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A Characterization of Single-Peaked Preferences via Random Social Choice Functions

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Abstract

The paper proves the following result: every path-connected domain of preferences that admits a strategy-proof, unanimous, tops-only random social choice function satisfying a compromise property, is single-peaked. Conversely, every single-peaked domain admits a random social choice function satisfying these properties. Single-peakedness is defined with respect to arbitrary trees. We also show that a maximal domain that admits a strategy-proof, unanimous, tops-only random social choice function satisfying a stronger version of the compromise property, is single-peaked on a line. A converse to this result also holds. The paper provides justification of the salience of single-peaked preferences and evidence in favour of the Gul conjecture (Barberà (2010)).

Keywords: Random Social Choice Functions; Strategy-proofness; Compromise; Single-peaked Preferences.

JEL Classification: D71.

1 Introduction

Single-peaked preferences are the cornerstone of several models in political economy and social choice theory. They were proposed initially by Black (1948) and Inada (1964) and can be informally described as follows. The set of alternatives is endowed with a structure that enables one to say for some triples of alternatives, say a, b and c that b is "closer" to

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a than c. On a preference order that is single-peaked, if an alternative b is closer to the maximal element in the preference than another alternative c, it must be the case that b is ranked above c in the preference. A domain of preferences is single-peaked if there is a *common* structure on alternatives such that all preference orders in the domain are single-peaked with respect to that structure. Single-peaked preferences arise naturally in various settings. However, their main attraction is that they allow successful preference aggregation both in the Arrovian and the strategic sense (Moulin (1980), Barberà (2010)), for instance, by the median voter aggregator/social choice function. Our goal in this paper is to provide a converse result with the following flavour: any "rich" preference domain that admits a suitably "well-behaved" solution to the strategic voting problem *must* be a single-peaked domain.[1]

Our model consists of a finite number of voters and alternatives.[2] We consider *random* social choice functions or RSCFs defined on a suitably rich but arbitrary domain of preference orders. A RSCF associates a probability distribution over alternatives with every profile of voters' preference orders in the domain. Following Gibbard (1977), a RSCF is *strategy-proof* if truth-telling by a voter results in a probability distribution that first-order stochastically dominates the probability distribution that arises from any misrepresentation by the voter. Moreover, this holds for every possible profile of other voters so that truth-telling is a (weakly) dominant strategy in the revelation game. In addition to strategy-proofness, we impose three other requirements on RSCFs under consideration. Two of these, *unanimity* and *tops-onlyness*, are standard in the literature on voting. The third assumption requires the RSCF to put a strictly positive probability on some element in the *compromise set* whenever it exists. The compromise set picks alternatives not top-ranked by any voter but which are ranked higher by every voter relative to the top-ranked alternative of any other voter. According to our main result, any rich domain that admits a strategy-proof, tops-only RSCF satisfying unanimity and the compromise property, must be single-peaked. Conversely, any single-peaked domain admits a strategy-proof, tops-only RSCF satisfying ex-post efficiency (a stronger version of unanimity) and the compromise property.

The single-peaked domain characterized by our result is more general than the usual one (for example, in Moulin (1980)). These preferences were introduced in Demange (1982) and Danilov (1994) and are defined on arbitrary *trees*. The more familiar notion of single-peakedness is the special case where the tree in the general definition is a line. We establish a second result according to which any rich and *maximal* domain that admits a strategy-proof, tops-only RSCF satisfying unanimity and a strong version of the compromise property, must be single-peaked on a line. Conversely, any single-peaked domain on a line admits a

[1]Claims of this nature have been referred to as the Gul Conjecture in Barberà (2010) and attributed to Faruk Gul. The precise formulation of the conjecture can take several forms. Our result can be regarded as further evidence in favour of the conjecture.

[2]The number of alternatives is assumed to be at least three.

strategy-proof, tops-only RSCF satisfying ex-post efficiency and the stronger version of the compromise property.

A consequence of considering RSCFs is that the anonymity requirement (anonymity implies that the names of voters do not matter and reshuffling preferences across voters does not affect the social outcome) imposed on *deterministic* social choice functions or DSCFs to rule out dictatorship must be replaced, because it is always possible to design a strategy-proof RSCF which satisfies anonymity. To see this, consider an arbitrary domain and the RSCF that picks the top-ranked alternative of voter i with probability $\frac{1}{N}$ at each profile (N is the number of voters). This RSCF is a particular instance of a random dictatorship (Gibbard (1977)) and is strategy-proof, anonymous, ex-post efficient and tops-only. However, it suffers from a well-recognized defect; it does not permit society to put positive probability on an alternative unless it is top-ranked for some voter, even though the alternative may be highly ranked (say second-ranked) for all voters. The present paper introduces the axiom of compromise which is a natural and systematic way of ensuring that social decisions give positive probability to the set of highly ranked alternatives when it exists, and asks what sort of domains allow the design of strategy-proof compromising RSCFs. In conjunction with the other assumptions on the RSCF, we find that it implies that the domain must be single-peaked.

A paper closely related to ours, is Chatterji et al. (2013). That paper investigated preference domains that admits well-behaved and strategy-proof DSCFs. In particular, it showed that every rich domain that admitted a strategy-proof, unanimous, anonymous and tops-only DSCF with an even number of voters, is *semi-single-peaked*.[3] These preferences are also defined on trees but are significantly less restrictive than single-peaked preferences. Our paper demonstrates that two important objectives can be met by considering RSCFs rather than DSCFs. The first is that a characterization of single-peaked rather than semi-single-peaked preferences can be obtained naturally. The second is that the awkward assumption regarding the even number of voters in Chatterji et al. (2013) can be removed.

An obvious question is whether the compromise axiom can be used in the context of DSCFs to obtain a single-peakedness result. We show that this is not possible. In particular, if there are at least four alternatives, domains of single-peaked preferences admit strategy-proof DCSFs satisfying the compromise property only in "exceptional circumstances". Consideration of RSCFs is therefore crucial for our results. It is of course, natural to allow for randomization whenever there are conflicts of interest among agents. Randomization also facilitates truth-telling because the evaluation of lotteries using the expected utility hypothesis imposes preference restrictions. Recently Chatterji et al. (2014) have shown that randomization can significantly enlarge the class of strategy-proof and unanimous rules in dictatorial domains. Results characterizing the class of strategy-proof and unanimous RSCFs for single-

[3]The notion of richness is exactly the same as that in our paper.

peaked domains (on the line) have been obtained in Ehlers et al. (2002), Peters et al. (2014) and Pycia and Unver (2014).

The paper is organized as follows. Section 2 and various subsections present the model, definitions and axioms. Sections 3 and 4 contain the characterization results for general single-peaked domains and single-peaked domains on a line respectively. Section 5 discusses the need for considering randomized social choice functions and Section 6 concludes.

2 Model and Notation

Let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of alternatives with $m \geq 3$. Let $\Delta(A)$ denote the lottery space induced by A. An element of $\Delta(A)$ is a *lottery* or probability distribution over the elements of A. For every $a_j \in A$, let $e_j \in \Delta(A)$ denote the degenerate lottery where alternative a_j gets probability one.

Let $I = \{1, \ldots, N\}$ be a finite set of voters with $|I| = N \geq 2$. Each voter i has a (strict preference) order P_i over A which is antisymmetric, complete and transitive, i.e., a linear order. For any $a_j, a_k \in A$, $a_j P_i a_k$ is interpreted as "a_j is *strictly* preferred to a_k according to P_i". Let $\mathbb{P}$ denote the set containing *all* linear orders over A. The set of all admissible orders is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the *preference domain.* A *preference profile* $P \equiv (P_1, P_2, \ldots, P_N) \in \mathbb{D}^N$ is an N-tuple of orders.

