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Jingyi XUE
Singapore Management University, JYXUE@smu.edu.sg

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Jingyi Xue
Singapore Management University
jyxue@smu.edu.sg

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Abstract

Imagine that agents have uncertain needs and a resource must be divided before uncertainty resolves. In this situation, waste typically occurs when the assignment to an agent turns out to exceed his realized need. How should the resource be divided in the face of possible waste? This is a question out of the scope of the existing rationing literature. Our main axiom to address the issue is *no domination*. It requires that no agent receive more of the resource than another while producing a larger expected waste, unless the other agent has been fully compensated. Together with *conditionally strict endowment monotonicity*, *consistency*, and *strong upper composition*, we characterize a class of rules which we call *expected-waste constrained uniform gains rules*. Such a rule is associated with a function that aggregates the two components of cost generated by an agent at an allocation: the amount of the resource assigned to him and the expected waste he generates. The rule selects the allocation that equalizes as much as possible the cost generated by each agent. Moreover, we characterize the subclasses of rules associated with homothetic and linear cost functions. Lastly, to appreciate the role of *no domination*, we establish all the characterizations with a decomposition of *no domination* into two axioms: *risk aversion* and *no reversal*. They respectively capture

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the ideas that a rule should not be unresponsive to need uncertainty, and neither should it be too sensitive to it.

Key-words: claims problems; need uncertainty; fair division; waste; expected-waste constrained uniform gains rule; rationing; bankruptcy

JEL classification: C71; D63; D74; D81

1 Introduction

Resource allocation in the face of *uncertain needs* is common in real life. For example, a government has to divide a budget among several districts to finance the development of infrastructure (roads, schools, etc.) with a rough knowledge of local public demands (Copas (1993)); an emergency management institute has to distribute rescue forces (search teams, medical supplies, etc.) to different areas struck by a natural disaster based on estimates of individual damage statistics (Rolland, Patterson, Ward, and Dodin (2010), Wex, Schryen, and Neumann (2012), Wex, Schryen, Feuerriegel, and Neumann (2014)); a network architect has to assign capacities (bandwidth, memories, etc.) to various nodes and links before the realization of random traffic flows (Meesublak (2008), Ukkusuri and Patil (2009)).

In those situations, a resource has to be divided ex ante, and ex post reallocation may be difficult. It could be costly to downsize an underutilized public facility, retrieve unused rescue forces, or remove idle capacities in a network. In such cases, division is *pre-committed*, and it generates *waste* when the allocated amount turns out to exceed the realized need.

To the best of our knowledge, this is the first paper providing a normative theory to address the issue of waste in a pre-committed division problem with uncertain needs. Besides the issue of waste, it is also a valid concern for assignments falling short of realized needs. We are interested in the former because how to divide a resource in the face of possible waste is a question that is specific to the context of need uncertainty — waste can always be avoided when needs are deterministic. It turns out that our work also inspires a dual approach to addressing the issue of unsatisfied needs. A discussion is provided in Section 6.

Our resource is one-dimensional and perfectly divisible. We model the uncertain needs of agents, called *claims*, as probability distributions. They are objectively verifiable. A "division problem" consists of a finite set of agents, the profile of their claims, and an endowment of the resource which does not exceed the aggregate maximal claims. An allocation for a problem specifies for each agent an assignment of the resource that is no larger than his maximal claim, the sum of all assignments equal to the endowment. A division rule picks an allocation for each problem. We search for desirable rules.

We use an example to illustrate the difficulty caused by uncertainty. There are 48 units of the resource and three agents. Agent 1 claims 50 for sure. Agent 2 claims 50, 0 and 100 with probability 0.98, 0.01 and 0.01 respectively. Agent 3 claims 0 and 100 with probability 0.5 respectively. Although all three agents have equal expected claims, agent 1's claim is deterministic while agent 3's is most risky. If the resource does not have to be divided before the realization of their claims, waste can be avoided. But we are interested in ex ante division which must be fully committed. A planner concerned with efficiency may want to minimize the expected amount of unused resource. Then the allocation $(48, 0, 0)$ should be adopted, since in any other allocation some amount of the resource will be wasted with a positive probability. The emphasis on efficiency makes the rule excessively averse to uncertainty, and this may well be judged unfair. In this problem, the small uncertainty in agent 2's claim ruins his opportunity to get anything, and the same thing is true for agent 3 although he claims a large amount with probability 0.5.

One simple solution might be to apply the proportional rule to the profile of expected claims. In this example, expected claims are equal, so the rule picks $(16, 16, 16)$. But this outcome is not desirable either, since it ignores how uncertain claims are — agents are assigned equal amounts although they generate different amounts of waste.

We study the problem from a normative perspective. Our key axiom that addresses the issue of waste is *no domination*. To illustrate it, note that an assignment to an agent induces two components of cost: the resource assigned to him and the expected waste generated by him. If a society has no bias toward any agent, it should not bear for any particular agent a larger cost in each component. Thus, our *no domination* axiom requires that no agent be assigned a larger amount while producing a larger expected waste than another, unless the other agent has been fully compensated.

The remaining axioms are all standard. *Symmetry* requires that if two agents have equal claims, they receive equal assignments. *Conditionally strict endowment monotonicity* says that if the endowment increases and if an agent had not been fully compensated, then his assignment should increase. *Positivity* requires that whenever the endowment is positive, an agent with a non-zero claim receive a positive assignment. For *consistency*, imagine that after an allocation is selected for a problem, some agents come first and take away their assignments. Then the division of the rest of the endowment among the remaining agents should yield the original assignments for them. *Upper composition* pertains to the possibility that after an allocation has been chosen for a problem, the endowment is found to have been overestimated. Then the initial allocation can be canceled and the smaller endowment divided, or the initial assignments can be used as upper bounds of agents' claims and the smaller endowment divided based on the claims truncated at these upper bounds. *Upper composition* requires that the two approaches yield the same allocation in the end. *Strong upper composition* strengthens *upper composition*. Consider in the same situation that some agents disagree with using their initial assignments as upper bounds and having their claims truncated. In such cases, agents should be allowed to use either the truncated claims or the initial claims when the smaller endowment is divided. *Strong upper composition* applies the same invariance requirement to all such cases no matter who use the truncated claims and who do not. That is, dividing the smaller endowment based on the initial claims should yield the same allocation as based on any revised claims with the claims of a subset of agents truncated at their initial assignments. *Lower composition* pertains to the "dual" situation that after an allocation has been chosen for a problem, the endowment is found to have been underestimated. Then the initial allocation can be canceled and the larger endowment divided, or the initial allocation can be implemented and the increment in the endowment divided based on the claims adjusted down by the initial assignments. *Lower composition* requires that the two approaches yield the same allocation in the end. *Claims truncation invariance* says that the part of claims exceeding the endowment should be regarded as irrelevant, so truncating claims at the endowment should not affect the allocation. Lastly, *scale invariance* says that if the claims and the endowment are multiplied by a common factor, then the allocation should be rescaled accordingly.

We first show that *no domination*, *conditionally strict endowment monotonicity*, *consistency*, and *strong upper composition* characterize a class of rules which we call expected-waste constrained uniform gains rules. To see how such a rule operates, recall that an assignment to an agent induces two components of cost: the amount of the resource assigned to him and the expected waste he generates. Each expected-waste constrained uniform gains rule is associated with a continuous and increasing cost function that aggregates the two components. For each problem, a cost level c^* is chosen so that assigning to each agent his maximal claim if it generates a cost less than c^* and otherwise an amount that induces a cost equal to c^* yields a feasible allocation. This allocation equalizes as much as possible the costs generated by all assignments to agents, and it is the one selected by our rule. When restricted to deterministic claims, this rule coincides with the classic "uniform gains rule".

Naturally, the properties of an expected-waste constrained uniform gains rule depends on the properties of the associated cost function. Adding *scale invariance*, we pin down the subclass of rules associated with homothetic cost functions. Moreover, we show that *no domination*, *positivity*, *consistency*, *lower composition*, and either *strong upper composition* or *claims truncation invariance* characterize the subclass of the rules associated with linear cost functions. For example, let U be the summation of assignment and expected waste. Then in the previous three-agent problem, the allocation is approximately $(18.067, 17.888, 12.045)$. The cost induced by agent 1 is 18.067, by agent 2 $17.888 + 17.888 \times 0.01$ and by agent 3 $12.045 + 12.045 \times 0.5$. Note that on one hand, unlike the proportional rule discussed before, this rule assigns a smaller amount of the resource to an agent who has a riskier claim. On the other hand, unlike the efficient rule, it avoids assigning a too small amount to such an agent.

In general, an expected-waste constrained uniform gains rule strikes a balance between being unresponsive to uncertainty and too sensitive to uncertainty. This is precisely the implication of *no domination*. In fact, we can decompose *no domination* into two axioms which capture respectively the two sides of the implication. The first is *risk aversion*. It says that if two agents have the same maximal claim and if the claim of one agent is riskier than the other's, then the agent who has a riskier claim should not be assigned a larger amount than the other agent. The second is *no reversal*. It says that in the same situation, the riskier agent should not be assigned so little as to generate a smaller expected waste. Although *no domination* is imposed for general populations of agents whose claims may not be comparable in terms of riskiness, combined with the other axioms, we show that *no domination* is equivalent to the combination of *risk aversion* and *no reversal*.

Lastly, we extend the characterization of the uniform gains rule from the domain of problems with deterministic claims (Dagan (1996), Martínez (2008)) to a subdomain of problems with uncertain claims. This subdomain, called the *strongly ordered domain*, consists of all problems in which for each pair of agents, the claim of one agent is given by a truncation of the other's. This subdomain is meaningful when agents only disagree on large needs.

Following the literature review, the remainder of the paper is organized as follows. Section 2 introduces the model and the axioms. Section 3 defines the class of expected-waste constrained uniform gains rules. Section 4 provides the characterizations. Section 5 checks the tightness of the characterizations. Section 6 remarks on a dual approach to addressing the issue of unsatisfied needs and summarizes the paper. All the proofs are in the Appendix.

1.1 Literature review

There is a rapidly growing literature on pre-committed resource allocation with uncertain needs in many practical fields such as emergency control, project management, and network design (e.g., Johansson and Sternad (2005), Rawls and Turnquist (2010), Turnquist and Nozick (2003), Wex, Schryen, and Neumann (2012)).

However, very few papers have addressed the problem from a normative perspective. We are only aware of the following axiomatic studies. In a model where uncertain claims are assumed to be intervals, a version of the proportional rule is characterized by Yager and Kreinovich (2000) (see also Branzei, Dimitrov, Pickl and Tijs (2004), Woeginger (2006)). When claims are assumed to be contingent on finitely many future states, the so-called "ex ante" and "ex post" proportional rules are characterized by Ertemel and Kumar (2017).[1] None of these papers address the issue of waste. As far as we know, we are the first to offer an axiomatic foundation for division in the face of possible waste due to claim uncertainty.

There is a rich literature on fair division with deterministic claims. For example, the uniform gains rule and its dual version, the uniform losses rule, are axiomatized by Dagan (1996), Herrero and Villar (2001, 2002), Yeh (2004, 2006, 2008), Martínez (2008), and Marchant (2008). The proportional rule is characterized by Banker (1981), O'Neill (1982), Moulin (1987), and Chun (1988). The class of "parametric rules" includes all the previous rules as special cases, and is characterized by Young(1987a). Moulin (2002) and Thomson (2003, 2015) provide excellent surveys on this subject.

There are two papers closely related to our work. First, our Theorem 1 is related to Moreno-Ternero and Roemer (2006) on a division problem without uncertainty. Moreno-Ternero and Roemer (2006) study how to divide a resource among a group of agents, each of whom is equipped with an output function that transforms the assigned resource into some output. Their "*priority*" axiom is a form of no domination. It requires that no agent receive more of the resource while producing a larger output than another agent. Together with their "*solidarity*" axiom, which can be decomposed into "*strict endowment monotonicity*"[2] and

[1] In a similar framework, Habis and Herings (2013) adopt a cooperative game approach to test the stability of a stochastic extension of well-known rules.

[2] They call it *endowment monotonicity*.

“*consistency*”, they characterize a class of “index-egalitarian” rules similar to ours. Each index-egalitarian rule is associated with an index that is continuous and increasing in resource assignment and output. It selects the allocation in which all assignments to agents generate the same index value.

Our generic agent, given his claim, can be thought of having a waste function (like their output function) that transforms the assigned resource to expected waste. Thus, our *no domination* corresponds in its structure to their *priority*. However, probabilistic claims and real output functions are different objects, and axioms dealing with the two objects do not in general correspond to each other. First, their output functions are all strictly increasing, whereas our waste functions, whenever induced by deterministic claims, always generate zero expected waste. Their characterization does not need to deal with agents with zero output functions, whereas ours relies on one additional axiom, *strong upper composition*, to pin down the division for deterministic claims. Second, their agents can receive an unlimited amount of the resource, while our agents cannot receive more than their maximal claims. Their *strict endowment monotonicity* therefore implies in our context that no agent could be fully compensated unless the endowment is sufficient to satisfy all agents’ maximal claims. This requirement is too strong for our rules. Our characterization utilizes the weaker *conditionally strict endowment monotonicity* axiom. Third, *lower composition* has different implications in the two models. It turns out to be quite restrictive in their model as it narrows down the whole class of their rules to two extreme ones (Moreno-Ternero and Roemer (2012)). In contrast, it implies an entire subclass of our rules, those associated with linear cost functions (see our Theorem 2).

There are also some technical differences between our model and Moreno-Ternero and Roemer (2006). First, they require that the union of the graphs of all output functions cover the whole positive quadrant in $\mathbb{R}^2$. In our model, an agent cannot waste more than what he is assigned, so the union of the graphs of all waste functions only contain points under the 45 degree line. Second, they assume that the output functions of their agents are unbounded, while the waste functions of our agents are bounded by their maximal claims. This further makes our proof different from theirs.

