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Autarkic Indeterminacy and Trade Determinacy

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Abstract:
Most existing evidences for indeterminacy are obtained from analyzing models that do not consider trade. This paper considers an extension of Nishimura and Shimomura (Journal of Economic Theory, 2002) Heckscher-Ohlin framework by removing sector-specific externalities in one country while maintaining all other assumptions previously made by the authors. We show that even though indeterminacy arises under autarky, it can be eliminated when trade takes place with another country exhibiting saddle-path stability. Consequently, support for indeterminacy from calibrating an autarkic framework should be treated with some degree of caution.

Key Words and Phrases: Indeterminacy, Trade, Two-Country, Heckscher-Ohlin.
JEL Classification Numbers: E32, F00, F11, F43.

1 Introduction

Recent macroeconomics literature has witnessed a growing interest in dynamic models that may feature indeterminacy, or a continuum of convergent equilibrium paths. One major focus is the study on production externalities as a contributing factor to indeterminacy. In their seminal work, Benhabib and Farmer (1994) introduce production externalities in a one-sector growth model and show that indeterminacy can arise when increasing returns to scale is large enough. More recently, Benhabib and Farmer (1996), Benhabib and Nishimura (1998), Benhabib, Meng and Nishimura (2000) reinforce the earlier findings by showing that the size of externalities needed diminishes with two or more sectors. These studies present a strong case for the existence of indeterminacy, since the theoretical criteria for indeterminacy to exist are consistent with the data highlighting small market imperfections and roughly constant returns to scale in production; see Burnside (1996), Basu and Fernald (1997).

The one and two-sector Benhabib-Farmer frameworks have stimulated various research along similar directions. Harrison (2001) and Harrison and Weder (2002) investigate the types of externalities necessary to generate indeterminacy in a model with aggregate and sector-specific externalities. Weder (2001) studies a small open-economy version of two-sector Benhabib-Farmer model and argues that indeterminacy is more easily attainable when foreign borrowing is permitted. Guo and Harrison (2001) and Guo and Lansing (1998) examine the effectiveness of different tax schemes in stabilizing the one and two-sector economy against sunspot fluctuations.

It should be clear that existing works in indeterminacy research are mostly framed in the autarkic or the small-open economy context. One exception is Nishimura and Shimomura (2002), who examine a two-country Heckscher-Ohlin model under sector-specific externalities. The authors show that when private and social factor intensity rankings are the reverse of the other, indeterminacy of the equilibrium path in the world market is possible. Consequently, the long-run Heckscher-Ohlin prediction of trade is uncertain, as the realized equilibrium path is indeterminate.

This paper is motivated by Nishimura and Shimomura's work on trade and indeterminacy, together with the observation that trade is largely ignored in the literature. We pose the following question: if a country's equilibrium path is indeterminate under autarky, can trade with another whose equilibrium path is unique under autarky lead to the uniqueness of the world equilibrium path? We follow this line of inquiry by relaxing Nishimura and Shimomura's original assumption that the production process in each sector is the same across countries from both private and social perspectives. The experiment involves removing the sector-specific externalities in one country while keeping all other assumptions committed by Nishimura and Shimomura. However, doing so destroys the integrated equilibrium that allows one to analyze to model more easily, but the problem can be overcome by imposing that factors are internationally immobile. Under autarky, it can be shown that indeterminacy arises with sector-specific externalities given certain parameter restrictions, while the equlibrium path is always unique when externalities are absent. However, the introduction of trade between the two can easily overturn the indeterminacy property that previously exists under autarky.

The purpose of the paper primarily serves to highlight an important implication on the choice of autarkic models used to demonstrate the plausibility of indeterminacy. We note that most evidences in support of indeterminacy are obtained from calibrating the U.S. economy. Through these exercises, Benhabib and Farmer (1994, 1996), Benhabib and Nishimura (1998) claim that the empirical data fall within the parameter requirements for indeterminacy implied by their model. However, this paper contends that trade with another saddle-path stable country may remove the indeterminacy that shows up in the case of autarky. Furthermore, the Heckscher-Ohlin assumption of identical technology across countries is demanding, although it allows indeterminacy to arise under both autarky and free trade. For instance, one may expect that the industrial attributes are different between the U.S. and Japan. Based on the Heckscher-Ohlin framework, indeterminacy can only take place in both countries if sector-specific externalities exist and factor intensity reversal occurs in a similar fashion. Therefore, when examining the U.S. economy, one ought to take into account the technological characteristics of Japan as well. In this respect, supporting evidences for indeterminacy that are derived from calibrating an autarkic framework such as one describing the U.S. should be treated with some degree of caution.