For any $P_i \in \mathbb{D}$, $r_k(P_i)$ denotes the kth ranked alternative in P_i, $k = 1, \ldots, m$. For any $P \in \mathbb{D}^N$, $r_1(P) = \cup_{i \in I}\{r_1(P_i)\}$ denotes the set of voters' peaks or first-ranked alternatives.

2.1 Random Social Choice Functions and Their Properties

A *Random Social Choice Function* (or RSCF) is a map $\varphi : \mathbb{D}^N \to \Delta(A)$. At every profile P, $\varphi(P)$ is the "socially desirable" lottery. For any $a_j \in A$, $\varphi_j(P)$ is the probability with which a_j will be chosen in the lottery $\varphi(P)$. Thus, $\varphi_j(P) \geq 0$ and $\sum_{j=1}^{m} \varphi_j(P) = 1$.

A *Deterministic Social Choice Function* (or DSCF) is a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ where the outcome at every preference profile is a degenerated probability distribution, i.e., $\varphi(P) = e_j$ for some j at each P.

An RSCF satisfies *unanimity* if it assigns probability one to any alternative that is ranked first by all voters, i.e., RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies unanimity if $[r_1(P_i) = a_j$ for all $i \in I] \Rightarrow [\varphi(P) = e_j]$ for all $a_j \in A$ and $P \in \mathbb{D}^N$.

An axiom stronger than unanimity is *ex-post efficiency*. It requires all Pareto-dominated outcomes to never be chosen. Formally, the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is ex-post efficient if for all $a_j, a_k \in A$ and $P \in \mathbb{D}^N$, $[a_j P_i a_k$ for all $i \in I] \Rightarrow [\varphi_k(P) = 0]$.

Voters' preferences are assumed to be private information. It is important therefore for voters to have appropriate incentives for revealing their private information *truthfully*.

An RSCF is *strategy-proof* if truth-telling is a dominant strategy for every voter, i.e., the truth-telling lottery first-order stochastically dominates the lotteries arising from misrepresentation. Formally, the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is *strategy-proof* if, $\sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) \geq \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P'_i, P_{-i})$, $t = 1, \ldots, m$, holds for all $i \in I$; $P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$. This notion of strategy-proofness was first formulated in Gibbard (1977). It is equivalent to requiring a voter's expected utility from truth-telling to be no less than her expected utility from misrepresentation for *any* cardinal respresentation of her true preferences. We omit these details which may be found in Gibbard (1977).

A prominent class of RSCFs is the class of *tops-only* RSCFs. The value of these RSCFs at any profile depends only on voter peaks at that profile. The RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the *tops-only* property if $[r_1(P_i) = r_1(P'_i)$ for all $i \in I] \Rightarrow [\varphi(P) = \varphi(P')]$ for all $P, P' \in \mathbb{D}^N$. Tops-only RSCFs have obvious informational and computational advantages - for this reason, they (more accurately, DSCFs) have received a great deal of attention in the literature (see Weymark (2008); Chatterji and Sen (2011)).

The notions of unanimity, ex-post efficiency, strategy-proofness and tops-onlyness are standard axioms in the literature on mechanism design in voting environments. Below, we introduce an axiom that is relatively novel.

Consider the RSCF known as *random dictatorship*. Each voter is assigned a non-negative weight with the sum of weights across voters being one. At any profile, the probability with which an arbitrary alternative a_j is chosen is sum of the probability weights of voters for whom a_j is the first-ranked alternative. Random dictatorships satisfy all the properties discussed above - they are ex-post efficient, strategy-proof and tops-only. If the weights are $\frac{1}{N}$, they also satisfy the property of *anonymity* , i.e., they do not depend on the "names" of voters.[4] Yet they suffer from an important and well-known infirmity - they do not admit *compromise*. Imagine a two-voter world with several alternatives (say, a thousand). Consider a profile where voter 1's first-ranked and thousandth-ranked alternatives are a_j and a_k respectively. On the other hand, voter 2's first-ranked and thousandth-ranked alternative are a_k and a_j respectively. Suppose, in addition that there is an alternative say a_r that is highly-ranked by *both* voters - for instance, ranked second by both. Any reasonable RSCF should put at least some probability weight on a_r. However, no random dictatorship would. The property introduced below ensures that a_r would indeed receive strictly positive probability.

Let P be a preference profile. The *Compromise Set* at P denoted by $C(P)$ is the set of all alternatives that *every* voter i strictly prefers to *every* other alternative that is first-ranked by some other voter i', i.e., $C(P) = \{a_r \in A | a_r P_i a_s$ for all $i \in I$ and all $a_s \in r_1(P) \setminus r_i(P)\}$. It is clear that $C(P) = \emptyset$ is possible, for instance, $[|r_1(P)| = 1] \Rightarrow [C(P) = \emptyset]$.

The RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the *weak compromise* property if for all $P \in \mathbb{D}^N$,

[4] A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is anonymous if for every permutation function $\sigma : I \to I$ and $P \in \mathbb{D}^N$, $\varphi(P_1, \ldots, P_N) = \varphi(P_{\sigma(1)}, \ldots, P_{\sigma(N)})$.

$[C(P) \neq \emptyset] \Rightarrow [\varphi_j(P) > 0$ for *some* $a_j \in C(P)]$. Similarly, $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the *strong compromise* property if for all $P \in \mathbb{D}^N$, $[C(P) \neq \emptyset] \Rightarrow [\varphi_j(P) > 0$ for *all* $a_j \in C(P)]$.

In the context of our earlier example and discussion, the set of all alternatives other than a_j and a_k constitutes the compromise set. The strong compromise property would require that all such alternatives receive strictly positive probability and the weak compromise property would require the same for at least one such alternative.

We investigate domains that admit RSCFs that are strategy-proof, tops-only, unanimous and satisfy various versions of the compromise property. A subtle point is that it may not be desirable to impose *only a subset* of these requirements. Suppose for instance, we imposed only tops-onlyness and the strong compromise property. Consider a profile where all voters have a common second-ranked alternative a_j but their first ranked alternatives are all distinct. By the strong compromise property, the RSCF must put a strictly positive probability on a_j. Now consider another profile where all the first-ranked alternatives are the same as earlier but a_j is bottom-ranked for all voters' preferences. The tops-only property would require the RSCF to put the same probability on a_j again. This would be clearly unsatisfactory. However, this RSCF would not be strategy-proof, i.e. these preferences cannot be admissible if all four axioms are to be satisfied.

2.2 Domains

Our goal in this paper is to characterize preference domains that admit RSCFs satisfying the properties described in the previous subsection. However, we need to restrict attention to domains that satisfy a regularity condition which we call *path-connectedness*.

The path-connectedness condition was introduced in Chatterji et al. (2013).[5] Fix a domain $\mathbb{D}$. A pair of distinct alternatives $a_j, a_k \in A$ satisfies the *Free Pair at the Top* (or FPT) property denoted by $a_j \approx a_k$, if there exist $P_i, P_i' \in \mathbb{D}$ such that (i) $r_1(P_i) = a_j$ and $r_2(P_i) = a_k$ (ii) $r_1(P_i') = a_k$ and $r_2(P_i') = a_j$ and (iii) $r_k(P_i) = r_k(P_i')$, $k = 3, \ldots, m$. In other words, two alternatives satisfy the FPT property if there exists a pair of admissible orders where the alternatives are at the top of both orders and are locally switched, i.e., all alternatives other than the specified pair are ranked in the same way in both orders. Let $FPT(\mathbb{D})$ denote the set of alternative pairs that satisfy the FPT property. The domain $\mathbb{D}$ is *path-connected* if for every pair of alternatives $a_j, a_k \in A$, there exists a sequence $\{x_t\}_{t=1}^T \subset A$, $T \geq 2$, such that $x_1 = a_j$, $x_T = a_k$ and $(x_t, x_{t+1}) \in FPT(\mathbb{D})$, $t = 1, \ldots, T-1$.

