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Abstract

In this paper provides, we study two extensions of Gilboa and Schmeidler (1989)'s maxmin expected utility decision rule to accommodate a decision maker's changing ambiguity attitude. The two rules are respectively a weighted maxmin rule and a variant constraint rule. The former evaluates an act by a weighted average of its worst and best possible expected utilities over a set of priors, with the weight on the worst case depending on the act. The latter evaluates an act by its worst expected utility over a neighborhood of a set of approximating priors, with the size of the neighborhood depending on the act. Canonical representations of the two rules are provided for classes of preferences that exhibit respectively ambiguity aversion of Schmeidler (1989) and ambiguity aversion of Ghirardato and Marinacci (2002). When restricted to the class of preferences exhibits both versions of ambiguity aversion, our results provide two alternative representations in addition to the ambiguity averse representation provided by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011).

In the second part of this paper, we study the wealth effect under ambiguity. We propose axioms on absolute and relative ambiguity aversion and derive the three representations for the ambiguity averse preferences displaying decreasing (increasing) absolute ambiguity aversion. In particular, decreasing absolute ambiguity aversion implies that as baseline utility increases, a weighted maxmin decision maker puts less weight on the worst case, and a variant constraint decision maker considers a smaller neighborhood of approximating priors.

Key-words: ambiguity; ambiguity averse preferences; weighted maxmin representation; variant constraint representation; decreasing absolute ambiguity aversion; increasing relative ambiguity aversion; wealth effect

JEL classification: D81

1 Introduction

A decision maker makes choices in the face of both unknown states of the world and unknown probability distributions of the states. Such a situation is called Knightian uncertainty (Knight (1921)) or ambiguity, in contrast with risk which refers to a situation with unknown states but a known probability distribution. The decision maker has a preference relation over acts that yield state-contingent outcomes. A well-known decision rule axiomatized by Gilboa and Schmeidler (1989) is the maxmin expected utility, or simply MEU, rule (see also Wald (1950a,b)). A MEU decision maker evaluates an act f by

$$\min_{p \in D} E_p u(f)$$

where D is a set of priors over the states, u is a utility function over outcomes, and $E_p u(f)$ denotes the expected utility of the act f with respect to a prior p. The decision maker behaves as if he regards the priors in D as possible and pessimistically evaluates an act by its worst possible expected utility.

The MEU decision rule is a prominent rule that incorporates a decision maker's aversion to ambiguity. However, by always considering the worst case, it does not accommodate the possibility that a decision maker's ambiguity attitude changes across acts. Even assuming that the decision maker is always averse to ambiguity, it does not accommodate the possibility that his degree of ambiguity aversion changes across acts. For example, as analogous to the wealth effect under risk, people may tend to be less averse to ambiguity when the baseline payoff of an act increases. This phenomenon is well evidenced by the recent experimental study of Baillon and Placido (2015). To accommodate changing ambiguity attitude or changing ambiguity aversion of a decision maker, we consider in this paper two extensions of the MEU decision rule.

The first extension is a generalized Hurwicz rule, or a weighted maxmin decision rule. A weighted maxmin decision maker evaluates an act f by

$$\lambda(u(f)) \min_{p \in D} E_p u(f) + (1 - \lambda(u(f))) \max_{p \in D} E_p u(f)$$

where D and u are the same as in the MEU rule, and λ is a function that depends on utility profiles induced by acts and takes values in $[0, 1]$. We call D an admissible set of priors, or simply an **admissible set**, and λ a weight function. A weighted maxmin decision maker behaves as if he considers not only the worst possible expected utility but also the best, and evaluates an act by a weighted average of the worst and the best expected utilities. Different from the MEU rule, the weight on the worst case is not a constant, but dependent on acts. Smaller weights correspond to less ambiguity aversion. When the weight goes to 1, the decision maker becomes optimistic and thus displays ambiguity loving.

One important issue with a weighted maxmin decision rule is the non-uniqueness of its representations. In fact, for each set D' of priors that includes D, there is a weight function λ' such that for each act f,

$$\begin{aligned}&\lambda(u(f))\min_{p\in D}E_p u(f)+(1-\lambda(u(f)))\max_{p\in D}E_p u(f)\\=&\lambda'(u(f))\min_{p\in D'}E_p u(f)+(1-\lambda'(u(f)))\max_{p\in D'}E_p u(f).\end{aligned}$$

Thus, each superset of an admissible set is an admissible set. In light of the nesting property of admissible sets, we are interested in finding the smallest admissible set to normalize the representation. This is a good way of normalization because while an admissible set provides for each act both an upper bound of its possible expected utility and a lower bound, only the bounds provided by the smallest admissible set provides are tight. We shall call a weighted maxmin representation of a preference relation a **canonical weighted representation** if its admissible set is the smallest admissible set in the set inclusion sense.

Ghirardato, Maccheroni and Marinacci (2004) also introduce a weighted maxmin decision rule with a unique representation. Instead of searching for the smallest admissible set, they impose a requirement on their admissible set in terms of an "unambiguous" preference relation induced from the initial preference relation. That is, the admissible set must be an component of a representation à la Bewley (2002) of the unambiguous preference relation. We shall call their admissible set a **Bewley set** and their representation a **Bewley weighted maxmin representation**.

Our first main result provides a canonical weighted maxmin representation for a class of so-called ambiguity averse preferences in the literature. This class of preferences is known to admit a Bewley weighted maxmin representation.[1] We further show that for each preference in this class, the smallest admissible set is in fact the Bewley set, so that the two types of weighted maxmin representations are the same.

In general, a canonical weighted maxmin representation is different from a Bewley weighted maxmin representation. Ambiguity averse preferences satisfy, besides some basic

[1] See Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci and Siniscalchi (2011).

axioms, the well-known ambiguity aversion axiom of Schmeidler (1989), called S-ambiguity aversion in this paper. Without imposing S-ambiguity aversion, we show by examples that when a Bewley weighted maxmin representation of a preference relation exists, a canonical weighted maxmin representation may not, and when both representations exist, the smallest admissible set may not be the Bewley set, so that the two types of weighted maxmin representations are different.

Another existing representation of the class of ambiguity averse preferences, provided by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011), is called an ambiguity averse representation. It ranks an act f according to

$$\min_{p \in \Delta} G(E_p u(f), p) \tag{1}$$

where u is the same as before, Δ is the probability space over the states, and $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ satisfying some properties is regarded as an ambiguity aversion index. They relate an ambiguity averse representation with a Bewley weighted maxmin representation by showing that the set $cl(\{p \in \Delta | G(t, p) < \infty \text{ for some } t\})$ is the Bewley set. In view of our result, it can also be related to a canonical weighted maxmin representation since $cl(\{p \in \Delta | G(t, p) < \infty \text{ for some } t\})$ is also the smallest admissible set.

The second extension of the MEU rule is in spirit a constraint decision rule introduced by Hansen and Sargent (2001) as one of their two robust decision rules. A variant constraint decision maker evaluates an act f by

$$\min_{p \in \Delta : d(p,K) \leq \sigma(u(f))} E_p u(f)$$

where u and Δ are the same as before, K is set of priors, $d(p, K)$ is the Euclidean distance between a prior p and the set K, and σ is a function that depends on utility profiles induced by acts and takes values in $\mathbb{R}_+$. We call K an essential set of priors, or simply an **essential set**, and σ a constraint function. A variant constraint decision maker behaves as if he considers the priors in the essential set K as best approximations of the true prior, and is also concerned with potential misspecification of approximating priors. Thus, the decision maker considers a neighborhood of the essential set and evaluates an act by its worst expected utility with respect to the priors in the neighborhood. Importantly, the size of the neighborhood, specified by σ, depends on the act in consideration. Larger values of σ correspond to less concern about misspecification of the essential set. When σ goes to 0, the decision maker only considers priors in essential set and is not at all concerned about its misspecification. When the constraint function σ is constant, it reduces to a MEU decision rule.

Similar to a weighted maxmin decision rule, a variant constraint decision rule has typically no unique representation. In fact, for each set K' of priors that is included in K, there is a constraint function σ' such that for each act f,

$$\min_{p \in \Delta : d(p,K) \leq \sigma(u(f))} E_p u(f) = \min_{p \in \Delta : d(p,K') \leq \sigma'(u(f))} E_p u(f).$$

Thus, each subset of an essential set is an essential set. This nesting property of essential sets motivate us to find the largest essential set to normalize the representation, as in the case of a weighted maxmin representation. We shall call a variant constraint representation of a preference relation a *canonical variant constraint representation* if its essential set is the largest essential set.

Our second main result axiomatizes the class of preferences that admits a canonical variant constraint representation. The characterizing axioms of this class of preferences are the ambiguity aversion axiom of Ghirardato and Marinacci (2002) and some other basic axioms. We shall call their ambiguity aversion axiom GM-ambiguity aversion. While there is an existing representation for the class of preferences satisfying S-ambiguity aversion (and other basic axioms),[2] our second main result provides the first representation for the class of preferences satisfying GM-ambiguity aversion.

Moreover, we fully characterize the largest essential set. To describe the set, we first recall the definition of GM-ambiguity aversion. GM-ambiguity aversion is defined based on a notion of comparative ambiguity aversion proposed by Ghirardato and Marinacci (2002). A preference relation is said to be **more ambiguity averse than** another preference relation if for each (ambiguous) act f and each (deterministic) outcome x, whenever x is preferred to f by the second preference relation, this is also the case by the first preference relation. Intuitively, if the ambiguity of an act is intolerable by a preference relation, it should also be intolerable by a more ambiguity averse preference relation. A preference relation is said to exhibit GM-ambiguity aversion if there is a subjective expected utility, or simply SEU, preference relation such that it is more ambiguity averse than the SEU preference relation. Each SEU preference relation is associated with a subjective prior, using which it reduces each act to a lottery and ranks acts by comparing reduced lotteries. Given a preference relation that admits a canonical variant constraint representation, we show that the largest essential set is exactly the set of all priors associated with which the SEU preferences are less ambiguity averse than the preference relation.

[2]It is provided by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011).

When S-ambiguity aversion is additionally imposed, we further relate a canonical variant constraint representation of a preference relation with an ambiguity aversion representation. We show that the largest essential set is in fact $\{p \in \Delta|$ for each $t, G(t, p) = t\}$.

The class of preferences admitting a canonical weighted maxmin representation and the class admitting a canonical variant constraint representation do not include each other. In view of our two representation results, there are preferences admitting canonical weighted maxmin representations but not canonical variant constraint representations, since S-ambiguity aversion does not imply GM-ambiguity aversion. Conversely, we also provide an example of a preference relation that admits a canonical variant constraint representation but not a canonical weighted maxmin representation. When a preference relation admits both canonical representations, we show that the largest essential set is always a subset of the smallest admissible set. Moreover, the MEU preferences are exactly characterized by the coincidence of the largest essential set with the smallest admissible set.

In the first part of the paper, we investigate two extensions of the MEU decision rule that accommodate a decision maker's changing ambiguity attitude. In the second part of the paper, we focus on studying a particular class of ambiguity averse preferences that display a monotonic pattern of changing ambiguity aversion. More precisely, we study the wealth effect on the class of preferences satisfying S-ambiguity aversion and some other basic axioms. As mentioned before, as evidenced by Baillon and Placido (2015), people tend to be less averse to ambiguity when they become better off overall. We propose an axiom of decreasing absolute ambiguity aversion to capture this type of behavior. Roughly, the axiom says that if an (ambiguous) act is preferred to a constant act, then it should be still preferred after a common improvement for both acts in every state. Intuitively, if the ambiguity of an act is tolerable before, it should be even more tolerable after an ensured improvement of the act.

Our third main result provides three representations for the subclass of ambiguity averse preferences displaying decreasing absolute ambiguity aversion — a weighted maxmin representation, a variant constraint representation, and an ambiguity averse representation. The wealth effect has straightforward implications on the weighted maxmin representation and the variant constraint representation. It implies that for each act f, $\lambda(u(f) + t\mathbf{1})$ and $\sigma(u(f) + t\mathbf{1})$ is weakly decreasing in t, where $t\mathbf{1}$ is a constant utility profile that yields utility t in each state. Intuitively, as the ensured utility of an act increases, a weighted maxmin decision maker behaves as if he becomes less pessimistic and puts a smaller weight on the worst case, and a variant constraint decision maker behaves as if he is less concerned with priors misspecification and considers a smaller neighborhood of the essential set. When reflected in the ambiguity averse presentation, the wealth effect amounts to that for each $p \in \Delta$, $G(t, p) - t$ is weakly increasing in t.

We emphasize that our third representation result is not a corollary of our first two main results. Instead of searching for two types of canonical representations, we impose two limit conditions on respective representations. First, we require that a weighted maxmin decision maker tend to put the whole weight on the worst case in the extremely bad situation in which the baseline utility of an act is sufficiently low and the scale of its ambiguous part is sufficiently large. Second, we require that a variant constraint decision maker tend to consider only the essential set in the extremely good situation in which the baseline utility of an act is sufficiently high and the scale of its ambiguous part is sufficiently small. It turns out that under respective conditions, the admissible set in the weighted maxmin representation is actually the smallest admissible set, and the essential set in the variant constraint representation is the largest essential set. Thus, the two types of representations that we derive are in fact canonical representations.

Analogous representations are obtained for the subclass of ambiguity averse preferences displaying increasing and constant absolute ambiguity aversion. As a corollary, we get two alternative representations for variational preferences studied by Maccheroni, Marinacci and Rustichini (2006), for variational preferences constitute the subclass of ambiguity averse preferences displaying constant absolute ambiguity aversion. Variational preferences also display increasing relative ambiguity aversion defined in a similar way as increasing absolute ambiguity aversion. This implies that for each act f, $\lambda(ku(f))$ and $\sigma(ku(f))$ increases in k on $(0, \infty)$. That is, as the scale of the ambiguity of an act increases, a weighted maxmin decision maker behaves as if he is more pessimistic and puts more weight on the worst case, and a variant constraint decision maker behaves as if he is more concerned about priors misspecification and considers a larger neighborhood of the essential set.

Lastly, we discuss the related literature. There are quite a few works studying different versions of the two generalized MEU decision rules in different settings, e.g., Ghirardato, Maccheroni and Marinacci (2004), Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci and Siniscalchi (2011), Olszewski (2007), Gajdosa, Hayashib, Tallona and Vergnaud (2008), Kopylov (2009), Chateauneuf and Faro (2009) and Hill (2013). Among them, the most related works are Ghirardato, Maccheroni and Marinacci (2004) and Hill (2013). Ghirardato, Maccheroni and Marinacci (2004) introduce a weighted maxmin decision rule and derive a representation with the certainty independence axiom of Gilboa and Schmeidler (1989) being imposed. The same type of representation is also obtained, after dropping certainty independence, by Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci and Siniscalchi (2011). The relation between their weighted maxmin representation with ours is elaborated in details in Section 3.

Hill (2013) also axiomatizes a class of preferences that display changing ambiguity aversion across acts. In his model, each act is evaluated by the worst expected utility over a set of priors that depends on the "stakes" involved in choosing the act. This class of preferences satisfies the axioms of our Theorem 2, so it constitutes a subclass of preferences that admits a canonical variant constraint representation. This subclass of preferences satisfies a stronger independence axiom, a stronger monotonicity axiom and a stronger ambiguity aversion axiom. These axioms together imply that as the stakes get bigger, the decision maker evaluates an act by its worst expected utility over a larger set of priors, as if he becomes less confident.

Regarding the wealth effect under ambiguity, Cherbonnier and Gollier (2015) propose a definition of decreasing aversion under ambiguity within the smooth ambiguity model and the α-MEU model. They consider the change in monetary wealth but their definition does not distinguish the the effect on risk aversion and ambiguity aversion. Cerreia-Vioglio, Maccheroni, and Marinacci (2017) propose a definition of decreasing absolute ambiguity aversion in a general setting. They also consider the change of monetary wealth, but their definition implies that a decision maker displays decreasing absolute ambiguity aversion must display constant absolute risk aversion. In contrast with these two works, our definition captures the effect of the change in baseline utility on ambiguity aversion and there is no restriction on a decision maker's risk attitude. Chambers, Grant, Polak and Quiggin (2014) provide a similar definition as ours but is stronger. Our axiom is closer in spirit to Klibanoff, Marinacci and Mukerji (2005)'s definition, although they only define constant absolute ambiguity aversion. We show that in their model, our axiom is equivalent to the decreasing concavity of a second-order utility index. This is consistent with the claim of Klibanoff, Marinacci and Mukerji (2005) that a second-order utility index summarizes one's ambiguity attitude in the same way as a von-Neumann-Morgenstern utility function summarizes one's risk attitude. Chateauneuf and Faro (2009) assume the existence of a worst outcome and propose the "worst independence" axiom. This axiom amounts to our constant relative ambiguity aversion axiom under the assumption of constant absolute ambiguity aversion, but in general it is weaker.

The rest of the paper is organized as follows. Section 2 introduces the model and axioms. Section 3 introduces two extensions of the MEU decision rule — a weighted maxmin rule and a variant constraint rule. Characterizations of classes of preferences that admit respective canonical representations are provided. As an application, we present the two canonical representations of multiplier preferences introduced by Hansen and Sargent (2001). Section 4 studies the wealth effect under ambiguity and proposes an axiom of decreasing absolute ambiguity aversion. Representations are provided for the subclass of ambiguity averse preferences displaying decreasing absolute ambiguity aversion. Analogous results are provided for preferences displaying increasing and constant absolute ambiguity aversion. Section 5 concludes. All the proofs are in the Appendix.

2 The model

Let S be a finite set of **states of the world** with $|S| \geq 2$, and Δ the probability space over S. Let X be a set of **outcomes**. Following Maccheroni, Marinacci and Rustichini (2006), we assume that X is a convex subset of some vector space. For example, X is an interval range of monetary payoffs, or the set of all lotteries over a set of prizes as in Anscombe and Aumann (1963). An **act** is a function $f : S \to X$ that yields in each state an outcome. Let $\mathcal{F}$ be the set of all acts. With a slight abuse of notation, for each $x \in X$, we denote by x the **constant act** in $\mathcal{F}$ that yields x in all states, and we identify X with the set of all constant acts. Given $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, let $\alpha f + (1 - \alpha)g$ be the mixed act that yields in each $s \in S$ the mixed outcome $\alpha f(s) + (1 - \alpha)g(s)$. Note that mixed acts are well-defined due to the convexity of X. A decision maker's **preference relation** is a binary relation $\succsim$ on $\mathcal{F}$, and let $\succ$ and $\sim$ denote respectively the asymmetric and symmetric parts of the preference relation $\succsim$. For each $f \in \mathcal{F}$, we denote by x_f a **certainty equivalent** of f which is a constant act such that $f \sim x_f$.

We impose the following basic properties on the preference relation $\succsim$.

A.1. Weak Order. The preference relation $\succsim$ is complete and transitive.

A.2. Risk Independence. For all $x, y, z \in X$ and all $\alpha \in (0, 1)$,

$$x \sim y \Rightarrow \alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z.$$

A.3. Continuity. For all $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] | \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] | h \succsim \alpha f + (1-\alpha)g\}$ are closed.

A.4. Monotonicity. For all $f, g \in \mathcal{F}$, if for all $s \in S$, $f(s) \succsim g(s)$, then $f \succsim g$.

Axiom A.1 says that the preference relation $\succsim$ should be rational. Axiom A.2 imposes Von-Neumann and Morgenstern's independence requirement on constant acts — acts that involve no state ambiguity. Axiom A.3 requires the preference relation $\succsim$ to be continuous with respect to mixture coefficients. Axiom A.4 says that an act should be preferred to another act if it yields a better outcome in each state, where the ranking of outcomes is assumed to be state independent and induced by the preference relation over constant acts.

Besides the basic requirements, we are interested in studying preferences that exhibit ambiguity aversion. There are two prominent definitions of ambiguity aversion in the literature. The first is due to Schmeidler (1989), which formulates ambiguity aversion as convexity. The interpretation is that for two acts that are indifferent, a mixture of them could be viewed as a hedge against ambiguity, so the mixed act should be preferred to either of the two acts for an ambiguity averse decision maker. We shall call this definition S-ambiguity aversion.

A.5.1. S-Ambiguity Aversion. For all $f, g \in \mathcal{F}$ and all $\alpha \in (0,1)$, if $f \sim g$, then $\alpha f + (1-\alpha)g \succsim f$.

An alternative definition of ambiguity aversion is provided by Ghirardato and Marinacci (2002). Their definition is based on a notion of comparative ambiguity aversion that they propose: A preference relation $\succsim_1$ is said to be **more ambiguity averse** than another preference relation $\succsim_2$ if for each $f \in \mathcal{F}$ and each $x \in X$, $x \succsim_2 f \implies x \succsim_1 f$.[3] Intuitively, the preference relation $\succsim_1$ is more ambiguity averse than the preference relation $\succsim_2$ if whenever the ambiguity is intolerable according to the preference relation $\succsim_2$, it is also intolerable according to the more ambiguity averse preference relation $\succsim_1$.

Based on the above notion, Ghirardato and Marinacci (2002) say that a preference relation $\succsim$ is ambiguity averse if it is more ambiguity averse than a SEU preference relation. We shall call this definition GM-ambiguity aversion. Formally, recall that a preference relation is a SEU preference relation if there is a subjective prior $p \in \Delta$ such that each act $f \in \mathcal{F}$

[3]For preferences satisfying Axioms A.1 - A.4, this definition is equivalent to the original definition in Ghirardato and Marinacci (2002).

and the reduced outcome $\sum_{s \in S} p_s f(s)$ with respect to p are indifferent. We denote the SEU preference relation associated with $p \in \Delta$ by $\geq_p$.

A.5.2. GM-Ambiguity Aversion. There is $p \in \Delta(S)$ such that the preference relation $\succsim$ is more ambiguity averse than $\geq_p$.

The two definitions of ambiguity aversion do not imply each other in general. But under the other axioms and a mild strengthening of Axiom A.2, S-ambiguity aversion implies GM-ambiguity aversion (see section 4).

Lastly, we impose an unboundedness requirement, which says that there exist arbitrarily good and arbitrarily bad outcomes.

A.6. Unboundedness. There are $x, y \in X$ such that for all $\alpha \in (0, 1)$, there are $z, z' \in X$ satisfying $\alpha z + (1 - \alpha) y \succsim x \succ y \succsim \alpha z' + (1 - \alpha) x$.

Axiom A.6 is stronger than the usual non-degeneracy axiom, requiring only the existence of two outcomes, one of which is strictly preferred to the other. It implies that the utility function representing the preference relation restricted to X is onto (see e.g. Kopylov (2001), Maccheroni, Marinacci and Rustichini (2006)). In the literature, Axiom A.6 is assumed sometimes to simplify the analysis, and sometimes to guarantee the uniqueness of a representation.[4] Throughout this paper, Axiom A.6 is imposed to simplify our presentation and it is also indispensable for some of our results.[5]

3 Two generalized MEU decision rules

3.1 Weighted maxmin rule

For each $u : X \to \mathbb{R}$ and each $f \in \mathcal{F}$, let $u(f) : S \to \mathbb{R}$ be the function given by composing u with f. For each $\varphi \in \mathbb{R}^S$ and each $p \in \Delta$, let $E_p \varphi$ denote the expected value of φ with respect to p. For each $\varphi \in \mathbb{R}^S$ and each nonempty closed convex set $D \subseteq \Delta$, let $l(\varphi; D) := \max_{p \in D} E_p \varphi - \min_{p \in D} E_p \varphi$.

[4] See e.g. Kopylov (2001), Maccheroni, Marinacci and Rustichini (2006), Strzalecki (2011b), Grant and Polak (2011), and Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011).

[5] Our Theorem 3 relies on how a preference relation ranks the "limiting" acts that yield arbitrarily good or bad outcomes in all the states.

Definition 1. *A preference relation* $\succsim$ *admits a* ***weighted maxmin representation*** *if there exist an affine onto function* $u : X \to \mathbb{R}$, *a non-empty closed convex set* $D \subseteq \Delta$, *and a function* $\lambda : \mathbb{R}^S \to [0,1]$ *continuous on* $\{\varphi \in \mathbb{R}^S | l(\varphi; D) \neq 0\}$ *such that for each pair* $f, g \in \mathcal{F}$,

$$\begin{aligned} f \succsim g \iff & \lambda(u(f)) \min_{p\in D} E_p u(f) + (1-\lambda(u(f))) \max_{p\in D} E_p u(f) \\ \geq & \lambda(u(g)) \min_{p\in D} E_p u(g) + (1-\lambda(u(g))) \max_{p\in D} E_p u(g). \end{aligned} \tag{2}$$

We denote the representation by $\langle u, D, \lambda \rangle$. *We call* D *an admissible set of priors, or simply an* ***admissible set****, and* λ *a* ***weight function****. We call the preference relation a* ***weighted maxmin preference relation****.*

For example, Hurwicz's α-pessimism decision rule (Hurwicz (1951)), also known as the α-MEU rule (e.g., Marinacci (2002)), admits a weighted maxmin representation with the weight function being constantly equal to α. When $\alpha = 1$, it reduces to Gilboa and Schmeidler (1989)'s MEU decision rule. MEU preferences exhibit S-ambiguity aversion (Axiom A.5.1). But in general, α-MEU preferences and thus weighted maxmin preferences do not S-ambiguity aversion.

