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Asymptotic Dynamics and Value-at-Risk of Large Diversified Portfolios in a Jump-Diffusion Market

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Abstract

This paper studies the modelling of large diversified portfolios in a financial market with jump-diffusion risks. The portfolios considered include three categories: equal money-weighted portfolios, risk minimizing portfolios, and market indices. Reduced-form dynamics driven jointly by one Brownian Motion and one Poisson process are derived for the asymptotics of such portfolios. We prove that derivatives written on a portfolio can be priced by treating the asymptotic dynamics as the underlying process if the number of assets in the portfolio is sufficiently large. Analytical and Monte Carlo Value-at-Risk (VaR) can be computed for the portfolios based on their asymptotic dynamics.

Keywords: Brownian Motion, Portfolio Process, Asymptotic Dynamics, Jump-Diffusion

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1 Introduction

Reducing risk by diversification as introduced in H. Markowitz (1959)'s seminal work is the basic idea behind modern portfolio theory (MPT). W. Sharpe (1964) and J. Lintner (1965) considered economic equilibrium in MPT and developed the Capital Asset Pricing Model (CAPM) in which asset risks are dichotomized as non-systematic (or diversifiable) risk and systematic (or non-diversifiable) risk. The CAPM has a powerful implication that only systematic risk matters in asset pricing. Under the CAPM model, non-systematic risk is eliminated as the number of assets grows. The single-period CAPM was later extended to an intertemporal model through the use of lognormal asset price diffusion processes by R. Merton (1973). T. Björk and B. Näslund (1998) introduced asymptotic assets, or infinitely diversified portfolios, into the model and studied the absence of asymptotic arbitrage and the resulting market completeness.

The asymptotic assets in T. Björk and B. Näslund (1998) are idealized instruments for research purpose. In reality they refer to large-scale portfolios held by mutual funds or banks, and financial indices (eg. S&P 500, Nikkei 225, NASDAQ 100) constructed as proxies of market movements and so on. The study of the rate of convergence of such portfolios and indices to their asymptotic forms under different probabilistic meanings is presented in a rigorous mathematical framework in N. Hofmann and E. Platen (2000). They considered a financial market with asset price dynamics modelled by a system of lognormal stochastic differential equation (SDE) and derived an SDE, also lognormal, for the asymptotic dynamics of large diversified portfolios of such assets. However, the lognormal assumption about asset price is not adequate to describe the well-known fat-tail empirical characteristic of asset returns. The latter is believed to be caused by jump risks due to rare events in the market. M. S. Gibson (2001) recently developed a methodology to incorporate event risk into VaR and implemented it in an empirical application. D. Duffie and J. Pan (2001) also employed assumptions of portfolios with multi-factor jump-diffusion asset returns, and derived analytical approximation of the VaR.

Our paper extends the results of N. Hofmann and E. Platen (2000) in several dimensions. We present a more realistic financial model that incorporates jump risks of the market or individual assets. The portfolios may include basic or derivative instruments with defaultable counterparties. The jump-diffusion of individual assets are then shown to aggregate to similar portfolio characteristics that accommodate fat-tail return distributions. We also investigate three important types of portfolios with definite economic significance. They are equal money-weighted portfolios, risk-minimizing portfolios, and financial indices. We are able to characterize directly the asymptotic processes of these portfolios, and thus provide a convenient and tractable way of measuring Value-at-Risk for purpose of bank risk management.

Hofmann and Platen (2000) provide results of convergence in the logarithm of portfolio values. This is useful for the study of the long-term behavior of large diversified portfolios as the processes of the logarithms are approximately Gaussian and exhibit the regular properties of linear growth in trend and variance. However the payoff functions for derivatives are conventionally defined on the values of the underlying variable instead of their logarithms. It is not straightforward to apply the convergence theorems by Hofmann and Platen to the approximation of derivative prices. We tackle this problem by directly working on the strong convergence and also weak convergence in portfolio values rather than in their logarithms.

In addition to testing the pathwise convergence, we also investigate the effect of replacing large diversified portfolios by their respective asymptotic dynamics in the valuation of exotic options and the estimation of portfolio VaR by Monte Carlo method.

The rest of the paper is organized as follows. Section 2 describes the market specifications and defines three categories of large-scale portfolios with distinct economic significance. Section 3 provides derivations of the asymptotic dynamics of the three categories of portfolios. Proofs of the different orders of convergence for large diversified portfolios to their asymptotic values in the weak and strong senses are provided. Section 4 derives an analytical formula for the tail probability of portfolio returns. Section 5 tests the theoretical results in the previous sections by simulation. Section 6 contains the conclusions.

2 Market specification of the problem

We consider a financial market of d assets, whose prices at time t are denoted by $X_t^i,\ i = 1, \cdots, d$. Suppose that the asset prices are modelled by the following stochastic differential equations on a complete probability space (Ω, F, P) equipped with a filtration $(F_t)_{t\geq 0}$:

$$\begin{aligned} dX_t^i &= \alpha_i(t)X_t^i dt + \beta_i(t)X_t^i dW_t^0 + \gamma_i(t)X_t^i dW_t^i \qquad (1) \\ &\quad + \delta_i(t-)X_t^i dN_t^0 + \theta_i(t-)X_t^i dN_t^i, \quad t \in [0, T], \end{aligned}$$

where W_t^i and N_t^i , $i = 1, \cdots, d$, are Brownian motions and Poisson processes with intensities λ_i respectively, and these $2(d+1)$ stochastic processes are independent. The coefficient functions in (1) are assumed to be sufficiently smooth and subject to an upper bound and a lower bound[1] independent of d on finite interval $[0, T]$. In the above characterization, we may think of the Brownian motions W_t^0 and W_t^i as representing the sources of systematic and non-systematic white noises. As a non-trivial extension to the existing literature, we introduce the Poisson processes N_t^0 and N_t^i in this paper to represent sources of systematic and non-systematic jump risks.

Let $g_t^i > 0,\ i = 1, \cdots, d$, be finite numbers of shares invested in assets i at time t. Then the value of this portfolio is:

$$V_t = \sum_{i=1}^{d} g_t^i X_t^i \tag{2}$$

and the weighted cash amount invested in asset i at time t is:

$$\omega_t^i = \frac{g_t^i X_t^i}{V_t}.$$

Under the self-financing constraint:

$$dV_t = \sum_{i=1}^{d} g_t^i dX_{t,}^i \tag{3}$$

the dynamics of V_t is governed by the following stochastic differential equation:

$$dV_t = \hat{\alpha}_t V_t dt + \hat{\beta}_t V_t dW_t^0 + \sum_{i=1}^{d} \hat{\gamma}_t^i V_t dW_t^i + \hat{\delta}_{t-} V_t dN_t^0 + \sum_{i=1}^{d} \hat{\theta}_{t-}^i V_t dN_t^i, \tag{4}$$

where $\hat{\alpha}_t = \sum_{i=1}^{d} \alpha_i(t)\omega_t^i$, $\hat{\beta}_t = \sum_{i=1}^{d} \beta_i(t)\omega_t^i$, $\hat{\gamma}_t^i = \gamma_i(t)\omega_t^i$, $\hat{\delta}_{t-} = \sum_{i=1}^{d} \delta_i(t-)\omega_t^i$, and $\hat{\theta}_{t-}^i = \theta_i(t-)\omega_t^i$, represent the appreciation rate, the aggregated white-noise volatility, the asset i -specific white-noise volatility, the aggregated jump size, and the asset i-specific jump size respectively. Let $L_t = \ln(V_t/V_0)$. By Ito's formula on jump-diffusion

[1] Throughout the paper, this is essentially a boundedness condition as in Arnold (1974). Given smooth time-functions, boundedness is guaranteed within a finite time interval as is the case in this paper. The coefficients could grow at a rapid rate with time within the time interval. Boundedness and measurability also ensure that a unique continuous solution to the stochastic differential equation exists.

processes (see Ikeda and Watanabe(1981)), L_t satisfies the following stochastic differential equation:

$$dL_t = \left[\hat{\alpha}_t - \frac{1}{2}\hat{\beta}_t^2 - \frac{1}{2}\sum_{i=1}^{d}\left(\hat{\gamma}_t^i\right)^2 + \sum_{i=1}^{d}\lambda_i \ln\left(1+\hat{\theta}_{t-}^i\right)\right]dt \tag{5}$$
$$+\hat{\beta}_t dW_t^0 + \sum_{i=1}^{d}\hat{\gamma}_t^i dW_t^i + \ln\left(1+\hat{\delta}_{t-}\right)dN_t^0 + \sum_{i=1}^{d}\ln\left(1+\hat{\theta}_{t-}^i\right)d\tilde{N}_t^i,$$

where $\tilde{N}_t^i = N_t^i - \lambda_i t$ are the compensated martingale processes of N_t^i. In this paper, we study three important categories of large diversified portfolios.

(I) (Almost) equal money-weighted portfolios:

An almost equal money-weighted portfolio is a portfolio satisfying the following condition for $i = 1, \cdots, d$, $t \in [0, T]$, and some constant $\delta > 0$:

$$\left|\omega_t^i - \frac{1}{d}\right| \leq \frac{\delta}{d^2} \text{ a.s.} \tag{6}$$

This condition is proposed in Hofmann and Platen (2000).

