o_3 & \succ & o_2 & \succ & o_4 & \succ & o_5 \\ o_1 & \succ' & o_2 & \succ' & o_3 & \succ' & o_4 & \succ' & o_5 \\ o_2 & \succ'' & o_1 & \succ'' & o_3 & \succ'' & o_4 & \succ'' & o_5 \end{array}$$

This domain exhibits the elevating structure by Liu and Zeng (2016) and hence implies nonexistence of an acceptable rule. We borrow a table from Liu and Zeng (2016) to exhibit the elevating structure.

Ranking:			k		$k+1$		$k+2$	
$\bar{P}_i$:	$\underbrace{\cdots\cdots\cdots}_{B(\bar{P}_i,a)}$	$\succ$	a	$\succ$	c	$\succ$	b	$\succ \cdots\cdots\cdots$
	$\shortparallel$							
P_i:	$\overbrace{\underbrace{\cdots\cdots\cdots}_{B(P_i,a)}}^{B(P_i,a)}$	$\succ$	a	$\succ$	b	$\succ$	c	$\succ \cdots\cdots\cdots$
	$\shortparallel$							
$\hat{P}_i$:	$\overbrace{\cdots\cdots\cdots}^{B(\hat{P}_i,b)}$	$\succ$	b	$\succ$	a	$\succ$	c	$\succ \cdots\cdots\cdots$

Table 4: The Local Elevating Property

We say a domain $\mathbb{D}$ exhibits the elevating structure if there are three preferences $\bar{P}_0, P_0, \hat{P}_0 \in \mathbb{D}$, three objects $a, b, c \in A$, and three adjacent positions $k, k+1, k+2$ such that (1) a, b, c take positions $k, k+1, k+2$ in all three preferences, (2) a and b take positions k and $k+1$ in two preferences, and (3) in the third preference, a takes position k and b takes position $k+2$. For the formal definition of the elevating structure and impossibility, please refer to Liu and Zeng (2016).

From the description illustrated by Table 3, it's evident that a necessary condition for a description to induce a domain admitting an acceptable rule is that the last characteristic can not take more than two feasible values, conditional on the previous characteristics.

A simple way to meet this necessary condition is to reverse the ranking of c and c', as illustrated by Table 5.

Table 5: Another bad description

Characteristic \ Object	o_1	o_4	o_2	o_5	o_3
1: c'	a	a	b	b	c
2: c	1	2	1	2	1

Now the induced domain will not exhibit the structure illustrated by $\succ$, $\succ'$, and $\succ''$ above. Does this domain admits an acceptable rule?

To answer the question, we strengthen the impossibility of Liu and Zeng (2016) in a manner so that, if n satisfies two technical assumptions (Assumptions 1 and 2 in Appendix A), then the three preferences forming the structure below imply nonexistence of an acceptable rule.

$$\begin{array}{ccccccccc} E & \succ & B & \succ & D & \succ & C & \succ & F \\ E & \succ' & B & \succ' & C & \succ' & D & \succ' & F \\ E & \succ'' & C & \succ'' & B & \succ'' & D & \succ'' & F \end{array}$$

Specifically, let B, C, D be three nonempty blocks of objects and $\succ, \succ', \succ''$ three admissible preferences such that (1) B, C, D take consecutive positions in three preferences, (2) B and C take the first two positions in two preferences, and (3) in the third preference, B takes the first position and C takes the third position. E is the common upper contour set that can be empty. Notice that the sizes of three blocks are arbitrary and the ranking of objects within each block is allowed to be arbitrary across preferences.[4]

According to this impossibility, we know that the description illustrated by Table 5, i.e., c and c' are chosen and c' is ranked above c, admits no acceptable rule since the induced domain includes three preferences exhibiting the structure that leads to impossibility, as illustrated below.

$$\begin{array}{ccccc} \{o_1, o_4\} & \succ & \{o_3\} & \succ & \{o_2, o_5\} \\ \{o_1, o_4\} & \succ & \{o_2, o_5\} & \succ & \{o_3\} \\ \{o_2, o_5\} & \succ & \{o_1, o_4\} & \succ & \{o_3\} \end{array}$$

What's more, according to this impossibility, we have a stronger necessary condition on descriptions that induce domains admitting an acceptable rule. The condition is that, for each feasible combination of values of the top-t ranked characteristics, the following-up characteristic can take at most two feasible values. Whenever this condition is violated, the induced domain include three preferences that exhibit the block elevating structure that leads to impossibility.

For the object set illustrated in Table 2, a description that satisfies the necessary condition is as follows: all three characteristics are chosen and c ranked the first, c'' the second, and c' the last. This description is illustrated by Table 6 below. It's evident that the necessary condition is satisfied: the first characteristic c takes two values 1 and 2; conditional on c's value the

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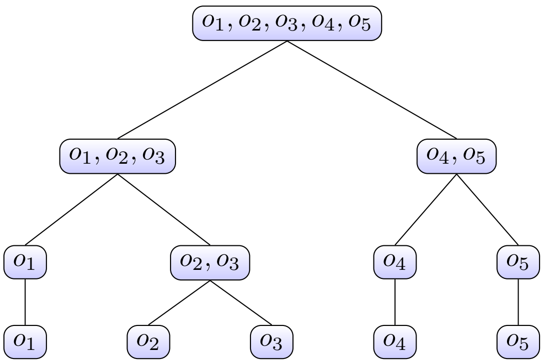
second characteristic c'' takes two values x and y; conditional on a combination of the first two characteristics, the last characteristic c' takes either one value or two values.

Table 6: A good description

Characteristic \ Object	o_1	o_2	o_3	o_4	o_5
1: c	1	1	1	2	2
2: c''	x	y	y	x	y
3: c'	a	b	c	a	b

We call a description satisfying this necessary condition a binary tree. The reason we name it in this way can be seen easily from the following representation of the above table. Particularly, according to the first characteristic c, objects are divided into two subsets, i.e., $\{o_1, o_2, o_3\}$ and $\{o_4, o_5\}$. Then according to the second characteristic, $\{o_1, o_2, o_3\}$ breaks into $\{o_1\}$ and $\{o_2, o_3\}$ and $\{o_4, o_5\}$ breaks into $\{o_4\}$ and o_5. Finally, according to the last characteristic, $\{o_2, o_3\}$ breaks into $\{o_2\}$ and $\{o_3\}$.



However since the impossibility justifies the binary tree as only a necessary condition, we still don't know whether there is an acceptable rule on the domain induced by the description in Table 6. To justify the condition as also a sufficient condition, we verify that the PS rule is sd-strategy-proof on the domain induced by a binary tree. Hence a description induces a preference domain on which there is an acceptable rule if and only if it is a binary tree, subject to two technical assumptions. To show sd-strategy-proofness of the PS rule on the domain induced by a binary tree, we utilize a result in Liu (2016), which shows that the PS rule is sd-strategy-proof on any sequentially dichotomous domain. It then suffices to show that as long as the description is a binary tree, the induced domain is covered by a sequentially dichotomous domain.

The remainder of the paper is organized as follows. The next section presents the model, both the classical random assignment model and our modeling of descriptions of objects. The third section presents the results and the fourth section concludes. Omitted proofs are gathered in the appendix.

2 Model and Definitions

The first subsection clarifies notations and definitions of the classic random assignment model and the second subsection defines the descriptions of objects and the domains induced by descriptions.

2.1 The Random Assignment Model

Let $I \equiv \{1, \cdots, n\}$ be the set of agents and A the set of objects. We assume $|I| = |A| = n \geqslant 4$. Each agent i is equipped with a strict preference P_i on A, i.e., a complete, transitive and antisymmetric binary relation on A. Let $\mathbb{P}$ denote the set of *all* strict preferences over A. The set of admissible preferences is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the *preference domain*. Thus, $\mathbb{P}$ is referred to as the universal domain. Given $P_i \in \mathbb{D}$ and $a \in A$, let $r_k(P_i)$, $k = 1, \ldots, n$, denote the k-th ranked object according to P_i. A *preference profile* $P \equiv (P_1, \ldots, P_n) \equiv (P_i, P_{-i}) \in \mathbb{D}^n$ is an n-tuple of admissible preferences.

Let $\Delta(A)$ denote the set of lotteries, or probability distributions, over A. Given $\lambda \in \Delta(A)$, λ_a denotes the probability of getting a. A (random) **assignment** is a bi-stochastic matrix $L \equiv [L_{ia}]_{i\in I, a\in A}$, namely a non-negative square matrix whose elements in each row and each column sum to unity, i.e., $L_{ia} \geqslant 0$ for all $i \in I$ and $a \in A$, $\sum_{a\in A} L_{ia} = 1$ for all $i \in I$, and $\sum_{i\in I} L_{ia} = 1$ for all $a \in A$. Evidently, in a bi-stochastic matrix L, each row is a lottery, i.e., $L_i \in \Delta(A)$ for all $i \in I$. Let $\mathcal{L}$ denote the set of all bi-stochastic matrices. Agents assess lotteries according to (first-order) stochastic dominance. Given $P_i \in \mathbb{D}$ and lotteries $\lambda, \lambda' \in \Delta(A)$, λ *stochastically dominates* λ' according to P_i, denoted $\lambda \;\; P_i^{sd} \;\; \lambda'$, if $\sum_{l=1}^{k} \lambda_{r_l(P_i)} \geqslant \sum_{l=1}^{k} \lambda'_{r_l(P_i)}$ for all $1 \leqslant k \leqslant n$. Analogously, given $P \in \mathbb{D}^n$, we say an assignment L stochastically dominates L' according to P, denoted $L \;\; P^{sd} \;\; L'$, if $L_i \;\; P_i^{sd} \;\; L'_i$ for all $i \in I$.

A *rule* is a mapping $\varphi : \mathbb{D}^n \to \mathcal{L}$. Given $P \in \mathbb{D}^n$, $\varphi_{ia}(P)$ denotes the probability of agent i receiving object a, and thus $\varphi_i(P)$ denotes the lottery assigned to agent i. Specifically the probabilistic serial rule is denoted as $PS : \mathbb{D}^n \to \mathcal{L}$.