One important issue is that the weighted maxmin representation of a preference relation is typically not unique, due to the nonuniqueness of admissible sets. In fact, given a weighted maxmin representation $\langle u, D, \lambda \rangle$ of a preference relation, each closed convex superset D' of D with $D' \subseteq \Delta$ is also an admissible set, that is, there is a weight function λ' such that $\langle u, D', \lambda' \rangle$ is also a weighted maxmin representation of the preference relation. This is because for each $f \in \mathcal{F}$, $[\min_{p\in D} E_p u(f), \max_{p\in D} E_p u(f)] \subseteq [\min_{p\in D'} E_p u(f), \max_{p\in D'} u(f)]$, and then there is $\lambda' : \mathbb{R}^S \to [0,1]$ such that for each $f \in \mathcal{F}$, $\lambda(u(f)) \min_{p\in D} E_p u(f) + (1-\lambda(u(f))) \max_{p\in D} E_p u(f) = \lambda'(u(f)) \min_{p\in D'} E_p u(f) + (1-\lambda'(u(f))) \max_{p\in D'} E_p u(f)$. Example 1 shows that a SEU preference relation admits for each $\epsilon \in [0,1]$, a weighted maxmin representation in which the admissible set is the set of ϵ-contaminated priors with respect to the subjective prior.

Example 1. [6] Suppose $X := \mathbb{R}$ for simplicity. Let $p^* \in \Delta$. Consider the SEU preference relation $\succsim$ over $\mathbb{R}^S$ with the subjective prior p^*: For each pair $f, g \in \mathbb{R}^S$, $f \succsim g \iff E_{p^*} f \geq E_{p^*} g$.

For each $\epsilon \in [0,1]$, let $D^\epsilon := \{(1-\epsilon)p^* + \epsilon p | p \in \Delta\}$ denote the set of ϵ-contaminated priors with respect to p^*. Note that $D^0 = \{p^*\}$, $D^1 = \Delta$, and for each pair $\epsilon, \epsilon' \in [0,1]$ with

[6] I thank the referee for suggesting this example.

$\epsilon \leq \epsilon'$, $D^\epsilon \subseteq D^{\epsilon'}$. Fix $\epsilon \in [0,1]$. Define $\lambda^\epsilon : \mathbb{R}^S \to \mathbb{R}$ by setting for each $f \in \mathbb{R}^S$,

$$\lambda^\epsilon(f) := \begin{cases} \dfrac{\max_{s\in S} f(s) - E_{p^*}f}{\max_{s\in S} f(s) - \min_{s\in S} f(s)} & \text{if } \min_{s\in S} f(s) \neq \max_{s\in S} f(s), \\ 1 & \text{if } \min_{s\in S} f(s) = \max_{s\in S} f(s). \end{cases}$$

Observe that for each $f \in \mathbb{R}^S$, λ^ϵ is continuous at f whenever $\min_{p\in D^\epsilon} E_p f \neq \max_{p\in D^\epsilon} E_p f$, and that

$$E_{p^*}f = \lambda^\epsilon(f) \min_{p\in D^\epsilon} E_p f + (1 - \lambda^\epsilon(f)) \max_{p\in D^\epsilon} E_p f,$$

since $\min_{p\in D^\epsilon} E_p f = (1-\epsilon)E_{p^*}f + \epsilon \min_{s\in S} f(s)$ and $\max_{p\in D^\epsilon} E_p f = (1-\epsilon)E_{p^*}f + \epsilon \max_{s\in S} f(s)$. Let u be the identity mapping on $\mathbb{R}$. Then, $\langle u, D^\epsilon, \lambda^\epsilon \rangle$ is a weighted maxmin representation of the preference relation $\succsim$.

In light of the nesting property of admissible sets, we are interested in finding the **smallest** one if it exists. This provides a natural way of normalizing weighted maxmin representations. Indeed, given a weighted maxmin representation $\langle u, D, \lambda \rangle$ of a preference relation, for each act, its maximum and minimum expected utilities over D are respectively an upper and lower bounds of its value, and the bounds will be tight if D is the smallest admissible set.

Definition 2. *A **canonical weighted maxmin representation** of a preference relation $\succsim$ is a weighted maxmin representation $\langle u, D, \lambda \rangle$ of the preference relation $\succsim$ such that D is the smallest admissible set, i.e., for each weighted maxmin representation $\langle u', D', \lambda' \rangle$ of the preference relation $\succsim$, $D \subseteq D'$.*

Ghirardato, Maccheroni and Marinacci (2004) introduce a well-known weighted maxmin representation different from a canonical weighted maxmin representation. Instead of seeking for the smallest admissible set, they require the admissible set in their representation to satisfy some property in terms of an "unambiguous" preference relation induced from the initial preference relation. More precisely, their admissible set is required to be one component of another representation à la Bewley (2002) of the unambiguous preference relation. We shall call their representation of the initial preference relation a **Bewley weighted maxmin representation**. To define it formally, we recall Ghirardato, Maccheroni and Marinacci (2004)'s notion of unambiguous preference relation (see also Nehring (2007)). Given a preference relation $\succsim$, for each pair $f, g \in \mathcal{F}$, f is said to be **unambiguously preferred to** g, denoted by $f \succsim^* g$, if for each $\alpha \in (0,1]$ and each $h \in \mathcal{F}$,

$$f \succsim^* g \iff \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Intuitively, if the consideration of either hedging against or speculating on ambiguity does not affect the ranking of acts f and g, then it is as if that the decision maker unambiguously prefers f to g (see Ghirardato, Maccheroni and Marinacci (2004) for more discussion).[7] Given a preference relation $\succsim$, a non-empty closed convex set $D \subseteq \Delta$ is called a Bewley set of priors, or simply a **Bewley set**, if there is an affine function $u : X \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$,

$$f \succsim^* g \iff \text{for each } p \in D,\ E_p u(f) \geq E_p u(g).$$

Definition 3. *A **Bewley weighted maxmin representation** of a preference relation $\succsim$ is a weighted maxmin representation $\langle u, D, \lambda \rangle$ of the preference relation $\succsim$ such that D is the Bewley set.*

It is known that a preference relation admits a Bewley weighted maxmin representation as long as it satisfies some basic properties.

Proposition 1. *A preference relation $\succsim$ satisfies Axioms A.1 - A.4 and A.6 if and only if it admits a Bewley weighted maxmin representation $\langle u, D, \lambda \rangle$. Moreover, u is unique up to a positive affine transformation, D is unique, and given u, λ is unique on $\{\varphi \in \mathbb{R}^S : l(\varphi; D) \neq 0\}$.*[8]

A version of the "only if" direction of Proposition 1 has been shown by Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci and Siniscalchi (2011). Ghirardato, Maccheroni and Marinacci (2004) also shows a version of the "only if" direction but with the certainty independence axiom of Gilboa and Schmeidler (1989) in addition, so that the weight function in their Bewley weighted maxmin represention has more structure.[9] Given the existing results, it is not hard to check the "if" direction and complete the characterization.

Our first main result says that if a preference relation satisfies S-ambiguity aversion in addition to Axioms A.1 - A.4 and A.6, then besides admitting a Bewley weighted maxmin representation, it also admits a canonical weighted maxmin representation. Moreover, the two types of weighted maxmin representations coincide.

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[7]Equivalently, the unambiguous preference relation $\succsim^*$ is the maximal restriction of the preference relation $\succsim$ that satisfies the independence axiom (Nehring (2007)).

[8]When $l(\varphi; D) = 0$, the choice of $\lambda(\varphi)$ does not matter.

[9]The weight function in their Bewley weighted maxmin representation is always constant linear, i.e., it is constant additive and positively homogeneous.

The converse holds if some additional conditions on the representation are imposed. For each nonempty closed convex set $D \subseteq \Delta$, let $\Lambda(D)$ denote the collection of functions $\lambda : \mathbb{R}^S \to [0,1]$ satisfying

(1) (monotonicity) for each pair $\varphi, \varphi' \in \mathbb{R}^S$ with $\varphi' \geq \varphi$,

$$\max_{p \in D} E_p \varphi' - \max_{p \in D} E_p \varphi \geq \lambda(\varphi') l(\varphi'; D) - \lambda(\varphi) l(\varphi; D); \tag{3}$$

(2) (quasi-concavity) for each pair $\varphi, \varphi' \in \mathbb{R}^S$ satisfying (3) and for $\varphi'' := \frac{\varphi+\varphi'}{2}$,

$$\max_{p \in D} E_p \varphi'' - \max_{p \in D} E_p \varphi \geq \lambda(\varphi'') l(\varphi''; D) - \lambda(\varphi) l(\varphi; D). \tag{4}$$

Theorem 1. *A preference relation $\succsim$ satisfies Axioms A.1 - A.4, A.5.1, and A.6 if and only if it admits a canonical weighted maxmin representation $\langle u, D, \lambda \rangle$ with $\lambda \in \Lambda(D)$. Moreover, u is unique up to a positive affine transformation, the smallest admissible set D is unique and coincides with the Bewley set, and given u, λ is unique on $\{\varphi \in \mathbb{R}^S : l(\varphi; D) \neq 0\}$.*

A well-known representation of the class of preferences satisfying Axioms A.1 - A.4, A.5.1, and A.6 in the literature, different from ours, is provided by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011). To define it formally, let $\mathcal{G}$ be the set of functions $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ such that (i) G is quasi-convex and lower semicontinuous on $\mathbb{R} \times \Delta$, (ii) for each $p \in \Delta$, $G(\cdot, p)$ is weakly increasing on $\mathbb{R}$, and (iii) for each $t \in \mathbb{R}$, $\min_{p \in \Delta} G(t, p) = t$. A function $G \in \mathcal{G}$ is said to be linearly continuous if the map $I : \mathbb{R}^S \to (-\infty, \infty]$, defined by $I(\varphi) = \inf_{p \in \Delta} G(E_p \varphi, p)$, is continuous.

Definition 4. *A preference relation $\succsim$ admits an* ***ambiguity averse representation*** *if there exist an affine onto function $u : X \to \mathbb{R}$ and a linearly continuous function $G \in \mathcal{G}$ such that for each pair $f, g \in \mathcal{F}$,*

$$f \succsim g \iff \min_{p \in \Delta} G(E_p u(f), p) \geq \min_{p \in \Delta} G(E_p u(g), p)$$

We denote the representation by $\langle u, G \rangle$.

It is shown by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) that a preference relation satisfies Axioms A.1 - A.4, A.5.1, and A.6 if and only if it admits an ambiguity averse representation $\langle u, G \rangle$, where u is unique up to a positive affine transformation, and given u, G is unique. Moreover, they show that given such a preference relation with an ambiguity averse representation $\langle u, G \rangle$, the set

$$D^* := cl(\{p \in \Delta | G(t, p) < \infty \text{ for some } t \in \mathbb{R}\}) \tag{5}$$

coincides with the Bewley set. Note that it can be readily shown that this set D^* is independent of the choice of ambiguity averse representations.[10]

Our result above, in view of theirs, not only provides an alternative representation of such a preference relation, but also shows that the set D^* derived from their representation is the smallest admissible set.

Corollary 1. *If a preference relation satisfies Axioms A.1 - A.4, A.5.1, and A.6, then it admits a canonical weighted maxmin representation* $\langle u, D^*, \lambda \rangle$.

3.1.1 Relation between a canonical weighted maxmin representation and a Bewley weighted maxmin representation

Restricted to preferences satisfying S-ambiguity aversion, we have shown that a canonical weighted maxmin representation of a preference relation is the same as a Bewley weighted maxmin representation. However, this is not the case for preferences that do not satisfy S-ambiguity aversion. We now provide two examples respectively showing that (1) a preference relation admitting a Bewley weighted maxmin representation may not admit a canonical weighted maxmin representation, and (2) even if a preference relation admits both types of representations, the Bewley set may not be the smallest admissible set. The proofs of our claims in the two examples are in Appendix.

Example 2 (The Bewley set exists, but the smallest admissible set does not)**.** Let $S := \{1, 2, 3\}$ and $X := \mathbb{R}$ for simplicity. Let $p' := (\frac{2}{3}, \frac{1}{12}, \frac{1}{4})$, $q' := (\frac{1}{12}, \frac{2}{3}, \frac{1}{4})$, and $p^* := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ be probabilities over S, where the i's coordinate denotes the probability of state i, $i = 1, 2, 3$. Let $D_1 := \{p', q'\}$ and $D_2 := \{p \in \Delta | d(p, p^*) \leq \frac{1}{\sqrt{6}}\}$. Define the function $V : \mathbb{R}^S \to \mathbb{R}$ by setting for each $f \in \mathbb{R}^S$,

$$V(f) := \begin{cases} \min\limits_{p \in D_1} E_p f & \text{if } \max\{f(1), f(2)\} < f(3), \\ \min\limits_{p \in D_2} E_p f & \text{if } \max\{f(1), f(2)\} \geq f(3). \end{cases}$$

Define the preference relation $\succsim$ over $\mathbb{R}^S$ by setting for each pair $f, g \in \mathbb{R}^S$, $f \succsim g \iff V(f) \geq V(g)$.

[10]It is shown by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) that if $\langle u', G' \rangle$ is another ambiguity averse representation of the preference, then there are $a > 0$ and $b \in \mathbb{R}$ such that $u' = au + b$, and for each $p \in \Delta$, $aG(t, p) + b = G'(at + b, p)$. Hence, for each $(t, p) \in \mathbb{R} \times \Delta$, $G(t, p) < \infty$ if and only if $G'(at + b, p) < \infty$.

The preference relation $\succsim$ satisfies Axioms A.1 - A.4 and A.6. By Proposition 1, it admits a Bewley weighted maxmin representation.[11] But it does not admit a canonical weighted maxmin representation.

Example 3 (The Bewley set is not the smallest admissible set)**.** Let S, X, p', q', p^*, and D_2 be defined as in Example 2. Let $p'' := (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ and $q'' := (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$ be probabilities over S. For each $f \in \mathbb{R}^S$ with $\max\{f(1), f(2)\} < f(3)$, let

$$\alpha(f) := \begin{cases} \text{med}\{0, \dfrac{1 - E_{p''} f}{1 - E_{p''} f + E_{p'} f}, 1\} & \text{if } f(1) \le f(2) < f(3), \\ \text{med}\{0, \dfrac{1 - E_{q''} f}{1 - E_{q''} f + E_{q'} f}, 1\} & \text{if } f(2) < f(1) < f(3), \end{cases}$$

and

$$D_1(f) := \{\alpha(f)p' + (1 - \alpha(f))p'', \alpha(f)q' + (1 - \alpha(f))q''\},$$

where med is the median operator.[12] Define the function $V : \mathbb{R}^S \to \mathbb{R}$ by setting for each $f \in \mathbb{R}^S$,

$$V(f) = \begin{cases} \min_{p \in D_1(f)} E_p f & \text{if } \max\{f(1), f(2)\} < f(3), \\ \min_{p \in D_2} E_p f & \text{if } \max\{f(1), f(2)\} \ge f(3). \end{cases}$$

Define the preference relation $\succsim$ over $\mathbb{R}^S$ by setting for each pair $f, g \in \mathbb{R}^S$, $f \succsim g \iff V(f) \ge V(g)$.

Like example 2, the preference relation $\succsim$ satisfies Axioms A.1 - A.4 and A.6, and thus admits a Bewley weighted maxmin representation.[13] Moreover, the preference relation $\succsim$ admits a canonical weighted maxmin representation in which the smallest admissible set is $\{p \in D_2 : p_3 \ge \frac{1}{6}\}$. However, this set is not the Bewley set.

3.2 Variant constraint rule

For each pair $p, q \in \Delta$, let $d(p, q)$ denote the Euclidean distance between p and q. For each $p \in D$ and each closed subset of A of Δ, define the distance, $d(p, A)$, between p and A to be $\min_{q \in A} d(p, q)$. Given a preference relation $\succsim$, for each $f \in \mathcal{F}$, let $x_{*f} \in X$ denote the worst outcome yielded by f, i.e., for some $s \in S$, $f(s) = x_{*f}$, and for each $s' \in S$, $f(s') \succsim x_{*f}$.

[11] It can be shown that the associated Bewley set is $cl(co(D_1 \cup \{p \in D_2 | p_3 \ge \frac{1}{6}\}))$.

[12] For each $t \in \mathbb{R}$, $\text{med}\{0, t, 1\}$ is the median of $0, t, 1$.

[13] It can be shown that the associated Bewley set is the same as in Example 2.

Definition 5. *A preference relation* $\succsim$ *admits a* ***variant constraint representation*** *if there exist an affine onto function* $u : X \to \mathbb{R}$*, a non-empty closed convex set* $K \subseteq \Delta$*, and a function* $\sigma : \mathbb{R}^S \to \mathbb{R}_+$ *continuous on* $\{u(f) \in \mathbb{R}^S | f \in \mathcal{F}, f \succ x_{*f}\}$ *and lower semicontinuous on* $\{u(f) \in \mathbb{R}^S | f \in \mathcal{F}, f \sim x_{*f}\}$ *such that for each pair* $f, g \in \mathcal{F}$*,*

$$f \succsim g \iff \min_{p \in \Delta : d(p,K) \leq \sigma(u(f))} E_p u(f) \geq \min_{p \in \Delta : d(p,K) \leq \sigma(u(f))} E_p u(g).$$

We denote the representation by $\langle u, K, \sigma \rangle$*. We call* K *an essential set of priors, or simply an* ***essential set****, and* σ *a* ***constraint function****. We call the preference relation a* ***variant constraint preference relation****.*

A MEU preference relation admits a variant constraint representation in which σ is constantly equal to 0. While MEU preferences exhibit S-ambiguity aversion, this is not true for variant constraint preferences in general. Instead, variant constraint preferences satisfy GM-ambiguity aversion, and later we shall characterize variant constraint preferences with GM-ambiguity aversion and other basic properties.

Like a weighted maxmin representation, a variant constraint representation of a preference relation is typically not unique, due to the nonuniqueness of essential sets. In fact, given a variant constraint representation $\langle u, K, \sigma \rangle$ of a preference relation, each non-empty convex closed subset K' of K is an essential set, since there is another constraint function σ' such that $\langle u, K', \sigma' \rangle$ is also a variant constraint representation of the preference relation. Example 4 shows the non-uniqueness of variant constraint representations of even a MEU preference relation.

Example 4. Let S, X, p^*, D_2 be defined as in Example 2. Consider the MEU preference relation $\succsim$ over $\mathbb{R}^S$ defined by, for each pair $f, g \in \mathbb{R}^S$, $f \succsim g \iff \min_{p \in D_2} E_p f \geq \min_{p \in D_2} E_p g$.

For each $\epsilon \in [0, \frac{1}{\sqrt{6}}]$, let $K^\epsilon := \{p \in \Delta | d(p, p^*) \leq \epsilon\}$. Note that $K^0 = \{p^*\}$, $K^{\frac{1}{\sqrt{6}}} = D_2$, and for each pair $\epsilon, \epsilon' \in [0, \frac{1}{\sqrt{6}}]$ with $\epsilon \leq \epsilon'$, $K^\epsilon \subseteq K^{\epsilon'}$. Fix $\epsilon \in [0, \frac{1}{\sqrt{6}}]$. Define $\sigma^\epsilon : \mathbb{R}^S \to \mathbb{R}_+$ by setting for each $f \in \mathbb{R}^S$, $\sigma^\epsilon(f) = \frac{1}{\sqrt{6}} - \epsilon$. Observe that σ^ϵ is continuous, and that for each $f \in \mathbb{R}^S$,

$$\min_{p \in D_2} E_p f = \min_{p \in \Delta : d(p, K^\epsilon) \leq \sigma^\epsilon(f)} E_p f,$$

since $D_2 = \{p \in \Delta | d(p, K^\epsilon) \leq \sigma^\epsilon(f)\}$. Let u be the identity mapping on $\mathbb{R}$. Then, $\langle u, K^\epsilon, \sigma^\epsilon \rangle$ is a variant constraint representation of the preference relation $\succsim$.

In light of the nesting property of essential sets, we can define a canonical variant constraint representation in a similar way of defining a canonical weighted maxmin representation.

Definition 6. *A **canonical variant constraint representation** of a preference relation is a variant constraint representation $\langle u, K, \sigma \rangle$ of the preference relation such that K is the largest essential set, i.e, for each variant constraint representation $\langle u', K', \sigma' \rangle$ of the preference relation, $K' \subseteq K$.*

Our second main result says that if a preference relation $\succsim$ satisfies GM-ambiguity aversion in addition to Axioms A.1 - A.4 and A.6, then it admits a canonical variant constraint representation. Moreover, the largest essential set is the set of all priors associated with which the SEU preferences are less ambiguity averse than the preference relation $\succsim$.

The converse holds with an additional condition on the variant constraint representation. For each nonempty closed convex set $K \subseteq \Delta$, let $\Sigma(K)$ denote the collection of functions $\sigma : \mathbb{R}^S \to \mathbb{R}_+$ such that $\forall \varphi, \varphi' \in \mathbb{R}^S$ with $\varphi' \geq \varphi$,

$$\min_{p \in \Delta : d(p,K) \leq \sigma(\varphi')} E_p \varphi' \geq \min_{p \in \Delta : d(p,K) \leq \sigma(\varphi)} E_p \varphi. \tag{6}$$

Theorem 2. *A preference relation satisfies Axioms A.1 - A.4, A.5.2, and A.6 if and only if it admits a canonical variant constraint representation $\langle u, K, \sigma \rangle$ with $\sigma \in \Sigma(K)$. Moreover, u is unique up to a positive affine transformation, K is unique, which is given by*

$$K = \{p \in \Delta | \textit{the preference relation } \succsim \textit{ is more ambiguity averse than the SEU preference relation } \geq_p\},$$

and given u, σ is unique on on $\{u(f) \in \mathbb{R}^S | f \in \mathcal{F}, f \not\sim x_{*f}\}$.[14]

While Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) provides the first characterization of preferences exhibiting S-ambiguity aversion, Theorem 2 provides the first characterization of preferences exhibiting GM-ambiguity aversion. Note that Axiom A.6 is indispensable for establishing the uniqueness of the representation of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011), whereas this is not the case for our representation. Axiom A.6 is imposed here for the ease of presentation.

[14]If $x_{*f} \sim f$, then $\min_{s \in S} u(f(s)) = u(x_f)$. Thus as long as $\sigma(u(f))$ is sufficiently large, $\min_{p \in D : d(p,K) \leq \sigma(u(f))} E_p u(f) = u(x_f)$.

The class of preferences admitting canonical variant constraint representations is different from the class admitting canonical weighted maxmin representations. In view of Theorems 1 and 2, there are preferences admitting canonical weighted maxmin representations but not canonical variant constraint representations, since S-ambiguity aversion does not imply GM-ambiguity aversion. On the other hand, there are preferences admitting canonical variant constraint representations but not canonical weighted maxmin representations, as shown by Example 5.

Example 5. Consider the preference relation defined in Example 2. As discussed in Example 2, it does not admit a canonical weighted maxmin representation. As seen in Example 2, the preference relation satisfies Axioms A.1 - A.4, and A.6. It also satisfies Axiom A.5.2, since for each $f \in \mathbb{R}^S$, $E_{p^*} f \geq V(f)$, so that it is more ambiguity averse than the SEU preference relation $\geq_{p^*}$. Then, by Theorem 2, it admits a canonical variant constraint representation.[15]

When a preference relation admits both a canonical weighted maxmin representation and a canonical variant constraint representation, we show that the largest essential set is always a subset of the smallest admissible set, and the two sets coincide if and only if the preference relation is a MEU preference relation.

Proposition 2. *If a preference relation admits both a canonical weighted maxmin represesentation $\langle u, D, \lambda \rangle$ and a canonical variant constraint representation $\langle u', K, \sigma \rangle$, then $K \subseteq D$.*

Proposition 3. *A preference relation is a MEU preference relation if and only if it admits both a canonical weighted maxmin represesentation $\langle u, D, \lambda \rangle$ and a canonical variant constraint representation $\langle u', K, \sigma \rangle$, and $K = D$. Moreover, λ is constantly* 1*, and σ is constantly* 0*.*

Recall that if a preference relation satisfies Axioms A.1 - A.4, A.5.1, and A. 6, then it admits an ambiguity averse representation $\langle u, G \rangle$ (Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011)). In the last section, we show that the preference relation admits a canonical weighted maxmin representation in which D^* defined in (5) is the smallest admissible set. A similar result can be obtained here. Formally, let

$$K^* := \{p \in \Delta | \text{ for each } t \in \mathbb{R}, G(t, p) = t\}. \tag{7}$$

[15] It can be shown that the largest essential set is $\{p \in D_2 | p_3 \geq \frac{1}{4}\}$.

Like D^*, the set K^* is independent of the choice of ambiguity averse representations of the preference relation. It can be readily shown that $p \in K^*$ if and only if the preference relation $\succsim$ is more ambiguity averse than the SEU preference relation $\geq_p$. Thus, the preference relation satisfies GM-ambiguity aversion if and only if K^* is non-empty, and in this case, by Theorem 2, the preference relation admits a canonical variant constraint representation in which K^* is the largest essential set.

Corollary 2. *If a preference relation satisfies Axioms A.1 - A.4, A.5.1, A.5.2, and A.6, then it admits a canonical variant constraint representation* $\langle u, K^*, \sigma \rangle$.

3.3 Different representations of the multiplier decision rule

There is a large class of preferences that satisfy both S-ambiguity aversion and GM-ambiguity aversion as well as other basic properties. Thus, besides admitting an ambiguity aversion representation of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011), this class of preferences also admits a canonical weighted maxmin representation and a canonical variant constraint representation. Although the three representations allow different interpretations, they are behaviorally equivalent.

Finding different representations of a decision rule facilitates our understanding of it. As an example, we consider two important robust decision rules introduced by Hansen and Sargent (2001): the constraint rule and the multiplier rule.[16] We shall apply our representation results to the multiplier rule and compare it with the constraint rule. The multiplier rule is defined in the form of an ambiguity averse representation. We shall argue that the other two representations provide a more straightforward comparison between the multiplier rule and the constraint rule.

For each pair $p, q \in \Delta$, we write $p \ll q$ if p is absolutely continuous with respect to q, and denote by $R(p\|q)$ the relative entropy of p with respect to q, that is,

$$R(p\|q) = \begin{cases} \sum_S p_i \log \dfrac{p_i}{q_i} & \text{if } p \ll q, \\ \infty & \text{otherwise.} \end{cases}$$

The relative entropy, known as Kullback-Leibler divergence, is a measure of "distance" between two probabilities.