The path-connectedness assumption imposes structure on the domain. It allows the construction of paths between admissible orders by switching preferences at the top of the orders. Very similar conditions have been identified in Carroll (2012) and Sato (2013) as being

[5] Slightly different names were used in Chatterji et al. (2013) for the Free Pair at the Top property and path-connectedness. We believe that the new names are more apposite.

critical for the purpose of identifying domains where *local incentive-compatibility* ensures strategy-proofness.[6]

Chatterji et al. (2013) provides extensive discussion of well-known domains that satisfy the path-connectedness assumption. The complete domain and the single-peaked domain are path-connected. Maximal single-crossing domains (Saporiti (2009)) are path-connected provided that every alternative is first-ranked in some order in the domain. A generalized single-peaked domain (Nehring and Puppe (2007)) may or may not be path-connected. On the other hand, the separable domain (Barberà et al. (1991), Le Breton and Sen (1999)) and the multi-dimensional single-peaked domain (Barberà et al. (1993)) are not path-connected. For details the reader is referred to Examples 1, 2 and 3 in Chatterji et al. (2013).

A domain of central importance in collective choice theory is the *single-peaked* domain. It was originally introduced in Black (1948) and Inada (1964). Here we consider a generalization due to Demange (1982) and Danilov (1994).

An undirected graph $G = \langle V, E\rangle$ is a set of vertices V and a set of edges E. The set E consists of pairs vertices, i.e., $E \subseteq \{(u, v)|u, v \in V \text{ and } u \neq v\}$. If $(u, v) \in E$, we say that (u, v) is an edge in G.[7] A path in G is a sequence $\{v_k\}_{k=1}^{s} \subseteq V$ where $s \geq 2$ and $(v_k, v_{k+1}) \in E$, $k = 1, \dots, s-1$. The graph G is *connected* if there exists a path between every pair of vertices, i.e., for all $u, v \in V$ with $u \neq v$, there exists a path $\{v_k\}_{k=1}^{s}$ such that $u = v_1$ and $v = v_s$. The connected graph G is a tree if the path between every pair of vertices is unique. Let G be a tree and $u, v \in V$ be a pair of vertices. We denote the unique path between them by $\langle u, v\rangle$.[8]

In what follows, we shall consider graphs G of the kind $G = \langle A, E\rangle$, i.e., whose vertex set is the set of alternatives.

Definition 1 *Let $G = \langle A, E\rangle$ be a tree. The order P_i is single-peaked on G if for all $a_j, a_k \in A$,*

$$\big[a_j \in \langle r_1(P_i), a_k\rangle \backslash \{a_k\}\big] \Rightarrow [a_j P_i a_k].$$

Pick a preference P_i and an arbitrary alternative a_k. Since the graph is a tree, there is a unique path between $r_1(P_i)$ and a_k. The order P_i is single-peaked if every alternative a_j on this path, distinct from a_k is strictly preferred to a_k according to P_i.

A domain $\mathbb{D}$ is single-peaked if there exists a tree G such that $P_i \in \mathbb{D}$ implies P_i is single-peaked on G.

A case of special interest is the one where the graph $G = \langle A, E\rangle$ is a *line*. Formally, G is a line if there exists a permutation $\sigma : \{1, \dots, m\} \to \{1, \dots, m\}$ such that

[6] Assume that every alternative is first-ranked in some preference. Then domains of ordinal preferences studied in both Carroll (2012) and Sato (2013) are path-connected.

[7] In an undirected graph, (u, v) and (v, u) represent a same edge.

[8] In particular, if $u = v$, $\langle u, v\rangle = \{u\}$ is a singleton set.

$E = \{(a_{\sigma(k)}, a_{\sigma(k+1)}), k = 1, \ldots, m-1\}$. The standard definition of a single-peaked domain is one where the underlying graph is a line.

We illustrate these notions with some examples.

EXAMPLE **1** Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\bar{\mathbb{D}}$ is described below:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_2	a_1	a_4	a_3	a_2	a_2	a_2	a_2
a_4	a_3	a_4	a_3	a_4	a_4	a_1	a_3	a_1
a_3	a_4	a_3	a_1	a_1	a_1	a_4	a_1	a_3

Table 1: Domain $\bar{\mathbb{D}}$

The domain $\bar{\mathbb{D}}$ is single-peaked on the tree G^T shown in Figure 1 below.

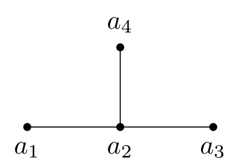

Figure 1: Tree G^T

Note that there are orders that are single-peaked on G^T but not included in $\bar{\mathbb{D}}$ - for instance, $a_2 P_{10} a_1 P_{10} a_3 P_{10} a_4$. The largest single-peaked domain on $\bar{\mathbb{D}}$ contains 12 orders. $\square$

EXAMPLE **2** Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\hat{\mathbb{D}}$ is described below

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4
a_2	a_1	a_3	a_3	a_2	a_2	a_4	a_3
a_3	a_3	a_4	a_1	a_1	a_4	a_2	a_2
a_4	a_4	a_1	a_4	a_4	a_1	a_1	a_1

Table 2: Domain $\hat{\mathbb{D}}$

The domain $\hat{\mathbb{D}}$ is single-peaked on the line G^L shown in Figure 2 below.

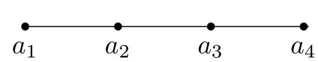

Figure 2: Line G^L

In contrast to $\bar{\mathbb{D}}$, domain $\hat{\mathbb{D}}$ includes *all* orders that are single-peaked on G^L. Observe also that $\bar{\mathbb{D}}$ is not single-peaked on a line, nor is $\hat{\mathbb{D}}$ single-peaked on G^T. In order to verify the former claim, observe that any domain that is single-peaked on a line must have at least two alternatives which have unique orders where these alternatives are peaks (these are the alternatives at either end of the line) - there are no alternatives with this property in $\bar{\mathbb{D}}$. On the other hand, the maximal number of alternatives that can be second-ranked to a given alternative on any domain that is single-peaked on a line, is two; on any domain that is single-peaked on a tree such as G^T, there must exist an alternative that has three distinct second-ranked ranked alternatives. $\square$

3 Main Result: Single-Peakedness

Our main result characterizes single-peaked domains.

Theorem **1** *Every path-connected domain that admits a unanimous, tops-only and strategy-proof RSCF satisfying the weak compromise property, is single-peaked. Conversely, every single-peaked domain admits an ex-post efficient, tops-only and strategy-proof RSCF satisfying the weak compromise property.*

Proof: We first prove necessity. Assume that $\mathbb{D}$ is path-connected. In addition, there exists a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ which is tops-only, strategy-proof, unanimous and satisfies the weak compromise property. We will show that there exists a tree G such that $\mathbb{D}$ is single-peaked on G.

The first four lemmas establish critical properties of the RSCF φ.