The other closely related work is Chun, Jang and Ju (2014). They propose a decomposition of Moreno-Ternero and Roemer (2006)’s *priority* into two axioms: “*order preservation*” and “*no reversal*”. Both axioms consider pairs of agents such that one agent is *less disabled* than the other agent in the sense that he produces a no smaller output for all assignments than the other. *Order preservation* says that the less disabled agent should not be assigned a larger amount of the resource than the more disabled. Their *no reversal* says that the less disabled agent should not be assigned so little as to produce a smaller output than the more disabled. On a well-ordered domain where agents are pairwise comparable in terms of ability, the combination of their *order preservation* and *no reversal* is amount to *priority*. On such a domain, they characterize the class of index-egalitarian rules by *order preservation*, *no reversal*, and "*agreement*", where *agreement* is equivalent to the combination of *strict endowment monotonicity* and a solidarity axiom that plays a similar role as *consistency*.[3]

In our model, if an agent has a riskier claim than another agent, he generates a no smaller expected waste than the other for all assignments. Thus, an agent who has a riskier claim in our model is like a less disabled agent in their model. Hence, our decomposition of *no domination* into *risk aversion* and our *no reversal* corresponds to their decomposition of *priority* into *order preservation* and their *no reversal*.

There are three main differences between our decomposition result and theirs. First and foremost, unlike their domain, ours is not well-ordered. The claims of our agents may not be comparable in terms of riskiness. We show without restriction on agents' claims that *no domination* can be replaced with *risk aversion* and our *no reversal* in all of our characterizations. Second, our two separate axioms are not as restrictive as theirs. Their two axioms are imposed on each pair of agents one of whom always produces a no smaller output than the other. In our model, even if one of two agents always produces a no smaller expected waste than the other, their claims may not be comparable in terms of riskiness,[4] so that our axioms impose no restriction on them. Third, each of their two axioms directly implies *symmetry*, while it is not trivial to show that our two axioms together imply *symmetry* under *strong upper composition* and *consistency* (see our Proposition 4, and also Proposition 5). This is because two agents with the same output functions are comparable in terms of ability by their definition, whereas the same claims are not comparable in riskiness by our definition.

Lastly, note that distributive justice under uncertainty has also been discussed in other contexts. For example, cost sharing of risky projects is studied by Hougaard and Moulin

[3]This solidarity axiom that they consider is called "*separability*". It says that after a shock on agents' output functions and the endowment, if the output functions of some agents are unaffected and if the sum of their assignments is unchanged, then each of them should receive his initial assignment. Moreover, they also provide a characterization of the class of index-egalitarian rules on a "rich" domain with a monotonicity axiom regarding the change in an agent's ability.

[4]It could be that the claim of the first agent is first-order stochastically dominated by that of the second.

(2016); general approaches to assessing risky social situations are studied by Fleurbaey (2010) (see also Fleurbaey, Gajdos and Zuber (2015)).

2 The model and the axioms

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{N}$ the set of positive integers, and $\mathcal{N}$ the set of all finite subsets of $\mathbb{N}$. For each pair $x, x' \in \mathbb{R}^n$, $n \in \mathbb{N}$, we write $x \leq x'$ if for each $m \in \{1, ..., n\}$, $x_m \leq x'_m$, and $x < x'$ if for each $m \in \{1, ..., n\}$, $x_m < x'_m$.

A one-dimensional and perfectly divisible resource is to be divided among some group of agents. Agents are denoted by elements in $\mathbb{N}$. An agent's (uncertain) **claim** of the resource is a cumulative distribution function with compact support in $\mathbb{R}_+$.[5] Let $\mathcal{F}$ be the set of claims. For each agent $i \in \mathbb{N}$, we denote by $F_i \in \mathcal{F}$ a typical claim of agent i, and $\underline{\text{supp}}\, F_i$ and $\overline{\text{supp}}\, F_i$ the minimal and maximal values in the support of F_i. For each population $I \in \mathcal{N}$, we denote by F a typical profile of claims, i.e., a vector $(F_i)_{i \in I}$ such that for each $i \in I$, $F_i \in \mathcal{F}$. Let $\mathcal{F}^I$ be the set of claim profiles with population I.

Given $I \in \mathcal{N}$, a claims problem, or simply a **problem**, with population I, consists of a claim profile $F \in \mathcal{F}^I$ and an endowment $T \in [0, \sum \overline{\text{supp}}\, F_i]$. We denote by (F, T) a typical problem and $\boldsymbol{C^I}$ the set of problems with population I. An **allocation** for $(F, T) \in C^I$ is a vector $t \in \mathbb{R}^I_+$ such that $\sum t_i = T$, and for each $i \in I$, $t_i \leq \overline{\text{supp}}\, F_i$. We call t_i the **assignment** to agent i. A division rule, or simply a **rule**, is a function r that specifies for each problem in $\bigcup_{I \in \mathcal{N}} C^I$ an allocation. We denote by $r_i(F, T)$ the assignment to agent i.

Given assignment t_i to agent i, if his realized need is x_i, then the difference $\max\{t_i - x_i, 0\}$ is wasted. If his claim is F_i, then $\int \max\{t_i - x_i, 0\} dF_i(x_i)$ is the **expected waste**. For each $F_i \in \mathcal{F}$ and $t_i \in [0, \overline{\text{supp}}\, F_i]$, let $w(F_i, t_i)$ denote the expected waste that results when agent i, claiming F_i, is assigned t_i. Note that the expected waste generated by an agent is independent of the probability distribution of his need that is above his assignment. That is, for each pair $F_i, F'_i \in \mathcal{F}$ and each $t_i \in [0, \min\{\overline{\text{supp}}\, F_i, \overline{\text{supp}}\, F'_i\}]$, if F_i and F'_i agree on $(-\infty, t_i)$, then $w(F_i, t_i) = w(F'_i, t_i)$.

A basic example of a rule is the rule that divides the resource proportionally to expected claims. In the three-agent problem of the Introduction, since all agents have equal expected claims, the outcome is simply $(16, 16, 16)$. But this is not desirable since this division com-

[5] It is possible to allow unbounded claims, but our axioms essentially ignore the tails of claims.

pletely ignores differences in the riskiness of the three claims.

Another example is the rule that picks, for each problem, the allocation that minimizes total expected waste. Unfortunately, this rule is too sensitive to uncertainty. In our three-agent problem, as long as agent 2 claims 0 with positive probability, no matter how small the probability is, the rule assigns the whole of the resource to agent 1, who has a deterministic claim of 50. The tiny bit of uncertainty in an agent's claim entirely destroys his chance to obtain anything. This may not be judged fair.

We seek to identify well-behaved rules which address the issue of waste. In particular, we are interested in rules which are responsive to the riskiness of claims but not too sensitive to it. For this purpose, we consider the following axioms.

2.1 No domination

Our key axiom is *no domination.* It formulates the principle that no agent should dominate another in every dimension. In our model, although the resource is one-dimensional, a society bears for each agent two components of cost: the resource assigned to him and the expected waste generated by him. Waste is costly since the resource would have been better utilized in other places (possibly outside the problem). In the case of dividing public funds, it is a reminiscent of the shadow cost in Laffont and Tirole (1993) due to the distortions generated in an economy. If a society has no bias toward any agent, it should not bear for any particular agent a larger cost in each component than for another agent. Thus, we require that no agent receive a larger assignment while generating a larger expected waste than another agent, unless the other agent has been fully compensated.

No domination: For each $I \in \mathcal{N}$, each $(F, T) \in C^I$, and each pair $\{i, j\} \in I$, if $r_i(F, T) > r_j(F, T)$, $w(F_i, r_i(F, T)) > w(F_j, r_j(F, T))$, then $r_j(F, T) = \overline{\text{supp}}\ F_j$.

The principle of "no domination in every dimension" has also appeared in earlier works. In the classic division problem, a bundle of multiple commodities is divided among agents who have individual preferences. A weak version of *envy-freeness* (Thomson (1983a), Thomson and Varian (1985), Moulin and Thomson (1988)) takes the commodities as these dimensions, and requires that no agent receive more of each commodity than another agent.[6] In the division problem studied by Moreno-Ternero and Roemer (2006), a one-dimensional

[6]It is equivalent to the standard envy-freeness when agents have Leontief preferences (Li and Xue (2013)).

resource is divided among agents who are equipped with individual output functions transforming the assigned resource into some output. Their *priority* axiom takes resource assignment as one dimension and output as the other dimension. They require that no agent receive more of the resource and produce a larger output than another agent.

The difference of our formulation of this principle from the earlier works is that the dimensions considered in the earlier works are aspects of an agent's welfare, whereas waste is not related to an agent's welfare. But from a broader perspective, these dimensions can be considered as aspects in which an agent could be treated in favor, no matter whether these aspects directly affect an agent's welfare or not. In our model, a society can favor an agent by bearing for him a larger cost in both resource assignment and expected waste. Our axiom rules out the domination of one agent over another in these cost components.

No domination has two essential implications which capture respectively the lessons of the two examples in the beginning of this section: A rule should be responsive to riskiness, but not too sensitive to it. Before we discuss the implications, let us formally introduce a notion of comparative riskiness. Following Rothschild and Stiglitz (1970), for each pair $\{i, j\} \subseteq \mathbb{N}$ and each $F \in \mathcal{F}^{\{i,j\}}$, we say F_i is **riskier** than F_j, or F_j is **more deterministic** than F_i, if F_i is a mean-preserving spread of F_j, i.e., F_i and F_j have the same mean, for each $c \in \mathbb{R}$,

$$\int_{-\infty}^{c} F_i(x_i)dx_i \geq \int_{-\infty}^{c} F_j(x_j)dx_j, \tag{1}$$

and (1) holds with strict inequality at some $c \in \mathbb{R}$. Note that if agent i, who claims F_i, is assigned t_i, then his expected waste $w(F_i, t_i)$ is equal to $\int_{-\infty}^{t_i} F_i(x_i)dx_i$. Hence, if agent i has a riskier claim than agent j, agent i would generate a no smaller expected waste than agent j whenever they receive the same assignments. In this sense, a riskier agent is more wasteful.

Now consider a two-agent problem in which both agents have the same maximal claims, and one agent has a riskier claim than the other. If the riskier agent receives a larger assignment, he must generate a no smaller expected waste. In a generic case, he generates a larger expected waste, and thus *no domination* is violated. Hence, in such cases, *no domination* implies that the riskier agent receives a no larger assignment. We strengthen this implication as a requirement applied to all cases and call it *risk aversion*.

Risk aversion: For each pair $\{i, j\} \subseteq \mathbb{N}$ and each $(F, T) \in \mathcal{C}^{\{i,j\}}$, if $\overline{\text{supp}}\ F_i = \overline{\text{supp}}\ F_j$ and F_i is riskier than F_j, then $r_i(F, T) \leq r_j(F, T)$.

Note that *risk aversion* is not an efficiency requirement. Assigning a larger amount of the resource to the riskier agent might generate a smaller total expected waste. *Risk aversion* simply imposes a punishment on the riskier agent for his being more wasteful.

On the other hand, *no domination* also implies that there is a limit to the punishment. Observe that if the riskier agent is assigned too little of the resource, he could generate a smaller expected waste than the other agent, which again violates *no domination*. Thus, *no domination* implies that the riskier agent is not assigned so little as to generate a smaller expected waste. We call this property *no reversal*.

No reversal: For each pair $\{i, j\} \subseteq \mathbb{N}$ and each $(F, T) \in C^{\{i,j\}}$, if $\overline{\text{supp}}\ F_i = \overline{\text{supp}}\ F_j$ and F_i is riskier than F_j, then $w(F_i, r_i(F, T)) \geq w(F_j, r_j(F, T))$.

When restricted to pairs of agents who have the same maximal claims and whose claims are comparable in terms of riskiness,[7] *no domination* is essentially equivalent to the combination of *risk aversion* and *no reversal*.[8] The surprising fact we find is that even without the restriction on agents' claims, under some standard axioms, *no domination* is equivalent to the combination of *risk aversion* and *no reversal* (Proposition 2). All of our characterizations hold when *no domination* is replaced with *risk aversion* and *no reversal*.

Our decomposition of *no domination* is related to a decomposition of Moreno-Ternero and Roemer (2006)'s *priority* by Chun, Jang and Ju (2014). Recall that Moreno-Ternero and Roemer (2006) study the division of a one-dimensional resource among agents who are endowed with individual output functions. An agent can be considered *less disabled* than another agent if he produces a no smaller output than the other agent for all assignments. Fixing a population of agents who are pairwise comparable in terms of ability, Chun, Jang and Ju (2014) propose a decomposition of *priority* into *order preservation* and *no reversal*. Their *order preservation* requires that a less disabled agent receive a no larger assignment. Their *no reversal* requires that a less disabled agent not be assigned so little as to produce a smaller output. An agent who has a riskier claim in our model is like an agent who is less

[7]The condition that two agents have equal maximal claims is indispensable for *risk aversion*. This is because when there is a sufficient amount of the endowment, every rule assigns a larger amount of the resource to agent i if he has a larger maximum claim. But this condition can be dropped in *no reversal*. Our results hold no matter which version of *no reversal* is imposed.

[8]*No domination* does not imply *risk aversion* in the case when both agents have non-zero minimal claims and the endowment is so small that any allocation induces a zero expected waste for both agents. In such cases, *no domination* has no restriction on the allocation.

disabled in their model. The main difference between our decomposition and theirs is that we establish a general equivalence result with no restriction on the comparability of agents' claims. We provide a detailed discussion in the Literature review.

2.2 Other axioms

We now introduce some axioms from the literature of deterministic claims problems to our uncertain problems. A minimal fairness requirement is *symmetry* (Thomson (2003)). It says that agents who have equal claims should receive equal assignments, i.e., a rule should not discriminate agents on the basis of their names.