[16]Strzalecki (2011a) axiomatizes the multiplier rule.

Definition 7. *A preference relation* $\succsim$ *admits a* ***constraint representation*** *if there exist an affine onto function* $u : X \to \mathbb{R}$*, a prior* $q \in \Delta$*, and a constant* $\eta \in \mathbb{R}_+$ *such that for each pair* $f, g \in \mathcal{F}$*,*

$$f \succsim g \iff \min_{p\in\Delta:R(p\|q)\leq\eta} E_p u(f) \geq \min_{p\in\Delta:R(p\|q)\leq\eta} E_p u(g). \tag{8}$$

We denote the representation by $\langle u, q, \eta\rangle$*. We call the preference relation a* ***constraint preference relation****.*

A constraint representation $\langle u, q, \eta\rangle$ is in fact a MEU representation $\langle u, D\rangle$ in which $D = \{p \in D | R(p\|q) \leq \eta\}$. It can also be viewed in spirit as a special variant constraint representation, with a singleton essential set, a different measure of "distance" between two probabilities, and a constant constraint function.

Definition 8. *A preference relation* $\succsim$ *admits a* ***multiplier representation*** *if there exist an affine onto function* $u : X \to \mathbb{R}$*, a prior* $q \in \Delta$*, and a constant* $\theta \in (0, \infty]$ *such that for each pair* $f, g \in \mathcal{F}$*,*

$$f \succsim g \iff \min_{p\in\Delta}[E_p u(f) + \theta R(p\|q)] \geq \min_{p\in\Delta}[E_p u(g) + \theta R(p\|q)]. \tag{9}$$

We denote the representation by $\langle u, q, \theta\rangle$*. We call the preference relation a* ***multiplier preference relation****.*[17]

A multiplier representation $\langle u, q, \theta\rangle$ is in fact an ambiguity averse representation $\langle u, G\rangle$ in which $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is given by, for each $(t, p) \in \mathbb{R} \times \Delta$, $G(t, p) = t + \theta R(p\|q)$. A multiplier preference relation satisfies both S-ambiguity aversion and GM-ambiguity aversion, as well as the other basic properties. By Theorems 1 and 2, it admits both a canonical weighted maxmin representation and a canonical variant constraint representation. Moreover, we can explicitly write down the components in both representations.

Proposition 4. *Suppose that a preference relation admits a multiplier representation* $\langle u, q, \theta\rangle$*. Then, it admits a canonical weighted maxmin representation* $\langle u, D, \lambda\rangle$*. When* $\theta < \infty$*,* $D = \{p \in \Delta : p \ll q\}$ *and* $\lambda : \mathbb{R}^S \to [0, 1]$ *is given by, for each* $\varphi \in \mathbb{R}^S$*,*

$$\lambda(\varphi) = \begin{cases} \dfrac{1}{\overline{\varphi} - \underline{\varphi}}(\overline{\varphi} + \theta \log E_q e^{-\frac{\varphi}{\theta}}) & \text{if } \underline{\varphi} < \overline{\varphi}, \\ 1 & \text{if } \underline{\varphi} = \overline{\varphi}, \end{cases} \tag{10}$$

[17] The preference relation in Definition 8 is called a multiplier preference relation since the parameter θ in the unconstrained minimization problem in (9) can be viewed as a Lagrange multiplier in the Lagrangian of the constrained minimization problem in (8).

where $\overline{\varphi} = \max_{s \in S: q_s > 0} \varphi(s)$ *and* $\underline{\varphi} = \min_{s \in S: q_s > 0} \varphi(s)$. *When* $\theta = \infty$, $D = \{q\}$ *and* $\lambda \equiv 1$.

The preference relation also admits a canonical variant constraint representation $\langle u, K, \sigma \rangle$. *When* $\theta < \infty$, $K = \{q\}$ *and* $\sigma : \mathbb{R}^S \to \mathbb{R}_+$ *is given by, for each* $\varphi \in \mathbb{R}^S$,

$$\sigma(\varphi) = \min_{p \in \Delta: E_p \varphi = -\theta \log E_q e^{-\frac{\varphi}{\theta}}} d(p, q). \tag{11}$$

When $\theta = \infty$, $K = \{q\}$ *and* $\sigma \equiv 0$.

The connection between the multiplier rule and the constraint rule is established in a dynamic resource allocation problem by Hansen and Sargent (2001, 2008). They show under some conditions that fixing $u : X \to \mathbb{R}$ and $q \in \Delta$, for each η, there is θ such that the constraint rule $\langle u, q, \eta \rangle$ and the multiplier rule $\langle u, q, \theta \rangle$ generate the same optimal allocation, and vise versa. But in general, as they point out, the two decision rules induce totally different preference rankings.

In view of our alternative representation results, the relationship between the two decision rules becomes more transparent. First, constraint and multiplier preferences are both special cases of weighted maxmin preferences with different types of weight functions in general. As mentioned before, a constraint preference relation that admits a representation $\langle u, q, \eta \rangle$ evaluates each act by its worst expected utility over a set of priors determined by q and η. In comparison, by Proposition 4, a multiplier preference relation that admits a representation $\langle u, q, \theta \rangle$ evaluates each act by a weighted average of its best and worst expected utilities over a set of priors determined by q and θ. While the weight put on the worst expected utilities of acts is constantly 1 for a constraint preference relation, it varies across acts for a multiplier preference relation.

Second, constraint and multiplier preferences are both special cases of variant constraint preferences with different constraint functions in general. In fact, for a more transparent comparison, we can replace the Euclidean distance measure with relative entropy in the variant constraint representation for the multiplier rule. That is, under the same assumption of Proposition 4, when $\theta < \infty$, for each pair $f, g \in \mathcal{F}$,

$$f \succsim g \iff \min_{p \in \Delta: R(p \| q) \leq \sigma'(u(f))} E_p u(f) \geq \min_{p \in \Delta: R(p \| q) \leq \sigma'(u(g))} E_p u(g) \tag{12}$$

where $\sigma' : \mathbb{R}^S \to \mathbb{R}$ is given by, for each $\varphi \in \mathbb{R}^S$,

$$\sigma'(\varphi) = \min_{p \in \Delta: E_p \varphi = -\theta \log E_q e^{-\frac{\varphi}{\theta}}} R(p \| q) \tag{13}$$

From comparing (8) and (12), we can see that both a constraint preference relation and a multiplier preference relation evaluate an act by its worst expected utility over a "neighborhood" of priors around q, but the size of the neighborhood is fixed for a constraint preference relation while it depends on the act for a multiplier preference relation.

Note that the key implication of using relative entropy instead of the Euclidean distance in a variant constraint representation is that it excludes the priors which are not absolutely continuous with respect to a central prior. Indeed, only when $p \ll q$, $R(p\|q) < \infty$. But this is not a problem for a multiplier preference relation, since it disregards all the priors that are not absolutely continuous with respect to its central prior q. Thus, relative entropy can be used instead of the Euclidean distance in its variant representation as long as the constraint function is adjusted correspondingly.

4 Changing ambiguity aversion

Within the class of preferences satisfying S-ambiguity aversion, GM-ambiguity aversion, and the other basic properties, we are interested, in this section, to study those displaying particular patterns of changing ambiguity aversion with respect to the change of wealth. We shall provide a definition of decreasing/increasing absolute ambiguity aversion and investigate its implication on the representations that we study in the previous section.

4.1 Wealth effect and decreasing absolute ambiguity aversion

To motivate the study of the wealth effect, consider the following variation of Ellsberg (1961)'s thought experiment.

Example 6. An urn contains 100 balls, of which 33 are red, and 67 are either black or white. A ball is drawn from the urn. For each $t \in \mathbb{R}_+$, r_t denotes the act "betting on red". It pays $100 + t$ dollars if the ball is red and t dollars otherwise. Let b_t denote the act "betting on black", and its payoff is similarly given. See the payoff table below.

Suppose that a decision maker's preference relation $\succsim$ satisfies Axioms A.1 - A.4, A.5.1, and A.6, and assume for simplicity that he is risk neutral. For each $t \in \mathbb{R}_+$, r_t is an unambiguous act which yields $100 + t$ with probability 0.33, while b_t is an ambiguous act which gives either $100 + t$ or t. The decision maker may prefer r_0 to b_0 if he is averse to ambiguity, but the degree of his ambiguity aversion may decrease with the increase of the baseline prize t.

Table 1: Payoffs of r_t and b_t

$t \in \mathbb{R}_+$	Red	Black	White
r_t	100+t	t	t
b_t	t	100+t	t

Table 2: Payoffs of r_0 and b_0

$t = 0$	Red	Black	White
r_0	100	0	0
b_0	0	100	0

Table 3: Payoffs of r_{10^4} and b_{10^4}

$t = 10^4$	Red	Black	White
r_{10^4}	10,100	10,000	10,000
b_{10^4}	10,000	10,100	10,000

It can be expected that when t is sufficiently large, he is willing to take the ambiguity bearing behavior, and prefers say b_{10^4} to r_{10^4}.

Such a behavioral pattern is evidenced by the laboratory experiments of Baillon and Placido (2015) with subjects that may or may not be risk neutral. In the following, we shall propose behavioral axioms to capture this and other analogous phenomena. The tests designed by Baillon and Placido (2015) are consistent with the implications of our axioms.

A.2.1. Decreasing Absolute Ambiguity Aversion.[18] For all $f \in \mathcal{F}$, $x, y, z \in X$ and $\alpha \in$

[18] After completing this paper, I learned that Ghirardato and Siniscalchi independently propose a very similar axiom of decreasing absolute ambiguity aversion in their work "Symmetric preferences", which is presented in RUD 2015 and D-TEA 2015.

(0, 1), if either f is a constant act or $y \succsim x$, then

$$\alpha f + (1-\alpha)x \succsim \alpha z + (1-\alpha)x$$
$$\Rightarrow \alpha f + (1-\alpha)y \succsim \alpha z + (1-\alpha)y. \tag{14}$$

When f is constant, (14) essentially imposes von Neumann-Morgenstern's independence requirement on constant acts. When f is not constant, then (14) says that if an (ambiguous) act $\alpha f + (1-\alpha)x$ is preferred to a constant act $\alpha z + (1-\alpha)x$, then it is still the case after improving the certainty part from x to y for both acts. In other words, if the ambiguity is tolerable before, it is even more tolerable after a common improvement in the certainty part. Axiom A.2.1 implies that if an act induces a larger utility than another act by an ensured amount t in each state, then the value of the former act is larger than the later by at least t.

Similarly, if we replace $y \succsim x$ in Axiom A.2.1 by $x \succsim y$, then the preference relation $\succsim$ displays increasing absolute ambiguity aversion.

A.2.2. Increasing Absolute Ambiguity Aversion. For all $f \in \mathcal{F}$, $x, y, z \in X$ and $\alpha \in (0, 1)$, if either f is a constant act or $x \succsim y$, then

$$\alpha f + (1-\alpha)x \succsim \alpha z + (1-\alpha)x$$
$$\Rightarrow \alpha f + (1-\alpha)y \succsim \alpha z + (1-\alpha)y.$$

If we require both Axiom A.2.1 and Axiom A.2.2 to hold, then the preference relation $\succsim$ satisfies constant absolute ambiguity aversion proposed by Grant and Polak (2013).

A.2.3. Constant Absolute Ambiguity Aversion. (Grant and Polak (2013)) For all $f \in \mathcal{F}$, $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)x \succsim \alpha z + (1-\alpha)x$$
$$\Rightarrow \alpha f + (1-\alpha)y \succsim \alpha z + (1-\alpha)y. \tag{15}$$

While the above axioms are about the effect of an absolute change in the certainty part of an act, one can imagine a similar effect of a relative change in the proportion of the certainty part of an act. We refer the readers to Maccheroni, Marinacci and Rustichini (2006) for such a thought experiment. As analogous to Axiom A.2.2, we propose the following axiom of increasing relative ambiguity aversion.

A.2.4. Increasing Relative Ambiguity Aversion. For all $f \in \mathcal{F}$, $x, z \in X$ and $\alpha, \beta \in (0, 1)$, if $\alpha \geq \beta$, then

$$\alpha f + (1-\alpha)x \succsim \alpha z + (1-\alpha)x \\ \Rightarrow \beta f + (1-\beta)x \succsim \beta z + (1-\beta)x. \tag{16}$$

Axiom A.2.4 says that if $\alpha f + (1-\alpha)x$ is preferred to a constant act $\alpha z + (1-\alpha)x$, then this is still the case after the proportion of the certainty part increases in both acts. That is, the degree of ambiguity aversion increases as the relative proportion of the uncertain part of an act increases. Similarly, a preference relation exhibits **decreasing relative ambiguity aversion** if (16) holds for each $\alpha, \beta \in (0, 1)$ with $\alpha \leq \beta$, and **constant relative ambiguity aversion** if (16) holds for each $\alpha, \beta \in (0, 1)$. Chateauneuf and Faro (2009) propose a so-called worst independence axiom under the assumption of the existence of a worst outcome in X. Their axiom amounts to our constant relative ambiguity aversion for the class of preferences satisfying Axioms A.1 - A.4, A.5.1, and constant absolute ambiguity aversion. In general, for preferences satisfying Axioms A.1 - A.4, their axiom is implied by ours.

Under Axioms A.1, A.3 - A.6, preferences satisfying constant absolute ambiguity aversion display increasing relative ambiguity aversion. We shall show later that decreasing absolute ambiguity aversion also implies increasing relative ambiguity aversion in some limit forms.

4.2 Characterizations

For each $t \in \mathbb{R}$, we denote by $t\mathbf{1}$ an element in $\mathbb{R}^S$ such that for each $s \in S$, $t\mathbf{1}(s) = t$, and when $t = 1$, we simply write $\mathbf{1}$. For each non-empty convex closed $K \subseteq D$, let $\bar{\Sigma}(K)$ be the collection of functions $\sigma \in \Sigma(K)$ such that for each pair $\varphi, \varphi' \in \mathbb{R}^S$ satisfying (6) and for $\varphi'' := \frac{\varphi+\varphi'}{2}$,

$$\min_{p \in \Delta : d(p,K) \leq \sigma(\varphi'')} E_p \varphi'' \geq \min_{p \in \Delta : d(p,K) \leq \sigma(\varphi)} E_p \varphi.$$

That is, the constraint functions in $\bar{\Sigma}(K)$ satisfy not only the monotonicity property required for the functions in $\Sigma(K)$, but also a quasi-concavity property.

Theorem 3. *Let a preference relation $\succsim$ be given. The following statements are equivalent.*

1. *The preference relation $\succsim$ satisfies Axioms A.1, A.2.1, A.3, A.4, A.5.1, and A.6.*

2. *The preference relation* $\succsim$ *admits a weighted maxmin representation* $\langle u, D, \lambda\rangle$ *such that* $\lambda \in \Lambda(D)$*, and for each* $\varphi \in \mathbb{R}^S$*,* $\lambda(\varphi + t\mathbf{1})$ *weakly decreases in t, and* $\lim_{k\to\infty}\lim_{t\to\infty}\lambda(k\varphi - t\mathbf{1}) = 1$.

3. *The preference relation* $\succsim$ *admits a variant constraint representation* $\langle u, K, \sigma\rangle$ *such that* $\sigma \in \bar{\Sigma}(K)$*, and for each* $\varphi \in \mathbb{R}^S$*,* $\sigma(\varphi + t\mathbf{1})$ *weakly decreases in t, and* $\lim_{k\searrow 0}\lim_{t\to\infty}\sigma(k\varphi + t\mathbf{1}) = 0$.

4. *The preference relation* $\succsim$ *admits an ambiguity averse representation* $\langle u, G\rangle$ *such that for each* $p \in \Delta$*,* $G(t, p) - t$ *weakly increases in t.*

Moreover, $\langle u, D, \lambda\rangle$ *is a canonical weighted maxmin representation,* $\langle u, K, \sigma\rangle$ *is a canonical variant constraint representation, and for each* $\varphi \in \mathbb{R}^S$*,* $\lim_{t\to\infty}\lambda(k\varphi - t\mathbf{1})$ *and* $\lim_{t\to\infty}\sigma(k\varphi + t\mathbf{1})$ *weakly increase in k on* $(0, \infty)$.

Decreasing absolute ambiguity aversion has straightforward behavioral implications on the weighted maxmin representation and the variant constraint representation of a preference relation. As the baseline utility of an act increases, it is as if that a weighted maxmin decision maker puts less weight on the worst expected utility, and a variant constraint decision maker considers a smaller neighborhood of priors around the essential set.

Note that under Axioms A.1, A.2.1, A.3, and A.4, S-ambiguity aversion (A.5.1) implies GM-ambiguity aversion (A.5.2). Thus, by Theorems 1 and 2, the preference relation admits a canonical weighted maxmin representation and a canonical variant constraint representation. However, the equivalence of statement 1, 2 and 3 is **not** a corollary of Theorems 1 and 2. In Theorem 3, the weighted maxmin representation and the variant constraint representation are uniquely determined not by the smallest admissible set and the largest essential set respectively, but by two limit conditions. It turns out that the two limit conditions are characterizing conditions of the smallest admissible set and the largest essential set respectively. Hence, both representations are in fact canonical representations.

The two limit conditions have natural interpretations. The condition $\lim_{k\to\infty}\lim_{t\to\infty}\lambda(k\varphi - t\mathbf{1}) = 1$ says that a weighted maxmin decision maker shall tend to consider only the worst expected utility in the extremely bad situation where the baseline utility of an act is sufficiently low and the scale of its uncertain part is sufficiently large. The condition of $\lim_{k\searrow 0}\lim_{t\to\infty}\sigma(k\varphi + t\mathbf{1}) = 0$ says that a variant constraint decision maker shall tend to consider only the priors in the essential set K in the extremely good situation where the baseline utility of an act is sufficiently high and the scale of its uncertain part is sufficiently small.

Theorem 3 also shows that decreasing absolute ambiguity aversion implies increasing relative ambiguity aversion in some limit form under the other axioms. The fact that $\lim_{t\to\infty} \lambda(k\varphi - t\mathbf{1})$ weakly increases in k on $(0, \infty)$ suggests that when the baseline utility of an act is sufficiently low, if the scale of its uncertain part increases, a weighted maxmin decision maker behaves as if he is more pessimistic and puts a larger weight on the worst expected utility. Similarly, $\lim_{t\to\infty} \sigma(k\varphi + t\mathbf{1})$ being weakly increasing in k on $(0, \infty)$ suggests that when the baseline utility of an act is sufficiently high, if the scale of its uncertain part increases, a variant constraint decision maker behaves as if he is more cautious and considers a larger neighborhood of priors around the essential set.

Analogous characterizations can be obtained for preferences displaying increasing absolute ambiguity aversion: Theorem 3 holds if Axiom A.2.1 is replaced by A.2.2 and t by $-t$. Since Axiom A.2.3 is equivalent to the combination of Axioms A.2.1 and A.2.2, then we further obtain the characterizations for preferences displaying constant absolute ambiguity aversion.

Corollary 3. *Let a preference relation $\succsim$ be given. The following statements are equivalent.*

1. *The preference relation $\succsim$ satisfies Axioms A.1, A.2.3, A.3, A.4, A.5.1, and A.6.*
2. *The preference relation $\succsim$ admits a weighted maxmin representation $\langle u, D, \lambda \rangle$ such that $\lambda \in \Lambda(D)$, and for each $\varphi \in \mathbb{R}^S$, $\lambda(\varphi + t\mathbf{1})$ is constant in t, and $\lim_{k\to\infty} \lambda(k\varphi) = 1$.*
3. *The preference relation $\succsim$ admits a variant constraint representation $\langle u, K, \sigma \rangle$ such that $\sigma \in \bar{\Sigma}(K)$, and for each $\varphi \in \mathbb{R}^S$, $\sigma(\varphi + t\mathbf{1})$ is constant in t, and $\lim_{k\searrow 0} \sigma(k\varphi) = 0$.*
4. *The preference relation $\succsim$ admits an ambiguity averse representation $\langle u, G \rangle$ such that for each $p \in \Delta$, $G(t, p) - t$ is constant in t.*

Moreover, $\langle u, D, \lambda \rangle$ is a canonical weighted maxmin representation, $\langle u, K, \sigma \rangle$ is a canonical variant constraint representation, and for each $\varphi \in \mathbb{R}^S$, $\lambda(k\varphi)$ and $\sigma(k\varphi)$ weakly increase in k on $(0, \infty)$.

As shown by Corollary 3, constant absolute ambiguity aversion implies that the weight put on the worst expected utility by a weighted maxmin decision maker is independent of the baseline utility of an act, and so is the size of the neighborhood of the essential set considered by a variant constraint decision maker.

It is known that constant absolute ambiguity aversion implies increasing relative ambiguity aversion under the other axioms. Corollary 3 shows that as the scale of an (uncertain) act increases, it is as if that a weighted maxmin decision maker puts a larger weighted on the worst expected utility and a variant constraint decision makers considers a larger neighborhood of the essential set.

The equivalence of statements 1 and 4 in Corollary 3 partially reproduces some existing results in the literature. Maccheroni, Marinacci and Rustichini (2006) propose a weak certainty independence axiom and call a preference relation a **variational preference relation** if it satisfies Axioms A.1, A.3 - A.6, and weak certainty independence. They show that a preference relation is a variational preference relation if and only if there exist an affine onto function $u : X \to \mathbb{R}$ and a lower semicontinuous convex function $c : \Delta \to [0, \infty]$ with $\min\limits_{p \in \Delta} c(p) = 0$ such that

$$f \succsim g \iff \min_{p \in \Delta}[E_p u(f) + c(p)] \geq \min_{p \in \Delta}[E_p u(g) + c(p)]. \tag{17}$$

Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) show that this represenation amounts to an ambiguity averse representation $\langle u, G \rangle$ in which G is additively separable.[19] Since weak certainty independence is known to be equivalent to constant absolute ambiguity aversion under the other axioms,[20] our result can be expected for G is additively separable if for each $p \in \Delta$, $G(t, p) - t$ is constant in t.

We close this subsection by providing a differential characterization of the smallest admissible set for preferences displaying decreasing absolute ambiguity aversion. Let a preference relation $\succsim$ be given. Following Rigotti, Shannon and Strazalecki (2008), define a correspondence $\pi : \mathcal{F} \rightrightarrows \Delta$ by setting for each $f \in \mathcal{F}$,

$$\pi(f) = \{p \in \Delta | \sum_{s \in S} p_s f(s) \succsim \sum_{s \in S} p_s g(s) \implies f \succsim g\}. \tag{18}$$

It is interpreted that the set $\pi(f)$ consists of all the prior beliefs that rationalize the choice of f over other acts (see Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011)). These beliefs are used in the study of ambiguity averse preferences and their applications

[19] See their Proposition 12.

[20] Grant and Polak (2013) show the equivalence under Axioms A.1 and A.3, and a weaker version of Axioms A.4 and A.6.

(see e.g., Rigotti, Shannon and Strazalecki (2008) and Lang (2016)). Mathematically, they correspond to the supporting hyperplanes of the upper contour set of f.

For a general preference relation satisfying Axioms A.1 - A.4, A.5.1, and A.6, Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) show that $D^* = cl(co(\cup_{f\in\mathcal{F}}\pi(f)))$. Thus, by our Corollary 1, $cl(co(\cup_{f\in\mathcal{F}}\pi(f)))$ is the smallest admissible set. Proposition 5 strengthens this result for preferences displaying decreasing absolute ambiguity aversion.

Proposition 5. *Suppose that a preference relation $\succsim$ satisfies Axioms A.1, A.2.1, A.3, A.4, A.5.1, and A.6. Let $\langle u, D, \lambda\rangle$ a canonical weighted maxmin representation of the preference relation $\succsim$. Then, for each $x \in X$, $D = cl(co(\cup_{f\sim x}\pi(f)))$.*

Thus, to identify the smallest admissible set, it suffices to find the collection of beliefs that rationalize the decision maker's choices of the acts that lie on the same indifference curve.

For each $x \in X$, the set $\pi(x)$ is interpreted as the set of beliefs that rationalize the choice of the constant act x over ambiguous acts. GM-ambiguity aversion is equivalent to the non-emptyness of $\cap_{x\in X}\pi(x)$ (see e.g., Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011)). By Theorem 2, it can be readily seen that if a preference relation satisfies Axioms A.1 - A.4, A.5.2, and A.6, then the largest essential set is $\cap_{x\in X}\pi(x)$.

Assuming that X is a set of simple lotteries over $\mathbb{R}$, Lang (2016) proposes definitions, for an ambiguity averse preference, to distinguish between first-order and second-order ambiguity aversion at a wealth level $w \in \mathbb{R}$. Lang (2016) characterizes second-order ambiguity aversion at w by $\pi(w)$ being a singleton. In view of Lang (2016)'s result, for an ambiguity averse preference that exhibits second-order ambiguity aversion at each $w \in \mathbb{R}$, it admits a canonical variant constraint representation only if there is $p \in \Delta$ such that for each $w \in \mathbb{R}$, $\pi(w) = \{p\}$. In this case, the largest essential set is exactly the singleton $\{p\}$.

4.3 Comparison with other definitions

Based on a notion of comparative "dispersion", Chambers, Grant, Polak and Quiggin (2014) also propose the definitions of decreasing, increasing and constant absolute ambiguity aversion. According to them, an act f is considered *at least as dispersed as* an act g, denoted by $f \trianglerighteq g$, if there exist $x \in X$ and $\lambda \in [0, 1]$ such that $g = \lambda f + (1 - \lambda)x$. They say that a preference relation $\succsim$ displays decreasing (increasing) absolute ambiguity aversion if for each pair $f, g \in \mathcal{F}$ with $f \trianglerighteq g$, each pair $x, y \in X$ with $y \succsim x$ ($x \succsim y$), and each $\alpha \in (0, 1)$,

$$\begin{aligned} &\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \\ &\qquad \Rightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y. \end{aligned} \tag{19}$$

They say that a preference relation satisfies constant absolute ambiguity aversion if it satisfies both their decreasing and increasing absolute ambiguity aversion.