Lemma **1** *Let $a_j, a_k \in A$ with $a_j \approx a_k$. Let $P_i, P_i' \in \mathbb{D}$ be such that* (i) $r_1(P_i) = r_2(P_i') = a_j$ (ii) $r_2(P_i) = r_1(P_i') = a_k$ *and* (iii) $r_t(P_i) = r_t(P_i')$, $t = 3, \ldots, m$. *Then, for all* $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi_j(P_i, P_{-i}) + \varphi_k(P_i, P_{-i}) = \varphi_j(P_i', P_{-i}) + \varphi_k(P_i', P_{-i})$ *and* $\varphi_t(P_i, P_{-i}) = \varphi_t(P_i', P_{-i})$ *for all* $a_t \in A \backslash \{a_j, a_k\}$.

Suppose voter i switches her order from P_i to P_i', a move that involves the reshuffling of the top two alternatives, say a_j and a_k, while leaving all other alternatives unaffected. According to Lemma 1, the switch leaves the probabilities of alternatives other than a_j and a_k and the *sum* of probabilities of a_j and a_k, unchanged. The Lemma is a special case of the preliminary Lemma 2 in Gibbard (1977). It is a consequence of strategy-proofness and we omit its elementary proof.

Lemma **2** *If domain $\mathbb{D}$ admits a unanimous, tops-only and strategy-proof RSCF satisfying the weak compromise property, then it admits a two-voter unanimous, tops-only and strategy-proof RSCF satisfying the weak compromise property.*

Proof: Construct a two-voter RSCF $\phi : \mathbb{D}^2 \to \Delta(A)$ as follows: $\phi(P_1, P_2) = \varphi(P_1, P_2, \ldots, P_2)$ for all $P_1, P_2 \in \mathbb{D}$. In other words, ϕ is constructed by "merging" voters 2 through N in φ. Evidently, ϕ is a RSCF satisfying unanimity and the tops-only property. It is also strategy-proof (see the proof of Lemma 3 in Sen (2011)). We show that ϕ satisfies the weak compromise property.

For all $P \equiv (P_1, P_2) \in \mathbb{D}^2$, it is clear that $C(P_1, P_2, \ldots, P_2) = C(P)$. Pick $P \in \mathbb{D}^2$ such that $C(P) \neq \emptyset$. Therefore $C(P_1, P_2, \ldots, P_2) \neq \emptyset$. Since φ satisfies the weak compromise property, there exists $a_j \in C(P_1, P_2, \ldots, P_2)$ such that $\varphi_j(P_1, P_2, \ldots, P_2) > 0$. It follows that $\phi_j(P) = \varphi_j(P_1, P_2, \ldots, P_2) > 0$. Hence ϕ satisfies the weak compromise property. This completes the proof of the Lemma. ■

In view of Lemma 2, we can assume without loss of generality that the set of voters is $\{1, 2\}$ and φ is an RSCF $\varphi : \mathbb{D}^2 \to \Delta(A)$ that is strategy-proof, tops-only, unanimous and satisfies the weak compromise property. We make a further simplification in notation. Since φ is tops-only, we can represent a profile $P \in \mathbb{D}^2$ by a pair of alternatives a_j and a_k where $r_1(P_1) = a_j$ and $r_1(P_2) = a_k$. We also mix the notation of alternative and preference, for instance, (a_j, P_2) denotes a profile of preferences where $r_1(P_1) = a_j$.

Lemma **3** *Let $a_j, a_k \in A$ with $a_j \approx a_k$. There exists $\beta \in [0, 1]$ such that $\varphi(a_j, a_k) = \beta e_j + (1 - \beta)e_k$.*

Proof: Let $P_i, P'_i \in \mathbb{D}$ be such that $r_1(P_i) = r_2(P'_i) = a_j$ and (ii) $r_2(P_i) = r_1(P'_i) = a_k$ (such two preferences exist since $a_j \approx a_k$). We then have

$$\begin{aligned}
\varphi_j(a_j, a_k) + \varphi_k(a_j, a_k) &= \varphi_j(P_i, a_k) + \varphi_k(P_i, a_k) \quad \text{(by the tops-only property)} \\
&= \varphi_j(P'_i, a_k) + \varphi_k(P'_i, a_k) \quad \text{(by Lemma 1)} \\
&= \varphi_k(a_k, a_k) = 1 \quad \text{(by unanimity).}
\end{aligned}$$

Let $\varphi_j(a_j, a_k) = \beta$. Thus, $\varphi(a_j, a_k) = \beta e_j + (1 - \beta)e_k$ as required. ■

The next lemma considers situations more general than those considered in the previous one. We illustrate it with an example. Suppose $a_1 \approx a_2$ and $a_2 \approx a_3$. We know from Lemma 3 that there exists $\beta_1, \beta_2 \in [0, 1]$ such that $\varphi(a_1, a_2) = \beta_1 e_1 + (1-\beta_1)e_2$ and $\varphi(a_2, a_3) = \beta_2 e_2 + (1-\beta_2)e_3$. The next lemma shows that $\beta_2 > \beta_1$ and $\varphi(a_1, a_3) = \beta_1 e_1 + (\beta_2 - \beta_1)e_2 + (1-\beta_2)e_3$.

Lemma **4** *Let $\{a_k\}_{k=1}^{s} \subseteq A$, $s \geq 3$, be such that $a_k \approx a_{k+1}$, $k = 1, \ldots, s-1$. Let $\beta_k = \varphi_k(a_k, a_{k+1})$, $k = 1, \ldots, s-1$. Then, the following two conditions hold.*

(i) $0 \leq \beta_k < \beta_{k+1} \leq 1$, $k = 1, \ldots, s-2$.

(ii) *for all* $1 \leq i < j \leq s$, $\varphi(a_i, a_j) = \beta_i e_i + \sum_{k=i+1}^{j-1}(\beta_k - \beta_{k-1})e_k + (1 - \beta_{j-1})e_j$.

Proof: We know from Lemma 3 that $\varphi(a_k, a_{k+1}) = \beta_k e_k + (1-\beta_k)e_{k+1}$, $k = 1, \ldots, s-1$. Pick k with $1 \leq k \leq s-2$. Since $a_{k+1} \approx a_{k+2}$ and $a_k \notin \{a_{k+1}, a_{k+2}\}$, Lemma 1 implies $\varphi_{k+1}(a_k, a_{k+2}) + \varphi_{k+2}(a_k, a_{k+2}) = \varphi_{k+1}(a_k, a_{k+1}) + \varphi_{k+2}(a_k, a_{k+1}) = \varphi_{k+1}(a_k, a_{k+1}) = 1 - \beta_k$ and $\varphi_k(a_k, a_{k+2}) = \varphi_k(a_k, a_{k+1}) = \beta_k$. Also, since $a_k \approx a_{k+1}$, Lemma 1 implies $\varphi_k(a_k, a_{k+2}) + \varphi_{k+1}(a_k, a_{k+2}) = \varphi_k(a_{k+1}, a_{k+2}) + \varphi_{k+1}(a_{k+1}, a_{k+2}) = \varphi_{k+1}(a_{k+1}, a_{k+2}) = \beta_{k+1}$. Therefore, $\varphi_{k+1}(a_k, a_{k+2}) = \beta_{k+1} - \varphi_k(a_k, a_{k+2}) = \beta_{k+1} - \beta_k$ and $\varphi_{k+2}(a_k, a_{k+2}) = 1 - \beta_k - \varphi_{k+1}(a_k, a_{k+2}) = 1 - \beta_{k+1}$. Therefore, $\varphi_k(a_k, a_{k+2}) + \varphi_{k+1}(a_k, a_{k+2}) + \varphi_{k+2}(a_k, a_{k+2}) = 1$, and $\varphi(a_k, a_{k+2}) = \beta_k e_k + (\beta_{k+1} - \beta_k)e_{k+1} + (1 - \beta_{k+1})e_{k+2}$. Therefore $\beta_{k+1} \geq \beta_k$. We conclude the argument by showing that the inequality must be strict.