We impose three axioms on a rule. First, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-efficient** if, for all $P \in \mathbb{D}^n$ and all $L' \in \mathcal{L}$, $[L' \;\; P^{sd} \;\; \varphi(P)] \Rightarrow [L' = \varphi(P)]$. Second, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-strategy-proof** if for all $i \in I$, $P_i, P'_i \in \mathbb{D}$, and $P_{-i} \in \mathbb{D}^{n-1}$, $\varphi_i(P_i, P_{-i}) \;\; P_i^{sd} \;\; \varphi_i(P'_i, P_{-i})$. Last, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ satisfies **equal treatment of equals** if for all $P \in \mathbb{D}^n$, $[P_i = P_j] \Rightarrow [\varphi_i(P) = \varphi_j(P)]$. In summary, we call a rule **acceptable** if it satisfies all three axioms above.

2.2 Descriptions of Objects and Induced Preference Domains

Let $\mathcal{C}$ denote the collection of characteristics, according to each of which an object can be evaluated. Hence the object set is a subset of a Cartesian product $A \subset \prod_{c\in\mathcal{C}} A_c$ where A_c is the collection of all possible values of characteristic c. For each object $a \in A$ and each characteristic $c \in \mathcal{C}$, we denote object's value of characteristic c by a_c. Without loss of generality, we assume no unused characteristic value, i.e., for each $c \in \mathcal{C}$ and $v \in A_c$ there is an object $a \in A$ such that $a_c = v$.

Given $A \subset \prod_{c \in \mathcal{C}} A_c$, the planner chooses a subset of the characteristics and specifies a ranking of these chosen characteristics. We call the pair of the chosen subset and the ranking of characteristics a description of the objects. Formally it is defined as below.

Definition 1. *A* ***description*** *of the object set $A \subset \prod_{c \in \mathcal{C}} A_c$ is a pair (C, σ) where*

- *$C \subset \mathcal{C}$ is a subset of characteristics such that for each pair of distinct objects $x, y \in A$ there is a characteristic $c \in C$ that gives $x_c \neq y_c$, and*
- *σ is an one-to-one mapping $\sigma : C \to \{1, \cdots, |C|\}$.*

The condition imposed on the chosen set of characteristics is requiring that the chosen characteristics be sufficiently informative so that an agent can differentiate each object from the others according to the description given by the planner.

Given a description (C, σ), an $t \in \{1, \cdots, |C|\}$, and an object $a \in A$, let $a_t^\sigma \equiv a_{\sigma^{-1}(t)}$ be a's value of the characteristic which is t-th ranked according to σ and let $A_t^\sigma \equiv A_{\sigma^{-1}(t)}$ be the admissible value set of the t-th ranked characteristic. In addition, for any $t \in \{2, \cdots, |C|\}$ and any combination of values of the top-$(t-1)$ ranked characteristics $(v_1, \cdots, v_{t-1}) \in \prod_{\tau=1}^{t-1} A_\tau^\sigma$, let $A_t^\sigma|(v_1, \cdots, v_{t-1}) \equiv \{v_t \in A_t^\sigma | \exists a \in A \text{ s.t. } (a_1^\sigma, \cdots, a_{t-1}^\sigma, a_t^\sigma) = (v_1, \cdots, v_{t-1}, v_t)\}$. For notational convenience, we write $A_1^\sigma | v_0 \equiv A_1^\sigma$.

After choosing a description (C, σ), the planner disclose the information of objects to the agents. Facing the chosen description of objects, an agent compares each pair of objects lexicographically according to the characteristics in C and the ranking specified by σ. Formally the preference and the collection of these preferences are defined as below.

Definition 2. *Given a description (C, σ) of $A \subset \prod_{c \in C} A_c$, a preference $P_0 \in \mathbb{P}$ is* ***lexicographically separable with respect to*** *(C, σ) if there is a strict preference P_0^c on each A_c such that $x \; P_0 \; y$ if and only if there exists $t \in \{1, \cdots, |C|\}$ s.t. $\left(x_1^\sigma, \cdots, x_{t-1}^\sigma\right) = \left(y_1^\sigma, \cdots, y_{t-1}^\sigma\right)$ and $x_t^\sigma \; P_0^{\sigma^{-1}(t)} \; y_t^\sigma$.*

In addition, let ***the domain induced by the description*** *(C, σ) be the collection of* all *lexicographically separable preferences with respect to (C, σ) and denoted as $\mathbb{D}_{(C,\sigma)}$.*

The preferences on A_c's that spell a lexicographically separable preference are called marginal preferences. We now present the example discussed in the introduction in the language just defined.

Example 1. *Consider the object set illustrated by Table 2. Let $\mathcal{C} \equiv \{c, c', c''\}$, $A_c \equiv \{1, 2\}$, $A_{c'} \equiv \{a, b, c\}$, and $A_{c''} \equiv \{x, y\}$. Now the object set in Table 2 can be expressed as a subset of $\prod_{c \in \mathcal{C}} A_c$.*

Consider the description illustrated in Table 5. The characteristic subset chosen is $C \equiv \{c, c'\}$ and the ranking is $\sigma(c') = 1$ and $\sigma(c) = 2$. The domain induced by description (C, σ) is such one that includes P_1 to P_{24} and is described as follows.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_3	o_3	o_3	o_3
o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_1	o_4	o_1	o_4
o_3	o_3	o_3	o_3	o_2	o_2	o_5	o_5	o_4	o_1	o_4	o_1
o_2	o_2	o_5	o_5	o_5	o_5	o_2	o_2	o_2	o_2	o_5	o_5
o_5	o_5	o_2	o_2	o_3	o_3	o_3	o_3	o_5	o_5	o_2	o_2

P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}	P_{23}	P_{24}
o_3	o_3	o_3	o_3	o_2	o_2	o_5	o_5	o_2	o_2	o_5	o_5
o_2	o_2	o_5	o_5	o_5	o_5	o_2	o_2	o_5	o_5	o_2	o_2
o_5	o_5	o_2	o_2	o_3	o_3	o_3	o_3	o_1	o_4	o_1	o_4
o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_4	o_1	o_4	o_1
o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_3	o_3	o_3	o_3

Consider another description, illustrated in Table 6. Now the description is $(\bar{C}, \bar{\sigma})$ where $\bar{C} \equiv \{c, c', c''\}$ and $\bar{\sigma}$ is such that $\bar{\sigma}(c) = 1$, $\bar{\sigma}(c') = 3$, and $\bar{\sigma}(c'') = 2$. Then the domain induced by this description is such that includes all preferences $\bar{P}_1$ to $\bar{P}_{16}$ below.

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$	$\bar{P}_6$	$\bar{P}_7$	$\bar{P}_8$	$\bar{P}_9$	$\bar{P}_{10}$	$\bar{P}_{11}$	$\bar{P}_{12}$	$\bar{P}_{13}$	$\bar{P}_{14}$	$\bar{P}_{15}$	$\bar{P}_{16}$
o_1	o_1	o_1	o_1	o_2	o_3	o_2	o_3	o_4	o_5	o_4	o_5	o_4	o_5	o_4	o_5
o_2	o_3	o_2	o_3	o_3	o_2	o_3	o_2	o_5	o_4	o_5	o_4	o_5	o_4	o_5	o_4
o_3	o_2	o_3	o_2	o_1	o_1	o_1	o_1	o_1	o_1	o_1	o_1	o_2	o_2	o_3	o_3
o_4	o_4	o_5	o_5	o_4	o_4	o_5	o_5	o_2	o_2	o_3	o_3	o_3	o_3	o_2	o_2
o_5	o_5	o_4	o_4	o_5	o_5	o_4	o_4	o_3	o_3	o_2	o_2	o_1	o_1	o_1	o_1

3 Result

This subsection presents the following two results. Firstly, whenever the problem size satisfies the two technical assumptions 1 and 2, the objects should be described as a binary tree since this is the only way the induced domain admits an acceptable rule.

Theorem 1. *Let (C, σ) be a description and $\mathbb{D}_{(C,\sigma)}$ the corresponding induced domain. In addition, let n satisfy the Assumptions 1 and 2. If there is an sd-strategy-proof sd-efficient and equal-treatment-of-equals rule defined on $\mathbb{D}_{(C,\sigma)}$, then $\left| A_t^\sigma|(v_1, \cdots, v_{t-1}) \right| \leqslant 2$ for all $t \in \{1, \cdots, |C|\}$ and $(v_1, \cdots, v_{t-1}) \in \prod_{\tau=1}^{t-1} A_\tau^\sigma$, i.e., the description is a binary tree*

In order to prove Theorem 1, we show an impossibility result, which states that whenever a domain exhibits the ”block elevating” property and two technical assumptions are satisfied, there is no possibility of finding an acceptable rule. We first formally define the block elevating property and then the impossibility.

A domain $\mathbb{D}$ satisfies the **block elevating property** if there are three admissible preferences and three nonempty blocks such that the block (can be empty) ranked above these three blocks in all three preferences is the same, three blocks are ranked next to each other in all three preferences, one block is ranked last among the three blocks in two of the three preferences and the second among the three in the third preference; formally:

Definition 3. *A domain $\mathbb{D}$ satisfies the **block elevating property** if there are three preferences $\bar{P}_0, P_0, \hat{P}_0 \in \mathbb{D}$, three nonempty blocks $B, C, D \subset A$ and two blocks $E, F \subset A$ which can be empty such that $B \cup C \cup D \cup E \cup F = A$ and three preferences are as follows*

$$E \;\; \bar{P}_0 \;\; B \;\; \bar{P}_0 \;\; D \;\; \bar{P}_0 \;\; C \;\; \bar{P}_0 \;\; F$$
$$E \;\; P_0 \;\; B \;\; P_0 \;\; C \;\; P_0 \;\; D \;\; P_0 \;\; F$$
$$E \;\; \hat{P}_0 \;\; C \;\; \hat{P}_0 \;\; B \;\; \hat{P}_0 \;\; D \;\; \hat{P}_0 \;\; F$$

Table 7: The Block Elevating Property

The block elevating property is a generalization of the elevating property by Liu and Zeng (2016), which requires all three blocks to be singletons. The impossibility with respect to the block elevating property is as follows.