Symmetry: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{C}^I$, and each pair $\{i, j\} \subseteq I$, if $F_i = F_j$, then $r_i(F, T) = r_j(F, T)$.

A solidarity requirement regarding change of the endowment is *endowment monotonicity* (Curiel, Maschler, and Tijs (1987), Chun and Thomson (1988), Roemer (1986a,b), Moulin (1999), Young (1988)). It says that when the endowment increases, no agent should get less than his initial assignment.

Endowment monotonicity: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{C}^I$, and each $T' \in [0, T)$, $r(F, T') \leq r(F, T)$.

A strengthening of *endowment monotonicity*, known as *strict endowment monotonicity*, requires that as the endowment increases, each agent with a non-zero claim get more than his initial assignment. *Strict endowment monotonicity* is a strong requirement. In particular, it does not allow any agent who has a non-zero claim to be fully compensated unless the endowment is sufficient to satisfy all agents' maximal claims. One conditional weakening of *strict endowment monotonicity* that gets rid of this restriction is to require an agent's assignment to increase with the endowment only if the agent had not been fully compensated (Thomson (2003)).

Conditionally strict endowment monotonicity: For each $I \in \mathcal{N}$, each $(F, T) \in \mathcal{C}^I$, each $T' \in [0, T)$, and each $i \in I$, if $r_i(F, T') < \overline{\text{supp}}\, F_i$, then $r_i(F, T') < r_i(F, T)$.

Conditionally strict endowment monotonicity implies a basic lower bound requirement: If an agent has a non-zero claim, then he should receive a positive assignment whenever the endowment is positive.

Positivity: For each $I \in \mathcal{N}$, each $(F, T) \in C^I$, and each $i \in I$, if $\overline{\text{supp}}\, F_i > 0$ and $T > 0$, then $r_i(F, T) > 0$.

The familiar *consistency* axiom is also a solidarity requirement (Aumann and Maschler (1985), Young (1987a), Thomson (1988, 2012)). Imagine that after an allocation has been chosen for some (F, T), some agents come first to take away their assignments. Consider now the problem of dividing among the remaining agents the rest of the resource. We require that for this problem, each agent be assigned the same amount as initially.[9]

Consistency: For each $I \in \mathcal{N}$, each $(F, T) \in C^I$, each $J \subseteq I$, and each $i \in J$, $r_i(F, T) = r_i(F_J, \sum_{j \in J} r_j(F, T))$, where F_J is the restriction of F onto J.

The rest of our axioms are all invariance properties. First, imagine that after an allocation has been chosen, the endowment is found to have been overestimated. In this case, the initial allocation can be canceled and the smaller endowment divided, or the initial assignments can be used as upper bounds of agents' claims and the smaller endowment divided for the claims truncated at these upper bounds. Both ways of dealing with decrease of the endowment are reasonable. *Upper composition* (Moulin (2000)) requires that they lead to the same allocation. *Strong upper composition* (Martínez (2008), Thomson (2015)),[10] which strengthens *upper composition*, considers the possibility that some agents disagree with using the initial assignments as upper bounds and having their claims truncated. In such a situation, an agent should have the option of using either the truncated claim or the initial claim when the smaller endowment is divided. *Strong upper composition* further applies the invariance requirement to all such situations no matter who use the truncated claims and who do not. That is, dividing the smaller endowment based on the initial claim profile should be the same as based on any revised claim profile with the claims of a subset of agents truncated at their initial assignments. This helps to avoid the potential dispute among agents about the influence of their different choices on the allocation.

To formally define the truncation of a claim, let $i \in \mathbb{N}$, $F_i \in \mathcal{F}$, and $c \in \mathbb{R}_+$ be given. The truncation of F_i at c is a claim in $\mathcal{F}$, denoted by $\overline{F_i|^c}$, that assigns the same probability to

[9] If a rule is *endowment monotonic* and *consistent*, then it is *population monotonic* — dividing an endowment among a subgroup of agents cannot lead to a decrease in their assignments (Thomson (1983b,c)).

[10] Martínez (2008) uses the name "*strong composition down*". We follow the terminology of Moulin (2000) and call it *strong upper composition*.

each $x_i \in [0, c)$ as F_i and assigns all the remaining probability to c. That is, for each $x_i \in \mathbb{R}$,

$$\overline{F_i|^c}(x_i) = \begin{cases} F_i(x_i) & \text{if } x_i \in (-\infty, c) \\ 1 & \text{if } x_i \in [c, \infty). \end{cases}$$

Note that when $c \geq \overline{\text{supp}}\ F_i$, $\overline{F_i|^c} = F_i$. Although only the case of $c \leq \overline{\text{supp}}\ F_i$ is relevant in *upper composition* and *strong upper composition*, the general definition will be useful in some other axiom later.

Strong upper composition: For each $I \in \mathcal{N}$, each $(F, T) \in C^I$, each $T' \in [0, T)$, and each $J \subseteq I$, $r(F, T') = r(((\overline{F_i|^{r_i(F,T)}})_{i \in J}, F_{I \setminus J}), T')$.

Upper composition simply requires the invariance of allocation for the case of $J = I$. Our characterization results rely on *strong upper composition*. When restricted to problems with deterministic claims, it is known that *symmetry*, *consistency*, and *strong upper composition* characterize the uniform gains rule (Martínez (2008), Thomson (2015)). We extend the result to a subdomain of problems with uncertain claims (Proposition 3) and show that *consistency* is redundant in the existing characterization.

Now imagine the following "dual" situation: after an allocation has been chosen, the endowment is found to have been underestimated. In this case, the initial allocation can be canceled and the larger endowment divided, or the initial allocation can be implemented and the increment in the endowment divided based on the claims reduced by the initial assignments. *Lower composition* (Kalai (1977), Young (1988), Moulin (2000)) requires the two-step division to be equivalent to the one-time division of the entire endowment, so that agents will not dispute which is the better way of proceeding.

To formally define a reduced claim, let $i \in \mathbb{N}$, $F_i \in \mathcal{F}$, and $t_i \in [0, \overline{\text{supp}}\ F_i]$ be given. If agent i's realized need is $x_i \in \mathbb{R}$, then $\max\{x_i - t_i, 0\}$ is the unsatisfied need. Thus, the reduced claim of agent i, denoted by $F_i|^{t_i}$, is the distribution of $\max\{x_i - t_i, 0\}$, $x_i \in \mathbb{R}$, induced by F_i. That is, for each $x_i \in \mathbb{R}$,

$$F_i|^{t_i}(x_i) = \begin{cases} 0 & \text{if } x_i \in (-\infty, 0) \\ F_i(x_i + t_i) & \text{if } x_i \in [0, \infty). \end{cases}$$

Lower composition: For each $I \in \mathcal{N}$, each $(F, T) \in C^I$, and each $T' \in [0, T)$, $r(F, T) = r(F, T') + r((F_i|^{r_i(F,T')})_{i \in I}, T - T')$.

Note that both *upper composition* (and thus *strong upper composition*) and *lower composition* imply *endowment monotonicity*.

Another invariance requirement is *claims truncation invariance* (Dagan and Volij (1993), Thomson (2015)). It says that the part of a claim that exceeds the endowment should be regarded as irrelevant, so truncating claims at the endowment should not affect the allocation.

Claims truncation invariance: For each $I \in \mathcal{N}$ and each $(F, T) \in C^I$, $r(F, T) = r((\overline{F_i|^T})_{i \in I}, T)$.

When restricted to problems with deterministic claims, it is known that *symmetry*, *lower composition*, and *claims truncation invariance* characterized the uniform gains rule (Dagan (1996)). We also extend this result to a subdomain of problems with uncertain claims (Proposition 3).

Lastly, *scale invariance* (Moulin (1987), Young (1988)) requires invariance with respect to uniform rescalings of problems. That is, if individual claims and the endowment are rescaled by a common factor, then assignments should be rescaled by the same factor.

Scale invariance: For each $I \in \mathcal{N}$, each $(F, T) \in C^I$, each $F' \in \mathcal{F}^I$, and each $c > 0$, if for each $i \in I$ and each $x_i \in \mathbb{R}$, $F'_i(cx_i) = F_i(x_i)$, then $cr(F, T) = r(F', cT)$.

3 Expected-waste constrained uniform gains rules

3.1 An illustrating example

As discussed before, an assignment to an agent induces two components of cost to a society: the resource assigned to him and the expected waste he generates. Suppose that the (total) cost of an agent is a weighted sum of the two components, and imagine that an egalitarian planner seeks to equalize among agents their costs. Thus, for each problem, the planner would set a *common* cost, and each agent can obtain as much of the resource as possible as long as the induced cost is no more than the common cost and the obtained resource is no more than his maximal claim. More precisely, if assigning to an agent his maximal claim induces a cost lower than the common cost, he receives his maximal claim. Otherwise, he is assigned the amount that exactly induces the common cost. The common cost is determined by the binding feasibility constraint.

Formally, let $\lambda \in \mathbb{R}_+$. If agent i, who claims F_i, is assigned t_i, his cost is $t_i + \lambda w(F_i, t_i)$. Note that the larger the weight λ is, the more the planner is concerned about waste. For each $I \in \mathcal{N}$ and each $(F, T) \in C^I$, a common cost c^* is chosen. For each $i \in I$, agent i can obtain his maximal claim $\overline{\text{supp}}\ F_i$ if the induced cost $\overline{\text{supp}}\ F_i + \lambda w(F_i, \overline{\text{supp}}\ F_i)$ is less than c^*, otherwise he obtains $t_i \in [0, \overline{\text{supp}}\ F_i]$ such that his cost $t_i + \lambda w(F_i, t_i)$ is equal to c^*. The assignment to each agent is non-decreasing as the common cost increases, and c^* is determined by the condition that all agents' assignments sum up to T. We denote this rule by r^λ.

Recall the three-agent problem in the Introduction. The allocation (t_1, t_2, t_3) chosen by r^λ is given by solving the following system of equations:

$$\begin{cases} t_1 = c^* \\ t_2 + \lambda \cdot 0.01 t_2 = c^* \\ t_3 + \lambda \cdot 0.5 t_3 = c^* \\ t_1 + t_2 + t_3 = 48. \end{cases}$$

We obtain $((100\frac{1}{\lambda^2}+51\frac{1}{\lambda}+0.5)a, (100\frac{1}{\lambda^2}+50\frac{1}{\lambda})a, (100\frac{1}{\lambda^2}+\frac{1}{\lambda})a)$ where $a = \frac{0.96}{6\frac{1}{\lambda^2}+2.04\frac{1}{\lambda}+0.01}$. When $\lambda = 1$, the allocation is approximately $(18.067, 17.888, 12.045)$. Note that the assignment of an agent is smaller if his claim is riskier, and on the other hand the assignment is not excessively small. In general, the larger the value of λ, the smaller the assignment of a riskier agent. When λ goes to 0, the planner tends to ignore waste and he equalizes assignments. When λ goes to the infinity, the planner tends to care only about waste and he equalizes expected waste. It can be shown that for each $\lambda \in \mathbb{R}_+$, r^λ satisfies all of our axioms.

3.2 The general definition

In the illustrating example, the two components of cost generated by an agent are assumed to be aggregated in a linear way. In general, the two components could be aggregated in any way that satisfies some basic properties. Given a general cost function, we can define a rule in the same way as before. That is, in each problem, the rule sets a common cost, and each agent receives as much of the resource as possible as long as his cost, given by the cost function, is no more than the common cost and the obtained resource is no more than his maximal claim.

To define a general cost function, let $D := \{(t_i, w(F_i, t_i)) : F_i \in \mathcal{F}, t_i \in [0, \overline{\text{supp}}\ F_i]\}$ denote the domain of the two components of cost: resource assignment and expected waste. (Equivalently, $D = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 > x_2\} \cup \{(0,0)\}$.) In general, a **cost function** is a continuous function $U : D \to \mathbb{R}$ satisfying that (i) for each pair $(x_1, x_2), (x'_1, x'_2) \in D$ such that either $(x_1, x_2) < (x'_1, x'_2)$ or $x_1 < x'_1$ and $x_2 = x'_2 = 0$, $U(x_1, x_2) < U(x'_1, x'_2)$, and (ii) $U(0, 0) = 0$, a normalization condition. Note that when an agent's assignment increases, his expected waste either increases or remains to be zero, and condition (i) requires that in both cases his cost increases. Let $\mathcal{U}$ denote the set of all cost functions.

Given a cost function, an agent's assignment is determined by a common cost c^* in each problem. An agent obtains his maximal claim if the induced cost is lower than c^*. Otherwise he is assigned the amount which induces a cost of c^*. To formally define this rule, we introduce an individual "assignment" function over all possible values of cost. For each $U \in \mathcal{U}$ and each $F_i \in \mathcal{F}$, define $U^{-1}_{F_i} : U(D) \to [0, \overline{\text{supp}}\, F_i]$ by

$$U^{-1}_{F_i}(c) = \begin{cases} \overline{\text{supp}}\, F_i & \text{if } c > U(\overline{\text{supp}}\, F_i, w(F_i, \overline{\text{supp}}\, F_i)), \\ t_i & \text{if } c = U(t_i, w(F_i, t_i)) \text{ where } t_i \in [0, \overline{\text{supp}}\, F_i]. \end{cases}$$

To see that $U^{-1}_{F_i}$ is well-defined, recall that by condition (i) in the definition of a cost function, $U(\cdot, w(F_i, \cdot))$ is increasing on $[0, \overline{\text{supp}}\, F_i]$, and thus when $c \leq U(\overline{\text{supp}}\, F_i, w(F_i, \overline{\text{supp}}\, F_i))$, there is a unique $t_i \in [0, \overline{\text{supp}}\, F_i]$ satisfying $c = U(t_i, w(F_i, t_i))$. Moreover, one can check that $U^{-1}_{F_i}$ is continuous and increasing on $[0, \overline{\text{supp}}\, F_i]$.