Pick $P_1, P_2 \in \mathbb{D}$ such that $r_1(P_1) = a_k$, $r_1(P_2) = a_{k+2}$ and $r_2(P_1) = r_2(P_2) = a_{k+1}$ (such two preferences exist since $a_k \approx a_{k+1}$ and $a_{k+1} \approx a_{k+2}$). Observe that $a_{k+1} \in C(P_1, P_2)$. Since φ satisfies the weak compromise property, there exists $a_r \in C(P_1, P_2)$ such that $\varphi_r(P_1, P_2) > 0$. However, we have shown $\varphi(P_1, P_2) = \varphi(a_k, a_{k+2}) = \beta_k e_k + (\beta_{k+1} - \beta_k)e_{k+1} + (1-\beta_{k+1})e_{k+2}$. Consequently, it must be the case that $a_r = a_{k+1}$ and $\varphi_r(P_1, P_2) = \beta_{k+1} - \beta_k > 0$ as required. This completes the verification of part (i) of the lemma.

Pick a_i, a_j in the sequence $\{a_k\}_{k=1}^s$ such that $i < j$. We will prove part (ii) by induction on the value of $l = j - i$. Observe that part (ii) has already been proved for the cases $l = 1$ (Lemma 3) and $l = 2$ (in the proof of part (i)). Assume therefore that $3 \leq l \leq s-1$. We impose the following induction hypothesis: for all $1 \leq \underline{i} < \underline{j} \leq s$,

$$\left[\underline{j} - \underline{i} < l\right] \Rightarrow \left[\varphi(x_{\underline{i}}, x_{\underline{j}}) = \beta_{\underline{i}} e_{x_{\underline{i}}} + \textstyle\sum_{k=\underline{i}+1}^{\underline{j}-1} (\beta_k - \beta_{k-1}) e_{x_k} + (1 - \beta_{\underline{j}-1}) e_{x_{\underline{j}}}\right].$$

We complete the proof by showing that part (ii) holds for all i, j with $1 \leq i < j \leq s$ and $j - i = l$.

Since $j - i = l \geq 3$, we know that $i < i+1 < j-1 < j$. Also $(j-1) - i = l - 1 < l$ and $j - (i+1) = l - 1 < l$. The induction hypothesis can then be applied to the profiles (a_i, a_{j-1}) and (a_{i+1}, a_j). Hence

$$\begin{aligned} \varphi(a_i, a_{j-1}) &= \beta_i e_i + \sum_{k=i+1}^{j-2} (\beta_k - \beta_{k-1}) e_k + (1 - \beta_{j-2}) e_{j-1} \quad \text{and} \\ \varphi(a_{i+1}, a_j) &= \beta_{i+1} e_{i+1} + \sum_{k=i+2}^{j-1} (\beta_k - \beta_{k-1}) e_k + (1 - \beta_{j-1}) e_j \end{aligned}$$

Since $a_j \approx a_{j-1}$ and $a_i, \ldots, a_{j-2}$ are distinct from a_{j-1} and a_j, Lemma 1 implies $\varphi_i(a_i, a_j) = \varphi_i(a_i, a_{j-1}) = \beta_i$ and $\varphi_k(a_i, a_j) = \varphi_k(a_i, a_{j-1}) = \beta_k - \beta_{k-1}$, $k = i+1, \ldots, j-2$. Similarly, since $a_i \approx a_{i+1}$, a_{j-1} and a_j are distinct from a_i and a_{i+1}, Lemma 1 implies $\varphi_{j-1}(a_i, a_j) = \varphi_{j-1}(a_{i+1}, a_j) = \beta_{j-1} - \beta_{j-2}$ and $\varphi_j(a_i, a_j) = \varphi_j(a_{i+1}, a_j) = 1 - \beta_{j-1}$. Thus, $\sum_{k=i}^{j} \varphi_k(a_i, a_j) = 1$ and $\varphi(a_i, a_j) = \beta_i e_i + \sum_{k=i+1}^{j-1} (\beta_k - \beta_{k-1}) e_k + (1 - \beta_{j-1}) e_j$ as required. ■

In order to demonstrate that $\mathbb{D}$ is single-peaked, we need to construct a tree $G = \langle A, E \rangle$ and show that $P_i \in \mathbb{D}$ implies P_i is single-peaked on G.

Let $G(\mathbb{D}) = \langle A, FPT(\mathbb{D})\rangle$ be a graph, i.e., $a_j, a_k \in A$ constitute an edge in $G(\mathbb{D})$ only if they satisfy the FPT property. Since $\mathbb{D}$ is path-connected, graph $G(\mathbb{D})$ is connected. The following lemma shows that $G(\mathbb{D})$ is a tree.

LEMMA **5** *$G(\mathbb{D})$ is a tree.*

Proof: Suppose not, i.e., there exists a sequence $\{a_k\}_{k=1}^{s} \subseteq A$, $s \geq 3$, such that $a_k \approx a_{k+1}$, $k = 1, \ldots, s$, where $a_{s+1} = a_1$. Let $\beta_k = \varphi_k(a_k, a_{k+1})$, $k = 1, \ldots, s-1$. Since $a_1 \approx a_2$, $a_2 \approx a_3$, …,$a_{s-1} \approx a_s$, Lemma 4 implies $\varphi(a_1, a_s) = \beta_1 e_1 + \sum_{k=2}^{s-1}(\beta_k - \beta_{k-1})e_k + (1 - \beta_{s-1})e_s$ where $0 \leq \beta_k < \beta_{k+1} \leq 1$, $k = 1, \ldots, s-2$. However, since $a_1 \approx a_s$, Lemma 3 implies $\varphi_k(a_1, a_s) = 0$ for all $a_k \neq a_1, a_s$. We have a contradiction. ■

LEMMA **6** *$P_i \in \mathbb{D} \Rightarrow P_i$ is single-peaked on $G(\mathbb{D})$.*

Proof: Suppose $a_t, a_j, a_k \in A$ are such that $r_1(P_i) = a_t$ and $a_j \in \langle a_t, a_k\rangle \backslash \{a_k\}$. Let $\langle a_t, a_k\rangle = \{x_r\}_{r=1}^{T}$ where $x_1 = a_t$, $x_T = a_k$ and $a_j = x_l$ for some $1 \leq l < T$. If $a_j = a_t$, $a_j P_i a_k$ follows trivially. Assume therefore that $a_j \neq a_t$. Thus, $T \geq 3$. Suppose $a_k P_i a_j$. Consider the profile $P = (a_t, a_k)$ and $\varphi(P)$. According to Lemma 4, all alternatives in the sequence $\{x_r\}_{r=2}^{T-1}$ get strictly positive probability. Hence $\varphi_j(a_t, a_k) > 0$. Since φ satisfies unanimity, $\varphi(a_k, a_k) = 1$. Then voter i can obtain a strictly higher probability on the set of alternatives at least as preferred to a_j under P_i (this set includes a_k by hypothesis) by putting a_k on top of her order. This contradicts the strategy-proofness of φ. Therefore, $a_j P_i a_k$ as required. ■

This completes the verification of the necessity part of Theorem.