Proposition 1. *Let $\mathbb{D}$ be a domain satisfying block elevating property. If n satisfies Assumptions 1 and 2, then $\mathbb{D}$ admits no sd-strategy-proof, sd-efficient, and equal-treatment-of-equals rule.*

The proof of Proposition 1 is in Appendix B. The proof is by contradiction, i.e., suppose $\mathbb{D}$ satisfies block elevating property and admits an acceptable rule, then we specify a series of preference profiles consisting of only three preferences illustrated in Table 7. We characterize the random assignments of these profiles according to the three axioms. Finally a contradiction is identified.

The identification of the contradiction relies on two assumptions, each of which compares zero with an expression of a floor function[5], conditional on that the real number is strictly larger than the integer identified by the floor function for this real number. We are unable to verify the comparisons analytically. However, we provide two Matlab codes to verify, for a specific n, whether these two assumptions hold, and we have verified them to be true for all n no larger than 1000.

Proof of Theorem 1. In stead of showing it directly, we show its contrapositive statement. Let (C, σ) be a description and $\mathbb{D}_{(C,\sigma)}$ its induced domain, in addition let $t^* \in \{1, \cdots, |C|\}$ and $(v_1, \cdots, v_{t^*-1}) \in \prod_{\tau=1}^{t^*-1} A^\sigma_\tau$ be such that $\left| A^\sigma_{t^*} | (v_1, \cdots, v_{t^*-1}) \right| \geqslant 3$, we show that $\mathbb{D}_{(C,\sigma)}$ satisfies the block elevating property.

Pick any three values $u_{t^*}, u'_{t^*}, u''_{t^*} \in A^\sigma_{t^*}|(v_1, \cdots, v_{t^*-1})$ and let $B \equiv \{b \in A | (b^\sigma_1, \cdots, b^\sigma_{t^*-1}) = (v_1, \cdots, v_{t^*-1}) \text{ and } b^\sigma_{t^*} = u_{t^*}\}$, $C \equiv \{c \in A | (c^\sigma_1, \cdots, c^\sigma_{t^*-1}) = (v_1, \cdots, v_{t^*-1}) \text{ and } c^\sigma_{t^*} = u'_{t^*}\}$, and $D \equiv \{d \in A | (d^\sigma_1, \cdots, d^\sigma_{t^*-1}) = (v_1, \cdots, v_{t^*-1}) \text{ and } d^\sigma_{t^*} = u''_{t^*}\}$. Consider the marginal preferences $(\bar{P}^c_0)_{c \in C}$, $(P^c_0)_{c \in C}$, and $(\hat{P}^c_0)_{c \in C}$ such that

$$\begin{aligned}
&r_1(\bar{P}_0^{\sigma^{-1}(\tau)}) = r_1(P_0^{\sigma^{-1}(\tau)}) = r_1(\hat{P}_0^{\sigma^{-1}(\tau)}) = v_\tau && \text{for all } \tau \leqslant t^* - 1 \\
&u_{t^*} \; \bar{P}_0^{\sigma^{-1}(t^*)} \; u''_{t^*} \; \bar{P}_0^{\sigma^{-1}(t^*)} \; u'_{t^*} \; \bar{P}_0^{\sigma^{-1}(t^*)} \; v_{t^*} && \text{for all } v_{t^*} \in A^\sigma_{t^*} \backslash \{u_{t^*}, u'_{t^*}, u''_{t^*}\} \\
&u_{t^*} \; P_0^{\sigma^{-1}(t^*)} \; u'_{t^*} \; P_0^{\sigma^{-1}(t^*)} \; u''_{t^*} \; P_0^{\sigma^{-1}(t^*)} \; v_{t^*} && \text{for all } v_{t^*} \in A^\sigma_{t^*} \backslash \{u_{t^*}, u'_{t^*}, u''_{t^*}\} \\
&u'_{t^*} \; \hat{P}_0^{\sigma^{-1}(t^*)} \; u_{t^*} \; \hat{P}_0^{\sigma^{-1}(t^*)} \; u''_{t^*} \; \hat{P}_0^{\sigma^{-1}(t^*)} \; v_{t^*} && \text{for all } v_{t^*} \in A^\sigma_{t^*} \backslash \{u_{t^*}, u'_{t^*}, u''_{t^*}\}
\end{aligned}$$

[5] A floor function identifies for a real number the largest integer no larger than the real number itself.

It's evident that these marginal preferences give the following preferences $\bar{P}_0, P_0, \hat{P}_0 \in \mathbb{D}_{(C,\sigma)}$, which completes the proof.

$$\begin{array}{ccccccc} B & \bar{P}_0 & D & \bar{P}_0 & C & \bar{P}_0 & A\backslash(B \cup C \cup D) \\ B & P_0 & C & P_0 & D & P_0 & A\backslash(B \cup C \cup D) \\ C & \hat{P}_0 & B & \hat{P}_0 & D & \hat{P}_0 & A\backslash(B \cup C \cup D) \end{array}$$

□

Given Theorem 1, an interesting question arises: what rule can we use on a domain induced by a binary tree? The following result shows that the PS rule is *sd-strategy-proof* on the domain induced by a binary tree.

Theorem 2. *Let (C, σ) be a description and $\mathbb{D}_{(C,\sigma)}$ the corresponding induced domain. If (C, σ) is a binary tree, then the PS rule is sd-strategy-proof on $\mathbb{D}_{(C,\sigma)}$.*

The proof of Theorem 2 is in Appendix C.

We prove the theorem by showing that when the description is a binary tree, the induced domain is a sub-domain of a sequentially dichotomous domain. Hence Liu (2016) implies the desired conclusion.

4 Conclusion

The result in this paper suggests that, if the planner believes that agents report preferences that are lexicographically separable according to the ranking of the chosen characteristics, she should make her choice and the ranking of characteristics a binary tree, i.e., given any feasible values of the top-t ranked characteristics, the following up characteristic can take at most two feasible values. In addition, due to the fact that the PS rule is *sd-strategy-proof* and hence acceptable on the domain induced by a binary tree, she should use the PS rule to allocate the objects after agents report their preferences.[6]

However since I assume explicitly that the problem size n satisfies two technical assumptions, before following the above suggestions, the planner needs to check these two assumptions. Although I can not prove them to be true analytically, I conjecture that they are true. For applications, a planner can use the Matlab code I provide to check whether these two assumptions hold. In addition, if the problem size is smaller than 1000, I have already checked them to be true so the suggestion above can be adopted directly.

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[6]In addition to be acceptable, PS rule is favored also in the sense that it satisfies sd-envy-freeness, a fairness axiom stronger than *equal treatment of equals*.

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Appendix

A Two Technical Assumptions

Assumption 1.

$$\begin{aligned} f(m_1, m_2) \equiv & - n(m_1 + m_2) \left(\left[\frac{m_2 n}{m_1 + m_2} \right]_- \right)^2 \\ & + \left[(n^2 - n + 1)m_1 + (2n^2 - n)m_2 \right] \left(\left[\frac{m_2 n}{m_1 + m_2} \right]_- \right) - n^2(n-1)m_2 \leqslant 0 \end{aligned}$$

*for all positive integers m_1, m_2 such that $m_1 + m_2 < n$ and $\frac{m_2 n}{m_1 + m_2}$ is **not an integer**, where $[x]_-$ denotes for a real number the largest integer which is no greater than x.*

Assumption 2.

$$g(m_1, m_2, m_3) \equiv \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n - (\bar{n}_5 - 1)}) - (\bar{n}_5 - 1) \times \gamma(\bar{n}_5)}{n - (\bar{n}_5 + 1)} \neq \frac{m_3}{n}$$

*for all positive integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 \leqslant n$ and $\frac{m_2 n}{m_1+m_2}$ is **not an integer**.*

$$\text{where} \quad \bar{n}_5 = \left[\frac{m_2 n}{m_1 + m_2}\right]_- + 1,$$

$$\gamma(k) = \frac{\Gamma_1(k)m_1 + \Gamma_2(k)m_2 + \Gamma_3(k)m_3}{n[n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)]}$$

$$\text{and} \quad \Gamma_1(k) = 2(k-2)n^2 - 2(k^2-k-2)n + 2(k^2-k-2)$$

$$\Gamma_2(k) = -2n^3 + 4kn^2 - 2(k^2+k-1)n + 2(k^2-k)$$

$$\Gamma_3(k) = n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)$$

I attach the Matlab code which can be used to check both assumptions given a fixed n.

- Check $f(m_1, m_2) \leqslant 0$ for all positive integers m_1, m_2 such that $m_1 + m_2 < n$ and $\frac{m_2 n}{m_1+m_2}$ is **not an integer**.

```
clear;clc
syms n m1 m2 x
a=sym('-n*(m1+m2)');
b=sym('(n^2-n+1)*m1+(2*n^2-n)*m2');
c=sym('-n^2*(n-1)*m2');
eqn = a*x^2+b*x+c == 0;
solx = solve(eqn,x); %solve equation ax^2+b+c=0.

n=1000; % Fix an n

flag=1;
for m1=1:n-2
  for m2=1:(n-m1-1)
    fprintf('n=%d m1=%d m2=%d',[n m1 m2])
    fprintf('\n')
    x=floor(m2*n/(m1+m2));
    if x<m2*n/(m1+m2)
      if x<max(eval(solx)) && x>min(eval(solx))
        flag=0;
        fprintf('n=%d m1=%d m2=%d x_1^* x_2^* x',[n m1 m2 min(eval(solx))...,
          max(eval(solx)) floor(m2*n/(m1+m2)) m2*n/(m1+m2)])
        fprintf('\n')
        break
      end
    end
  end
  if flag==0
    break
  end
end
```