2 The Model

The model is outlined as follows. Consider first the home country. The home country is populated by an infinitely-lived representative agent having an instantaneous utility given by

$$U(C) = \frac{C^{1-\eta}}{1-\eta}$$

C denotes the home country's consumption and $1/\eta \in (1, \infty)$ represents the intertemporal elasticity of substitution in consumption. Labor is inelastically supplied. Let the consumption good and the investment good, denoted by I, be produced using Cobb-Douglas technology. The agent's optimization problem is

$$\underset{\mathrm{C}}{\mathrm{Max}} \int_0^\infty \frac{C^{1-\eta}}{1-\eta} e^{-\rho t} dt \tag{1}$$

subject to

$$\hat{I} = L_I^{a_I} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} \quad \text{and} \quad \hat{C} = L_C^{a_C} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{2}$$

$$K = K_I + K_C \quad \text{and} \quad L = L_I + L_C \tag{3}$$

$$\dot{K} = \hat{I} + p\hat{C} - \delta K - pC \tag{4}$$

$$K(0) \quad \text{given}$$

where $\rho \in (0,\infty)$ is the subjective rate of time discount and $K(0)$ is the initial capital stock. Equation (2) describes the production frontier based on Benhabib and Nishimura (1998), where $\hat{C}$ and $\hat{I}$ is output level of consumption and investment goods. Since this is a two-country model, the distinction between the actual consumption and investment levels with the actual output levels has to be made because only under autarky can we immediately conclude that market clearing implies $C = \hat{C}$ and $I = \hat{I}$. The private factor shares of labor and capital in sector $i = I, C$ are measured by a_i and b_i and the externalities associated with labor and capital are represented by $\overline{L_i}$ and $\overline{K_i}$. In both sectors, production exhibits constant returns to scale from the social perspective, i.e. $a_i + \alpha_i + b_i + \beta_i = 1$, but decreasing returns to scale from the private perspective, i.e. $a_i + b_i < 1$. Equation (3) describes the resource constraints for capital and labor, both of which are perfectly mobile across sectors. Equation (4) is capital accumulation process where $\delta \in (0,1)$ is the capital depreciation allowance. The current value Hamiltonian is

$$\begin{aligned}\mathrm{H} = {} & \frac{C^{1-\eta}}{1-\eta} + \lambda(\hat{I} + p\hat{C} - \delta K - pC) + P_I(L_I^{a_I} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} - \hat{I}) \\ & + P_C(L_C^{a_C} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} - \hat{C}) + \overline{r}(K - K_I - K_C) + \overline{w}(L - L_I - L_C)\end{aligned}$$

where P_I, P_C, $\overline{r}$ and $\overline{w}$ are the Lagrange multipliers representing the shadow price of investment, consumption, capital and labor respectively. Let p be the price of consumption in terms of investment. Differentiating the Hamiltonian with respect to $\hat{I}$ and $\hat{C}$ yield

$$\lambda = P_I$$

$$\lambda p = P_C$$

Combining the above, one has $p = P_C/P_I$. Next, define $r = \overline{r}/P_I$ and $w = \overline{w}/P_I$. The remaining necessary conditions are

$$C^{-\eta} = \lambda p \tag{5}$$

$$w = a_I L_I^{a_I - 1} K_I^{b_I} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} = p a_C L_C^{a_C - 1} K_C^{b_C} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{6}$$

$$r = b_I L_I^{a_I} K_I^{b_I - 1} \overline{L}_I^{\alpha_I} \overline{K}_I^{\beta_I} = p b_C L_C^{a_C} K_C^{b_C - 1} \overline{L}_C^{\alpha_C} \overline{K}_C^{\beta_C} \tag{7}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r) \tag{8}$$

$$\lim_{t\to 0} K(t)\lambda(t)e^{-\rho t} = 0 \tag{9}$$

Equation (5) expresses the equality between the marginal utility and the marginal cost of consumption while (8) represents the intertemporal arbitrage condition. Equations (6) and (7) state the equalization of the marginal revenue product of labor and of capital across sectors. Equation (9) is the transversality condition. Note that (6) and (7) consist of four equations in four unknowns, namely L_I, L_C, K_I and K_C. They are implicitly solved in terms of w and r, which are in turn determined by pinning down p. The relationship between total income from the private and the social perspectives can be expressed as