In order to demonstrate sufficiency, let $\mathbb{D}$ be a single-peaked domain on a tree $G = \langle A, E\rangle$. We construct a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ that is strategy-proof, tops-only, ex-post efficient and satisfies the weak compromise property. We proceed as follows: in the first step, we use the idea in Chatterji et al. (2013) to construct a specific DSCF (see the proof of the sufficiency part of the Theorem in Chatterji et al. (2013)); in the second step, we consider randomization over such DSCFs.

For any set $B \subseteq A$, let $G(B)$ be the *minimal subgraph* of G that contains B as vertices. More formally, $G(B)$ is the unique graph that satisfies the following properties.

1. The set of vertices in $G(B)$ contains B.
2. Let $a_j, a_k \in B$. Graph $G(B)$ has an edge (a_j, a_k) only if (a_j, a_k) is an edge in G.
3. $G(B)$ is connected.
4. $a_k \in G(B)$ if and only if $a_k \in \langle a_r, a_j\rangle$ where $a_r, a_j \in B$.

Fix a profile $P \in \mathbb{D}^N$ and an alternative $a_k \in A$. Consider the graph $G(r_1(P))$. Suppose $a_k \notin G(r_1(P))$. Since G is a tree and contains no cycles, there exists a *unique* alternative in $G(r_1(P))$ that belongs to *every* path from a_k to any vertex in $G(r_1(P))$. Let this alternative be denoted by $a_{\beta(a_k,P)}$.[9] Then, define the alternative $a_{\pi(a_k,P)}$ as follows:

$$a_{\pi(a_k,P)} = \begin{cases} a_k & \text{if } a_k \in G(r_1(P)) \\ a_{\beta(a_k,P)} & \text{if } a_k \notin G(r_1(P)) \end{cases}$$

Consider Example 1. Suppose $N = \{1,2,3\}$. Let a_k be the alternative a_4 and let P be a profile such that $r_1(P) = \{a_1, a_2, a_3\}$. Then $G(r_1(P))$ is the graph consisting of the vertices $\{a_1, a_2, a_3\}$ and the edges (a_1, a_2) and (a_2, a_3). Then $a_{\pi(a_k,P)} = a_{\beta(a_4,P)} = a_2$. Further examples can be found in Chatterji et al. (2013).

For every $a_k \in A$, the RSCF $\phi^{a_k} : \mathbb{D}^N \to \Delta(A)$ is defined as follows: for all $P \in \mathbb{D}^N$, $\phi^{a_k}(P) = e_{\pi(a_k,P)}$. Evidently, ϕ^{a_k} is a DSCF. Its outcome at profile P is the "projection" of a_k on the minimal subgraph of G generated by the set of the first-ranked alternatives in P.

In the next step, we construct the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ as follows: for all $P \in \mathbb{D}^N$, $\varphi(P) = \sum_{a_k \in A} \lambda^{a_k} \phi^{a_k}(P)$, where $\lambda^{a_k} > 0$ for all $a_k \in A$ and $\sum_{a_k \in A} \lambda^{a_k} = 1$. The RSCF is obtained by choosing over the DSCFs ϕ^{a_k}, $k = 1, \ldots, m$, according to a fixed probability distribution where the probability of choosing each such DSCF is strictly positive. We claim that the φ satisfies all the required properties.

Lemma **7** *The RSCF φ is strategy-proof, tops-only and satisfies unanimity.*

Proof: According to Proposition 1 in Chatterji et al. (2013), a single-peaked domain is semi-single-peaked where every alternative can be taken to be a threshold in the definition of semi-single-peakedness. The sufficiency part of the Theorem in Chatterji et al. (2013) shows that for any threshold $a_k \in A$, ϕ^{a_k} is strategy-proof, tops-only and satisfies unanimity over a semi-single-peaked domain. Consequently, each ϕ^{a_k} is strategy-proof, tops-only and satisfies unanimity. Therefore, φ which is a convex combination of distinct unanimous, tops-only and strategy-proof RSCFs is also a unanimous, tops-only and strategy-proof RSCF.[10] ■

Lemma **8** *The RSCF φ is ex-post efficient.*

Proof: Suppose the Lemma is false, i.e., there exists $P \in \mathbb{D}^N$ and $a_j, a_k \in A$ such that $a_j P_i a_k$ for all $i \in I$ and $\varphi_k(P) > 0$. Evidently, $a_k \notin r_1(P)$. Since φ satisfies unanimity, $\varphi_k(P) > 0$ implies that $|r_1(P)| > 1$. Observe that $a_{\pi(a_t,P)} \in G(r_1(P))$ for all $a_t \in A$.

[9] It would have been more appropriate to write $a_{\beta(a_k, G(r_1(P)))}$ but we choose to suppress the dependence of this alternative on G for notational convenience.

[10] These arguments are routine and therefore omitted.

Hence, by construction of φ, if a_r is not included in the vertex set of $G(r_1(P))$, $\varphi_r(P) = 0$. Therefore, a_k belongs to the vertex set of $G(r_1(P))$.

Let $Ext\big(G(r_1(P))\big)$ denote the set of vertices in $G(r_1(P))$ with degree one, i.e., $a_t \in Ext\big(G(r_1(P))\big)$ if there exists a unique $a_s \in A$ such that (a_t, a_s) is an edge in $G(r_1(P))$. Observe that $Ext\big(G(r_1(P))\big) \subseteq r_1(P)$. (Suppose $a_t \in Ext\big(G(r_1(P))\big)$ but $a_t \notin r_1(P)$. Then a_t can be deleted as a vertex in $G(r_1(P))$ contradicting the assumption that $G(r_1(P))$ is minimal). In other words, the vertices at the ends of every maximal path in $G(r_1(P))$ must be some elements of $r_1(P)$.

It follows from the arguments in the two previous paragraphs that $a_k \in G(r_1(P)) \backslash Ext\big(G(r_1(P))\big)$. Consequently, there exist $i, i' \in I$ such that $r_1(P_i) \neq r_1(P'_i)$, $a_k \in \langle r_1(P_i), r_1(P_{i'})\rangle$ and $a_k \neq r_1(P_i), r_1(P_{i'})$. Let a_t be the projection of a_j on the interval $\langle r_1(P_i), r_1(P_{i'})\rangle$. By assumption, $a_t \in \langle r_1(P_i), r_1(P_{i'})\rangle$. Hence, either $a_k \in \langle r_1(P_i), a_t\rangle$ or $a_k \in \langle r_1(P_{i'}), a_k\rangle$ must hold. Therefore either $a_k \in \langle r_1(P_i), a_j\rangle$ or $a_k \in \langle r_1(P_{i'}), a_j\rangle$ must hold, i.e., either $a_k P_i a_j$ or $a_k P_{i'} a_j$ must hold by single-peakedness of $\mathbb{D}$. We have a contradiction to our initial hypothesis that $a_j P_i a_k$ for all $i \in I$. Therefore, φ is ex-post efficient. ■

Lemma **9** *The RSCF φ satisfies the weak compromise property.*

Proof: Pick an arbitrary $P \in \mathbb{D}^N$ such that $C(P) \neq \emptyset$. Clearly $|r_1(P)| \geq 2$. For notational convenience, let $\bar{A} = r_1(P)$ and $\bar{G} = G\big(r_1(P)\big)$. Pick $a_k \in C(P)$ - note that by definition, $a_k \notin \bar{A}$. We proceed via several claims.

Claim 1: $\bar{A} = Ext(\bar{G})$.