- Check $g(m_1, m_2, m_3) \neq \frac{m_3}{n}$ for all positive integers m_1, m_2, m_3 such that $m_1+m_2+m_3 \leqslant n$ and $\frac{m_2 n}{m_1+m_2}$ is **not an integer**.

```
n=1000; % Fix an n
for m1=1:n-2
  for m2=1:n-m1-1
    for m3=1:n-m1-m2
      fprintf('n=%d m1=%d m2=%d m3=%d',[n m1 m2 m3])
      fprintf('\n')
      if floor(m2*n/(m1+m2))<m2*n/(m1+m2)
        nbar5=floor(m2*n/(m1+m2))+1;
        A1=2*(nbar5 -2)*n^2-2*(nbar5^2-nbar5 -2)*n+2*(nbar5^2-nbar5 -2);
        A2=-2*n^3+4*nbar5*n^2-2*(nbar5^2+nbar5 -1)*n+2*(nbar5^2-nbar5);
        A3=n^4-2*(nbar5+1)*n^3+(nbar5^2+5*nbar5 -1)*n^2...,
          -(3*nbar5^2+nbar5 -2)*n+2*(nbar5^2-nbar5);
        gamma=(A1*m1+A2*m2+A3*m3)/(n^(n^4 -2*(nbar5+1)^n^3...,
          +(nbar5^2+5*nbar5 -1)*n^2-(3*nbar5^2+nbar5 -2)*n +2*(nbar5^2-nbar5)));
        m=m1+m2+m3;
        f=(m3-2*(m/n-m1/(n-(nbar5 -1)))-(nbar5 -1)*gamma)/(n-(nbar5+1));
        if abs(f - m3/n) < eps
          fprintf('n=%d m1=%d m2=%d m3=%d nbar5=%f gamma=%f f=%f'...,
            ,[n m1 m2 m3 nbar5 f])
          flag=0;
          break
        end
      end
    end
    if flag==0
      break
    end
end
  if flag==0
    break
  end
end
```

B Proof of Proposition 1

Let $E, B, C, D, F \subset A$ with $m_1 \equiv |B| \geqslant 1$, $m_2 \equiv |C| \geqslant 1$, $m_3 \equiv |D| \geqslant 1$. Let $m \equiv m_1 + m_2 + m_3$. In addition, given a real number x, $[x]_-$ denotes the largest integer which is smaller or equal to x. Finally given a random assignment L and a subset of objects $B \subset A$, we denote $L_{i,B} = \sum_{x \in B} L_{i,x}$.

Let $\mathbb{D} \equiv \{\bar{P}_i, P_i, \hat{P}_i\}$ where the preferences are from Table 7. To prove the theorem, it suffices to prove $\mathbb{D}$ admits no good rule. Suppose not, and let $\varphi : \mathbb{D}^n \longrightarrow \mathcal{L}$ be a good rule.

Lemma 1. *For any $P \in \mathbb{D}^n$, $\varphi_{i,B}(P) + \varphi_{i,C}(P) + \varphi_{i,D}(P) = \frac{m}{n}$ for all $i \in I$.*

This lemma can be proved by applying repeatedly *equal treatment of equals* and *sd-strategy-proofness*. The proof is standard and hence omitted.

Notice that, since $\frac{m_1 n}{m_1+m_2} + \frac{m_2 n}{m_1+m_2} = n$, it's either both $\frac{m_1 n}{m_1+m_2}$ and $\frac{m_2 n}{m_1+m_2}$ are integers or neither one of them is an integer. I'll show two contradictions, one for each case. When $\frac{m_2 n}{m_1+m_2}$ is an integer, the contradiction is identified. While Assumptions 1 and 2 are needed to identify the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is not an integer.

In the following ,I'll construct six groups of profiles and characterize the random assignments of B, C, D for each of these profiles. The contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is an integer can be found using profile groups I to IV. To find the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is not an integer, we need in addition profile groups V and VI.

Firstly I list all profiles.

Profile group I:

$$
\begin{aligned}
P^{1,0} &= (P_1, \cdots, P_n) \\
P^{1,1} &= (\hat{P}_1, P_2, \cdots, P_n) \\
&\vdots \\
P^{1,k} &= (\hat{P}_1, \cdots, \hat{P}_k, P_{k+1}, \cdots, P_n) \\
&\vdots \\
P^{1,\bar{n}_1} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_1}, P_{\bar{n}_1+1}, \cdots, P_n)
\end{aligned}
$$

where $\bar{n}_1 = \frac{m_2 n}{m_1+m_2}$ when $\frac{m_2 n}{m_1+m_2}$ is an integer and $\bar{n}_1 = \left[\frac{m_2 n}{m_1+m_2}\right]_-$ otherwise.

Profile group II:

$$
\begin{aligned}
P^{2,1} &= (P_1, \cdots, P_{n-1}, \bar{P}_n) \\
P^{2,2} &= (\hat{P}_1, P_2 \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{2,k} &= (\hat{P}_1, \cdots, \hat{P}_{k-1}, P_k, \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{2,\bar{n}_2} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_2-1}, P_{\bar{n}_2}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}
$$

where $\bar{n}_2 = \frac{m_2 n}{m_1+m_2}$ when $\frac{m_2 n}{m_1+m_2}$ is an integer and $\bar{n}_2 = \left[\frac{m_2 n}{m_1+m_2}\right]_- + 1$ otherwise.

Profile group III:

$$
\begin{aligned}
P^{3,0} &= (\hat{P}_1, \cdots, \hat{P}_n) \\
P^{3,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-1}, P_n) \\
&\vdots \\
P^{3,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_n) \\
&\vdots \\
P^{3,\bar{n}_3} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_3}, P_{n-\bar{n}_3+1}, \cdots, P_n)
\end{aligned}
$$

where $\bar{n}_3 = \frac{m_1 n}{m_1+m_2}$ when $\frac{m_1 n}{m_1+m_2}$ is an integer and $\bar{n}_3 = \left[\frac{m_1 n}{m_1+m_2}\right]_-$ otherwise.

Profile group IV:

$$\begin{aligned}
P^{4,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-1}, \bar{P}_n) \\
P^{4,2} &= (\hat{P}_1, \cdots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{4,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{4,\bar{n}_4} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_4}, P_{n-\bar{n}_4+1}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}$$

where $\bar{n}_4 = \frac{m_1 n}{m_1+m_2}$ when $\frac{m_1 n}{m_1+m_2}$ is an integer and $\bar{n}_4 = \left[\frac{m_1 n}{m_1+m_2}\right]_-$ otherwise.

Profile group V:

$$\begin{aligned}
P^{5,1} &= (P_1, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
P^{5,2} &= (\hat{P}_1, P_2 \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{5,k} &= (\hat{P}_1, \cdots, \hat{P}_{k-1}, P_k, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{5,\bar{n}_5} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_5-1}, P_{\bar{n}_5}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)
\end{aligned}$$

where $\bar{n}_5 = \left[\frac{m_2 n}{m_1+m_2}\right]_- + 1$ and $\frac{m_2 n}{m_1+m_2}$ is not an integer.

Profile group VI:

$$\begin{aligned}
P^{6,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n) \\
P^{6,2} &= (\hat{P}_1, \cdots, \hat{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
P^{6,3} &= (\hat{P}_1, \cdots, \hat{P}_{n-3}, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{6,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{6,\bar{n}_6} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_6}, P_{n-\bar{n}_6+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)
\end{aligned}$$

where $\bar{n}_6 = \left[\frac{m_1 n}{m_1+m_2}\right]_-$ and $\frac{m_2 n}{m_1+m_2}$ is not an integer.

Now we characterize the random assignments for the preference profiles through a series of claims.

Claim 1. *For each preference profile $P^{1,k}$, $\varphi(P^{1,k})$ specifies probabilities on B, C, and D as follows*

	B	C	D
1	0	$\frac{m_1+m_2}{n}$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	0	$\frac{m_1+m_2}{n}$	$\frac{m_3}{n}$
$k+1$	$\frac{m_1}{n-k}$	$\frac{m_1+m_2}{n} - \frac{m_1}{n-k}$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{m_1}{n-k}$	$\frac{m_1+m_2}{n} - \frac{m_1}{n-k}$	$\frac{m_3}{n}$

Proof. Verification of the claim consists of three steps.

Step 1: We show $\varphi_{i,D}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$ and all $k = 0, 1, \cdots, \bar{n}_1$.

First, by *equal treatment of equals*, $\varphi_{i,D}(P^{1,0}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n$. Second, we show for all $k = 1, \cdots, \bar{n}_1$ if $\varphi_{i,D}(P^{1,k-1}) = \frac{m_3}{n}$ for all $i \in I$, then $\varphi_{i,D}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$. Notice that $P^{1,k}$ and $P^{1,k-1}$ are different only in agent k's preference, i.e., $P^{1,k}_k = \hat{P}_i$ and $P^{1,k-1}_k = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{k,D}(P^{1,k}) = \varphi_{k,D}(P^{1,k-1}) = \frac{m_3}{n}$. Hence by feasibility and *equal treatment of equals*, $\varphi_{i,D}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$.

Step 2: We show $\varphi_{i,B}(P^{1,k}) = 0$ for all $i = 1, \cdots, k$ and all $k = 0, 1, \cdots, \bar{n}_1$. Fix an k and suppose without loss of generality $\varphi_{1,B}(P^{1,k}) = \beta > 0$. Then *sd-efficiency* implies $\varphi_{i,C}(P^{1,k}) = 0$ for all $i = k+1, \cdots, n$ and *equal treatment of equals* implies $\varphi_{i,C}(P^{1,k}) = \frac{m_2}{k}$ for all $i = 1, \cdots, k$.

$$\begin{aligned}\varphi_{1,B}(P^{1,k}) + \varphi_{1,C}(P^{1,k}) + \varphi_{1,D}(P^{1,k}) &= \beta + \frac{m_2}{k} + \frac{m_3}{n} \\ &> \frac{m_2}{k} + \frac{m_3}{n} \\ &\geqslant \frac{m}{n}\end{aligned}$$

where the last inequality comes from $k \leqslant \bar{n}_1 \leqslant \frac{m_2 n}{m_1+m_2}$: a contradiction against Lemma 1.