(II) (Almost) risk-minimizing portfolios:

A risk-minimizing fully invested portfolio is achieved by the following risk-averse strategy $\left(\omega^1(t), \cdots, \omega^d(t)\right)$:

$$\begin{cases} \min f(\omega^1(t), \cdots, \omega^d(t)) \\ \sum_{i=1}^{d}\omega^i(t) = 1, \end{cases} \tag{7}$$

where the quadratic objective function,

$$f\left(\omega^1(t), \cdots, \omega^d(t)\right) = \left[\sum_{i=1}^{d}\beta_i(t)\omega^i(t)\right]^2 + \lambda_0\left[\sum_{i=1}^{d}\delta_i(t-)\omega^i(t)\right]^2 \tag{8}$$
$$+\sum_{i=1}^{d}\left[\gamma_i(t)\omega^i(t)\right]^2 + \sum_{i=1}^{d}\lambda_i\left[\theta_i(t-)\omega^i(t)\right]^2,$$

reflects the magnitude of the overall risk at t. This quadratic function is the variance of the portfolio return in (5). By solving the Lagrangian multiplier equations

$$\begin{cases} \dfrac{\partial}{\partial \omega^i(t)} \left[f\left(\omega^1(t), \cdots, \omega^d(t)\right) + \zeta \left(\sum_{i=1}^{d} \omega^i(t) - 1 \right) \right] = 0, \ i = 1, \cdots, d \\ \sum_{i=1}^{d} \omega^i(t) - 1 = 0, \end{cases} \tag{9}$$

we obtain the solution to the optimization problem in (7) to (9):

$$\omega^i(t) = -\frac{\zeta(t) + 2\beta_i(t)c_1(t) + 2\lambda_0 \delta_i(t-)c_2(t)}{2a_i(t)}, \tag{10}$$

where

$$a_i(t) = \gamma_i^2(t) + \lambda_i \theta_i^2(t-),$$

and $\zeta(t)$, $c_1(t)$, $c_2(t)$ solve the following simultaneous linear equations:

$$\begin{cases} \sum_{k=1}^{d} \dfrac{\beta_k(t)}{a_k(t)} c_1(t) + \lambda_0 \sum_{k=1}^{d} \dfrac{\delta_k(t-)}{a_k(t)} c_2(t) + \sum_{k=1}^{d} \dfrac{1}{2a_k(t)} \zeta(t) = -1 \\ \left(1 + \sum_{k=1}^{d} \dfrac{\beta_k^2(t)}{a_k(t)}\right) c_1(t) + \lambda_0 \sum_{k=1}^{d} \dfrac{\beta_k(t)\delta_k(t-)}{a_k(t)} c_2(t) + \sum_{k=1}^{d} \dfrac{\beta_k(t)}{2a_k(t)} \zeta(t) = 0 \\ \sum_{k=1}^{d} \dfrac{\beta_k(t)\delta_k(t-)}{a_k(t)} c_1(t) + \left(1 + \lambda_0 \sum_{k=1}^{d} \dfrac{\delta_k^2(t-)}{a_k(t)}\right) c_2(t) + \sum_{k=1}^{d} \dfrac{\delta_k(t-)}{2a_k(t)} \zeta(t) = 0. \end{cases}$$

In particular, if we let $\beta_i(t), \delta_i(t-) \equiv 0$ for $i = 1, \cdots, d$, then

$$\omega^i(t) = \left(a_i(t) \sum_{k=1}^{d} \frac{1}{a_k(t)} \right)^{-1}, \tag{11}$$

which corresponds to the strategy that minimizes the total non-systematic risk.

An almost risk-minimizing fully invested portfolio is a portfolio $\left(\omega_t^1, \cdots, \omega_t^d\right)$ that satisfies the following conditions for $i = 1, \cdots, d$, $t \in [0, T]$, and some constant $\delta > 0$:

$$\left|\omega_t^i - \omega^i(t)\right| \leq \frac{\delta}{d^2} \text{ a.s.} \tag{12}$$

Lemma 1: Suppose there exists a constant $\delta > 0$ independent of d, such that

$$\frac{a_i(t)}{a_j(t)} \leq \delta \tag{13}$$

for $i < j,\ i,\ j \in \{1, \cdots, d,\},\ t \in [0, T]$. Then we have

$$\omega_t^i \leq \frac{C}{d},\ \ i = 1, \cdots, d,\ t \in [0, T], \tag{14}$$

where $C > 0$ is a constant independent[2] of d, and ω_t^i's are given in (11).

The proof of the lemma is straightforward. We can show using the triangle inequality via (12) that $\omega_t^i \leq \omega^i(t) + \frac{\delta}{d^2}$. (11) and (13) together imply $\omega^j(t) \leq \delta\omega^i(t)$. If we let $\omega^1(t) = \frac{C'}{d}$ where C' is some finite constant, then all the $\omega^j(t)$'s will be bounded by $\frac{\delta^j C'}{d}$. Then, $\omega_t^i \leq \frac{\delta^i C' + \delta}{d}$. Put C as the generic constant $\delta^i C' + \delta$. The result of this lemma is useful for proving the convergence of the portfolios to their respective asymptotics. One can arrive at conclusions similar to Lemma 1 for more general strategies in category II by imposing suitable boundedness conditions on the coefficient functions in (1).

(III) Financial indices:

Indices can be regarded as portfolios with time-constant strategy $g_t^i \equiv g^i$. They are often used as proxies of market portfolios in financial studies and as underlying benchmarks for derivative products.

Lemma 2: Suppose (i) there exists a constant $\delta > 0$ independent of d such that

$$\frac{\omega_0^i}{\omega_0^j} \leq \delta, \tag{15}$$

for $i < j,\ i,\ j \in \{1, \cdots, d,\}$, and (ii) all the coefficient functions in (1) have upper and lower bounds independent of d on $[0, T]$. Then there exists a constant C independent of d such that the following estimate holds for $i, j = 1, \cdots, d$:

$$E\left\{\frac{\omega_t^i}{\omega_t^j}\right\} \leq C. \tag{16}$$

[2]In this article we use C to denote generic constants. They may represent different values in different contexts.

Proof of Lemma 2: Let $Y_t^{i,j} = \frac{X_t^i}{X_t^j}$ and $E_{i,j}(t) = E\{Y_t^{i,j}\}$. By Ito's formula, we have

$$\begin{aligned}\frac{dY_t^{i,j}}{Y_t^{i,j}} &= \left[\alpha_i(t) - \alpha_j(t) + \beta_j^2(t) + \gamma_j^2(t) - \beta_i(t)\beta_j(t)\right] dt \\ &\quad + (\beta_i(t) - \beta_j(t))\, dW_t^0 + \gamma_i(t) dW_t^i - \gamma_j(t) dW_t^j \\ &\quad + \frac{\delta_i(t-) - \delta_j(t-)}{1+\delta_j(t-)} dN_t^0 + \theta_i(t-) dN_t^i - \frac{\theta_j(t-)}{1+\theta_j(t-)} dN_t^j.\end{aligned}$$

Since stochastic integrals with respect to W_t^i or N_t^i are martingales, it is easy to show that

$$dE_{i,j}(t) = H_{i,j}(t) E_{i,j}(t) dt,$$

where

$$\begin{aligned}H_{i,j}(t) &= \alpha_i(t) - \alpha_j(t) + \beta_j^2(t) + \gamma_j^2(t) - \beta_i(t)\beta_j(t) \\ &\quad + \lambda_0 \frac{\delta_i(t-) - \delta_j(t-)}{1+\delta_j(t-)} + \lambda_i \theta_i(t-) - \lambda_j \frac{\theta_j(t-)}{1+\theta_j(t-)}\end{aligned}$$

Thus by (ii) we have

$$E_{i,j}(t) = \left(\frac{X_0^i}{X_0^j}\right) e^{\int_0^t H_{i,j}(s) ds} \le C' \frac{X_0^i}{X_0^j} \tag{17}$$

for some constant $C' > 0$ independent of d. It follows from (15) and (17) that

$$E\left\{\frac{\omega_t^i}{\omega_t^j}\right\} = \frac{g^i}{g^j} E_{i,j}(t) \le C' \frac{\omega_0^i}{\omega_0^j} \le C$$

with $C = C'\delta$.

Q.E.D.

Lemma 2 shows that the weighted values of different components of an index will be of a similar order of magnitude if they start from a comparable state. It follows from Holder's inequality that

$$1 = \left(E\left\{\left(\frac{\omega_t^i}{\omega_t^j}\right)^{\frac{1}{2}} \left(\frac{\omega_t^j}{\omega_t^i}\right)^{\frac{1}{2}}\right\}\right)^2 \le E\left\{\frac{\omega_t^i}{\omega_t^j}\right\} E\left\{\frac{\omega_t^j}{\omega_t^i}\right\},$$

or

$$E\left\{\frac{\omega_t^j}{\omega_t^i}\right\} \geq \frac{1}{E\left\{\frac{\omega_t^i}{\omega_t^j}\right\}}.$$

By (16), we have

$$E\left\{\frac{1}{\omega_t^i}\right\} = \sum_{j=1}^{d} E\left\{\frac{\omega_t^j}{\omega_t^i}\right\} \geq \sum_{j=1}^{d} \frac{1}{E\left\{\frac{\omega_t^i}{\omega_t^j}\right\}} \geq \frac{d}{C}.$$

This result is analogous to Lemma 1.