We have shown in the proof of the previous lemma that $Ext(\bar{G}) \subseteq \bar{A}$. Suppose that $Ext(\bar{G})$ is a strict subset of $\bar{A}$. In particular, suppose $a_t \notin Ext(\bar{G})$ but $a_t = r_1(P_{i''})$ for some $i'' \in I$. Since $a_t \notin Ext(\bar{G}) \subset \bar{A}$, there exists $i, i' \in I$ such that $a_t \in \langle r_1(P_i), r_1(P_{i'})\rangle$ and $a_t \neq r_1(P_i), r_1(P_{i'})$. It must therefore be the case that either $a_t \in \langle r_1(P_i), a_k\rangle$ or $a_t \in \langle r_1(P_{i'}), a_k\rangle$, holds. The single-peakedness of P_i and $P_{i'}$ implies that either $a_t P_i a_k$ or $a_t P_{i'} a_k$ holds. Since $a_t = r_1(P_{i''})$, we have a contradiction to the assumption that $a_k \in C(P)$. This completes the verification of the claim.

Let $a_{\pi(a_k,P)} = a_r$. Evidently, a_r is contained in the vertex set of $\bar{G}$. The construction of φ makes it clear that $\varphi_r(P) > 0$. In order to show that φ satisfies the weak compromise property, it will therefore suffice to show that $a_r \in C(P)$. If $a_r = a_k$, this follows immediately. Assume therefore that $a_r \neq a_k$.

Claim 2: $a_r \notin Ext(\bar{G})$.

Suppose not, i.e., $a_r \in Ext(\bar{G})$. By Claim 1, there exists $i \in I$ such that $r_1(P_i) = a_r$. Since $|\bar{A}| \geq 2$, there exists another voter say i' such that $r_1(P_{i'}) = a_s \neq a_r$. Note that $a_r \in \langle a_s, a_k\rangle$

by the definition of a_r. The single-peakedness of $P_{i'}$ implies $a_r P_{i'} a_k$ which contradicts the assumption that $a_k \in C(P)$. This completes the verification of the claim.

Claim 3: $a_r \in C(P)$.

By the definition of a_r, we must have $a_r \in \langle a_j, a_k \rangle$ for all $a_j \in \bar{A}$. Using Claims 1 and 2, it follows that $a_r \notin \bar{A}$. The single-peakedness of P_i implies $a_r P_i a_k$ for all $i \in I$. Since $a_k \in C(P)$ by assumption, it must be the case that $a_r \in C(P)$. This completes the verification of the claim and completes the proof of the Lemma. ■

This completes the proof of the sufficiency part of the Theorem. ■

3.1 Discussion: The Compromise Property

A natural question is whether the weak compromise property can be strengthened to its strong counterpart. We claim that doing so has implications for the underlying tree with respect to which single-peakedness is defined.

Consider domain $\bar{\mathbb{D}}$ in Example 1 and suppose we look for a strategy-proof, unanimous, tops-only RSCF satisfying the strong compromise property in a two-voter society. Since $a_4 \in C(P_1, P_6)$, we must have $\varphi_4(P_1, P_6) > 0$. Strategy-proofness implies $\varphi_4(P_1, P_6) = \varphi_4(P_3, P_6) = \varphi_4(P_3, P_5)$ (referring to Lemma 1). Hence $\varphi_4(P_3, P_5) > 0$ which contradicts the assumption that φ satisfies unanimity. Hence $\bar{\mathbb{D}}$ (or any of it's supersets) does not admit a strategy-proof, unanimous and tops-only RSCF that also satisfies the strong unanimity property. Note that the weak compromise property can be satisfied in the same domains in conjunction with the other axioms because $a_2 \in C(P_1, P_6)$. Requiring $\varphi_2(P_1, P_6) > 0$ is consistent with the other axioms.

On the other hand, consider domain $\hat{\mathbb{D}}$ in Example 2. We claim that this domain does admit strategy-proof, unanimous, tops-only RSCFs satisfying the strong compromise property. For instance, in profile (P_1, P_8), both alternatives a_2 and a_3 are compromises. Both a_2 and a_3 lie on the path $\langle a_1, a_4 \rangle$ so that Lemma 4 implies that a strategy-proof, unanimous, tops-only RSCF φ on this domain must satisfy $\varphi_2(P_1, P_8), \varphi_3(P_1, P_8) > 0$.[11]

These considerations suggest that the line structure of the underlying tree may be important if the strong compromise property has to be satisfied. The next section shows that a richness condition of preferences together with the strong compromise property (and the other axioms) characterizes single-peakedness on a line.

We conclude this section by the following remark. One might be concerned that some alternative which is not a peak of any voter in a profile $P \in \mathbb{D}^N$ and gets positive probability by the weak compromise property is unanimously ranked poorly in another tops-equivalent

[11] A formal argument will be provided in the next section.

profile $P' \in \mathbb{D}^N$, i.e., $r_1(P'_i) = r_1(P_i)$ for all $i \in I$. Our axioms on RSCFs as specified in Theorem 1 rule out this concern by ensuring that every alternative that belongs to the support of $\varphi(P)$ and is not the peak of some voter, either belongs to $C(P')$, or is ranked relatively high by every voter in P' in the following sense: if $a_j \notin r_1(P)$ and $\varphi_j(P) > 0$, then for each $i \in I$, $a_j P'_i a_k$ for some $a_k \in r_1(P')$.[12] In particular, if $|r_1(P)| = |r_1(P')| = 2$, $a_j \in C(P')$.

4 Single-Peakedness on A Line

We say that the domain $\mathbb{D}$ is *c-maximal* if there does not exist a superset of $\mathbb{D}$ that admits a strategy-proof, unanimous, tops-only RSCF satisfying the weak compromise property. Observe that c-maximality is defined with reference to the weak rather than the strong compromise property.

Theorem **2** *Every path-connected domain that admits a unanimous, tops-only, strategy-proof RSCF satisfying the strong compromise property and is c-maximal, is single-peaked on a line. Conversely, a single-peaked domain on a line admits an ex-post efficient, tops-only and strategy-proof RSCF satisfying the strong compromise property.*

Proof: We first show necessity. Let $\mathbb{D}$ be a path-connected domain. In addition, $\mathbb{D}$ admits a strategy-proof, tops-only and unanimous RSCF satisfying the strong compromise property. Finally, $\mathbb{D}$ is c-maximal, i.e., there does not exist a domain $\bar{\mathbb{D}} \supset \mathbb{D}$ that admits a strategy-proof, tops-only, unanimous RSCF satisfying the weak compromise property.

Since the strong compromise property implies the weak compromise property, Lemmas 1 - 6 continue to hold.[13] Therefore there exists a two-voter RSCF $\varphi : \mathbb{D}^2 \to \Delta(A)$ that is strategy-proof unanimous, tops-only, strategy-proof and satisfies the strong compromise property. Moreover, Theorem 1 implies that domain $\mathbb{D}$ is single-peaked on a tree G. The proof is going to be completed by showing that G is in fact, a line.

Suppose G not a line. Therefore there must exist a vertex with degree three, i.e., there exist $a_j, a_k, a_r, a_t \in A$ such that $(a_j, a_t), (a_k, a_t)$ and (a_r, a_t) are edges in G. By the construction of G, $a_j \approx a_t$, $a_k \approx a_t$, and $a_r \approx a_t$. Therefore $a_j \notin \langle a_k, a_r\rangle$, $a_k \notin \langle a_r, a_j\rangle$ and $a_r \notin \langle a_j, a_k\rangle$. Consequently, Lemma 4 implies $\varphi_j(a_k, a_r) = \varphi_k(a_r, a_j) = \varphi_r(a_j, a_k) = 0$.