Since for each $f \in \mathcal{F}$ and each $z \in X$, $f \trianglerighteq z$, then their decreasing absolute ambiguity aversion is stronger than ours. If we apply their definition, we could get similar result as Theorem 3 with modified monotonicity conditions of λ, σ and G.[21] This is also the case for increasing absolute ambiguity aversion. As for constant absolute ambiguity aversion, it turns out that their seemingly stronger definition is in fact equivalent to Axiom A.2.3.

Proposition 6. *Suppose that a preference relation $\succsim$ satisfies A.1 - A.4 and A.6. Then for each pair $f, g \in \mathcal{F}$ with $f \trianglerighteq g$, each pair $x, y \in X$, and each $\alpha \in (0, 1)$, (19) holds if and only if the preference relation satisfies Axiom A.2.3.*

The validity of our axioms can be seen from their implications in the smooth ambiguity model of Klibanoff, Marinacci and Mukerji (2005). A preference relation $\succsim$ admits a *smooth ambiguity representation* is if there exist an affine onto function $u : X \to \mathbb{R}$, a strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$, a countably additive Borel probability measure μ over Δ such that for each pair $f, g \in \mathcal{F}$,

$$f \succsim g \Leftrightarrow \int_{p \in \Delta} \phi(E_p u(f)) d\mu(p) \geq \int_{p \in \Delta} \phi(E_p u(g)) d\mu(p). \tag{20}$$

We denote such a representation by $\langle u, \phi, \mu \rangle$.

Klibanoff, Marinacci and Mukerji (2005) show that in this model "attitudes towards pure risk are characterized by the shape of u, as usual, while attitudes towards ambiguity are characterized by the shape of ϕ," and "one advantage of this model is that the well-developed machinery for dealing with risk attitudes can be applied as well to ambiguity attitudes."

[21] If a preference relation $\succsim$ satisfies Axioms A.1 - A.4, and A.6, then there exist an affine and onto function $u : X \to \mathbb{R}$, and an increasing and continuous functional $I : \mathbb{R}^S \to \mathbb{R}$ such that $f \succsim g \iff I(u(f)) \geq I(u(g))$. Their decreasing absolute ambiguity aversion amounts to that for each pair $\varphi, \varphi' \in \mathbb{R}^S$, if $\varphi' = \lambda\varphi + (1-\lambda)t'$ for some $\lambda \in [0, 1]$ and some $t' \in \mathbb{R}$, and if $I(\varphi) = I(\varphi')$, then for each $t > 0$, $I(\varphi + t\mathbf{1}) \geq I(\varphi' + t\mathbf{1})$, which means for example in the weighted maxmin representation that $\lambda(\varphi + t\mathbf{1}) \leq \lambda(\varphi' + t\mathbf{1})$.

We focus on the case where ϕ is concave so that $\langle u, \phi, \mu \rangle$ represents a preference relation satisfying S-ambiguity aversion. In this case, our definitions of decreasing and increasing absolute ambiguity aversion correspond exactly to the usual monotonicity properties of the Arrow-Pratt coefficient of absolute risk aversion of ϕ (Arrow (1963), Pratt (1964)).

Proposition 7. *Fix an affine onto function $u : \mathbb{R} \to \mathbb{R}$ and a strictly increasing concave and twice differentiable function $\phi : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{P}$ be the collection of preferences represented by $\langle u, \phi, \mu \rangle$ for some countably additive Borel probability measure μ on Δ. Then, for each preference relation in $\mathcal{P}$, Axiom A.2.1 (A.2.2) is satisfied if and only if ϕ displays decreasing (increasing) absolute risk aversion, i.e., $-\frac{\phi''}{\phi'}$ is weakly decreasing (increasing).*

Since variational preferences are those displaying constant absolute ambiguity aversion, as a corollary of Proposition 7, we obtain the following result of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) (see their Theorem 23).

Corollary 4 (Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011))**.** *Let $u : X \to \mathbb{R}$ be an affine onto function and $\phi : \mathbb{R} \to \mathbb{R}$ a strictly increasing and concave function. The triplet $\langle u, \phi, \mu \rangle$ represents a variational preference relation for all countably additive Borel probability measures μ on Δ if and only if ϕ displays constant absolute risk aversion.*

Cherbonnier and Gollier (2015) propose a definition of decreasing aversion under ambiguity in the smooth ambiguity model. They assume that the decision maker with initial wealth $z \in \mathbb{R}_+$ is facing N possible monetary lotteries $(\tilde{x}_1, ..., \tilde{x}_N)$. For each $n \in \{1, ..., N\}$, $\tilde{x}_n$ occurs with probability q_n. The value function of the decision maker obeys the smooth ambiguity rule:

$$\sum_{n=1}^{N} q_n \phi(Eu(z + \tilde{x}_n)), \tag{21}$$

where ϕ is strictly increasing and concave. According to them, the decision maker exhibits decreasing aversion if

$$\begin{aligned} &\phi^{-1}(\sum_{n=1}^{N} q_n \phi(Eu(z + \tilde{x}_n))) = u(z) \\ &\qquad \Rightarrow \sum_{n=1}^{n} q_n \phi'(Eu(z + \tilde{x}_n)) Eu'(z + \tilde{x}_n) \geq \phi'(u(z))u'(z). \end{aligned} \tag{22}$$

The key difference between their definition and ours is that their definition does not distinguish the effect of wealth on risk aversion and ambiguity aversion, while ours captures the effect only on ambiguity aversion. More precisely, their definition says that an ambiguous lottery becomes more desirable at a higher *monetary wealth* level, while our axioms essentially says that it becomes more desirable at a higher *baseline utility* level. When the comparison of behavior is based on the change in baseline utilities, we do not confound the wealth effect on risk aversion with that on ambiguity aversion.

Indeed, they show that (22) holds if and only if both u and $\phi \circ u$ exhibit decreasing concavity, where u, according to Klibanoff, Marinacci and Mukerji (2005), summarizes the decision maker's risk attitude. Instead, as shown in Proposition 7, our Axiom A.2.1 corresponds only to the decreasing concavity of ϕ, the measure of the ambiguity attitude.

Cherbonnier and Gollier (2015) also provide an analogous definition for decreasing aversion in the α-maxmin expected utility model studied by Ghirardato, Maccheroni and Marinacci (2004). That is,

$$\alpha \min_n Eu(z + \tilde{x}_n) + (1 - \alpha) \max_n Eu(z + \tilde{x}_n) \leq u(z)$$
$$\Rightarrow \forall z' \leq z,\ \alpha \min_n Eu(z' + \tilde{x}_n) + (1 - \alpha) \max_n Eu(z' + \tilde{x}_n) \leq u(z'). \tag{23}$$

While the weight α is fixed, their definition only imposes restriction on the function u which, according to Ghirardato, Maccheroni and Marinacci (2004), describes one's risk attitude. In contrast, our axiom captures decreasing aversion towards ambiguity, which is reflected by assigning less weight on the worst expected utility as the baseline utility increases (see Theorem 3).

Focusing also on the effect of changing monetary wealth, Cerreia-Vioglio, Maccheroni, and Marinacci (2017) provide a definition of decreasing/increasing absolute ambiguity aversion in a general setting in which X is assumed to be a set of monetary lotteries. Given a lottery x and a wealth level w, the transformed lottery at w, x^w, is defined as a lottery that yields a payoff of $c + w$ with the same probability as x yields c. Intuitively, x^w is the "real" lottery faced by a decision maker at the wealth level w. Given a preference relation $\succsim$ and a wealth level w, they define the induced preference relation $\succsim^w$ at w as a preference relation that ranks acts as the initial preference relation $\succsim$ ranks "real" acts that yield in each state transformed lotteries at w. Then, in terms of the notion of comparative ambiguity aversion by Ghirardato and Marinacci (2002), a preference relation $\succsim$ is said to display decreasing absolute ambiguity aversion if for each pair w, w' with $w' > w$, $\succsim^w$ is more ambiguity averse than $\succsim^{w'}$.

One implication of their definition is that if a preference relation displays decreasing absolute ambiguity aversion, then it must display constant absolute risk aversion. Their definition does not allow, for example, a decision maker to exhibit both decreasing absolute ambiguity aversion and decreasing absolute risk aversion. In contrast, our definition does not impose restriction on a decision maker's risk attitude and captures the changing pattern of ambiguity aversion with respect to the increase in baseline utility level. Thus, we can accommodate the possibility that a decision maker exhibits both decreasing absolute ambiguity aversion and decreasing absolute risk aversion. In case a decision maker is risk neutral, their definition has the same implication as ours on the representations, since the change in monetary wealth translates directly to the change in baseline utility.

5 Conclusion

In this paper, we study two extensions of the well-known MEU decision rule to accommodate a decision maker's changing ambiguity attitude: a weighted maxmin rule and a variant constraint rule. Due to the non-uniqueness of their representations, we are interested in finding canonical representations of the two rules in terms of the smallest admissible set and the largest essential set respectively. We characterize a class of preferences that admits a canonical weighted maxmin representation as well as a class that admits a canonical variant constraint representation. The first class of preferences exhibits S-ambiguity aversion while the second exhibits GM-ambiguity aversion. In the second part of this paper, we study the wealth effect under ambiguity. We propose axioms of decreasing (increasing and constant) absolute and relative ambiguity aversion. Representations are provided for the subclass of ambiguity averse preferences displaying decreasing (increasing and constant) absolute ambiguity aversion. The monotonic pattern of changing ambiguity aversion is reflected in a intuitive way in both the weighted maxmin representation and the variant constraint representation.

6 Appendix: proofs

We denote by $\mathbb{N}$ the set of positive integers. For each $\varphi \in \mathbb{R}^S$, let $\varphi^* = \max_{s\in S} \varphi(s)$ and $\varphi_* = \min_{s\in S} \varphi(s)$. We say that $I : \mathbb{R}^S \to \mathbb{R}$ is *normalized* if for each $t \in \mathbb{R}$, $I(t\mathbf{1}) = t$; I is monotonic if for each pair $\varphi, \varphi' \in \mathbb{R}^S$ with $\varphi \leq \varphi'$, $I(\varphi) \leq I(\varphi')$; I is *constant superadditive* if for each $\varphi \in \mathbb{R}^S$ and each $t \in \mathbb{R}_+$, $I(\varphi + t\mathbf{1}) \geq I(\varphi) + t$; I is *constant additive* if for each $\varphi \in \mathbb{R}^S$ and each $t \in \mathbb{R}$, $I(\varphi + t\mathbf{1}) = I(\varphi) + t$; I is *superadditive* if for each $\varphi, \varphi' \in \mathbb{R}^S$, $I(\varphi + \varphi') \geq I(\varphi) + I(\varphi')$.

6.1 Proofs in Section 3

Lemma 1. *A preference relation satisfies Axioms A.1 - A.4 and A.6 if and only if there exist an affine onto function $u : X \to \mathbb{R}$ and a normalized, monotonic, and continuous functional $I : \mathbb{R}^S \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \iff I(u(f)) \geq I(u(g))$. Moreover, u is unique up to a positive affine transformation, and given u, there is a unique normalized functional $I : \mathbb{R}^S \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \Leftrightarrow I(u(f)) \geq I(u(g))$.*

Proof. Let a preference relation $\succsim$ satisfying Axioms A.1 - A.4 and A.6 be given. Note that A.6 implies the non-degeneracy axiom: There are $f, g \in \mathcal{F}$ such that $f \succ g$. Applying the same argument in Lemma 28 of Marinacci, Maccheroni and Rustichini (2006), it can be shown that there exist a non-constant affine function $u : X \to \mathbb{R}$ and a normalized monotonic functional $I : u(X)^S \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \iff I(u(f)) \geq I(u(g))$. Since the preference relation satisfies A.6, by Lemma 29 of Marinacci, Maccheroni and Rustichini (2006), $u(X) = \mathbb{R}$. Since the preference relation satisfies A.3, by Proposition 43 of Cerreia-Vioglio, Maccheroni, Marinacci and Siniscalchi (2011), I is lower-semicontinuous. By an analogous argument, I is also upper semicontinuous.

The proof of the "if" direction can be readily seen and thus is omitted. The uniqueness property follows from routine arguments. □

Proof of Proposition 1. Let a preference relation $\succsim$ satisfying A.1 - A.4 and A.6 be given. By Lemma 1, there exist an affine onto function $u : X \to \mathbb{R}$ and a normalized, monotonic, and continuous functional $I : \mathbb{R}^S \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \iff I(u(f)) \geq I(u(g))$.

By Propositions 1 and 2 of Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci and

Siniscalchi (2011)[22], there exist a non-empty closed convex set $D \subseteq \Delta$ such that for each pair $f, g \in \mathcal{F}$,

$$f \succsim^* g \iff \text{for each } p \in D,\ E_p u(f) \geq E_p u(g).$$

Let $\lambda : \mathbb{R}^S \to \mathbb{R}$ be defined by setting for each $\varphi \in \mathbb{R}^S$,

$$\lambda(\varphi) := \begin{cases} \dfrac{\max_{p\in D} E_p\varphi - I(\varphi)}{\max_{p\in D} E_p\varphi - \min_{p\in D} E_p\varphi} & \text{if } \min_{p\in D} E_p\varphi \neq \max_{p\in D} E_p\varphi, \\ 1 & \text{if } \min_{p\in D} E_p\varphi = \max_{p\in D} E_p\varphi. \end{cases}$$

It can be readily verified that for each $\varphi \in \mathbb{R}^S$,

$$I(\varphi) = \lambda(\varphi) \min_{p\in D} E_p\varphi + (1 - \lambda(\varphi)) \max_{p\in D} E_p\varphi.$$ [23]

Thus,

$$\begin{aligned} f \succsim g \iff & \lambda(u(f)) \min_{p\in D} E_p u(f) + (1 - \lambda(u(f))) \max_{p\in D} E_p u(f) \\ & \geq \lambda(u(g)) \min_{p\in D} E_p u(g) + (1 - \lambda(u(g))) \max_{p\in D} E_p u(g). \end{aligned}$$

Moreover, since I is continuous, then λ is continuous on $\{\varphi \in \mathbb{R}^S | \min_{p\in D} E_p\varphi \neq \max_{p\in D} E_p\varphi\}$. Hence, $\langle u, D, \lambda\rangle$ is a Bewley weighted maxmin representation of the preference relation $\succsim$ by definition.

Conversely, let $\succsim$ be a preference relation that admits a Bewley weighted maxmin representation $\langle u, D, \lambda\rangle$. Clearly, the preference relation satisfies A.1. Since u is affine, the preference relation satisfies A.2. To see that A.3 is satisfied, let $f, g, h \in \mathcal{F}$, $\alpha \in [0, 1]$, and let $\{\alpha^n\}_{n=1}^{\infty}$ be a sequence of elements in $[0, 1]$ such that $\lim_{n\to\infty} \alpha^n = \alpha$. Suppose that for each $n \in \mathbb{N}$, $\alpha^n f + (1 - \alpha^n)g \succsim h$. We want to show that $\alpha f + (1 - \alpha)g \succsim h$. Let $I : \mathbb{R}^S \to \mathbb{R}$ be defined by setting for each $\varphi \in \mathbb{R}^S$,

$$I(\varphi) := \lambda(\varphi) \min_{p\in D} E_p\varphi + (1 - \lambda(\varphi)) \max_{p\in D} E_p\varphi.$$

It suffices to show that $\lim_{n\to\infty} I(\alpha^n u(f) + (1 - \alpha^n)u(g)) = I(\alpha u(f) + (1 - \alpha)u(g))$. Suppose that $\min_{p\in D} E_p[\alpha u(f) + (1 - \alpha)u(g)] \neq \max_{p\in D} E_p[\alpha u(f) + (1 - \alpha)u(g)]$. Then, λ is continuous at

[22] See also Propositions 4 and 5 of Ghirardato, Maccheroni and Marinacci (2004).

[23] See also Proposition 5 of Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci and Siniscalchi (2011).

$\alpha u(f) + (1-\alpha)u(g)$, and thus $\lim_{n\to\infty} I(\alpha^n u(f) + (1-\alpha^n)u(g)) = I(\alpha u(f) + (1-\alpha)u(g))$. Suppose that $\min_{p\in D} E_p[\alpha u(f)+(1-\alpha)u(g)] = \max_{p\in D} E_p[\alpha u(f)+(1-\alpha)u(g)]$. Then, $I(\alpha u(f)+(1-\alpha)u(g)) = \min_{p\in D} E_p[\alpha u(f) + (1-\alpha)u(g)] = \lim_{n\to\infty} \min_{p\in D} E_p[\alpha^n u(f) + (1-\alpha^n)u(g)] = \lim_{n\to\infty} \max_{p\in D} E_p[\alpha^n u(f) + (1-\alpha^n)u(g)]$. Thus, $\lim_{n\to\infty} I(\alpha^n u(f) + (1-\alpha^n)u(g)) = I(\alpha u(f) + (1-\alpha)u(g))$, as desired.

To see that the preference relation satisfies A.4, let $f, g \in \mathcal{F}$ be such that for each $s \in S$, $f(s) \succsim g(s)$. Thus, $u(f) \geq u(g)$, and hence for each $p \in D$, $E_p u(f) \geq E_p u(g)$. By the definition of a Bewley weighted maxmin representation, $f \succsim^* g$, which implies $f \succsim g$, as desired. Lastly, since u is onto, by Lemma 29 of Marinacci, Maccheroni and Rustichini (2006), the preference relation satisfies A.6.

To show the uniqueness of the representation, let $\langle u, D, \lambda\rangle$ and $\langle u', D', \lambda'\rangle$ be two Bewley weighted maxmin representations of the preference relation $\succsim$. Since both u and u' are affine functions representing the preference relation restricted to X, by a routine argument, u' is a positive affine transformation of u. By Proposition 5 of Ghirardato, Maccheroni and Marinacci (2004), $D = D'$. Lastly, suppose that $u = u'$. Let $\varphi \in \mathbb{R}^S$ be such that $l(\varphi; D) \neq 0$, i.e., $\max_{p\in D} E_p\varphi \neq \min_{p\in D} E_p\varphi$. Let $f \in \mathcal{F}$ be such that $u(f) = \varphi$. Since both $\langle u, D, \lambda\rangle$ and $\langle u', D', \lambda'\rangle$ are weighted maxmin representations of the preference relation $\succsim$, then

$$u(x_f) = \lambda(u(f)) \min_{p\in D} E_p u(f) + (1 - \lambda(u(f))) \max_{p\in D} E_p u(f), \tag{24}$$

and

$$u'(x_f) = \lambda(u'(f)) \min_{p\in D'} E_p u'(f) + (1 - \lambda(u'(f))) \max_{p\in D'} E_p u'(f). \tag{25}$$

Since $u = u'$, $u(f) = \varphi$, and $D = D'$, then by (24) and (25),

$$\lambda(\varphi) \min_{p\in D} E_p\varphi + (1-\lambda(\varphi)) \max_{p\in D} E_p\varphi = \lambda'(\varphi) \min_{p\in D} E_p\varphi + (1-\lambda'(\varphi)) \max_{p\in D} E_p\varphi.$$

Since $\max_{p\in D} E_p\varphi \neq \min_{p\in D} E_p\varphi$, then $\lambda(\varphi) = \lambda'(\varphi)$, as desired. □

Proof of Theorem 1. Let a preference relation $\succsim$ be given. To show the "only if" direction, suppose that the preference relation $\succsim$ satisfies Axioms A.1 - A.6. By Proposition 1, the preference relation $\succsim$ admits a Bewley weighted maxmin representation $\langle u, D, \lambda\rangle$. We show that D is the smallest admissible set.

By Theorems 3 and 5 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), the preference relation $\succsim$ admits an ambiguity averse representation $\langle v, G\rangle$, where $G : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is given by, for each $(t, p) \in \mathbb{R} \times \Delta$,

$$G(t, p) = \sup\{v(x_f) | f \in \mathcal{F}, E_p v(f) \leq t\}. \tag{26}$$

Recall that, defined in (5), $D^* = cl(\{p \in \Delta | G(t,p) < \infty \text{ for some } t \in \mathbb{R}\})$. By Theorem 10 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), $D = D^*$. Thus, it is equivalent to show that D^* is the smallest admissible set.

Let $\langle u', D', \lambda' \rangle$ be another weighted maxmin representation of the preference relation $\succsim$, and we want to show that $D^* \subseteq D'$. Suppose to the contrary that $D^* \nsubseteq D'$. Since D' is closed, there is $q \in \Delta \setminus D'$ such that $G(t,q) < \infty$ for some $t \in \mathbb{R}$. Since $q \notin D'$, by a standard separation theorem, there is $\varphi \in \mathbb{R}^S$ such that $E_q\varphi < 0 < \min_{p \in D'} E_p\varphi$. Since v, u' are affine functions that represent the preference relation $\succsim$ restricted to X, by a routine argument, v is a positive affine transformation of u'. Without loss of generality, we assume that $v = u'$. Let $x \in X$ be such that $v(x) = t$. Since $E_q v(x) = t$, by (26), $G(t,q) \geq v(x) = t$. Let $n \in \mathbb{N}$ be such that $E_q n\varphi < t \leq G(t,q) < \min_{p \in D'} E_p n\varphi$. Let $g \in \mathcal{F}$ be such that $v(g) = n\varphi$. On one hand, since $E_q v(g) = E_q n\varphi < t$, by (26), $v(x_g) \leq G(t,q)$. On the other hand, since $G(t,q) < \min_{p \in D'} E_p n\varphi$, then $G(t,q) < \min_{p \in D'} E_p v(g) = \min_{p \in D'} E_p u'(g) \leq \lambda'(u'(g)) \min_{p \in D'} E_p u'(g) + (1 - \lambda'(u'(g))) \max_{p \in D'} E_p u'(g) = u'(x_g) = v(x_g)$, which contradicts $v(x_g) \leq G(t,q)$. Hence, $D^* \subseteq D'$, as desired.

We complete the proof of the "only if" direction by showing that $\lambda \in \Lambda(D)$. Define $I : \mathbb{R}^S \to \mathbb{R}$ by setting for each $\varphi \in \mathbb{R}^S$,

$$I(\varphi) = \lambda(\varphi) \min_{p \in D} E_p\varphi + (1 - \lambda(\varphi)) \max_{p \in D} E_p\varphi. \tag{27}$$

Since $\langle u, D, \lambda \rangle$ is a Bewley weighted maxmin representation of the preference relation $\succsim$, then by the proof of Proposition 1, I is monotonic and continuous. Let $\varphi, \varphi' \in \mathbb{R}^S$. Suppose that $\varphi' \geq \varphi$. Since I is monotonic, then $I(\varphi') \geq I(\varphi)$, and thus, (3) holds. Suppose that φ, φ' satisfy (3), and let $\varphi'' := \frac{\varphi + \varphi'}{2}$. Since φ, φ' satisfy (3), then $I(\varphi') \geq I(\varphi)$. Since I is monotonic and continuous, there is $t \in \mathbb{R}_+$ such that $I(\varphi' - t\mathbf{1}) = I(\varphi)$. Let $f, g \in \mathcal{F}$ be such that $u(f) = \varphi$ and $u(g) = \varphi' - t\mathbf{1}$. Then, $f \sim g$. By Axiom A.5, $\frac{1}{2}f + \frac{1}{2}g \succsim f$. Thus, $I(\frac{1}{2}\varphi + \frac{1}{2}(\varphi' - t\mathbf{1})) \geq I(\varphi)$. Since I is monotonic, then $I(\frac{1}{2}\varphi + \frac{1}{2}\varphi') \geq I(\frac{1}{2}\varphi + \frac{1}{2}(\varphi' - t\mathbf{1}))$. Thus, $I(\varphi'') \geq I(\varphi)$, which implies that (4) holds.

To show the "if" direction, suppose that the preference relation $\succsim$ admits a canonical weighted maxmin representation $\langle u, D, \lambda \rangle$ with $\lambda \in \Lambda(D)$. It can be readily seen that the preference relation satisfies Axioms A.1, A.2, A.4 and A.6. To show that it satisfies Axiom A.3, it suffices to show that the functional I defined in (27) is continuous. Let $\varphi \in \mathbb{R}^S$ and $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim_{n \to \infty} \varphi_n = \varphi$. Then, $\lim_{n \to \infty} \min_{p \in D} E_p\varphi_n = \min_{p \in D} E_p\varphi$ and $\lim_{n \to \infty} \max_{p \in D} E_p\varphi_n = \max_{p \in D} E_p\varphi$. If $l(\varphi; D) > 0$, then λ is continuous at φ, and thus

$\lim_{n\to\infty} I(\varphi_n) = I(\varphi)$. If $l(\varphi; D) = 0$, then $\lim_{n\to\infty} \min_{p\in D} E_p\varphi_n = \lim_{n\to\infty} \max_{p\in D} E_p\varphi_n = \min_{p\in D} E_p\varphi$, and thus $\lim_{n\to\infty} I(\varphi_n) = \min_{p\in D} E_p\varphi = I(\varphi)$.

Lastly, to show the uniqueness of the representation, let $\langle u, D, \lambda\rangle$ and $\langle u', D', \lambda'\rangle$ be two canonical weighted maxmin representations of the preference relation $\succsim$. Since both D and D' are the smallest admissible set, then $D = D'$. By the proof of the "if" direction of the characterization, both D and D' are the Bewley set, so that both $\langle u, D, \lambda\rangle$ and $\langle u', D', \lambda'\rangle$ are Bewley weighted maxmin representations of the preference relation $\succsim$. Thus, the uniqueness property follows from Proposition 1. □

Proposition 8. *The preference relation $\succsim$ in Example 2 admits a Bewley weighted maxmin representation but not a canonical weighted maxmin representation.*

Proof of Proposition 8. We first prove that the preference relation $\succsim$ admits a Bewley weighted maxmin representation. By Proposition 1, it suffices to show that the preference relation $\succsim$ satisfies Axioms A.1 - A.4 and A.6. Clearly, it satisfies A.1. Since for each $x \in X$, $V(x) = x$, then it satisfies Axioms A.2 and A.6.