3 Asymptotic dynamics of the portfolios

It has long been accepted that non-systematic risks are fully diversifiable when the number of assets d tends to infinity, but it was only until recently that Hofmann and Platen (2000) gave a rigorous proof to this conclusion in a setting of lognormal prices. They showed the asymptotic convergence in terms of the logarithm values of the portfolios. The results are suited for long-term investigation of the portfolios. But if we want to study the effect of replacing large diversified portfolios with their asymptotic dynamics in derivative pricing, we have to consider the convergence in the portfolio values rather than in their logarithms. In fact, the difference between the convergence in values and convergence in log-values is not mathematically trivial. For example, it is proved in Hofmann and Platen (2000) for the diffusion case that, if $f \in \Phi_P^3$, where

$$\Phi_P^3 = \left\{ f : \mathbf{R} \to \mathbf{R} \,\middle|\, \exists n, \text{ s.t. } \left|\frac{d^k f}{dx^k}(x)\right| \leq C(1+|x|^n), \text{for } k = 0, \cdots, 3. \right\},$$

then the following estimate holds for the log-value of a portfolio L and that of the respective discrete asymptotic portfolio $\bar{L}$ with step size $\triangle$:

$$|E(f(\bar{L}_T)) - E(f(L_T))| \leq C_f(T)\left(\frac{1}{d} + \Delta\right).$$

But if h are the payoff functions for derivatives such as futures and options, then the function f implied by the relationship $h(V_0 \exp(L_T)) = f(L_T)$ does not satisfy the polynomial growth condition of Φ_P^3. The aim of this section is to derive strong convergence and weak convergence results on the portfolio values directly so as to ensure the corresponding convergence in derivative pricing.

In order to simplify (4) for analyses involving V_t and for computation of VaR as well as derivative pricing, we introduce a more tractable SDE driven merely by W_t^0 and N_t^0:

$$\frac{d\bar{V}_t}{\bar{V}_t} = \bar{\mu}(t)dt + \bar{\beta}(t)dW_t^0 + \bar{\delta}(t-)dN_t^0 \tag{18}$$

with $\bar{V}_0 = V_0$ and the coefficients corresponding to the three categories of portfolios given as follows.

(I) (Almost) equal money-weighted portfolios:

$$\begin{aligned} \bar{\alpha}(t) &= \frac{1}{d}\sum_{i=1}^{d}\alpha_i(t), \quad \bar{\mu}(t) = \bar{\alpha}(t) + \frac{1}{d}\sum_{i=1}^{d}\lambda_i\theta_i(t-), \\ \bar{\beta}(t) &= \frac{1}{d}\sum_{i=1}^{d}\beta_i(t), \quad \bar{\delta}(t-) = \frac{1}{d}\sum_{i=1}^{d}\delta_i(t-). \end{aligned} \tag{19}$$

(II) (Almost) risk-minimizing portfolios:

$$\begin{aligned} \bar{\alpha}(t) &= \sum_{i=1}^{d}\omega^i(t)\alpha_i(t), \quad \bar{\mu}(t) = \bar{\alpha}(t) + \sum_{i=1}^{d}\omega^i(t)\lambda_i\theta_i(t-), \\ \bar{\beta}(t) &= \sum_{i=1}^{d}\omega^i(t)\beta_i(t), \quad \bar{\delta}(t-) = \sum_{i=1}^{d}\omega^i(t)\delta_i(t-). \end{aligned} \tag{20}$$

where $\omega^i(t),\ i = 1, \cdots, d$, are given in (10). If all the coefficient functions in (1) have upper and lower bounds independent of d on $[0, T]$, then by (6) and (12), we have

$$\left|\bar{\phi}(t) - \hat{\phi}_t\right| \leq \frac{C}{d} \tag{21}$$

for categories I and II, where $C > 0$ is a constant independent of d, $\phi(t)$ stands for $\alpha(t), \beta(t)$, and $\delta(t-)$, and ϕ_t stands for α_t, β_t, and δ_{t-}.

(III) Financial indices:

Following the same treatment on the previous two categories of portfolios, one can formulate the coefficient functions for the SDE of index portfolios as follows:

$$\begin{aligned}
\bar{\alpha}(t) &= \sum_{i=1}^{d} \frac{g^i X_t^i}{V_t} \alpha_i(t), \quad \bar{\mu}(t) = \bar{\alpha}(t) + \sum_{i=1}^{d} \frac{g^i X_t^i}{V_t} \lambda_i \theta_i(t-), \\
\bar{\beta}(t) &= \sum_{i=1}^{d} \frac{g^i X_t^i}{V_t} \beta_i(t), \quad \bar{\delta}(t-) = \sum_{i=1}^{d} \frac{g^i X_t^i}{V_t} \delta_i(t-).
\end{aligned} \tag{22}$$

However this does not ensure that the SDE (18) is self-containing as X_t^i are involved in the coefficient functions in (22). Therefore, equation (18) cannot serve as a satisfactory simplification of the equation (4). To derive an SDE which is free of all the asset-specific risks for the asymptotic dynamics of the portfolios, we introduce the following jump-diffusion processes $\bar{X}_t^i$ as follows to replace X_t^i:

$$d\bar{X}_t^i = \alpha_i(t)\bar{X}_t^i dt + \beta_i(t)\bar{X}_t^i dW_t^0 + \delta_i(t-)\bar{X}_t^i dN_t^0 \tag{23}$$

with $\bar{X}_0^i = X_0^i,, \ i = 1, \cdots, d$, and a portfolio

$$\bar{V}_t = \sum_{i=1}^{d} g^i \bar{X}_t^i. \tag{24}$$

$\bar{V}_t$ satisfies (18) with

$$\begin{aligned}
\bar{\alpha}(t) &= \sum_{i=1}^{d} \frac{g^i \bar{X}_t^i}{\bar{V}_t} \alpha_i(t), \quad \bar{\mu}(t) = \bar{\alpha}(t) + \sum_{i=1}^{d} \frac{g^i \bar{X}_t^i}{\bar{V}_t} \lambda_i \theta_i(t-), \\
\bar{\beta}(t) &= \sum_{i=1}^{d} \frac{g^i \bar{X}_t^i}{\bar{V}_t} \beta_i(t), \quad \bar{\delta}(t-) = \sum_{i=1}^{d} \frac{g^i \bar{X}_t^i}{\bar{V}_t} \delta_i(t-).
\end{aligned} \tag{25}$$

We shall prove that $\bar{V}_t$ approximates V_t at the same convergence rate as the portfolios in the other two categories approximate their counterparts.

Let $\bar{L}_t = \ln(\bar{V}_t/\bar{V}_o)$. By (18) and Ito's formula, $\bar{L}_t$ satisfies the following SDE:

$$d\bar{L}_t = \left[\bar{\mu}(t) - \frac{1}{2}\bar{\beta}^2(t)\right] dt + \bar{\beta}(t) dW_t^0 + \ln\left(1 + \bar{\delta}(t-)\right) dN_t^0. \tag{26}$$

For a semimartingale Z_t given by the SDE:

$$dZ_t = A_t dt + \sum_{i=1}^{d} B_t dW_t^i + \sum_{i=1}^{d} C_{t-}^i dN_t^i, \tag{27}$$

we define

$$\Lambda\left(Z_t\right)=A_t+\frac{1}{2}\sum_{i=1}^{d}\left(B_t^i\right)^2+\sum_{i=1}^{d}\lambda_i\left(e^{C_{t-}^i}-1\right).$$

Lemma 3: If there exists a constant $\delta>0$ such that

$$\left|\Lambda\left(Z_t\right)\right|\le\delta \quad \text{a.s.},$$

then the semimartingle Z_t in (27) satisfies

$$E\left\{e^{Z_0}\right\}e^{-\delta T}\le E\left\{e^{Z_T}\right\}\le E\left\{e^{Z_0}\right\}e^{\delta T}. \tag{28}$$

Proof of Lemma 3: Let $Y_t=e^{Z_t}$. By Ito's formula,

$$\begin{aligned} dY_t &= Y_t\left\{\left[A_t+\frac{1}{2}\sum_{i=1}^{d}\left(B_t^i\right)^2+\sum_{i=1}^{d}\lambda_i\left(e^{C_{t-}^i}-1\right)\right]dt\right.\\ &\quad \left.+\sum_{i=1}^{d}B_t^i dW_t^i+\sum_{i=1}^{d}\lambda_i\left(e^{C_{t-}^i}-1\right)d\tilde{N}_t^i\right\}. \end{aligned}$$

Thus

$$\begin{aligned} E\left\{Y_T\right\} &= E\left\{Y_0\right\}+E\left\{\int_0^T Y_t\left[A_t+\frac{1}{2}\sum_{i=1}^{d}\left(B_t^i\right)^2+\sum_{i=1}^{d}\lambda_i\left(e^{C_{t-}^i}-1\right)\right]dt\right\}\\ &\le E\left\{Y_0\right\}+\delta\int_0^T E\left\{Y_s\right\}ds. \end{aligned}$$

By Gronwall's inequality we have

$$E\left\{e^{Z_T}\right\}\le E\left\{e^{Z_0}\right\}e^{\delta T}.$$

The other half of (28) can be proved similarly.

Q.E.D.

Theorem 1: Suppose (i) all the coefficient functions in (1) have upper and lower bounds independent of d on $[0,T]$ and (ii) portfolios of categories I-III satisfy the conditions (6), (13) and (15) respectively. Then there exists a constant $C > 0$ such that

$$E\left\{\left|\bar{V}_T - V_T\right|\right\} \leq \frac{C}{\sqrt{d}}. \tag{29}$$

Proof of Theorem 1 for categories I and II: We take category I as an example.

The proof for category II can be accomplished by using the same approach and Lemma 2.