Step 3: Lemma 1 and *equal treatment of equals* imply all other entries. □

Claim 2. *For each preference profile $P^{2,k}$, $\varphi(P^{2,k})$ specifies probabilities on B, C, and D as follows*

	B	C	D
1	0	$\frac{m}{n} - \alpha(k)$	$\alpha(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	0	$\frac{m}{n} - \alpha(k)$	$\alpha(k)$
k	$\frac{m_1}{n-(k-1)}$	$\frac{m_2-(k-1)\times(\frac{m}{n}-\alpha(k))}{n-k}$	$\frac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\alpha(k)}{n-k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\frac{m_1}{n-(k-1)}$	$\frac{m_2-(k-1)\times(\frac{m}{n}-\alpha(k))}{n-k}$	$\frac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\alpha(k)}{n-k}$
n	$\frac{m_1}{n-(k-1)}$	0	$\frac{m}{n} - \frac{m_1}{n-(k-1)}$

where $\alpha(k) = \frac{(k-2)m_1-(n-(k-1))m_2+(n-1)(n-(k-1))m_3}{n(n-1)(n-(k-1))}$.

Proof. Verification of the claim consists of six steps.
Step 1: We show $\varphi(P^{2,1})$ specifies probabilities on B, C, and D as follows

	B	C	D
1	$\frac{m_1}{n}$	$\frac{m_2}{n-1}$	$\frac{m_2+m_3}{n} - \frac{m_2}{n-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\frac{m_1}{n}$	$\frac{m_2}{n-1}$	$\frac{m_2+m_3}{n} - \frac{m_2}{n-1}$
n	$\frac{m_1}{n}$	0	$\frac{m_2+m_3}{n}$

First notice that $P^{2,1}$ and $P^{1,0}$ are different only in agent n's preference, i.e., $P^{2,1}_n = \bar{P}_1$ and $P^{1,0} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n,B}(P^{2,1}) = \varphi_{n,B}(P^{1,0}) = \frac{m_1}{n}$. Hence feasibility and *equal treatment of equals* imply $\varphi_{i,B}(P^{2,1}) = \frac{m_1}{n}$ for all $i \in I$.

Second $\varphi_{n,C}(P^{2,1}) = 0$. Suppose not, then *sd-efficiency* implies $\varphi_{i,D}(P^{2,1}) = 0$ for all $i = 1, \cdots, n-1$. Hence feasibility implies $\varphi_{n,D}(P^{2,1}) = m_3 \geqslant 1$: a contradiction against Lemma 1.

Last, feasibility and *equal treatment of equals* imply all other entries.
Step 2: We show $\varphi_{n,B}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ for all $k = 2, \cdots, \bar{n}_2$. Fix an k. Notice that $P^{2,k}$ and $P^{1,k-1}$ are different only in agent n's preference, i.e., $P^{2,k}_n = \bar{P}_i$ and $P^{2,k}_n = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n,B}(P^{2,k}) = \varphi_{n,B}(P^{1,k-1}) = \frac{m_1}{n-(k-1)}$.
Step 3: We show $\varphi_{n,C}(P^{2,k}) = 0$ for all $k = 2, \cdots, \bar{n}_2$. Fix an k and suppose $\varphi_{n,C}(P^{2,k}) > 0$. Then *sd-efficiency* implies $\varphi_{i,D}P^{2,k} = 0$ for all $i = 1, \cdots, n-1$ and hence $\varphi_{n,D}P^{2,k} = m_3$: a contradiction against Lemma 1.
Step 4: We show $\varphi_{i,D}(P^{2,k}) = \alpha(k)$ for all $i = 1, \cdots, k-1$ and all $k = 2, \cdots, \bar{n}_2$.

First we show $\varphi_{1,D}(P^{2,2}) = \alpha(2)$. Notice that $P^{2,2}$ and $P^{2,1}$ are different only in agent 1's preference, i.e., $P^{2,2}_1 = \hat{P}_i$ and $P^{2,1}_1 = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{1,D}(P^{2,2}) = \varphi_{1,D}(P^{2,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-1}$.

$$\begin{aligned}\alpha(2) &= \frac{(2-2)m_1 - (n-(2-1))m_2 + (n-1)(n-(2-1))m_3}{n(n-1)(n-(2-1))} \\ &= \frac{(n-1)m_3 - m_2}{n(n-1)} \\ &= \frac{m_2+m_3}{n} - \frac{m_2}{n-1}.\end{aligned}$$

Second, we show an induction: If $\varphi_{i,D}(P^{2,k}) = \alpha(k)$ for all $i = 1, \cdots, k-1$ and an $k \in \{2, \cdots, \bar{n}_2 - 1\}$, then $\varphi_{i,D}(P^{2,k+1}) = \alpha(k+1)$ for all $i = 1, \cdots, k$. Notice that $P^{2,k+1}$ and $P^{2,k}$ are different only in agent k's preference, i.e., $P^{2,k+1}_k = \hat{P}_i$ and $P^{2,k}_k = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{k,D}(P^{2,k+1}) = \varphi_{k,D}(P^{2,k})$. Hence for all $i = 1, \cdots, k$

$$
\begin{aligned}
\varphi_{i,D}(P^{2,k+1}) &= \varphi_{k,D}(P^{2,k+1}) && \text{by } \textit{equal treatment of equals}\\
&= \varphi_{k,D}(P^{2,k}) && \text{by } \textit{sd-strategy-proofness}\\
&= \tfrac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\varphi_{k-1,D}(P^{2,k})}{n-k} && \text{by feasibility and } \textit{equal treatment of equals}\\
&= \tfrac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\alpha(k)}{n-k} && \text{by induction hypothesis}\\
&= \alpha(k+1) && \text{by simplifying expression.}
\end{aligned}
$$

Step 5: We show $\varphi_{i,B}(P^{2,k}) = 0$ for all $i = 1, \cdots, k-1$ and all $k = 2, \cdots, \bar{n}_2$. Fix an k. Suppose without loss of generality $\varphi_{1,B}(P^{2,k}) = \beta > 0$. Then *equal treatment of equals* implies $\varphi_{i,B}(P^{2,k}) = \beta$ for all $i = 1, \cdots, k-1$. Hence Lemma 1 and Step 4 imply $\varphi_{i,C}(P^{2,k}) = \frac{m}{n} - \alpha(k) - \beta$ for all $i = 1, \cdots, k-1$ and *sd-efficiency* implies $\varphi_{i,C}(P^{2,k}) = 0$ for all $i = k, \cdots, n-1$.

Now we show $(k-1) \times (\frac{m}{n} - \alpha(k) - \beta) < m_2$: a contradiction against feasibility.

$$
\begin{aligned}
&(k-1) \times (\frac{m}{n} - \alpha(k) - \beta) < m_2\\
\Leftarrow&(k-1) \times (\frac{m}{n} - \alpha(k)) \leqslant m_2\\
\Leftrightarrow&(k-1) \times \left[\frac{m}{n} - \frac{(k-2)m_1 - (n-(k-1))m_2 + (n-1)(n-(k-1))m_3}{n(n-1)(n-(k-1))}\right] - m_2 \leqslant 0\\
\Leftrightarrow&(k-1) \times [(n-1)(n-(k-1))(m_1+m_2) - (k-2)m_1 + (n-(k-1))m_2]\\
&- n(n-1)(n-(k-1))m_2 \leqslant 0\\
\Leftrightarrow& - n(m_1+m_2)(k-1)^2 + \left[(n^2-n+1)m_1 + (2n^2-n)m_2\right](k-1) - n^2(n-1)m_2 \leqslant 0
\end{aligned}
$$

Let $f(\theta) = -n(m_1+m_2)(\theta-1)^2 + [(n^2-n+1)m_1 + (2n^2-n)m_2](\theta-1) - n^2(n-1)m_2$. To verify the Step, it suffices to show $f(\theta) \leqslant 0$ for all $k = 2, \cdots, \bar{n}_2$.

From the functional form of $f(\theta)$, we have first-order derivative and the second order derivative as follows

$$
\begin{aligned}
f'(\theta) &= -2n(m_1+m_2)(\theta-1) + (n^2-n+1)m_1 + (2n^2-n)m_2\\
f''(\theta) &= -2n(m_1+m_2)
\end{aligned}
$$

When $\frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}$ is an integer, $\bar{\mathbf{n}}_\mathbf{2} = \frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}$.

$$
\begin{aligned}
f(\bar{n}_2) =& - n(m_1+m_2)\left(\frac{m_2 n}{m_1+m_2} - 1\right)^2\\
&+ \left[(n^2-n+1)m_1 + (2n^2-n)m_2\right]\left(\frac{m_2 n}{m_1+m_2} - 1\right) - n^2(n-1)m_2\\
=&\frac{1}{m_1+m_2}\{-n[(n-1)m_2 - m_1]^2\\
&+ \left[(n^2-n+1)m_1 + (2n^2-n)m_2\right][(n-1)m_2 - m_1] - n^2(n-1)m_2(m_1+m_2)\}\\
=&\frac{1}{m_1+m_2}\left[-(n^2+1)m_1^2 - \left((n-\frac{1}{2})^2 + \frac{3}{4}\right)m_1 m_2\right] < 0.
\end{aligned}
$$

$$
\begin{aligned}
f'(\bar{n}_2) =& -2n(m_1+m_2)(\frac{m_2 n}{m_1+m_2}-1)+(n^2-n+1)m_1+(2n^2-n)m_2 \\
=& \frac{1}{m_1+m_2}[-2n(m_1+m_2)((n-1)m_2-m_1)+(n^2-n+1)m_1(m_1+m_2) \\
&+(2n^2-n)m_2(m_1+m_2)] \\
=& \frac{1}{m_1+m_2}[(n^2+n+1)m_1^2+nm_2^2+(n^2+2n+1)m_1m_2]>0
\end{aligned}
$$

By $f''(\theta)<0$ and $f'(\bar{n}_2)>0$, $f'(\theta)>0$ for all $\theta \leqslant \bar{n}_2$, that is $f(\theta)$ is increasing through 2 to $\bar{n}_2$. Then $f(\bar{n}_2)<0$ implies $f(\theta)<0$ for all $\theta \leqslant \bar{n}_2$, which is what we want.