$$wL + rK + \Pi = W_I L_I + R_I K_I + W_C L_C + R_C K_C \left(= \hat{I} + p\hat{C}\right) \tag{10}$$

where Π is the firm's profit due to private decreasing returns to scale, W_i and R_i are the social wage and rental income in sector i respectively. Using the fact that production functions exhibit constant returns to scale, L_i and K_i can be expressed as functions of W_i and R_i. This is done by setting $L_i = \overline{L_i}$, $K_i = \overline{K_i}$, $a_i + \alpha_i = \theta_i$ and $b_i + \beta_i = 1 - \theta_i$ in (2) and observing that

$$W_i = p_i \theta_i L_i^{\theta_i - 1} K_i^{1-\theta_i} \tag{11}$$

$$R_i = p_i \left(1 - \theta_i\right) L_i^{\theta_i} K_i^{-\theta_i} \tag{12}$$

where $p_i = 1$ for $i = I$ and $p_i = p$ for $i = C$. Note that W_i and R_i may differ across sectors. Equations (11) and (12) can be equivalently represented by

$$W_i l_i = p_i \theta_i \tag{13}$$

$$R_i k_i = p_i \left(1 - \theta_i\right) \tag{14}$$

where l_i and k_i are the unit labor and capital requirement respectively. Using (13) and (14), the two social zero profit conditions in the goods markets are

$$1 = W_I l_I + R_I k_I \tag{15}$$

$$p = W_C l_C + R_C k_C \tag{16}$$

Due to social constant returns to scale, the unit input requirements are functions of social income alone. Equations (11) and (12), combined with (6) and (7) yield two useful relationships:

$$W_i = \frac{\theta_i}{a_i} w \tag{17}$$

$$R_i = \frac{1 - \theta_i}{b_i} r \tag{18}$$

By substituting the social income levels from (17) and (18) into (15) and (16), and log-differentiating the social zero profit functions with respect to p, we obtain the Stolper-Samuelson conditions as

$$\frac{pr'(p)}{r(p)} = \frac{\theta_I}{\theta_I - \theta_C} \tag{19}$$

$$\frac{pw'(p)}{w(p)} = \frac{1-\theta_I}{\theta_C - \theta_I} \tag{20}$$

where $\theta_C - \theta_I$ measures the social factor intensity ranking. $\theta_C - \theta_I > 0$ holds if labor is socially more intensive in the consumption sector, and the inequality reverses if the converse is true. It turns out that the Stolper-Samuelson conditions, which express the price elasticities of wage and rental income, are the same even when production externalities are absent.

Applying Shephard's Lemma to (15) and (16), the factor clearing conditions can be expressed as

$$\begin{pmatrix} l_I & l_C \\ k_I & k_C \end{pmatrix} \begin{pmatrix} \hat{I} \\ \hat{C} \end{pmatrix} = \begin{pmatrix} L \\ K \end{pmatrix} \tag{21}$$

The system of equations in (21) contains two unknowns $\hat{I}$ and $\hat{C}$, which can be solved using (13), (14), (17) and (18) to obtain

$$\hat{I} = \frac{b_C w(p)L - a_C r(p)K}{\Delta} \tag{22}$$

$$\hat{C} = \frac{a_I r(p)K - b_I w(p)L}{p\Delta} \tag{23}$$

where $\Delta = a_I b_C - a_C b_I$ measures the private factor intensity ranking. In particular, $\Delta > 0$ can be obtained with $a_I > a_C$, where labor is privately more intensive in the investment sector, together with $b_C > b_I$, where capital is privately more intensive in the consumption sector. Substituting $\hat{I}$ and $\hat{C}$ from (22) and (23) into (4), we have

$$\dot{K} = \frac{a_I r(p)K - b_I w(p)L}{\Delta} + \frac{b_C w(p)L - a_C r(p)K}{\Delta} - \delta K - pC \tag{24}$$

Equations (8) and (24) govern the law of motion for the home country.