It follows from Holder's inequality that

$$\begin{aligned} E\{\left|\bar{V}_T - V_T\right|\} &= E\left\{\left|\bar{V}_T\left(1 - \frac{V_T}{\bar{V}_T}\right)\right|\right\} \qquad (30) \\ &= E\left\{\left|V_0 e^{\bar{L}_T}\left(1 - e^{L_T - \bar{L}_T}\right)\right|\right\} \\ &\leq V_0\left(E\left\{e^{2\bar{L}_T}\right\}\right)^{\frac{1}{2}}\left(1 - 2E\left\{e^{L_T - \bar{L}_T}\right\} + E\left\{e^{2(L_T - \bar{L}_T)}\right\}\right)^{\frac{1}{2}}. \end{aligned}$$

Since $\bar{L}_t$ satisfies (26), by condition (i) we have

$$\Lambda\left(2\bar{L}_t\right) = \left|2\bar{\mu}(t) + \bar{\beta}^2(t) + \lambda_0\bar{\delta}(t-)\left(\bar{\delta}(t-) + 2\right)\right| \leq C$$

for some constant $C > 0$. Thus we obtain the following boundedness by applying Lemma 3:

$$E\{e^{2\bar{L}_T}\} \leq C, \tag{31}$$

where C is dependent on T but independent of d. (5) and (26) yield that,

$$\begin{aligned} d(L_t - \bar{L}_t) &= \left[\hat{\alpha}_t - \bar{\alpha}(t) - \frac{1}{2}\left(\hat{\beta}_t^2 - \bar{\beta}^2(t)\right) - \frac{1}{d}\sum_{i=1}^{d}\lambda_i \ln\left(1 + \hat{\theta}_i(t-)\right) - \frac{1}{2}\sum_{i=1}^{d}\left(\hat{\gamma}_t^i\right)^2\right] dt \\ &\quad + \left(\hat{\beta}_t - \bar{\beta}(t)\right) dW_t^0 + \sum_{i=1}^{d}\hat{\gamma}_t^i dW_t^i \\ &\quad + \left[\ln\left(1 + \hat{\delta}_{t-}\right) - \ln\left(1 + \bar{\delta}(t-)\right)\right] dN_t^0 + \sum_{i=1}^{d}\ln\left(1 + \hat{\theta}_{t-}^i\right) dN_t^i. \end{aligned}$$

By (6), (21) and condition (i), it is not difficult to show that,

$$\begin{aligned}\left|\Lambda\left(L_t-\bar{L}_t\right)\right| &= \left|\hat{\alpha}_t-\bar{\alpha}(t)+\bar{\beta}(t)\left(\bar{\beta}(t)-\hat{\beta}_t\right)+\lambda_0\frac{\hat{\delta}_{t-}-\bar{\delta}(t-)}{1+\bar{\delta}(t-)}\right.\\ &\quad \left.+\sum_{i=1}^{d}\lambda_i\left[\hat{\theta}^i_{t-}-\frac{1}{d}\ln\left(1+\hat{\theta}_i(t-)\right)\right]\right|\le\frac{\delta}{d}\end{aligned}$$

and

$$\begin{aligned}\left|\Lambda\left(2\left(L_t-\bar{L}_t\right)\right)\right| &= \left|2\left(\hat{\alpha}(t)-\bar{\alpha}(t)\right)+\left(\hat{\beta}_t-3\bar{\beta}(t)\right)\left(\hat{\beta}(t)-\bar{\beta}(t)\right)\right.\\ &\quad +\sum_{i=1}^{d}\left(\hat{\gamma}^i_t\right)^2+\lambda_0\frac{\left(2+\hat{\delta}_{t-}+\bar{\delta}(t-)\right)\left(\hat{\delta}_{t-}-\bar{\delta}_{t-}\right)}{\left(1+\bar{\delta}(t-)\right)^2}\\ &\quad \left.+\sum_{i=1}^{d}\lambda_i\left[\left(\hat{\theta}^i_{t-}\right)^2+2\left(\hat{\theta}^i_{t-}-\frac{1}{d}\ln\left(1+\hat{\theta}_i(t-)\right)\right)\right]\right|\le\frac{\delta}{d}\end{aligned}$$

for some constant $\delta>0$. Then it follows from Lemma 3 that

$$\begin{aligned}E\left\{e^{L_T-\bar{L}_T}\right\} &\ge e^{-\frac{\delta T}{d}},\\ E\left\{e^{2(L_T-\bar{L}_T)}\right\} &\le e^{\frac{\delta T}{d}}.\end{aligned}$$

When d is sufficiently large, there exist a constant $C>0$ dependent on T but independent of d such that

$$1-2E\left\{e^{L_T-\bar{L}_T}\right\}+E\left\{e^{2(L_T-\bar{L}_T)}\right\}\le 1-2e^{-\frac{\delta T}{d}}+e^{\frac{\delta T}{d}}\le\frac{C}{d}. \tag{32}$$

The result (29) follows from (30) to (32).

Q.E.D.

Proof of Theorem 1 for category III: For parsimony in notations, we rewrite the portfolios (2) as follows:

$$V_t=\sum_{k=1}^{d}\alpha^k A^k_t B^k_t,$$

where

$$\begin{aligned}
\alpha^k &= \omega_0^k V_0, \\
A_t^k &= \exp\left[\int_0^t \left(\alpha_k(s) - \frac{1}{2}\beta_k^2(s) + \lambda_k\theta_k(s-)\right) ds + \int_0^t \beta_k(s) dW_s^0 + \int_0^t \ln\left(1+\delta_k(s-)\right) dN_s^0\right], \\
B_t^k &= \exp\left[\int_0^t \left(-\frac{1}{2}\gamma_k^2(s) - \lambda_k\theta_k(s-)\right) ds + \int_0^t \gamma_k(s) dW_s^k + \int_0^t \ln\left(1+\theta_k(s-)\right) dN_s^k\right].
\end{aligned}$$

It follows from Ito's formula that B_t^k solves the SDE:

$$\begin{cases} B_0^k = 1 \\ dB_t^k = \gamma(t)B_t^k dW_t^k + \theta_k(t-)B_t^k d\tilde{N}_t^k, \end{cases}$$

and is a martingale. Therefore, it is easy to show that

$$E\{B_t^k\} = 1,\ t \in [0,T].$$

Let $Y_t = \left(B_t^k\right)^2$. By Ito's formula, we have

$$dY_t = \left(\gamma_k^2(t) + \lambda_k\theta_k^2(t-)\right) Y_t dt + 2\gamma(t)Y_t dW_t^k + \left(\theta_k^2(t-) + 2\theta_k(t-)\right) Y_t d\tilde{N}_t^k.$$

Solving the resulting ordinary differential equation for $E\{Y_t\}$, we obtain

$$E\left\{\left(B_T^k\right)^2\right\} = \exp\left(\int_0^T \left(\gamma_k^2(t) + \lambda_k\theta_k^2(t-)\right) dt\right).$$

Likewise,

$$E\left\{\left(A_T^k\right)^2\right\} = \exp\left(\int_0^T \left[2\left(\alpha_k(t) + \lambda_k\theta_k(t-)\right) + \beta_k^2(t) + \lambda_0\left(\delta_k^2(t-) + 2\delta_k(t-)\right)\right] dt\right).$$

Since $\left\{N_t^k, W_t^k\right\}_{k=0}^d$ are independent, we can estimate the difference between V_T and $\bar{V}_T$ as follows.

$$\begin{aligned}
& E\left\{(V_T - \bar{V}_T)^2\right\} \\
= \ & E\left\{\left[\sum_{k=1}^d \alpha^k A_T^k \left(B_T^k - 1\right)\right]^2\right\}
\end{aligned}$$

$$= \sum_{k=1}^{d} \left(\alpha^k\right)^2 E\left\{\left(A_T^k\right)^2\right\} E\left\{\left(B_T^k - 1\right)^2\right\}$$

$$+2 \sum_{k \neq k'} \alpha^k \alpha^{k'} E\left\{A_T^k A_T^{k'}\right\} E\left\{B_T^k - 1\right\} E\left\{B_T^{k'} - 1\right\}$$

$$= \sum_{k=1}^{d} (\alpha^k)^2 \exp\left(\int_0^T \left[2\left(\alpha_k(t) + \lambda_k \theta_k(t-)\right) + \beta_k^2(t) + \lambda_0 \left(\delta_k^2(t-) + 2\delta_k(t-)\right)\right] dt\right)$$

$$\times \left[\exp\left(\int_0^T \left(\gamma_k^2(t) + \lambda_k \theta_k^2(t-)\right) dt\right) - 1\right]$$

$$\leq C' \sum_{k=1}^{d} \left(\omega_0^k\right)^2,$$

where $C' > 0$ is a constant dependent on T but independent of d. Since (15) implies that $\omega_0^i \leq \frac{\delta}{d}, \;\; i = 1, \cdots, d$, we have

$$E\left\{(V_T - \bar{V}_T)^2\right\} \leq \frac{C'\delta^2}{d},$$

or

$$E\left\{\left|V_T - \bar{V}_T\right|\right\} \leq \frac{C}{\sqrt{d}}$$

with $C = \delta\sqrt{C'}$.

Q.E.D.