Definition 1. *A rule r is an* ***expected-waste constrained uniform gains rule****, if there is $U \in \mathcal{U}$ such that for each $I \in \mathcal{N}$, each $(F, T) \in C^I$, and each $i \in I$,*

$$r_i(F, T) = U^{-1}_{F_i}(c^*), \text{ where } c^* \text{ solves } \sum U^{-1}_{F_j}(c) = T.$$

When restricted to problems with deterministic claims, the cost of an agent only depends on his assignment, and thus an expected-waste constrained uniform gains rule agrees with the classic uniform gains rule. That is, for each $I \in \mathcal{N}$ and each $(F, T) \in C^I$, if for each $i \in I$, F_i assigns probability one to some $x_i \in \mathbb{R}_+$, then for each $i \in I$,

$$r_i(F, T) = \min\{x_i, c^*\}, \text{ where } c^* \text{ solves } \sum \min\{x_j, c\} = T.$$

Expected-waste constrained uniform gains rules belong to the well-known parametric family (Young (1987a)) appropriately generalized to the domain of problems with uncertain claims. To be precise, a rule r is a **generalized parametric rule** if there is a real-valued function f on a closed interval $[a, b]$ such that (1) for each $F_i \in \mathcal{F}$, $f(F_i, \cdot)$ is non-decreasing and continuous on $[a, b]$ with $f(F_i, a) = 0$ and $f(F_i, b) = \overline{\text{supp}}\, F_i$, and (2) for each $I \in \mathcal{N}$, each $(F, T) \in C^I$, and each $i \in I$,

$$r_i(F,T) = f(F_i, c^*), \text{ where } c^* \text{ solves } \sum f(F_j, c) = T.$$

The expected-waste constrained uniform gains rule associated with U is the generalized parametric rule with $f(F_i, c) = U_{F_i}^{-1}(c)$.

4 Characterizations

Our axioms characterize the class of expected-waste constrained uniform gains rules and some subclasses.

Theorem 1. *A rule satisfies* no domination, conditionally strict endowment monotonicity, consistency, *and* strong upper composition *if and only if it is an expected-waste constrained uniform gains rule.*[11]

Proposition 1. *An expected-waste constrained uniform gains rule satisfies* scale invariance *if and only if it is associated with a homothetic cost function.*

If *lower composition* is added in Theorem 1, we can weaken *conditionally strict endowment monotonicity* to *positivity* and characterize the rules illustrated by the example of Section 3.1. Moreover, *strong upper composition* is equivalent to *claims truncation invariance* under the other axioms.

Theorem 2. *A rule satisfies (i)* no domination, positivity, consistency, strong upper composition, *and* lower composition, *or (ii)* no domination, positivity, consistency, lower composition, *and* claims truncation invariance *if and only if it is an expected-waste constrained uniform gains rule associated with a linear cost function.*

No domination turns out to be equivalent to the combination of *risk aversion* and *no reversal* under other axioms in Theorem 1 and Theorem 2.

Proposition 2. *Let a rule satisfy either (i)* conditionally strict endowment monotonicity, consistency, *and* strong upper composition, *or (ii)* positivity, consistency, lower composition, *and* claims truncation invariance. *Then, it satisfies* no domination *if and only if it satisfies* risk aversion *and* no reversal.

[11] This result is related to Moreno-Ternero and Roemer (2006). See the Literature review for a detailed discussion.

It follows immediately from Proposition 2 that Theorem 1 holds with *no domination* replaced by *risk aversion* and *no reversal*. Since *positivity* and *lower composition* together imply *conditionally strict endowment monotonicity* (see Lemma 3 in the Appendix), Proposition 2 also implies that the same is true for Theorem 2.

Corollary 1. *Theorem 1 and Theorem 2 hold with no domination replaced by risk aversion and no reversal.*

Finally, we extend the characterizations of the classic uniform gains rule by Dagan (1996) and Martínez (2008) from the domain of problems with deterministic claims to a subdomain of problems with uncertain claims. This subdomain contains all problems in which for each pair of agents, the claim of one agent is given by a truncation of the other's. Formally, for each $I \in \mathcal{N}$ and each $F \in \mathcal{F}^I$, F is said to be **strongly ordered** if for each pair $\{i, j\} \subseteq I$ with $\overline{\text{supp}}\ F_i \leq \overline{\text{supp}}\ F_j$, $F_i = \overline{F_j|^{\overline{\text{supp}}\ F_i}}$. For each $I \in \mathcal{N}$, let $\bar{C}^I := \{(F, T) \in C^I : F \text{ is strongly ordered}\}$. Let $\bar{C} := \bigcup_{I \in \mathcal{N}} \bar{C}^I$, and we call it the **strongly ordered domain**. Since each profile of deterministic claims is strongly ordered, $\bar{C}$ contains all problems with deterministic claims.

Proposition 3. *Let r be a rule satisfying either (i) symmetry and strong upper composition, or (ii) symmetry, lower composition, and claims truncation invariance. Then, for each $I \in \mathcal{N}$, each $(F, T) \in \bar{C}^I$, and each $i \in I$,*

$$r_i(F, T) = \min\{\overline{\text{supp}}\ F_i, c^*\}, \text{ where } c^* \text{ solves } \sum \min\{\overline{\text{supp}}\ F_j, c\} = T.$$

Note that the converse is true if the axioms are restricted to the strongly ordered domain. On the domain of problems with deterministic claims, Martínez (2008) characterizes the classic uniform gains rule by the axioms in (i) together with *consistency*, and Dagan (1996) characterizes it by the axioms in (ii). Our result shows that *consistency* is redundant in Martínez (2008)'s characterization.

5 Tightness

The characterizations in Theorems 1, 2, and Corollary 1 are tight.

Dropping *no domination* and *risk aversion*, define a rule in the same way as an expected-waste constrained uniform gains rule except that the associated "cost function" U is decreasing in the second coordinate: For each $(x_1, x_2) \in D$, $U(x_1, x_2) = x_1 - \frac{x_2}{2}$. Since an increase in expected waste leads to a decrease in U, this rule violates *no domination* and *risk aversion*. It can be readily seen that it satisfies *no reversal*, *conditionally strict endowment monotonicity*, *positivity*, *consistency*, *strong upper composition*, and *lower composition*.

Dropping *no reversal*, define a rule in the same way as an expected-waste constrained uniform gains rule except that the associated "cost function" is claim-specific. Formally, let $\hat{D} := \{(F_i, t_i) : F_i \in \mathcal{F}, t_i \in [0, \overline{\text{supp}}\ F_i]\}$ be the domain of the new cost function U, and define $U : \hat{D} \to \mathbb{R}_+$ by setting for each $(F_i, t_i) \in \hat{D}$,

$$U(F_i, t_i) = \begin{cases} \frac{1}{10} t_i & \text{if } t_i \leq \underline{\text{supp}}\ F_i, \\ \frac{1}{10} \underline{\text{supp}}\ F_i + [t_i - \underline{\text{supp}}\ F_i + w(F_i|^{\underline{\text{supp}}\ F_i}, t_i - \underline{\text{supp}}\ F_i)] & \text{if } t_i > \underline{\text{supp}}\ F_i. \end{cases}$$

Note that if $\underline{\text{supp}}\ F_i = 0$, U is simply the sum of assignment and expected waste, like a usual cost function. If $\underline{\text{supp}}\ F_i > 0$, U is sum of the cost of receiving an amount that is surely needed ($\min\{t_i, \underline{\text{supp}}\ F_i\}$) and the cost of receiving an additional amount that is not surely needed ($\max\{t_i - \underline{\text{supp}}\ F_i, 0\}$), and the former cost is one-tenth the latter cost. The rule associated with such U assigns more of the resource to agents with larger ensured needs. To see that it violates *no reversal*, let $(F, T) \in \mathcal{F}^{\{1,2\}}$ be such that F_1 assigns probability 0.9 to 20 and 0.1 to 45, F_2 assigns probability 0.5 to 0 and 0.5 to 45, and $T = 44$. The allocation selected by this rule is $(30, 14)$ since $U(F_1, 30) = \frac{1}{10} \cdot 20 + (30 - 20) + 0.9 \cdot (30 - 20) = 21$ and $U(F_2, 14) = 14 + 0.5 \cdot 14 = 21$. Since F_2 is riskier than F_1 and $w(F_1, 30) = 0.9 \cdot 10 > 0.5 \cdot 14 = w(F_2, 14)$, *no reversal* is violated. Moreover, since $30 > 14$, $w(F_1, 30) > w(F_2, 14)$, and $14 < \overline{\text{supp}}\ F_2$, *no domination* is also violated. It can be readily seen that the rule satisfies *risk aversion*, *conditionally strict endowment monotonicity*, *positivity*, *consistency*, *strong upper composition*, and *claims truncation invariance*. To see that it satisfies *lower composition*, simply notice that for each $F_i \in \mathcal{F}$ and each pair $t_i, t_i' \in [0, \overline{\text{supp}}\ F_i]$ with $t_i < t_i'$, $U(F_i, t_i') = U(F_i, t_i) + U(F_i|^{t_i}, t_i' - t_i)$.

Dropping *strong upper composition* and *claims truncation invariance*, consider a rule that is defined in the same way as in the previous paragraph expect that it is associated with a different claim-specific cost function $U : \hat{D} \to \mathbb{R}_+$: For each $(F_i, t_i) \in \hat{D}$,

$$U(F_i, t_i) = \begin{cases} \frac{1}{10} t_i & \text{if } \underline{\text{supp}}\ F_i = \overline{\text{supp}}\ F_i, \\ t_i + w(F_i, t_i) & \text{if } \underline{\text{supp}}\ F_i < \overline{\text{supp}}\ F_i. \end{cases}$$

Note that if a claim is uncertain ($\underline{\text{supp}}\ F_i < \overline{\text{supp}}\ F_i$), U is simply the sum of assignment and expected waste, like a usual cost function. If a claim is deterministic ($\underline{\text{supp}}\ F_i = \overline{\text{supp}}\ F_i$), then the same assignment generates one-tenth the cost generated in the case of an uncertain claim. Since a truncation of an uncertain claim can make it deterministic, *strong upper composition* and *claims truncation invariance* will be violated. It can be readily seen that this rule satisfies *no domination*, *risk aversion*, *no reversal*, *conditionally strict endowment monotonicity*, *positivity*, *consistency*, and *lower composition*.

Dropping *lower composition*, consider an expected-waste constrained uniform gains rule associated with a non-linear cost function. Such a rule satisfies *no domination*, *risk aversion*, *no reversal*, *conditionally strict endowment monotonicity*, *positivity*, *consistency*, *strong upper composition*, and *claims truncation invariance*.

Dropping *conditionally strict endowment monotonicity* and *positivity*, consider the rule r that equalizes agents' assignments as much as possible when the endowment is small and equalizes their expected waste as much as possible when the endowment is large. Formally, for each $I \in \mathcal{N}$ and each $(F, T) \in C^I$, when $T \leq \sum \underline{\text{supp}}\, F_i$, $r(F, T) = t$ if and only if for some $c^* \in \mathbb{R}_+$ and for each $j \in I$,

$$t_j = \min\{\underline{\text{supp}}\, F_j, c^*\} \text{ and } \sum t_i = T,$$

and when $T > \sum \underline{\text{supp}}\, F_i$, $r(F, T) = t$ if and only if for some $c^* \in \mathbb{R}_+$ and for each $j \in I$,

$$w(F_j, t_j) = \min\{w(F_j, \overline{\text{supp}}\, F_j), c^*\} \text{ and } \sum t_i = T.$$

This rule satisfies *no domination*, *risk aversion*, *no reversal*, *consistency*, *strong upper composition*, and *lower composition*.

Dropping *consistency*, consider an expected-waste constrained uniform gains rule relating to one linear cost function for two-agent problems and another linear cost function for three-or-more-agent problems. Such a rule satisfies *no domination*, *risk aversion*, *no reversal*, *conditionally strict endowment monotonicity*, *positivity*, *strong upper composition*, *lower composition*, and *claims truncation invariance*.

6 Concluding remarks

Besides the concern for waste, there is also a valid concern for assignments falling short of realized needs. Our approach to addressing the issue of waste inspires a "dual" approach to addressing the issue of unsatisfied needs. To discuss the dual approach, let us first introduce some terminologies. Suppose that agent i claims F_i and receives t_i. Recall that if his realized need is x_i, $\max\{x_i - t_i, 0\}$ is his unsatisfied need, and $F_i|^{t_i}$ is the distribution of his unsatisfied needs. Let us call $\max\{x_i - t_i, 0\}$ his **deficit** for simplicity and the expectation of $F_i|^{t_i}$ his **expected deficit**, and let us denote the latter $d(F_i, t_i)$. Following the convention in the deterministic claims problems, we call the difference between agent i's maximal claim and his assignment, $\overline{\text{supp}}\ F_i - t_i$, the **loss** of agent i, which is equal to his maximal deficit. While the expected waste of agent i increases with his assignment, the expected deficit increases with his loss.

As in the deterministic claims problems, the dual approach focuses on how to divide the shortfall, i.e., the difference between the sum of the maximal claims and the endowment. A dual allocation specifies for each agent a loss such that the sum of their losses is equal to the shortfall. One can impose dual axioms on dual allocations. For example, *dual no domination* requires that no agent suffer a larger loss than another agent while getting a larger expected deficit, unless the other agent has suffered the maximal loss (i.e., he has been assigned nothing). *Dual risk aversion* requires that if two agents have the same maximal claims and the claim of one agent is riskier than the other's, then the riskier agent suffer a no smaller loss than the other agent. *Dual no reversal* requires that in the same situation, the riskier agent not suffer so large a loss as to get a larger expected deficit than the other agent. Similarly, one can formulate dual requirements based on the other axioms.

