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Precautionary Saving with Changing Income Ambiguity

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Precautionary saving with changing income ambiguity*

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Abstract

We study a two-period saving model where the agent's future income might be ambiguous. Our agent has a version of the smooth ambiguity decision criterion (Klibanoff, Marinacci and Mukerji (2005)), where the agent's perception about ambiguity is described by a second-order belief over first-order risks. We model increasing ambiguity as a spreading-out of the second-order belief. We show that under a "Risk Comonotonicity" condition, our agent saves more when ambiguity in future income increases. We argue that the condition is indispensable for our result.

JEL classification numbers: D80, D81, D91, E21

Key words: Precautionary Saving; Smooth Ambiguity; Increasing Ambiguity; Risk Comonotonicity; Informativeness

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1 Introduction and Summary

We study a saving problem of an ambiguity averse agent facing ambiguity in future income. The agent has a version of the smooth ambiguity decision criterion axiomatized by Klibanoff, Marinacci and Mukerji (2005). When the agent is ambiguity neutral, our problem reduces to the classic one of Kimball (1990). Our main result is that under an appealing condition, the agent has a stronger precautionary saving motive when the ambiguity in future income increases. The condition roughly says that when "first-order beliefs" about future income change, the expected utility and the expected *marginal* utility from future income move in the opposite directions. Since utility is increasing and marginal utility is decreasing in income, this condition holds intuitively in a variety of contexts. We also argue that the condition is in fact indispensable, so our main result is tight in this sense.

Our contributions are twofold. First, we propose a notion of increasing ambiguity for future income, elaborating on the idea in Snow (2010).[1] It relates directly to the informativeness of signals in Blackwell's information theory. The notion admits various equivalent interpretations and could be useful in a variety of applications. Second, we demonstrate that the notion is plausible at least in our saving context. When the future looks more ambiguous in our notion, an ambiguity averse agent is shown to save more as expected.

Our model is similar to, but different from, Berger (2014) and Osaki and Schlesinger (2014), where the agent has the recursive smooth ambiguity decision criterion (Klibanoff, Marinacci and Mukerji (2009)). Notably, Berger (2014) reports an important result that an ambiguity averse agent saves more in the case of ambiguity in future income than the case of no ambiguity, under a condition similar to ours in spirit. The result however is silent if an initially ambiguous future income gets more ambiguous, which our model can neatly handle. We speculate that such intermediate cases, though important, are hard to characterize in the recursive smooth ambiguity decision model.

2 Precautionary Saving under Income Ambiguity

An *environment* is a pair (S, Y) of random variables jointly distributed on $\mathbb{R}^2$, where S is a signal and Y an income level. It summarizes an agent's perception about income ambiguity. In Klibanoff, Marinacci and Mukerji (2005)'s terms, each conditional distribution $Y|S = s$,

[1] Snow (2010) proposes a notion of increasing ambiguity in a general abstract setup.

$s \in \mathbb{R}$, is a *first-order belief* about future income, and the distribution of all first-order beliefs $\{Y|S = s, s \in \mathbb{R}\}$ induced by S is his *second-order belief*. Consider the saving problem of an agent who has a sure income $e \in \mathbb{R}_+$ today, an ambiguous income (S, Y) tomorrow, and faces a per unit saving cost $q \in (0, \infty)$. The agent solves

$$\max_z U(v(e - qz), \mathbf{E}[\phi(\mathbf{E}[u(z + Y)|S])]) \tag{1}$$

where $z \in \mathbb{R}$ is an amount of saving, and v, u, ϕ are increasing and smooth real-valued functions over $\mathbb{R}$ with $v'' < 0$,[2] $u'' \leq 0$, $u''' \geq 0$, $\phi'' \leq 0$, and $\phi''' \geq 0$.

$$\max_z v(e - qz) + \beta\mathbf{E}[\phi(\mathbf{E}[u(z + Y)|S])] \tag{2}$$

In period 1, the utility from net income is measured by v. In period 2, the utility is calculated first by finding the expected values of u conditional on various first-order beliefs, and then these conditional expected values, after transformed by ϕ, are averaged with respect to the second-order belief. The utility function conforms with the smooth ambiguity decision criterion.[3] When ϕ is strictly concave, the agent is strictly ambiguity averse. When ϕ is linear, the agent is ambiguity neutral, and (2) reduces to Kimball (1990)'s classical problem.