Since $a_r \notin \langle a_j, a_k\rangle$, there exists a single-peaked order $\bar{P}_i$ on G such that $r_1(\bar{P}_i) = a_k$ and $a_r \bar{P}_i a_j$. Similarly, since $a_j \notin \langle a_r, a_k\rangle$, there exists a single-peaked order P'_i on G such

[12] The necessity part of Theorem 1 shows that the domain $\mathbb{D}$ is single-peaked on a tree G. Then, it is true that for every $P \in \mathbb{D}^N$ and $a_j \in A$, if $a_j \notin r_1(P)$ and a_j is included in the vertex set of $G(r_1(P))$, then for each $i \in I$, $a_j P_i a_k$ for some $a_k \in r_1(P)$. Consider an arbitrary RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfying axioms specified in Theorem 1. It is easy to verify that the support of $\varphi(P)$ is included in the vertex set of $G(r_1(P))$ for all $P \in \mathbb{D}^N$, i.e., for all $a_j \in A$, if the vertex set of $G(r_1(P))$ does not contain it, then $\phi_j(P) = 0$.

[13] Applying the same verification, it is easy to extend the argument in Lemma 2 from the weak compromise property to the strong compromise property.

that $r_1(P_i') = a_r$ and $a_k P_i' a_j$. The sufficiency part of Theorem 1 shows that *any* single-peaked domain on G admits a strategy-proof, tops-only, unanimous RSCF satisfying the weak compromise property. Since $\mathbb{D}$ is c-maximal, we must have $\bar{P}_i, P_i' \in \mathbb{D}$.

Pick an arbitrary $P_i \in \mathbb{D}$ with $r_1(P_i) = a_j$. Either $a_r P_i a_k$ or $a_k P_i a_r$ must hold. Suppose $a_r P_i a_k$. Since $a_r \in C(P_i, \bar{P}_i)$, the strong compromise property requires $\varphi_r(P_i, \bar{P}_i) > 0$, i.e., $\varphi_r(a_j, a_k) > 0$. However, this contradicts our earlier conclusion that $\varphi_r(a_j, a_k) = 0$. If $a_k P_i a_r$, we have $a_k \in C(P_i, P_i')$ and $\varphi_k(a_j, a_r) > 0$ by the strong compromise property. On the other hand we have earlier shown that $\varphi_k(a_j, a_r) = 0$. We have another contradiction. Hence G is a line as required.

We now establish the sufficiency part of the Theorem. Let $\mathbb{D}$ be a single-peaked domain on the line G^L. We employ the same construction as in the proof of the sufficiency part of Theorem 1. Using the same arguments as before, it follows that the constructed RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is strategy-proof, tops-only and ex-post efficient. We conclude the proof by showing the φ satisfies the strong compromise property.

Let $P \in \mathbb{D}^N$ be a profile such that $C(P) \neq \emptyset$. Clearly $|r_1(P)| \geq 2$. Since G^L is a line, all voters' peaks must lie on a path, i.e., there exist voters i, i' such that $r_1(P_{i''}) \in \langle r_1(P_i), r_1(P_{i'})\rangle$ for all $i'' \in I$. Thus, $G(r_1(P)) = \langle r_1(P_i), r_1(P_{i'})\rangle$. Denote $r_1(P_i)$ and $r_1(P_{i'})$ by a_j and a_k respectively. Pick an arbitrary $a_t \in C(P)$. We claim that $a_t \in \langle a_j, a_k\rangle$ and $a_t \neq a_j, a_k$. The latter follows from the observation that no element of $C(P)$ can be a peak of any voter by definition. Suppose $a_t \notin \langle a_j, a_k\rangle$. Accordingly, either $a_k \in \langle a_j, a_t\rangle$ or $a_j \in \langle a_t, a_k\rangle$ must hold. In the former case, the single-peakedness of P_i implies $a_k P_i a_t$; in the latter, the single-peakedness of $P_{i'}$ implies $a_j P_{i'} a_t$. In either case, we have a contradiction to $a_t \in C(P)$. Therefore, $a_t \in \langle a_j, a_k\rangle = G(r_1(P))$ and $a_{\pi(a_t,P)} = a_t$. By construction, $\varphi(P) = \sum_{a_k \in A} \lambda^{a_k} e_{\pi(a_k,P)}$ implies that $\varphi_t(P) \geq \lambda^{a_t} > 0$. Hence, φ satisfies the strong compromise property as required. ■

5 Is It Necessary to Consider Random Social Choice Functions?

A natural question is whether our characterization results can be obtained by only considering deterministic social choice functions. We show below that this is not possible.

Example **3** Let $A = \{a_1, a_2, a_3, a_4\}$ and let $\hat{\mathbb{D}}$ be the domain specified in Example 2, i.e., it is the largest single-peaked domain on the line G^L. Let $I = \{1, 2\}$. Consider a DSCF $\phi : \hat{\mathbb{D}}^2 \to \Delta(A)$. The counterpart of the weak compromise property in this setting would require ϕ to pick a compromise alternative at every profile where such an alternative exists, i.e., for every profile P, $[C(P) \neq \emptyset] \Rightarrow [\phi(P) = e_j$ for some $a_j \in C(P)]$. We claim that there does not exist a strategy-proof DSCF satisfying the deterministic weak compromise property.

Let $P_1, P_2, \bar{P}_1, \bar{P}_2 \in \hat{\mathbb{D}}$ be the orders described below.

P_1	P_2	$\bar{P}_1$	$\bar{P}_2$
a_1	a_3	a_2	a_4
a_2	a_2	a_3	a_3
a_3	a_1	a_4	a_2
a_4	a_4	a_1	a_1

The deterministic weak compromise property requires $\phi(P_1, P_2) = e_2$ and $\phi(\bar{P}_1, \bar{P}_2) = e_3$. Also, $\phi(P_1, P_2) = e_2$ and strategy-proofness implies $\phi(\bar{P}_1, P_2) = e_2$. Similarly $\phi(\bar{P}_1, \bar{P}_2) = e_3$ and strategy-proofness implies $\phi(\bar{P}_1, P_2) = e_3$, leading to a contradiction. □

The arguments in Example 3 can be generalized easily.

Proposition 1 *Let G be an arbitrary tree containing a path with at least four vertices. Let $\bar{\mathbb{D}}^G$ be the largest collection of single-peaked orders on G. Then there does not exist a strategy-proof DSCF $\phi : [\bar{\mathbb{D}}^G]^N \to \Delta(A)$ satisfying the deterministic weak compromise property.*

We omit the proof of this Proposition. We show by means of an example that the Proposition does not hold if the maximal path length in a tree is three.

Example 4 Let G^S be a star tree with a_1 as the hub (for instance, G^T in Figure 1 is a star tree and a_2 is the hub accordingly). Let $\bar{\mathbb{D}}^S$ be the largest single-peaked domain on G^S. Define a DSCF $\phi : [\bar{\mathbb{D}}^S]^2 \to \Delta(A)$ as follows: for all profiles (a_i, a_j),

$$\phi(a_i, a_j) = \begin{cases} e_i & \text{if } a_j = a_i; \\ e_1 & \text{otherwise.} \end{cases}$$

All single-peaked preferences on G^S have the feature that a_1 is ranked either first or second. The compromise set is non-empty at all profiles where the voters first-ranked alternatives are distinct from each other and from a_1. Thus, a_1 must be a compromise alternative at these profiles. Since a_1 is the outcome of ϕ at these profiles, the deterministic weak compromise property is satisfied. It is also strategy-proof, tops-only and unanimous. □

6 Conclusion

We have characterized domains of single-peaked preferences as the only domains that admit "well-behaved" random social choice functions.

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