To show that the preference relation satisfies Axiom A.3, it suffices for us to prove the continuity of V. Let $f \in \mathbb{R}^S$ and $\{f^n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathbb{R}^S$ that converges to f. We check that $\lim_{n\to\infty} V(f^n) = V(f)$ in each of the following two cases.

Case 1: $\max\{f(1), f(2)\} \neq f(3)$. Suppose that $\max\{f(1), f(2)\} < f(3)$. Then, $V(f) = \min_{p\in D_1} E_p f$. Moreover, for sufficiently large n, $\max\{f^n(1), f^n(2)\} < f^n(3)$, so $V(f^n) = \min_{p\in D_1} E_p f^n$. Hence, $\lim_{n\to\infty} V(f^n) = \lim_{n\to\infty} \min_{p\in D_1} E_p f^n = \min_{p\in D_1} E_p f = V(f)$. Similarly, when $\max\{f(1), f(2)\} > f(3)$, it can be shown that $\lim_{n\to\infty} V(f^n) = V(f)$.

Case 2: $\max\{f(1), f(2)\} = f(3)$. Suppose that $f(1) < f(2)$. Then, $f(2) = f(3)$. Let $p'' := (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ as in Example 2. Note that $p'' \in D_2$, and for each $p \in D_2$, $p_1 \le p''_1$ since

$$\frac{1}{6} \ge (p_1 - \frac{1}{3})^2 + (p_2 - \frac{1}{3})^2 + (p_3 - \frac{1}{3})^2 \ge (p_1 - \frac{1}{3})^2 + 2(\frac{1-p_1}{2} - \frac{1}{3})^2 = \frac{3}{2}(p_1 - \frac{1}{3})^2. \quad (28)$$

Since $f(1) < f(2) = f(3)$, then $V(f) = \min_{p\in D_2} E_p f = E_{p''} f = E_{p'} f$. For sufficiently large n, $f^n(1) < f^n(2)$, so either $V(f^n) = \min_{p\in D_1} E_p f^n = E_{p'} f^n$ or $V(f^n) = \min_{p\in D_2} E_p f^n$. Since $\lim_{n\to\infty} E_{p'} f^n = E_{p'} f = V(f)$ and $\lim_{n\to\infty} \min_{p\in D_2} E_p f^n = \min_{p\in D_2} E_p f = V(f)$, then $\lim_{n\to\infty} V(f^n) = V(f)$. Similarly, when $f(1) > f(2)$, it can be shown that $\lim_{n\to\infty} V(f^n) = V(f)$. Suppose that $f(1) = f(2)$. Then, f is a constant act, so $\lim_{n\to\infty} \min_{p\in D_1} E_p f^n = \lim_{n\to\infty} \min_{p\in D_2} E_p f^n = V(f)$. Since for each n, either $V(f^n) = \min_{p\in D_1} E_p f^n$ or $V(f^n) = \min_{p\in D_2} E_p f^n$, then $\lim_{n\to\infty} V(f^n) = V(f)$.

To show that the preference relation satisfies Axiom A.4, let $f, g \in \mathbb{R}^S$ be such that $f \geq g$. We check that $V(f) \geq V(g)$ in each of the following three cases.

Case 1: Either $\max\{f(1), f(2)\} < f(3)$ and $\max\{g(1), g(2)\} < g(3)$, or $\max\{f(1), f(2)\} \geq f(3)$ and $\max\{g(1), g(2)\} \geq g(3)$. Then, either $V(f) = \min_{p \in D_1} E_p f$ and $V(g) = \min_{p \in D_1} E_p g$, or $V(f) = \min_{p \in D_2} E_p f$ and $V(g) = \min_{p \in D_2} E_p g$. Since $f \geq g$, then $V(f) \geq V(g)$ in either scenario.

Case 2: $\max\{f(1), f(2)\} < f(3)$ and $\max\{g(1), g(2)\} \geq g(3)$. Then, $V(f) = \min_{p \in D_1} E_p f$ and $V(g) = \min_{p \in D_2} E_p g$. Let $f' \in \mathbb{R}^S$ be such that

$$f'(1) = f(1), \quad f'(2) = f(2), \quad f'(3) = \max\{f(1), f(2)\}.$$

Since $\max\{f(1), f(2)\} < f(3)$, then $f'(3) < f(3)$, and thus $f' \leq f$. Since $f \geq g$, then $f'(3) = \max\{f(1), f(2)\} \geq \max\{g(1), g(2)\} \geq g(3)$, and thus $f' \geq g$. Consider the sequence $\{\frac{1}{n} f + \frac{n-1}{n} f'\}_{n=1}^{\infty}$ of elements of $\mathbb{R}^S$. For each n, since $\max\{\frac{1}{n} f(1) + \frac{n-1}{n} f'(1), \frac{1}{n} f(2) + \frac{n-1}{n} f'(2)\} = \max\{f(1), f(2)\} < \frac{1}{n} f(3) + \frac{n-1}{n} f'(3)$, then $V(\frac{1}{n} f + \frac{n-1}{n} f') = \min_{p \in D_1} E_p(\frac{1}{n} f + \frac{n-1}{n} f')$. As shown before, V is continuous, so $V(f') = \lim_{n \to \infty} V(\frac{1}{n} f + \frac{n-1}{n} f') = \lim_{n \to \infty} \min_{p \in D_1} E_p(\frac{1}{n} f + \frac{n-1}{n} f') = \min_{p \in D_1} E_p f'$. Since $f' \leq f$, then $\min_{p \in D_1} E_p f' \leq \min_{p \in D_1} E_p f$, and thus $V(f') \leq V(f)$. Since $\max\{f'(1), f'(2)\} = f'(3)$ and $f' \geq g$, then $V(f') = \min_{p \in D_2} E_p f' \geq \min_{p \in D_2} E_p g = V(g)$. Since $V(f) \geq V(f') \geq V(g)$, then $V(f) \geq V(g)$.

Case 3: $\max\{f(1), f(2)\} \geq f(3)$ and $\max\{g(1), g(2)\} < g(3)$. Then, $V(f) = \min_{p \in D_2} E_p f$ and $V(g) = \min_{p \in D_1} E_p g$. Let $f' \in \mathbb{R}^S$ be such that

$$f'(1) = \begin{cases} f(3) & \text{if } f(1) \geq f(2), \\ f(1) & \text{if } f(1) < f(2), \end{cases} \qquad f'(2) = \begin{cases} f(2) & \text{if } f(1) \geq f(2), \\ f(3) & \text{if } f(1) < f(2), \end{cases} \qquad f'(3) = f(3).$$

Since $\max\{f(1), f(2)\} \geq f(3)$, then $f'(1) \leq f(1)$ and $f'(2) \leq f(2)$, and thus $f' \leq f$. By the definition of f', $\max\{f'(1), f'(2)\} \geq f'(3)$. Thus, $V(f') = \min_{p \in D_2} E_p f'$. Since $f' \leq f$, then $\min_{p \in D_2} E_p f' \leq \min_{p \in D_2} E_p f$. Hence, $V(f') \leq V(f)$. Let $g' \in \mathbb{R}^S$ be such that

$$g'(1) = \begin{cases} g(3) & \text{if } f(1) \geq f(2), \\ g(1) & \text{if } f(1) < f(2), \end{cases} \qquad g'(2) = \begin{cases} g(2) & \text{if } f(1) \geq f(2), \\ g(3) & \text{if } f(1) < f(2), \end{cases} \qquad g'(3) = g(3).$$

Since $f \geq g$, then $f' \geq g'$. Since $\max\{g(1), g(2)\} < g(3)$, then $g' \geq g$ and $\max\{g'(1), g'(2)\} = g'(3)$. Thus, $V(f') = \min_{p \in D_2} E_p f' \geq \min_{p \in D_2} E_p g' = V(g')$. Consider the sequence $\{\frac{1}{n} g + \frac{n-1}{n} g'\}_{n=1}^{\infty}$ of elements of $\mathbb{R}^S$. Since $\max\{g(1), g(2)\} < g(3)$, then for each n, $\max\{\frac{1}{n}g(1) + \frac{n-1}{n}g'(1), \frac{1}{n}g(2) + \frac{n-1}{n}g'(2)\} < \frac{1}{n}g(3) + \frac{n-1}{n}g'(3)$, and thus, $V(\frac{1}{n}g + \frac{n-1}{n}g') = \min_{p \in D_1} E_p(\frac{1}{n}g + \frac{n-1}{n}g')$. As shown before, V is continuous, so $V(g') = \lim_{n\to\infty} V(\frac{1}{n}g + \frac{n-1}{n}g') = \lim_{n\to\infty} \min_{p\in D_1} E_p(\frac{1}{n}g + \frac{n-1}{n}g') = \min_{p\in D_1} E_p g'$. Since $g' \geq g$, then $\min_{p\in D_1} E_p g' \geq \min_{p\in D_1} E_p g$. Thus, $V(g') \geq V(g)$. Since $V(f) \geq V(f') \geq V(g') \geq V(g)$, then $V(f) \geq V(g)$.

Next, we prove that the preference relation $\succsim$ does not admit a canonical weighted maxmin representation. Suppose to the contrary that $\langle u, D, \lambda\rangle$ is a canonical weighted maxmin representation of the preference relation $\succsim$. Let $D_3 := \{p \in \Delta | p_3 \geq \frac{1}{4}\}$. We shall show that both D_2 and D_3 are admissible sets of priors for the preference relation $\succsim$.

Let $f \in \mathbb{R}^S$. We check that $V(f) \leq \max_{p\in D_2} E_p f$ and $V(f) \leq \max_{p\in D_3} E_p f$. Suppose that $f(1) \leq f(2) < f(3)$. Then, $V(f) = \min_{p\in D_1} E_p f = E_{p'} f \leq E_{p^*} f$. Similarly, if $f(2) < f(1) < f(3)$, $V(f) \leq E_{p^*} f$. Suppose that $\max\{f(1), f(2)\} \geq f(3)$. Then, $V(f) = \min_{p\in D_2} E_p f$. Since $p^* \in D_2$, then $V(f) \leq E_{p^*} f$. In all cases, $V(f) \leq E_{p^*} f$, and since $p^* \in D_2 \cap D_3$, then $V(f) \leq \max_{p\in D_2} E_p f$ and $V(f) \leq \max_{p\in D_3} E_p f$.

We now check that $V(f) \geq \min_{p\in D_2} E_p f$. Suppose that $f(1) \leq f(2) < f(3)$. Then, $V(f) = \min_{p\in D_1} E_p f = E_{p'} f \geq E_{p''} f$. Since $p'' \in D_2$, then $V(f) \geq \min_{p\in D_2} E_p f$. Similarly, if $f(2) < f(1) < f(3)$, $V(f) \geq \min_{p\in D_2} E_p f$. Lastly, if $\max\{f(1), f(2)\} \geq f(3)$, $V(f) = \min_{p\in D_2} E_p f$. In all cases, $V(f) \geq \min_{p\in D_2} E_p f$.

We then check that $V(f) \geq \min_{p\in D_3} E_p f$. Suppose that $\max\{f(1), f(2)\} < f(3)$. Then, $V(f) = \min_{p\in D_1} E_p f$. Since $D_1 \subseteq D_3$, $V(f) \geq \min_{p\in D_3} E_p f$. Suppose that $f(3) < \min\{f(1), f(2)\}$. Then, $V(f) = \min_{p\in D_2} E_p f \geq f(3) = \min_{p\in D_3} E_p f$. Suppose that $f(1) \leq f(3) \leq f(2)$. Then, $V(f) = \min_{p\in D_2} E_p f$. Recall that for each $p \in D_2$, by (28), $p_1 \leq \frac{2}{3}$. Thus, for each $p \in D_2$, $p_1 < \frac{3}{4}$ so that $E_p f \geq \frac{3}{4} f(1) + \frac{1}{4} f(3)$. Since $(\frac{3}{4}, 0, \frac{1}{4}) \in D_3$, then $\min_{p\in D_2} E_p f \geq \min_{p\in D_3} E_p f$. Hence, $V(f) \geq \min_{p\in D_3} E_p f$. Similarly, if $f(2) \leq f(3) \leq f(1)$, $V(f) \geq \min_{p\in D_3} E_p f$. In all cases, $V(f) \geq \min_{p\in D_3} E_p f$.

Define the function $\lambda_2 : \mathbb{R}^S \to \mathbb{R}$ by setting for each $f \in \mathbb{R}^S$,

$$\lambda_2(f) := \begin{cases} \dfrac{\max_{p\in D_2} E_p f - V(f)}{\max_{p\in D_2} E_p f - \min_{p\in D_2} E_p f} & \text{if } \min_{p\in D_2} E_p f \neq \max_{p\in D_2} E_p f, \\ 1 & \text{if } \min_{p\in D_2} E_p f = \max_{p\in D_2} E_p f. \end{cases}$$

For each $f \in \mathbb{R}^S$, we have just checked that $V(f) \in [\min_{p\in D_2} E_p f, \max_{p\in D_2} E_p f]$, so $\lambda_2(f) \in [0, 1]$.

As shown before, V is continuous, and hence, λ_2 is continuous on $\{f \in \mathbb{R}^S | \min_{p\in D_2} E_p f \neq \max_{p\in D_2} E_p f\}$. By the definition of λ_2, it can be readily seen that for each $f \in \mathbb{R}^S$, $V(f) = \lambda_2(f) \min_{p\in D_2} E_p f + (1 - \lambda_2(f)) \max_{p\in D_2} E_p f$. Let v be the identity mapping on $\mathbb{R}$. Then, (v, D_2, λ_2) is a weighed maxmin representation of the preference relation $\succsim$. Similarly, one can define $\lambda_3 : \mathbb{R}^S \to [0,1]$ so that (v, D_3, λ_3) is a weighted maxmin representation of the preference relation $\succsim$. Since $\langle u, D, \lambda\rangle$ is a canonical weighted maxmin representation of the preference relation $\succsim$, then $D \subseteq D_2 \cap D_3$.

Let $f \in \mathbb{R}^S$ be such that $f(1) < f(2) = f(3)$. Then, $V(f) = \min_{p\in D_2} E_p f$. Recall that for each $p \in D_2$, by (28), $p_1 \leq \frac{2}{3}$, and moreover, it can be readily seen that $p_1 = \frac{2}{3}$ if only if $p_2 = p_3 = \frac{1}{6}$. Thus, $V(f) = E_{p''} f$, and for each $p \in D_2 \setminus \{p''\}$, $E_p f > V(f)$. Since $p'' \notin D_3$ and $D \subseteq D_2 \cap D_3$, then $D \subseteq D_2 \setminus \{p''\}$. Thus, $\min_{p\in D} E_p f > V(f) = x_f$. Since both v and u are affine functions that represent the preference relation $\succsim$ restricted to X, by a routine argument, u is a positive affine transformation of v. Then, $\min_{p\in D} E_p u(f) > u(x_f)$, and thus $\lambda(u(f)) \min_{p\in D} E_p u(f) + (1 - \lambda(u(f))) \max_{p\in D} E_p u(f) > u(x_f)$. Since $\langle u, D, \lambda\rangle$ is a weighted maxmin representation of the preference relation $\succsim$, then $f \succ x_f$, which is a contradiction. Therefore, there is no canonical weighted maxmin representation of the preference relation $\succsim$. □

Proposition 9. *The preference relation $\succsim$ in Example 3 admits both a Bewley weighted maxmin representation and a canonical weighted maxmin representation, whereas the Bewley set of priors for the preference relation $\succsim$ is not the smallest admissible set.*

Proof of Proposition 9. We first prove that the preference relation $\succsim$ admits a Bewley weighted maxmin representation. By Proposition 1, it suffices to show that the preference relation $\succsim$ satisfies Axioms A.1 - A.4 and A.6. Clearly, it satisfies Axiom A.1. Since for each $x \in \mathbb{R}$, $V(x) = x$, then it satisfies Axioms A.2 and A.6.

To show that the preference relation satisfies Axiom A.3, it suffices for us to prove the continuity of V. For that, we first check the continuity of α. Let $f \in \mathbb{R}^S$ be such that $\max\{f(1), f(2)\} < f(3)$. Let $\{f^n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathbb{R}^S$ that converges to f and such that for each n, $\max\{f^n(1), f^n(2)\} < f^n(3)$. We want to show that $\lim_{n\to\infty} \alpha(f^n) = \alpha(f)$. Suppose that $f(1) < f(2) < f(3)$. Then, $\alpha(f) = \text{med}\{0, \frac{1-E_{p''}f}{1-E_{p''}f+E_{p'}f}, 1\}$, and for sufficiently large n, $f^n(1) < f^n(2) < f^n(3)$, so that $\alpha(f^n) = \text{med}\{0, \frac{1-E_{p''}f^n}{1-E_{p''}f^n+E_{p'}f^n}, 1\}$. Since the median operator is continuous, then $\lim_{n\to\infty} \alpha(f^n) = \alpha(f)$. Similarly, when $f(2) < f(1) < f(3)$, it can be shown that $\lim_{n\to\infty} \alpha(f^n) = \alpha(f)$. Suppose that $f(1) = f(2) < f(3)$. Then,

$\lim_{n\to\infty} \frac{1-E_{p''}f^n}{1-E_{p''}f^n+E_{p'}f^n} = \frac{1-E_{p''}f}{1-E_{p''}f+E_{p'}f} = \frac{1-E_{q''}f}{1-E_{q''}f+E_{q'}f} = \lim_{n\to\infty} \frac{1-E_{q''}f^n}{1-E_{q''}f^n+E_{q'}f^n}$. For each n, either $\alpha(f^n) = \text{med}\{0, \frac{1-E_{p''}f^n}{1-E_{p''}f^n+E_{p'}f^n}, 1\}$, or $\alpha(f^n) = \text{med}\{0, \frac{1-E_{q''}f^n}{1-E_{q''}f^n+E_{q'}f^n}, 1\}$. Thus, $\lim_{n\to\infty} \alpha(f^n) = \text{med}\{0, \frac{1-E_{p''}f}{1-E_{p''}f+E_{p'}f}, 1\} = \alpha(f)$.

To show the continuity of V, let $f \in \mathbb{R}^S$ and $\{f^n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathbb{R}^S$ that converges to f. We show that $\lim_{n\to\infty} V(f^n) = V(f)$ in each of the following two cases.

Case 1: $\max\{f(1), f(2)\} \neq f(3)$. Suppose that $\max\{f(1), f(2)\} < f(3)$. Then, $V(f) = \min_{p\in D_1(f)} E_p f$, and for sufficiently large n, $\max\{f^n(1), f^n(2)\} < f^n(3)$, so that $V(f^n) = \min_{p\in D_1(f^n)} E_p f^n$. Since α is continuous at f, then D_1 is continuous (i.e., both upper and lower hemicontinuous) at f. By the maximum theorem, $\lim_{n\to\infty} V(f^n) = V(f)$. Suppose that $\max\{f(1), f(2)\} > f(3)$. Then, $V(f) = \min_{p\in D_2} E_p f$, and for sufficiently large n, $\max\{f^n(1), f^n(2)\} > f^n(3)$, so that $V(f^n) = \min_{p\in D_2} E_p f^n$. Hence, $\lim_{n\to\infty} V(f^n) = V(f)$.

Case 2: $\max\{f(1), f(2)\} = f(3)$. Then, $V(f) = \min_{p\in D_2} E_p f$. Suppose that $f(1) < f(2)$. Then, $f(2) = f(3)$. As argued in the proof of Proposition 8, for each $p \in D_2$, by (28), $p_1 \leq \frac{2}{3} = p''_1$. Thus, $\min_{p\in D_2} E_p f = E_{p''} f = E_{p'} f$. For sufficiently large n, $f^n(1) < f^n(2)$, so either $V(f^n) = \alpha(f^n) E_{p'} f^n + (1 - \alpha(f^n)) E_{p''} f^n$, or $V(f^n) = \min_{p\in D_2} E_p f^n$. Since $\lim_{n\to\infty} E_{p'} f^n = E_{p'} f = E_{p''} f = \lim_{n\to\infty} E_{p''} f^n$, and for each n, $\alpha(f^n) E_{p'} f^n + (1 - \alpha(f^n)) E_{p''} f^n \in [\min\{E_{p'} f^n, E_{p''} f^n\}, \max\{E_{p'} f^n, E_{p''} f^n\}]$, then $\lim_{n\to\infty} \alpha(f^n) E_{p'} f^n + (1 - \alpha(f^n)) E_{p''} f^n = E_{p''} f$. Moreover, $\lim_{n\to\infty} \min_{p\in D_2} E_p f^n = \min_{p\in D_2} E_p f$. Since $E_{p''} f = \min_{p\in D_2} E_p f = V(f)$, then $\lim_{n\to\infty} V(f^n) = V(f)$. Similarly, when $f(1) > f(2)$, it can be shown that $\lim_{n\to\infty} V(f^n) = V(f)$. Lastly, suppose that $f(1) = f(2)$. Then, f is a constant act and $\lim_{n\to\infty} \min_{p\in D_1(f^n)} E_p f^n = \lim_{n\to\infty} \min_{p\in D_2} E_p f^n = V(f)$. For each n, either $V(f^n) = \min_{p\in D_1(f^n)} E_p f^n$, or $V(f^n) = \min_{p\in D_2} E_p f^n$. Thus, $\lim_{n\to\infty} V(f^n) = V(f)$.

Now we show that the preference relation satisfies Axiom A.4. It helps to rewrite α and V in decomposed form. Note that for each $f \in \mathbb{R}^S$ with $f(1) \leq f(2) < f(3)$,

$$\alpha(f) = \begin{cases} 1 & \text{if } E_{p'} f \leq 0, \\ \dfrac{1 - E_{p''} f}{1 - E_{p''} f + E_{p'} f} & \text{if } E_{p'} f \geq 0, \text{ and } E_{p''} f \leq 1,^{24} \\ 0 & \text{if } E_{p''} f \geq 1, \end{cases}$$

and similarly, for each $f \in \mathbb{R}^S$ with $f(2) < f(1) < f(3)$,

$$\alpha(f) = \begin{cases} 1 & \text{if } E_{q'} f \leq 0, \\ \dfrac{1 - E_{q''} f}{1 - E_{q''} f + E_{q'} f} & \text{if } E_{q'} f > 0 \text{ and } E_{q''} f < 1, \\ 0 & \text{if } E_{q''} f \geq 1. \end{cases}$$

Thus, for each $f \in \mathbb{R}^S$,

$$V(f) = \begin{cases} E_{p'}f & \text{if } f(1) \leq f(2) < f(3) \text{ and } E_{p'}f \leq 0, \\ \dfrac{E_{p'}f}{1 - E_{p''}f + E_{p'}f} & \text{if } f(1) \leq f(2) < f(3), E_{p'}f > 0, \text{ and } E_{p''}f < 1, \\ E_{p''}f & \text{if } f(1) \leq f(2) < f(3) \text{ and } E_{p''}f \geq 1, \\ E_{q'}f & \text{if } f(2) < f(1) < f(3) \text{ and } E_{q'}f \leq 0, \\ \dfrac{E_{q'}f}{1 - E_{q''}f + E_{q'}f} & \text{if } f(2) < f(1) < f(3), E_{q'}f > 0, \text{ and } E_{q''}f < 1, \\ E_{q''}f & \text{if } f(2) < f(1) < f(3) \text{ and } E_{q''}f \geq 1, \\ \min_{p \in D_2} E_p f & \max\{f(1), f(2)\} \geq f(3). \end{cases}$$

Let $f, g \in \mathbb{R}^S$ be such that $f \geq g$. We check that $V(f) \geq V(g)$ in each of the following four cases.

Case 1: $f(1) \leq f(2) < f(3)$ and $g(1) \leq g(2) < g(3)$. Suppose that $E_{p'}f \leq 0$. Then, $E_{p'}g \leq E_{p'}f \leq 0$, and thus

$$V(f) = E_{p'}f \geq E_{p'}g = V(g).$$

Suppose that $E_{p'}f > 0$ and $E_{p''}f < 1$. Then, $E_{p''}g \leq E_{p''}f < 1$. If $E_{p'}g \leq 0$, then

$$V(f) = \frac{E_{p'}f}{1 - E_{p''}f + E_{p'}f} > 0 \geq E_{p'}g = V(g).$$

If $E_{p'}g > 0$, then

$$V(f) = \frac{E_{p'}f}{1 - E_{p''}f + E_{p'}f} = \frac{1}{\dfrac{1 - E_{p''}f}{E_{p'}f} + 1} \geq \frac{1}{\dfrac{1 - E_{p''}g}{E_{p'}g} + 1} = \frac{E_{p'}g}{1 - E_{p''}g + E_{p'}g} = V(g),$$

where the inequality holds since $1 - E_{p''}f \leq 1 - E_{p''}g$ and $E_{p'}f \geq E_{p'}g$. Suppose that $E_{p''}f \geq 1$. If $E_{p'}g \leq 0$, then

$$V(f) = E_{p''}f \geq 1 > 0 \geq E_{p'}g = V(g).$$

If $E_{p'}g > 0$ and $E_{p''}g < 1$, then

$$V(f) = E_{p''}f \geq 1 > \frac{E_{p'}g}{1 - E_{p''}g + E_{p'}g} = V(g).$$

If $E_{p''}g \geq 1$, then

$$V(f) = E_{p''}f \geq E_{p''}g = V(g).$$

Case 2: $f(2) < f(1) < f(3)$ and $g(2) < g(1) < g(3)$, or $\max\{f(f), f(2)\} \geq f(3)$ and $\max\{g(1), g(2)\} \geq g(3)$. In the former scenario, by a similar argument as in Case 1, it can be shown that $V(f) \geq V(g)$. In the latter scenario, $V(f) = \min_{p \in D_2} E_p f \geq \min_{p \in D_2} E_p g = V(g)$.