We can define the class of expected-deficit constrained uniform losses rules analogously to the way we define expected waste constrained uniform gains rules. Let $D' := \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 \in [0, \overline{\text{supp}}\ F_i], x_2 = d(F_i, \overline{\text{supp}}\ F_i - x_1) \text{ where } F_i \in \mathcal{F}\}$ be the set of all loss-and-expected-deficit pairs, and V a continuous and increasing real-valued function on D'. In each problem, the expected-deficit constrained uniform losses rule associated with V assigns to agent i his maximal loss $\overline{\text{supp}}\ F_i$ if $V(\overline{\text{supp}}\ F_i, d(F_i, \overline{\text{supp}}\ F_i - \overline{\text{supp}}\ F_i))$ is smaller than some common value $c^* \in \mathbb{R}_+$, and otherwise a loss $s_i \in [0, \overline{\text{supp}}\ F_i]$ such that $V(s_i, d(F_i, \overline{\text{supp}}\ F_i - s_i)) = c^*$. The value of c^* is determined by the binding feasibility constraint. When restricted to problems with deterministic claims, all expected-deficit constrained uniform losses rules agree with the classic uniform losses rule (Herrero and Villar (2001)).

We conjecture that the class of expected-deficit constrained uniform losses rules and its homothetic and linear subclasses can be characterized by the corresponding sets of dual axioms. There are two main issues that require further investigation. First, some dual axioms may be redundant in the dual characterization due to some technical differences between waste and deficit. For example, expected deficit is always increasing in loss, whereas expected waste could be constantly zero when assignment increases. Thus, axioms dealing with the case in which agents have zero expected waste (like *strong upper composition*) could be redundant in the dual characterization. Second and more importantly, the duality between two problems, two rules and two axioms need to be systematically established, and their definitions may be different from their deterministic counterparts. For example, when restricted to deterministic claims, a problem is said to be dual to another if the profiles of claims in two problems are the same and the endowment of the former is the shortfall of the latter (Thomson (2015)). When extended to uncertain claims, the definition of a dual problem may require an appropriate adjustment of agents' claims. This is because it might be ideal if the deficit of an agent in a problem corresponds to his waste for his adjusted claim in the dual problem, and this cannot be true if his claim is unchanged. For instance, if an agent has a deterministic claim and it remains unchanged in the dual problem, then in both problems, waste does not exist and only deficit is relevant.

To summarize, in this paper, we study the pre-committed division of a resource in the face of uncertain needs. Unlike division with deterministic needs, waste is a common issue under uncertainty. We axiomatize a class of rules that strikes a balance between being unresponsive to uncertainty and too sensitive to it. More studies should be done in this field, for example, regarding different approaches to addressing the issue of waste, axioms that incorporate the considerations of both waste and deficit, and the duality theory under uncertainty.

7 Appendix

Given $I, I' \in \mathcal{N}$, $F \in \mathcal{F}^I$ and $F' \in \mathcal{F}^{I'}$ such that $I \cap I' = \emptyset$, we denote by (F, F') the claim profile in $\mathcal{F}^{I \cup I'}$ defined by setting for each $i \in I$, agent i's claim is F_i, and for each agent $j \in I'$, agent j's claim is F'_j.

Observe that *symmetry* and *consistency* together imply *anonymity*, i.e., the names of agents should have no impact on allocation.[12]

Anonymity: For each $I \in \mathcal{N}$, each $(F, T) \in C^I$ and each $\pi : I \to \mathbb{N}$ which is injective, if $(F', T) \in C^{\pi(I)}$ is such that for each $i \in I$, $F'_{\pi(i)} = F_i$, then for each $i \in I$, $r_i(F, T) = r_{\pi(i)}(F', T)$.

Some intermediate results will be useful for our proofs. We state these results below and leave their proofs to the online appendix.

[12]For a proof, see Lemma 3 in Chambers and Thomson (2002).

Lemma 1. *If r is endowment monotonic, then (1) for each $I \in \mathcal{N}$, each $F \in \mathcal{F}^I$, each $i \in I$, and each $t_i \in [0, \overline{\text{supp}}\ F_i]$, there is a smallest $T \in [0, \sum \overline{\text{supp}}\ F_j]$ such that $r_i(F,T) = t_i$; (2) for each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{C}^I$, and each $i \in I$, if $T > 0$, then $r_i(F,T) = \sup\{r_i(F,T') : T' \in [0,T)\}$.*

Lemma 2. *Let r be a rule satisfying either (i) symmetry and strong upper composition, or (ii) symmetry, consistency, lower composition, and claims truncation invariance. For each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{C}^I$, and each pair $\{i,j\} \subseteq I$, if there is $c \in [0, \min\{\overline{\text{supp}}\ F_i, \overline{\text{supp}}\ F_j\}]$ such that F_i and F_j agree on $(-\infty, c)$ and $r_i(F,T) < c$, then $r_i(F,T) = r_j(F,T)$.*

Lemma 3. *If a rule satisfies positivity and lower composition, then it is conditionally strict endowment monotonic.*

Proposition 4. *Let a rule satisfy consistency and strong upper composition. If in addition, it satisfies either no domination and conditionally strict endowment monotonicity, or risk aversion and no reversal, then it is symmetric.*

Proposition 5. *Let a rule satisfy consistency, lower composition, and claims truncation invariance. If in addition, it satisfies either no domination and positivity, or risk aversion, then it is symmetric.*

Proof of Theorem 1. The "if" direction can be readily verified, so the proof is omitted. To show the "only if" direction, let r be a rule satisfying *no domination*, *conditionally strict endowment monotonicity*, *consistency*, and *strong upper composition*. By *strong upper composition*, r is *endowment monotonic*. By Proposition 4, r is *symmetric*. Recall that by *symmetry* and *consistency*, r is *anonymous*.

Step 1. For each $I \in \mathcal{N}$, each $(F,T) \in \mathcal{C}^I$, each $J \subseteq I$, and each $G \in \mathcal{F}^J$, if for each $T' \in [0,T)$ and each $i \in J$, $r_i(F,T') < r_i(F,T)$, and for each $i \in J$, $\overline{\text{supp}}\ G_i \geq r_i(F,T)$ and $w(F_i, r_i(F,T)) = w(G_i, r_i(F,T))$, then $r((G, F_{I \setminus J}), T) = r(F,T)$.

Let $I \in \mathcal{N}$, $(F,T) \in \mathcal{C}^I$, $J \subseteq I$, and $G \in \mathcal{F}^J$ satisfy the required conditions. Let $t := r(F,T)$. Let $\pi : J \to \mathbb{N} \setminus I$ be an injective mapping, and $F_{\pi(J)}$ a claim profile such that for each $i \in J$, $F_{\pi(i)} = G_i$. Consider $((F, F_{\pi(J)}), T + \sum_{i \in J} t_i) \in \mathcal{C}^{I \cup \pi(J)}$ and let $t' := r((F, F_{\pi(J)}), T + \sum_{i \in J} t_i)$. By *consistency* and *anonymity*, it suffices to show that for each $i \in J$, $t'_{\pi(i)} = t_i$. Let $j \in J$. Suppose that $t'_{\pi(j)} > t_j$. Then, $t'_j \geq t_j$. To see this, assume that $t'_j < t_j$. Then, $t'_j < t'_{\pi(j)}$ and $w(F_j, t'_j) \leq w(F_j, t_j) = w(G_j, t_j) \leq w(F_{\pi(j)}, t'_{\pi(j)})$. If $w(F_j, t'_j) < w(F_{\pi(j)}, t'_{\pi(j)})$, then by *no domination*, $t'_j = \overline{\text{supp}}\ F_j$. However, $t'_j < t_j \leq \overline{\text{supp}}\ F_j$, which is a contradiction. If $w(F_j, t'_j) = w(F_{\pi(j)}, t'_{\pi(j)})$, then $w(F_j, t'_j) = w(F_j, t_j) = 0$ and thus $w(G_j, t_j) = 0$. Hence, F_j and $F_{\pi(j)}$ agree on $(-\infty, t_j)$. Since $t'_j < t_j$, then by Lemma 2, $t'_j = t'_{\pi(j)}$, which is a contradiction. Thus, $t'_j \geq t_j$. Then by *consistency*, $r_j(F, \sum_{i \in I} t'_i) = t'_j \geq t_j$. Since for each $T' \in [0, T)$, $r_j(F, T') < t_j$, then $\sum_{i \in I} t'_i \geq T$. Hence, $\sum_{i \in J} t'_{\pi(i)} \leq \sum_{i \in J} t_i$. Since $t'_{\pi(j)} > t_j$, then there is $k \in J$ such that $t'_{\pi(k)} < t_k$. By *endowment monotonicity* and *consistency*, $t_k \leq r_k(F, \sum_{i \in I} t'_i) = t'_k$. Thus, $t'_{\pi(k)} < t_k$ and $w(F_{\pi(k)}, t'_{\pi(k)}) \leq w(G_k, t_k) = w(F_k, t_k) \leq w(F_k, t'_k)$. If $w(F_{\pi(k)}, t'_{\pi(k)}) < w(F_k, t'_k)$, then by *no domination*, $t'_{\pi(k)} = \overline{\text{supp}}\ F_{\pi(k)}$. However, $t'_{\pi(k)} < t_k \leq \overline{\text{supp}}\ G_k = \overline{\text{supp}}\ F_{\pi(k)}$, which is a contradiction. If $w(F_{\pi(k)}, t'_{\pi(k)}) = w(F_k, t'_k)$, then $w(F_{\pi(k)}, t'_{\pi(k)}) = w(G_k, t_k) = 0$ and $w(F_k, t_k) = 0$. Hence, F_k and $F_{\pi(k)}$ agree on $(-\infty, t_k)$. Since $t'_{\pi(k)} < t_k$, then $t'_{\pi(k)} = t'_k$, which is again a contradiction. Hence, $t'_{\pi(j)} > t_j$ is not possible. Similarly, $t'_{\pi(j)} < t_j$ is not possible, either.

Step 2. Define a binary relation $\simeq$ of "as costly as" between vectors in D as follows. For each pair $(x_1, x_2), (x'_1, x'_2) \in D$, $(x_1, x_2) \simeq (x'_1, x'_2)$ if there are $I \in \mathcal{N}$, $(F, T) \in C^I$ and $\{i, j\} \subseteq I$ with $w(F_i, x_1) = x_2$ and $w(F_j, x'_1) = x'_2$, and satisfying

(a) $r_i(F, T) = x_1$, $r_j(F, T) = x'_1$;

(b) for each $T' \in [0, T)$, $r_i(F, T') < x_1$, $r_j(F, T') < x'_1$.

Intuitively, (x_1, x_2) seems revealed to be as costly as (x'_1, x'_2) if in a two-agent problem, one agent is assigned x_1, generating expected waste x_2, and the other agent is assigned x'_1, generating expected waste x'_2. This is actually what condition (a) says. But this condition is not sufficient. If the endowment in the problem is decreased and one agent's assignment, say x_1, remains unchanged, then his expected waste, x_2, is also unchanged, whereas the other agent's assignment must decrease and expected waste not increase. In this case, if we imposed only condition (a) when defining the relation $\simeq$, (x_1, x_2) would then be as costly as two vectors, one of which dominates the other, violating the monotonicity property that we intend to have. To avoid that, condition (b) further requires that as the endowment decreases, neither agent's assignment remain unchanged, which is equivalent, in light of *endowment monotonicity*, to that both of their assignments decrease.

We claim that $\simeq$ is an equivalence relation. Moreover, for each pair (x_1, x_2) and (x'_1, x'_2) in D, if $x_1 < x'_1$ and either $x_2 < x'_2$ or $x_2 = x'_2 = 0$, then $(x_1, x_2) \not\simeq (x'_1, x'_2)$.

By *symmetry*, $\simeq$ is reflexive. By definition, $\simeq$ is symmetric. To show $\simeq$ is transitive, let $(x_1, x_2), (x'_1, x'_2), (x''_1, x''_2) \in D$ be such that $(x_1, x_2) \simeq (x'_1, x'_2)$ and $(x'_1, x'_2) \simeq (x''_1, x''_2)$. By *endowment monotonicity*, *consistency*, and *anonymity*, there are $(F,T) \in C^{\{1,2\}}$ and $(F',T') \in C^{\{3,4\}}$ such that $w(F_1, x_1) = x_2$, $w(F_2, x_1') = w(F_3, x_1') = x_2'$, $w(F_4, x_1'') = x_2''$, and condition (a) and (b) hold for both (F,T) and (F',T').

Let $((F,F'), T+T') \in C^{\{1,2,3,4\}}$ and $t := r((F,F'), T+T')$. We claim that $t_1 = x_1$, $t_2 = t_3 = x_1'$, and $t_4 = x_1''$. Suppose that $t_1 < x_1$. Then by *endowment monotonicity*, *consistency*, and condition (b), $t_2 < x_1' \leq t_3$. Thus, $t_2 < \overline{\text{supp}}\ F_2$ and $w(F_2, t_2) \leq w(F_2, x_1') = w(F_3, x_1') \leq w(F_3, t_3)$. If $w(F_2, t_2) < w(F_3, t_3)$, then *no domination* is violated. If $w(F_2, t_2) = w(F_3, t_3)$, then $w(F_2, t_2) = w(F_2, x_1') = 0$, and thus $w(F_3, x_1') = 0$. Hence, F_2 and F_3 agree on $(-\infty, x_1')$. Since $t_2 < x_1'$, then by Lemma 2, $t_2 = t_3$, which contradicts that $t_2 < t_3$. Suppose that $t_1 > x_1$. Then by *endowment monotonicity*, *consistency*, and condition (b), $t_3 < x_1' \leq t_2$, which will result in similar contradictions. Hence, $t_1 = x_1$. Analogously, $t_4 = x_1''$. Moreover, since $t_2 + t_3 = x_1 + 2x_1' + x_1'' - t_1 - t_4 = 2x_1'$, then by the previous arguments, $t_2 = t_3 = x_1'$.