Remark: The convergence stated in Theorem 1 is called strong convergence in the terminology of numerical SDE. The strong convergence ensures the rationality of replacing V_t by $\bar{V}_t$ in applications such as scenario simulation and stress testing. But it is not always necessary in all situations. For instance, in pricing vanilla European options, we are only concerned about the weak convergence, i.e. the convergence of expected functionals of V_t to those of $\bar{V}_t$. It is straightforward to derive a weak convergence of order $O\left(\frac{1}{\sqrt{d}}\right)$. However, the following theorem shows that the weak convergence can be of higher order.

Theorem 2: Suppose (i) all the coefficient functions in (1) have upper and lower bounds independent of d on $[0,T]$ and (ii) portfolios of categories I-III satisfy the conditions (6), (13) and (15) respectively. Then for any f in the set[3]

$$\Phi_P^2 \equiv \left\{ f : \mathbf{R} \to \mathbf{R} \,\middle|\, \exists n, \text{ s.t. } \left| \frac{d^k f}{dx^k}(x) \right| \le C(1+|x|^n) \text{ for } k = 0, \cdots, 2. \right\},$$

there exists a constant $C > 0$, dependent on f and T but independent of d, such that the following estimate holds:

$$\left| E\{f(\bar{V}_T) - f(V_T)\} \right| \le \frac{C}{d}.$$

Proof of Theorem 2 for categories I and II: We show the proof of category I. The proof can be easily adapted to category II.

For $f \in \Phi_P^2$, the functional $u(t,x) \equiv E\left\{f(\bar{V}_T)|\bar{V}_t = x\right\}$ solves the Kolmogorov backward integro-differential equation

$$\begin{cases} \mathcal{L}^0 u(t,x) = 0 \\ u(T,x) = f(x), \end{cases} \tag{33}$$

where

$$\mathcal{L}^0 u(t,x) = \frac{\partial u}{\partial t} + \bar{\mu}(t) x \frac{\partial u}{\partial x} + \frac{1}{2}\bar{\beta}^2(t) x^2 \frac{\partial^2 u}{\partial x^2} + \lambda_0 \left[u\left(t, x + \bar{\delta}(t-)x\right) - u(t,x)\right].$$

Clearly, we have $E\left\{f(\bar{V}_T)\right\} = u(0,\bar{V}_0) = u(0,V_0)$ and $u\left(T,V_T\right) = f\left(V_T\right)$. By Ito's formula we have

$$\begin{aligned} & E\left\{f(V_T) - f(\bar{V}_T)\right\} \qquad (34) \\ = \;& E\left\{u\left(T,V_T\right) - u\left(0,V_0\right)\right\} \\ = \;& E\left\{ \int_0^T \left[\frac{\partial u(t,V_t)}{\partial t} + \hat{\alpha}_t V_t \frac{\partial u(t,V_t)}{\partial x} + \frac{1}{2}\left(\hat{\beta}_t^2 + \sum_{i=1}^d \left(\hat{\gamma}_t^i\right)^2\right) V_t^2 \frac{\partial^2 u(t,V_t)}{\partial x^2} \right.\right. \\ & \left.\left. + \lambda_0 \left(u\left(t, V_t + \hat{\delta}_{t-} V_t\right) - u(t,V_t)\right) + \sum_{i=1}^d \lambda_i \left(u\left(t, V_t + \hat{\theta}_{t-}^i V_t\right) - u(t,V_t)\right) \right] dt \right\}. \end{aligned}$$

[3]This relaxes the respective requirement on f in the theorem on weak convergence in Hofmann and Platen (2000).

It follows from (33) and (34) that,

$$
\begin{aligned}
& \left|E\{f(V_T)-f(\bar{V}_T)\}\right| \qquad (35)\\
= & \left|E\left\{\int_0^T \left[(\hat{\alpha}_t-\bar{\alpha}(t))\,V_t\frac{\partial u(t,V_t)}{\partial x}\right.\right.\right.\\
& +\frac{1}{2}\left(\hat{\beta}_t^2-\bar{\beta}^2(t)+\sum_{i=1}^d\left(\hat{\gamma}_t^i\right)^2\right)V_t^2\frac{\partial^2 u(t,V_t)}{\partial x^2}\\
& +\lambda_0\left(u\left(t,V_t+\hat{\delta}_{t-}V_t\right)-u\left(t,V_t+\bar{\delta}(t-)V_t\right)\right)\\
& \left.\left.\left.+\sum_{i=1}^d\lambda_i\left(u\left(t,V_t+\hat{\theta}_{t-}^iV_t\right)-u(t,V_t)-\frac{\theta_i(t-)}{d}V_t\frac{\partial u(t,V_t)}{\partial x}\right)\right]dt\right\}\right|.
\end{aligned}
$$

We denote by I_t^4 the last term of the integrand on the right hand side of (35) and estimate it as follows:

$$
\begin{aligned}
\left|E\left\{I_t^4\right\}\right| \leq & \sum_{i=1}^d\lambda_iE\left\{\left|u\left(t,V_t+\hat{\theta}_{t-}^iV_t\right)-u(t,V_t)-\frac{\theta_i(t-)}{d}V_t\frac{\partial u(t,V_t)}{\partial x}\right|\right\} \qquad (36)\\
= & \sum_{i=1}^d\lambda_iE\left\{\left|\frac{\partial u}{\partial x}(t,V_t)\left(\hat{\theta}_{t-}^i-\frac{\theta_i(t-)}{d}\right)V_t+\right.\right.\\
& \left.\left.+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(t,V_t+\tau(V_t)\hat{\theta}_{t-}^iV_t\right)\left(\hat{\theta}_{t-}^i\right)^2V_t^2\right|\right\}\\
\leq & \sum_{i=1}^d\lambda_i\left[\left(E\left\{\left(\hat{\theta}_i(t-)-\frac{\theta_i(t-)}{d}\right)^2\right\}\right)^{\frac{1}{2}}\left(E\left\{\left(\frac{\partial u}{\partial x}(t,V_t)V_t\right)^2\right\}\right)^{\frac{1}{2}}\right.\\
& \left.+\left(E\left\{\left(\hat{\theta}_{t-}^i\right)^4\right\}\right)^{\frac{1}{2}}\left(E\left\{\left(\frac{\partial^2 u}{\partial x^2}\left(t,V_t+\tau(V_t)\hat{\theta}_{t-}^iV_t\right)V_t^2\right)^2\right\}\right)^{\frac{1}{2}}\right],
\end{aligned}
$$

where $\tau(V_t)\in(0,1)$. By the polynomial growth constraint on Φ_P^2 and (6), we have the following estimate:

$$\left|E\left\{I_t^4\right\}\right|\leq\frac{C}{d},\ \ t\in[0,T],$$

such that

$$\left|\int_0^T E\left\{I_t^4\right\} dt\right| \le \frac{CT}{d}.$$

The estimation of the other terms on the right hand side of (35) is easier and can be done in a similar way. Hence the theorem is proved.

Q.E.D.

Proof of Theorem 2 for category III: As introduced in the proof of Theorem 1, the processes $A_t^k,\ k=1,\cdots,d$, and $\bar{V}_t$ are independent of $B_t^k,\ k=1,\cdots,d$. Therefore, since $E\left\{B_T^k-1\right\}=0$ for $k=1,\cdots,d$, we have the following estimate:

$$\begin{aligned}
& \left|E\left\{f(V_T)-f(\bar{V}_T)\right\}\right| \\
= & \left|E\left\{f'(\bar{V}_T)(V_T-\bar{V}_T)+\frac{1}{2}f''\left(\tau\left(V_T,\bar{V}_T\right)V_T+\left(1-\tau(V_T,\bar{V}_T)\right)\bar{V}_T\right)(V_T-\bar{V}_T)^2\right\}\right| \\
\le & \left|\sum_{k=1}^d E\left\{f'(\bar{V}_T)\alpha^k A_T^k\right\}E\left\{B_T^k-1\right\}\right| \\
& +\frac{1}{2}\left|E\left\{f''\left(\tau\left(V_T,\bar{V}_T\right)V_T+\left(1-\tau(V_T,\bar{V}_T)\right)\bar{V}_T\right)(V_T-\bar{V}_T)^2\right\}\right| \\
\le & \frac{1}{2}\left(E\left\{\left(f''\left(\tau\left(V_T,\bar{V}_T\right)V_T+\left(1-\tau(V_T,\bar{V}_T)\right)\bar{V}_T\right)^2\right)\right\}\right)^{\frac{1}{2}}\left(E\left\{(V_T-\bar{V}_T)^4\right\}\right)^{\frac{1}{2}},
\end{aligned}$$

where $\tau\left(V_\tau,\bar{V}_\tau\right)\in(0,1)$. As f'' is at most of polynomial growth, it is not difficult to see that there exists a constant $C'>0$ dependent on T and f but independent of d, such that

$$\begin{aligned}
& \left|E\left\{f(V_T)-f(\bar{V}_T)\right\}\right| \\
\le & C'\left(E\left\{(V_T-\bar{V}_T)^4\right\}\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= C'\left(E\left\{\left[\sum_{k=1}^{d}\alpha^k A_T^k\left(B_T^k-1\right)\right]^4\right\}\right)^{\frac{1}{2}} \\
&= C'\left(E\left\{\sum_{k=1}^{d}\left(\alpha^k A_T^k\left(B_T^k-1\right)\right)^4+12\sum_{k\neq k'}\left(\alpha_T^k\alpha_T^{k'}A_T^kA_T^{k'}\left(B_T^k-1\right)\left(B_T^{k'}-1\right)\right)^2\right\}\right)^{\frac{1}{2}} \\
&\leq C'\left(\sum_{k=1}^{d}\left(\omega_0^k\right)^4+\sum_{k\neq k'}\left(\omega_0^k\omega_0^{k'}\right)^2\right)^{\frac{1}{2}} \\
&\leq \frac{C}{d},
\end{aligned}
$$

where $C > 0$ is a constant dependent on T and f but independent of d.