Case 3: $f(1) \leq f(2) < f(3)$ and $g(2) < g(1) < g(3)$, or $f(2) < f(1) < f(3)$ and $g(1) \leq g(2) < g(3)$. Consider the former scenario. Let $f' \in \mathbb{R}^S$ be such that

$$f'(1) = f(1), \quad f'(2) = f(1), \quad f'(3) = f(3).$$

Then, $f \geq f'$, and by the result in Case 1, $V(f) \geq V(f')$. Moreover, $f'(1) = f'(2) < f'(3)$ and $f' \geq g$. Consider the sequence $\{\frac{n-1}{n}f' + \frac{1}{n}g\}_{n=1}^{\infty}$ of elements in $\mathbb{R}^S$. For each n, $\frac{n-1}{n}f'(2) + \frac{1}{n}g(2) < \frac{n-1}{n}f'(1) + \frac{1}{n}g(1) < \frac{n-1}{n}f'(3) + \frac{1}{n}g(3)$ and $\frac{n-1}{n}f' + \frac{1}{n}g \geq g$. By the result in Case 2, for each n, $V(\frac{n-1}{n}f' + \frac{1}{n}g) \geq V(g)$. As shown before, V is continuous. Then, $V(f') = \lim_{n\to\infty} V(\frac{n-1}{n}f' + \frac{1}{n}g) \geq V(g)$. Thus, $V(f) \geq V(f') \geq V(g)$. By a similar argument, it can be shown in the later scenario that $V(f) \geq V(g)$.

Case 4: $\max\{f(1), f(2)\} < f(3)$ and $\max\{g(1), g(2)\} \geq g(3)$, or $\max\{f(1), f(2)\} \geq f(3)$ and $\max\{g(1), g(2)\} < g(3)$. Consider the former scenario. Let $f' \in \mathbb{R}^S$ be such that

$$f'(1) = f(1), \quad f'(2) = f(2), \quad f'(3) = \max\{f(1), f(2)\}.$$

Then, $f \geq f' \geq g$ and $\max\{f'(1), f'(2)\} = f'(3)$. By the result in Case 2, $V(f') \geq V(g)$. Consider the sequence $\{\frac{n-1}{n}f' + \frac{1}{n}f\}_{n=1}^{\infty}$ of elements in $\mathbb{R}^S$. For each n, $\max\{\frac{n-1}{n}f'(1) + \frac{1}{n}f(1), \frac{n-1}{n}f'(2) + \frac{1}{n}f(2)\} = \max\{f(1), f(2)\} < \frac{n-1}{n}f'(3) + \frac{1}{n}f(3)$, and $f \geq \frac{n-1}{n}f' + \frac{1}{n}f$. By the results in Cases 1, 2, and 3, for each n, $V(f) \geq V(\frac{n-1}{n}f' + \frac{1}{n}f)$. As shown before, V is continuous. Thus, $V(f) \geq \lim_{n\to\infty} V(\frac{n-1}{n}f' + \frac{1}{n}f) = V(f')$. Hence, $V(f) \geq V(f') \geq V(g)$. By a similar argument, it can be shown in the later scenario that $V(f) \geq V(g)$.

Next, we prove that the preference relation $\succsim$ admits a canonical weighted maxmin representation with $D := \{p \in D_2 : p_3 \geq \frac{1}{6}\}$ being the smallest admissible set. Let $f \in \mathbb{R}^S$. We claim that $V(f) \in [\min_{p \in D} E_p f, \max_{p \in D} E_p f]$. To see this, assume first that $f(1) \leq f(2) < f(3)$. Then, $E_{p''}f < E_{p'}f < E_{p^*}f$ and $V(f) = \alpha(f)E_{p'}f + (1 - \alpha(f))E_{p''}f$. Since $p'', p^* \in D$, then $[E_{p''}f, E_{p'}f] \subseteq [\min_{p \in D} E_p f, \max_{p \in D} E_p f]$. Thus, $V(f) \in [\min_{p \in D} E_p f, \max_{p \in D} E_p f]$. When $f(2) <$

$f(1) < f(3)$, by a similar argument, it can be shown that $V(f) \in [\min_{p\in D} E_p f, \max_{p\in D} E_p f]$. Lastly, assume that $\max\{f(1), f(2)\} \geq f(3)$. Then, $V(f) = \min_{p\in D_2} E_p f$. Since $p^* \in D_2$ and $p \in D$, then $V(f) \leq E_{p^*} f \leq \max_{p\in D} E_p f$. To show that $V(f) \geq \min_{p\in D} E_p f$, suppose to the contrary that $V(f) < \min_{p\in D} E_p f$. Then, f is not constant. Let $\bar{p} \in \arg\min_{p\in D_2} E_p f$. Then, $\bar{p} \notin D$, i.e., $\bar{p}_3 < \frac{1}{6}$. Suppose that $f(1) \leq f(2)$. Then, $f(3) \leq f(2)$. As argued in the proof of Proposition 8, for each $p \in D_2$, by (28), $p_1 \leq \frac{2}{3}$, and moreover $p_1 = \frac{2}{3}$ if and only if $p_2 = p_3 = \frac{1}{6}$. Thus, $\bar{p}_1 < \frac{2}{3}$. Since $\bar{p}_1 < p'_1$, $\bar{p}_3 < p'_3$, $f(1) \leq f(2)$, $f(3) \leq f(2)$, and since f is not constant, then $E_{\bar{p}} f > E_{p'} f$. Since $p' \in D_2$, then $E_{\bar{p}} f > \min_{p\in D_2} E_p f$, which is a contradiction. By a similar argument, it can be shown that when $f(1) > f(2)$, $V(f) \geq \min_{p\in D} E_p f$.

Since for each $f \in \mathbb{R}^S$, $V(f) \in [\min_{p\in D} E_p f, \max_{p\in D} E_p f]$, then there is $\lambda : \mathbb{R}^S \to [0, 1]$ such that for each $f \in \mathbb{R}^S$, $V(f) = \lambda(f) \min_{p\in D} E_p f + (1 - \lambda(f)) \max_{p\in D} E_p f$. By the continuity of V, it can be readily verified that λ is continuous on $\{f \in \mathbb{R}^S | \min_{p\in D} E_p f \neq \max_{p\in D} E_p f\}$. Let u be the identity mapping on $\mathbb{R}$. Then, $\langle u, D, \lambda\rangle$ is a weighted maxmin representation of the preference relation $\succsim$.

To show that D is the smallest admissible set, let $\langle u', D', \lambda'\rangle$ be another weighted maxmin representation of the preference relation $\succsim$, and we want to check that $D \subseteq D'$. Suppose to the contrary that there is $\bar{p} \in D \setminus D'$. By a standard separation theorem, there is $\bar{f} \in \mathbb{R}^S \setminus \mathbb{R}\mathbf{1}$ such that $E_{\bar{p}} \bar{f} < \min_{p\in D'} E_p \bar{f}$. Let $t \in \mathbb{R}$ be such that $E_{p''}(\bar{f} + t\mathbf{1}) \geq 1$ and $E_{q''}(\bar{f} + t\mathbf{1}) \geq 1$.

We claim that $V(\bar{f}+t\mathbf{1}) \leq E_{\bar{p}}(\bar{f}+t\mathbf{1})$. This is clearly true if $\max\{\bar{f}(1)+t, \bar{f}(2)+t\} \geq \bar{f}(3)+t$, since $V(\bar{f}+t\mathbf{1}) = \min_{p\in D_2} E_p(\bar{f}+t\mathbf{1})$ and $\bar{p} \in D \subseteq D_2$. Suppose that $\bar{f}(1)+t \leq \bar{f}(2)+t < \bar{f}(3)+t$. Then, $V(\bar{f} + t\mathbf{1}) = E_{p''}(\bar{f} + t\mathbf{1})$. Recall that as argued in the proof of Proposition 8, for each $p \in D_2$, $p_1 \leq \frac{2}{3}$. Since $\bar{p} \in D_2$, then $\bar{p}_1 \leq \frac{2}{3} = p''_1$. Since $\bar{p} \in D$, then $\bar{p}_3 \geq \frac{1}{6} = p''_3$. Since $\bar{f}(1) + t \leq \bar{f}(2) + t < \bar{f}(3) + t$, then $E_{p''}(\bar{f} + t\mathbf{1}) \leq E_{\bar{p}}(\bar{f} + t\mathbf{1})$. Thus, $V(\bar{f} + t\mathbf{1}) \leq E_{\bar{p}}(\bar{f} + t\mathbf{1})$. By a similar argument, it can be shown that when $\bar{f}(2) + t < \bar{f}(1) + t < \bar{f}(3) + t$, $V(\bar{f} + t\mathbf{1}) \leq E_{\bar{p}}(\bar{f} + t\mathbf{1})$.

Since $V(\bar{f}+t\mathbf{1}) \leq E_{\bar{p}}(\bar{f}+t\mathbf{1})$ and $E_{\bar{p}} \bar{f} < \min_{p\in D'} E_p \bar{f}$, then $V(\bar{f}+t\mathbf{1}) < \min_{p\in D'} E_p(\bar{f}+t\mathbf{1})$, and thus $x_{\bar{f}+t\mathbf{1}} < \min_{p\in D'} E_p(\bar{f} + t\mathbf{1})$. Since both u and u' are affine functions that represent the preference relation $\succsim$ restricted to X, by a routine argument, u' is a positive affine transformation of u. Then, $u'(x_{\bar{f}+t\mathbf{1}}) < \min_{p\in D'} E_p u'(\bar{f} + t\mathbf{1}) \leq \lambda'(u'(\bar{f} + t\mathbf{1})) \min_{p\in D'} E_p u'(\bar{f} + t\mathbf{1}) + (1 - \lambda'(u'(\bar{f} + t\mathbf{1}))) \max_{p\in D'} E_p u'(\bar{f} + t\mathbf{1})$. Since $\langle u', D', \lambda'\rangle$ represents the preference relation $\succsim$, then $\bar{f} + t\mathbf{1} \succ x_{\bar{f}+t\mathbf{1}}$, which is a contradiction. Hence, $D \subseteq D'$, as desired.

Finally, we show that D is not the Bewley set. Since $d(p', p^*) > \frac{1}{\sqrt{6}}$, then $p' \notin D$.

By a standard separation theorem, there is $f' \in \mathbb{R}^S \setminus \mathbb{R}\mathbf{1}$ such that $E_{p'} f' < \min_{p \in D} E_p f'$. Let $x := \min_{p \in D} E_p f'$. Let $g \in \mathbb{R}^S$ be such that $g(1) < g(2) < g(3)$, $f'(1) + g(1) < f'(2) + g(2) < f'(3) + g(3)$, and $E_{p'}(x\mathbf{1} + g) \leq 0$. Since $E_{p'} f' < x$, then $E_{p'}(\frac{1}{2} f' + \frac{1}{2} g) < E_{p'}(\frac{1}{2} x\mathbf{1} + \frac{1}{2} g) \leq 0$. Hence, $V(\frac{1}{2} f' + \frac{1}{2} g) = E_{p'}(\frac{1}{2} f' + \frac{1}{2} g) < E_{p'}(\frac{1}{2} x\mathbf{1} + \frac{1}{2} g) = V(\frac{1}{2} x\mathbf{1} + \frac{1}{2} g)$. Thus, $\frac{1}{2} x\mathbf{1} + \frac{1}{2} g \succ \frac{1}{2} f' + \frac{1}{2} g$. Suppose that D is the Bewley set. Since for each $p \in D$, $E_p f' \geq \min_{p \in D} E_p f' = x$, then $f' \succsim^* x$. Thus, $\frac{1}{2} f'\mathbf{1} + \frac{1}{2} g \succsim \frac{1}{2} x + \frac{1}{2} g$, which is a contradiction. Therefore, D is not the Bewley set. □

Lemma 2. *Let $\varphi \in \mathbb{R}^S$, $t \in [\varphi_*, \varphi^*]$, and $B := \{p \in \Delta | E_p \varphi = t\}$. Let K be a non-empty closed subset of Δ such that for each $p \in K$, $E_p \varphi \geq t$. Let $c := \min_{p \in B} d(p, K)$. Then, $\min_{p \in \Delta : d(p,K) \leq c} E_p \varphi = t$.*

Proof. Since both B and K are non-empty, closed, and bounded, then there is $\bar{p} \in B$ such that $d(\bar{p}, K) = c$. Since $\bar{p} \in B$, then $E_{\bar{p}} \varphi = t$. Since $d(\bar{p}, K) = c$, then $E_{\bar{p}} \varphi \geq \min_{p \in \Delta : d(p,K) \leq c} E_p \varphi$. Thus, $t \geq \min_{p \in \Delta : d(p,K) \leq c} E_p \varphi$. Suppose that $t > \min_{p \in \Delta : d(p,K) \leq c} E_p \varphi$. Then, there is $p' \in \Delta$ such that $d(p', K) \leq c$ and $t > E_{p'} \varphi$. Let $q' \in K$ be such that $d(p', q') = d(p', K)$. Since $q' \in K$, then $E_{q'} \varphi \geq \min_{p \in K} E_p \varphi$. Since for each $p \in K$, $E_p \varphi \geq t$, then $\min_{p \in K} E_p \varphi \geq t$. Thus, $E_{q'} \varphi \geq t > E_{p'} \varphi$. Hence, there is $\alpha \in [0, 1)$ such that $t = E_{\alpha p' + (1-\alpha) q'} \varphi$, i.e., $\alpha p' + (1 - \alpha) q' \in B$. Since $p' \neq q'$ and $\alpha \in [0, 1)$, then $d(\alpha p' + (1 - \alpha) q', q') < d(p', q')$. Thus,

$$c = \min_{p \in B} d(p, K) \leq d(\alpha p' + (1 - \alpha) q', q') < d(p', q') = d(p', K) \leq c,$$

which is not possible. Hence, $t = \min_{p \in \Delta : d(p,K) \leq c} E_p \varphi$. □

Proof of Theorem 2. Let a preference relation $\succsim$ be given. To show that "only if" direction, suppose that the preference relation $\succsim$ satisfies Axioms A.1 - A.4, A.5', and A.6. By Lemma 1, there exist an affine onto function $u : X \to \mathbb{R}$ and a normalized, monotonic, and continuous functional $I : \mathbb{R}^S \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \iff I(u(f)) \geq I(u(g))$.

Let $K^* := \{p \in \Delta : \text{for each } \varphi \in \mathbb{R}^S, I(\varphi) \leq E_p \varphi\}$. Since I is continuous, K^* is closed. By the definition of K^*, it is convex. We claim that it is non-empty. Since the preference relation $\succsim$ satisfies Axiom A.5', then there is $q \in \Delta$ such that for each $f \in \mathcal{F}$ and each $x \in X$ with $f \succsim x$, $\sum_{s \in S} q_s f(s) \succsim x$. Thus, for each $f \in \mathcal{F}$, $\sum_{s \in S} p_s f(s) \succsim x_f$, or equivalently, $I(u(\sum_{s \in S} q_s f(s))) \geq I(u(x_f))$, which implies $E_q u(f) \geq I(u(f))$ since u is affine and I is normalized. For each $\varphi \in \mathbb{R}^S$, there is $f \in \mathcal{F}$ such that $u(f) = \varphi$, so $E_q \varphi = E_q u(f) \geq I(u(f)) = I(\varphi)$. By the definition of K^*, $q \in K^*$, and thus K^* is non-empty.

Define $B : \mathbb{R}^S \rightrightarrows \Delta$ by setting for each $\varphi \in \mathbb{R}^S$,

$$B(\varphi) = \{p \in \Delta | E_p\varphi = I(\varphi)\}.$$

For each $\varphi \in \mathbb{R}^S$, $B(\varphi)$ is non-empty since I is normalized and monotonic so that $I(\varphi) \in [\varphi_*, \varphi^*]$, and $B(\varphi)$ is closed since the expected value of φ is continuous in p over Δ. Define $\sigma : \mathbb{R}^S \to \mathbb{R}_+$ by setting for each $\varphi \in \mathbb{R}^S$,

$$\sigma(\varphi) = \min_{p \in B(\varphi)} d(p, K^*).$$

By Lemma 2, for each $\varphi \in \mathbb{R}^S$, $I(\varphi) = \min_{p \in \Delta : d(p,K^*) \le \sigma(\varphi)} E_p\varphi$.

We now check the continuity property of σ. Let $f \in \mathcal{F}$ and $\varphi := u(f)$. Let $\{\varphi_n\}_{n=1}^\infty$ be a sequence of elements in $\mathbb{R}^S$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. Consider first the case that $f \nsim x_{*f}$, and we want to show that $\lim_{n\to\infty} \sigma(\varphi_n) = \sigma(\varphi)$. Since $f \nsim x_{*f}$, then $I(\varphi) \neq I(\varphi_* \mathbf{1})$. Since I is normalized and monotonic, then $I(\varphi) > \varphi_*$. First, suppose that $I(\varphi) = \varphi^*$. Let $q^* \in K^*$. By the definition of K^*, $I(\varphi) \le E_{q^*}\varphi$. Since $E_{q^*}\varphi \le \varphi^* = I(\varphi)$, then $I(\varphi) = E_{q^*}\varphi$. Hence, $q^* \in B(\varphi)$, and thus, $\sigma(\varphi) = \min_{p \in B(\varphi)} d(p, K^*) = d(q^*, q^*) = 0$. To show that $\lim_{n\to\infty} \sigma(\varphi_n) = 0$, suppose to the contrary that there exist $\epsilon > 0$ and a subsequence $\{\varphi_{n_m}\}_{m=1}^\infty$ of $\{\varphi_n\}_{n=1}^\infty$ such that for each $m \in \mathbb{N}$, $\sigma(\varphi_{n_m}) > \epsilon$. Let $q_* \in \Delta$ be such that $E_{q_*}\varphi = \varphi_*$. Since $I(\varphi) > \varphi_*$, then $I(\varphi) > E_{q_*}\varphi$, and thus $q_* \notin K^*$. Let $\lambda \in (0, 1)$ be such that $d(\lambda q_* + (1-\lambda)q^*, q^*) < \epsilon$. Since $q^* \in K$, then $d(\lambda q_* + (1-\lambda)q^*, K^*) < \epsilon$. Thus, for each $m \in \mathbb{N}$, $d(\lambda q_* + (1-\lambda)q^*, K^*) < \sigma(\varphi_{n_m})$, and hence $\min_{p \in \Delta : d(p,K^*) \le \sigma(\varphi_{n_m})} E_p\varphi_{n_m} \le E_{\lambda q_* + (1-\lambda)q^*}\varphi_{n_m}$. For each $m \in \mathbb{N}$, by the result in the previous paragraph, $I(\varphi_{n_m}) = \min_{p \in \Delta : d(p,K^*) \le \sigma(\varphi_{n_m})} E_p\varphi_{n_m}$, so $I(\varphi_{n_m}) \le E_{\lambda q_* + (1-\lambda)q^*}\varphi_{n_m}$. Since I is continuous, then $I(\varphi) = \lim_{m\to\infty} I(\varphi_{n_m})$. Since $\lim_{m\to\infty} I(\varphi_{n_m}) \le \lim_{m\to\infty} E_{\lambda q_* + (1-\lambda)q^*}\varphi_{n_m} = \lambda E_{q_*}\varphi + (1-\lambda)E_{q^*}\varphi = \lambda\varphi_* + (1-\lambda)\varphi^* < \varphi^*$, then $I(\varphi) < \varphi^*$, which contradicts our assumption that $I(\varphi) = \varphi^*$. Hence, in this case, $\lim_{n\to\infty} \sigma(\varphi_n) = \sigma(\varphi)$.

Second, suppose that $I(\varphi) < \varphi^*$. By the maximum theorem, it suffices to show that B is a continuous correspondence at φ. To show the upper hemicontinuity of B, let $\bar{p} \in \Delta$ and let $\{p_n\}_{n=1}^\infty$ be a sequence of elements in Δ such that $\lim_{n\to\infty} p^n = \bar{p}$ and for each $n \in \mathbb{N}$, $p^n \in B(\varphi^n)$. Then, $E_{p^n}\varphi^n = I(\varphi^n)$. Since $\lim_{n\to\infty} E_{p^n}\varphi^n = E_{\bar{p}}\varphi$ and since I is continuous, then $E_{\bar{p}}\varphi = I(\varphi)$. Thus, $\bar{p} \in B(\varphi)$, as desired. To show the lower hemicontinuity of B, let $\hat{p} \in B(\varphi)$. Then, $E_{\hat{p}}\varphi = I(\varphi) \in (\varphi_*, \varphi^*)$. For each $\epsilon > 0$, define $A(\epsilon) = \{p \in \Delta | d(p, \hat{p}) \le \epsilon\}$. For each $\epsilon > 0$, since $E_{\hat{p}}\varphi = I(\varphi) \in (\varphi_*, \varphi^*)$, then $\min_{p \in A(\epsilon)} E_p\varphi < I(\varphi) < \max_{p \in A(\epsilon)} E_p\varphi$, and since I is continuous, for sufficiently large n, $\min_{p \in A(\epsilon)} E_p\varphi_n < I(\varphi_n) < \max_{p \in A(\epsilon)} E_p\varphi_n$. Thus, for each $\epsilon > 0$ and for each $n \in \mathbb{N}$ that is sufficiently large, there is $p_n \in A(\epsilon)$ such that $I(\varphi_n) = E_{p_n}\varphi_n$, i.e., $p_n \in B(\varphi_n)$.

Let $\{p_{n_m}\}_{m=1}^{\infty}$ be a subsequence of $\{p_n\}_{n=1}^{\infty}$ such that for each $m \in \mathbb{N}$, $p_{n_m} \in A(\frac{1}{m}) \cap B(\varphi_{n_m})$. Since for each $m \in \mathbb{N}$, $p_{n_m} \in A(\frac{1}{m})$, then $\lim_{m\to\infty} p_{n_m} = \hat{p}$, as desired.

Consider now the case that $f \sim x_{*f}$, and we want to show that $\liminf_{n\to\infty} \sigma(\varphi_n) \geq \sigma(\varphi)$. Since $f \sim x_{*f}$ and I is normalized, then $I(\varphi) = I(\varphi_* \mathbf{1}) = \varphi_*$. Suppose that φ is constant. Then, $B(\varphi) = \Delta$ and $\sigma(\varphi) = \min_{p\in B(\varphi)} d(p, K^*) = 0$. Since $\sigma(\varphi) = 0$ and for each $n \in \mathbb{N}$, $\sigma(\varphi_n) \geq 0$, then $\liminf_{n\to\infty} \sigma(\varphi_n) \geq \sigma(\varphi)$. Suppose that φ is not constant. Suppose to the contrary that there is a convergent subsequence $\{\varphi_{n_m}\}_{m=1}^{\infty}$ of $\{\varphi_n\}_{n=1}^{\infty}$ such that $\lim_{m\to\infty} \sigma(\varphi_{n_m}) < \sigma(\varphi)$. Let $t \in \mathbb{R}$ be such that $\lim_{m\to\infty} \sigma(\varphi_{n_m}) < t < \sigma(\varphi)$. For each $p \in \Delta$ with $d(p, K^*) \leq t$, if $E_p\varphi = I(\varphi)$, then $p \in B(\varphi)$, and hence, by the definition of σ, $\sigma(\varphi) \leq d(p, K^*)$, so that $\sigma(\varphi) \leq t < \sigma(\varphi)$ which is not possible. Thus, for each $p \in \Delta$ with $d(p, K^*) \leq t$, $E_p\varphi \neq I(\varphi)$, and since $E_p\varphi \geq \varphi_* = I(\varphi)$, then $E_p\varphi > I(\varphi)$. Therefore, $\min_{p\in\Delta: d(p,K^*)\leq t} E_p\varphi > I(\varphi)$. Since $\lim_{m\to\infty} \sigma(\varphi_{n_m}) < t$, then for sufficiently large m, $\min_{p\in\Delta: d(p,K^*)\leq \sigma(\varphi_{n_m})} E_p\varphi_{n_m} \geq \min_{p\in\Delta: d(p,K^*)\leq t} E_p\varphi_{n_m}$. Since I is continuous, and since for each $m \in \mathbb{N}$, $I(\varphi_{n_m}) = \min_{p\in\Delta: d(p,K^*)\leq \sigma(\varphi_{n_m})} E_p\varphi_{n_m}$, then

$$I(\varphi) = \lim_{m\to\infty} I(\varphi_{n_m}) = \lim_{m\to\infty} \min_{p\in\Delta: d(p,K^*)\leq \sigma(\varphi_{n_m})} E_p\varphi_{n_m} \geq \lim_{m\to\infty} \min_{p\in\Delta: d(p,K^*)\leq t} E_p\varphi_{n_m} = \min_{p\in\Delta: d(p,K^*)\leq t} E_p\varphi,$$

which contradicts that $\min_{p\in\Delta: d(p,K^*)\leq t} E_p\varphi > I(\varphi)$.

Since u, K^*, and σ satisfy the requirements in the definition of a variant constraint representation, and since for each pair $f, g \in \mathcal{F}$,

$$f \succsim g \iff \min_{p\in\Delta: d(p,K^*)\leq \sigma(u(f))} E_p u(f) = I(u(f)) \geq I(u(g)) = \min_{p\in\Delta: d(p,K^*)\leq \sigma(u(g))} E_p u(g),$$

then $\langle u, K^*, \sigma\rangle$ is a variant constraint representation of the preference relation $\succsim$. Moreover, since I is monotonic, $\sigma \in \Sigma(K^*)$.