Next, let $T'' \in [0, T+T')$ and $t' := r((F,F'), T'')$. Since $\sum_{i=1}^{4} t_i' = T'' < T + T'$, then either $t_1' + t_2' < T$ or $t_3' + t_4' < T'$. In the former case, by *consistency* and condition (b), for $i \in \{1,2\}$, $t_i' < t_i$. Since $t_2' < t_2 = x_1'$, then by the arguments in the last paragraph, $t_3' < x_1' = t_3$. By *endowment monotonicity*, *consistency*, and condition (b), $t_4' < t_4$. Similarly, in the latter case, we can show that $t_1' < t_1$ and $t_4' < t_4$.

Lastly, abusing notation, let $(x_1, x_2), (x_1', x_2') \in D$ be such that $x_1 < x_1'$ and $(x_1, x_2) \simeq (x_1', x_2')$. By the definition of $\simeq$, $x_1 > 0$. Let $(F,T) \in C^{\{1,2\}}$ be as above. Let $G_1 \in \mathcal{F}$ assign probability $\frac{x_2}{x_1}$ to 0 and $1 - \frac{x_2}{x_1}$ to x_1'. Note that $\overline{\text{supp}}\ G_1 > x_1$ and $w(G_1, x_1) = x_2$. By Step 1, $r((G_1, F_2), T) = r(F,T)$. If $x_2 < x_2'$, then it is a violation of *no domination*. If $x_2 = x_2' = 0$, then $w(G_1, x_1) = w(F_2, x_1') = 0$. Since $x_1 < x_1'$, then G_1 and F_2 agree on $(-\infty, x_1)$. Let $\epsilon > 0$ be such that $x_1 < x_1' - \epsilon$. By *conditionally strict endowment monotonicity*, $r_1((G_1, F_2), T-\epsilon) < x_1 < x_1' - \epsilon \leq r_2((G_1, F_2), T-\epsilon)$. However, by Lemma 2, $r_1((G_1, F_2), T-\epsilon) = r_2((G_1, F_2), T-\epsilon)$, which is a contradiction.

Step 3. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be such that for each $c \in \mathbb{R}_+$, $f(c) = \frac{c}{2}$. For each $(x_1, x_2) \in D$, there is a unique $c \in \mathbb{R}_+$ such that $(x_1, x_2) \simeq (c, f(c))$.

Let $(x_1, x_2) \in D$. Let $F \in \mathcal{F}^{\{1,2\}}$ be such that $\overline{\text{supp}}\ F_1 > x_1$, $w(F_1, x_1) = x_2$, and F_2 assigns probability 0.5 respectively to 0 and some $\bar{c} \in \mathbb{R}_+$ satisfying $(\bar{c}, f(\bar{c})) > (x_1, x_2)$. Note that for each $t_2 \in [0, \overline{\text{supp}}\ F_2]$, $w(F_2, t_2) = f(t_2)$. By Lemma 1, there is $T \in [0, \overline{\text{supp}}\ F_1 + \overline{\text{supp}}\ F_2]$ such that $r_1(F,T) = x_1$. Let $c := r_2(F,T)$. By *no domination*, $c < \bar{c}$. By *conditionally strict endowment monotonicity*, for each $T' < T$, $r(F,T) < r(F,T')$. Thus, $(x_1, x_2) \simeq (c, f(c))$. By

Step 2, such a c is unique.

Step 4. Define $U : D \to \mathbb{R}_+$ by setting for each $(x_1, x_2) \in D$, $U(x_1, x_2) = c$ if $(x_1, x_2) \simeq (c, f(c))$ for some $c \in \mathbb{R}_+$. The function U belongs to $\mathcal{U}$.

Clearly, $U(0,0) = 0$. To show the monotonicity property of U, let $(x_1, x_2), (x_1', x_2') \in D$ be such that $x_1 < x_1'$ and either $x_2 < x_2'$ or $x_2 = x_2' = 0$. Let $c := U(x_1, x_2)$ and $c' := U(x_1', x_2')$. Suppose to the contrary that $c \geq c'$. Let $F \in \mathcal{F}^{\{1,2\}}$ be as in the proof of Step 3, so $c < \overline{\text{supp}}\ F_2$ and by the definition of U, $r_1(F, x_1 + c) = x_1$ and $r_2(F, x_1 + c) = c$. By Lemma 1, there is $T \in [0, \overline{\text{supp}}\ F_1 + \overline{\text{supp}}\ F_2]$ such that $r_2(F, T) = c'$. Let $x_1'' := r_1(F, T)$. By *endowment monotonicity* and *conditional endowment monotonicity*, $x_1'' \leq x_1$ and $(c', w(F_2, c')) \simeq (x_1'', w(F_1, x_1''))$. Since $w(F_2, c') = f(c')$, then $(c', f(c')) \simeq (x_1'', w(F_1, x_1''))$. Since $\simeq$ is transitive, $(x_1', x_2') \simeq (x_1'', w(F_1, x_1''))$. But $x_1'' \leq x_1 < x_1'$, and either $w(F_1, x_1'') \leq x_2 < x_2'$ or $w(F_1, x_1'') = x_2 = x_2' = 0$, which contradicts Step 2.

Lastly, we show that U is continuous. To see upper semi-continuity, let $\epsilon > 0$ and $(x_1, x_2) \in D$. We claim that there is an open set O in D such that $(x_1, x_2) \in O$ and $O \subseteq U^{-1}((-\infty, U(x_1, x_2) + \epsilon))$. Abusing notation, let $c := U(x_1, x_2)$. Consider $F \in \mathcal{F}^{\{1,2\}}$ as in the proof of Step 3 and such that $x_1 \geq \underline{\text{supp}}\ F_1$, so $w(F_1, \cdot)$ is increasing on $[x_1, \overline{\text{supp}}\ F_1]$. Let $\delta \in (0, \min\{\epsilon, \overline{\text{supp}}\ F_1 - x_1, \overline{\text{supp}}\ F_2 - c\})$, $x_1' := r_1(F, x_1 + c + \delta)$ and $c' := r_2(F, x_1 + c + \delta)$. By *endowment monotonicity* and *conditionally strict endowment monotonicity*, $x_1' \in (x_1, \overline{\text{supp}}\ F_1)$, $c' \in (c, \min\{c + \epsilon, \overline{\text{supp}}\ F_2\})$, and $(x_1', w(F_1, x_1')) \simeq (c', w(F_2, c'))$. Since $w(F_2, c') = f(c')$, then $(x_1', w(F_1, x_1')) \simeq (c', f(c'))$. Thus, $U(x_1', w(F_1, x_1')) = c' \in (c, c + \epsilon)$. Since $(x_1, x_2) < (x_1', w(F_1, x_1'))$, then there is an open neighborhood O of (x_1, x_2) such that for each $(x_1'', x_2'') \in O$, $(x_1'', x_2'') < (x_1', w(F_1, x_1'))$. By the monotonicity property of U, $O \subseteq U^{-1}((-\infty, U(x_1, x_2) + \epsilon))$. Lower semicontinuity follows from a similar argument.

Step 5. The rule r is an expected-waste constrained uniform gains rule associated with U.

Let $I \in \mathcal{N}$, $(F, T) \in C^I$, and $c^* \in \mathbb{R}_+$ be such that c^* solves $\sum U_{F_i}^{-1}(c) = T$. Suppose to the contrary that for some $\{j, k\} \subseteq I$, $r_j(F, T) > U_{F_j}^{-1}(c^*)$ and $r_k(F, T) < U_{F_k}^{-1}(c^*)$. Thus, $T > 0$, $c^* > 0$, $0 < U_{F_j}^{-1}(c^*) < r_j(F, T) \leq \overline{\text{supp}}\ F_j$ and $r_k(F, T) < \overline{\text{supp}}\ F_k$. By *endowment monotonicity* and Lemma 1, there is $T' \in (0, T)$ such that $r_j(F, T') = U_{F_j}^{-1}(c^*)$. By *endowment monotonicity* and *conditionally strict endowment monotonicity*, $r_k(F, T') < r_k(F, T)$ and $(r_j(F, T'), w(F_j, r_j(F, T'))) \simeq (r_k(F, T'), w(F_k, r_k(F, T')))$. By Step 2 and the definition of U, $U(r_j(F, T'), w(F_j, r_j(F, T'))) = U(r_k(F, T'), w(F_k, r_k(F, T')))$. However, $U(r_j(F, T'), w(F_j, r_j(F, T'))) = c^* > U(r_k(F, T), w(F_k, r_k(F, T))) >$

$U(r_k(F,T'), w(F_k, r_k(F,T')))$, which is a contradiction. $\square$

Proof of Proposition 1. The "if" direction can be readily verified, so the proof is omitted. To show the "only if" direction, let r be the expected-waste constrained uniform gains rule associated with $U \in \mathcal{U}$ and suppose that r satisfies *scale invariance*. Let $a > 0$, and $(x_1, x_2), (x'_1, x'_2) \in D$ be such that $(x_1, x_2) \neq (x'_1, x'_2)$ and $U(x_1, x_2) = U(x'_1, x'_2)$. We claim that $U(ax_1, ax_2) = U(ax'_1, ax'_2)$. By the monotonicity property of U, $x_1 > 0$ and $x'_1 > 0$. Let $(F,T) \in C^{\{1,2\}}$ be such that F_1 assigns probability $\frac{x_2}{x_1}$ to 0 and $1 - \frac{x_2}{x_1}$ to $x_1 + x'_1 + 1$, F_2 assigns probability $\frac{x'_2}{x'_1}$ to 0 and $1 - \frac{x'_2}{x'_1}$ to $x_1 + x'_1 + 1$, and $T = x_1 + x'_1$. Thus, $r_1(F,T) = x_1$ and $r_2(F,T) = x'_1$. Let $F' \in \mathcal{F}^{\{1,2\}}$ be such that for each $i \in \{1,2\}$ and each $y_i \in \mathbb{R}$, $F'_i(ay_i) = F_i(y_i)$. By *scale invariance*, $r_1(F', aT) = ax_1$ and $r_2(F', aT) = ax'_1$. Note that for each $i \in \{1,2\}$, $ax_i < \overline{\text{supp}}\ F'_i$. Hence, $U(ax_1, ax_2) = U(ax_1, w(F'_1, ax_1)) = U(ax'_1, w(F'_2, ax'_1)) = U(ax'_1, ax'_2)$. $\square$

Proof of Theorem 2. The "if" direction can be readily verified, so the proof is omitted. To show the "only if" direction, let r be a rule satisfying *no domination*, *positivity*, *consistency*, *lower composition*, and either *strong upper composition* or *claims truncation invariance*. By *lower composition*, r is *endowment monotonic*. Since r satisfies *positivity* and *lower composition*, by Lemma 3, it is *conditionally strict endowment monotonic*. By Proposition 4 and 5, r is *symmetric*. Recall that by *symmetry* and *consistency*, r is *anonymous*. Since r satisfies the axioms required in Lemma 2, the conclusion of Lemma 2 holds. Note that our proof of the "only if" direction of Theorem 1 only relies on *no domination*, *symmetry*, *endowment monotonicity*, *conditionally strict endowment monotonicity*, *consistency*, and Lemma 2. Thus, by applying the same proof, we know that there is $U \in \mathcal{U}$ such that r is the expected-waste constrained uniform gains rule associated with U.

Let $\succsim$ be the weak order on D represented by U, i.e, for each pair $(x_1, x_2), (x'_1, x'_2) \in D$, $(x_1, x_2) \succsim (x'_1, x'_2)$ if and only if $U(x_1, x_2) \geq U(x'_1, x'_2)$. We claim that $\succsim$ is homothetic. Let $a > 0$, and $(x_1, x_2), (x'_1, x'_2) \in D$ be such that $(x_1, x_2) \neq (x'_1, x'_2)$ and $U(x_1, x_2) = U(x'_1, x'_2)$. We need to show that $U(ax_1, ax_2) = U(ax'_1, ax'_2)$. By the monotonicity property of U, $x_1 > 0$ and $x'_1 > 0$. Let $(F,T), (F', aT) \in C^{\{1,2\}}$ be as in the proof of Proposition 1. Thus, $r_1(F,T) = x_1$ and $r_2(F,T) = x'_1$. Suppose that $a = n$ where $n \in \mathbb{N}$. For each $m \in \mathbb{N}$ with $m \leq n$, let F^m_1 assign probability $\frac{x_2}{x_1}$ to 0 and $1 - \frac{x_2}{x_1}$ to $n(x_1 + x'_1 + 1) - (m-1)x_1$, and F_2 assign probability $\frac{x'_2}{x'_1}$ to 0 and $1 - \frac{x'_2}{x'_1}$ to $n(x_1 + x'_1 + 1) - (m-1)x'_1$. By *lower composition*, $r(F', nT) = \sum_{m=1}^{n} r(F^m, T) = nr(F,T)$.

Thus, $U(ax_1, ax_2) = U(ax'_1, ax'_2)$. A similar argument applies when $a = \frac{1}{n}$, $n \in \mathbb{N}$, and hence when $a = \frac{m}{n}$, $m, n \in \mathbb{N}$. The case of general a follows from the continuity of U.