Q.E.D.

Remark So far, we have proved both the weak convergence and strong convergence in a continuous-time setting. One can easily extend Theorems 1 and 2 to the case of discrete-time strategies by applying the results on the two types of convergence for the discretization of jump diffusions in Liu and Li (1999, 2000). In practice, the step size for time discretization could be very small as real-time re-adjustments of portfolio positions can be facilitated by high performance computers. Therefore, the approximation error between $\bar{V}_t$ and V_t is primarily determined by the asset number d.

4 Asymptotic Value-at-Risk

Duffie and Pan (2001) developed an analytical formula for computing the VaR of multi-factor jump-diffusion portfolios. But the computation becomes tedious when the asset number becomes too large. It is also intractable if we use Monte Carlo method to estimate the VaR of large diversified portfolios because too many Brownian motions and Poisson paths have to be simulated. However, if we use the asymptotic dynamics (18) instead, then the analytical calculation and Monte Carlo estimation of the VaR will become much easier.

Suppose we observe that $X^i_{t_n} = X^i_n$, such that $V_n = \sum_{i=1}^d g^i_{t_n} X^i_n$. By (26), the change in the logarithm of the respective asymptotic portfolio values $\bar{V}_t$ at $t_{n+1} = t_n + \tau$ can be written as:

$$\Delta\bar{L}_n \equiv \bar{L}_{n+1} - \bar{L}_n = \left(\bar{\alpha}(t_n) - \frac{1}{2}\bar{\beta}^2(t_n)\right)\tau + \bar{\beta}(t_n)\tau z + \ln\left(1 + \bar{\delta}(t_n-)\right) N_\tau, \quad (37)$$

where z is a random normal variable and N_t is a standard Poisson process with intensity λ_0. $\bar{\alpha}(t_n)$, $\bar{\beta}(t_n)$, and $\bar{\delta}(t_n-)$ can be evaluated as specified in (19),(20), and (25) respectively. The calculation of the tail probability of $\Delta\bar{L}_n$ is straightforward:

$$P(\Delta\bar{L}_n \le c) = \sum_{n=0}^{\infty} p_n F(d_n) \quad (38)$$

for a given number c, where

$$p_n \equiv P(N_\tau = n) = e^{-\lambda_0\tau}\frac{(\lambda_0\tau)^n}{n!},$$

$$d_n = \frac{c - \tau\left(\bar{\alpha}(t_n) - \frac{1}{2}\bar{\beta}^2(t_n)\right) - n\ln\left(1 + \bar{\delta}(t_n-)\right)}{\tau\bar{\beta}(t_n)},$$

and F is the cumulative density function for a standard normal distribution. The series in (38) converges rapidly, so that only the first few terms need to be computed to achieve an accurate approximation of the tail probability. Finally, the VaR of V_{n+1} can be approximated by inverting the tail distribution function (38).

5 Simulation Studies

In this section, we investigate numerically the performance of $\bar{V}_t$ in mimicking the pathwise evolution of V_t and in proxying as a substitute of V_t for VaR estimation and derivative pricing.

5.1 Pathwise approximation

Theorem 1 implies that sample paths of the portfolio value V_t approach those of its asymptotic dynamics $\bar{V}_t$ with respect to the same realizations of the market white-noise W^0_t and market jump N^0_t as the asset number d increases. We test the result on all the three categories of portfolios with respect to the parameters as in Table 1. For simulation purpose, the parameters have been set to hold across each $i = 1, \cdots, d$,

Table 1: Parameters for simulations

d	α_i	β_i	γ_i	δ_i	θ_i	λ_i	X_0^i	g_0^i	Δt
200	-0.3	random in $[0,1]$	0.01	0.05	0.05	1.0	random in$[0,1]$	$(\frac{1}{d})(\frac{1}{X_0^i})$	$\frac{1}{365}$

where d is the number of assets, e.g. stocks, in the portfolio. Also, the portfolio value is normalized to start at $V_0 = 1$.

Figures 1 to 3 plot the sample paths of the jump-diffusion processes V_t of portfolios in the categories I-III and their asymptotic processes $\bar{V}_t$ respectively. See figures at the end of paper. It turns out that $\bar{V}_t$ almost replicates V_t in every case even if there were occurrences of dramatic jumps. This evidence of pathwise approximation ensures the feasibility of replacing the large diversified portfolios with their corresponding asymptotic dynamics in pricing path-dependent options on V_t. We shall further investigate this issue in the next subsection.

5.2 Valuation of Vanilla and Exotic Options

Financial indices are special portfolios that function not only as indicators of the economy, but also as the underlying variable for derivatives such as index futures and options. Prices of such derivatives can be expressed as expectations of functionals of the index V_t. According to Theorem 2, we can use $\bar{V}_T$ given in (24) and (25) to substitute for V_T in valuing vanilla European options on V_T. Furthermore, since $\bar{V}_t$ approximates V_t in a pathwise sense as demonstrated in the previous subsection, we can also replace the processes V_t with $\bar{V}_t$ in the valuation of path-dependent options.

Table 2 compares the prices of vanilla, barrier, Asian and lookback call options obtained by using V_t as the underlying with those obtained by using $\bar{V}_t$. For simplicity, we assume that the equation (4) already represents the evolution of V_t under a prescribed risk-neutral probability measure. The option prices are produced by Monte Carlo method with 1000 sample paths under the same setup as described in the previous subsection. Table 3 presents similar results on put options. From Tables 2 and 3, we can see that the two sets of prices are close enough in all cases. This confirms that $\bar{V}_t$ provides a computationally parsimonious proxy of V_t for pricing vanilla and exotic options.

Table 2: Prices of Call Options on V_t and $\bar{V}_t$ respectively

$S_0 = 1.0$	Call Option		Barrier Option		Asian Option		Lookback Option	
	on V_t	on $\bar{V}_t$	on V_t	on $\bar{V}_t$	on V_t	on $\bar{V}_t$	on V_t	on $\bar{V}_t$
K=0.7, H=1.4	0.4555	0.4551	1.58E-02	1.64E-02	0.3705	0.3694	0.4791	0.4771
K=0.8, H=1.4	0.3575	0.3571	1.19E-02	1.24E-02	0.2725	0.2714	0.3810	0.3790
K=0.9, H=1.4	0.2594	0.2590	7.97E-03	8.55E-03	0.1744	0.1733	0.2830	0.2810
K=1.0, H=1.4	0.1614	0.1610	4.05E-03	4.63E-03	7.76E-02	7.62E-02	0.1849	0.1829
K=1.1, H=1.4	7.97E-02	7.89E-02	1.03E-03	9.77E-04	1.08E-02	1.01E-02	9.47E-02	9.21E-02
K=1.2, H=1.4	3.21E-02	3.05E-02	0	0	1.54E-04	1.02E-04	3.63E-02	3.40E-02
K=1.3, H=1.4	7.04E-03	6.72E-03	0	0	0	0	8.49E-03	7.41E-03

Table 3: Prices of Put Options on V_t and $\bar{V}_t$ respectively

$S_0 = 1.0$	Put Option		Barrier Option		Asian Option		Lookback Option	
	on V_t	on $\bar{V}_t$	on V_t	on $\bar{V}_t$	on V_t	on $\bar{V}_t$	on V_t	on $\bar{V}_t$
K=0.7, H=1.4	0	0	0	0	0	0	0	0
K=0.8, H=1.4	0	0	0	0	0	0	0	0
K=0.9, H=1.4	0	0	0	0	0	0	0	0
K=1.0, H=1.4	0	0	0	0	1.28E-03	9.44E-04	0	0
K=1.1, H=1.4	1.64E-02	1.59E-02	9.01E-04	2.60E-04	3.25E-02	3.29E-02	7.83E-03	7.25E-03
K=1.2, H=1.4	6.68E-02	6.56E-02	3.78E-03	3.20E-03	0.1198	0.1209	4.75E-02	4.72E-02
K=1.3, H=1.4	0.1398	0.1398	7.70E-03	7.12E-03	0.2177	0.2188	0.1177	0.1186

5.3 Asymptotic VaR

Another application of the asymptotic portfolios $\bar{V}_t$ is that they can be used for computing the tail probability of $\Delta\bar{V}_t$ (or $\Delta\bar{L}_t$) so as to obtain approximate VaR of the actual portfolios V_t. For example, to estimate one-month [4] values-at-risk for the large diversified portfolios specified in subsection 5.1, we simulate 1000 sample paths of V_t and $\bar{V}_t$ respectively and compute the cumulative density function (cdf) of ΔV_t and $\Delta\bar{V}_t$. The 1-month cdf curves for ΔV_t and $\Delta\bar{V}_t$ are plotted in Figures 4 to 6 for portfolios in the three categories respectively. See figures at end of paper. The two cdf curves, for V_t and $\bar{V}_t$ respectively, are almost identical in all the situations, whereas the estimation based on $\bar{V}_t$ can provide for a great deal of economy in terms of computing resources in simulating multi-noise diffusion processes, V_t. This is a significant improvement in real time VaR estimation for large scale portfolios.