The foreign country differs from the home country through the absence of sector-specific externalities. Distinguishing the foreign country's variables by the asterisk, the production functions are

$$\hat{I}^* = L_I^{*\theta_I} K_I^{*1-\theta_I} \quad \text{and} \quad \hat{C}^* = L_C^{*\theta_C} K_C^{*1-\theta_C} \tag{25}$$

Assume that the size of the labor force is the same in both countries. The dynamic equations governing the evolution of the foreign country are

$$\dot{K}^* = w^*(p)L + r^*(p)K^* - \delta^* K^* - pC^* \tag{26}$$

$$\dot{\lambda}^* = \lambda^* \left(\rho^* + \delta^* - r^*(p)\right) \tag{27}$$

where $p = p^*$ given that free trade takes place. Equations (26) and (27), together with (8) and (24) determine the dynamics of the two-country world economy. Since social technologies are the same as the home country's, the Stolper-Samuelson conditions are

$$\frac{pw^*(p)}{w^*(p)} = \frac{1-\theta_I}{\theta_C - \theta_I} \tag{28}$$

$$\frac{pr^*(p)}{r^*(p)} = \frac{\theta_I}{\theta_I - \theta_C} \tag{29}$$

In equilibrium, the trade balance for consumption good is given by

$$(\lambda p)^{-1/\eta} + (\lambda^* p)^{-1/\eta} = \frac{a_I r(p)K - b_I w(p)L}{p\Delta} + \frac{\theta_I r^*(p)K^* - (1-\theta_I)w^*(p)L}{p(\theta_I - \theta_C)} \tag{30}$$

To simplify the analysis, the system is reduced by one dimension. The following lemma and assumption are useful for this purpose.

Lemma 1. *There is a constant $\xi > 1$, determined from $r^*(0) = \xi r(0)$, that solves $r^*(t) = \xi r(t)$ for $t \in [0, \infty)$.*

Proof. Available upon request. □

Assumption 1. $\frac{\rho^*+\delta^*}{\xi} = \rho + \delta$

Assumption 1 is time invariant since ξ is a constant. Given the above, we have

Lemma 2. *Under Lemma 1 and Assumption 1, $\lambda^* = m\lambda^\xi$ holds for $t \in [0, \infty)$.*

Proof. By Lemma 1, Equation (27) is equivalent to

$$\frac{\dot{\lambda}^*}{\lambda^*}\frac{1}{\xi} = \frac{\rho^* + \delta^*}{\xi} - r(p)$$

which together with Assumption 1 can be written as

$$\frac{\dot{\lambda}^*}{\lambda^*} = \xi \frac{\dot{\lambda}}{\lambda}$$

This expression is integrated to obtain $\lambda^* = m\lambda^\xi$, where $m > 0$ is a constant. □

Using Lemma 2, the model's behavior is described by the following system of three equations

$$\dot{K} = \frac{a_I r(p)K - b_I w(p)L}{\Delta} + \frac{b_C w(p)L - a_C r(p)K}{\Delta} - \lambda^{-1/\eta}p^{1-1/\eta} - \delta K \tag{31}$$

$$\dot{K}^* = w^*(p)L + r^*(p)K - (m\lambda^\xi)^{-1/\eta}p^{1-1/\eta} - \delta^* K^* \tag{32}$$

$$\dot{\lambda} = \lambda(\rho + \delta - r(p)) \tag{33}$$

The steady state price level is obtained by setting $\dot{\lambda} = 0$ in (33). Totally differentiating the trade balance equation for consumption yield

$$\frac{dp}{dK} = -\frac{\Sigma a_I(\rho+\delta)}{\Delta\left((\lambda p)^{-1/\eta} + (m\lambda^{\xi} p)^{-1/\eta}\right)} \tag{34}$$

$$\frac{dp}{dK^*} = -\frac{\Sigma \theta_I(\rho^*+\delta^*)}{(\theta_I - \theta_C)\left((\lambda p)^{-1/\eta} + (m\lambda^{\xi} p)^{-1/\eta}\right)} \tag{35}$$

$$\frac{dp}{d\lambda} = -\frac{p\Sigma\left((\lambda p)^{-1/\eta} + \xi(m\lambda^{\xi} p)^{-1/\eta}\right)}{\eta\lambda\left((\lambda p)^{-1/\eta} + (m\lambda^{\xi} p)^{-1/\eta}\right)} \tag{36}$$

where

$$\frac{1}{\Sigma} = \frac{1}{\eta} - 1 + \frac{\frac{\theta_I r'^*(p)K^* - (1-\theta_I)w'^*(p)L}{\theta_I - \theta_C} + \frac{a_I r'(p)K - b_I w'(p)L}{\Delta}}{(\lambda p)^{-1/\eta} + (m\lambda^{\xi} p)^{-1/\eta}} \tag{37}$$