When $\frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}$ is not an integer, $\bar{\mathbf{n}}_\mathbf{2}=\left[\frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}\right]_- + \mathbf{1}$.

$$
\begin{aligned}
f(\bar{n}_2) =& -n(m_1+m_2)\left(\left[\frac{m_2 n}{m_1+m_2}\right]_-\right)^2 \\
&+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right]\left(\left[\frac{m_2 n}{m_1+m_2}\right]_-\right)-n^2(n-1)m_2 \leqslant 0
\end{aligned}
$$

where the last inequality comes from Assumption 1 in Appendix A.

$$
\begin{aligned}
f'(\bar{n}_2) =& -2n(m_1+m_2)(\frac{m_2 n}{m_1+m_2}-\delta)+(n^2-n+1)m_1+(2n^2-n)m_2 \\
=& \frac{1}{m_1+m_2}[-2n(m_1+m_2)((n-\delta)m_2-\delta m_1)+(n^2-n+1)m_1(m_1+m_2) \\
&+(2n^2-n)m_2(m_1+m_2)] \\
=& \frac{1}{m_1+m_2}[m_1 n(n-1)+m_2 n(m_1(n-2)-m_2)+m_1m_2+m_1^2 \\
&+2\delta(m_1^2 n+m_2^2 n+2m_1m_2n)]>0
\end{aligned}
$$

where the last inequality comes from $m_2 \leqslant (n-2)$ and $m_1 \geqslant 1$.

Step 6: Lemma 1 and *equal treatment of equals* imply all other entries. □

Claim 3. *For each preference profile $P^{3,k}$, $\varphi(P^{3,k})$ specifies probabilities on B, C, and D as follows*

	B	C	D
1	$\frac{m_1+m_2}{n}-\frac{m_2}{n-k}$	$\frac{m_2}{n-k}$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-k$	$\frac{m_1+m_2}{n}-\frac{m_2}{n-k}$	$\frac{m_2}{n-k}$	$\frac{m_3}{n}$
$n-k+1$	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$

Proof. This claim can be verified by the similar arguments that verify Claim 1. □

Claim 4. *For each preference profile $P^{4,k}$, $\varphi(P^{4,k})$ specifies probabilities on B, C, and D as follows*

	B	C	D
1	$\frac{m_1+m_2}{n} - \frac{m_2}{n-k}$	$\frac{m_2}{n-k}$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-k$	$\frac{m_1+m_2}{n} - \frac{m_2}{n-k}$	$\frac{m_2}{n-k}$	$\frac{m_3}{n}$
$n-k+1$	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$
n	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$

Proof. Verification of the claim consists of five steps.
Step 1: We show $\varphi_{n,B}(P^{4,k}) = \frac{m_1+m_2}{n}$ for all $k = 1, \cdots, \bar{n}_4$. Fix an k. Notice that $P^{4,k}$ and $P^{3,k}$ are different only in agent n's preference, i.e., $P^{4,k}_n = \bar{P}_i$ and $P^{3,k}_n = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n,B}(P^{4,k}) = \varphi_{n,B}(P^{3,k}) = \frac{m_1+m_2}{n}$.
Step 2: We show $\varphi_{n,C}(P^{4,k}) = 0$ and $\varphi_{n,D}(P^{4,k}) = \frac{m_3}{n}$ for all $k = 1, \cdots, \bar{n}_4$. Fix an k. Suppose $\varphi_{n,C}(P^{4,k}) > 0$, then *sd-efficiency* implies $\varphi_{i,D}(P^{4,k}) = 0$ for all $i = 1, \cdots, n-1$ and hence $\varphi_{n,D}(P^{4,k}) = m_3$: a contradiction against Lemma 1. Given $\varphi_{n,C}(P^{4,k}) = 0$, Lemma 1 implies $\varphi_{n,D}(P^{4,k}) = \frac{m_3}{n}$.
Step 3: We show $\varphi_{i,D}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$ and all $k = 1, \cdots, \bar{n}_4$. First *equal treatment of equals* and Step 2 imply $\varphi_{i,D}(P^{4,1}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$. Second we prove an induction: For any $k = 2, \cdots, \bar{n}_4$, if $\varphi_{i,D}(P^{4,k-1}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$, then $\varphi_{i,D}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$. Notice that $P^{4,k-1}$ and $P^{4,k}$ are different only in agent $(n-k+1)$'s preference, i.e., $P^{4,k-1}_{n-k+1} = \hat{P}_i$ and $P^{4,k}_{n-k+1} = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-k+1,D}(P^{4,k}) = \varphi_{n-k+1,D}(P^{4,k-1}) = \frac{m_3}{n}$. Hence feasibility and *equal treatment of equals* imply $\varphi_{i,D}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$.
Step 4: We show $\varphi_{i,C}(P^{4,k}) = 0$ for all $i = n-k+1, \cdots, n-1$ and all $k = 2, \cdots, \bar{n}_4$. Fix an k and suppose without loss of generality $\varphi_{n-1,C}(P^{4,k}) = \beta > 0$. By *equal treatment of equals*, $\varphi_{i,C}(P^{4,k}) = \beta$ for all $i = n-k+1, \cdots, n-1$. Then Lemma 1 implies $\varphi_{i,B}(P^{4,k}) = \frac{m_1+m_2}{n} - \beta$ for all $i = n-k+1, \cdots, n-1$ and *sd-efficiency* implies $\varphi_{i,B}(P^{4,k}) = 0$ for all $i = 1, \cdots, n-k$. Then we have a contradiction against feasibility

$$\begin{aligned} m_1 &= (n-k) \times 0 + (k-1) \times (\frac{m_1+m_2}{n} - \beta) + \frac{m_1+m_2}{n} \\ &< k \times \frac{m_1+m_2}{n} \leqslant m_1 \end{aligned}$$

where the last inequality comes from $k \leqslant \bar{n}_4 \leqslant \frac{m_1 n}{m_1+m_2}$.
Step 5: Lemma 1 and *equal treatment of equals* imply all other entries. □

Now we have the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is an integer.

$$
\begin{aligned}
P^{2,\bar{n}_2} =& (\hat{P}_1, \cdots, \hat{P}_{\frac{m_2 n}{m_1+m_2}-1}, P_{\frac{m_2 n}{m_1+m_2}}, \cdots, P_{n-1}, \bar{P}_n) \\
P^{4,\bar{n}_4} =& (\hat{P}_1, \cdots, \hat{P}_{n-\frac{m_1 n}{m_1+m_2}}, P_{n-\frac{m_1 n}{m_1+m_2}+1}, \cdots, P_{n-1}, \bar{P}_n) \\
=& (\hat{P}_1, \cdots, \hat{P}_{\frac{m_2 n}{m_1+m_2}}, P_{\frac{m_2 n}{m_1+m_2}+1}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}
$$

Hence $P^{2,\bar{n}_2}$ and $P^{4,\bar{n}_4}$ are different only in agent $\frac{m_2 n}{m_1+m_2}$'s preference, i.e., $P^{2,\bar{n}_2}_{\frac{m_2 n}{m_1+m_2}} = P_i$ and $P^{2,\bar{n}_2}_{\frac{m_2 n}{m_1+m_2}} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 1. Then *sd-strategy-proofness* implies $\varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{2,\bar{n}_2}) = \varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{4,\bar{n}_4})$. Now we have the contradiction as the following elaboration:

$$
\begin{aligned}
& \varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{2,\bar{n}_2}) = \varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{4,\bar{n}_4}) \\
\Leftrightarrow & \frac{m_3 - (\frac{m}{n} - \frac{m_1}{n-(\bar{n}_2-1)}) - (\bar{n}_2 - 1) \times \alpha(\bar{n}_2)}{n - \bar{n}_2} = \frac{m_3}{n} \\
\Leftrightarrow & - m_1 n(m_1 + m_2)((n+1)m_1 + m_2) = 0\text{: contradiction!}
\end{aligned}
$$

To find the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is not an integer, we characterize the assignment of D for the profiles in groups V and VI.

Let k^* be such that $k^* - 1 = n - \frac{m_1}{\frac{m}{n} - \frac{m_3}{2}}$, which is equivalent to $\frac{m}{n} - \frac{m_1}{n-(k^*-1)} - \frac{m_3}{2} = 0$. I first present two types of assignments and later I will show that both assignments are possible for profiles in group V by Claim 5, 6, and 7.

Assignment 1:

	B	C	D
1	$-$	$-$	$\gamma(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$-$	$-$	$\gamma(k)$
k	$-$	$-$	$\frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\gamma(k)}{n-(k+1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	$\frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\gamma(k)}{n-(k+1)}$
$n-1$	$\frac{m_1}{n-(k-1)}$	0	$\frac{m}{n} - \frac{m_1}{n-(k-1)}$
n	$\frac{m_1}{n-(k-1)}$	0	$\frac{m}{n} - \frac{m_1}{n-(k-1)}$

where $\gamma(k) = \dfrac{\Gamma_1(k)m_1 + \Gamma_2(k)m_2 + \Gamma_3(k)m_3}{n[n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)]}$

and $\Gamma_1(k) = 2(k-2)n^2 - 2(k^2-k-2)n + 2(k^2-k-2)$

$\Gamma_2(k) = -2n^3 + 4kn^2 - 2(k^2+k-1)n + 2(k^2-k)$

$\Gamma_3(k) = n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)$

Assignment 2:

	B	C	D
1	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$-$	$-$	0
k	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	0
$n-1$	$\frac{m_1}{n-(k-1)}$	$\frac{m}{n}-\frac{m_1}{n-(k-1)}-\frac{m_3}{2}$	$\frac{m_3}{2}$
n	$\frac{m_1}{n-(k-1)}$	$\frac{m}{n}-\frac{m_1}{n-(k-1)}-\frac{m_3}{2}$	$\frac{m_3}{2}$

Claim 5. *If $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$, $\varphi(P^{5,k})$ specifies probabilities on B, C, and D as assignment 1 for all $k = 1, \cdots, \bar{n}_5$.*

Proof. Verification of the claim consists of four steps.

Step 1: We show, if $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$, $\varphi(P^{5,1})$ specifies probabilities on B, C, and D as follows

	B	C	D
1	$\frac{m_1}{n}$	$\frac{m_2}{n-2}$	$\frac{m_2+m_3}{n}-\frac{m_2}{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$\frac{m_1}{n}$	$\frac{m_2}{n-2}$	$\frac{m_2+m_3}{n}-\frac{m_2}{n-2}$
$n-1$	$\frac{m_1}{n}$	0	$\frac{m_2+m_3}{n}$
n	$\frac{m_1}{n}$	0	$\frac{m_2+m_3}{n}$

First notice that $P^{5,1}$ and $P^{2,1}$ are different only in agent $(n-1)$'s preference, i.e., $P^{5,1}_{n-1} = \bar{P}_i$ and $P^{2,1}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{5,1}) = \varphi_{n-1,B}(P^{2,1}) = \frac{m_1}{n}$ and hence feasibility and *equal treatment of equals* imply $\varphi_{i,B}(P^{5,1}) = \frac{m_1}{n}$ for all $i \in I$.