6 Conclusions

This paper derives continuous-time asymptotic dynamics in terms of a Brownian motion and a Poisson process for large diversified portfolios with jump-diffusion asset prices, such as equal money-weighted portfolios, risk-minimizing portfolios, and financial indices. The jump-diffusion process provides for better and more appropriate fit of empirically observed price processes displaying fat-tails. Different orders of strong convergence and weak convergence for the portfolio values toward their asymptotic values are proved. The prices of vanilla and path-dependent options can be approximated by using the respective asymptotic portfolio as the underlying security if the number of assets in a large diversified portfolio is sufficiently large.

Our results allow for portfolio VaR to be estimated analytically or else by Monte Carlo simulations based on the asymptotic dynamics. The theoretical results in this paper are verified by numerical evidence. The direct and explicit characterization of the asymptotic processes of these portfolios, and the analytical computation of VaR thus provide a convenient and tractable way of measuring Value-at-Risk for purpose of bank risk management.

[4]To be convincing, here we provide the results of long-term VaR to illustrate the effectiveness of using the asymptotic dynamics. Our other simulations show that they perform equally well or even better in the estimation of short-term VaR, which is popular in the practice of portfolio management.

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Figure 1: Sample Path of Portfolio I and Respective Asymptotics

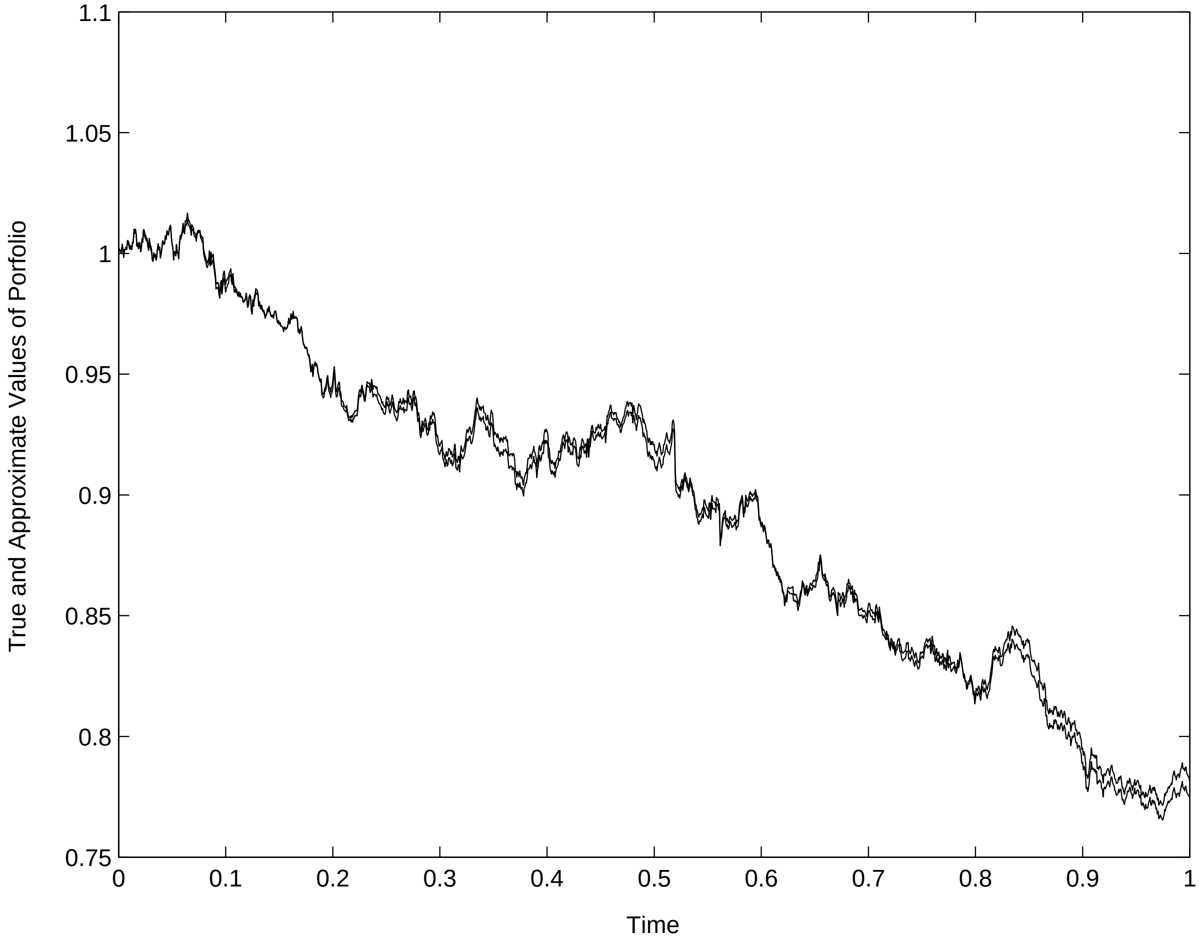

Figure 2: Sample Path of Portfolio II and Respective Asymptotics

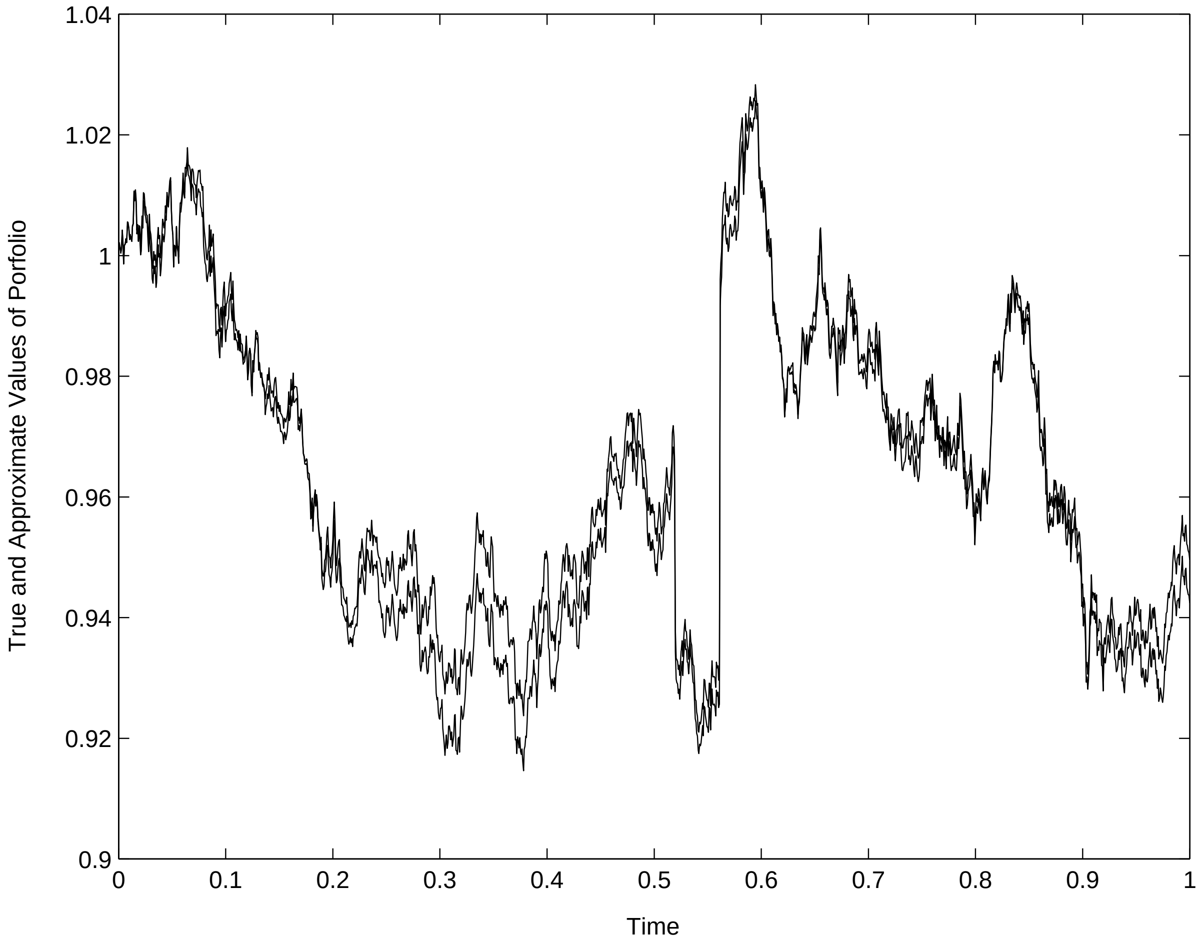

Figure 3: Sample Path of Portfolio III and Respective Asymptotics

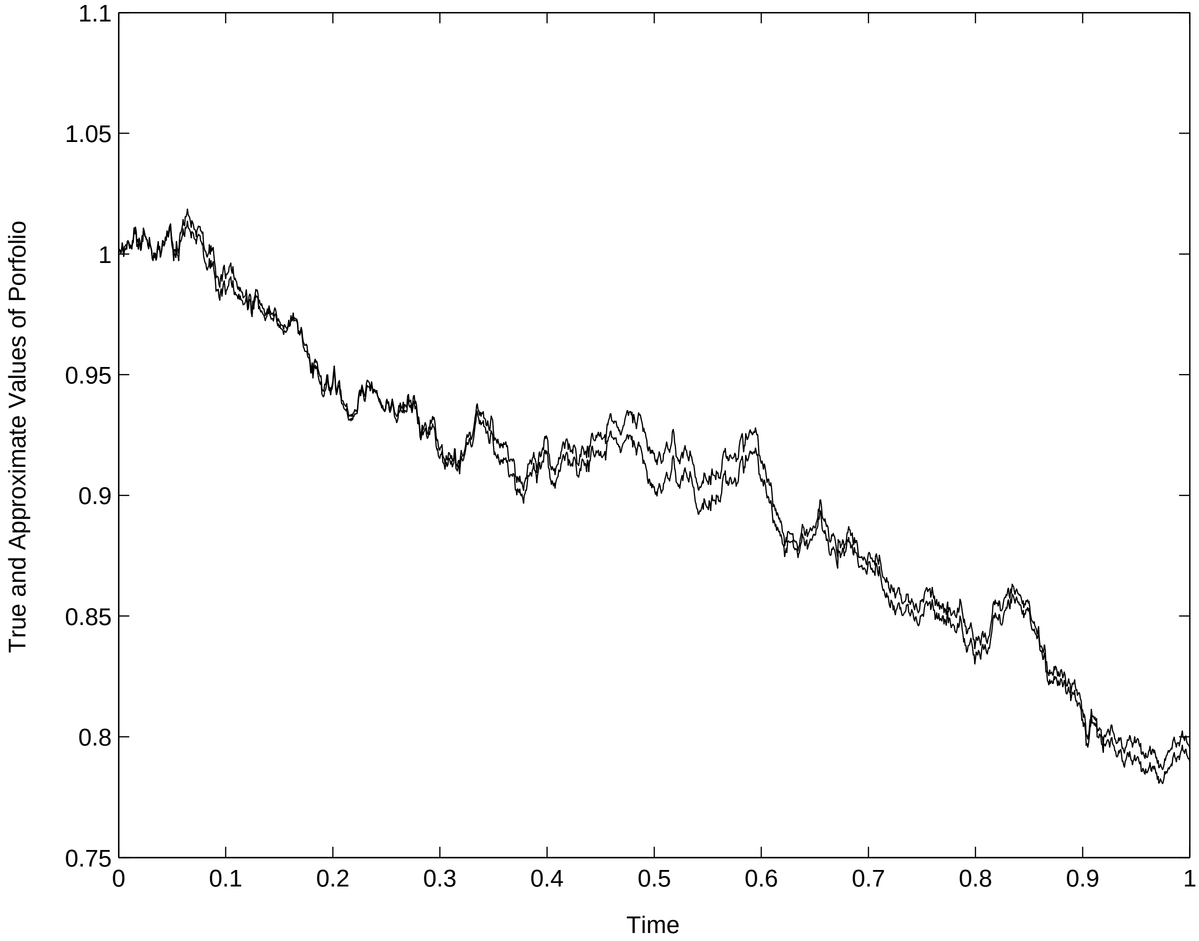

Figure 4: Distribution of One-month Value of Portfolio I

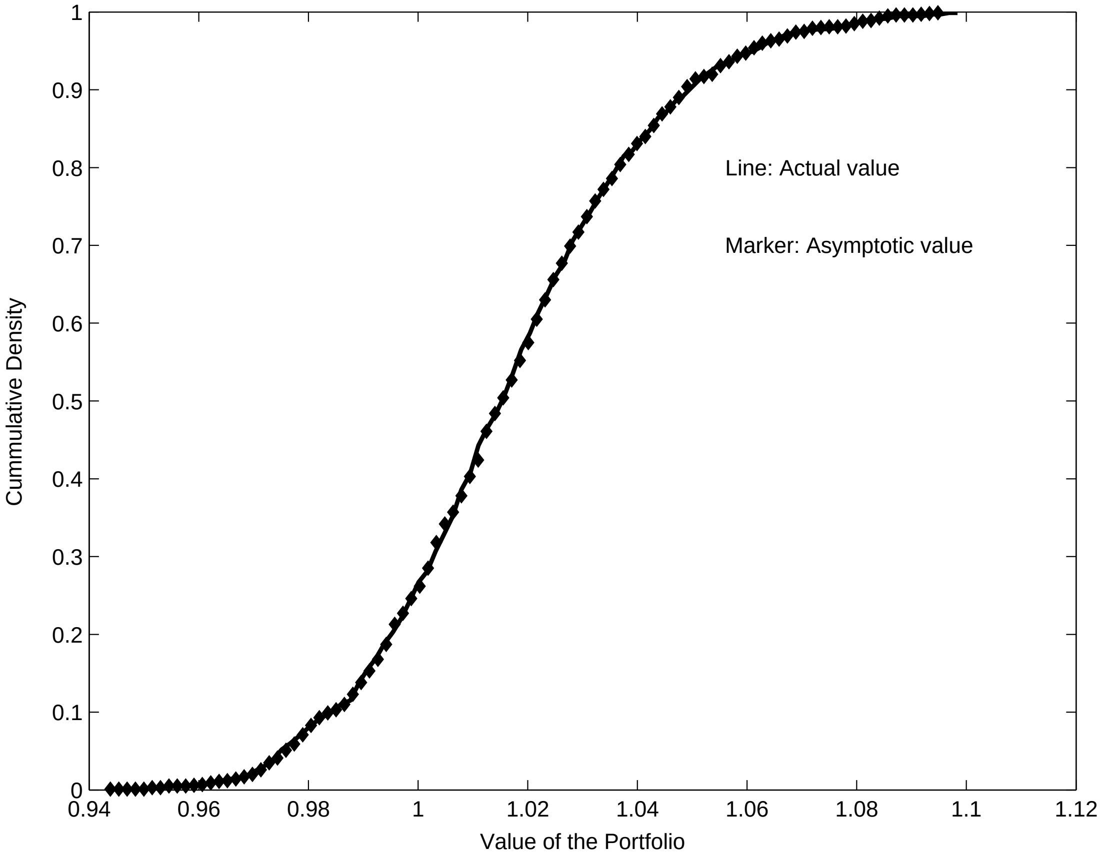

Figure 5: Distribution of One-month Value of Portfolio II

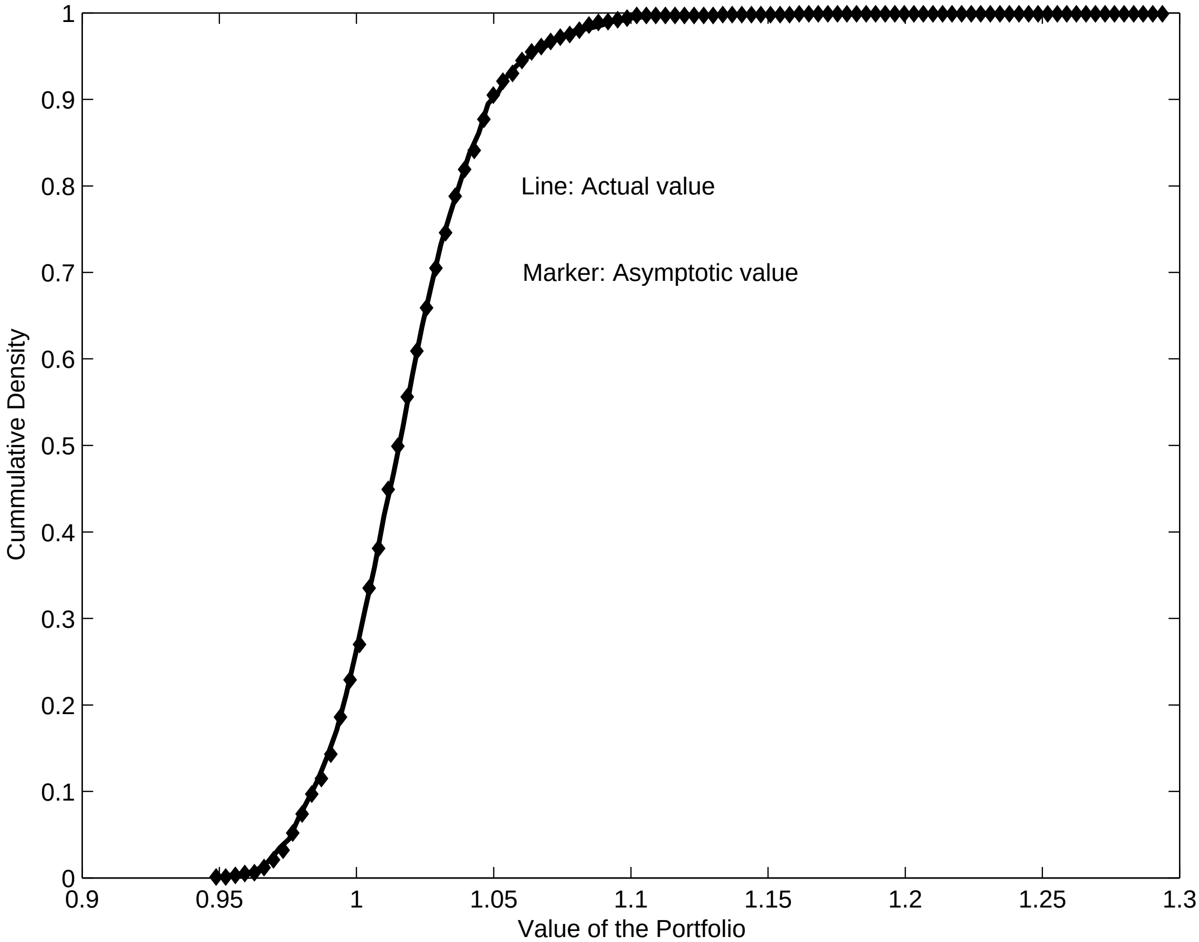

Figure 6: Distribution of One-month Value of Portfolio III