In the special case of $v = \phi \circ u$, the objective function, $\phi \circ u(\cdot) + \beta\mathbf{E}[\phi(\mathbf{E}[u(\cdot)|S])]$, represents an additively time separable preference.[4] If Y equals to a constant y, then $\mathbf{E}[\phi(\mathbf{E}[u(z + Y)|S])] = \phi \circ u(z + y) = v(z + y)$, so v measures non-random income in each period. If S is a constant, then $\mathbf{E}[\phi(\mathbf{E}[u(z + Y)|S])] = \phi(\mathbf{E}[u(z + Y)])$, which is a monotonic transformation of $\mathbf{E}[u(z + Y)]$. In other words, the preference restricted to the second-period risks is represented by the vNM function u.

Different from the recursive smooth ambiguity preference model (Klibanoff, Marinacci and Mukerji (2009)), our model distinguishes the ambiguity neutral preference and the ambiguity averse preference in the absence of ambiguity. For example, suppose that $v = \phi \circ u$.

[2]The strict concavity of v is assumed to guarantee the uniqueness of the solution. It simplifies the presentation but is not necessary for our result.

[3]See Klibanoff, Marinacci and Mukerji (2005) for an axiomatization. However, except for the trivial case, our preferences are not recursive in the sense of Klibanoff, Marinacci and Mukerji (2009).

[4]One can extend this type of preference to the sum of an infinite series of discounted utilities, which admits the standard dynamic programming techniques in principle.

When φ is the identity function (ambiguity neutrality), our objective function becomes

$$\begin{aligned} & u\,(e-qz)+\beta\mathbf{E}\,[\mathbf{E}\,[u\,(z+Y)\,|S\,]] \\ =& u\,(e-qz)+\beta\mathbf{E}\,[u\,(z+Y)]\,. \end{aligned}$$

On the other hand, when φ is strictly concave and S is not random, our objective function becomes

$$\varphi(u\,(e-qz))+\beta\varphi(\mathbf{E}\,[u\,(z+Y)]).$$

Fixing v, u and ϕ as well as e, β and q, we study how optimal saving changes with environment. Throughout, we assume that probability distributions in consideration are well-behaved so that we can apply differentiation under expectation operators. Differentiating (2) with respect to z gives

$$\Psi\,(z;(S,Y)) := -qv'\,(e-qz)+\beta\mathbf{E}\,[\phi'\,(\mathbf{E}\,[u\,(z+Y)\,|S\,])\cdot\mathbf{E}\,[u'\,(z+Y)\,|S\,]]\,. \tag{3}$$

Since $-qv'\,(e-qz)$ is decreasing in z, and $\phi'\,(\mathbf{E}\,[u\,(z+Y)\,|S\,])$ and $\mathbf{E}\,[(u'\,(z+Y))\,|S\,]$ are non-increasing in z, then $\Psi\,(z;(S,Y))$ is decreasing in z. Hence, our problem is a well-defined concave problem. Write $z^*\,(S,Y)$ for the optimal saving under (S,Y). Then $z^*\,(S,Y) \leq z^*\,(S',Y')$ if $\Psi\,(z^*(S,Y);(S',Y')) \geq 0$.

3 Comparative Statics on Environment

3.1 Increasing Background Risks

For an illustrative purpose, we first compare the optimal amounts of saving under (S,Y) and (S,Y'), where signals are identically distributed. Suppose that $Y'|S$ is riskier than $Y|S$ with probability one, i.e., based on almost all the first-order beliefs, the agent perceives a greater income risk under the latter environment.

When ϕ is linear, Kimball (1990) shows that an ambiguity neutral agent saves more when income risk increases. In our general setup, an ambiguity averse agent also saves more. Indeed, for each z, $\phi'\,(\mathbf{E}\,[u\,(z+Y)\,|S\,]) \leq \phi'\,(\mathbf{E}\,[(u\,(z+Y'))\,|S\,])$ with probability one, and $\mathbf{E}\,[(u'\,(z+Y))\,|S\,] \leq \mathbf{E}\,[(u'\,(z+Y'))\,|S\,]$ with probability one. Hence, $\Psi\,(z;(S,Y)) \leq \Psi\,(z;(S,Y'))$ holds at each z, and a fortiori at $z^*\,(S,Y)$.

3.2 Risk and Ambiguity Trade-off

Assume that S, S' and Y are jointly distributed. We compare (S, Y) and (S', Y), i.e., the income distribution is the same, but the signals are different, generating different ambiguity. Recall that when S is a constant, (2) reduces to $v(e - qz) + \beta\phi(\mathbf{E}[u(z + Y)])$. Thus, tomorrow's income is purely risky, not ambiguous at all. At the other extreme, when $S' = Y$ with probability one, (2) reduces to $v(e - qz) + \beta\mathbf{E}[\phi(u(z + Y))]$. So the final income tomorrow is evaluated with a compound function $\phi \circ u$, i.e., it is purely ambiguous rather than risky.