Due to the intractable nature of the problem, we choose ρ^* such that

Assumption 2. $\rho^* = \frac{(a_I - a_C)r(p) - \delta\Delta}{\Delta} > 0$

Define $\gamma = \left(\frac{(a_I - a_C)r(p) - \delta\Delta}{\Delta}\right)$. Using (34), (35), (36) and (37), the linearization of (31), (32), and (33) yield Jacobian matrix J with determinant

$$Det(J) = \frac{pr'(p)}{\eta}\left(-\rho^*\gamma\frac{\lambda\eta}{p}\frac{dp}{d\lambda} + \rho^*(\lambda p)^{-1/\eta}\frac{dp}{dK} + \xi\gamma(m\lambda^{\xi}p)^{-1/\eta}\frac{dp}{dK^*}\right) \tag{38}$$

Now consider two cases. First, suppose $\Delta < 0$ and $\theta_I - \theta_C < 0$. In this case, factor intensity rankings from the private and social perspectives are the same.

Definition. *The equilibrium path is a sequence* $\{p(t), w(t), w^*(t), r(t), r^*(t)\}_{t=0}^{\infty}$ *and* $\{K(t), K^*(t), \lambda(t)\}_{t=0}^{\infty}$ *such that for each t, the sequence i) solves the representative agent's problem, ii) satisfies (31), (32) and (33) given the initial stock of capital and the transversalilty condition for each country, and iii) clears all factor markets and trade balances.*

Proposition 1. *Under Assumptions 1 and 2, the equilibrium path is unique.*

Proof. The model's behavior can be analyzed by straightforward application of the Routh Theorem.[1] Since $\Delta < 0$ and $\theta_I - \theta_C < 0$, we know that $\Sigma > 0$, which implies that both dp/dK and dp/dK^* are positive and $dp/d\lambda$ is negative. Moreover, $\theta_I - \theta_C < 0$ implies that $r'(p) < 0$ and therefore $Det(J) < 0$. Since $Trace(J)$ contains both positive and negative terms, we consider two

[1] The Routh Theorem states that the number of changes in signs in the following scheme

$$-1 \qquad Trace(J) \qquad -F + \frac{Det(J)}{Trace(J)} \qquad Det(J)$$

indicates the number of eigenvalues with positive parts, where F is the sum of the minor matrices $c_{11}c_{22} - c_{21}c_{12}$, $c_{11}c_{33} - c_{31}c_{13}$, $c_{22}c_{33} - c_{23}c_{32}$.

cases:

1. $Trace(J) > 0$. The signs must change twice, regardless of the sign of $-F + \frac{Det(J)}{Trace(J)}$, and hence there are two eigenvalues with positive real part and one with negative real part. Since λ is the only non-predetermined variable, the steady state is saddle-path stable.

2. $Trace(J) < 0$. Appendix 1 demonstrates that $-F + \frac{Det(J)}{Trace(J)}$ is always positive if $Trace(J) < 0$. Once again, the signs must change twice, Hence, the steady state is always saddle-path stable. $\square$

Next, consider $\Delta > 0$, $\theta_I - \theta_C < 0$ and $\Sigma > 0$. In this case, private and social factor intensity rankings differ.

Proposition 2. *Under Assumptions 1 and 2, the equilibrium path is unique if*

$$\frac{C^*}{C} \in \left(\frac{(a_I(\rho + \delta) - \rho^* \Delta)(\theta_C - \theta_I)}{\xi(\rho^* \theta_C + \delta^* \theta_I)\Delta}, \infty \right) \tag{39}$$

Proof. With $r'(p) < 0$ and Assumption 2, the determinant is negative as long as

$$-\rho^* \frac{\lambda \eta}{p} \frac{dp}{d\lambda} + (\lambda p)^{-1/\eta} \frac{dp}{dK} + \xi (m \lambda^{\xi} p)^{-1/\eta} \frac{dp}{dK^*} > 0 \tag{40}$$

If the trace is positive, the steady state is saddle-path stable. Otherwise, saddle-path stability arises for $-F + \frac{Det(J)}{Trace(J)} > 0$. This is satisfied as long as (40) holds.[2] Using (34), (35), (36), and the fact that $C = (\lambda p)^{-1/\eta}$ and $C^* = (m\lambda^{\xi} p)^{-1/\eta}$, we can express (40) as (39). $\square$