Moreover, $\succsim$ is quasi-linear in the first coordinate. Let $a > 0$, $(x_1, x_2), (x'_1, x'_2) \in D$ be such that $(x_1, x_2) \neq (x'_1, x'_2)$ and $U(x_1, x_2) = U(x'_1, x'_2)$. Thus, $x_1 > 0$ and $x'_1 > 0$. Let $(F, T) \in \mathcal{C}^{\{1,2\}}$ be as in the proof of Proposition 1. Thus, $r_1(F, T) = x_1$ and $r_2(F, T) = x'_1$. Abusing notation, let $F' \in \mathcal{F}^{\{1,2\}}$ be such that for each $i \in \{1, 2\}$ and each $y_i \in \mathbb{R}$, $F'_i(y_i + a) = F_i(y_i)$. By *lower composition*, for each $i \in \{1, 2\}$, $r_i(F', T + 2a) = r_i(F, T) + a$. Since $w(F'_1, x_1 + a) = x_2$ and $w(F'_2, x'_1 + a) = x'_2$, then $U(x_1 + a, x_2) = U(x'_1 + a, x'_2)$. By the monotonicity and continuity properties of U, the same result holds if $a < 0$ and $(x_1 + a, x_2), (x'_1 + a, x'_2) \in D$.

Let $U' : D \to \mathbb{R}_+$ be defined by setting for each $(x_1, x_2) \in D$, $U'(x_1, x_2) = c$ if and only if $U(x_1, x_2) = U(c, 0)$. By the monotonicity, continuity, and homotheticity properties of U, U' is well-defined and $\succsim$ is represented by U'. Since only the ordinal properties of U matter when defining r, then r is the expected-waste constrained uniform gains rule with respect to U'. We claim that U' is linear. Define $u : \mathbb{R}_+ \to \mathbb{R}$ by setting for each $y \in \mathbb{R}_+$, $u(y) = U'(2y, y) - 2y$. Let $(x_1, x_2) \in D$. By the definition of U', $U(x_1, x_2) = U(U'(x_1, x_2), 0)$ and $U(2x_2, x_2) = U(U'(2x_2, x_2), 0)$. By quasi-linearity of $\succsim$ and the monotonicity property of U, $2x_2 - x_1 = U'(2x_2, x_2) - U'(x_1, x_2)$. Thus, $U'(x_1, x_2) = x_1 + u(x_2)$. Since $\succsim$ is homothetic, when $x_2 > 0$, $u(x_2) = U'(2x_2, x_2) - 2x_2 = x_2[U'(2, 1) - 2] = u(1)x_2$. When $x_2 = 0$, $u(x_2) = U'(0, 0) - 0 = 0 = u(1)x_2$. Hence, $U'(x_1, x_2) = x_1 + u(1)x_2$. Lastly, note that by the monotonicity property of U, $u(1) \geq 0$. □

Proof of Proposition 2. Let r be a rule satisfying the axioms either in (*i*) or (*ii*). Then, r is *endowment monotonic*. Moreover, if r satisfies *positivity* and *lower composition*, then by Lemma 3, it is *conditionally strict endowment monotonic*.

Suppose that r satisfies *no domination*. By Proposition 4 and 5, r is *symmetric*. Recall that by *symmetry* and *consistency*, r is *anonymous*. To see that it is *risk averse*, let $I \in \mathcal{N}$, $(F, T) \in \mathcal{C}^I$, and $\{i, j\} \subseteq I$ be such that $\overline{\text{supp}}\, F_i = \overline{\text{supp}}\, F_j$ and F_i is riskier than F_j. Suppose to the contrary that $r_i(F, T) > r_j(F, T)$. Then, $T > 0$, $r_j(F, T) < \overline{\text{supp}}\, F_j$ and

$$w(F_i, r_i(F, T)) \geq w(F_i, r_j(F, T)) = \int_{-\infty}^{r_j(F,T)} F_i(x_i)dx_i \geq \int_{-\infty}^{r_j(F,T)} F_j(x_j)dx_j = w(F_j, r_j(F, T)).$$

If $w(F_i, r_i(F, T)) > w(F_j, r_j(F, T))$, then *no domination* is violated. If $w(F_i, r_i(F, T)) = w(F_j, r_j(F, T))$, then $w(F_i, r_i(F, T)) = w(F_i, r_j(F, T)) = 0$ and thus $w(F_j, r_j(F, T)) = 0$. Hence, F_i and F_j agree on $(-\infty, r_j(F, T))$. Since $T > 0$ and $\overline{\text{supp}}\, F_j > 0$, by *conditionally strict endowment monotonicity*, $r_j(F, T) > 0$. By *endowment monotonicity* and

Lemma 1, there is $T' \in [0, T)$ such that $r_i(F, T') = \frac{1}{2}[r_i(F, T) + r_j(F, T)]$. By *endowment monotonicity* and *conditionally strict endowment monotonicity*, $r_j(F, T') < r_j(F, T)$. Thus, $r_j(F, T') < r_i(F, T')$. Since F_i and F_j agree on $(-\infty, r_j(F, T))$ and $r_j(F, T') < r_j(F, T)$, then by Lemma 2, $r_j(F, T') = r_i(F, T')$, which contradicts that $r_j(F, T') < r_i(F, T')$. Hence, $r_i(F, T) \le r_j(F, T)$ as desired.

To see that r satisfies *no reversal*, let $I \in \mathcal{N}$, $(F, T) \in C^I$, and $\{i, j\} \subseteq I$ be such that $\overline{\text{supp}}\ F_i = \overline{\text{supp}}\ F_j$ and F_i is riskier than F_j. Suppose to the contrary that $w(F_i, r_i(F, T)) < w(F_j, r_j(F, T))$. If $r_i(F, T) \ge r_j(F, T)$, then $w(F_i, r_i(F, T)) \ge w(F_i, r_j(F, T))$. Since F_i is riskier than F_j, then $w(F_i, r_j(F, T)) \ge w(F_j, r_j(F, T))$. Thus, $w(F_i, r_i(F, T)) \ge w(F_j, r_j(F, T))$, which contradicts that $w(F_i, r_i(F, T)) < w(F_j, r_j(F, T))$. Hence, $r_i(F, T) < r_j(F, T)$ and $r_i(F, T) < \overline{\text{supp}}\ F_j = \overline{\text{supp}}\ F_i$. This contradicts *no domination*.

Conversely, suppose that r satisfies *risk aversion* and *no reversal*. By Proposition 4 and 5, r is *symmetric*. Recall that by *symmetry* and *consistency*, r is *anonymous*. Suppose to the contrary that it violates *no domination*. By *consistency* and *anonymity*, there is $(F, T) \in C^{\{1,2\}}$ such that $r_1(F, T) > r_2(F, T)$, $w(F_1, r_1(F, T)) > w(F_2, r_2(F, T))$, and $r_2(F, T) < \overline{\text{supp}}\ F_2$. Then, $T > 0$ and $w(F_1, r_1(F, T)) > 0$. Let $t := r(F, T)$. By *endowment monotonicity* and Lemma 1, there is $\bar{T} \in (0, T]$ such that $r_1(F, \bar{T}) = t_1$ and for each $T' \in [0, \bar{T})$, $r_1(F, T') < t_1$. By *endowment monotonicity*, there is $T' \in (0, \bar{T})$ such that $r_1(F, T') > r_2(F, T')$, $w(F_1, r_1(F, T')) > w(F_2, r_2(F, T'))$, $r_1(F, T') < \overline{\text{supp}}\ F_1$ and $r_2(F, T') < \overline{\text{supp}}\ F_2$. Hence, it is without loss of generality to assume that $t_1 < \overline{\text{supp}}\ F_1$. We shall derive a contradiction in six steps.

Step 1. For each $F' \in \mathcal{F}^{\{1,2\}}$ such that for each $i \in \{1, 2\}$, F'_i and F_i agree on $(-\infty, t_i)$, $r(F', T) = t$.

Let $G \in \mathcal{F}^{\{3,4\}}$ be such that $G_3 = F'_1$ and $G_4 = F'_2$. Let $t' := r((F, G), 2T)$. We claim that $t'_3 = t_1$ and $t'_4 = t_2$. To see this, suppose first that $t'_3 < t_1$. Then, by Lemma 2, $t'_1 = t'_3 < t_1$. By *endowment monotonicity*, *conditionally strict endowment monotonicity*, and *consistency*, $t'_2 < t_2$. Thus, by Lemma 2, $t'_4 = t'_2 < t_2$. Hence, $\sum_{i=1}^{4} t'_i < 2T$, which is a contradiction. Suppose now that $t'_3 > t_1$. Then, by Lemma 2, $t'_1 \ge t_1$. By *endowment monotonicity*, *conditionally strict endowment monotonicity*, and *consistency*, $t'_2 \ge t_2$. Thus, by Lemma 2, $t'_4 \ge t_2$. Hence, $\sum_{i=1}^{4} t'_i > 2T$, which is a contradiction. Thus, $t'_3 = t_1$. Similarly, $t'_4 = t_2$. By *consistency* and *anonymity*, $r(F', T) = t$.

Step 2. For each $I \in \mathcal{N}$, each $(F', T) \in \mathcal{F}^I$, and each pair $\{i, j\} \subseteq I$ such that $\overline{\text{supp}}\ F'_i =$

$\overline{\text{supp}}\ F'_j$ and F'_i is riskier than F'_j, if $w(F'_i, r_i(F', T)) = w(F'_j, r_i(F', T))$, then $r_j(F', T) = r_i(F', T)$.

By *risk aversion*, $r_j(F', T) \geq r_i(F', T)$. Suppose to the contrary that $r_j(F', T) > r_i(F', T)$. Then $T > 0$. Assume first that $w(F'_j, r_j(F', T)) > 0$. Then, $w(F'_j, r_i(F', T)) < w(F'_j, r_j(F', T))$. Since $w(F'_i, r_i(F', T)) = w(F'_j, r_i(F', T))$, then $w(F'_i, r_i(F', T)) < w(F'_j, r_j(F', T))$, which is a violation of *no reversal*. Now, assume that $w(F'_j, r_j(F', T)) = 0$. Then, $w(F'_j, r_i(F', T)) = 0$. Since $w(F'_i, r_i(F', T)) = w(F'_j, r_i(F', T))$, then $w(F'_i, r_i(F', T)) = 0$. Hence, F'_i and F'_j agree on $(-\infty, r_i(F', T))$. By *endowment monotonicity* and Lemma 1, there is $T' \in [0, T)$ such that $r_j(F', T') \in (r_i(F', T), r_j(F', T))$. Since $r_i(F', T) < r_j(F', T) \leq \overline{\text{supp}}\ F'_j = \overline{\text{supp}}\ F'_i$, by *endowment monotonicity* and *conditionally strict endowment monotonicity*, $r_i(F', T') < r_i(F', T)$. By Lemma 2, $r_j(F', T') = r_i(F', T')$. But $r_i(F', T') < r_i(F', T) < r_j(F', T')$, which is a contradiction.

Step 3. Let $p_1 \in (F_1(t_1), 1)$ and $p_2 \in (F_2(t_2), 1)$ be such that when $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} < 1$, $p_1 > p_2 > \frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2}$, and when $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} \geq 1$, $p_2 > p_1$. Then, for each $i \in \{1, 2\}$, $t_i - \frac{w(F_i,t_i)}{p_i} \geq 0$, and $\frac{1}{p_1-p_2}[t_1 p_1 - t_2 p_2 - w(F_1, t_1) + w(F_2, t_2)] > t_1$.

For each $i \in \{1, 2\}$, $w(F_i, t_i) \leq t_i F_i(t_i) \leq t_i p_i$, and thus $t_i - \frac{w(F_i,t_i)}{p_i} \geq 0$. By our conditions on p_1 and p_2,

$$\begin{aligned}&\frac{1}{p_1 - p_2}[t_1 p_1 - t_2 p_2 - w(F_1, t_1) + w(F_2, t_2)] - t_1 \\ =&\frac{t_1 - t_2}{p_1 - p_2}(p_2 - \frac{w(F_1, t_1) - w(F_2, t_2)}{t_1 - t_2}) > 0.\end{aligned}$$

Hence, $\frac{1}{p_1-p_2}[t_1 p_1 - t_2 p_2 - w(F_1, t_1) + w(F_2, t_2)] > t_1$.

Step 4. Let $F' \in \mathcal{F}^{\{1,2\}}$ be such that F'_1 assigns probability p_1 to $t_1 - \frac{w(F_1,t_1)}{p_1}$ and probability $1 - p_1$ to $\frac{1}{p_1-p_2}[t_1 p_1 - t_2 p_2 - w(F_1, t_1) + w(F_2, t_2)]$, and F'_2 assigns probability p_2 to $t_2 - \frac{w(F_2,t_2)}{p_2}$ and probability $1 - p_2$ to $\frac{1}{p_1-p_2}[t_1 p_1 - t_2 p_2 - w(F_1, t_1) + w(F_2, t_2)]$. When $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} < 1$, F'_2 is riskier than F'_1, and when $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} \geq 1$, F'_1 is riskier than F'_2.