Notice that signal S is completely uninformative for Y in the first extreme case, while S' is perfectly informative in the second. This observation suggests the following criterion to compare ambiguous environments. It is essentially equivalent to that proposed by Snow (2010).

Definition 1. *An environment (S', Y) is* ***no less ambiguous than*** *another environment (S, Y) if S' is at least as informative as S for Y, i.e., for each integrable function f, $\mathbf{E}[\mathbf{E}[f(Y)|S']|S] = \mathbf{E}[f(Y)|S]$.*

In other words, an environment is more ambiguous if the agent learns more from the signal. Since the agent cannot choose an action contingent on the signal, an additional piece of information is useless per se, and it will even hurt an ambiguity averse agent who cares about first-order beliefs.[5]

This condition is the same as the informativeness in Blackwell's information theory. In the case of discrete random variables, it is equivalent to that for each y and s in the support, $\Pr(Y = y|S = s) = \sum_{s'} \Pr(Y = y|S' = s')\Pr(S' = s'|S = s)$. In general, it says that for each s in the support, the conditional distribution $Y|S = s$ is an average of conditional distributions $E[(Y|S')|S = s]$.[6]

We shall show that our agent saves more facing the same income risks but more ambiguous environment, under a condition below. Let

$$W_0(S'; z) := \mathbf{E}[u(z + Y)|S'], \tag{4}$$

$$W_1(S'; z) := \mathbf{E}[u'(z + Y)|S']. \tag{5}$$

[5] See Grant, Kajii and Polak (1998) for their general discussion on Blackwell's theorem without contingent action choice.

[6] Snow (2010) uses this version, essentially.

Since u is increasing and u' is non-increasing, intuitively, W_0 and W_1 move in the opposite directions with S'. If S' brings good news so that Y tends to be high, then W_0 tends to be high and W_1 tends to be low. But this is not necessarily true because Y is random conditional on S', and this is what we need to assume.

Definition 2. ***Risk Comonotonicity*** *at z is satisfied if W_0 and $-W_1$ are comonotonic random variables at z, i.e., for each pair of realizations s'_1 and s'_2 of S',*

$$(W_0(s'_1;z) - W_0(s'_2;z)) \cdot (W_1(s'_1;z) - W_1(s'_2;z)) \leq 0$$

At each z, Risk Comonotonicity holds immediately if $Y = S'$ with probability one, i.e., tomorrow's income is purely ambiguous, or if u' is constant, i.e., the agent is risk neutral. It also holds if the distributions in $\{Y|S' = s' : s' \in \mathbb{R}\}$ are ordered by the first-order or second-order stochastic dominance.[7]

Proposition 1. *Suppose that (S', Y) is no less ambiguous than (S, Y), and that Risk Comonotonicity holds at $z^*(S, Y)$. Then $z^*(S, Y) \leq z^*(S', Y)$.*

Proof. Let $z := z^*(S, Y)$. We shall show that $\Psi(z; (S', Y)) \geq 0$. Write for simplicity $W_i(S') = W_i(S'; z)$, $i = 0, 1$. Since (S', Y) is no less ambiguous than (S, Y), then $\mathbf{E}[\mathbf{E}[u(z+Y)|S']|S] = \mathbf{E}[u(z+Y)|S]$ and $\mathbf{E}[\mathbf{E}[u'(z+Y)|S']|S] = \mathbf{E}[u'(z+Y)|S]$. Thus,

$$\mathbf{E}[W_0(S')|S] = \mathbf{E}[u(z+Y)|S], \tag{6}$$

$$\mathbf{E}[W_1(S')|S] = \mathbf{E}[u'(z+Y)|S]. \tag{7}$$

Since ϕ' is non-increasing, by Risk Comonotonicity, for each pair of realizations s'_1 and s'_2 of S',

$$(\phi'(W_0(s'_1)) - \phi'(W_0(s'_2))) \cdot (W_1(s'_1) - W_1(s'_2)) \geq 0.$$

Since s'_1 and s'_2 are arbitrary, we can take the expectation of the above, conditional on S, first letting $s'_1 = S'$ and then $s'_2 = S'$. Thus,

$$\mathbf{E}[\phi'(W_0(S')) \cdot W_1(S')|S] \geq \mathbf{E}[\phi'(W_0(S'))|S] \cdot \mathbf{E}[W_1(S')|S] \tag{8}$$

[7] Actually, the techniques to establish propositions 1 and 2 in Berger (2014) can be applied almost directly to assure Risk Comonotonicity. So we refer the reader to them for more conditions that guarantee Risk Comonotonicity.

with probability one.