We state without proof that under autarky, indeterminacy arises in the home country for $\Delta > 0$, $\theta_I - \theta_C < 0$ and $\Sigma > 0$ while the equilibrium path is always unique for the foreign country.[3] By introducing trade, saddle-path stability exists in both countries whenever (39) is satisfied. In this respect, Proposition 2 asserts that even though indeterminacy may arise under autarky, determinacy may be effected through trade. If such situation is ruled out categorically, examining an autarkic model to ascertain the dynamic properties of an economy will lead us to appropriate conclusions about the existence of indeterminacy. Otherwise, results obtained from such a study could be misleading, as indeterminacy may show up under autarky when determinacy is the actual outcome.

To determine the restrictiveness of the sufficient condition, consider the benchmark model:

[Insert Table 1 here]

Consistent with empirical data, the numerical values assumed show that the quantities of externalities are very small. In addition, we impose $a_I > a_C$ given $b_I < b_C$ so that $\Delta > 0$, otherwise

[2] Derivation available upon request.
[3] Details available upon request.

indeterminacy may not arise. Hence, the benchmark parameters imply that $\theta_I - \theta_C = -0.0001$ and $\Delta = 0.01247$. Furthermore, by setting $\rho = 0.05$ and $\delta = 0.05$, a foreign discount rate of $\rho^* = 0.03019$ satisfies Assumption 2. Finally, the lower bound for sufficiency is larger the closer ξ is to one since $\xi > 1$. In the limiting case, setting $\xi = 1$ implies that if C^*/C satisfies the sufficient condition for this lower bound, it will satisfy the sufficient condition for any $\xi > 1$. Given these values, the sufficient condition is met as long as C^*/C is greater than approximately 7.8915×10^{-3}. For example, if sector-specific externalities are absent in the U.S. major trading partners, say Canada and Japan, then the private consumption ratios (in 1997) of 0.05738 with Canada and 0.50633 with Japan indicate that indeterminacy may not emerge in the U.S. after all, contrary to the earlier conclusions made.[4]

3 Conclusion

The motivation of this paper is to question the use of an autarkic framework to examine the plausibility of indeterminacy. We consider an extension of Nishimura and Shimomura by removing externalities in one country while maintaining all other original assumptions of the authors. Under autarky, indeterminacy may arise with sector-specific externalities, but uniqueness is always the case when externalities are absent. Indeterminacy will continue to hold in two-country world economy if both countries share the same production specifications, externalities are present, and factor intensity rankings from the private and social perspectives are the reverse of the other. However, relaxing the assumption about the existence of externalities in one country may remove indeterminacy in the world economy. Therefore, the paper cautions against the use of an autarkic model in indeterminacy research since it may not be sufficient in helping us obtain the desired conclusion. For example, examining an autarkic framework may lead us to believe that indeterminacy is plausible, while in reality, the equilibrium path could be determinate depending on the production characteristics of the other countries. The Heckscher-Ohlin assumption of identical technology across countries is non-trivial, notwithstanding the fact that a common production specification for both countries, both from the private and the social perspectives, is a difficult one to envisage. In this respect, the criteria for indeterminacy are more stringent than one might imagine and the plausibility of indeterminacy may be overstated by evidences based on theoretical models that do not take trade into account.

[4] Based on our calculation from the OECD Business Sector Database.

Appendix 1

The Jacobian matrix is

$$J = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

where

$$\begin{aligned}
c_{11} =& \Lambda \frac{dp}{dK} & c_{12} =& \Lambda \frac{dp}{dK^*} & c_{13} =& \Lambda \frac{dp}{d\lambda} + \frac{p(\lambda p)^{-1/\eta}}{\eta\lambda} \\
&+ \frac{(a_I - a_C) r(p) - \delta\Delta}{\Delta} & & & & \\
c_{21} =& \Gamma \frac{dp}{dK} & c_{22} =& \rho^* + \Gamma \frac{dp}{dK^*} & c_{23} =& \Gamma \frac{dp}{d\lambda} + \frac{\xi p (m\lambda^\xi p)^{-1/\eta}}{\eta\lambda} \\
c_{31} =& -\lambda r'(p) \frac{dp}{dK} & c_{32} =& -\lambda r'(p) \frac{dp}{dK^*} & c_{33} =& -\lambda r'(p) \frac{dp}{d\lambda}
\end{aligned}$$

and

$$\Lambda = \frac{a_I r'(p) K - b_I w'(p) L}{\Delta} + \frac{b_C w'(p) L - a_C r'(p) K}{\Delta} + \frac{(1-\eta)(\lambda p)^{-1/\eta}}{\eta}$$

$$\Gamma = w^{*\prime}(p) L + r^{*\prime}(p) K^* + \frac{(1-\eta)(m\lambda^\xi p)^{-1/\eta}}{\eta}$$