Since $t_1 > t_2$, by Step 3, $\overline{\text{supp}}\ F'_1 = \overline{\text{supp}}\ F'_2 = \frac{1}{p_1-p_2}[t_1 p_1 - t_2 p_2 - w(F_1, t_1) + w(F_2, t_2)] > \max\{t_i - \frac{w(F_i,t_i)}{p_i} : i \in \{1, 2\}\}$. For each $i \in \{1, 2\}$ and each $c \in \mathbb{R}$,

$$\int_{-\infty}^{c} F'_i(x_i) dx_i = \begin{cases} 0 & \text{if } c \in (-\infty, t_i - \frac{w(F_i,t_i)}{p_i}), \\ p_i[c - (t_i - \frac{w(F_i,t_i)}{p_i})] & \text{if } c \in [t_i - \frac{w(F_i,t_i)}{p_i}, \overline{\text{supp}}\ F'_i), \\ p_i[\overline{\text{supp}}\ F'_i - (t_i - \frac{w(F_i,t_i)}{p_i})] + c - \overline{\text{supp}}\ F'_i & \text{if } c \in [\overline{\text{supp}}\ F'_i, \infty). \end{cases}$$

Moreover,

$$\begin{aligned}
&p_1[\overline{\text{supp}}\, F_1' - (t_1 - \frac{w(F_1,t_1)}{p_1})] \\
=&\frac{p_1}{p_1-p_2}[t_1p_1 - t_2p_2 - w(F_1,t_1) + w(F_2,t_2) - t_1(p_1-p_2) + (1-\frac{p_2}{p_1})w(F_1,t_1)] \\
=&\frac{p_1p_2}{p_1-p_2}[t_1 - \frac{w(F_1,t_1)}{p_1} - t_2 + \frac{w(F_2,t_2)}{p_2}],
\end{aligned}$$

and

$$\begin{aligned}
&p_2[\overline{\text{supp}}\, F_2' - (t_2 - \frac{w(F_2,t_2)}{p_2})] \\
=&\frac{p_2}{p_1-p_2}[t_1p_1 - t_2p_2 - w(F_1,t_1) + w(F_2,t_2) - t_2(p_1-p_2) + (\frac{p_1}{p_2}-1)w(F_2,t_2)] \\
=&\frac{p_1p_2}{p_1-p_2}[t_1 - \frac{w(F_1,t_1)}{p_1} - t_2 + \frac{w(F_2,t_2)}{p_2}].
\end{aligned}$$

Thus, $p_1[\overline{\text{supp}}\, F_1' - (t_1 - \frac{w(F_1,t_1)}{p_1})] = p_2[\overline{\text{supp}}\, F_2' - (t_2 - \frac{w(F_2,t_2)}{p_2})]$. Since $\overline{\text{supp}}\, F_1' = \overline{\text{supp}}\, F_2'$, then for each $c \in [\overline{\text{supp}}\, F_1', \infty)$, $\int_{-\infty}^{c} F_1'(x_1)dx_1 = \int_{-\infty}^{c} F_2'(x_2)dx_2$, and F_1' and F_2' have the same mean.

Suppose that $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} < 1$. Then, by the conditions on p_1 and p_2,

$$t_1 - \frac{w(F_1,t_1)}{p_1} - (t_2 - \frac{w(F_2,t_2)}{p_2}) > t_1 - t_2 - (\frac{w(F_1,t_1)}{p_2} - \frac{w(F_2,t_2)}{p_2}) > 0.$$

Thus, $t_1 - \frac{w(F_1,t_1)}{p_1} > t_2 - \frac{w(F_2,t_2)}{p_2}$, and for each $c \in (-\infty, t_1 - \frac{w(F_1,t_1)}{p_1})$, $\int_{-\infty}^{c} F_1'(x_1)dx_1 \leq \int_{-\infty}^{c} F_2'(x_2)dx_2$. Since $\overline{\text{supp}}\, F_1' = \overline{\text{supp}}\, F_2'$ and $p_1 > p_2$, then for each $c \in [t_1 - \frac{w(F_1,t_1)}{p_1}, \overline{\text{supp}}\, F_1')$,

$$p_1[c - (t_1 - \frac{w(F_1,t_1)}{p_1})] - p_2[c - (t_2 - \frac{w(F_2,t_2)}{p_2})] = (p_1 - p_2)(c - \overline{\text{supp}}\, F_1') < 0,$$

and thus $\int_{-\infty}^{c} F_1'(x_1)dx_1 < \int_{-\infty}^{c} F_2'(x_2)dx_2$. Hence, when $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} < 1$, F_2' is riskier than F_1'.

Suppose that $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} \geq 1$. Then, by the conditions on p_1 and p_2,

$$\begin{aligned}
&t_1 - \frac{w(F_1,t_1)}{p_1} - (t_2 - \frac{w(F_2,t_2)}{p_2}) < t_1 - t_2 - (\frac{w(F_1,t_1)}{p_2} - \frac{w(F_2,t_2)}{p_2}) \\
<&t_1 - t_2 - [w(F_1,t_1) - w(F_2,t_2)] \leq 0.
\end{aligned}$$

Thus, $t_1 - \frac{w(F_1,t_1)}{p_1} < t_2 - \frac{w(F_2,t_2)}{p_2}$, and for each $c \in (-\infty, t_2 - \frac{w(F_2,t_2)}{p_2})$, $\int_{-\infty}^{c} F_2'(x_2)dx_2 \leq \int_{-\infty}^{c} F_1'(x_1)dx_1$. Since $\overline{\text{supp}}\, F_1' = \overline{\text{supp}}\, F_2'$ and $p_1 < p_2$, then for each $c \in [t_2 - \frac{w(F_2,t_2)}{p_2}, \overline{\text{supp}}\, F_2')$,

$$p_1[c - (t_1 - \frac{w(F_1,t_1)}{p_1})] - p_2[c - (t_2 - \frac{w(F_2,t_2)}{p_2})] = (p_1 - p_2)(c - \overline{\text{supp}}\, F_2') > 0,$$

and thus $\int_{-\infty}^{c} F_2'(x_2)dx_2 < \int_{-\infty}^{c} F_1'(x_1)dx_1$. So when $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} \geq 1$, F_1' is riskier than F_2'.

Step 5. Let $I := \{1,2,3,4\}$ and $F' \in \mathcal{F}^I$ be such that for each $i \in \{1,2\}$, F_i' is defined as in Step 4, and F_{i+2}' agrees with F_i on $(-\infty, t_i)$ and agrees with F_i' on $[t_i, \infty)$.[13] Then, for each $i \in \{1,2\}$, $r_i(F', 2T) = r_{i+2}(F', 2T) = t_i$.

Let $t' := r(F', 2T)$. By Step 3, for each $i \in \{1,2\}$ and each $c \in \mathbb{R}$,

$$\int_{-\infty}^{c} F_i'(x_i)dx_i = \begin{cases} 0 & \text{if } c \in (-\infty, t_i - \frac{w(F_i,t_i)}{p_i}), \\ w(F_i,t_i) - (t_i - c)p_i & \text{if } c \in [t_i - \frac{w(F_i,t_i)}{p_i}, t_i), \\ w(F_i,t_i) + \int_{t_i}^{c} F_i'(x_i)dx_i & \text{if } c \in [t_i, \infty). \end{cases}$$

Besides, for each $i \in \{1,2\}$, by the definition of F_{i+2}',

$$\int_{-\infty}^{c} F_{i+2}'(x_{i+2})dx_{i+2} = \begin{cases} w(F_i,t_i) - \int_{c}^{t_i} F_i(x_i)dx_i & \text{if } c \in (-\infty, t_i), \\ w(F_i,t_i) + \int_{t_i}^{c} F_i'(x_i)dx_i & \text{if } c \in [t_i, \infty). \end{cases}$$

By Step 3, for each $i \in \{1,2\}$, $\overline{\text{supp}}\ F_i' = \overline{\text{supp}}\ F_{i+2}' > t_i$, so $\int_{-\infty}^{\overline{\text{supp}}\ F_i'} F_i'(x_i)dx_i = \int_{-\infty}^{\overline{\text{supp}}\ F_{i+2}'} F_{i+2}'(x_{i+2})dx_{i+2}$, and thus F_i' and F_{i+2}' have the same mean. For each $i \in \{1,2\}$, since $p_i > F_i(t_i)$, then for each $c \in \mathbb{R}$,

$$\int_{-\infty}^{c} F_i'(x_i)dx_i \leq \int_{-\infty}^{c} F_{i+2}'(x_{i+2})dx_{i+2}, \tag{2}$$

and (2) holds with strict inequality when $c \in [t_i - \frac{w(F_i,t_i)}{p_i}, t_i)$. Since $w(F_1,t_1) > 0$, $[t_1 - \frac{w(F_1,t_1)}{p_1}, t_1) \neq \emptyset$. Hence, F_3' is riskier than F_1'. Similarly, if $w(F_2,t_2) > 0$, then F_4' is riskier than F_2'. If $w(F_2,t_2) = 0$, then $\underline{\text{supp}}\ F_2 \geq t_2 = \underline{\text{supp}}\ F_2'$, and thus $F_4' = F_2'$.

By *anonymity* and Step 1, for each $i \in \{1,2\}$, $r_{i+2}((F_3', F_4'), T) = t_i$. Suppose that $t_3' < t_1$. Then, by *endowment monotonicity* and *consistency*, $t_4' \leq t_2$. Since $\sum_{i=1}^{4} t_i' = 2T = 2(t_1 + t_2)$, then either $t_1' > t_1$ or $t_2' > t_2$. Let $j \in \{1,2\}$ be such that $t_j' > t_j$. Then, by *endowment monotonicity* and Lemma 1, there is $T' \in [2T, 2\overline{\text{supp}}\ F_1' + 2\overline{\text{supp}}\ F_2']$ such that $r_{j+2}(F', T') = t_j$. By *endowment monotonicity*, $r_j(F', T') > t_j$. Thus, $r_j(F', T') > r_{j+2}(F', T')$. Moreover, $w(F_{j+2}', r_{j+2}(F', T')) = w(F_{j+2}', t_j) = w(F_j, t_j) = w(F_j', t_j) = w(F_j', r_{j+2}(F', T'))$. By the conclusion in the previous paragraph, either F_{j+2}' is riskier than F_j' or $F_{j+2}' = F_j'$. If F_{j+2}' is riskier than F_j', since $\overline{\text{supp}}\ F_j' = \overline{\text{supp}}\ F_{j+2}'$ and $w(F_{j+2}', r_{j+2}(F', T')) = w(F_j', r_{j+2}(F', T'))$, then by Step 2, $r_j(F', T') = r_{j+2}(F', T')$. This contradicts that $r_j(F', T') > r_{j+2}(F', T')$. If

[13] For each $i \in \{1,2\}$, F_{i+2}' is well-defined since $F_i'(t_i) = p_i > F_i(t_i)$.

$F'_{j+2} = F'_j$, by *symmetry*, $r_j(F', T') = r_{j+2}(F', T')$, which is again a contradiction. Hence, $t'_3 \geq t_1$. Similarly, $t'_4 \geq t_2$. Suppose that $t'_1 < t_1$. Then, $t'_3 \geq t_1 > t'_1$. Since $\overline{\text{supp}}\ F'_1 = \overline{\text{supp}}\ F'_3$ and F'_3 is riskier than F'_1, then by *risk aversion*, $t'_3 \leq t'_1$. This contradicts that $t'_3 > t'_1$. Hence, $t'_1 \geq t_1$. Similarly, if $t'_2 < t_2$ and F'_4 is riskier than F'_2, there is a contradiction. If $t'_2 < t_2$ and $F'_4 = F'_2$, then $t'_4 \geq t_2 > t'_2$ and by *symmetry*, $t'_4 = t'_2$, which is also not possible. Hence, $t'_2 \geq t_2$. Since for each $i \in \{1, 2\}$, $t'_i \geq t_i$ and $t'_{i+2} \geq t_i$, then for each $i \in \{1, 2\}$, $t'_i = t'_{i+2} = t_i$.

Step 6. When $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} < 1$, *no reversal* is violated. When $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} \geq 1$, *risk aversion* is violated.

Let $I \in \mathcal{N}$ and $F' \in \mathcal{F}^I$ be defined as in Step 5. By Step 5, $r_1(F', 2T) = t_1$ and $r_2(F', 2T) = t_2$. Then, $r_1(F', 2T) > r_2(F', 2T)$ and $w(F'_1, r_1(F', 2T)) = w(F_1, t_1) > w(F_2, t_2) = w(F'_2, r_2(F', 2T))$. When $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} < 1$, by Step 3 and Step 4, $\overline{\text{supp}}\ F'_1 = \overline{\text{supp}}\ F'_2$ and F'_2 is riskier than F'_1. Thus, *no reversal* is violated. When $\frac{w(F_1,t_1)-w(F_2,t_2)}{t_1-t_2} \geq 1$, by Step 3 and Step 4, $\overline{\text{supp}}\ F'_1 = \overline{\text{supp}}\ F'_2$ and F'_1 is riskier than F'_2. Thus, *risk aversion* is violated. □

Proof of Proposition 3. Let r be a rule satisfying *symmetry* and *strong upper composition*. Then, r is *endowment monotonic*. Let $I \in \mathcal{N}$ and $(F, T) \in \bar{C}^I$. Let $t := r(F, T)$ and $c^* \in \mathbb{R}_+$ be such that $\sum \min\{\overline{\text{supp}}\ F_i, c^*\} = T$. We claim that for each $i \in I$, $t_i = \min\{\overline{\text{supp}}\ F_i, c^*\}$. Suppose to the contrary that there are $j, k \in I$ such that $t_j < \min\{\overline{\text{supp}}\ F_j, c^*\}$ and $t_k > \min\{\overline{\text{supp}}\ F_k, c^*\}$. Then, $c^* \leq \overline{\text{supp}}\ F_k$ and $t_j < t_k$. Thus, $\min\{\overline{\text{supp}}\ F_j, c^*\} \leq \overline{\text{supp}}\ F_k$, and F_j and F_k agree on $(-\infty, \min\{\overline{\text{supp}}\ F_j, c^*\})$. Since $t_j < \min\{\overline{\text{supp}}\ F_j, c^*\}$, then by Lemma 2, $t_j = t_k$, which contradicts that $t_j < t_k$.

Let r be a rule satisfying *symmetry*, *lower composition*, and *claims truncation invariance*. By a similar argument as in the first paragraph of Case 2 in the proof of Lemma 2, it can be shown that for each $I \in \mathcal{N}$ and each $(F, T) \in \bar{C}^I$, if $c \in [0, \min\{\overline{\text{supp}}\ F_i : i \in I\}]$ is such that all the claims agree on $(-\infty, c)$ and $T \leq |I|c$, then for each $i \in I$, $r_i(F, T) = \frac{T}{|I|}$. Lastly, by *lower composition*, for each $i \in I$, $r_i(F, T) = \min\{\overline{\text{supp}}\ F_i, c^*\}$ where $c^* \in \mathbb{R}_+$ is such that $\sum \min\{\overline{\text{supp}}\ F_i, c^*\} = T$. □

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