Second we show $\varphi_{n-1,C}(P^{5,1}) = \varphi_{n,C}(P^{5,1}) = 0$. Suppose not, let $\beta \equiv \varphi_{n-1,C}(P^{5,1}) = \varphi_{n,C}(P^{5,1}) > 0$, then *sd-efficiency* implies $\varphi_{i,D}(P^{5,1}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1,D}(P^{5,1}) = \varphi_{n,D}(P^{5,1}) = \frac{m_3}{2}$. Then Lemma 1 requires $\frac{m_1+m_2+m_3}{n} = \frac{m_1}{n} + \beta + \frac{m_3}{2}$. Then $\beta > 0$ implies $\frac{m_1+m_2+m_3}{n} - \frac{m_1}{n} - \frac{m_3}{2} > 0$ which is equivalent to $\frac{m_3}{m_2} < \frac{2}{n-2}$: contradiction!

All the other entries are implied by Lemma 1 and *equal treatment of equals*.

Step 2: We show $\varphi_{n-1,D}(P^{2,k}) = \varphi_{n,D}(P^{2,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)}$ for all $k = 1, \cdots, \bar{n}_5$.

Fix an k. First notice that $P^{5,k}$ and $P^{2,k}$ are different only in agent $(n-1)$'s preference, i.e., $P^{5,k}_{n-1} = \bar{P}_i$ and $P^{2,k}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{5,k}) = \varphi_{n-1,B}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ and hence *equal treatment of equals* implies $\varphi_{n,B}(P^{5,k}) = \varphi_{n-1,B}(P^{5,k}) = \frac{m_1}{n-(k-1)}$.

Second we show $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = 0$. Suppose not, let $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = \beta > 0$, *sd-efficiency* implies $\varphi_{i,D}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1,D}P^{5,k} = \varphi_{n,D}P^{5,k} = \frac{m_3}{2}$. Then we have a contradiction:

$$\varphi_{n,B}(P^{5,k}) + \varphi_{n,C}(P^{5,k}) + \varphi_{n,D}(P^{5,k}) = \varphi_{n,B}(P^{5,1}) + \varphi_{n,C}(P^{5,1}) + \varphi_{n,D}(P^{5,1})$$

$$\Leftrightarrow \frac{m_1}{n-(k-1)} + \beta + \frac{m_3}{2} = \frac{m_1}{n} + 0 + \frac{m_2+m_3}{n}\text{: contradiction!}$$

where the contradiction comes from $\frac{m_1}{n-(k-1)} \geqslant \frac{m_1}{n}$, $\beta > 0$, and that $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$ implies $\frac{m_3}{2} \geqslant \frac{m_2+m_3}{n}$.

Lastly, Lemma 1 implies what we want.

Step 3: We show $\varphi_{1,D}(P^{5,2}) = \gamma(2)$. Notice that $P^{5,2}$ and $P^{5,1}$ are different only in agent 1's preference, i.e., $P_1^{5,2} = \hat{P}_i$ and $P_1^{5,2} = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{1,D}(P^{5,2}) = \varphi_{1,D}(P^{5,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-2} = \frac{-2m_2}{n(n-2)} + \frac{m_3}{n}$. Notice that $B(2) = 0$, $C(2) = -2n^3 + 8n^2 - 10n + 4$, and $D(2) = n^4 - 6n^3 + 13n^2 - 12n + 4$. Then

$$\gamma(2) = \frac{0m_1 + (-2n^3 + 8n^2 - 10n + 4)(k)m_2 + (n^4 - 6n^3 + 13n^2 - 12n + 4)(k)m_3}{n[n^4 - 6n^3 + 13n^2 - 12n + 4]}$$

$$= \frac{-2m_2}{n(n-2)} + \frac{m_3}{n}.$$

Step 4: We show an induction: For any $2 \leqslant k < \bar{n}_5$, if $\varphi_{i,D}(P^{5,k}) = \gamma(k)$ for all $i = 1, \cdots, k-1$, then $\varphi_{i,D}(P^{5,k+1}) = \gamma(k+1)$ for all $i = 1, \cdots, k$. By *equal treatment of equals*, it suffices to show $\varphi_{k,D}(P^{5,k+1}) = \gamma(k+1)$. Notice that $P^{5,k+1}$ and $P^{5,k}$ are different only in agent k's preference, i.e., $P_k^{5,k+1} = \hat{P}_i$ and $P_k^{5,k} = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then

$\varphi_{k,D}(P^{5,k+1})$	$= \varphi_{k,D}(P^{5,k})$	by *sd-strategy-proofness*
	$= \frac{m_3 - 2\times\varphi_{n-1,D}(P^{5,k}) - (k-1)\times\varphi_{k-1,D}(P^{5,k})}{n-(k+1)}$	by feasibility and *equal treatment of equals*
	$= \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\gamma(k)}{n-(k+1)}$	by Step 2 and hypothesis assumption
	$= \gamma(k+1)$	by simplifying the expression

□

Claim 6. *If $\frac{m_3}{m_2} < \frac{2}{n-2}$ and $\bar{n}_5 < k^*$, $\varphi(P^{5,k})$ specifies probabilities on B, C, and D as assignment 2 for each $k = 1, \cdots, \bar{n}_5$.*

Proof. Verification of the claim consists of four steps.

Step 1: We show, if $\frac{m_3}{m_2} < \frac{2}{n-2}$, $\varphi(P^{5,1})$ specifies probabilities on B, C, and D as follows

	B	C	D
1	$\frac{m_1}{n}$	$\frac{m_2+m_3}{n}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$\frac{m_1}{n}$	$\frac{m_2+m_3}{n}$	0
$n-1$	$\frac{m_1}{n}$	$\frac{m_2-(n-2)\times\frac{m_2+m_3}{n}}{2}$	$\frac{m_3}{2}$
n	$\frac{m_1}{n}$	$\frac{m_2-(n-2)\times\frac{m_2+m_3}{n}}{2}$	$\frac{m_3}{2}$

First by the same argument showing the Step 1 in Claim 5, $\varphi_{i,B}(P^{5,1}) = \frac{m_1}{n}$.

Second we show $\varphi_{n-1,C}(P^{5,1}) = \varphi_{n-1,C}(P^{5,1}) > 0$. Suppose not, $\varphi(P^{5,1})$ is specified as by the Step 1 in Claim 5. Then $\varphi_{1,D}(P^{5,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-2} \geqslant 0$: contradicting against $\frac{m_3}{m_2} < \frac{2}{n-2}$.

Lastly, *sd-efficiency* implies $\varphi_{i,D}(P^{5,1}) = 0$ for all $i = 1, \cdots, n-2$. All the other entries are implied by Lemma 1 and *equal treatment of equals*.

Step 2: We show $\varphi_{n-1,B}(P^{5,k}) = \varphi_{n,B}(P^{5,k}) = \frac{m_1}{n-(k-1)}$ for all $k = 1, \cdots, \bar{n}_5$. Fix an k. Notice that $P^{5,k}$ and $P^{2,k}$ are different only in agent $(n-1)$'s preference, i.e., $P^{5,k}_{n-1} = \bar{P}_i$ and $P^{2,k}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{5,k}) = \varphi_{n-1,B}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ and hence *equal treatment of equals* implies $\varphi_{n,B}(P^{5,k}) = \varphi_{n-1,B}(P^{5,k}) = \frac{m_1}{n-(k-1)}$.

Step 3: For any $k < k^*$, if $\varphi_{i,D}(P^{5,k-1}) = 0$ for all $i = 1, \cdots, n-2$, then $\varphi_{i,D}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$. By *sd-efficiency*, it suffices to show $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) > 0$. Suppose not. First, by Step 2 and Lemma 1, $\varphi_{n-1,D}(P^{5,k}) = \varphi_{n,D}(P^{5,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)}$. Second, notice that $P^{5,k}$ and $P^{5,k-1}$ are different only in agent k's preference, i.e., $P^{5,k}_k = \hat{P}_i$ and $P^{5,k-1}_k = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* and *equal treatment of equals* imply $\varphi_{i,D}(P^{5,k}) = \varphi_{k,D}(P^{5,k-1}) = 0$ for all $i = 1, \cdots, k$. Last, feasibility and *equal treatment of equals* imply $\varphi_{i,D}(P^{5,k}) = \frac{m_3 - 2\left(\frac{m}{n} - \frac{m_1}{n-(k-1)}\right)}{n-(k+1)}$. Then by $k < k^*$, we have a contradiction: $\varphi_{i,D}(P^{5,k}) < \frac{m_3 - 2\left(\frac{m}{n} - \frac{m_1}{n-(k^*-1)}\right)}{n-(k+1)} = 0$.

□

Claim 7. *If* $\frac{m_3}{m_2} < \frac{2}{n-2}$ *and* $\bar{n}_5 \geqslant k^*$, $\varphi(P^{5,k})$ *specifies probabilities on* B, C, *and* D *as assignment 2 for each* $k = 1, \cdots, k^*$ *and as assignment 1 for each* $k = k^* + 1, \cdots, \bar{n}_5$.

Proof. Verification of the claim consists of two steps.

By Claim 6, $\varphi(P^{5,k})$ specifies probabilities on B, C, and D as assignment 2 for each $k = 1, \cdots, k^*$.

Step 1: $\varphi(P^{5,k^*})$ specifies probabilities on B, C, and D as follows.