Since ϕ' is convex, then by Jensen's inequality

$$\mathbf{E}\left[\phi'\left(W_0\left(S'\right)\right)|S\right] \geq \phi'\left(\mathbf{E}\left[W_0\left(S'\right)|S\right]\right)$$

with probability one. Since W_1 is a positive random variable, then

$$\mathbf{E}\left[\phi'\left(W_0\left(S'\right)\right)|S\right] \cdot \mathbf{E}\left[W_1\left(S'\right)|S\right] - \phi'\left(\mathbf{E}\left[W_0\left(S'\right)|S\right]\right) \cdot \mathbf{E}\left[W_1\left(S'\right)|S\right] \geq 0 \tag{9}$$

with probability one.

Since $z = z^*(S, Y)$, in view of (6) and (7),

$$-qv'\left(e - qz\right) + \beta\mathbf{E}\left[\phi'\left(\mathbf{E}\left[W_0\left(S'\right)|S\right]\right) \cdot \mathbf{E}\left[W_1\left(S'\right)|S\right]\right] = 0.$$

Hence,

$$\begin{aligned}
\frac{1}{\beta}\Psi\left(z;\left(S', Y\right)\right) &= -\frac{q}{\beta}v'\left(e - qz\right) + \mathbf{E}\left[\phi'\left(\mathbf{E}\left[u\left(z + Y\right)|S'\right]\right) \cdot \mathbf{E}\left[u'\left(z + Y\right)|S'\right]\right] \\
&= \mathbf{E}\left[\phi'\left(W_0\left(S'\right)\right) \cdot W_1\left(S'\right)\right] - \mathbf{E}\left[\phi'\left(\mathbf{E}\left[W_0\left(S'\right)|S\right]\right) \cdot \mathbf{E}\left[W_1\left(S'\right)|S\right]\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\phi'\left(W_0\left(S'\right)\right) \cdot W_1\left(S'\right)|S\right] - \phi'\left(\mathbf{E}\left[W_0\left(S'\right)|S\right]\right) \cdot \mathbf{E}\left[W_1\left(S'\right)|S\right]\right] \\
&\geq \mathbf{E}\left[\mathbf{E}\left[\phi'\left(W_0\left(S'\right)\right)|S\right] \cdot \mathbf{E}\left[W_1\left(S'\right)|S\right] - \phi'\left(\mathbf{E}\left[W_0\left(S'\right)|S\right]\right) \cdot \mathbf{E}\left[W_1\left(S'\right)|S\right]\right] \\
&\geq 0
\end{aligned}$$

where the third equality holds because $\mathbf{E}[\cdot] = \mathbf{E}[\mathbf{E}[\cdot|S]]$, the first inequality holds by (8), and the last inequality by (9). □

Remark 1. *Risk Comonotonicity is used to establish (8): $\phi'\left(W_0\left(S'\right)\right)$ and $W_1\left(S'\right)$ are positively correlated conditional on S. So we could strengthen Proposition 1 by simply assuming (8). But as discussed in the next section, Risk Comonotonicity is indispensable for a robust comparative statics result.*

Remark 2. *A similar analysis can be done with the recursive decision criterion, where the relevant first-order effect corresponding to (3) is:*

$$-qv'\left(e - qz\right) + \frac{\beta\mathbf{E}\left[\phi'\left(\mathbf{E}\left[u\left(z + Y\right)|S\right]\right) \cdot \mathbf{E}\left[u'\left(z + Y\right)|S\right]\right]}{\phi'\left\{\phi^{-1}\left(E\left[\phi\left(E\left[\mathbf{E}\left[u\left(z + Y\right)|S\right]\right]\right)\right]\right)\right\}}$$

If S is constant, i.e., there is no ambiguity, then the denominator of the fraction cancels out. This is the property Berger (2014) takes advantage of. If S is not constant, i.e., both (S, Y) and (S', Y) are ambiguous environments, the comparison of the first-order effects appears to be complicated, and there does not seem to be an analogous result as Proposition 1.

3.3 Tightness

We shall argue that if Risk Comonotonicity fails at some z, then the saving implication is reserved in some problem.

Let q, u and v be as assumed. Suppose that $u'' < 0$, and for ease of exposition that $u > 0$ so that when constructing ϕ, we only need to define it for positive numbers. Moreover, assume that $\lim_{x\to 0} v'(x) = \infty$ and $\lim_{x\to\infty} v'(x) = 0$ so that an optimal consumption level in period 1 is positive.