We complete the proof of the "only if" direction by showing that K^* is the largest essential set, so that $\langle u, K^*, \sigma\rangle$ is a canonical variant constraint representation of the preference relation $\succsim$. Let $\langle u', K', \sigma'\rangle$ be another variant constraint representation of the preference relation $\succsim$. We want to show that $K' \subseteq K^*$. Since u, u' are both affine functions representing the preference relation restricted to X, by a routine argument, u' is a positive affine transformation of u. Let $p' \in K'$ and $\varphi \in \mathbb{R}^S$. There is $f \in \mathcal{F}$ such that $u(f) = \varphi$. Since $\langle u', K', \sigma'\rangle$ is a variant constraint representation of the preference relation $\succsim$, then $u'(x_f) = \min_{p\in\Delta: d(p,K')\leq \sigma'(u'(f))} E_p u'(f)$. Since $p' \in K'$, then $\min_{p\in\Delta: d(p,K')\leq \sigma'(u'(f))} E_p u'(f) \leq E_{p'} u'(f)$. Thus, $u'(x_f) \geq E_{p'} u'(f)$. Since u' is a positive affine transformation of u, then $u(x_f) \geq E_{p'} u(f)$, and thus $I(\varphi) \geq E_{p'}\varphi$. By the definition of K^*, $p' \in K^*$, as desired.

To show the "if" direction, suppose that the preference relation $\succsim$ admits a canonical variant constraint representation $\langle u, K, \sigma \rangle$ with $\sigma \in \Sigma(K)$. It can be readily seen that the preference relation satisfies Axioms A.1, A.2, A.4, and A.6.

We show that the preference relation satisfies Axiom A.3. Define $C : \mathbb{R}^S \rightrightarrows \Delta$ by setting for each $\varphi \in \mathbb{R}^S$,

$$C(\varphi) = \{p \in \Delta | d(p, K) \leq \sigma(\varphi)\}.$$

Define $I : \mathbb{R}^S \to \mathbb{R}$ by setting for each $\varphi \in \mathbb{R}^S$,

$$I(\varphi) = \min_{p \in C(\varphi)} E_p \varphi.$$

It suffices to show that I is continuous. Fix $\varphi \in \mathbb{R}^S$ and a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of elements in $\mathbb{R}^S$ such that $\lim_{n\to\infty} \varphi_n = \varphi$. If φ is constant, then $\lim_{n\to\infty} |I(\varphi_n) - I(\varphi)| \leq \lim_{n\to\infty} \sup_{s\in S} |\varphi_n(s) - \varphi(s)| = 0$, and thus $\lim_{n\to\infty} I(\varphi_n) = I(\varphi)$.

Suppose that φ is not constant and $I(\varphi) > \varphi_*$. Let $f \in \mathcal{F}$ be such that $u(f) = \varphi$. Since $I(\varphi) > \varphi_*$, then $f \succ x_{*f}$, and thus σ is continuous at φ. By the maximum theorem, to show that I is continuous at φ, it suffices to show that C is continuous at φ. To show the upper hemicontinuity, let $p \in \Delta$ and $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements in Δ such that $\lim_{n\to\infty} p_n = p$ and for each $n \in \mathbb{N}$, $p_n \in C(\varphi_n)$. We want to show that $p \in C(\varphi)$. For each $n \in \mathbb{N}$, since $p_n \in C(\varphi_n)$, then $d(p_n, K) \leq \sigma(\varphi_n)$. Since $d(\cdot, K)$ is continuous on Δ and σ is continuous at φ, then $d(p, K) = \lim_{n\to} d(p_n, K) \leq \lim_{n\to\infty} \sigma(\varphi_n) = \sigma(\varphi)$. Hence, $p \in C(\varphi)$. To show the lower hemicontinuity, let $p \in C(\varphi)$ and let $q \in K$ be such that $d(p, q) = d(p, K)$. For each $n \in \mathbb{N}$, if $d(p, q) \leq \sigma(\varphi_n)$, let $p_n := p$, and if $\sigma(\varphi_n) < d(p, q)$, let $\epsilon_n \in [0, 1]$ be such that $d(\epsilon_n q + (1 - \epsilon_n)p, q) = \sigma(\varphi_n)$ and let $p_n := \epsilon_n q + (1 - \epsilon_n)p$. Thus, for each $n \in \mathbb{N}$, $d(p_n, K) \leq d(p_n, q) \leq \sigma(\varphi_n)$, and thus $p_n \in C(\varphi_n)$. We want to show that $\lim_{n\to\infty} p_n = p$. For each $\epsilon > 0$, let $m \in \mathbb{N}$ be such that $|\sigma(\varphi) - \sigma(\varphi_n)| < \epsilon$. Then, for each $n \in \mathbb{N}$ such that $n \geq m$, either $p_n = p$, or $p_n = \epsilon_n q + (1 - \epsilon_n)p$ and $\sigma(\varphi_n) = d(p_n, q) < d(p, q) < \sigma(\varphi_n) + \epsilon$, so that $d(p_n, p) < \epsilon$ in either case. Hence, $\lim_{n\to\infty} p_n = p$.

Suppose that φ is not constant and $I(\varphi) = \varphi_*$. Let $f \in \mathcal{F}$ be such that $u(f) = \varphi$. Since $I(\varphi) = \varphi_*$, then $f \sim x_{*f}$, and thus σ is lower semicontinuous at φ. To show that I is continuous at φ, suppose to the contrary that there exist $\epsilon > 0$ and a subsequence $\{\varphi_{n_m}\}_{m=1}^{\infty}$ of $\{\varphi_n\}_{n=1}^{\infty}$ such that for each $m \in \mathbb{N}$, $|I(\varphi_{n_m}) - I(\varphi)| > \epsilon$. Since $\lim_{m\to\infty} \varphi_{n_m *} = \varphi_* = I(\varphi)$ and I is monotonic, then when m is sufficiently large, $I(\varphi_{n_m}) - \varphi_{n_m *} > \frac{\epsilon}{2}$. Thus, there is $\lambda \in (0, 1)$ such that for sufficiently large m, $\lambda(\varphi^*_{n_m} - \varphi_{n_m *}) < \frac{\epsilon}{2} < I(\varphi_{n_m}) - \varphi_{n_m *}$, so that $(1 - \lambda)\varphi_{n_m *} + \lambda \varphi^*_{n_m} <$

$I(\varphi_{n_m})$. For each $m \in \mathbb{N}$, let $B_m := \{p \in \Delta | E_p\varphi_{n_m} = (1-\lambda)\varphi_{n_m *} + \lambda\varphi^*_{n_m}\}$, and let $B_\infty := \{p \in \Delta | E_p\varphi = (1-\lambda)\varphi_* + \lambda\varphi^*\}$. Note that for each $m \in \mathbb{N}$, B_m is non-empty and closed, and so is B_∞. It can be readily shown that $\lim_{m\to\infty} \min_{p\in B_m} d(p,K) = \min_{p\in B_\infty} d(p,K)$. Let $c := \min_{p\in B_\infty} d(p,K)$. When m is sufficiently large, for each $p \in B_m$, $E_p\varphi_{n_m} = (1-\lambda)\varphi_{n_m *} + \lambda\varphi^*_{n_m} < I(\varphi_{n_m})$, so by the definition of I, $p \notin C(\varphi_{n_m})$, and by the definition of C, $d(p,K) > \sigma(\varphi_{n_m})$. Thus, for sufficiently large m, $\min_{p\in B_m} d(p,K) > \sigma(\varphi_{n_m})$. Therefore, $c = \lim_{m\to\infty} \min_{p\in B_m} d(p,K) \geq \liminf_{n\to\infty} \sigma(\varphi_n)$. Since σ is lower semicontinuous at φ, then $\liminf_{n\to\infty} \sigma(\varphi_n) \geq \sigma(\varphi)$. Thus, $c \geq \sigma(\varphi)$, so that $\min_{p\in\Delta:d(p,K)\leq c} E_p\varphi \leq \min_{p\in C(\varphi)} E_p\varphi = I(\varphi) = \varphi_* < (1-\lambda)\varphi_* + \lambda\varphi^*$. On the other hand, for each $p \in K$ and each $m \in \mathbb{N}$, since $p \in C(\varphi_{n_m})$, then $I(\varphi_{n_m}) \leq E_p\varphi_{n_m}$, and when m is sufficiently large, since $(1-\lambda)\varphi_{n_m *} + \lambda\varphi^*_{n_m} < I(\varphi_{n_m})$, then $(1-\lambda)\varphi_{n_m *} + \lambda\varphi^*_{n_m} < E_p\varphi_{n_m}$. Therefore, for each $p \in K$, $E_p\varphi = \lim_{m\to\infty} E_p\varphi_{n_m} \geq \lim_{m\to\infty}(1-\lambda)\varphi_{n_m *} + \lambda\varphi^*_{n_m} = (1-\lambda)\varphi_* + \lambda\varphi^*$. By lemma 2, $\min_{p\in\Delta:d(p,K)\leq c} E_p\varphi = (1-\lambda)\varphi_* + \lambda\varphi^*$, which contradicts that $\min_{p\in\Delta:d(p,K)\leq c} E_p\varphi < (1-\lambda)\varphi_* + \lambda\varphi^*$.

To show that the preference relation $\succsim$ satisfies Axiom A.5', let $q \in K$. Let $f \in \mathcal{F}$ and $x \in X$ be such that $f \succsim x$. Thus, $E_q u(f) \geq \min_{p\in\Delta:d(p,K)\leq\sigma(u(f))} E_p u(f) \geq u(x)$. Hence, $\sum_{s\in S} q_s f(s) \succsim x$, as desired.

Lastly, we check the uniqueness of the representation. By a routine arguement, u is unique up to a positive affine transformation. By its definition, the largest essential set K^* is unique. Given u, to show the uniqueness of the constraint function, let $\langle u, K^*, \sigma' \rangle$ be another variant constraint representation of the preference relation $\succsim$. Let $f \in \mathcal{F}$ be such that $f \nsim x_{*f}$, and let $\varphi := u(f)$. We want to show that $\sigma(\varphi) = \sigma'(\varphi)$. Since $f \nsim x_{*f}$, then $I(\varphi) \neq I(\varphi_* \mathbf{1})$. Since I is normalized and monotonic, then $I(\varphi) > \varphi_*$. Let $p_*, q, q' \in \Delta$ be such that $E_{p_*}\varphi = \varphi_*$, $q \in \arg\min_{p\in\Delta:d(p,K^*)\leq\sigma(\varphi)} E_p\varphi$, and $q' \in \arg\min_{p\in\Delta:d(p,K^*)\leq\sigma'(\varphi)} E_p\varphi$. Since $I(\varphi) = \min_{p\in\Delta:d(p,K^*)\leq\sigma(\varphi)} E_p\varphi$, then $I(\varphi) = E_q\varphi$. Since $I(\varphi) = I(u(f)) = I(u(x_f)) = u(x_f)$, and since $u(x_f) = \min_{p\in\Delta:d(p,K^*)\leq\sigma'(\varphi)} E_p\varphi = E_{q'}\varphi$, then $I(\varphi) = E_{q'}\varphi$. Suppose that $\sigma(\varphi) < \sigma'(\varphi)$. Then, $d(q, K^*) < \sigma'(\varphi)$, and thus, there is $\epsilon \in (0,1)$ such that $d(\epsilon p_* + (1-\epsilon)q, K^*) \leq \sigma'(\varphi)$. Hence,

$$I(\varphi) = E_{q'}\varphi = \min_{p\in\Delta:d(p,K^*)\leq\sigma'(\varphi)} E_p\varphi \leq E_{\epsilon p_* + (1-\epsilon)q}\varphi = \epsilon\varphi_* + (1-\epsilon)I(\varphi) < I(\varphi),$$

which is not possible. Suppose that $\sigma(\varphi) > \sigma'(\varphi)$. Then, $d(q', K^*) < \sigma(\varphi)$. Since $E_{q'}\varphi = I(\varphi)$, then $q' \in B(\varphi)$. By the definition of σ, $\sigma(\varphi) \leq d(q', K^*)$, which contradicts that $d(q', K^*) < \sigma(\varphi)$. Therefore, $\sigma(\varphi) = \sigma'(\varphi)$. □

Proof of Proposition 2. Suppose that a preference relation $\succsim$ admits both a canonical weighted maxmin representation $\langle u, D, \lambda \rangle$ and a canonical variant constraint representation

$\langle u', K, \sigma \rangle$. To show that $K \subseteq D$, suppose to the contrary that there is $\bar{p} \in K \setminus D$. Since $\bar{p} \notin D$, by a standard separation theorem, there is $\varphi \in \mathbb{R}^S$ such that $E_{\bar{p}}\varphi < \min_{p \in D} E_p \varphi$. Let $f \in \mathcal{F}$ and $x \in X$ be such that $u(f) = \varphi$ and $u(x) = \min_{p \in D} E_p \varphi$. Since $\langle u, D, \lambda \rangle$ is a weighted maxmin representation of the preference relation $\succsim$, and since

$$u(x) = \min_{p \in D} E_p \varphi \leq \lambda(\varphi) \min_{p \in D} E_p \varphi + (1 - \lambda(\varphi)) \max_{p \in D} E_p \varphi,$$

then $f \succsim x$. Since $\bar{p} \in K$, then $E_{\bar{p}} u'(f) \geq \min_{p \in \Delta : d(p,K) \leq \sigma(u'(f))} E_p u'(f)$. Since both u and u' are affine functions that represent the preference relation $\succsim$ restricted to X, by a routine argument, u' is a positive affine transformation of u. Thus, $u'(x) = \min_{p \in D} E_p u'(f) > E_{\bar{p}} u'(f)$. Hence, $u'(x) > \min_{p \in \Delta : d(p,K) \leq \sigma(u'(f))} E_p u'(f)$. Since $\langle u', K, \sigma \rangle$ is a variant constraint representation of the preference relation $\succsim$, then $x \succ f$, which contradicts that $f \succsim x$, as desired. □

Proof of Proposition 3. Let a preference relation $\succsim$ be given. To show the "only if" direction, suppose that the preference relation is a MEU preference relation, that is, there exist an affine onto function $u : \mathbb{R}^S \to \infty$ and a non-empty closed convex set $D \subseteq \Delta$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \iff \min_{p \in D} E_p u(f) \geq \min_{p \in D} E_p u(g)$. It can be readily seen that the preference relation satisfies Axioms A.1 -A.6, and A.5'. By Theorem 1, it admits a canonical weighted maxmin representation and the smallest admissible set is the Bewley set. By Theorem 2, it admits a canonical variant constraint representation and the largest essential set is the Bewley set. By Ghirardato, Maccheroni, and Marinaccia (2004),[25] the preference relation admits a Bewley weighted maxmin representation and D is the Bewley set. Let $\lambda : \mathbb{R}^S \to [0, 1]$ be such that for each $\varphi \in \mathbb{R}^S$, $\lambda(\varphi) = 1$. Let $\sigma : \mathbb{R}^S \to \mathbb{R}_+$

To show the "if" direction, suppose that the preference relation admits both a canonical weighted maxmin representation $\langle u, D, \lambda \rangle$ and a canonical variant constraint representation $\langle u', K, \sigma \rangle$, and $K = D$. To show that the preference relation $\succsim$ is a MEU preference relation, it suffices to show that for each $f \in \mathcal{F}$, $\lambda(u(f)) = 1$. Fix $f \in \mathcal{F}$. Since $\langle u', K, \sigma \rangle$ is a variant constraint representation of the preference relation $\succsim$, then $u'(x_f) = \min_{p \in \Delta : d(p,K) \leq \sigma(u'(f))} E_p u'(f)$. Since u' is a positive affine transformation of u, then $u(x_f) = \min_{p \in \Delta : d(p,K) \leq \sigma(u'(f))} E_p u(f)$. Since $\langle u, D, \lambda \rangle$ is a weighted maxmin representation of the preference relation $\succsim$ and $K = D$, then $u(x_f) = \lambda(u(f)) \min_{p \in D} E_p u(f) + (1 - \lambda(u(f))) \max_{p \in D} E_p u(f) = \lambda(u(f)) \min_{p \in K} E_p u(f) + (1 -$

[25] See Theorem 11 and the first paragraph on Page 151 of their paper.

$\lambda(u(f))) \max_{p\in K} E_p u(f)$. Thus,

$$\begin{aligned}
&\min_{p\in\Delta: d(p,K)\leq\sigma(u'(f))} E_p u(f) = u(x_f) \\
=&\lambda(u(f)) \min_{p\in K} E_p u(f) + (1-\lambda(u(f))) \max_{p\in K} E_p u(f) \\
\geq& \min_{p\in K} E_p u(f) \geq \min_{p\in\Delta: d(p,K)\leq\sigma(u'(f))} E_p u(f).
\end{aligned}$$

Hence, $\lambda(u(f)) \min_{p\in K} E_p u(f) + (1-\lambda(u(f))) \max_{p\in K} E_p u(f) = \min_{p\in K} E_p u(f)$, and thus $\lambda(u(f)) = 1$, as desired. □

Proof of Corollary 2. Suppose that a preference relation $\succsim$ satisfies Axioms A.1 -A.4, A.5.1, A.5.2, and A.6. By Theorem 2, the preference relation admits a canonical variant constraint representation $\langle u, K, \sigma\rangle$. It suffices to show that $K = K^*$.

By Theorems 3 and 5 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), the preference relation admits an ambiguity averse representation $\langle u, G\rangle$. By Proposition 11 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), $K^* = \cap_{x\in X}\pi(x)$ where $\pi(\cdot)$ is defined in (18). Thus, $p \in K^*$ if and only if for each $x \in X$ and each $f \in \mathcal{F}$, $x \succsim \sum_{s\in S} p_s f(s) \implies x \succsim f$. For each $p \in \Delta$, the condition that for each $x \in X$ and $f \in \mathcal{F}$, $x \geq_p f \implies x \succsim f$ implies that for each pair $x, y \in X$, $x \geq_p y \iff x \succsim y$. Thus, for each $p \in \Delta$, the preference relation $\succsim$ is more ambiguity averse than the SEU preference relation $\geq_p$ if and only if for each $x \in X$ and $f \in \mathcal{F}$, $x \succsim \sum_{s\in S} p_s f(s) \implies x \succsim f$. Hence, $p \in K^*$ if and only if the preference relation $\succsim$ is more ambiguity averse than the SEU preference relation $\geq_p$. By Theorem 2, $K = K^*$. □

Proof of Proposition 4. Suppose that a preference relation admits a multiplier representation $\langle u, q, \theta\rangle$. Then the preference admits an ambiguity averse representation $\langle u, G\rangle$ such that for each $(t, p) \in \mathbb{R} \times \Delta$, $G(t, p) = t + \theta R(p\|q)$. By Theorems 3 and 5 of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), the preference relation $\succsim$ satisfies Axioms A.1 - A.4, A.5.1, and A.6. It can be readily seen that the preference relation $\succsim$ is more ambiguity averse than the SEU preference relation $\geq_q$, so that it satisfies Axiom A.5.2. By Theorems 1 and 2, the preference relation $\succsim$ admits both a canonical weighted maxmin representation and a canonical variant constraint representation.

If $\theta = \infty$, then clearly $D^* = K^* = \{q\}$, and thus the result naturally follows. Suppose that $\theta < \infty$. Then it can be readily seen that $D^* = \{p \in \Delta : p \ll q\}$ and $K^* = \{q\}$. Thus

$\overline{\varphi} = \max_{p \in D^*} E_p \varphi$ and $\underline{\varphi} = \min_{p \in D^*} E_p \varphi$. By the following variational formula (see, e.g., Proposition 1.4.2 of Dupuis and Ellis (1997))

$$\min_{p \in \Delta} [E_p \varphi + \theta R(p \| q)] = -\theta \log E_q e^{-\frac{\varphi}{\theta}} \tag{29}$$

and the construction of λ and σ in the proofs of Theorems 1 and 2, we can get the desired results. □

6.2 Proofs in Section 4

Lemma 3. *A preference relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A.1, A.2.1, A.3 - A.6 if and only if there exists an affine onto function $u : X \to \mathbb{R}$ and a normalized, weakly increasing, quasi-concave, continuous and constant superadditive functional $I : \mathbb{R}^S \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \Leftrightarrow I(u(f)) \geq I(u(g))$. Moreover, u is unique up to a positive affine transformation, and given u, there is a unique normalized functional $I : \mathbb{R}^S \to \mathbb{R}$ such that for each pair $f, g \in \mathcal{F}$, $f \succsim g \Leftrightarrow I(u(f)) \geq I(u(g))$.*

Proof. The "if" direction is easy. For the "only if" direction, the existence and uniqueness of the required u and I follow from Lemma 1, except the constant superadditivity of I. We now check this property.

Let $\varphi \in \mathbb{R}^S$ and $t \in \mathbb{R}_+$ be arbitrarily given. Let $x, x_0 \in X$ and $f \in \mathcal{F}$ be such that $u(x) = 2t$, $u(x_0) = 0$ and $u(f) = 2\varphi$. Then $u(\frac{1}{2}f + \frac{1}{2}x) = \varphi + t\mathbf{1}$ and $u(\frac{1}{2}f + \frac{1}{2}x_0) = \varphi$. Since u is an affine onto function, then there exists $z \in X$ such that $\frac{1}{2}f + \frac{1}{2}x_0 \sim \frac{1}{2}z + \frac{1}{2}x_0$ for some $z \in X$. Since $t \in \mathbb{R}_+$, then $x \succsim x_0$. By Axiom A.2, we know that $\frac{1}{2}f + \frac{1}{2}x \succsim \frac{1}{2}z + \frac{1}{2}x$. Thus,

$$\begin{aligned}
& I(\varphi + t\mathbf{1}) = I(u(\frac{1}{2}f + \frac{1}{2}x)) \geq I(u(\frac{1}{2}z + \frac{1}{2}x)) \\
& = \frac{1}{2}u(z) + \frac{1}{2}u(x) = \frac{1}{2}u(z) + \frac{1}{2}u(x_0) + \frac{1}{2}u(x) = u(\frac{1}{2}z + \frac{1}{2}x_0) + t \\
& = I(u(\frac{1}{2}f + \frac{1}{2}x_0)) + t = I(\varphi) + t
\end{aligned}$$

as desired.

□

Proof of Theorem 3. We shall only check that statement 1 implies statement 2, 3 and 4. The necessity of A.2.1 for statement 2, 3 and 4 is easy to check. The necessity of A.1, A.3, A.4, A.5.1, and A.6 for statement 2 can be readily verified, and that for statement 3 and 4 follows respectively from the proof of Corollary 2 and Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011)'s characterization result.

Suppose that statement 1 holds. We shall first prove statement 2. Let $u : X \to \mathbb{R}$ and $I : \mathbb{R}^S \to \mathbb{R}$ be given as in Lemma 3. We define $J : \mathbb{R}^S \to \mathbb{R}$ by for each $\varphi \in \mathbb{R}^S$,

$$J(\varphi) = \lim_{k\to\infty} \lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]. \tag{30}$$

We check the following properties of J.

Let $\varphi \in \mathbb{R}^S$ be arbitrarily given. First, for each $k > 0$ and each $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]$ is bounded in $[\min_S \varphi(s), \max_S \varphi(s)]$. This is because I is normalized and weakly increasing. Second, for each $k > 0$, $\frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]$ weakly decreases in t on $\mathbb{R}$ since I is constant superadditive. Third, $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]$ weakly decreases in k on $(0, \infty)$. To see it, suppose the contrary that $k' > k > 0$ and $\lim_{t\to\infty} \frac{1}{k'}[I(k'\varphi - t\mathbf{1}) + t] > \lim_{t\to\infty} \frac{1}{k}[I(k\varphi - t\mathbf{1}) + t]$. Hence, there exists $\bar{t}$ such that for each pair $t, t' \geq \bar{t}$, $\frac{1}{k}[I(k\varphi - t\mathbf{1}) + t] > \frac{1}{k'}[I(k'\varphi - t'\mathbf{1}) + t']$. That is,

$$I(k\varphi - t\mathbf{1}) < \frac{k}{k'}I(k'\varphi - t'\mathbf{1}) - t + \frac{k}{k'}t'. \tag{31}$$

Pick $t, t' \geq \bar{t}$ such that $\frac{k}{k'}(-t') + (1 - \frac{k}{k'})I(k'\varphi - t'\mathbf{1}) = -t$. Thus, $k\varphi - t\mathbf{1} = \frac{k}{k'}(k'\varphi - t'\mathbf{1}) + (1 - \frac{k}{k'})I(k'\varphi - t'\mathbf{1})\mathbf{1}$. since I is normalized and quasi-concave, and by the choice of t, t', we have

$$I(k\varphi - t\mathbf{1}) \geq I(k'\varphi - t'\mathbf{1}) = \frac{k}{k'}I(k'\varphi - t'\mathbf{1}) - t + \frac{k}{k'}t', \tag{32}$$

which contradicts (31) as desired.

Thus it can be readily verifiable that for each $\varphi \in \mathbb{R}^S$, $I(\varphi) \leq J(\varphi)$, and that J is normalized, weakly increasing, constant additive and positive homogeneous of degree 1. We now check that J is superadditive. Suppose the contrary that there exist $\varphi, \varphi' \in \mathbb{R}^S$ such that $J(\varphi + \varphi') < J(\varphi) + J(\varphi')$. Since J is positive homogeneous of degree 1, then $J(\frac{1}{2}\varphi + \frac{1}{2}\varphi') < \frac{1}{2}J(\varphi) + \frac{1}{2}J(\varphi')$. Thus, there exist $k > 0$ and $\bar{t}$ such that for each pair $t, t' \in \mathbb{R}$ and each $t'' \geq \bar{t}$,

$$\frac{1}{k}[I(k(\frac{1}{2}\varphi + \frac{1}{2}\varphi') - t''\mathbf{1}) + t''] < \frac{1}{2k}[I(k\varphi - t\mathbf{1}) + t] + \frac{1}{2k}[I(k\varphi' - t'\mathbf{1}) + t'],$$

i.e.,

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbf{1}) < \frac{1}{2}I(k\varphi - t\mathbf{1}) + \frac{1}{2}I(k\varphi' - t'\mathbf{1}) - t'' + \frac{t + t'}{2}. \tag{33}$$

Pick $t, t' \geq \bar{t}$ such that $I(k\varphi - t\mathbf{1}) = I(k\varphi' - t'\mathbf{1})$. Let $t'' = \frac{t+t'}{2}$ so that $t'' \geq \bar{t}$ and $\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbf{1} = \frac{1}{2}(k\varphi - t\mathbf{1}) + \frac{1}{2}(k\varphi' - t'\mathbf{1})$. Since I is quasi-concave, then

$$I(\frac{1}{2}k\varphi + \frac{1}{2}k\varphi' - t''\mathbf{1}) \geq \frac{1}{2}I(k\varphi - t\mathbf{1}) + \frac{1}{2}I(k\varphi' - t'\mathbf{1})$$

which contradicts (33) as desired.