The proof that $-F + \frac{Det(J)}{Trace(J)} > 0$ given $Trace(J) < 0$ is as follows. First, write $-F + \frac{Det(J)}{Trace(J)} > 0$ as $-\frac{1}{Trace(J)}(F\,Trace(J) - Det(J))$. Since $-\frac{1}{Trace(J)} > 0$, $-F + \frac{Det(J)}{Trace(J)} > 0$ if and only if $F\,Trace(J) - Det(J) > 0$. We define

$$\begin{aligned}
X =& c_{11}c_{22} - c_{21}c_{12} = \rho^*\gamma + \rho^*\Lambda \frac{dp}{dK} + \gamma\Gamma \frac{dp}{dK^*} \\
Y =& c_{22}c_{33} - c_{23}c_{32} = -\rho^*\lambda r'(p) \frac{dp}{d\lambda} + r'(p) \frac{\xi p}{\eta} (m\lambda^\xi p)^{-1/\eta} \frac{dp}{dK^*} \\
Z =& c_{11}c_{33} - c_{31}c_{13} = -\gamma\lambda r'(p) \frac{dp}{d\lambda} + r'(p) \frac{p}{\eta} (\lambda p)^{-1/\eta} \frac{dp}{dK}
\end{aligned}$$

where $F = X + Y + Z$. Write $F \, Trace(J)$ as

$$
\begin{aligned}
&(X+Y+Z)\,Trace(J)\\
&= Trace(J)\,X + \left(\rho^* + \Lambda\frac{dp}{\partial K} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) Y + \gamma Y\\
&- Trace(J)\,\gamma\lambda r'(p)\frac{dp}{d\lambda} + \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)\\
&+ \gamma r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}
\end{aligned}
$$

The second line is derived by expanding $Trace(J)$ in $Trace(J)\,Y$, the third and last line are derived by expanding both $Trace(J)$ and Z in $Trace(J)\,Z$. After rearranging, we obtain

$$
\begin{aligned}
&(X+Y+Z)\,Trace(J)\\
&= Trace(J)\,X + \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) Y\\
&- Trace(J)\,\gamma\lambda r'(p)\frac{dp}{d\lambda} + \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)\\
&+ \gamma\left(Y + r'(p)\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)
\end{aligned}
$$

By Assumption 2,

$$
\begin{aligned}
&(X+Y+Z)\,Trace(J) - Det(J)\\
&= Trace(J)\,(X - \gamma\lambda r'(p)\frac{dp}{d\lambda}) + \left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) Y\\
&+ r'(p)\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right)
\end{aligned}
\tag{41}
$$

In addition, $Trace(J) < 0$ and Assumption 2 imply $\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{d}{d\lambda}\right) < 0$ and

$$
\begin{aligned}
&Trace(J)\,(X - \gamma\lambda r'(p)\frac{dp}{d\lambda})\\
&= \gamma Trace(J)\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) > 0
\end{aligned}
$$

Since $Y < 0$,

$$
\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right) Y > 0
$$

Furthermore, $dp/dK > 0$ and $r'(p) < 0$ imply

$$r'(p)\left(\rho^* + \Lambda\frac{dp}{dK} + \Gamma\frac{dp}{dK^*} - \lambda r'(p)\frac{dp}{d\lambda}\right)\left(\frac{p}{\eta}(\lambda p)^{-1/\eta}\frac{dp}{dK}\right) > 0$$

Therefore, $F\,Trace(J) - Det(J) > 0$. $\square$

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Table 1: Benchmark specification of the model economy

$a_I = 0.3$	$\alpha_I = 0.001$	$b_I = 0.68$	$\beta_I = 0.019$
$a_C = 0.29$	$\alpha_C = 0.0111$	$b_C = 0.6989$	$\beta_C = 0$