Let (S', Y) be an environment where S' takes values from $\{s'_0, ..., s'_n\}$, s'_i with probability $p_i > 0$, $i = 0, ..., n$, and $\sum p_i = 1$. Suppose that Risk Comonotonicity fails at $z > 0$. Let $w_{0j} := \mathbf{E}[u(z + Y)|S' = s'_j]$ and $w_{1j} := \mathbf{E}[u'(z + Y)|S' = s'_j]$, $j = 0, 1$. Without loss of generality, say $w_{i1} < w_{i0}$ for $i = 0, 1$.

We shall construct ϕ, e and (S, Y) such that (S', Y) is no less ambiguous than (S, Y), but $z^*(S, Y) > z^*(S', Y)$.

Let random variables S, S', Y be generated as follows. First, draw a number from $\{s_1, ..., s_n\}$, s_1 with probability $p_0 + p_1$, s_i with probability p_i, $i = 2, ..., n$, and set S to be the drawn number. If s_1 is drawn, choose s'_0 with probability $\frac{p_0}{p_0+p_1}$, and s'_1 with probability $\frac{p_1}{p_0+p_1}$ and set S' to be the chosen number. If s_i, $i \neq 1$, is drawn, set $S' = s'_i$. Finally, choose Y according to the conditional probability distribution given S'. Clearly, the joint distribution of S' and Y is the same as in the given environment (S', Y), and (S', Y) is no less ambiguous than (S, Y).

To construct ϕ, let η be a smooth, positive, decreasing, convex, and integrable function such that

$$\begin{aligned}\frac{p_0}{p_0 + p_1}\eta(w_{00}) + \frac{p_1}{p_0 + p_1}\eta(w_{01}) - \eta(\frac{p_0}{p_0 + p_1}w_{00} + \frac{p_1}{p_0 + p_1}w_{01})\\ < \frac{p_0 p_1[\eta(w_{01}) - \eta(w_{00})](w_{10} - w_{11})}{(p_0 + p_1)(p_0 w_{10} + p_1 w_{11})}.\end{aligned}$$

This is possible since the left hand side can be made arbitrarily small by making η flat, keeping the right hand side unchanged. Let $\phi(t) := \int_0^t \eta(s)\, ds$, $t > 0$. Clearly, $\phi' > 0$, $\phi'' = \eta' < 0$ and $\phi''' = \eta'' > 0$, as required.

Finally, let e be such that

$$-qv'(e - qz) + \mathbf{E}[\phi'(\mathbf{E}[u(z + Y)|S]) \cdot \mathbf{E}[u'(z + Y)|S]] = 0,$$

so that $z = z^*(S, Y)$.

To show that $z^*(S', Y) < z$, it suffices to check that (3) is negative at (S', Y). Notice that $\mathbf{E}[u(z+Y)|S = s_1] = \frac{p_0}{p_0+p_1}w_{00} + \frac{p_1}{p_0+p_1}w_{01}$, $\mathbf{E}[u'(z+Y)|S = s_1] = \frac{p_0}{p_0+p_1}w_{10} + \frac{p_1}{p_0+p_1}w_{11}$, and that the expectations conditional on $S = s_i$ and on $S' = s'_i$ coincide for $i = 2, ..., n$. Therefore,

$$
\begin{aligned}
&\mathbf{E}\left[\phi'\left(\mathbf{E}\left[u\left(z+Y\right)|S'\right]\right)\cdot\mathbf{E}\left[u'\left(z+Y\right)|S'\right]\right] - \mathbf{E}\left[\phi'\left(\mathbf{E}\left[u\left(z+Y\right)|S\right]\right)\cdot\mathbf{E}\left[u'\left(z+Y\right)|S\right]\right]\\
=&p_0\phi'(w_{00})w_{10} + p_1\phi'(w_{01})w_{11}\\
&-(p_0+p_1)\phi'(\frac{p_0}{p_0+p_1}w_{00} + \frac{p_1}{p_0+p_1}w_{01})\cdot(\frac{p_0}{p_0+p_1}w_{10} + \frac{p_1}{p_0+p_1}w_{11})\\
=&-\frac{p_0p_1}{p_0+p_1}[\eta(w_{01}) - \eta(w_{00})]\cdot(w_{10} - w_{11})\\
&+[\frac{p_0}{p_0+p_1}\eta(w_{00}) + \frac{p_1}{p_0+p_1}\eta(w_{01}) - \eta(\frac{p_0}{p_0+p_1}w_{00} + \frac{p_1}{p_0+p_1}w_{01})]\cdot(p_0w_{10} + p_1w_{11})\\
<&0,
\end{aligned}
$$

where the last inequality holds by the construction of η. This proves that (3) is negative at (S', Y).

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