	B	C	D
1	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k^*-1	$-$	$-$	0
k^*	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	0
$n-1$	$\frac{m_1}{n-(k^*-1)}$	0	$\frac{m_3}{2}$
n	$\frac{m_1}{n-(k^*-1)}$	0	$\frac{m_3}{2}$

Step 2: For any $k > k^*$, if $\varphi_{n-1,C}(P^{5,k-1}) = \varphi_{n,C}(P^{5,k-1}) = 0$, then $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = 0$. Suppose not, then $\varphi_{i,D}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1,D}(P^{5,k}) = \varphi_{n,D}(P^{5,k}) = \frac{m_3}{2}$. By Step 2 and Lemma 1, $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)} - \frac{m_3}{2}$. Then by $k > k^*$, we have a contradiction: $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) < \frac{m}{n} - \frac{m_1}{n-(k^*-1)} - \frac{m_3}{2} = 0$.

□

Claim 8. *For each preference profile $P^{6,k}$, $\varphi(P^{6,k})$ specifies probabilities on B, C, and D as follows*

	B	C	D
1	$-$	$-$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-k$	$-$	$-$	$\frac{m_3}{n}$
$n-k+1$	$-$	$-$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	$\frac{m_3}{n}$
$n-1$	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$
n	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$

Proof. Verification of the claim consists of three steps.

Step 1: We show $\varphi_{n-1,B}(P^{6,k}) = \varphi_{n,B}(P^{6,k}) = \frac{m_1+m_2}{n}$ for all $k = 2, \cdots, \bar{n}_6$. Fix an k. Notice that $P^{6,k}$ and $P^{4,k}$ are different only in agent $(n-1)$'s preference, i.e., $P^{6,k}_{n-1} = \bar{P}_i$ and $P^{4,k}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{6,k}) = \varphi_{n-1,B}(P^{4,k}) = \frac{m_1+m_2}{n}$. Hence *equal treatment of equals* implies $\varphi_{n-1,B}(P^{6,k}) = \varphi_{n,B}(P^{6,k}) = \frac{m_1+m_2}{n}$.

Step 2: We show $\varphi_{n-1,C}(P^{6,k}) = \varphi_{n,C}(P^{6,k}) = 0$ and $\varphi_{n-1,D}(P^{6,k}) = \varphi_{n,D}(P^{6,k}) = \frac{m_3}{n}$ for all $k = 2, \cdots, \bar{n}_6$. Fix an k. By Lemma 1, it suffices to show $\varphi_{n-1,C}(P^{6,k}) = \varphi_{n,C}(P^{6,k}) = 0$. Suppose not, then *sd-efficiency* implies $\varphi_{i,D}(P^{6,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence feasibility and *equal treatment of equals* imply $\varphi_{n-1,D}(P^{6,k}) = \varphi_{n,D}(P^{6,k}) = \frac{m_3}{2}$. Then $\frac{m_1+m_2}{n} + 0 + \frac{m_3}{2} > \frac{m}{n}$: contradiction against Lemma 1.

Step 3: We show $\varphi_{i,D}(P^{6,k}) = \frac{m_3}{n}$ for all $i = 1 \in I$ and all $k = 3, \cdots, \bar{n}_6$.

We first show $\varphi_{i,D}(P^{6,3}) = \frac{m_3}{n}$ for all $i = 1 \in I$. Notice that, by Step 2 and *equal treatment of equals*, $\varphi_{n-2,D}(P^{6,2}) = \frac{m_3}{n}$. Notice also that $P^{6,3}$ and $P^{6,2}$ are different only in agent $(n-2)$'s preference, i.e., $P^{6,3}_{n-2} = P_i$ and $P^{6,2}_{n-2} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-2,D}(P^{6,3}) = \varphi_{n-2,D}(P^{6,2}) = \frac{m_3}{n}$. Then Step 2 and *equal treatment of equals* imply what we want.

Now we show an induction: for any $3 \leqslant k < \bar{n}_6$, if $\varphi_{i,D}(P^{6,k}) = \frac{m_3}{n}$ for all $i = 1 \in I$, then $\varphi_{i,D}(P^{6,k+1}) = \frac{m_3}{n}$ for all $i = 1 \in I$. Notice that $P^{6,k+1}$ and $P^{6,k}$ are different only in agent $(n-k)$'s preference, i.e., $P^{6,k+1}_{n-k} = P_i$ and $P^{6,k}_{n-k} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-k,D}(P^{6,k+1}) = \varphi_{n-k,D}(P^{6,k}) = \frac{m_3}{n}$. Hence Step 2 and *equal treatment of equals* imply what we want.

□

Now we have the contradiction to prove the theorem for the case where $\frac{m_2 n}{m_1+m_2}$ is not an integer.

$$P^{5,\bar{n}_5} = (\hat{P}_1, \cdots, \hat{P}_{\left[\frac{m_2 n}{m_1+m_2}\right]_-}, P_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$$

$$P^{6,\bar{n}_6} = (\hat{P}_1, \cdots, \hat{P}_{n-\left[\frac{m_1 n}{m_1+m_2}\right]_-}, P_{n-\left[\frac{m_1 n}{m_1+m_2}\right]_-+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$$

Notice that $\left[\frac{m_2 n}{m_1+m_2}\right]_- = \left[n - \frac{m_1 n}{m_1+m_2}\right]_- = \left(n - \left[\frac{m_1 n}{m_1+m_2}\right]_-\right) - 1$. Then $P^{5,\bar{n}_5}$ and $P^{6,\bar{n}_6}$ are different only in agent $\left(\left[\frac{m_2 n}{m_1+m_2}\right]_- + 1\right)$'s preference, i.e., $P^{5,\bar{n}_5}_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1} = P_i$ and $P^{6,\bar{n}_6}_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1} = \hat{P}_i$. Hence *sd-strategy-proofness* implies $\varphi_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1,D}(P^{5,\bar{n}_5}) = \varphi_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1,D}(P^{6,\bar{n}_6})$.

If $\varphi(P^{5,\bar{n}_5})$ is in the form of Assignment 2, the contradiction is evident: $0 \neq \frac{m_3}{n}$.

If $\varphi(P^{5,\bar{n}_5})$ is in the form of Assignment 1, the contradiction is verified by Assumption 2 in Appendix A.

$$\frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(\bar{n}_5-1)}) - (\bar{n}_5 - 1) \times \gamma(\bar{n}_5)}{n - (\bar{n}_5 + 1)} \neq \frac{m_3}{n}$$

C Proof of Theorem 2

Fix a description (C, σ) that satisfies the condition in Statement **??**, we show that the induced preference domain $\mathbb{D}_{(C,\sigma)}$ is covered by a sequentially dichotomous domain.

For each $t^* \in \{2, \cdots, |C|\}$, let

$$\mathbf{A}^{\sigma}_{t^*} \equiv \left\{ B \subset A \middle| \exists (v_1, \cdots, v_{t^*-1}) \in \prod_{\tau=1}^{t^*-1} A^{\sigma}_{\tau} \text{ s.t. } b \in B \text{ whenever } (b^{\sigma}_1, \cdots, b^{\sigma}_{t^*-1}) = (v_1, \cdots, v_{t^*-1}) \right\},$$

i.e., objects are grouped according to their values of the top-$(t^* - 1)$ ranked characteristics.

Now we construct a sequence of partitions $(\mathbf{A}_t)_{t=1}^n$ using $(\mathbf{A}_{t^*})_{t^*=2}^{|C|}$ as the backbones.

- $\mathbf{A}_1 \equiv \{A\}$
- $\mathbf{A}_{|\mathbf{A}_{t^*}|} \equiv \mathbf{A}_{t^*}$, for each $t^* \in \{2, \cdots, |C|\}$
- for each $t^* \in \{2, \cdots, |C| - 1\}$, the partitions $\mathbf{A}_t$ with $|\mathbf{A}_{t^*}| < t < |\mathbf{A}_{t^*+1}|$ are defined as follows
  - pick any $B \in \mathbf{A}_{t^*} \backslash \mathbf{A}_{t^*+1}$, due to statement **??**, there are two blocks $C, D \in \mathbf{A}_{t^*+1} \backslash \mathbf{A}_{t^*}$ such that $B = C \cup D$
  - label blocks in $\mathbf{A}_{t^*} \backslash \mathbf{A}_{t^*+1}$ as $\left(B^1, \cdots, B^{|\mathbf{A}_{t^*+1}| - |\mathbf{A}_{t^*}|}\right)$ and blocks in $\mathbf{A}_{t^*+1} \backslash \mathbf{A}_{t^*}$ as $\left(C^1, D^1, \cdots, C^{|\mathbf{A}_{t^*+1}| - |\mathbf{A}_{t^*}|}, D^{|\mathbf{A}_{t^*+1}| - |\mathbf{A}_{t^*}|}\right)$ such that $B^m = C^m \cup D^m$ for all $m = 1, \cdots, |\mathbf{A}_{t^*+1}| - |\mathbf{A}_{t^*}|$
  - define $\mathbf{A}_{|\mathbf{A}_{t^*}|+m} \equiv (\mathbf{A}_{t^*} \backslash \{B^m\}) \bigcup \{C^m, D^m\}$ for each $m = 1, \cdots, |\mathbf{A}_{t^*+1}| - |\mathbf{A}_{t^*}|$.

The following two claims prove that the induced domain is covered by a sequentially dichotomous domain and hence Liu (2016) implies what we want.

Claim 9. *The sequence $(\mathbf{A}_t)_{t=1}^n$ is a path.*

By definition, $\mathbf{A}_1 = \{A\}$ and $\mathbf{A}_n = \{\{a\} : a \in A\}$, it suffices to show for each $t \in \{1, \cdots, n-1\}$, $\mathbf{A}_{t+1}$ is a direct refinement of $\mathbf{A}_t$, i.e., there is exactly one block $A_k \in \mathbf{A}_t$ and two blocks $A_i, A_j \in \mathbf{A}_{t+1}$ such that $A_k = A_i \cup A_j$ and for each $A_l \in \mathbf{A}_t \backslash \{A_k\}$ there is $A_i \in \mathbf{A}_{t+1}$ such that $A_l = A_i$. This is obvious from the construction of $(\mathbf{A}_t)_{t=1}^n$.

Claim 10. *Every preference $P_0 \in \mathbb{D}_{(C,\sigma)}$ observes every partition in the path $(\mathbf{A}_t)_{t=1}^n$.*

This is evident from the definition of lexicographically separable preferences and the path $(\mathbf{A}_t)_{t=1}^n$.