Then by Gilboa and Schmeidler (1989)[26], there exists a unique non-empty closed convex set $D \subseteq \Delta$ such that $J(\varphi) = \min_{p\in D} E_p\varphi$ for all $\varphi \in \mathbb{R}^S$. Fix $\varphi \in \mathbb{R}^S$. We shall show that $I(\varphi) \in [\min_{p\in D} E_p\varphi, \max_{p\in D} E_p\varphi]$. Since $I(\varphi) \geq J(\varphi)$, thus $I(\varphi) \geq \min_{p\in D} E_p\varphi$. For the upper bound, let $t \in \mathbb{R}$ be given such that $I(\varphi) = I(-\varphi + t\mathbf{1})$. Since I is quasi-concave, then $I(\frac{\varphi}{2} + \frac{-\varphi+t\mathbf{1}}{2}) \geq \frac{1}{2}I(\varphi) + \frac{1}{2}I(-\varphi + t\mathbf{1}) \geq \frac{1}{2}I(\varphi) + \frac{1}{2}\min_{p\in D} E_p(-\varphi) + \frac{t}{2}$. Since I is normalized, then $\frac{t}{2} \geq \frac{1}{2}I(\varphi) + \frac{1}{2}\min_{p\in D} E_p(-\varphi) + \frac{t}{2}$. Thus, $I(\varphi) \leq -\min_{p\in D} E_p(-\varphi) = \max_{p\in D} E_p\varphi$.

Define $\lambda : \mathbb{R}^S \to [0, 1]$ by

$$\lambda(\varphi) = \begin{cases} 1 & \text{if } l(\varphi; D) = 0, \\ \frac{\max_{p\in D} E_p\varphi - I(\varphi)}{l(\varphi;D)} & \text{otherwise.} \end{cases}$$

Thus, for each $\varphi \in \mathbb{R}^S$, $I(\varphi) = \lambda(\varphi)\min_{p\in D} E_p\varphi + (1-\lambda(\varphi))\max_{p\in D} E_p\varphi$. By the properties of I, it can be readily verified that for each $\varphi \in \mathbb{R}^S$, $\lambda(\varphi+t\mathbf{1})$ weakly decreasing in t, $\lim_{t\to\infty}\lambda(k\varphi-t\mathbf{1})$ weakly increases in k on $(0, \infty)$, $\lim_{k\to\infty}\lim_{t\to\infty}\lambda(k\varphi - t\mathbf{1}) = 1$, and λ is continuous on $\{\varphi \in \mathbb{R}^S | l(\varphi; D) > 0\}$.

To show that D is the smallest admissible set, by Corollary 1, it suffices to show that $D \subseteq D^*$. Let $\langle u, G\rangle$ be an ambiguity averse representation of the preference relation $\succsim$. Let $D^\circ := \{p \in \Delta | G(t, p) < \infty$ for some $t \in \mathbb{R}\}$. Let $\varphi, \varphi' \in \mathbb{R}^S$ and $f, f' \in \mathcal{F}$ be given such that $\varphi = u(f)$, $\varphi' = u(f')$, and for each $p \in D^\circ$, $E_p\varphi \geq E_p\varphi'$. Since D° is convex, by Proposition A.1 of Ghirardato, Maccheroni and Marinacci (2004), as long as we show that for each $p \in D$, $E_p\varphi \geq E_p\varphi'$, we have that $D \subseteq cl(D^\circ) = D^*$.

Fix arbitrarily $\alpha \in [0, 1]$ and $h \in \mathcal{F}$. For each $k > 0$ and $t \in \mathbb{R}$, let $g_{k,t}, g'_{k,t} \in \mathcal{F}$ be such that $u(g_{k,t}) = ku(\alpha f + (1 - \alpha)h) - t\mathbf{1}$ and $u(g'_{k,t}) = ku(\alpha f' + (1 - \alpha)h) - t\mathbf{1}$. Then, for each $k > 0$, each $t \in \mathbb{R}$, and each $p \in \cup_{t\in\mathbb{R}}\text{dom } c_t$, $E_pu(g_{k,t}) \geq E_pu(g'_{k,t})$, and thus $\inf_{p\in\Delta} G(E_pu(g_{k,t}), p) \geq \inf_{p\in\Delta} G(E_pu(g'_{k,t}), p)$. That is, for each $k > 0$ and each $t \in \mathbb{R}$, $I(ku(\alpha f + (1 - \alpha)h) - t\mathbf{1}) \geq I(ku(\alpha f' + (1 - \alpha)h) - t\mathbf{1})$. Hence, $\lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}[I(ku(\alpha f + (1 - \alpha)h) - t\mathbf{1}) + t] \geq \lim_{k\to\infty}\lim_{t\to\infty}\frac{1}{k}[I(ku(\alpha f' + (1 - \alpha)h) - t\mathbf{1}) + t]$. By the proof of Theorem 3, we have

[26]See their Lemma 3.5.

$\min_{p\in D} E_p u(\alpha f+(1-\alpha)h) \geq \min_{p\in D} E_p u(\alpha f'+(1-\alpha)h)$. By Ghirardato, Maccheroni and Marinacci (2004) (p.151), for each $p \in D$, $E_p\varphi \geq E_p\varphi'$, as desired.

We proceed to prove statement 1 implies statement 3. Define $J' : \mathbb{R}^S \to \mathbb{R}$ by for each $\varphi \in \mathbb{R}^S$,

$$J'(\varphi) = \lim_{k\searrow 0}\lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]. \tag{34}$$

By a similar argument as before, one can show that (1) for each $k > 0$ and each $t \in \mathbb{R}$, $\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$ is bounded in $[\min_S \varphi(s), \max_S \varphi(s)]$, (2) for each $k > 0$, $\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$ weakly increases in t, and (3) $\lim_{t\to\infty} \frac{1}{k}[I(k\varphi+t\mathbf{1})-t]$ weakly decreases in k on $(0,\infty)$. Hence, J' is well-defined and for each $\varphi \in \mathbb{R}^S$, $I(\varphi) \leq J'(\varphi)$. It can also be analogously verified that J' is normalized, weakly increasing, constant additive, positive homogeneous of degree 1 and superadditive.

Again by Gilboa and Schmeidler (1989)'s Lemma 3.5, there exists a non-empty closed convex set $K \subseteq \Delta$ such that for each $\varphi \in \mathbb{R}^S$, $J'(\varphi) = \min_K E_p\varphi$. Define $B : \mathbb{R}^S \rightrightarrows \Delta$ by $B(\varphi) = \{p \in \Delta | I(\varphi) = E_p\varphi\}$, and define $\sigma : \mathbb{R}^S \to \mathbb{R}_+$ by $\sigma(\varphi) = d(B(\varphi), K)$. Since for each $\varphi \in \mathbb{R}^S$, $I(\varphi) \leq \min_K E_p\varphi$, then by the same argument as in the proof of Theorem 2, one can show that for each $\varphi \in \mathbb{R}^S$, $I(\varphi) = \min_{p\in\Delta: d(p,K)\leq\sigma(\varphi)} E_p\varphi$.

Now we turn to the properties of σ. First, σ is continuous on $\{u(f) \in \mathbb{R}^S | f \in \mathcal{F} \setminus \mathcal{F}_*\}$ and lower semicontinuous on $\{u(f) \in \mathbb{R}^S | f \in \mathcal{F}_*\}$. The proof utilizes the continuity of I and is the same as in the proof of Theorem 2.

Second, for each $\varphi \in \mathbb{R}^S$, $\sigma(\varphi + t\mathbf{1})$ weakly decreases in t. To see it, let $\varphi \in \mathbb{R}^S$ and $t \leq t'$ in $\mathbb{R}$ be given. Let $p \in B(\varphi + t\mathbf{1})$ and $q \in K$ be such that $\sigma(\varphi + t\mathbf{1}) = d(p, q)$. Since I is constant superadditive, then $E_p(\varphi + t'\mathbf{1}) \leq I(\varphi + t'\mathbf{1})$. If $E_p(\varphi + t'\mathbf{1}) < I(\varphi + t'\mathbf{1})$, then $\sigma(\varphi + t\mathbf{1}) = d(p, q) > \sigma(\varphi + t'\mathbf{1})$, otherwise $I(\varphi + t'\mathbf{1}) = \min_{p'\in\Delta: d(p',K)\leq\sigma(\varphi+t'\mathbf{1})} E_{p'}(\varphi + t'\mathbf{1}) \leq E_p(\varphi+t'\mathbf{1}) < I(\varphi+t'\mathbf{1})$, which is a contradiction. If $E_p(\varphi+t'\mathbf{1}) = I(\varphi+t'\mathbf{1})$, then $p \in B(\varphi+t'\mathbf{1})$, and thus $\sigma(\varphi + t\mathbf{1}) = d(p, q) \geq d(B(\varphi + t'\mathbf{1}), K) = \sigma(\varphi + t'\mathbf{1})$ as desired.

Third, for each $\varphi \in \mathbb{R}^S$, $\lim_{t\to\infty} \sigma(k\varphi + t\mathbf{1})$ weakly increases in k on $(0,\infty)$. Fix $\varphi \in \mathbb{R}^S$. For each $k > 0$, define $B_k : [0,\infty] \rightrightarrows \Delta$ by for each $t \in \mathbb{R}_+$, $B_k(t) = B(k\varphi + t\mathbf{1})$, and $B_k(\infty) = \{p \in \Delta | E_p\varphi = \lim_{t\to\infty} \frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]\}$. Fix $k > 0$, and we first check that $\lim_{t\to\infty} d(B_k(t), K) = d(B_k(\infty), K)$. By the maximum theorem, it suffices to prove that B_k is continuous at ∞. The upper hemicontinuity is easy. For the lower hemicontinuous, let $\{t_n\}_{n=1}^\infty$ be a sequence of elements in $\mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$. Fix $\bar{p} \in B_k(\infty)$. If $E_{\bar{p}}\varphi = \varphi_*$, then for each $n \in \mathbb{N}$,

$\bar{p} \in B_k(t_n)$ and we are done. Suppose $E_{\bar{p}}\varphi > \varphi_*$. Let $p_* \in \Delta$ be such that $E_{p_*}\varphi = \varphi_*$. Then, for each $n \in \mathbb{N}$, there exists a unique $\alpha_n \in [0,1]$ such that $\alpha_n \bar{p} + (1-\alpha_n)p_* \in B_k(t_n)$. It can be readily verified that $\lim_{n\to\infty}[\alpha_n \bar{p} + (1-\alpha_n)p_*] = \bar{p}$ as desired. Now let $k' \geq k$, $p \in B_k(\infty)$, $p' \in B_{k'}(\infty)$ and $q, q' \in K$ be given such that $d(p,q) = d(B_k(\infty), K)$ and $d(p', q') = d(B_{k'}(\infty), K)$. We would like to check that $d(p,q) \leq d(p',q')$. Note that $E_p\varphi \geq E_{p'}\varphi$. If $E_p\varphi = E_{p'}\varphi$, then $p' \in B_k(\infty)$, and we are done. If $E_p\varphi > E_{p'}\varphi$, since $E_{q'}\varphi \geq J'(\varphi) \geq E_p\varphi$, then there exists a unique $\alpha \in [0,1)$ such that $E_{\alpha p' + (1-\alpha)q'}\varphi = E_p\varphi$, i.e., $\alpha p' + (1-\alpha)q' \in B_k(\infty)$. Thus $d(p,q) \leq d(\alpha p' + (1-\alpha)q', q') = \alpha d(p',q') < d(p',q')$, as desired.

Fourth, $\lim_{k\searrow 0}\lim_{t\to\infty}\sigma(k\varphi + t\mathbf{1}) = 0$. Fix $\varphi \in \mathbb{R}^S$. Let $q \in K$ and $p_* \in \Delta$ be given such that $E_q\varphi = J'(\varphi)$ and $E_{p_*}\varphi = \varphi_*$. If $E_q\varphi = \varphi_*$, then for each $k > 0$ and each $t \in \mathbb{R}$, $q \in B(k\varphi + t\mathbf{1})$, and thus the result follows from the definition of σ. If $E_q\varphi > \varphi_*$, then for each $k > 0$ there exists a unique $\alpha_k \in [0,1]$ such that $E_{\alpha_k p_* + (1-\alpha_k)q}\varphi = \lim_{t\to\infty}\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t]$. Thus $\lim_{k\searrow 0}\alpha_k = \lim_{k\searrow 0}\frac{1}{E_{p_*}\varphi - E_q\varphi}[\lim_{t\to\infty}\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t] - E_q\varphi] = \frac{E_q\varphi - E_q\varphi}{E_{p_*}\varphi - E_q\varphi} = 0$. Then, for each $k > 0$,

$$\begin{aligned} 0 &\leq \lim_{t\to\infty}\sigma(k\varphi + t\mathbf{1}) = d(B_k(\infty), K) \\ &\leq d(\alpha_k p_* + (1-\alpha_k)q, q) = \alpha_k d(p_*, q), \end{aligned}$$

where the first equality and the second inequality follow from the proof in the last step. By taking the limit of k to 0, we get the desired result.

To show that K is the largest essential set, by Theorem 2 and Corollary 2, it suffices to show that $K^* \subseteq K$. Suppose the contrary that $q \in K^* \setminus K$. Thus there exists $\varphi \in \mathbb{R}^S$ such that $E_q\varphi < \min_{p\in K} E_p\varphi$. Let $f \in \mathcal{F}$ be such that $u(f) = \varphi$. For each $k > 0$ and $t \in \mathbb{R}$, let $f_{k,t} \in \mathcal{F}$ and $x_{k,t} \in X$ be such that $u(f_{k,t}) = k\varphi + t\mathbf{1}$ and $u(x_{k,t}) = kE_q\varphi + t$. Since $q \in K^*$, then for each $k > 0$ and each $t \in \mathbb{R}$, $G(u(x_{k,t}), q) = u(x_{k,t})$, and thus $u(x_{k,t}) \geq E_q u(f_{k,t})$ implies $kE_q\varphi + t \geq I(k\varphi + t\mathbf{1})$. Then, $\min_{p\in K} E_p\varphi = \lim_{k\searrow 0}\lim_{t\to\infty}\frac{1}{k}[I(k\varphi + t\mathbf{1}) - t] \leq E_q\varphi$, which gives a contradiction as desired.

Lastly, we check that statement 1 implies statement 4. By Lemma 3, we know that $\succsim$ satisfies A.2. Then by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) $\succsim$ admits an ambiguity averse representation $\langle u, G\rangle$ in which G is given by, for each $(t,p) \in \mathbb{R}\times\Delta$,

$$G(t,p) = \sup\{u(x_f) | f \in \mathcal{F}, E_p u(f) \leq t\}.$$

Let $I : \mathbb{R}^S \to \mathbb{R}$ be such that for each $\varphi \in \mathbb{R}^S$,

$$I(\varphi) = \min_{p \in \Delta} G(E_p \varphi, p). \tag{35}$$

Given $p \in \Delta$ and $t' \geq t$ in $\mathbb{R}$, we want to show that $G(t', p) - t' \geq G(t, p) - t$. For each $f \in \mathcal{F}$ such that $E_p u(f) \leq t$, there exists $f' \in \mathcal{F}$ such that $u(f') = u(f) + (t' - t)\mathbf{1}$ and thus $E_p u(f') \leq t'$. Note that by Lemma 3, I is constant superadditive. Hence $u(x_{f'}) - t' = I(u(f')) - t' = I(u(f) + (t' - t)\mathbf{1}) - t' \geq I(u(f)) + t' - t - t' = I(u(f)) - t = u(x_f) - t$. Then by definition $G(t', p) - t' \geq G(t, p) - t$. □

Proof of Proposition 5. By Theorem 10 and Proposition 11 of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011), $D^* = cl(co(\cup_{f \in \mathcal{F}} \pi(f)))$. To show that for each $x \in X$, $D^* = cl(co(\cup_{f \sim x} \pi(f)))$, fix $y \succ z$ in X and it suffices to prove that $cl(co(\cup_{f \sim y} \pi(f))) = cl(co(\cup_{f \sim z} \pi(f)))$. Let $\langle u, G \rangle$ be an ambiguity averse representation of $\succsim$, and let I be given as in (35). For each $x \in X$, define $J_x : \mathbb{R}^S \to \mathbb{R}$ by for each $\varphi \in \mathbb{R}^S$,

$$J_x(\varphi) = u(x) + t_x(\varphi),$$

where $t_x(\varphi)$ is given by $I(\varphi - t_x(\varphi)\mathbf{1}) = u(x)$, and let $\succsim_x$ be the preference relation induced by J_x, i.e., for each pair $f, g \in \mathcal{F}$, $f \succsim_x g \Leftrightarrow J_x(u(f)) \geq J_x(u(g))$, with $\sim_x$ denoting its indifference component. It can be readily verified that for each $x \in X$, $\succsim_x$ is a variational preference relation (Maccheroni, Marinacci and Rustichini (2006)), and for each $f \in \mathcal{F}$, $x \succsim f \Leftrightarrow x \succsim_x f$ and $x \sim f \Leftrightarrow x \sim_x f$. For each $x \in X$, define $\pi_x : \mathcal{F} \rightrightarrows \Delta$ by for each $f \in \mathcal{F}$,

$$\pi_x(f) = \{p \in \Delta | \sum_{s \in S} p_s f(s) \geq \sum_{s \in S} p_s g(s) \text{ implies } f \succsim_x g\},$$

and notice that for each $x \in X$ and each $f \in \mathcal{F}$ with $f \sim x$, $\pi_x(f) = \pi(f)$. It is easy to see that for all $x, y, z \in X$, $\cup_{f \sim_x y} \pi_x(f) = \cup_{f \sim_x z} \pi_x(f)$, and hence $\cup_{f \in \mathcal{F}} \pi_x(f) = \cup_{f \sim_x x} \pi_x(f) = \cup_{f \sim x} \pi(f)$. Moreover, for each $\varphi \in \mathbb{R}^S$,

$$\begin{aligned}
&u(y) - u(z) = I(\varphi - t_y(\varphi)\mathbf{1}) - I(\varphi - t_z(\varphi)\mathbf{1}) \\
&= I(\varphi + (t_z(\varphi) - t_z(\varphi) - t_y(\varphi))\mathbf{1}) - I(\varphi - t_z(\varphi)\mathbf{1}) \\
&\geq I(\varphi - t_z(\varphi)\mathbf{1}) + t_z(\varphi) - t_y(\varphi) - I(\varphi - t_z(\varphi)\mathbf{1}) = t_z(\varphi) - t_y(\varphi),
\end{aligned}$$

where the inequality follows from the constant superadditivity of I and the fact that $t_z(\varphi) > t_y(\varphi)$. Thus, for each $\varphi \in \mathbb{R}^S$, $J_y(\varphi) - J_z(\varphi) = u(y) - u(z) + t_y(\varphi) - t_z(\varphi) \geq 0$. Therefore, $\succsim_z$

is more ambiguity averse than $\succsim_y$ in the sense of Maccheroni, Marinacci and Rustichini (2006). By Proposition 6, 11 and Theorem 10 of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011), $cl(co(\cup_{f\in\mathcal{F}}\pi_y(f))) \subseteq cl(co(\cup_{f\in\mathcal{F}}\pi_z(f)))$. Combined with the above observation, we have $cl(co(\cup_{f\sim y}\pi(f))) \subseteq cl(co(\cup_{f\sim z}\pi(f)))$.

Conversely, suppose that there exists $g \sim z$ and $q \in \pi(g) \setminus cl(co(\cup_{f\sim y}\pi(f)))$. Hence, there exists $\varphi \in \mathbb{R}^S$ such that for each $p \in cl(co(\cup_{f\sim y}\pi(f)))$, $E_q\varphi < 0 < E_p\varphi$. Pick $n \in N$ such that $E_q n\varphi < E_q u(g)$ and for each $p \in cl(co(\cup_{f\sim y}\pi(f)))$, $E_p n\varphi > u(y)$. Let $h \in \mathcal{F}$ be such that $u(h) = n\varphi$. Thus $z \sim g \succsim h$. Applying Theorem 10 and Proposition 11 of Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) for $\succsim_y$, we have $h \succsim_y y$ and thus $h \succsim y$, which contradicts $y \succ z \succsim h$. Hence, $\cup_{f\sim z}\pi(f) \subseteq cl(co(\cup_{f\sim y}\pi(f)))$, and then $cl(co(\cup_{f\sim z}\pi(f))) \subseteq cl(co(\cup_{f\sim y}\pi(f)))$. □

Proof of Proposition 6. The "only if" direction is clear. For the "if" direction, suppose that A.2.3 holds. By Lemma 1, then there exists an affine onto utility function $u : X \to \mathbb{R}$ and a normalized monotone continuous functional $I : \mathbb{R}^S \to \mathbb{R}$ such that $f \succsim g$ in $\mathcal{F}$ if and only if $I(u(f)) \geq I(u(g))$. A.2.3 implies that for each $\varphi \in \mathbb{R}^S$ and each $t \in \mathbb{R}$, $I(\varphi + \mathbf{1}) = I(\varphi) + t$.

Let $\alpha \in (0, 1)$, $x, y \in X$ and $f, g \in \mathcal{F}$ with $f \trianglerighteq g$ be given such that $\alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x$. Pick $f' \in \mathcal{F}$ with $u(f') = u(f) - t\mathbf{1}$ for some $t \in \mathbb{R}_+$ and $I(\alpha u(f') + (1 - \alpha)u(x)\mathbf{1}) = I(\alpha u(g) + (1 - \alpha)u(x)\mathbf{1})$. By the implication of A.2.3, $I(\alpha u(f') + (1 - \alpha)u(y)\mathbf{1}) = I(\alpha u(g) + (1 - \alpha)u(y)\mathbf{1})$. Since $I(\alpha u(f) + (1 - \alpha)u(y)\mathbf{1}) \geq I(\alpha u(f') + (1 - \alpha)u(y)\mathbf{1})$, then $I(\alpha u(f) + (1 - \alpha)u(y)\mathbf{1}) \geq I(\alpha u(g) + (1 - \alpha)u(y)\mathbf{1})$. Thus $\alpha f + (1 - \alpha)y \succsim \alpha g + (1 - \alpha)y$ as desired. □

Proof of Proposition 7. Fix u and φ as required. By Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) (Section 5.2), if $\succsim$ admits a smooth ambiguity representation $\langle u, \phi, \mu \rangle$, then it also admits an ambiguity averse representation $\langle u, G \rangle$, and for $I_\mu : \mathbb{R}^S \to \mathbb{R}$ defined in (35),

$$I_\mu(\varphi) = \phi^{-1}\Big(\int_{p\in\Delta} \phi(E_p\varphi)d\mu(p)\Big).$$

Thus by the proof of Lemma 3, $\succsim$ satisfies A.2.1 if and only if I_μ is constant superadditive. If $-\frac{\phi''}{\phi'}$ is weakly decreasing, then one can show that I_μ is constant superadditive[27] and thus A.2.1 is satisfied.

[27] The result follows from the counterparts of Lemma 52 and the proof of Proposition 53 in Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011).

Conversely, suppose that each $\succsim \in \mathcal{P}$ satisfies A.2.1. To show that $-\frac{\phi''}{\phi'}$ is weakly decreasing, it is equivalent to check that for each $t > 0$, $J_t : \phi(\mathbb{R}) \to \mathbb{R}$, defined by for each $r \in \phi(\mathbb{R})$,

$$J_t(r) = \phi[\phi^{-1}(r) + t],$$

is convex on $\phi(\mathbb{R})$. Let $x, x' \in X$, $r, r' \in \phi(\mathbb{R})$ be given such that $r = \phi(u(x))$ and $r' = \phi(u(x'))$. Let $f \in \mathcal{F}$ be such that $f(s) = x$ and $f(s') = x'$ for some $s, s' \in S$. Let δ_s and $\delta_{s'}$ be the degenerate measures which assign the whole probability respectively on s and s'. For each $\alpha \in [0, 1]$, let μ_α be the second-order belief which assigns probability α to δ_s and $1 - \alpha$ to $\delta_{s'}$. Then, for each $t > 0$ and each $\alpha \in [0, 1]$,

$$\begin{aligned}
J_t(\alpha r + (1-\alpha)r') &= J_t(\int_{p\in\Delta} \phi(E_p u(f)) d\mu_\alpha(p)) = \phi[I_{\mu_\alpha}(u(f)) + t] \\
&\leq \phi[I_{\mu_\alpha}(u(f) + t\mathbf{1})] = \int_{p\in\Delta} \phi(E_p(u(f) + t\mathbf{1})) d\mu_\alpha(p) \\
&= \alpha\phi[\phi^{-1}(r) + t] + (1-\alpha)\phi[\phi^{-1}(r') + t] = \alpha J_t(r) + (1-\alpha)J_t(r'),
\end{aligned}$$

where the first inequality comes from the constant superadditivity of I_{μ_α}.

Similarly, one can prove that for each preference relation $\succsim \in \mathcal{P}$, it satisfies Axiom A.2.2 if and only if $-\frac{\phi''}{\phi'}$ is weakly increasing. □

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